UNIVERSITÄT BAYREUTH

Master Thesis

Stabilizing Global Simulation in GKW by the Introduction of the Parallel Electric Field $E_{1\parallel}$

Manuel Lippert

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Physics Department at the University of Bayreuth

Supervisors:
Prof. Arthur G. Peeters
Dr. Florian Rath

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CHAPTER

1

Plasma Physics Basics

1.1 Charged Particle Motion in Magnetic and Electric Field

In magnetic confinement devices like the tokamak reactor, the charged particles experience forces caused by magnetic and electric fields which results in distinct motion under the associated force. Charged particles can be separated in species, e.g. electrons and ions, which will be later on not displayed in the governing equation. Throughout this thesis the charge q, the mass m or the temperature T indicate the quantities of a specific species, i.e., electrons or ions.

1.1.1 Particle Motion perpendicular to the Magnetic Field

Due to the Lorentz force, particles with a velocity component perpendicular to the homogenous magnetic field $v_\perp$ undergo a circular motion in the plane perpendicular to the magnetic field [Fig. **??**(a)]. This type of motion has circular frequency, which is often referred to as *cyclotron frequency* and is defined as

$$\omega_\mathrm{c} = \frac{|q|B}{m} \ , \tag{1}$$

where m and q are the mass and the charge of the particle and B the strength of the magnetic field. The radius, the so called *Larmor radius*, of this motion is given by

$$\rho = \frac{mv_\perp}{|q|B} \tag{2}$$

with the center often being referred to as *gyrocenter*. Note that since the Lorentz force depends on the species charge of the particle, the circulation direction is the opposite between electron in ions.

Due to Coulomb collisions the plasma gets thermalized. Together with the Maxwell-Boltzmann distribution the typical thermal velocity is

$$v_\mathrm{th} = \sqrt{\frac{2T}{m}} \ , \tag{3}$$

where T represents the species temperature. Based on the thermal velocity v_th the *thermal Larmor radius* gets introduced as**?**

$$\rho_\mathrm{th} = \frac{mv_\mathrm{th}}{|q|B} \ . \tag{4}$$

1.1.2 Particle Motion parallel to the Magnetic Field

In absence of forces in the direction parallel to the magnetic field the particles can move freely in parallel direction to the homogenous magnetic field. The velocity of this motion is of order of the thermal velocity v_{th} and is dominated by electrons due to their lighter mass compared to ions ($v_{\mathrm{th,e}}/v_{\mathrm{th,i}} = 60$).

When an electric field with a component parallel to the magnetic field $E_{\|}$ influences the plasma the charged particles are accelerated by the electric force

$$F_{\|,E} = qE_{\|} \ . \tag{5}$$

The parallel motion follows then from the equation of motion. Here the direction of the motion also depends on the species type [Fig. **??**(b)].

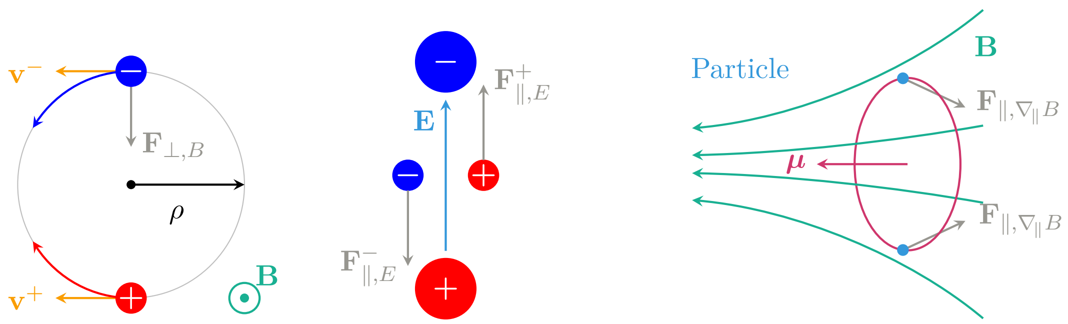
Since magnetic fields are not always homogenous, an inhomogeneous magnetic field with its gradient ∇B containing a component parallel to the magnetic field which is given by

$$\nabla_{\|} B = \frac{\mathbf{B}}{B} \cdot \nabla B \tag{6}$$

causes the force

$$F_{\|,\nabla_{\|} B} = -\frac{mv_{\perp}^2}{2B} \nabla_{\|} B = -\mu \nabla_{\|} B \ ; \qquad \mu = \frac{mv_{\perp}^2}{2B} \tag{7}$$

with *magnetic moment* μ. The magnetic moment μ is an adabatic invariant (constant of motion) if the variation of the magnetic field over time is smaller than the inverse of the cyclotron frequency ω_{c}^{-1} and the spatial variation is larger the Larmor radius ρ_{L}. The resulting force has its application in the mirror effect where a charged particle gets reflected due to this force [Fig. **??**(c)].[?]



(a) **(b)** **(c)**

Figure 1.1: Forces acting on a charged particle:

(a) Lorentz force $\mathbf{F}_{\perp,B}$ perpendicular to velocity $\mathbf{v}^{\pm}$ and magnetic field $\mathbf{B}$ which causes, circular motion with different directions for electron and ions, Lamor radius ρ_{L} and cyclotron frequency ω_{c},

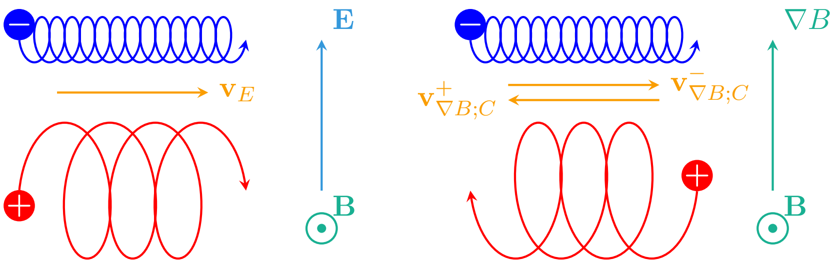
(b) Electric force $\mathbf{F}^{\pm}_{\parallel,E}$ with electric field $\mathbf{E}$,

(c) Mirror effect with force $\mathbf{F}_{\parallel,\nabla_\parallel B}$ and magnetic moment μ caused by an inhomogeneous magnetic field $\mathbf{B}$.

1.1.3 Drifts in the Gyrocenter

In the presence of a magnetic field (homogenous, inhomogeneous or perturbed) and electric fields the gyrocenter undergoes slow (compared to the thermal velocity v_{th}) drift motions perpendicular to the magnetic field. There are several examples for this drift motion. According to this thesis topic only the main three drift types will be covered in the following.

1. **$\mathbf{E} \times \mathbf{B}$ Drift:**
   If an electric field $\mathbf{E}$ with a perpendicular component together with the magnetic field $\mathbf{B}$ (both fields are homogenous) is present the acting Coulomb force and Lorentz force results into a drift of the gyrocenter with
   $$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \tag{8}$$
   which is called the $\mathrm{E} \times \mathrm{B}$ drift. Since both acting forces direction depends on the species type the direction of the $\mathrm{E} \times \mathrm{B}$ drift is for every species the same [Fig. **??**(a)].
2. **$\boldsymbol{\nabla} B$ Drift:**
   Inhomogeneous magnetic field causes a gradient ∇B of the magnetic field. Because of that gradient the gyrocenter undergoes a ∇B drift defined by
   $$\mathbf{v}_{\nabla B} = \frac{m v_\perp^2}{2q} \frac{\mathbf{B} \times \nabla B}{B^3} \; . \tag{9}$$
   The gradient of the magnetic field ∇B varies thereby on scales larger compared to the Larmor radius. The direction of the ∇B drift depends on the species type [Fig. **??**(b)].
3. **Curvature Drift:**
   Due to centrifugal force acting on the particle in a curved magnetic field the gyrocenter experiences a curvature drift according to
   $$\mathbf{v}_C = \frac{m v_\parallel^2}{q} \frac{\mathbf{B} \times \mathbf{C}}{B^2} = \frac{m v_\parallel^2}{q} \frac{\mathbf{B} \times \nabla B}{B^3} \; ; \qquad \mathbf{C} = -(\mathbf{b} \cdot \nabla)\mathbf{b} = \frac{\nabla B}{B} \; , \tag{10}$$
   where $\mathbf{b}$ is the unit vector along the magnetic field. To obtain the result for the curvature $\mathbf{C}$ in Eq. (**??**) the plasma pressure has to be small compared to the magnetic field strength B. In the form of Eq. (**??**) ∇B and curvature drift can be treated similarly.[?]



(a) **(b)**

Figure 1.2: Dirft motion in gyrocenter:

(a) E $\times$ B Drift with drift velocity $\mathbf{v}_E$, electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$,

(b) ∇B Drift/Curvature Drift with drift velocity $\mathbf{v}^{\pm}_{\nabla B;\kappa}$, magnetic field $\mathbf{B}$ and gradient of the magnetic field ∇B.

1.2 Magnetic Confinement and Plasma Rotation

In tokamak devices strong magnetic fields confine the hot plasma. As mentioned in Chapter **??** a magnetic field forces a perpendicular particle motion and a motion which contains the gyro motion and slow perpendicular gyro center drifts. Because of the much smaller size of the Larmor radius compared to the device size R the particle and energy losses are caused by the gyro center drift. To avoid additional loss of particles because of the parallel motion the field lines of the magnetic field in the tokamak devices is shaped like a torus. This type of geometry has nested surfaces with constant magnetic flux, so-called *flux surfaces*, and magnetic field lines which lie on these surfaces. To maintain stability the magnetic field has a toroidal and a poloidal component. According to the force balance the magnetic field is equivalent to the plasma pressure which means on flux-surfaces the plasma pressure is constant.[?] [?] The toroidal component is produced by external coils whereas the poloidal component is provided by the toroidal plasma current. Together the components result in a magnetic field which follows helical trajectories [Fig **??**]. To characterize the quality of confinement the so-called *plasma beta* is used and is given as

$$\beta = \frac{nT}{B^2/2\mu_0} \ , \tag{11}$$

with n the plasma density, T as temperature, μ_0 the permeability in vacuum and the magnetic field strength B. Respectively, the plasma beta compares the thermal plasma pressure nT to the ambient magnetic field pressure $B^2/2\mu_0$. For fusion devices the plasma beta has to be a bit smaller than 1 ($\beta < 1$) for optimal confinement. In a tokamak reactor the plasma beta has a typical order of a few percent.[?]

The rotation of the plasma can be described in a co-rotating frame of reference, which is rigidly rotating with the velocity $\mathbf{u}_0$ and will be used later on in the derivation of the gyrokinetic equations in Chapter **??**. It assumend that the poloidal component of the plasma rotation is much smaller compared to the toridial component and will be neglected. With this assumption in mind the reference frame is chosen to move in the toridial direction exclusivly and its velocity $\mathbf{u}_0$ can be expressed as

$$\mathbf{u}_0 = \mathbf{\Omega} \times \mathbf{x} = R^2 \Omega \nabla \varphi \ , \tag{12}$$

where $\mathbf{\Omega}$ is the constant angular frequency, φ is the toroidal angle and $R\nabla\varphi$ is the unit vector in the toroidal direction.

Since the rotation of the plasma in the labortory frame is not a rigid body rotation, it will be characterized by the radial profile of the angular velocity $\hat{\Omega}(\psi)$. Then the angular frequency of the rotating frame Ω is chosen to match the plasma rotation on a certain point, i.e. $\Omega = \hat{\Omega}(\psi_r)$. The plasma rotation in the co-rotating frame of reference will be denoted as

$$\omega_\varphi(\psi) = \hat{\Omega}(\psi) - \Omega. \tag{13}$$

with the rotation speed along the magnetic field line

$$u_\parallel = \frac{RB_\mathrm{t}}{B}\omega_\varphi(\psi)\ , \tag{14}$$

where B_t is the toridial component of the magnetic field.**?**

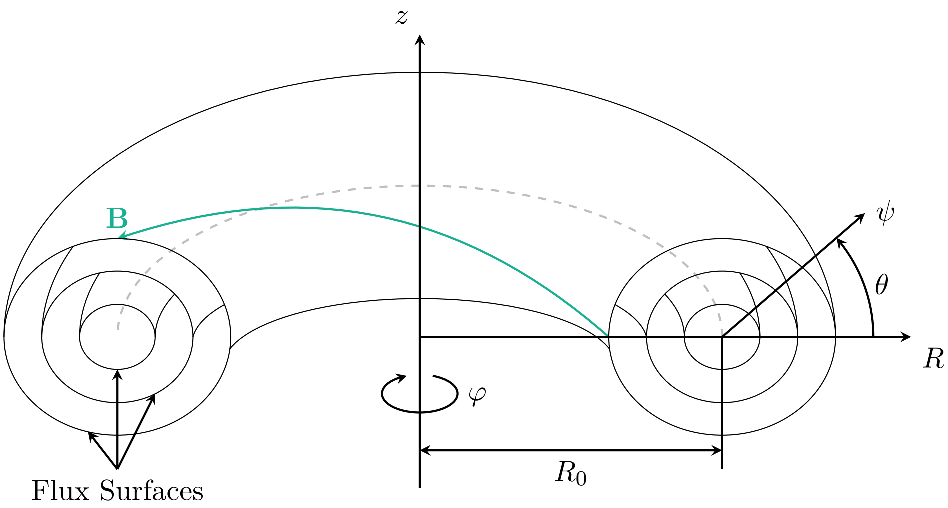


Figure 1.3: Toroidal flux surfaces in tokamak plasma with helical magnetic field (green line) in torus coordinates (ψ (radial), φ (toroidal), θ (poloidal)) or cylindrical coordinates (z, R, $-\varphi$).**?**

CHAPTER

2 Derivation of Gyrokinetic Equation

Plasma can be described in various theoretical models. The main two models are the Magnetohydrodynamics and the kinetic model. The key differences will be briefly explained

- **Magnetohydrodynamics:**
  In Magnetohydrodynamics the plasma will be described as an electric conductiv fluid which carries current. Here, the electrons and ions are two mixed fluids expressed in the two-fluid theory. In both description macroscopic quantities will be used, i.e. density, velocity of the fluid and temperature.
- **Kinetic Model:**
  In the kinetic model the plasma will be described in the sixdimensional phase space through the Vlasov equation. In combination with the Maxwell's equations is it possible to discribe the dynamics of the plasma as the Vlasov-Maxwell system.

In this section the kinetic model will be covered in greater detail for that the following scheme will be used:

1. The Lagrangian L for a particle in a magnetic field will reformulate in the fundamental one-form γ according to
$$\int \mathrm{d}t\ L = \int \gamma\ . \tag{1}$$
   From this point on the fundamental one-form and Lagrangian referres to the quantity γ, which will only be used in this thesis. Then, the Lagrangian γ will be transformed in guiding center phase space and separated in its equilibrium and perturbated part. Through the Lietransformation the Lagrangian gets transformed intop the gyrtocenter phase space by eliminating the gyro phase.
2. The Lagrangian gets plugged in to the Euler-Lagrangian equation, which results in the equations of motions. From the equations of motion the Vlasov equation can be derived.
3. The Vlasov equation solves for the density distribution function f, which will be used to express the particle density n and current $\mathbf{j}$ with the moments of the distrubution function.
4. Particle density n and current $\mathbf{j}$ will be plugged into the Maxwell's equations and the field equations of the potentials will be derived.

This part of the thesis is based on the Disseratation of Tilman Dannert[?] and the derivation document provided from the GKW group[?] . For the introduction of the inductive electric field the Disseratation of Paul Charles Crandall[?] will be used to formulate the electromagnetic gyrokinetic model for GKW.

2.1 Gyrokinetic Ordering

In the derivation of the gyrokinetic theory the aim is to decouple the effect of small-scale, small amplitude fluctuations of the plasma in the Langrangian. For this it is chosen to take the properties of fluctuations as small parameter, which will result in the ordering assumptions applied in gyrokinetic theory. This section is based on Ref. **?** and **?** .

- **Low Frequency**:
  The characteristic fluctuation frequency is small compared to the cyclotron frequency
  $$\Rightarrow \frac{\omega}{\omega_\mathrm{c}} \ll 1 \ .$$
- **Anisotropy**:
  The length scales of the turbulence are associated with the wave vector $\mathbf{k}$ which can be seperated into a perpendicular component $k_\perp = |\mathbf{k} \times \mathbf{b}|$ and a parallel component $k_\| = |\mathbf{k} \cdot \mathbf{b}|$ where $\mathbf{b}$ is parallel to the poloidal component of the magnetic field. The perpendicular correlation length of the turbulance has a length scale of around $10 - 100$ gyroradii while the parallel length scales can be of the order of meters, which can be expressed in wavenaumber as
  $$\Rightarrow \frac{k_\|}{k_\perp} \ll 1 \ .$$
- **Strong Magnetization**:
  The Larmor radius ρ is small compared to the gradient length scales for
  - Background Density: $L_n = n_0 \left(\frac{\mathrm{d}n_0}{\mathrm{d}x}\right)^{-1}$
  - Background Temperature: $L_T = T_0 \left(\frac{\mathrm{d}T_0}{\mathrm{d}x}\right)^{-1}$
  - Background Magnetic Field: $L_B = B_0 \left(\frac{\mathrm{d}B_0}{\mathrm{d}x}\right)^{-1}$

  $$\Rightarrow \frac{\rho}{L_n} \sim \frac{\rho}{L_T} \sim \frac{\rho}{L_B} \ll 1 \ .$$
- **Small Fluctuations**:
  The fluctuating part of the gyrocenter distribution function F_1 is assumend to be small compared to the background distribution function F_0
  $$\Rightarrow \frac{F_1}{F_0} \ll 1 \ .$$
  Furthermore, the fluctuations of the vector potential $\mathbf{A}_1$ and scalar potentials Φ_1 are small compared to there background part
  $$\Rightarrow \frac{\mathbf{A}_1}{\mathbf{A}_0} \sim \frac{\Phi_1}{\Phi_0} \ll 1 \ .$$

Taken together these assumptions leads to the gyrokinetic ordering

$$\frac{\omega}{\omega_{\mathrm{c}}} \sim \frac{k_{\parallel}}{k_{\perp}} \sim \frac{\rho}{L_n} \sim \frac{\rho}{L_T} \sim \frac{\rho}{L_B} \sim \frac{F_1}{F_0} \sim \frac{\mathbf{A}_1}{\mathbf{A}_0} \sim \frac{\Phi_1}{\Phi_0} \sim \epsilon_\delta \ , \tag{2}$$

where ϵ_δ is a small parameter. For the derivation of the gyrokinetic equations of GKW all derived equations are evaluated up to the first order of the ratio of the reference thermal Larmor radius $\rho_{\mathrm{th,ref}}$ and the equilibrium magnetic length scale L_B as small parameter and is defined as

$$\rho_\star = \frac{\rho_{\mathrm{th,ref}}}{L_B} = \frac{m_{\mathrm{ref}} v_{\mathrm{th,ref}}}{e B_{\mathrm{ref}}} \sim \epsilon_\delta \ . \tag{3}$$

2.2 Gyrokinetic Lagrangian (Fundamental One-Form)

2.2.1 Lagrangian in Patricle Phase Space

The Lagrangian of a particle γ with mass m and charge number Z in the electro magnetic field will be described through the particle position $\mathbf{x}$ and the velocity $\mathbf{v}$ as coordinates $\{\mathbf{x}, \mathbf{v}\}$ and can be written as

$$\gamma = \gamma_\nu \mathrm{d}z^\nu = \underbrace{(m\mathbf{v} + Ze\mathbf{A}(\mathbf{x}))}_{\text{Sympletic Part}} \cdot \mathrm{d}\mathbf{x} - \underbrace{\left(\frac{1}{2}mv^2 + Ze\Phi(\mathbf{x})\right)}_{\text{Hamiltonian } H(\mathbf{x},\mathbf{v})} \mathrm{d}t \ , \tag{4}$$

where $\mathbf{A}$ and Φ are the vector and scalar potential, ν indexes the six coordinates, and Einstein notation is applied. This form is also known as fundamental one-form.

The defined Lagrangian γ will then be transformed in the rotating frame of reference [Ch. **??**], which can be achieved the following Lorentz transformation

$$\mathbf{v} \to \mathbf{v} + \mathbf{u}_0 \qquad \mathbf{E} \to \mathbf{E} + \mathbf{u}_0 \times \mathbf{B} \qquad \Phi \to \Phi + \mathbf{A} \cdot \mathbf{u}_0 \ . \tag{5}$$

After performing the transformation outlined in Ref. **?** the Lagrangian γ becomes

$$\gamma = (m\mathbf{v} + m\mathbf{u}_0 + Ze\mathbf{A}(\mathbf{x})) \cdot \mathrm{d}\mathbf{x} - \left(\frac{1}{2}mv^2 - \frac{1}{2}mu_0^2 + Ze\Phi(\mathbf{x})\right) \mathrm{d}t \ . \tag{6}$$

In the next step small scale perturbations of the electromagnetic field gets introduced as following

$$\Phi = \Phi_0 + \Phi_1 \qquad \mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1 \ . \tag{7}$$

Here, it is assumend that the equilibrium electric field is zero in a stationary plasma, but it will be kept in case for finite plasma rotation. According to the gyrokinetic ordering [Ch. **??**] the perturbations are in the first order of $\rho_\star$. Taking everything into account the Lagrangian in the particle phase space with perturbations can be written as

$$\begin{aligned}
\gamma &= \gamma_0 + \gamma_1 \\
\gamma_0 &= (m\mathbf{v} + m\mathbf{u}_0 + Ze\mathbf{A}_0(\mathbf{x})) \cdot \mathrm{d}\mathbf{x} - \left(\frac{1}{2}mv^2 - \frac{1}{2}mu_0^2 + Ze\Phi_0(\mathbf{x})\right) \mathrm{d}t \\
\gamma_1 &= Ze\mathbf{A}_1(\mathbf{x}) \cdot \mathrm{d}\mathbf{x} - Ze\Phi_1(\mathbf{x}) \ \mathrm{d}t \ .
\end{aligned} \tag{8}$$

2.2.2 Lagrangian in Guiding Center Phase Space

For the description of charged particle behaviour in the tokamak device the *guiding center coordinates* are used [Fig. **??**]. This set of coordinates are defined as the following

$$\mathbf{X}(\mathbf{x},\mathbf{v}) = \mathbf{x} - \mathbf{r} \qquad v_\parallel = \mathbf{v}\cdot\mathbf{b}(\mathbf{x})$$
$$\mu(\mathbf{x},\mathbf{v}) = \frac{m v_\perp^2(\mathbf{x})}{2B(\mathbf{x})} \qquad \theta(\mathbf{x},\mathbf{v}) = \arccos\left(\frac{1}{v_\perp}\left(\mathbf{b}(\mathbf{x})\times\mathbf{v}\right)\cdot\hat{\mathbf{e}}_1\right) \;, \tag{9}$$

where the guiding center follows the magnetic field with the parallel velocity $v_\parallel$. The gyromotion is described together with the magnetic moment μ, the guiding center $\mathbf{X}$ and the gyro phase θ which gives a parameter set of six quantities $\{\mathbf{X}, v_\parallel, \mu, \theta\}$. Vector $\mathbf{b}(\mathbf{x})$ is the unit vector in the direction of the equilibrium magnetic field and $\mathbf{r} = \rho(\mathbf{x},\mathbf{v})\mathbf{a}(\mathbf{x},\mathbf{v})$ is the vector pointing from the guiding center to the particles postion, which is defined by the unit vector $\mathbf{a}(\mathbf{x},\mathbf{v})$ and its length is the Lamor radius $\rho(\mathbf{x},\mathbf{v})$. The unit vector $\mathbf{a}(\mathbf{x},\mathbf{v})$ can be expressed in a local orthonormal basis as the function of the gyroangle θ

$$\mathbf{a}(\theta) = \hat{\mathbf{e}}_1\cos\theta + \hat{\mathbf{e}}_2\sin\theta \;. \tag{10}$$

The vectors $\mathbf{b}$, $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ form a local Cartesian coordinate system at the guiding center position.

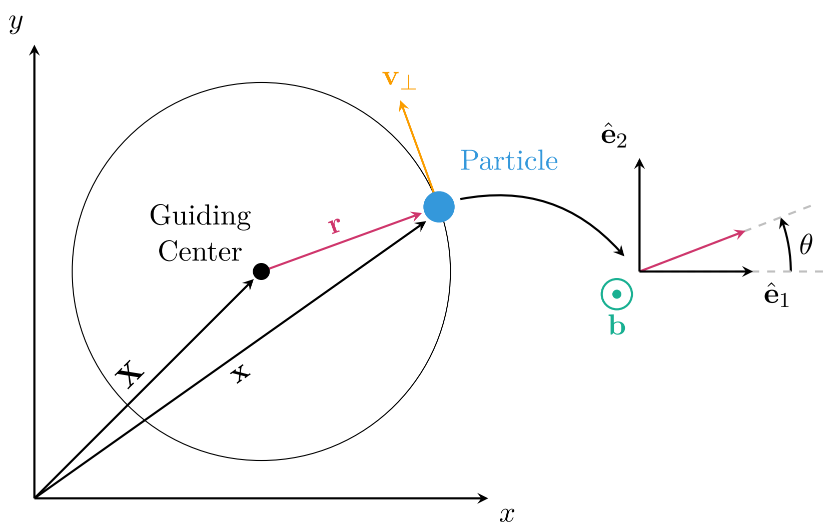


Figure 2.1: Sketch of guiding center coordinates where the charged particle performs a circular motion around the guiding center.?

To transform the fundamental one-from into the guiding center coordinates the following relation will be used

$$\Gamma_\eta = \gamma_\nu \frac{\mathrm{d}z^\nu}{\mathrm{d}Z^\eta} \ , \tag{11}$$

where Γ_η is a component of the guiding center fundamental one-form. To calculate the new coordinates the transformation [Eq. (**??**)] have to be inverted to provide the old coordinates as function of the new one $z(Z)$. Here, the direct transformation is clearly uniquely determined if the magnetic field is known at the particle postion. However, the inverse transformation is not uniquely due to the dependence of the Larmor radius ρ on magnetic field at the particle position $\mathbf{x}$. Taylor expansion of the Larmor radius ρ around the guiding center $\mathbf{X}$ yields $\rho(\mathbf{x}) \approx \rho(\mathbf{X})$. Note, that terms of order ρ^2, which leads to second order terms in $\rho_\star$, will get neglected due to the gyrokinetic ordering. The Larmor radius ρ also depends on the velocity $\mathbf{v}$ in particle phase space [Eq. (**??**)], or the magnetic moment μ in the guiding center phase space through the formular

$$\rho(\mathbf{X}, \mu) = \frac{1}{Ze}\sqrt{\frac{2\mu m}{B(\mathbf{X})}} \ . \tag{12}$$

This dependence will be only used if greater clarity is needed. With result of the taylor expansion the particle position $\mathbf{x}$ can be expressed with the guiding center coordinates as

$$\mathbf{x}(\mathbf{X}, \theta) \approx \mathbf{X} + \rho(\mathbf{X})\mathbf{a}(\theta) \ . \tag{13}$$

The particle velocity $\mathbf{v}$ is the sum of the velocity along the magnetic field $v_\parallel$, the gyration velocity $\mathbf{v}_\perp$ and the drift velocity, which will be neglected because the particle drifts can be described by the motion of the guiding center. So to summarize the velocity $\mathbf{v}$ in the guiding center frame can be expressed as

$$\mathbf{v} = v_\parallel \mathbf{b}(\mathbf{x}) + \mathbf{v}_\perp = v_\parallel \mathbf{b}(\mathbf{x}) + \rho(\mathbf{x})\dot{\mathbf{a}}(\theta) \ . \tag{14}$$

Applying Taylor expansion again around the guding center $\mathbf{X}$ teh following expression can be obtained

$$\mathbf{v}(\mathbf{X}, v_\parallel, \mu, \theta) \approx v_\parallel \left[\mathbf{b}(\mathbf{X}) + \partial_{\mathbf{X}}\mathbf{b}(\mathbf{X}) \cdot \mathbf{a}(\theta)\rho(\mathbf{X}, \mu)\right] + \rho(\mathbf{X}, \mu)\dot{\mathbf{a}}(\theta) \ . \tag{15}$$

Now, the transformation [Eq. (**??**)] can be applied to express the fundamental one-form in the new coordinates with the following components

$$\begin{aligned}
\Gamma_{X^i} &= \gamma_{x^j}\frac{\mathrm{d}x^j}{\mathrm{d}X^i} + \gamma_{v^j}\frac{\mathrm{d}v^j}{\mathrm{d}X^i} + \gamma_t\frac{\mathrm{d}t}{\mathrm{d}X^i} = \gamma_{x^j}\frac{\mathrm{d}x^j}{\mathrm{d}X^i} \qquad & \Gamma_{v_\parallel} &= \gamma_{x^j}\frac{\mathrm{d}x^j}{\mathrm{d}v_\parallel} + \gamma_{v^j}\frac{\mathrm{d}v^j}{\mathrm{d}v_\parallel} = 0 \\
\Gamma_\mu &= \gamma_{x^j}\frac{\mathrm{d}x^j}{\mathrm{d}\mu} & \Gamma_\theta &= \gamma_{x^j}\frac{\mathrm{d}x^j}{\mathrm{d}\theta} \\
\Gamma_t &= \gamma_t\frac{\mathrm{d}t}{\mathrm{d}t} + \mu B(\mathbf{X}) = \gamma_t + \mu B(\mathbf{X}) \ .
\end{aligned} \tag{16}$$

Note, that to the Hamiltonian part Γ_t the energy term of the magnetic field B at the guiding center $\mathbf{X}$ has to be added, due to the circular motion of the particle around the center.

In Equation (**??**) the components of the fundamental one-form of the particle phase space γ_ν and the equation for $\mathbf{v}$ in guiding center coordinates [Eq. (**??**)] will be inserted and the Taylor expansion up to the first order applied for terms containing the particle position $\mathbf{x}$ as argument. After that, the gyroaveraging operator $\mathcal{G}$ will be used, which is defined as the integral over the gyrophase θ

$$\mathcal{G}\{G(\mathbf{x})\} = \bar{G}(\mathbf{X}) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \; G(\mathbf{X} + \mathbf{r}(\theta)) \; , \tag{17}$$

with an example field $G(\mathbf{x})$. Due to the definition of the vector $\mathbf{a}$ [Eq. (**??**)] the first order terms in $\mathbf{a}$ and $\dot{\mathbf{a}}$ disappear under gyroaveraging. Following all the previous steps, one can obtain

$$\begin{aligned}
\bar{\Gamma}_{\mathbf{X}} &= m v_\parallel b_i(\mathbf{X}) + m u_{0i} + Ze\mathbf{A}(\mathbf{X}) & \bar{\Gamma}_{v_\parallel} &= 0 \\
\bar{\Gamma}_\mu &= 0 & \bar{\Gamma}_\theta &= \frac{2\mu m}{Ze} \\
\bar{\Gamma}_t &= -\left(\frac{1}{2} m v_\parallel^2 - \frac{1}{2} m u_0^2 + Ze\Phi(\mathbf{X}) + \mu B(\mathbf{X})\right) \; ,
\end{aligned} \tag{18}$$

which results in the fundamental one-form in guiding center coordinates

$$\begin{aligned}
\bar{\Gamma} = &\left(m v_\parallel \mathbf{b}(\mathbf{X}) + m\mathbf{u}_0 + Ze\mathbf{A}(\mathbf{X})\right) \cdot \mathrm{d}\mathbf{X} + \frac{2\mu m}{Ze}\mathrm{d}\theta \\
&- \left(\frac{1}{2} m v_\parallel^2 - \frac{1}{2} m u_0^2 + Ze\Phi(\mathbf{X}) + \mu B(\mathbf{X})\right) \mathrm{d}t \; .
\end{aligned} \tag{19}$$

Note that as a consequence of the Lagrangian being independent of the gyrophase θ, the magnetic moment μ (the associated conjugated coordinate pair of θ) becomes an invariant of the motion ($\dot{\mu} = 0$).

As in Chapter **??** perturbations [Eq. (**??**)] will get introduced to the guiding center Lagrangian. The transformation of the equilibrium part is already performed above, so only the pertrubation part with the pertubated Lagrangian in the particle phase space γ_1 has to be transfromed to the guiding center phase space. The transformation is analogous to the calculation before, the key difference is that the fluctuations quantities vary on a small length scale and Taylor expansion around the guiding center $\mathbf{X}$ can not be applied advantageous. Their values have to be taken at the particle position, which is a function of the gyroangle in guiding center coordinates. After this clearification the components of the perturbated Lagrangian in the guiding center phase space can be written as

$$\begin{aligned}
\Gamma_{1,X^i} &= \gamma_{1,x^j}\frac{\mathrm{d}x^j}{\mathrm{d}X^i} & \Gamma_{1.v_\parallel} &= 0 \\
\Gamma_{1,\mu} &= \gamma_{1,x^j}\frac{\mathrm{d}x^j}{\mathrm{d}\mu} & \Gamma_{1,\theta} &= \gamma_{1,x^j}\frac{\mathrm{d}x^j}{\mathrm{d}\theta} \\
\Gamma_{1,t} &= \gamma_{1,t} \; .
\end{aligned} \tag{20}$$

After inserting the components of the perturbated Lagrangian γ_1 and neglecting terms of order ρ^2, due to gyrokinetic ordering, the pertrubated compentents in the guiding center coordinates can be expressed as

$$\begin{aligned}
\Gamma_{1,\mathbf{X}} &\approx Ze\mathbf{A}_1(\mathbf{x}) & \Gamma_{1.v_\parallel} &= 0 \\
\Gamma_{1,\mu} &= \frac{Z}{|Z|}\frac{1}{v_\perp(\mathbf{X},\mu)}\mathbf{A}_1(\mathbf{x})\cdot\mathbf{a}(\theta) & \Gamma_{1,\theta} &= \frac{Z}{|Z|}\frac{2\mu}{v_\perp(\mathbf{X},\mu)}\mathbf{A}_1(\mathbf{x})\cdot\frac{\mathrm{d}\mathbf{a}(\theta)}{\mathrm{d}\theta} \\
\Gamma_{1,t} &= -Ze\Phi_1(\mathbf{x})\ .
\end{aligned} \tag{21}$$

Finally, the fundamental one-form in the guiding center phase space Γ with perturbation can be written as

$$\begin{aligned}
\Gamma &= \bar{\Gamma}_0 + \Gamma_1 \\
\bar{\Gamma}_0 &= \big(mv_\parallel\mathbf{b}(\mathbf{X}) + m\mathbf{u}_0 + Ze\mathbf{A}_0(\mathbf{X})\big)\cdot\mathrm{d}\mathbf{X} + \frac{2\mu m}{Ze}\mathrm{d}\theta \\
&\quad - \left(\frac{1}{2}mv_\parallel^2 - \frac{1}{2}mu_0^2 + Ze\Phi_0(\mathbf{X}) + \mu B_0(\mathbf{X})\right)\mathrm{d}t \\
\Gamma_1 &= Ze\mathbf{A}_1(\mathbf{x})\cdot\mathrm{d}\mathbf{X} + \frac{Z}{|Z|}\frac{1}{v_\perp}\mathbf{A}_1(\mathbf{x})\cdot\mathbf{a}\mathrm{d}\mu + \frac{Z}{|Z|}\frac{2\mu}{v_\perp}\mathbf{A}_1(\mathbf{x})\cdot\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}\theta}\mathrm{d}\theta - Ze\Phi_1(\mathbf{x})\mathrm{d}t\ .
\end{aligned} \tag{22}$$

2.2.3 Lagrangian in Gyrocenter Phase Space

The transformation of the guiding center Lagrangian Γ into the Lagrangian in gyrocenter phase space $\bar{\Gamma}$ aims to remove the gyroangle θ dependence resulting from the introduction of fluctuations. To destinguish between the guiding center and the gyrocenter coordinates all quantities associated with the gyrocenter are getting an overbar, i.e. $\bar{\Gamma}$. The new set of gyrocenter coordinates are given by $\{\bar{\mathbf{X}}, \bar{v}_{\|}, \bar{\mu}\}$, but these coordinates will not been used in this thesis. Since the derivation of fundamental the one-form in gyrocenter phase space $\bar{\Gamma}$ uses the Lie transform pertubation method, which is beyond the scopes of this thesis, the reader is referred to the Refs. **?** and **?** for more details. The Lagrangian in the gyrocenter phase space can be expressed as

$$\begin{aligned}\bar{\Gamma} &= \bar{\Gamma}_0 + \bar{\Gamma}_1 \\ &= \left(mv_{\|}\mathbf{b}_0(\mathbf{X}) + m\mathbf{u}_0 + Ze\left(\mathbf{A}_0(\mathbf{X}) + \bar{\mathbf{A}}_1(\mathbf{X})\right)\right) \cdot \mathrm{d}\mathbf{X} + \frac{2\mu m}{Ze}\mathrm{d}\theta \\ &\quad - \left(\frac{1}{2}m\left(v_{\|}^2 - u_0^2\right) + Ze\left(\Phi_0(\mathbf{X}) + \bar{\Phi}_1(\mathbf{X})\right) + \mu\left(B_0(\mathbf{X}) + \bar{B}_{1\|}(\mathbf{X})\right)\right)\,\mathrm{d}t\ ,\end{aligned} \tag{23}$$

where B_0 is the equilibrium magnetic field and $\bar{B}_{1\|}$ is the magnetic field introduced by the vector potential $\mathbf{A}_1$. Note, that the pertubations of the scalar and vector potential will be seperated into an oscillating and a gyroaveraged part which will be expressed as

$$\Phi_1 = \widetilde{\Phi}_1 + \bar{\Phi}_1 \qquad \mathbf{A}_1 = \widetilde{\mathbf{A}}_1 + \bar{\mathbf{A}}_1\ , \tag{24}$$

although the oscillating parts will be added to the gauge function of the Lie transformation and will not be included in the Lagrangian of the gyrocenter. The quantity $\bar{B}_{1\|}$ is the shorter notation of following gyroaveraged quantity defined as

$$\mathcal{G}\{Ze\mathbf{A}_1(\mathbf{x}) \cdot \mathbf{v}_\perp(\mathbf{X}, \mu, \theta)\} = \mu\bar{B}_{1\|}(\mathbf{X})\ . \tag{25}$$

2.3 Gyrokinetic Equation

2.3.1 Vlasov Equation

Because of the large number of particles in the fusion plasma a prediction on the basis of Newton-Maxwell dynamics results in an impossible task for simulation, but this problem can be solved with a statistical approach. For that the distribution function $f(\mathbf{x}, \mathbf{v}, t)$ in the particle phase space $\{\mathbf{x},\ \mathbf{v}\}$ will be considered. Because collisions are happening at much smaller frequencies than the characteristic frequencies connected to turbulence, the collisionless model is often preferred[?] which results through evolution of the particle density distribution function in the *Vlasov equation*

$$\frac{\partial f}{\partial t} + \dot{\mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{x}} + \dot{\mathbf{v}} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0\ . \tag{26}$$

In the gyrocenter phase space $\{\mathbf{X}, v_\|, \mu\}$, the overbar introduced in Chapter **??** gets droped for simplicity for all quantities, the Vlasov equation with the gyrocenter distribution function F takes the following form

$$\frac{\partial F}{\partial t} + \dot{\mathbf{X}} \cdot \frac{\partial F}{\partial \mathbf{X}} + \dot{v}_\| \cdot \frac{\partial F}{\partial v_\|} = 0\ , \tag{27}$$

where the gyrophase θ is still an ignorable coordinate and the time derivative of the magnetic moment μ is zero, beacause the magnetic moment μ is an exact invariant. In Equation (**??**) the terms of the time derivative of the gyrocenter $\dot{\mathbf{X}}$ and the parallel velocity $v_\|$ have to be expressed through the gyrocenter Lagrangian with the Euler-Lagrange equation. The Euler-Lagrange equation can be written as

$$\left(\frac{\partial \gamma_j}{\partial z^i} - \frac{\partial \gamma_i}{\partial z^j}\right)\frac{\mathrm{d}z^j}{\mathrm{d}t} = \frac{\partial H}{\partial z^i} + \frac{\partial \gamma_i}{\partial t}\ . \tag{28}$$

Inserting Equation (**??**) into the Euler-Lagrange equation and apply multiple calculations detailed in Ref. **?** the equations of motion can be obtained as

$$\begin{aligned}
\dot{\mathbf{X}} &= v_\|\mathbf{b}_0 + \mathbf{v}_\chi + \mathbf{v}_\mathrm{D} \qquad & \dot{v}_\| &= \frac{\dot{\mathbf{X}}}{m v_\|} \cdot \Big(Ze\bar{\mathbf{E}} - \mu\nabla(B_0 + \bar{B}_{1\|}) + \underbrace{\frac{1}{2} m \nabla u_0^2}_{mR\Omega^2\nabla R}\Big) \\
\dot{\mu} &= 0 & \dot{\theta} &= \omega_\mathrm{c} - \frac{Ze}{m}\partial_\mu \Big(Ze\bar{\mathbf{A}}_1 \cdot \dot{\mathbf{X}} - Ze\bar{\Phi}_1 - \mu\bar{B}_{1\|}\Big)\ ,
\end{aligned} \tag{29}$$

with the drift velocity $\mathbf{v}_\chi$ defined as the sum of the streaming velocity perpendicular to the pertubated magnetic field $\mathbf{v}_{\bar{B}_{1\perp}}$, the $\mathrm{E} \times \mathrm{B}$ drift in the total electric field $\mathbf{v}_{\bar{E}}$ and the grad-B dirft of the parallel perturbed magnetic field $\mathbf{v}_{\nabla \bar{B}_{1\|}}$. The drift velocity

$\mathbf{v}_{\mathrm{D}}$ containing the sum of the curvature drift $\mathbf{v}_C$, the grad-B drift of the equilibrium magnetic field $\mathbf{v}_{\nabla B_0}$ and the drifts due to the Coriolis force $\mathbf{v}_{\mathrm{Co}}$ and centrifugal force $\mathbf{v}_{\mathrm{Ce}}$. The quantity χ can be expressed as

$$\chi = \underbrace{(\Phi_0 + \bar{\Phi}_1)}_{\bar{\Phi}_1} - v_\| \bar{A}_{1\|} + \frac{\mu}{Ze} \bar{B}_{1\|} \tag{30}$$

with which follows the drift velocity $\mathbf{v}_\chi$

$$\mathbf{v}_\chi = \frac{\mathbf{b} \times \nabla \chi}{B_0} = \mathbf{v}_{\bar{B}_{1\perp}} + \mathbf{v}_{\bar{E}} + \mathbf{v}_{\nabla \bar{B}_{1\|}} \ . \tag{31}$$

Note that the term containing u_0^2 got replaced with Equation (**??**) with $u_0^2 = R^2 \Omega^2$ and the total electric field $\bar{\mathbf{E}}$ is defined as

$$\bar{\mathbf{E}} = -\nabla \bar{\Phi}_1 - \partial_t \bar{\mathbf{A}}_1 \approx -\nabla \bar{\Phi}_1 - \partial_t \bar{A}_{1\|} \ , \tag{32}$$

since the time derivative of the vector potential $\partial_t \bar{A}_{1\perp}$ is one order smaller as the gradient of the electrostatic potential $\nabla \bar{\Phi}_1$ due to normalization assumptions in gyrokinetics**?** .

2.3.2 The delta-f Approximation

The delta-f approximation separates the density distrubution function F into an equilibrium part F_0 and pertubation part F_1, i.e. $F = F_0 + F_1$. Applying the delta-f approximation on the gyrocenter Vlasov equation leads to

$$\frac{\partial F_1}{\partial t} + \dot{\mathbf{X}} \cdot \nabla F_1 + \dot{v}_\parallel \cdot \frac{\partial F_1}{\partial v_\parallel} = \underbrace{-\dot{\mathbf{X}} \cdot \nabla F_0 - \dot{v}_\parallel \frac{\partial F_0}{\partial v_\parallel}}_{S} , \tag{33}$$

with the source term S. Substituting from Equation (**??**) the equations for $\dot{\mathbf{X}}$ and $\dot{v}_\parallel$ into the delta-f approximated Vlasov equation results in

$$\frac{\partial F_1}{\partial t} + \dot{\mathbf{X}} \cdot \nabla F_1 - \frac{\mathbf{b}_0}{m} \cdot \left(Ze\nabla\Phi_0 + \mu\nabla B_0 - mR\Omega^2\nabla R \right) \cdot \frac{\partial F_1}{\partial v_\parallel} = S \ . \tag{34}$$

Note that only the terms of order $\rho_\star$ has to be kept in $\dot{v}_\parallel \partial_{v_\parallel} F_1$, which results in neglecting the drift velocities $\mathbf{v}_\chi$ and $\mathbf{v}_\mathrm{D}$ and the contribution of $\bar{B}_{1\parallel}$ and $\bar{\Phi}_1$, since these terms are after calculation of order $\rho_\star^2$.

The equilibrium distribution fuction F_0 is assumed to be a Maxwellian which includes a finite equilibrium electric field Φ_0 to balance the centrifugal force (in the co-rotating frame) due toridial rotation of the plasma.
In the rotating frame the included energy term can be written as

$$\mathcal{E} = Ze\langle\Phi_0\rangle - \frac{1}{2} m\omega_\varphi^2 \left(R^2 - R_0^2 \right) \ , \tag{35}$$

where $\langle \cdot \rangle$ denote flux-surface averaging, ω_φ the plasma rotation frequency [Eq. (**??**)], R the local major radius and R_0 is an integration constant which can be chosen, i.e. major radius of the plasma or flux surface average of the major radius. The Maxwellian is given by the following expression

$$F_0 = F_\mathrm{M}(\mathbf{X}, v_\parallel, \mu) = \frac{n_{R_0}}{(2\pi T/m)^{3/2}} \exp\left(-\frac{\frac{1}{2}m(v_\parallel - u_\parallel)^2 + \mu B_0 + \mathcal{E}}{T} \right) \ , \tag{36}$$

where n_{R_0} is the particle density at the position $R = R_0$ and is related to equilibrium particle density through the relation $n_0 = n_{R_0} \exp(-\mathcal{E}/T)$.?
Furthermore, $u_\parallel$ is the rotation speed of the plasma in the rotating frame parallel to the magnetic field [Eq. (**??**)]. The Maxwellian can seperated in

$$F_\mathrm{M}(\mathbf{X}, v_\parallel, \mu) = F_\mathrm{M}(v_\parallel) F_\mathrm{M}(\mu) e^{-\mathcal{E}/T}, \tag{37}$$

$$F_\mathrm{M}(v_\parallel) = \frac{n_{R_0}}{(2\pi T/m)^{3/2}} \exp\left(-\frac{\frac{1}{2}m(v_\parallel - u_\parallel)^2}{T} \right) \qquad F_\mathrm{M}(\mu) = \exp\left(-\frac{\mu B_0}{T} \right) \ . \tag{38}$$

The derivatives of the Maxwellian can be expressed as

$$\begin{aligned}
\nabla F_{\mathrm{M}} &= \left[\frac{\nabla n_{R_0}}{n_{R_0}} + \left(\frac{\frac{1}{2}mv_\parallel^2 + \mu B_0 + \mathcal{E}}{T} - \frac{3}{2}\right)\frac{\nabla T}{T} - \frac{\mu B_0}{T}\frac{\nabla B_0}{B_0}\right. \\
&\qquad \left. + \left(\frac{mv_\parallel R B_{\mathrm{t}}}{BT} + m\omega\left(R^2 - R_0^2\right)\right)\nabla\omega_\varphi\right] F_{\mathrm{M}} \\
\partial_{v_\parallel} F_{\mathrm{M}} &= -\frac{mv_\parallel}{T} F_{\mathrm{M}} \\
\partial_\mu F_{\mathrm{M}} &= -\frac{B_0}{T} F_{\mathrm{M}} \ ,
\end{aligned} \tag{39}$$

where the $\nabla\omega_\varphi$ terms are the result of the derivatives of the parallel rotation velocity $u_\parallel$ and rotation energy $\mathcal{E}$ evaluated at zero rotation speed locally in the co-rotating frame.? It can be shown with Equations (**??**) and (**??**) that the ∇B_0 term in $-\dot{\mathbf{X}} \cdot \nabla F_{\mathrm{M}}$ cancels with $(\dot{\mathbf{X}}/mv_\parallel)\mu\nabla B_0 \partial_{v_\parallel} F_{\mathrm{M}}$ for purely torodial rotation. Finally, the source term can than be written as

$$\begin{aligned}
S = &-(\mathbf{v}_\chi + \mathbf{v}_{\mathrm{D}}) \cdot \widetilde{\nabla} F_{\mathrm{M}} - \frac{Zev_\parallel}{T}\partial_t \bar{A}_{1\parallel} F_{\mathrm{M}} \\
&- \frac{F_{\mathrm{M}}}{T}(v_\parallel \mathbf{b}_0 + \mathbf{v}_{\mathrm{D}} + \mathbf{v}_{\bar{B}_{1\perp}}) \cdot (Ze\nabla\bar{\Phi}_1 + \mu\nabla\bar{B}_{1\parallel}) \ ,
\end{aligned} \tag{40}$$

where $\widetilde{\nabla}$ referres to only the ∇n_{R_0}, ∇T and $\nabla\omega_\varphi$ terms of ∇F_{M}.

2.4 Gyrokinetic Field Equations

2.4.1 Maxwell's Equations

To obtain a closed system the Vlasov equation gets combined with the Maxwell equations to calculate the pertubated electromagnetic fields. As usual in fusion plasma the Gauss law gets replaced by the quasi neutrality condition which implies that any deviation from neutrality can only happen on small length scales within the Debye lenght and on a timescale much shorter than that of the fluctuations. Due to non-relativistic timescale of the turbulence the dissplacement current in Ampere's law gets also neglected. Taking everything into account the Maxwell's equations can be written as

$$\begin{aligned} \sum_s Z_s e\, n_s &= 0 \qquad & \nabla \times \mathbf{E}_1 &= -\frac{\partial \mathbf{B}_1}{\partial t} \\ \nabla \cdot \mathbf{B}_1 &= 0 \qquad & \nabla \times \mathbf{B}_1 &= \mu_0 \sum_s \mathbf{j}_s \ , \end{aligned} \tag{41}$$

where the index s refers to the species of particles, i.e. proton or electron and $\sum_s$ means that all species will be taken into account. For simplicity of the derivation the "s" index gets droped if not explicitly needed. The Maxwell's equation contain densities n and currents $\mathbf{j}$ of part icles which can be expressed through the moments of the particle phase space distribution function f as follows

$$n(\mathbf{x}) = \int \mathrm{d}\mathbf{v}\ f(\mathbf{x}, \mathbf{v}) \tag{42}$$

$$j_\parallel = Ze \int \mathrm{d}\mathbf{v}\ v_\parallel f(\mathbf{x}, \mathbf{v}) \qquad \mathbf{j}_\perp = Ze \int \mathrm{d}\mathbf{v}\ \mathbf{v}_\perp f(\mathbf{x}, \mathbf{v}) \ . \tag{43}$$

2.4.2 Pull Back Operation into the Guiding Center Phase Space

Since the Vlasov equations [Eq. (**??**)] describes the evolution of the distribution function in the gyrocenter phase space F, the particle moments will be expressed with the guiding center phase space distribution function F^{gc} and which itself will be described through the gyrocenter distrubution function F by performing a pull back from F to the guiding center phase space. A schemamtic about the general idea can be seen in Figure **??**. The pull back will be performed with the pull back operator $\mathcal{P}$ which results in

$$F^{\mathrm{gc}} = \mathcal{P}\left\{F\right\} = F \underbrace{- \frac{F_{\mathrm{M}}}{T}\left(Ze\widetilde{\Phi}_1 - \mu \bar{B}_{1\|}\right)}_{\text{Correction Term}} , \tag{44}$$

where $\widetilde{\Phi}_1$ donates to the oscillating part of the pertubation Φ_1. Here, the correction term contain the fluctuations of the electro-magnetic fields and describes physically the polarization and magnetization effects of the fluctuations on the gyro orbit.[?]

Figure 2.2: Idea of gyrokinetic Maxwell's equations: The particle density $n(\mathbf{x})$, the current density $\mathbf{j}(\mathbf{x})$ and the gyrocenter distribution function F are expressed in the guiding center phase space.

The particle density n and currents $\mathbf{j}$ of one species can be expressed with the guiding center distribution function F^{gc} as

$$\begin{aligned}
n(\mathbf{x}) &= \int \mathrm{d}\mathbf{v}\ f(\mathbf{x},\mathbf{v}) = \frac{B_0}{m}\int \mathrm{d}\mathbf{X}\mathrm{d}v_\|\mathrm{d}\theta\mathrm{d}\mu\ \delta(\mathbf{X}+\mathbf{r}-\mathbf{x})F^{\mathrm{gc}} \\
j_\| &= Ze\int \mathrm{d}\mathbf{v}\ v_\| f(\mathbf{x},\mathbf{v}) = \frac{ZeB_0}{m}\int \mathrm{d}\mathbf{X}\mathrm{d}v_\|\mathrm{d}\theta\mathrm{d}\mu\ \delta(\mathbf{X}+\mathbf{r}-\mathbf{x})v_\| F^{\mathrm{gc}} \\
\mathbf{j}_\perp &= Ze\int \mathrm{d}\mathbf{v}\ \mathbf{v}_\perp f(\mathbf{x},\mathbf{v}) = \frac{ZeB_0}{m}\int \mathrm{d}\mathbf{X}\mathrm{d}v_\|\mathrm{d}\theta\mathrm{d}\mu\ \delta(\mathbf{X}+\mathbf{r}-\mathbf{x})\mathbf{v}_\perp F^{\mathrm{gc}} ,
\end{aligned} \tag{45}$$

where B_0/m is the Jacobian of the guiding center coordinates. The delta function δ appears due to the change of coordinates and gurantees that the spatial region taken into account in the integral remains unchanged during the coordinate transformation. Physically, the delta function δ expresses that all particles which have a Larmor orbit crossing a given point $\mathbf{x}$ in the real space contribute to the particle density [Fig. **??**].

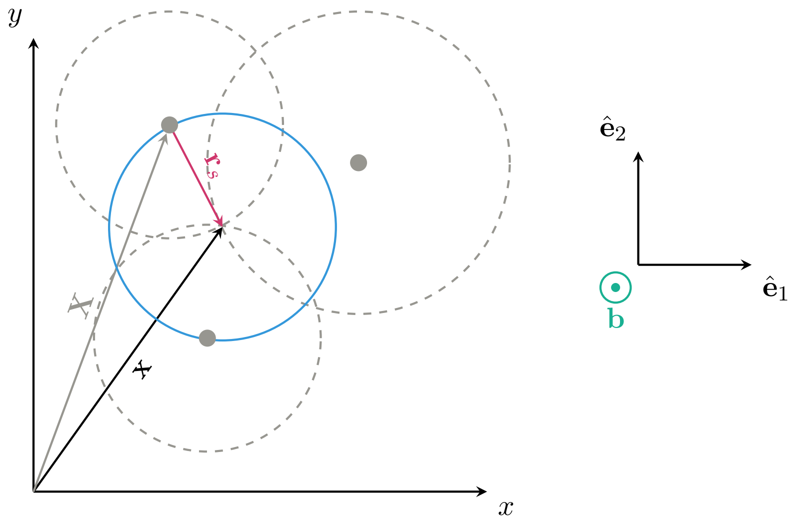


Figure 2.3: Connection between density of particles $n_s(\mathbf{x})$ and density of guiding centers: Gyro orbits with different guiding center $\mathbf{X}$ (gray dashed circles) can cross in postion $\mathbf{x}$, such that the respective gyrating particles add to the particle density n_s there. For a fixed Larmor radius $\rho_s = |\mathbf{r}_s| = |\rho_s \mathbf{a}|$ (red) the particle density n_s at $\mathbf{x}$ is obtained by collecting the contributions of all guiding centers on a circle with radius ρ_s centered at position $\mathbf{x}$ (blue circle).

2.4.3 Gyrooperator $\mathcal{G}$

As stated in Chapter **??** the gyrooperator $\mathcal{G}$ averages over the gyrophase θ which is mostly used in the derivation of the Vlasov equation and is defined as

$$\mathcal{G}\{G(\mathbf{x})\} = \bar{G}(\mathbf{X}) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta\ G(\mathbf{X} + \mathbf{r}(\theta)) \ . \tag{46}$$

To derive the field equations a second kind of gyrooperator will be introduced as

$$\mathcal{G}^\dagger\{G(\mathbf{X})\} = \langle G\rangle(\mathbf{x}) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\mathbf{X}\mathrm{d}\theta\ \delta(\mathbf{X} + \mathbf{r}(\theta) - \mathbf{x}) G(\mathbf{X}) \ , \tag{47}$$

where $\mathcal{G}^\dagger$ the hermitian conjugate of $\mathcal{G}$[?] and the delta function δ originates from the pull back operation from Chapter **??**.[?] Furthermore, the double gyroaverage operator is defined as

$$\mathcal{G}^\dagger\{\mathcal{G}\{G(\mathbf{x})\}\} = \langle\bar{G}\rangle(\mathbf{x}) = \frac{1}{(2\pi)^2}\int_0^{2\pi} \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\theta'\ G(\mathbf{X} - \mathbf{r}(\theta) + \mathbf{r}(\theta')) \ , \tag{48}$$

which performs a gyroaverage of the field value at all gyrocenter positions $\mathbf{X}$ with particle position $\mathbf{x}$ in their trajectory.[?]

2.4.4 Perturbated Electrostatic Potential Φ_1

To evaluate the pertrubated electrostatic potential Φ_1 the modified guiding center distribution function F^{gc} [Eq. (**??**)] gets inserted into the equation for the particle density $n(\mathbf{x})$ which results in

$$\begin{aligned} n(\mathbf{x}) &= \frac{B_0}{m} \int \mathrm{d}\mathbf{X} \mathrm{d}v_\parallel \mathrm{d}\theta \mathrm{d}\mu \ \delta(\mathbf{X}+\mathbf{r}-\mathbf{x}) \left(F - \frac{F_\mathrm{M}}{T} \left(Ze\widetilde{\Phi}_1 - \mu \bar{B}_{1\parallel} \right) \right) \\ &= \bar{n}(\mathbf{x}) + n_\mathcal{P}(\mathbf{x}) \ , \end{aligned} \tag{49}$$

with the density of the gyrocenter $\bar{n}(\mathbf{x})$ and the variations on the gyro orbit to the particle density $n_\mathcal{P}(\mathbf{x})$ which describes the polarization effects of the fluctuating fields on the gyro orbit **?** . The gyrocenter density can be simplyfied with the gyroperator $\mathcal{G}^\dagger$ [Eq. (**??**)] to

$$\bar{n}(\mathbf{x}) = \frac{B_0}{m} \int \mathrm{d}\mathbf{X} \mathrm{d}v_\parallel \mathrm{d}\theta \mathrm{d}\mu \ \delta(\mathbf{X}+\mathbf{r}-\mathbf{x}) F = \frac{2\pi B_0}{m} \int \mathrm{d}v_\parallel \mathrm{d}\mu \ \langle F \rangle \ . \tag{50}$$

The polarization density $n_\mathcal{P}$ is given by

$$n_\mathcal{P}(\mathbf{x}) = -\frac{2\pi B_0}{m} \int \mathrm{d}v_\parallel \mathrm{d}\mu \ \frac{F_\mathrm{M}}{T} \left(Ze \left(\Phi_1(\mathbf{x}) - \langle \bar{\Phi}_1 \rangle (\mathbf{x}) \right) - \mu \langle \bar{B}_{1\parallel} \rangle \right) \ , \tag{51}$$

where n_0 is a background density, T is a background temperature. To derive the term for the polarization density $n_\mathcal{P}$ the oscillating Part of the electro static potential got replaced with $\widetilde{\Phi}_1(\mathbf{X}+\mathbf{r}) = \Phi_1(\mathbf{X}+\mathbf{r}) - \bar{\Phi}_1(\mathbf{X})$ and the gyrooperator $\mathcal{G}^\dagger$ were used. Taking everything into account and insert it into the quaisneutrality equation $\sum_s Z_s e \, n_s = 0$ the field equation for the perturbated electrostatic potential Φ_1 is given by

$$\begin{aligned} &\sum_s \frac{Z_s^2 e^2}{m_s} \int \mathrm{d}v_\parallel \mathrm{d}\mu \ \frac{F_{\mathrm{M},s}}{T_s} \left(\Phi_1(\mathbf{x}) - \langle \bar{\Phi}_1 \rangle (\mathbf{x}) \right) = \\ &\sum_s \frac{Z_s e}{m_s} \int \mathrm{d}v_\parallel \mathrm{d}\mu \ \langle F_s \rangle + \frac{F_{\mathrm{M},s}}{T_s} \mu \langle \bar{B}_{1\parallel} \rangle \ . \end{aligned} \tag{52}$$

2.4.5 Perturbated Parallel Magnetic Field $B_{1\parallel}$

The perpendicular compentent of Ampere's law can be written as

$$(\nabla \times B_{1\parallel})_\perp = \begin{pmatrix} \partial_y B_{1\parallel} - \partial_z B_{1y} \\ \partial_z B_{1x} - \partial_x B_{1\parallel} \end{pmatrix} = \mu_0 \mathbf{j}_{1\perp} \;, \tag{53}$$

where z is the direction of the equilibrium magnetic field B_0. The parallel gradients of the pertrubated magnetic field can be neglegted since they are one order smaller than the perpendicular ones, which results in

$$\begin{pmatrix} \partial_y B_{1\parallel} \\ -\partial_x B_{1\parallel} \end{pmatrix} = \nabla_\perp B_{1\parallel} \times \mathbf{b} = \mu_0 \mathbf{j}_{1\perp} \;. \tag{54}$$

After performing the pull back operation the perpendicular current $\mathbf{j}_{1\perp}$ is given by

$$\mathbf{j}_{1\perp} = \frac{ZeB_0}{m} \int \mathrm{d}\mathbf{X} \mathrm{d}v_\parallel \mathrm{d}\theta \mathrm{d}\mu \; \delta(\mathbf{X} + \mathbf{r} - \mathbf{x}) \mathbf{v}_\perp \left(F - \frac{F_\mathrm{M}}{T} \left(Ze\widetilde{\Phi}_1 - \mu \bar{B}_{1\parallel} \right) \right) \;. \tag{55}$$

Inserting the pertrubated perpenedicular current $\mathbf{j}_{1\perp}$ into Ampere's law.

TODO:

- Derive Equation for pertrubated parallel magnetic Field $B_{1\parallel}$ with Gyrooperators
- Think about gettting $B_{1\parallel}$ without the use of the technic from Ref. **?**

2.4.6 Plasma Induction $\mathbf{A_{1\|}}$

To express the parallel pertrubation of the vector potential $A_{1\|}$, i.e. the plasma induction, the method is analogous to Chapter **??**. Using the Coulomb gauge $\nabla \cdot \mathbf{A}_1 = 0$ in the parallel compentent of Ampere's law results in

$$\nabla^2 A_{1\|} = -\mu_0 j_{1\|} = -\mu_0 \sum_s j_{1\|,s} \ . \tag{56}$$

Performing the pull back one last time the parallel perturbation of the current density $j_{1\|}$ is given by

$$\begin{aligned} j_{1\|} &= \frac{ZeB_0}{m} \int \mathrm{d}\mathbf{X} \mathrm{d}v_\| \mathrm{d}\theta \mathrm{d}\mu \ \delta(\mathbf{X} + \mathbf{r} - \mathbf{x}) v_\| \left(F - \frac{F_\mathrm{M}}{T} \left(Ze\widetilde{\Phi}_1 - \mu \bar{B}_{1\|} \right) \right) \\ &= \frac{ZeB_0}{m} \int \mathrm{d}\mathbf{X} \mathrm{d}v_\| \mathrm{d}\theta \mathrm{d}\mu \ \delta(\mathbf{X} + \mathbf{r} - \mathbf{x}) v_\| F \\ &= \frac{2\pi ZeB_0}{m} \int \mathrm{d}v_\| \mathrm{d}\mu \ \langle v_\| F \rangle \ , \end{aligned} \tag{57}$$

although the term $v_\| F_\mathrm{M}$ vanishes during the integration along $v_\|$, due to symmetry of the Maxwellian F_M. Inserting Equation (**??**) into Ampere's law yields the field equation for the plasma induction as follows

$$\nabla^2 A_{1\|} = -\sum_s \frac{2\pi Z_s e \mu_0 B_0}{m_s} \int \mathrm{d}v_\| \mathrm{d}\mu \ \langle v_\| F_s \rangle \ . \tag{58}$$

2.4.7 Inductive Electric Field $\mathbf{E}_{1\|}$

Global Gyrokinetic simulations are suffering from numerical problems, mainly from the cancellation problem**?** examined codes**?** which uses the paticle-in-cell or the Eulerian methods. This problem limits the electromagnetic investigations to extremely low β parameters.**?** As in Chapter **??** discussed gets the density distribution function f seperated into an equilibrium part f_0, i.e. a Maxwellian F_{M}, and a pertrubated part F_1. Through the total electric field $\bar{\mathbf{E}}$ in the time derivative from the parallel velocity $v_\|$ [Eq. (**??**) & (**??**)] the time derivative of the pertrubed vector potential $\partial_t \bar{A}_{1\|}$ appears in the Source term [Eq. (**??**)] which is computationally difficult to evaluate. To avoid further complications a modified distribution function g will be introduced as

$$g = F_1 + \frac{Zev_\|}{T} \bar{A}_{1\|} F_{\mathrm{M}} \ . \tag{59}$$

This is method is currently implemented in GKW**?** . The goal of this Chapter is to handle the $\partial_t \bar{A}_{1\|}$ term with the consideration of electromagnetic fields and rework the Vlasov equation and fiels equations for GKW. This section follows the work of P.C. Crandall in his Disseration**?** .

First the Vlasov equation gets written down in the F_1 framework with the source term [Eq. (**??**) & (**??**)] will get simplified into

$$\frac{\partial F_1}{\partial t} + \frac{Zev_\|}{T} \partial_t \bar{A}_{1\|} F_{\mathrm{M}} = \mathcal{V} \ , \tag{60}$$

where $\mathcal{V}$ represents all terms of the Vlasov Equation which excludes the time derivative of plasma induction $\partial_t \bar{A}_{1\|}$. The equation of the $\bar{A}_{1\|}$ is already astablished in Chapter **??** but for this derivation a recall will be made. The equation for $\bar{A}_{1\|}$ is given by

$$\nabla^2 A_{1\|} = -\mu_0 j_{1\|} = -\sum_s \frac{2\pi Z_s e \mu_0 B_0}{m_s} \int \mathrm{d}v_\| \mathrm{d}\mu \ \langle v_\| F_s \rangle \ . \tag{61}$$

Now the following formalism will be used

$$E_{1\|} = -\frac{\partial A_{1\|}}{\partial t} \ . \tag{62}$$

Taking the time derivative of Equation (**??**) results into the field equation for the induced electric field $E_{1\|}$ which can be expressed as

$$\nabla^2 E_{1\|} - \sum_s \frac{2\pi Z_s e \mu_0 B_0}{m_s} \int \mathrm{d}v_\| \mathrm{d}\mu \ \langle v_\| \frac{\partial F_s}{\partial t} \rangle = 0 \ . \tag{63}$$

In Equation (**??**) the time derivative of the gyrocenter distrubution function has to be further simplified for that the gyrokinetic equation shall be rewritten as

$$\frac{\partial F}{\partial t} = \mathcal{V} + \frac{Zev_\parallel}{T} \bar{E}_{1\parallel} F_\mathrm{M} \ . \tag{64}$$

Plugging Equation (**??**) into Equation (**??**), one can derive the equation for the inductive electric field

$$\begin{aligned} &\left(\nabla^2 - \sum_s \frac{2\pi (Z_s e)^2 \mu_0 B_0}{T_s m_s} \int \mathrm{d}v_\parallel \mathrm{d}\mu \ \mathcal{G}^\dagger v_\parallel^2 F_{\mathrm{M}s} \mathcal{G} \right) E_{1\parallel} = \\ &\sum_s \frac{2\pi Z_s e \mu_0 B_0}{m_s} \int \mathrm{d}v_\parallel \mathrm{d}\mu \ \mathcal{G}^\dagger \left\{ v_\parallel \mathcal{V}_s \right\} = \mu_0 \frac{\partial j_{1\parallel}}{\partial t} \ , \end{aligned} \tag{65}$$

although the relation of $\bar{E}_{1\parallel} = \mathcal{G}\left\{E_{1\parallel}\right\}$ and the definition of $\mathcal{G}^\dagger$ was used to simplyfied the integral on the right-hand side. To complete the derivation of this section the delta-f Vlasov equation will be recalled with the inductive filed $E_{1\parallel}$ in the source term. The delta-f Vlasov equation is given by

$$\frac{\partial F_1}{\partial t} + \dot{\mathbf{X}} \cdot \nabla F_1 - \frac{\mathbf{b}_0}{m} \cdot \left(Ze\nabla\Phi_0 + \mu\nabla B_0 - mR\Omega^2 \nabla R \right) \cdot \frac{\partial F_1}{\partial v_\parallel} = S \ , \tag{66}$$

with the source term

$$\begin{aligned} S = &-(\mathbf{v}_\chi + \mathbf{v}_\mathrm{D}) \cdot \tilde{\nabla} F_\mathrm{M} + \frac{Zev_\parallel}{T} \bar{E}_{1\parallel} F_\mathrm{M} \\ &- \frac{F_\mathrm{M}}{T} (v_\parallel \mathbf{b}_0 + \mathbf{v}_\mathrm{D} + \mathbf{v}_{\bar{B}_{1\perp}}) \cdot (Ze\nabla\bar{\Phi}_1 + \mu\nabla\bar{B}_{1\parallel}) \ . \end{aligned} \tag{67}$$

CHAPTER

3

Plasma Induction in Gyrokinetic Simulations

3.1 Local Simulations

3.1.1 Gyro-Operator in Local Simulations

In the case of local simulations the gyrooperators $\mathcal{G}$ and $\mathcal{G}^\dagger$ simplies to

$$\begin{aligned} \mathbf{G}\bar{(}\mathbf{x}) &= J_0(\lambda)\mathbf{G}(\mathbf{X}) \\ \langle \mathbf{G}(\mathbf{X}) \rangle &= J_0(\lambda)\mathbf{G}(\mathbf{x}) \end{aligned} \tag{1}$$

with J_0 as the zeroth order Bessel function. Note, that $\bar{B}_{1\|}(\mathbf{X}) = -I_1(\lambda)\mu B_{1\|}(\mathbf{X})$, where I_1 is the modified first order Bessel function of first kind defined as $I_1(\lambda) = 2/\lambda\, J_1(\lambda)$. To obtain Equation (**??**) the process of gyroaveraging will be performend in the Fourier space as follows

$$\begin{aligned} \bar{\mathbf{G}}(\mathbf{x}) = \bar{\mathbf{G}}(\mathbf{X}+\mathbf{r}) &= \mathcal{G}\left\{ \int \mathrm{d}\mathbf{k}\ \hat{\mathbf{G}}(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{X}+\mathbf{r})} \right\} \\ &= \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta \int \mathrm{d}\mathbf{k}\ \hat{\mathbf{G}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{X}} e^{ik_\perp \rho\cos\theta} \\ &= \int \mathrm{d}\mathbf{k}\ \hat{\mathbf{G}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{X}} \underbrace{\frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\theta\ e^{ik_\perp\rho\cos\theta}}_{J_0(\rho k_\perp)} \\ &= \int \mathrm{d}\mathbf{k}\ J_0(\rho k_\perp)\hat{\mathbf{G}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{X}} = J_0(\lambda)\mathbf{G}(\mathbf{X})\ , \end{aligned} \tag{2}$$

where the wave vector $\mathbf{k}$ is defined as $\mathbf{k} = \hat{e}_1 k_\perp$. The argument λ is given by $i\rho\nabla_\perp$ which is the inverse Fourier transformated expression of $\rho k_\perp$. The same routine can be applied for the gyrooperator $\mathcal{G}^\dagger$ with the same result. In general the Bessel function and modified Bessel function is defined as**?**

$$\begin{aligned} J_n(z) &= \left(\frac{z}{2}\right)^n \underbrace{\sum_{\nu=0}^{\infty} \frac{\left(\left(-1/4z^2\right)^\nu\right)}{\nu!(1+\nu)!}}_{I_n(z)}\ , \\ J_n(z) &= \frac{i^{-n}}{\pi}\int_0^{\pi}\mathrm{d}\theta\ e^{iz\cos\theta}\cos(n\theta)\ . \end{aligned} \tag{3}$$

Integrals with Bessel Function

In the upcoming section there will be often integrals which contain the zeroth Besselfunction $J_0(\lambda)$ and the modified Besselfunctions $I_1(\lambda)$. In general this types of integrals have the form

$$\int \mathrm{d}v_{\|}\mathrm{d}\mu\ \mu^n J_0^{2-n}(\lambda) I_1^n(\lambda) F_{\mathrm{M}}, \tag{4}$$

wuth the natural number $n = \{0, 1, 2\}$. Equation (**??**) can be seperated together with the Maxwellian [Eq. (**??**) with $u_{\|} = 0$, because of reference system] to

$$e^{-\mathcal{E}/T} \int \mathrm{d}v_{\|}\ F_{\mathrm{M}}(v_{\|}) \int \mathrm{d}\mu\ \mu^n J_0^{2-n}(\lambda) I_1^n(\lambda) F_{\mathrm{M}}(\mu)\ . \tag{5}$$

The first integral appears in the following types

$$\begin{aligned}
&1) \int \mathrm{d}v_{\|}\ F_{\mathrm{M}}(v_{\|}) && = \frac{n_{R_0} m}{2\pi T} \\
&2) \int \mathrm{d}v_{\|}\ v_{\|} F_{\mathrm{M}}(v_{\|}) && = 0 \text{ (Due to symmetry)} \\
&3) \int \mathrm{d}v_{\|}\ v_{\|}^2 F_{\mathrm{M}}(v_{\|}) && = \frac{n_{R_0}}{2\pi}
\end{aligned} \tag{6}$$

and the last integral occurs in three types

$$\begin{aligned}
&1) \int \mathrm{d}\mu\ J_0^2(\lambda) F_{\mathrm{M}}(\mu) && = \frac{T}{B_0}\Gamma_0(b) \\
&2) \int \mathrm{d}\mu\ \mu J_0(\lambda) I_1(\lambda) F_{\mathrm{M}}(\mu) && = \frac{T^2}{B_0^2}(\Gamma_0(b) - \Gamma_1(b)) \\
&3) \int \mathrm{d}\mu\ \mu^2 I_1^2(\lambda) F_{\mathrm{M}}(\mu) && = \frac{T^3}{B_0^3} 2(\Gamma_0(b) - \Gamma_1(b))\ ,
\end{aligned} \tag{7}$$

with the notation $\Gamma_n(b) = I_n(b)e^{-b}$ with the modified Bessel function I_n [Eq. (**??**)] and $b = -\rho_{\mathrm{th}}^2 \nabla^2{}_{\perp}$. ρ_{th} referres the thermal Larmor radius [Eq. (**??**)].[?]

3.1.2 Normalization

To implement the field equations from Chapter **??** into the local version of GKW, one has to normalize the quantities in the equations. This section is based on Ref. **?** . The reference values are indicated by the index "ref", the dimensionless noramlized by "N" and the relative dimensionless values by the index "R". This section is based on Ref **?** , **?** and **?** .

A reference mass m_{ref}, density n_{ref}, temperature T_{ref}, magnetic field B_{ref} and major radius R_{ref} is chosen. With these quantities the reference thermal velocity $v_{\text{th,ref}}$ and reference thermal Larmor radius $\rho_{\text{th,ref}}$ gets defines as

$$T_{\text{ref}} = \frac{1}{2} m_{\text{ref}} v_{\text{th,ref}}^2 \qquad \rho_{\text{th,ref}} = \frac{m_{\text{ref}} v_{\text{th,ref}}}{eB_{\text{ref}}} = \frac{2T_{\text{ref}}}{eB_{\text{ref}} v_{\text{th,ref}}} \qquad \rho_\star = \frac{\rho_{\text{th,ref}}}{R_{\text{ref}}} \tag{8}$$

and for convenience reason the small parameter $\rho_\star$ got redefined.

- **Relative Quantities:**

$$m = m_{\text{ref}} m_{\text{R}} \qquad n_{R_0} = n_{\text{ref}} n_{\text{R}} \qquad T = T_{\text{ref}} T_{\text{R}} \qquad v_{\text{th}} = v_{\text{th,ref}} v_{\text{thR}} \tag{9}$$

- **Normalized Quantities:**

$$R = R_{\text{ref}} R_{\text{N}} \qquad B_0 = B_{\text{ref}} B_{\text{N}} \qquad \mu = \frac{2T_{\text{ref}} T_{\text{R}}}{B_{\text{ref}}} \mu_{\text{N}} \qquad v_\parallel = v_{\text{th}} v_{\parallel\text{N}} \tag{10}$$

$$k = \frac{k_{\text{N}}}{\rho_\star} \qquad k_\parallel = \frac{k_{\parallel\text{N}}}{\rho_\star} \qquad k_\perp = \frac{k_{\perp\text{N}}}{\rho_{\text{th,ref}}} \tag{11}$$

$$\beta = \beta_{\text{ref}} \beta_{\text{N}} \qquad \beta_{\text{ref}} = \frac{2\mu_0 n_{\text{ref}} T_{\text{ref}}}{B_{\text{ref}}^2} \tag{12}$$

  - **Fluctuating Fields:**

$$\Phi_1 = \rho_\star \frac{T_{\text{ref}}}{e} \Phi_{1\text{N}} \qquad B_{1\parallel} = \rho_\star B_{\text{ref}} B_{1\parallel\text{N}} \tag{13}$$

$$A_{1\parallel} = B_{\text{ref}} R_{\text{ref}} \rho_\star^2 A_{1\parallel\text{N}} \qquad E_{1\parallel} = \frac{T_{\text{ref}}}{e} \frac{1}{R_{\text{ref}}} \rho_\star E_{1\parallel\text{N}} \tag{14}$$

  - **Time, Frequency and Centrifugal Energy:**

$$t = \frac{R_{\text{ref}}}{v_{\text{th,ref}}} t_{\text{N}} \qquad \Omega = \frac{v_{\text{th,ref}}}{R_{\text{ref}}} \Omega_{\text{N}} \qquad \mathcal{E} = T_{\text{ref}} \mathcal{E}_{\text{N}} \tag{15}$$

  - **Distrubution Function and Vlasov Equation:**

$$F = \rho_\star \frac{n_{R_0}}{v_{\text{th}}^3} F_{\text{N}} \qquad F_{\text{M}} = \frac{n_{R_0}}{v_{\text{th}}^3} F_{\text{MN}} \qquad \mathcal{V} = \rho_\star \frac{v_{\text{th,ref}}}{R_{\text{ref}}} \frac{n_{R_0}}{v_{\text{th}}^3} \mathcal{V}_{\text{N}} \tag{16}$$

  - **Gradients:**

$$\nabla_\perp = \frac{1}{R_{\text{ref}}} \nabla_{\perp\text{N}} \qquad \nabla_\parallel = \frac{1}{R_{\text{ref}}} \nabla_{\parallel\text{N}} \ . \tag{17}$$

3.1.3 Field Equations in Local Simulations

Perturbated Electrostatic Potential Φ_1

With the use of the local of the local gyrooperator the density of the gyrocenter $\bar{n}$ and the polarization density $n_{\mathcal{P}}$ can be expressed as

$$\bar{n}(\mathbf{x}) = \frac{2\pi B_0}{m} \int \mathrm{d}v_\| \mathrm{d}\mu \; J_0(\lambda) F(\mathbf{x}, v_\|, \mu) \; , \tag{18}$$

$$n_{\mathcal{P}}(\mathbf{x}) = \frac{Zen_{R_0}(\mathbf{x})}{T} e^{-\mathcal{E}/T} (\Gamma_0(b) - 1)\Phi_1(\mathbf{x}) + n_{R_0}(\mathbf{x}) e^{-\mathcal{E}/T} (\Gamma_0(b) - \Gamma_1(b)) \frac{B_{1\|}(\mathbf{x})}{B_0} \; , \tag{19}$$

where n_0 is a background density, T is a background temperature. To derive the term for the polarization density $n_{\mathcal{P}}$ the integrals mentioned in Chapter **??** were performed. The field equation for the perturbated electrostatic potential Φ_1 in the local simulation is given by

$$\begin{aligned} &\sum_s \frac{Z_s^2 e^2}{T_s} n_{R_0,s} e^{-\mathcal{E}_s/T_s} \left(1 - \Gamma_0(b_s)\right) \Phi_1(\mathbf{x}) = \\ &\sum_s Z_s e \left(\bar{n}_s + n_{R_0,s} e^{-\mathcal{E}_s/T_s} (\Gamma_0(b_s) - \Gamma_1(b_s)) \frac{B_{1\|}(\mathbf{x})}{B_0} \right) \end{aligned} \tag{20}$$

and takes the following form after performing the Fourier-transform and applying the normalizing expressions [Ch. **??**]

$$\begin{aligned} &\sum_s Z_s n_{\mathrm{R},s} e^{-\mathcal{E}_{\mathrm{N},s}/T_{\mathrm{R},s}} \left(1 - \Gamma_0(b_s)\right) \frac{Z_s}{T_{\mathrm{R},s}} \widehat{\Phi}_{1\mathrm{N}} = \\ &\sum_s Z_s n_{\mathrm{R},s} \left(2\pi B_\mathrm{N} \int \mathrm{d}v_{\|\mathrm{N}} \mathrm{d}\mu_\mathrm{N} \; J_0(k_\perp \rho_s) \widehat{F}_{\mathrm{N},s} + e^{-\mathcal{E}_{\mathrm{N},s}/T_{\mathrm{R},s}} (\Gamma_0(b_s) - \Gamma_1(b_s)) \frac{\widehat{B}_{1\|\mathrm{N}}}{B_\mathrm{N}} \right) \end{aligned} \tag{21}$$

Plasma Induction $\mathbf{A_{1\|}}$

The field equation for the plasma induction follows simply after the use of the local gyroperator and is given by

$$\nabla^2 A_{1\|} = -\sum_s \frac{2\pi Z_s e \mu_0 B_0}{m_s} \int \mathrm{d}v_\| \mathrm{d}\mu \; v_\| J_0(\lambda_s) F_s(\mathbf{x}, v_\|, \mu) \; . \tag{22}$$

Fourier transformation and normalization yields

$$k_{\perp\mathrm{N}}^2 \widehat{A}_{1\|\mathrm{N}} = 2\pi B_\mathrm{N} \beta_\mathrm{ref} \sum_s Z_s n_{\mathrm{R},s} v_{\mathrm{thR},s} \int \mathrm{d}v_{\|\mathrm{N}} \mathrm{d}\mu_\mathrm{N} \; v_{\|\mathrm{N}} J_0(k_\perp \rho_s) \widehat{F}_{\mathrm{N},s} \; . \tag{23}$$

Perturbated Parallel Magnetic Field $B_{1\|}$

The perpendicular compentent of Ampere's law can be written as

$$(\nabla \times B_{1\|})_\perp = \begin{pmatrix} \partial_y B_{1\|} - \partial_z B_{1y} \\ \partial_z B_{1x} - \partial_x B_{1\|} \end{pmatrix} = \mu_0 \mathbf{j}_{1\perp} \ , \tag{24}$$

where z is the direction of the equilibrium magnetic field B_0. The parallel gradients of the pertrubated magnetic field can be neglegted since they are one order smaller than the perpendicular ones, which results in

$$\begin{pmatrix} \partial_y B_{1\|} \\ -\partial_x B_{1\|} \end{pmatrix} = \nabla_\perp B_{1\|} \times \mathbf{b} = \mu_0 \mathbf{j}_{1\perp} \ . \tag{25}$$

After performing the pull back operation the perpendicular current $\mathbf{j}_{1\perp}$ is given by

$$\mathbf{j}_{1\perp} = \frac{ZeB_0}{m} \int \mathrm{d}\mathbf{X}\mathrm{d}v_\| \mathrm{d}\theta \mathrm{d}\mu \ \delta(\mathbf{X} + \mathbf{r} - \mathbf{x}) \mathbf{v}_\perp \left(F - \frac{F_\mathrm{M}}{T} \left(Ze\widetilde{\Phi}_1 - \mu \bar{B}_{1\|} \right) \right) \ . \tag{26}$$

Inserting the pertrubated perpenedicular current $\mathbf{j}_{1\perp}$ into Ampere's law and apply the same method as in Ref **?** results in the field equation for the pertrubated parallel magnetic field $B_{1\|}$ which can be expressed as

$$\begin{aligned} &\left(1 + \sum_s \beta_s (\Gamma_0(b_s) - \Gamma_1(b_s)) e^{-\mathcal{E}_s/T_s} \right) B_{1\|} = \\ &- \sum_s \frac{2\pi\mu_0 B_0}{m_s} \int \mathrm{d}v_\| \mathrm{d}\mu \ \mu I_n(\lambda_s) F_s \\ &- \sum_s (\Gamma_0(b_s) - \Gamma_1(b_s)) e^{-\mathcal{E}_s/T_s} \frac{Z_s e \mu_0 n_{R_0,s}}{B_0} \Phi_1 \ , \end{aligned} \tag{27}$$

with β_s as the plasma beta of a given species. Fourier transformation and normalization with $\beta = \beta_\mathrm{ref}\beta_\mathrm{N}$ yields bla bla

$$\begin{aligned} &\left(1 + \sum_s \frac{T_{\mathrm{R},s} n_{\mathrm{N},s}}{B_\mathrm{N}^2} \beta_\mathrm{ref} (\Gamma_0(b_s) - \Gamma_1(b_s)) e^{-\mathcal{E}_{\mathrm{N},s}/T_{\mathrm{R},s}} \right) \widehat{B}_{1\|\mathrm{N}} = \\ &- \sum_s \beta_\mathrm{ref} 2\pi B_\mathrm{N} T_{\mathrm{R},s} n_{\mathrm{R},s} \int \mathrm{d}v_{\|\mathrm{N}} \mathrm{d}\mu_\mathrm{N} \ \mu_\mathrm{N} I_1(k_\perp \rho_s) \widehat{F}_{\mathrm{N},s} \\ &- \sum_s \beta_\mathrm{ref} (\Gamma_0(b_s) - \Gamma_1(b_s)) e^{-\mathcal{E}_{\mathrm{N},s}/T_{\mathrm{R},s}} \frac{Z_s n_{\mathrm{R},s}}{2B_\mathrm{N}} \widehat{\Phi}_{1\mathrm{N}} \ . \end{aligned} \tag{28}$$

Inductive Electric Field $\mathbf{E}_{1\parallel}$

Using the new definition of the gyrooperator into Equation (**??**), one can derive the equation for the included electric field

$$\begin{aligned}&\left(\nabla^2-\sum_s\frac{2\pi(Z_se)^2\mu_0B_0}{T_sm_s}\int \mathrm{d}v_\parallel\mathrm{d}\mu\ v_\parallel^2J_0^2(\lambda_s)F_{\mathrm{M}s}\right)E_{1\parallel}=\\&\sum_s\frac{2\pi Z_se\mu_0B_0}{m_s}\int \mathrm{d}v_\parallel\mathrm{d}\mu\ v_\parallel J_0(\lambda_s)\mathcal{V}_s=\mu_0\frac{\partial j_{1\parallel}}{\partial t}\ ,\end{aligned} \tag{29}$$

although the relation of $\bar{E}_{1\parallel}=\mathcal{G}\left\{E_{1\parallel}\right\}=J_0(\lambda)E_{1\parallel}$ was used to simplyfied the integral on the right-hand side. The integral itself can be more simplified with performing the integral over $v_\parallel$ and μ

$$\begin{aligned}I&=\frac{2\pi(Ze)^2\mu_0B_0}{Tm}\int \mathrm{d}v_\parallel\mathrm{d}\mu\ v_\parallel^2J_0^2(\lambda)F_\mathrm{M}\\&=\frac{2\pi(Ze)^2\mu_0B_0}{Tm}e^{-\mathcal{E}/T}\underbrace{\int \mathrm{d}v_\parallel\ v_\parallel^2F_\mathrm{M}(v_\parallel)}_{n_{R_0}/2\pi}\underbrace{\int \mathrm{d}\mu\ J_0^2(\lambda)F_\mathrm{M}(\mu)}_{T/B_0\ \Gamma_0(b)}\\&=\frac{(Ze)^2\mu_0n_{R_0}}{m}\Gamma_0(b)e^{-\mathcal{E}/T}\ ,\end{aligned} \tag{30}$$

where the seperation of the Maxwellian was used [Eq. (**??**)]. Finally, the field equation for the induced electric field can be written as

$$\begin{aligned}&\left(\nabla^2-\sum_s\frac{Z_s^2e^2\mu_0n_{R_0,s}}{m_s}\Gamma_0(b_s)e^{-\mathcal{E}_s/T_s}\right)E_{1\parallel}=\\&\sum_s\frac{2\pi Z_se\mu_0B_0}{m_s}\int \mathrm{d}v_\parallel\mathrm{d}\mu\ v_\parallel J_0(\lambda_s)\mathcal{V}_s=\mu_0\frac{\partial j_{1\parallel}}{\partial t}\ .\end{aligned} \tag{31}$$

After performing the Fourier transform and normalize Equation (**??**) the final field equation for the inductive electric field is given by

$$\begin{aligned}&\left(k_{\perp\mathrm{N}}^2+\beta_\mathrm{ref}\sum_s\frac{Z_s^2n_{\mathrm{R},s}}{m_{\mathrm{R},s}}\Gamma_0(b_s)e^{-\mathcal{E}_{\mathrm{N},s}/T_{\mathrm{R},s}}\right)\frac{\widehat{E}_{1\parallel\mathrm{N}}}{2}=\\&-2\pi B_\mathrm{N}\beta_\mathrm{ref}\sum_s Z_sn_{\mathrm{R},s}v_{\mathrm{thR},s}\int \mathrm{d}v_{\parallel\mathrm{N}}\mathrm{d}\mu_\mathrm{N}\ v_{\parallel\mathrm{N}}J_0(k_\perp\rho_s)\widehat{\mathcal{V}}_{\mathrm{N},s}\ .\end{aligned} \tag{32}$$

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