

Gradualism Bellman Equation and Value Function

ECN 490

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Lobby

$$\max_{m^t, l^t} \sum_{t=1}^{\infty} \beta^{t-1} \{A(m_t)F^\alpha \cdot l_t^{1-\alpha} \cdot P^W - l_t - \mu_t\} \quad \text{s.t.} \quad m_{t+1} = (1 - \delta)m_t + \mu_t$$

$$\text{So } \mu_t = m_{t+1} - (1 - \delta)m_t$$

Bellman Equation

$$V_l(m_t) = \max_{m_t, l_t} \{A(m_t)F^\alpha \cdot (l_t)^{1-\alpha} \cdot P^W - l_t - (m_{t+1} - (1 - \delta)m_t) + \beta V_l(m_{t+1})\}$$

$$\text{In the optimal stage } m_t = m_t^* = m_{t+1}^*$$

Value Function

$$V_l(m_{t+1}^*) = \{A(m_{t+1}^*)F^\alpha \cdot (l_t^*)^{1-\alpha} \cdot P^W - l_t^* - (m_{t+1}^* - (1 - \delta)m_{t+1}^*) + \beta V_l(m_{t+1}^*)\}$$

Which implies that

$$V_l(m_{t+1}^*) = \frac{1}{1 - \beta} [A(m_{t+1}^*) \cdot F^\alpha \cdot (l_t^*)^{1-\alpha} \cdot P^W - l_t^* - \delta(m_{t+1}^*)]$$

FOCs

$$\frac{\partial V_l(m_{t+1}^*)}{\partial m_{t+1}^*} = \frac{1}{1 - \beta} \cdot \left[\frac{\partial A(m_{t+1}^*)}{\partial m_{t+1}^*} \cdot F^\alpha \cdot l_t^{*1-\alpha} \cdot P^W - \delta \right]$$

$$\frac{\partial V_l}{\partial m_t^*} = \frac{\partial A(m_t^*)}{\partial m_t^*} \cdot F^\alpha \cdot l_t^{*1-\alpha} \cdot P^W + \frac{\beta}{1 - \beta} \cdot \left[\frac{\partial A(m_{t+1}^*)}{\partial m_{t+1}^*} \cdot F^\alpha \cdot l_t^{*1-\alpha} \cdot P^W - \delta \right] \cdot (1 - \delta) = \delta$$

$$\frac{\partial V_L}{\partial l_t^*} = (1 - \alpha)A(m_t^*) \cdot F^\alpha \cdot l^{-\alpha} \cdot P^W = 1$$

$$\text{rearrange } l_t^{*1-\alpha} : (1 - \delta)A(m_t^*) \cdot F^\alpha \cdot P^W = l_t^{*\alpha}$$

$$\text{square } \left(\frac{1 - \alpha}{\alpha} \right) \text{ on both sides} = [(1 - \delta)A(m_t^*) \cdot F^\alpha \cdot P^W]^{\frac{1-\alpha}{\alpha}} = l_t^{*1-\alpha}$$

EulerEquation

$$\frac{\partial A(m_t^*)}{\partial m_t^*} \cdot F^\alpha \cdot [(1 - \delta)A(m_t^*) \cdot F^\alpha \cdot P^W]^{\frac{1-\alpha}{\alpha}} + \frac{\beta}{1 - \beta} \cdot (1 - \delta) \cdot \left[\frac{\partial A(m_{t+1}^*)}{\partial m_{t+1}^*} \cdot F^\alpha \cdot \left(\frac{1 - \alpha}{\alpha} \right) \cdot [(1 - \delta)A(m_{t+1}^*) \cdot F^\alpha \cdot P^W]^{\frac{1-\alpha}{\alpha}} \right]$$

t = this period

$t + 1$ = next period

β = discount factor

δ = deprecation factor