Q2. $\min \frac{1}{2}\|w\|^2$ sub to. $y^t(wx^t) \geqslant 1,\ t=1,\dots,N$

(a) bias $b=0$,

$x^{(1)} = (1,1)^T$, $x^{(2)} = (1,0)^T$;

$y^{(1)} = 1$, $y^{(2)} = -1$;

$w = ?$ $r = ?$

transfer to dual problem:

$$\min_{\alpha} \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j y_i y_j (x_i \cdot x_j) - \sum_{t=1}^{N} a^i$$

$$= \frac{1}{2}(2a_1^2 + a_2^2 - 2a_1 a_2) - a_1 a_2$$

s.t. $\alpha_i \geq 0$, $i=1,2$.

$$\min_{a} \quad a_1^2 + \frac{1}{2}a_2^2 - a_1 a_2 - a_1 - a_2 = A$$

Let $\frac{dA}{da_1} = 0$ $\frac{dA}{da_2} = 0$

$a_1 = 2$, $a_2 = 3$;

$a_1 > 0$, $a_2 > 0$, condition fulfilled

$w = \sum a_i y_i x_i = (-1, 2)$.

$r = \frac{1}{\|w\|} = \frac{1}{\sqrt{5}}$

Q2

(2) bias b can be non-zero:

transfer to dual problem:

$$\min_{\alpha} \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} a_i a_j y_i y_j (x_i \cdot x_j) - \sum_{i=1}^{N} a^i$$

$$= \frac{1}{2}(2a_1^2 + a_2^2 - 2a_1 a_2) - a_1 a_2$$

s.t. $\alpha_i \geq 0$, $i = 1,2$. $a_1 - a_2 = 0$

$$\min_{a} \quad \frac{1}{2}a_1^2 - 2a = B$$

$$\frac{dB}{da_1} = 0 \quad \frac{dB}{da_2} = 0$$

$a_1 = 2 = a_2 > 0$, condition satisfied

$$\omega = \sum a_i y_i x_i = (0, 2)$$

for $a_j > 0$:

$$b = y_j - \sum_{i=1}^{N} a_i y_i (x_i x_j) = -1$$

$$r = \frac{1}{\|\omega\|} = 1$$

Q3. 1. $k(x,z) = k_1(x,z)k_2(x,z)$

$k_1(x,y) = a(x)^T a(y)$, where $a(z) = [a_1(z) \dots a_m(z)]$

$k_2(x,y) = b(x)^T b(y)$, where $b(z) = [b_1(z) \dots b_n(z)]$

where $a(z), b(z)$ projects an n-d, m-d vector respectively.

$k(x,z) = k_1(x,z)k_2(x,z)$

$= a(x)^T a(y) \cdot b(x)^T b(y)$

$= \sum_{m=1}^{M} \sum_{n=1}^{N} [a_m(x) b_n(x)]^T [a_m(y) b_n(x)]$

det $c_{mn}(z) = a_m(z) b_n(z)$

where $c(z)$ projects an $m \times n$ d vector.

Hence, $k(x,z)$ is a kernel function.

2. $k(x,z) = a(k_1(x,z) + bk_2(x,z))$, $a, b > 0, \in R$

By Mercer's Theorem, if K is semi-definite, positive, symmetric continuous function,

there exists $\{\phi_i\}_{i \in N}$, $\{\lambda_i\}_{i \in N}$ that $k(x,z) = \sum_{i=1}^{\infty} \lambda_i \phi_i(x) \phi_i(z)$, a kernel func.

Since $k_1(x,z)$, $k_2(x,z)$ are kernel func, k_1, k_2 are symmetric, positive, semi definitive.

Now we need to prove k is also semi-definitive, positive, symmetric.

Since k_1, k_2 is symetric, a, b is constant

ak_1, bk_2 is symetric.

By theorem of symmetric matrix, the sum of 2 symetric matrix is still symetric:

proof: $(A+B)^T = A^T + B^T = (A+B)$

Hence: $k(x,z)$ is symmetric.

since, a, b are ~~real~~ positive real, $ak_1(x,z)$, $bk_2(x,z)$ also semi-positive

hence $h(a\ k_1(x,z))h^T + hb\ k_2(x,z)h^T$

$= h(k_1(x,z) + k_2(x,z))h^T$.

Hence. $a\ k_1(x,z) + b\ k_2(x,z) = k(x,z)$ positive semi-definitive

Hence. k is kernel func.

3. when $a=1$, $b=2$, $k_1 = k_2$

$k(x,z) = -k_2(x,z)$

k now has semi-negative kernel matrix

by counter example

k is not.

4. by kernel definition,

$k(x,z) = \phi(x)\phi(z)$, when $\phi(z): R^n \to R$,

if $f: R^n \to R$,

$k(x,z) = f(x)f(z)$, is a kernel