$$\frac{x-t}{\sqrt{1-2xt+t^2}} = (1-2xt+t^2)\sum_{n=0}^{\infty} n P_n(x) t^{n-1} \quad (1)$$

usando que: $\displaystyle\sum_{n=0}^{\infty} P_n(x) t^n = \frac{1}{\sqrt{1-2xt+t^2}}$ se tiene que (1) sería

$$(x-t)\sum_{n=0}^{\infty} P_n(x) t^n = \sum_{n=0}^{\infty} n P_n(x) t^{n-1} - \sum_{n=0}^{\infty} 2xn P_n(x) t^n + \sum_{n=0}^{\infty} n P_n(x) t^{n+1}$$

$$\sum_{n=0}^{\infty} x P_n(x) t^n - \sum_{n=0}^{\infty} P_n(x) t^{n+1} = \sum_{n=0}^{\infty} n P_n(x) t^{n-1} - \sum_{n=0}^{\infty} 2xn P_n(x) t^n + \sum_{n=0}^{\infty} n P_n(x) t^{n+1}$$

Acomodando:

$$\sum_{n=0}^{\infty} n P_n(x) t^{n-1} = \sum_{n=0}^{\infty} (2n+1) x P_n(x) t^n - \sum_{n=0}^{\infty} (1+n) P_n(x) t^{n+1}$$

introduciendo $m = n-1 \Rightarrow n = m+1$ y $k = n+1 \Rightarrow n = k-1$

$$\sum_{m=-1}^{\infty} (m+1) P_{m+1}(x) t^m = \sum_{n=0}^{\infty} (2n+1) x P_n(x) t^n - \sum_{k=1}^{\infty} k P_{k-1}(x) t^k$$

nótese que $m=-1$ da cero, por ende podemos empezar en $m=0$, de manera similar, k puede de empezar en $k=0$ pq' sería un cero. Por último m, k son labels y podemos llamarlos n

$$0+\sum_{n=0}^{\infty} (n+1) P_{n+1}(x) t^n = \sum_{n=0}^{\infty} (2n+1) x P_n(x) t^n - \sum_{n=0}^{\infty} n P_{n-1}(x) t^n - 0$$

Agrupando tendremos

$$\sum_{n=0}^{\infty} (n+1) P_{n+1}(x) t^n = \sum_{n=0}^{\infty} \left[(2n+1) x P_n(x) - n P_{n-1}(x)\right] t^n$$

Igualando potencias de t, tendremos que

$$\boxed{(n+1) P_{n+1}(x) = (2n+1) x P_n(x) - n P_{n-1}(x)} \quad (A)$$

Partiendo de:

$$t\sum_{n=0}^{\infty} P_n(x)\, t^n = (1-2xt+t^2)\sum_{n=0}^{\infty} P_n'(x)\, t^n$$

$$\sum_{n=0}^{\infty} P_n(x)\,t^{n+1} = \sum_{n=0}^{\infty} P_n'(x)\,t^n - \sum_{n=0}^{\infty} 2x P_n'(x)\,t^{n+1} + \sum_{n=0}^{\infty} P_n'(x)\,t^{n+2}$$

$$\sum_{n=0}^{\infty} P_n'(x)\,t^n + \sum_{n=0}^{\infty} P_n'(x)\,t^{n+2} = \sum_{n=0}^{\infty} \left(P_n(x)+2xP_n'(x)\right)t^{n+1}$$

$$\sum_{n=0}^{\infty} P_n'(x)\,t^n + \sum_{n=2}^{\infty} P_{n-2}'\,t^n = \sum_{n=1}^{\infty} \left[P_{n-1}(x)+2xP_{n-1}'(x)\right]t^n$$

definiendo $m=n-1$ tendremos

$$\sum_{m=-1}^{\infty} P_{m+1}'(x)\,t^{m+1} + \sum_{m=1}^{\infty} P_{m-1}'\,t^{m+1} = \sum_{m=0}^{\infty} \left[P_m(x)+2xP_m'(x)\right]t^{m+1}$$

Expandiendo algunos ordenes y recordando que $P_0(x)=1$, y $P_1(x)=x$

$$\underbrace{P_0'(x)\,t^0}_{0} + \underbrace{P_1'(x)\,t}_{t} + \sum_{m=1}^{\infty} \left(P_{m+1}'(x)+P_{m-1}'(x)\right)t^{m+1} = \overset{t}{\cancel{P_0(x)\,t}} + \overset{0}{\cancel{2xP_0'(x)}}\,t + \sum_{n=1}^{\infty} \left[P_m(x)+2xP_m'(x)\right]t^{m+1}$$

Simplificando las t, tendremos

$$\sum_{m=1}^{\infty} \left(P_{m+1}'(x)+P_{m-1}'(x)\right)t^{m+1} = \sum_{m=1}^{\infty} \left[P_m(x)+2xP_n'(x)\right]t^{m+1}$$

lo que igualando en ordenes de t conduce a:

$$\boxed{P_{n+1}'(x)+P_{n-1}'(x) = P_n(x)+2xP_n'(x)}$$ (B) donde $n=1,2,\ldots$

Para continuar derivemos la expresión (A) con respecto de x.

$$(n+1)P_{n+1}'(x) = (2n+1)\left[P_n(x)+xP_n'(x)\right] - nP_{n-1}'(x)$$

$$\boxed{(n+1)P_{n+1}'(x)+nP_{n-1}'(x) = (2n+1)P_n(x)+(2n+1)xP_n'(x)}$$ (C)

Realicemos las sgtes operaciones $2C-(2n+1)B$

$$2(n+1)P'_{n+1}(x) + 2nP'_{n-1}(x) = 2(2n+1)P_n(x) + 2(2n+1)xP'_n(x)$$

$$(2n+1)P'_{n+1}(x) + (2n+1)P'_{n-1}(x) = (2n+1)P_n(x) + 2(2n+1)xP'_n(x)$$ restando

$$\boxed{P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n} \quad \text{D}$$

Ahora realicemos $(B-D)/2$ y $(B+D)/2$ veamos

$$P'_{n+1}(x) + P'_{n-1}(x) = P_n(x) + 2xP'_n(x)$$

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$$

$$\text{E} \quad \boxed{P'_{n-1}(x) = -nP_n(x) + xP'_n(x)}$$ ← Restando y dividiendo entre 2

$$\text{F} \quad \boxed{P'_{n+1}(x) = (n+1)P_n(x) + xP'_n(x)}$$ ← Sumando y dividiendo entre 2

Reemplazando en F $m = n+1$ tendremos

$$\text{F}_2 \quad \boxed{P'_m(x) = mP_{m-1}(x) + xP'_{m-1}(x)}$$

donde $m = 2, 3, ..$ Acomodando

$$\begin{cases} mP_m(x) = xP'_m(x) - P'_{m-1}(x) \\ mP_{m-1}(x) = P'_m(x) - xP'_{m-1}(x) \end{cases}$$

← notar que $n \to m$ o lo que es lo mismo restringimos a $n = 2, 3$

$$m\begin{pmatrix} P_m(x) \\ P_{m-1}(x) \end{pmatrix} = \begin{pmatrix} x & -1 \\ 1 & -x \end{pmatrix}\begin{pmatrix} P'_m(x) \\ P'_{m-1}(x) \end{pmatrix} \Rightarrow \begin{pmatrix} P'_n(x) \\ P'_{m-1}(x) \end{pmatrix} = m\underbrace{\begin{pmatrix} x & -1 \\ 1 & -x \end{pmatrix}^{-1}}_{\frac{1}{x^2-1}\begin{pmatrix} x & -1 \\ 1 & -x \end{pmatrix}}\begin{pmatrix} P_m(x) \\ P_{m-1}(x) \end{pmatrix}$$

Tendremos entonces que:

$$\begin{pmatrix} P'_m(x) \\ P'_{m-1}(x) \end{pmatrix} = \frac{m}{x^2-1}\begin{pmatrix} x & -1 \\ 1 & -x \end{pmatrix}\begin{pmatrix} P_m(x) \\ P_{m-1}(x) \end{pmatrix}$$

$$\Rightarrow \boxed{P'_n(x) = \frac{nP_{n-1}(x) - nxP_n(x)}{1-x^2}} \quad n = 2, 3, \ldots$$