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ABSTRACT

Moment equations that model plasma transport require an ansatz distribution function to close the system of equations. The resulting transport is sensitive to the specific closure used, and several options have been proposed in the literature. Two different 8-moment distribution functions can be generalized to form a single-parameter family of distribution functions. The transport coefficients resulting from this generalized distribution function can be expressed in terms of this free parameter. This provides the flexibility of matching the 8-moment model to some validating result at a given magnetization value, such as Braginskii's transport, or the more recent results of Davies *et al.* [Phys. Plasma, **28**, 012305 (2021)]. This process can be thought of as a solution for the 8-moment distribution function that matches the value of a transport coefficient given by a Chapman–Enskog expansion while retaining the improved physical properties, such as finite propagation speeds and time dependence, which belong to the hyperbolic moment models. Since the presented generalized distribution function only has a single free parameter, only a single transport coefficient can be matched at a time. However, this generalization process may be extended to provide multiple free parameters. The focus of this Brief Communication is on the dramatically improved thermal conductivity of the proposed model compared to the two base moment models.

Prior work in Hamilton and Seyler presented the transport capabilities of 8-moment plasma models based on the distribution functions of Grad and Killie *et al.*[1–3] Those results showed that moment models can be used to give reasonably accurate transport coefficients compared to Chapman–Enskog expansion models such as Braginskii.[4] Among the various collisional transport effects considered, the electron's thermal conductivity was the most sensitive to the choice of distribution function closure used. This Brief Communication generalizes those prior results and the moment closures of Grad and Killie *et al.* into a single-parameter family of distribution functions. This free parameter can tune the new flexible distribution function to more accurately reproduce the values of transport coefficients given by the Chapman–Enskog expansion models. This analysis suggests that a more accurate moment model closure can be obtained by taking a specific linear combination of the 8-moment distribution functions.

An in-depth introduction to moment models and their application to plasma transport is given by Hamilton and Seyler and the thesis of Hamilton.[5] The advantages of using moment models over Chapman–Enskog models include their finite propagation speeds, explicit time dependence, and lower dependence on requiring high collisionality. However, moment models have drawbacks as well, such as possessing nonphysical sub-shock solutions at low collisionality and being prone to instability at a large Mach number.[6–8] These moment models are applied to the Boltzmann–Fokker–Planck equation for weakly coupled plasmas. Deriving the moment equations, thus, involves solving the Fokker–Planck collision integral using the proposed generalized distribution function. This results in a hyperbolic partial differential equation for each moment, forming a closed system of equations.

The work done in Hamilton[5] summarizes all of the resulting transport coefficients in terms of the free parameter and plots them for all magnetizations compared to Chapman–Enskog models such as Braginskii, Epperlein and Haines, and Davies *et al.*[4,9,10] The notation used here will follow that of Hamilton and Seyler which this work intimately builds on.[1]

The moment model's closure scheme is based on Grad, where each species' distribution function is expanded in the number of moments desired. An 8-moment expansion would model a plasma's density n_α, velocity vector with three components $\mathbf{u}_\alpha$, energy density ε_α, and heat flux vector with three components $\mathbf{q}_\alpha$, for each species α.

This work builds off of two possible distribution closures for these moments.

Grad's distribution function,

$$f_{G,\alpha} = f_{0,\alpha}\left[1 - \frac{m_\alpha}{T_\alpha P_\alpha}\left(1 - \frac{m_\alpha w_\alpha^2}{5T_\alpha}\right)\mathbf{q}_\alpha \cdot \mathbf{w}_\alpha\right], \tag{1}$$

is cubic in velocity $\mathbf{w}_\alpha$, where f_0 is the equilibrium Maxwellian distribution, m_α is the species' mass, T_α is the temperature, and P_α is the pressure. The distribution function of Killie *et al.*,

$$f_{K,\alpha} = f_{0,\alpha}\left[1 - \frac{m_\alpha^2 w_\alpha^2}{5T_\alpha^2 P_\alpha}\left(1 - \frac{m_\alpha w_\alpha^2}{7T_\alpha}\right)\mathbf{q}_\alpha \cdot \mathbf{w}_\alpha\right], \tag{2}$$

is quintic in velocity and lacks a linear term, resulting in the transport being more sensitive to the high-velocity tails of the distribution than with Grad's $f_{G,\alpha}$ which is weighted more by the central core of the distribution. Killie *et al.* argued that this would better approximate the Coulomb cross section's velocity dependence and, hence, improve the transport of collisional plasmas.[3] As will be shown below, this leads to a significant difference in transport coefficients and plasma behavior, in general. The answer to which distribution is ultimately more accurate depends on the given plasma regime, a result that is not particularly satisfactory.

These two closures can, however, be described by a single-parameter family of distribution functions. By taking a linear combination of f_G and f_K, this generalized 8-moment distribution function can be written as

$$f_{\delta,\alpha} = f_{0,\alpha}\left[1 - \frac{m_\alpha}{T_\alpha P_\alpha}\left(\mathcal{A} - \frac{m_\alpha w_\alpha^2}{T_\alpha}\mathcal{B} - \frac{m_\alpha^2 w_\alpha^4}{T_\alpha^2}\mathcal{C}\right)\mathbf{q}_\alpha \cdot \mathbf{w}_\alpha\right], \tag{3}$$

where the coefficients $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are restricted such that the moments have consistent definitions in terms of f_α. This constrains the three coefficients to a single free parameter δ_0,

$$\mathcal{A} = 1 - \frac{7}{5}\delta_0, \quad \mathcal{B} = \frac{1}{5} - \frac{14}{25}\delta_0, \quad \mathcal{C} = \frac{\delta_0}{25}, \tag{4}$$

where $\delta_0 = 0$ corresponds to $f_\delta = f_G$ and $\delta_0 = \frac{5}{7}$ corresponds to $f_\delta = f_K$. The single-parameter family of the 8-moment distribution functions can, thus, be written as

$$f_{\delta,\alpha} = f_{0,\alpha}\left[1 - \frac{m_\alpha}{T_\alpha P_\alpha}\left(\left(1 - \frac{7}{5}\delta_0\right) - \frac{m_\alpha w_\alpha^2}{T_\alpha}\left(\frac{1}{5} - \frac{14}{25}\delta_0\right) - \frac{m_\alpha^2 w_\alpha^4}{T_\alpha^2}\frac{1}{25}\delta_0\right)\mathbf{q}_\alpha \cdot \mathbf{w}_\alpha\right]. \tag{5}$$

To obtain a corresponding closed 8-moment model for f_δ, the Fokker–Planck collision integral must be solved. Since the ansatz distribution function remains a polynomial multiplied by a Maxwellian, analytical solutions are obtainable, which is the primary motivation for using a Grad-like expansion. However, these collision integrals are very tedious, so only the final solutions will be presented below (for the interested, Chap. 2 of Burgers shows how to solve these integrals).[11]

The system of 8-moment equations is similar to that presented in Hamilton and Seyler.[1] Since the closure [the choice of distribution function given by Eq. (5)] only alters the collisional terms in these equations, the only terms dependent on the parameter δ_0 are the collision force $\mathbf{R}_\alpha = \mathbf{R}_{\mathbf{u},\alpha} + \mathbf{R}_{\mathbf{T},\alpha}$, the collision heat flux production or loss term $\dot{\mathbf{W}}_\alpha$, and the thermoelectric coefficient H. Following a similar derivation from the Fokker–Planck collision integral as shown in Chap. 2 of Burgers,[11] these terms are given by

$$\mathbf{R}_{\mathbf{u},\alpha} = \sum_\beta n_\alpha m_\alpha \nu_{\alpha\beta}(\mathbf{u}_\beta - \mathbf{u}_\alpha), \tag{6}$$

$$\mathbf{R}_{\mathbf{T},\alpha} = \sum_\beta \frac{3}{5}\frac{m_\alpha m_\beta \nu_{\alpha\beta}}{m_\alpha T_\beta + m_\beta T_\alpha}\left[\mathbf{q}_\alpha\left(1 - \delta_0\frac{m_\beta T_\alpha}{m_\alpha T_\beta + m_\beta T_\alpha}\right) - \mathbf{q}_\beta\frac{m_\alpha n_\alpha}{m_\beta n_\beta}\left(1 - \delta_0\frac{m_\alpha T_\beta}{m_\alpha T_\beta + m_\beta T_\alpha}\right)\right], \tag{7}$$

$$\dot{\mathbf{W}}_\alpha = -\sum_{\beta\neq\alpha}\nu_{\alpha\beta}\left(a_{\alpha\beta}\mathbf{q}_\alpha - b_{\alpha\beta}\frac{m_\alpha n_\alpha}{m_\beta n_\beta}\mathbf{q}_\beta + c_{\alpha\beta}P_\alpha(\mathbf{u}_\alpha - \mathbf{u}_\beta)\right) - \frac{4}{5}\left(1 - \frac{3}{5}\delta_0\right)\nu_{\alpha\alpha}\mathbf{q}_\alpha, \tag{8}$$

$$H = \frac{3}{5}\frac{m_e \nu_{ei}}{P_e e}\left(1 - \delta_0\frac{m_i T_e}{m_e T_i + m_i T_e}\right), \tag{9}$$

with the coefficients in Eq. (8) given by

$$a_{ei} = -\frac{1}{5}\left(1 - \frac{3}{5}\delta_0\right), \quad a_{ie} = 3, \tag{10a}$$

$$b_{ei} = \frac{6}{5}, \quad b_{ie} = -\frac{3}{2}\frac{m_e}{m_i}(1 - \delta_0), \tag{10b}$$

$$c_{ei} = 1, \quad c_{ie} = \frac{5}{2}, \tag{10c}$$

and Maxwell–Ampère's Law and Faraday's Law are used to solve for $\mathbf{E}$ and $\mathbf{B}$, respectively.

In these solutions, the normalized relative velocity difference $V_J = \frac{|J|}{n_e e}\sqrt{\frac{1}{2}\frac{m_\alpha m_\beta}{m_\alpha T_\beta + m_\beta T_\alpha}}$ is assumed to be small to simplify the Fokker–Planck collision integral (in other words, the distribution functions of both species overlap significantly in phase space). In addition, the coefficients in Eq. (10) assume a small temperature difference between species $|T_e - T_i| \ll T_i$. The results of Hamilton and Seyler can be obtained by setting $\delta_0 = 0$ or $\delta_0 = \frac{5}{7}$ for Grad's or the distribution of Grad or Killie *et al.*, respectively, which verifies that the collision integral for f_δ was calculated correctly. This more general moment model has the advantage that the optimal δ_0 (and hence the optimal moment closure) can be solved by using a Chapman–Enskog solution such as Braginskii as a constraint.

Of course, other generalizations of the 8-moment distribution functions are possible, but the single-parameter family used here is sufficient to provide tunable transport coefficients, as will be shown below. However, the parameter δ_0 is restricted by the thermodynamic requirement of positive entropy production, which Appendix X of Hamilton derives as $-4.579 < \delta_0 < 0.937$ for an ionization of $Z = 1$.[5] Negative values of δ_0 are admissible but lead to loss of accuracy compared to the Chapman–Enskog expansion results if δ_0 is too negative.

The distribution function f_δ itself is not positive-definite. For any δ_0, the distribution given by Eq. (5) becomes negative at a critical velocity value that is inversely proportional to the heat flux. Thus, as the distribution function departs further from equilibrium and the heat flux increases, the region of the distribution's tail that is negative gets closer to the core. This is not a property unique to moment

TABLE I. Values for δ_0 that provide the same transport coefficients as Braginskii's or Epperlein and Haines's Chapman–Enskog results in the weakly magnetized limit $\chi_h = \omega_{c,e}\tau_{ei} \ll 1$ for a plasma with ionization $Z = 1$.[4,9] These values correspond to a specific moment model obtained from Eq. (5) that reproduces the Chapman–Enskog solution for each transport coefficient (electrical conductivity, thermal conductivity, and thermoelectricity, respectively).

Model used to solve for δ_0	$\sigma_\parallel$, $\sigma_\perp$	$\sigma_\wedge$	$\kappa_{e.\parallel}$, $\kappa_{e.\perp}$	$\kappa_{e.\wedge}$	$\kappa_{i.\parallel}$, $\kappa_{i.\perp}$	$\kappa_{i.\wedge}$	$\beta_\parallel$, $\beta_\perp$	$\beta_\wedge$
Braginskii	$\delta_0 = -0.1431$	0.7118	0.6250	0.7012	0.4536	0.5289	0.6281	0.5602
Epperlein and Haines	$\delta_0 = -0.4782$	0.5954	0.6307	0.7118			0.6546	0.6191

expansions, as distribution functions obtained from perturbative Chapman–Enskog expansions are also not positive-definite.[12] Note that the heat flux is bounded by its free-streaming limit, $|q| \le q_{fs} = P\sqrt{T/m}$, which is the approximate value it obtains when there are no collisions to randomize the particle directions. In a collisional regime, the heat flux is well below this limit. For Grad's distribution, for example, the root of the polynomial multiplying the Maxwellian in Eq. (1) has $w_{crit} \sim q^{-1/3}$. At $q = q_{fs}$, Grad's velocity distribution becomes negative at a critical velocity, $w_{crit} \approx -2.63\sqrt{T/m}$, which is well away from the core, and at lower values of the heat flux, this critical value will be much further away. Other choices of δ_0 affect the critical velocity value, but the distribution function remains positive near the core for all $|q| \le q_{fs}$.

The transport coefficients for electrical resistivity, electron and ion thermal conductivity, and thermoelectricity can be derived from the moment equations using the same process shown in Hamilton and Seyler.[1] These results are summarized in terms of the free parameter δ_0 in Chap. 3 of the thesis of Hamilton.[5] A summary of the values for δ_0 that correspond to various Chapman–Enskog results in the weakly magnetized limit is provided in Table I.

In this Brief Communication, the results for the electron's thermal conductivity will be highlighted. Defined as the heat flux's linear response to a temperature gradient in the local rest frame,

$$\mathbf{q}_e = -\boldsymbol{\kappa}_e \cdot \nabla T_e, \tag{11}$$

the thermal conductivity $\boldsymbol{\kappa}_e$ is a rank 2 symmetric tensor which shall be expressed using the standard coordinate basis oriented with respect to the magnetic field unit vector $\mathbf{b}$,

$$\boldsymbol{\kappa} \cdot \nabla T = \kappa_\parallel(\mathbf{b}\cdot\nabla T)\mathbf{b} + \kappa_\perp \mathbf{b} \times (\nabla T \times \mathbf{b}) + \kappa_\wedge \mathbf{b} \times \nabla T. \tag{12}$$

To obtain these values, an expression for $\mathbf{q}_e$ in the form of Eq. (11) must be derived from the moment model's system of equations. Following the same steps shown in Hamilton and Seyler, the components of $\boldsymbol{\kappa}_e$ are found to be

$$\kappa_{e,\parallel} = \kappa_{e,0} \equiv \frac{5}{2}\frac{P_e}{m_e \nu_{q,e}}, \tag{13a}$$

$$\kappa_{e,\perp} = \kappa_{e,0}\frac{(\nu_{q,e}/\nu_{ei})^2}{(\nu_{q,e}/\nu_{ei})^2 + \chi_h^2}, \tag{13b}$$

and

$$\kappa_{e,\wedge} = \kappa_{e,0}\frac{\nu_{q,e}}{\nu_{ei}}\frac{\chi_h}{(\nu_{q,e}/\nu_{ei})^2 + \chi_h^2}, \tag{13c}$$

where the collisional effects were collected into a single frequency,

$$\nu_{q,e} = \left(\frac{13}{10} - \frac{69}{50}\delta_0\right)\nu_{ei} + \frac{4}{5}\left(1 - \frac{3}{5}\delta_0\right)\nu_{ee}, \tag{14}$$

which can be considered the heat flux exchange rate (in analogy to ν_{ei} which is the momentum exchange rate). These solutions give the thermal conductivity coefficients for all distribution function closures parametrized by δ_0, for all magnetizations $\chi_h = \omega_{c,e}\tau_{ei}$.

Figure 1 shows the perpendicular thermal conductivity for various transport models for all magnetizations, and Fig. 2 similarly shows the cross components, using normalized units of $P_e/m_e\nu_{ei}$. The parallel components are simply the unmagnetized limit ($\chi_h = 0$) of the perpendicular components. Whereas Grad's distribution gives far too low conductivity and the distribution of Killie *et al.* gives far too large, these figures show that an optimal distribution function closure can be found that matches the Chapman–Enskog expansion results at a given magnetization. The distribution function with $\delta_0 = 0.6307$, for example, matches the solutions of Braginskii or Epperlein and Haines with far more accuracy than either of the base moment models f_G or f_K. This value of δ_0 was found by setting Eq. (13a) equal to Epperlein and Haines' solution for $\kappa_{e,0}$. Depending on the magnetization of the plasma, a different optimal moment model closure could be found by solving for δ_0 with some other solution for $\boldsymbol{\kappa}_e$ that is known to be more correct for that magnetization, for example, Davies *et al.* which shows that not all choices of fitting functions used in the Chapman–Enskog expansion models give the correct physical scaling at low magnetization.[10]

These results, thus, allow the moment model described by the closure of Eq. (5) to reproduce any Chapman–Enskog model solution for the thermal conductivity at a given magnetization. Of course, all of the other collisional transport coefficients can also be generalized in

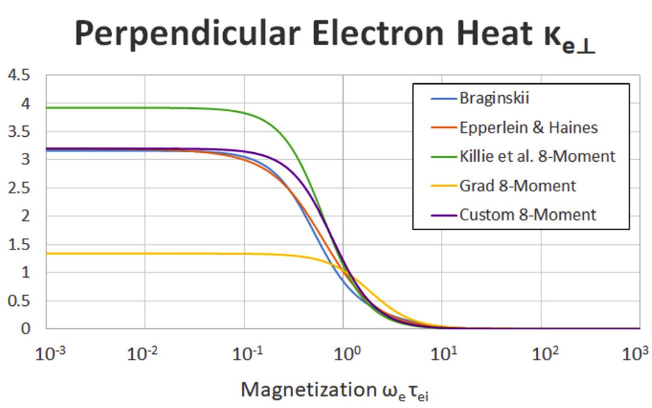

FIG. 1. Plot of the normalized perpendicular component of the electron's thermal conductivity tensor as a function of the magnetization $\omega_{c,e}\tau_{ei}$. Several different models are shown, including the results of Braginskii and Epperlein and Haines.[4,9] The custom distribution model was f_δ with $\delta_0 = 0.6307$ which was chosen to match the $\omega_{c,e}\tau_{ei} = 0$ result of Epperlein and Haines.

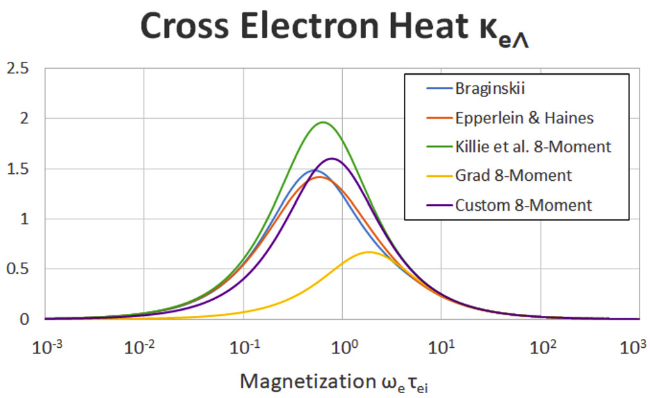

FIG. 2. Plot of the normalized cross component of the electron's thermal conductivity tensor as a function of the magnetization $\omega_{c,e}\tau_{ei}$. Several different models are shown, including the results of Braginskii and Epperlein and Haines.[4,9] The custom distribution model was identical to that of Fig. 1 where $\delta_0 = 0.6307$. To match Braginskii's value at low magnetization instead, $\delta_0 = 0.7012$ should be used.

terms of the parameter δ_0, with all results shown in Chap. 3 of Hamilton.[5]

In conclusion, this 8-moment model closure is based on the ansatz of a distribution function's expansion in these 8-moments, of which there are many choices of expansions. These distribution functions all result in the same system of equations except for the collision terms, where the Fokker–Planck term is dependent on the specific expansion used. This provides transport coefficients that are dependent on the choice of distribution. The ansatz of the distribution function is, therefore, related to an ansatz on how collisions should affect the plasma transport. Two velocity distributions by Grad[2] and Killie *et al.*[3] [Eqs. (1) and (2)] have been proposed to model the heat flux moments and differ from each other by weighing the core of the distribution by w or w^3, respectively. However, neither reproduces the Chapman–Enskog expansion results for a near-equilibrium plasma, and thus, the choice between the two seems somewhat arbitrary. These two options can be generalized into a single-parameter family of distribution functions f_δ [Eq. (5)], where the free parameter δ_0 can be tuned to vary between the results of Grad and Killie *et al.* to optimize the accuracy of the closure.

The calculation of the 8-moment model's thermal conductivity in terms of the parameter δ_0 and the magnetization $\omega_{c,e}\tau_{ei}$ is shown in Figs. 1 and 2, with greatly improved results using $\delta_0 = 0.6307$ compared to the closures of Grad and Killie *et al.* The remaining transport coefficients can similarly be calculated and are presented in the thesis of Hamilton.[5] The definition of these coefficients requires a steady-state near-equilibrium plasma where the time derivative terms and non-linear flux terms can be dropped (the moment model itself does not make these assumptions, however). All of the transport effects found from Chapman–Enskog expansions such as Braginskii, Epperlein and Haines, and Davies *et al.* are then found to be present.[4,9,10] The parameter δ_0 can be chosen to match the results of any of these models at a given magnetization although as a single parameter it can only match a single transport coefficient at a time (Table I shows optimal values in the unmagnetized limit). This can be construed as solving for the 8-moment distribution function that best approximates the perturbed distribution of the Chapman–Enskog expansion. In addition, Chap. 3.6 of Hamilton exhibits that this moment closure gives the correct scaling for all transport coefficients at low and high magnetization that was shown by Davies *et al.* Thus, the 8-moment model is suggested as an alternative to the Chapman–Enskog expansion models that have been used almost exclusively in the high energy density physics field for codes that calculate plasma transport.

To improve the accuracy of the transport, additional degrees of freedom are required for f_δ to replicate the Chapman–Enskog expansion results in the $\chi_h \sim 1$ regime. This can be done by expanding to a higher-order moment model or by generalizing the distribution function with more free parameters that can be solved by using the Chapman–Enskog expansion coefficients as necessary constraints. It is possible that the minimal number of additional free parameters should characterize the orientation of the magnetic field in order to most efficiently give the correct $\omega_{c,\alpha}\tau_\alpha$ dependence, but this has not been adequately investigated. The main obstacle to using a more complex distribution function is in the ability to calculate transport coefficients from the Fokker–Planck collision integral to retain explicit dependence on the moments themselves, allowing a feasible numerical scheme to solve the moment equations.

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AUTHOR DECLARATIONS

Conflict of Interest

Authors have no conflicts of interest.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

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