Gradualism Bellman Equation and Value Function

ECN 490

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Lobby

$$\max_{m^t, l^t} \sum_{t=1}^{\infty} \beta^{t-1} \left\{ A(m_t) F^{\alpha} \cdot l_t^{1-\alpha} \cdot P^W - l_t - \mu_t \right\} \quad \text{s.t.} \quad m_{t+1} = (1-\delta) m_t + \mu_t$$

So $\mu_t = m_{t+1} - (1-\delta) m_t$

Bellman Equation

$$V_l(m_t) = \max_{m_t, l_t} \left\{ A(m_t) F^{\alpha} \cdot (l_t)^{1-\alpha} \cdot P^W - l_t - (m_{t+1} - (1-\delta) m_t) + \beta V_l(m_{t+1}) \right\}$$

In the optimal stage $m_t = m_t^* = m_{t+1}^*$

Value Function

$$V_l(m_{t+1}^*) = \left\{ A(m_{t+1}^*) F^{\alpha} \cdot (l_t^*)^{1-\alpha} \cdot P^W - l_t^* - (m_{t+1}^* - (1-\delta) m_{t+1}^*) + \beta V_l(m_{t+1}^*) \right\}$$

Which implies that

$$V_l(m_{t+1}^*) = \frac{1}{1-\beta} \left[A(m_{t+1}^*) \cdot F^{\alpha} \cdot (l_t^*)^{1-\alpha} \cdot P^W - l_t^* - \delta(m_{t+1}^*) \right]$$

FOCs

$$\frac{\partial V_l(m_{t+1}^*)}{\partial m_{t+1}^*} = \frac{1}{1-\beta} \cdot \left[\frac{\partial A(m_{t+1}^*)}{\partial m_{t+1}^*} \cdot F^{\alpha} \cdot l_t^{*1-\alpha} \cdot P^W - \delta \right]$$

$$\frac{\partial V_l}{\partial m_t^*} = \frac{\partial A(m_t^*)}{\partial m_t^*} \cdot F^{\alpha} \cdot l_t^{*1-\alpha} \cdot P^W + \frac{\beta}{1-\beta} \cdot \left[\frac{\partial A(m_{t+1}^*)}{\partial m_{t+1}^*} \cdot F^{\alpha} \cdot l_t^{*1-\alpha} \cdot P^W - \delta \right] \cdot (1-\delta) = \delta$$

$$\frac{\partial V_L}{\partial l_t^*} = (1-\alpha) A(m_t^*) \cdot F^{\alpha} \cdot l^{-\alpha} \cdot P^W = 1$$

rearrange $l_t^{*1-\alpha}$: $(1-\delta) A(m_t^*) \cdot F^{\alpha} \cdot P^W = l_t^{*\alpha}$

square $\left(\frac{1-\alpha}{\alpha}\right)$ on both sides $= \left[(1-\delta) A(m_t^*) \cdot F^{\alpha} \cdot P^W \right]^{\frac{1-\alpha}{\alpha}} = l_t^{*1-\alpha}$

Euler Equation:

$$\frac{\partial A(m_t^*)}{\partial m_t^*} \cdot F^\alpha \cdot \left[(1-\delta)A(m_t^*) \cdot F^\alpha \cdot P^W\right]^{\frac{1-\alpha}{\alpha}} + \frac{\beta}{1-\beta} \cdot (1-\delta) \cdot \left[\frac{\partial A(m_{t+1}^*)}{\partial m_{t+1}^*} \cdot F^\alpha \cdot \left[(1-\delta)A(m_t^*) \cdot F^\alpha \cdot P^W\right]^{\frac{1-\alpha}{\alpha}} \cdot P^W - \delta\right] = \delta \quad (1)$$

$t =$ this period

$t + 1 =$ next period

$\beta =$ discount factor

$\delta =$ deprecation factor