Gradualism Bellman Equation and Value Function

ECN 490

April 16, 2018

Lobby

$$\max_{m^t, l^t} \sum_{t=1}^{\infty} \beta^{t-1} \left\{ A(m_t) F^{\alpha} \cdot l_t^{1-\alpha} \cdot P^W - l_t - (1-\delta)\mu_t \right\} \quad \text{s.t.} \quad m_{t+1} = m_t + \mu_t$$

So $\mu_t = m_{t+1} - m_t$

Bellman Equation

$$V_l(m_t) = \max_{m_t, l_t} \left\{ A(m_t) F^{\alpha} \cdot (l_t)^{1-\alpha} \cdot P^W - l_t - (m_{t+1} - (1-\delta) m_t) + \beta V_l(m_{t+1}) \right\}$$

In the optimal stage $m_t = m_t^* = m_{t+1}^*$

Value Function

$$V_l(m_{t+1}^*) = \left\{ A(m_{t+1}^*) F^{\alpha} \cdot (l_t^*)^{1-\alpha} \cdot P^W - l_t^* - (m_{t+1}^* - (1-\delta) m_{t+1}^*) + \beta V_l(m_{t+1}^*) \right\}$$

Which implies that

$$V_l(m_{t+1}^*) = \frac{1}{1-\beta} [A(m_{t+1}^*) \cdot F^{\alpha} \cdot (l_t^*)^{1-\alpha} \cdot P^W - \delta(m_{t+1}^*)]$$

So

$$V_l(m_t) = A(m_t) F^{\alpha} \cdot (l_t)^{1-\alpha} \cdot P^W - l_t - (m_{t+1}^* - (1-\delta) m_t) + \frac{\beta}{1-\beta} [A(m_{t+1}^*) \cdot F^{\alpha} \cdot (l_t^*)^{1-\alpha} \cdot P^W - \delta(m_{t+1}^*)]$$

FOCs

$$\frac{\partial V_L}{\partial m_t} = \frac{\partial A(m_t)}{\partial m_t} \cdot F^{\alpha} \cdot l_t^{1-\alpha} \cdot P^W - \delta + \frac{\beta}{1-\beta} [A(m_{t+1}) \cdot F^{\alpha} \cdot (l_t)^{1-\alpha} \cdot P^W - \delta(m_{t+1})] = 1$$

$$\frac{\partial V_L}{\partial l_t} = (1-\alpha) \cdot A(m_t) \cdot F^{\alpha} \cdot l^{-\alpha} \cdot P^W = 1$$

$t =$ this period

$t+1 =$ next period

$\beta =$ discount factor

$\delta =$ deprecation factor