



Master Thesis

Mitigation of Cancellation Problem & Introduction of GKW f version

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This thesis is dedicated to my cat **Leo**
who is part of our family since July 2020.

He is the kindest cat I ever own and very playful.
I will miss our walks together and you accompany in my life. (Not final version)

I love you.

Manuel Lippert

Declaration

The author states that every information, regarding this thesis can be found under the GitHub Repository with the link <https://github.com/ManeLippert/Masterthesis-Parallel-Electric-Field>.

Contents

1 Motivation	6
2 Plasma Physics Basics	8
2.1 Charged Particle Motion in Magnetic and Electric Field	9
2.1.1 Particle Motion in Electromagnetic Field	9
2.1.2 Drifts in the Guiding Center	12
2.2 Magnetic Confinement and Plasma Rotation	14
3 Derivation of Gyrokinetic Equation	16
3.1 Gyrokinetic Ordering	18
3.2 Gyrokinetic Lagrangian	20
3.2.1 Lagrangian in Particle Phase Space	20
3.2.2 Lagrangian in Guiding Center Phase Space	21
3.2.3 Lagrangian in gyrocenter Phase Space	24
3.3 Gyrokinetic Equation	25
3.3.1 Vlasov Equation	25
3.3.2 The delta- f Approximation	27
3.4 Maxwell's Equations	29
3.5 Pull Back Operation into the Guiding Center Phase Space	30
3.6 Gyrooperator \mathcal{G}	32
3.7 Normalization	34
3.8 Field Equations	35
3.8.1 Coulomb's Law - Perturbated Electrostatic Potential Φ_1	35
3.8.2 Plasma Compression - Perturbated Parallel Magnetic Field $B_{1\parallel}$. .	37
3.8.3 Ampere's Law - Plasma Induction $A_{1\parallel}$	38
3.8.4 Cancellation Problem	39
3.8.5 Faraday's Law - Inductive Electric Field $E_{1\parallel}$	41

Contents

4 Mitigation of Cancellation Problem in Local Gyrokinetic Simulations	43
4.1 Gyrokinetic Workshop (GKW)	44
4.2 Improvements	45
4.3 $E_{1\parallel}$ Field Equation	48
4.4 f Version of GKW	64
4.5 Nonlinear Terms	70
5 Conclusion	71
6 Appendix	73
6.1 Comparision between $E_{1\parallel}$ and $A_{1\parallel}$ for various plasma beta	75
6.1.1 For the g-version of GKW	75
6.1.2 For the f-version of GKW	79
7 Bibliography	84
Eidesstattliche Erklärung	87

CHAPTER

1

Motivation

1 Motivation

Plasma can be described in various theoretical models. The main two models are the Magnetohydrodynamics and the kinetic model. The key differences will be briefly explained in the following.

- **Magnetohydrodynamics (MHD):**

In Magnetohydrodynamics the plasma will be described as an electric conductive fluid which carries current. Here, the electrons and ions are two mixed fluids expressed in the two-fluid theory.

- **Kinetic Model:**

In the kinetic model the plasma will be described in the sixdimensional phase space through the Vlasov equation. In combination with the Maxwell's equations is it possible to describe the dynamics of the plasma as the Vlasov-Maxwell system.

In both description macroscopic quantities will be used, i.e. density, velocity of the fluid and temperature. For this thesis only the kinetic description will be used.

However, the plasma beta is considered to be one of the fundamental dimensionless parameters in plasma physics, due to its relevance for the outcome, the fusion rate ($\sim \beta^2$), bootstrap fraction ($\sim \beta$), MHD stability and confinement quality of a fusion device such as a Tokamak. Also, the plasma beta is an indicator for the relevance of electromagnetic effects in a gyrokinetic systems, since electromagnetic fields vanish in the limit $\beta \rightarrow 0$.

To investigate the kinetic model a numerical approach is often needed, due to the high dimensional partial differential equations. The numerical description can be separate in a local (flux-tube) and global description, which both have their numerical problems concerning the plasma beta β parameter. For local gyrokinetic simulations a high plasma beta results in a saturation of the heat fluxes at a high level of transport, which is known as the nonzonal transition (NZT) and was not found in realistic experiments. This behaviour limits flux-tube simulations the plasma beta under the critical value for MHD stability. The global electromagnetic gyrokinetic simulations are suffering mainly from the *cancellation problem*⁵ examined in codes which uses the particle-in-cell or the Eulerian methods⁷. Here, the cancellation problem scales with the quotient β/k_{\perp}^2 ¹⁶, which again limits investigations to very low plasma beta and high perpendicular wave vectors k_{\perp} . Overall, these problems limits the electromagnetic investigations to extremely low plasma beta.⁶

The goal of this thesis is to lay the groundwork for the global electromagnetic gyrokinetic model for GKW and to mitigate the cancellation problem in the local version of GKW. For that purpose, the field equation for the induced electric field $E_{1\parallel}$ gets implemented by using Faraday's law and the switch to *f-version* of GKW will be performed. The overall scheme of this thesis follows Chapter 5 of the Dissertation of Paul Charles Crandall⁶.

CHAPTER

Plasma Physics Basics

2

2 Plasma Physics Basics

This chapter is based on the introduction book to Tokamaks from John Wesson²⁵ For more books regarding plasma physics basics the reader is referred to

- KAUFMANN, MICHAEL 2013 *Plasmaphysik und Fusionsforschung*. Wiesbaden: Springer Fachmedien Wiesbaden
- STROTH, ULRICH 2018 *Plasmaphysik: Phänomene, Grundlagen und Anwendungen*. Berlin, Heidelberg: Springer Berlin Heidelberg.

2.1 Charged Particle Motion in Magnetic and Electric Field

In magnetic confinement devices like the tokamak reactor, the charged particles experience forces caused by magnetic and electric fields which results in distinct motion under the associated force. Charged particles can be separated in species, e.g. electrons and ions, which will be later on indicated by the index s in the governing equations in Chapter ???. Until then for simplicity reasons the index will be dropped.

2.1.1 Particle Motion in Electromagnetic Field

Due to the Lorentz force, particles with a velocity v_{\perp} perpendicular to the homogenous magnetic field, undergo a circular motion in the plane perpendicular to the magnetic field [Fig. 2.1(a)]. This type of motion has circular frequency, which is often referred to as *cyclotron frequency* and is defined as

$$\omega_c = \frac{|q|B}{m} , \quad (2.1)$$

where m and q are the mass and the charge of the particle and B the strength of the magnetic field. The radius, the so called *Larmor radius*, of this motion is given by

$$\rho = \frac{mv_{\perp}}{|q|B} \quad (2.2)$$

with the center often being referred to as *guiding center*. Note that since the Lorentz force depends on the species charge of the particle, the circulation direction is the opposite between electron in ions. Due to Coulomb collisions the plasma gets thermalized. Together with the Maxwell-Boltzmann distribution the typical thermal velocity is

$$v_{\text{th}} = \sqrt{\frac{2T}{m}} , \quad (2.3)$$

where T represents the species temperature. Based on the thermal velocity v_{th} the *thermal Larmor radius* gets introduced as

$$\rho_{\text{th}} = \frac{mv_{\text{th}}}{|q|B} . \quad (2.4)$$

In absence of forces in the direction parallel to the magnetic field the particles can move freely in parallel direction to the homogenous magnetic field. The velocity of this motion is of order of the thermal velocity v_{th} and is dominated by electrons due to their lighter mass compared to ions ($v_{\text{th,e}}/v_{\text{th,i}} = 60$). When an electric field E_{\parallel} with a component parallel to the magnetic field influences the plasma, the charged particles are accelerated by the electric force

$$F_{\parallel,E} = qE_{\parallel} . \quad (2.5)$$

The parallel motion follows then from the equation of motion. Here the direction of the motion also depends on the species type [Fig. 2.1(b)].

Since magnetic fields are not always homogenous, an inhomogeneous magnetic field with its gradient ∇B containing a component parallel to the magnetic field which is given by

$$\nabla_{\parallel}B = \frac{\mathbf{B}}{B} \cdot \nabla B \quad (2.6)$$

causes the force

$$F_{\parallel,\nabla_{\parallel}B} = -\frac{mv_{\perp}^2}{2B}\nabla_{\parallel}B = -\mu\nabla_{\parallel}B ; \quad \mu = \frac{mv_{\perp}^2}{2B} \quad (2.7)$$

with *magnetic moment* μ . The magnetic moment μ is an adabatic invariant (constant of motion) if the variation of the magnetic field over time is smaller than the inverse of the cyclotron frequency ω_c^{-1} and the spatial variation is larger than Larmor radius ρ . Note that, the force is not depending on the charge q , so the direction of the force $F_{\parallel,\nabla_{\parallel}B}$ is for electrons and ions the same. The resulting force has its application in the mirror effect where a charged particle gets reflected due to this force [Fig. 2.1(c)].

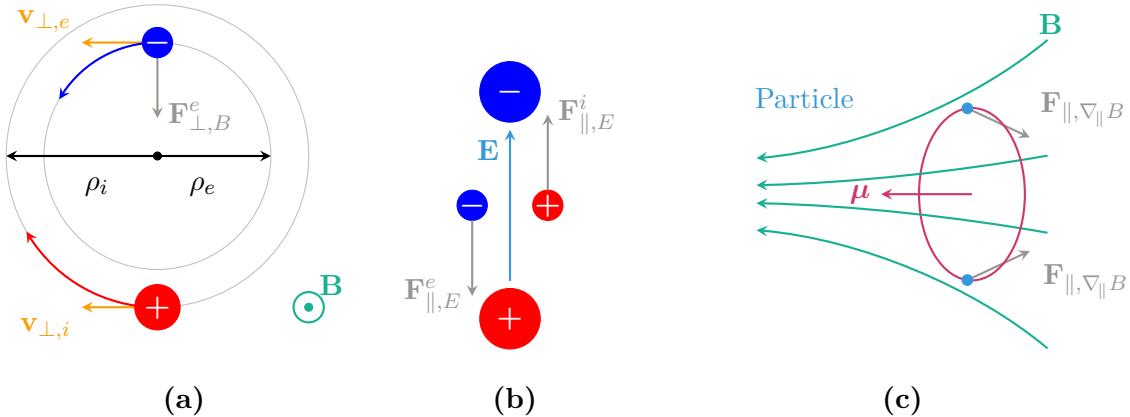


Figure 2.1: Forces acting on a charged particle:

- (a) Circular motion caused by Lorentz force $\mathbf{F}_{\perp,B}^s$ with perpendicular velocity $\mathbf{v}_{\perp,s}$ and Larmor radius ρ_s ,
- (b) Electric force $\mathbf{F}_{\parallel,E}^s$ with electric field \mathbf{E} ,
- (c) Mirror effect with force $\mathbf{F}_{\parallel,\nabla_{\parallel}B}$ and magnetic moment μ caused by an inhomogeneous magnetic field \mathbf{B} .

2.1.2 Drifts in the Guiding Center

In the presence of a magnetic field (homogenous, inhomogeneous or perturbed) and electric fields the guiding center undergoes slow (compared to the thermal velocity v_{th}) drift motions perpendicular to the magnetic field. There are several examples for this drift motion. Here the main three drift types will be covered in the following.

1. $\mathbf{E} \times \mathbf{B}$ Drift:

If an electric field \mathbf{E} with a perpendicular component together with the magnetic field \mathbf{B} (both fields are homogenous) is present the acting Coulomb force and Lorentz force results into a drift of the gyrocenter with

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2} \quad (2.8)$$

which is called the $\mathbf{E} \times \mathbf{B}$ drift. Since both acting forces direction depends on the species type the direction of the $\mathbf{E} \times \mathbf{B}$ drift is for every species the same [Fig. 2.2(a)].

2. ∇B Drift:

Inhomogeneous magnetic field causes a gradient ∇B of the magnetic field. Because of that gradient the gyrocenter undergoes a ∇B drift defined by

$$\mathbf{v}_{\nabla B} = \frac{mv_{\perp}^2}{2q} \frac{\mathbf{B} \times \nabla B}{B^3} . \quad (2.9)$$

The gradient of the magnetic field ∇B varies thereby on scales larger compared to the Larmor radius. The direction of the ∇B drift depends on the species type [Fig. 2.2(b)].

3. Curvature Drift:

Due to centrifugal force acting on the particle in a curved magnetic field the gyrocenter experiences a curvature drift according to

$$\mathbf{v}_C = \frac{mv_{\parallel}^2}{q} \frac{\mathbf{B} \times \mathbf{C}}{B^2} = \frac{mv_{\parallel}^2}{q} \frac{\mathbf{B} \times \nabla B}{B^3} ; \quad \mathbf{C} = -(\mathbf{b} \cdot \nabla)\mathbf{b} = \frac{\nabla B}{B} , \quad (2.10)$$

where \mathbf{b} is the unit vector along the magnetic field. To obtain the result for the curvature \mathbf{C} in Eq. (2.10) the plasma pressure has to be small compared to the magnetic field strength B . In the form of Eq. (2.10) ∇B and curvature drift can be treated similarly.

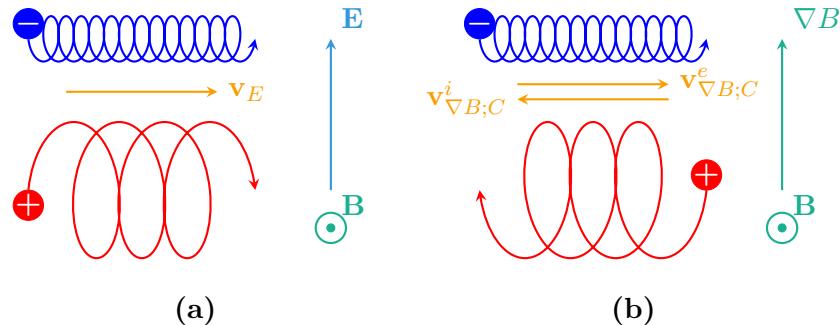


Figure 2.2: Drift motion in gyrocenter:

- (a) $E \times B$ Drift with drift velocity \mathbf{v}_E , electric field \mathbf{E} and magnetic field \mathbf{B} ,
- (b) ∇B Drift/Curvature Drift with drift velocity $\mathbf{v}_{\nabla B;C}^s$, magnetic field \mathbf{B} and gradient of the magnetic field ∇B .

2.2 Magnetic Confinement and Plasma Rotation

In tokamak devices strong magnetic fields confine the hot plasma. As mentioned in Chapter 2.1 a magnetic field forces a perpendicular particle motion and a motion which contains the gyro motion and slow perpendicular guiding center drifts. Because of the much smaller size of the Larmor radius compared to the device size R the particle and energy losses are caused by the guiding center drifts. To avoid additional loss of particles because of the parallel motion, the field lines of the magnetic field in the tokamak devices is shaped like a torus. This type of geometry has nested surfaces with constant magnetic flux, the so-called *flux surfaces*, and magnetic field lines which lie on these surfaces. To maintain stability the magnetic field has a toroidal and a poloidal component. According to the force balance the magnetic field is equivalent to the plasma pressure which means on flux-surfaces the plasma pressure is constant.^{22,25} The toroidal component is produced by external coils whereas the poloidal component is provided by the toroidal plasma current. Together the components result in a magnetic field which follows helical trajectories [Fig 2.3]. To characterize the quality of confinement the so-called *plasma beta* is used and is given as

$$\beta = \frac{nT}{B^2/2\mu_0} , \quad (2.11)$$

with n the plasma density, T as temperature, μ_0 the permeability in vacuum and the magnetic field strength B . Respectively, the plasma beta compares the thermal plasma pressure nT to the ambient magnetic field pressure $B^2/2\mu_0$. For fusion devices the plasma beta has to be a bit smaller than 1 ($\beta < 1$) for optimal confinement. In a tokamak reactor the plasma beta has a typical order of a few percent. The plasma beta is also considered to be one of the fundamental dimensionless parameters in plasma physics, due to its relevance for the outcome, the fusion rate ($\sim \beta^2$), bootstrap fraction ($\sim \beta$) and MHD stability of a fusion device. Furthermore, the plasma beta is an indicator for the relevance of electromagnetic effects in a gyrokinetic systems, since electromagentic fields vanish in the limit $\beta \rightarrow 0$.⁶

The rotation of the plasma can be described in a co-rotating frame of reference, which is rigidly rotating with the velocity \mathbf{u}_0 and will be used later on in the derivation of the gyrokinetic equations in Chapter 3.2. It assumens that the poloidal component of the plasma rotation is much smaller compared to the torodial component and will be neglected. With this assumption in mind the reference frame is chosen to move in the torodial direction exclusivly and its velocity \mathbf{u}_0 can be expressed as

$$\mathbf{u}_0 = \boldsymbol{\Omega} \times \mathbf{x} = R^2 \boldsymbol{\Omega} \nabla \varphi , \quad (2.12)$$

where $\boldsymbol{\Omega}$ is the constant angular frequency, φ is the toroidal angle and $R\nabla\varphi$ is the unit vector in the toroidal direction.

2 Plasma Physics Basics

Since the rotation of the plasma in the laboratory frame is not a rigid body rotation, it will be characterized by the radial profile of the angular velocity $\hat{\Omega}(\psi)$. The angular frequency of the rotating frame Ω is chosen to match the plasma rotation on a certain point, i.e. $\Omega = \hat{\Omega}(\psi_r)$. The plasma rotation in the co-rotating frame of reference will be denoted as

$$\omega_\varphi(\psi) = \hat{\Omega}(\psi) - \Omega. \quad (2.13)$$

with the rotation speed along the magnetic field line

$$u_{\parallel} = \frac{RB_t}{B}\omega_\varphi(\psi), \quad (2.14)$$

where B_t is the toroidal component of the magnetic field.²⁰

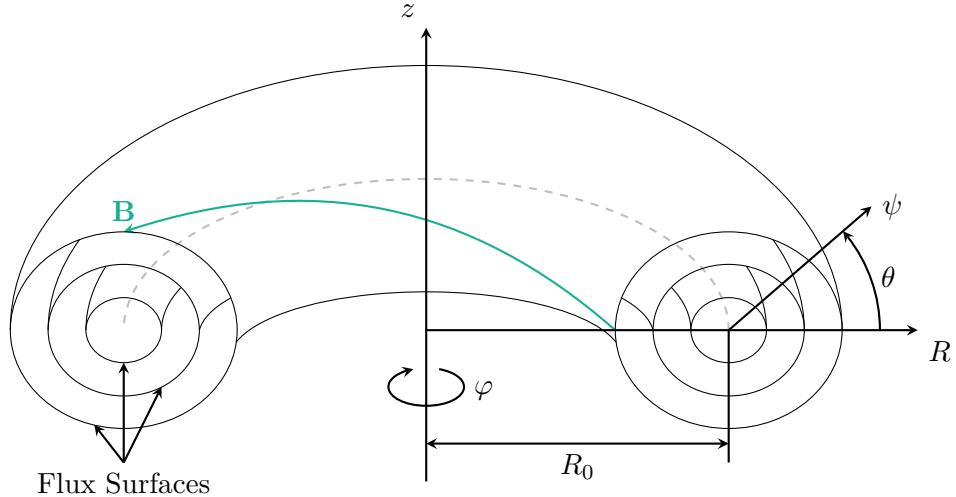


Figure 2.3: Toroidal flux surfaces in tokamak plasma with helical magnetic field (green line) in torus coordinates (ψ (radial), φ (toroidal), θ (poloidal)) or cylindrical coordinates (z , R , $-\varphi$).³

CHAPTER

Derivation of Gyrokinetic Equation

3

3 Derivation of Gyrokinetic Equation

In this chapter the following scheme will be discussed:

1. The Lagrangian L for a particle in a magnetic field will reformulate in the fundamental one-form γ according to

$$\int dt L = \int \gamma . \quad (3.1)$$

From this point on the fundamental one-form and Lagrangian refers to the quantity γ . Then, the Lagrangian γ will be transformed in guiding center phase space and separated in its equilibrium and perturbated part. Through the Lie transformation the Lagrangian gets transformed into the gyrocenter phase space by eliminating the gyro phase.

2. The Lagrangian gets plugged into the Euler-Lagrangian equation, which results in the equations of motions. From the equations of motion the Vlasov equation can be derived.
3. The Vlasov equation solves for the density distribution function, which will be used to express the particle density n and current \mathbf{j} with the moments of the distribution function.
4. Particle density n and current \mathbf{j} will then be plugged into the Maxwell's equations and the field equations.

This chapter is based on the Dissertation of Tilman Dannert⁸ and the derivation document provided by the GKW group²³. The derivation of Faraday's law in Chapter 3.8.5 is based on the dissertation of Paul Crandall⁶.

3.1 Gyrokinetic Ordering

In the derivation of the gyrokinetic theory the aim is to decouple the effect of small scale, small amplitude fluctuations of the plasma in the Langrangian. For that, the properties of fluctuations will be chosen as small parameter, which will result in the ordering assumptions applied in gyrokinetic theory. This section is based on Ref. 4 and 14. There are the following assumptions:

- **Low Frequency**

The characteristic fluctuation frequency is small compared to the cyclotron frequency

$$\Rightarrow \frac{\omega}{\omega_c} \ll 1 .$$

- **Anisotropy**

The length scales of the turbulence are associated with the wave vector \mathbf{k} which can be separated into a perpendicular component $k_{\perp} = |\mathbf{k} \times \mathbf{b}|$ and a parallel component $k_{\parallel} = |\mathbf{k} \cdot \mathbf{b}|$ where \mathbf{b} is parallel to the toroidal component of the magnetic field. The perpendicular correlation length of the turbulence has a length scale of around 10 – 100 gyroradii while the parallel length scales can be of the order of meters, which can be expressed in wavenumber as

$$\Rightarrow \frac{k_{\parallel}}{k_{\perp}} \ll 1 .$$

- **Strong Magnetization**

The Larmor radius ρ is small compared to the gradient length scales for

- Background Density: $L_n = n_0 \left(\frac{dn_0}{dx} \right)^{-1}$
- Background Temperature: $L_T = T_0 \left(\frac{dT_0}{dx} \right)^{-1}$
- Background Magnetic Field: $L_B = B_0 \left(\frac{dB_0}{dx} \right)^{-1}$

$$\Rightarrow \frac{\rho}{L_n} \sim \frac{\rho}{L_T} \sim \frac{\rho}{L_B} \ll 1 .$$

- **Small Fluctuations**

The fluctuating part of the gyrocenter distribution function F_1 is assumed to be small compared to the background distribution function F_0

$$\Rightarrow \frac{F_1}{F_0} \ll 1 .$$

Furthermore, the fluctuations of the vector potential \mathbf{A}_1 and scalar potentials Φ_1 are small compared to their background part

$$\Rightarrow \frac{\mathbf{A}_1}{\mathbf{A}_0} \sim \frac{\Phi_1}{\Phi_0} \ll 1 .$$

3 Derivation of Gyrokinetic Equation

Taken together these assumptions leads to the gyrokinetic ordering

$$\frac{\omega}{\omega_c} \sim \frac{k_{\parallel}}{k_{\perp}} \sim \frac{\rho}{L_n} \sim \frac{\rho}{L_T} \sim \frac{\rho}{L_B} \sim \frac{F_1}{F_0} \sim \frac{\mathbf{A}_1}{\mathbf{A}_0} \sim \frac{\Phi_1}{\Phi_0} \sim \epsilon_{\delta}, \quad (3.2)$$

where ϵ_{δ} is a small parameter. For the derivation of the gyrokinetic equations of GKW all derived equations are evaluated up to the first order of the ratio of the reference thermal Larmor radius $\rho_{\text{th,ref}}$ and the equilibrium magnetic length scale L_B as small parameter and is defined as

$$\rho_{\star} = \frac{\rho_{\text{th,ref}}}{L_B} = \frac{m_{\text{ref}} v_{\text{th,ref}}}{e B_{\text{ref}}} \sim \epsilon_{\delta}. \quad (3.3)$$

3.2 Gyrokinetic Lagrangian

3.2.1 Lagrangian in Particle Phase Space

The Lagrangian of a particle γ with mass m and charge number Z in the electro magnetic field will be described through the particle position \mathbf{x} and the velocity \mathbf{v} as coordinates $\{\mathbf{x}, \mathbf{v}\}$ and can be written as

$$\gamma = \gamma_\nu dz^\nu = \underbrace{(\mathbf{mv} + Ze\mathbf{A}(\mathbf{x})) \cdot d\mathbf{x}}_{\text{Symplectic Part}} - \underbrace{\left(\frac{1}{2}mv^2 + Ze\Phi(\mathbf{x}) \right)}_{\text{Hamiltonian } H(\mathbf{x}, \mathbf{v})} dt , \quad (3.4)$$

where \mathbf{A} and Φ are the vector and scalar potential, ν indexes the six coordinates and Einstein notation is applied. This form is also known as fundamental one-form.

The defined Lagrangian γ will then be transformed in the rotating frame of reference [Ch. 2.2], which can be achieved the following Lorentz transformation

$$\mathbf{v} \rightarrow \mathbf{v} + \mathbf{u}_0 \quad \mathbf{E} \rightarrow \mathbf{E} + \mathbf{u}_0 \times \mathbf{B} \quad \Phi \rightarrow \Phi + \mathbf{A} \cdot \mathbf{u}_0 . \quad (3.5)$$

After performing the transformation outlined in Ref. 20 the Lagrangian γ becomes

$$\gamma = (m\mathbf{v} + m\mathbf{u}_0 + Ze\mathbf{A}(\mathbf{x})) \cdot d\mathbf{x} - \left(\frac{1}{2}mv^2 - \frac{1}{2}mu_0^2 + Ze\Phi(\mathbf{x}) \right) dt . \quad (3.6)$$

In the next step small scale perturbations of the electromagnetic field gets introduced as following

$$\Phi = \Phi_0 + \Phi_1 \quad \mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1 . \quad (3.7)$$

Here, it is assumend that the equilibrium electric field is zero in a stationary plasma, but it will be kept in case for finite plasma rotation. According to the gyrokinetic ordering [Ch. 3.1] the perturbations are in the first order of ρ_\star . Taking everything into account the Lagrangian in the particle phase space with perturbations can be written as

$$\begin{aligned} \gamma &= \gamma_0 + \gamma_1 \\ \gamma_0 &= (m\mathbf{v} + m\mathbf{u}_0 + Ze\mathbf{A}_0(\mathbf{x})) \cdot d\mathbf{x} - \left(\frac{1}{2}mv^2 - \frac{1}{2}mu_0^2 + Ze\Phi_0(\mathbf{x}) \right) dt \\ \gamma_1 &= Ze\mathbf{A}_1(\mathbf{x}) \cdot d\mathbf{x} - Ze\Phi_1(\mathbf{x}) dt . \end{aligned} \quad (3.8)$$

3 Derivation of Gyrokinetic Equation

3.2.2 Lagrangian in Guiding Center Phase Space

For the description of charged particle behaviour in the tokamak device the *guiding center coordinates* are used [Fig. 3.1]. This set of coordinates are defined as the following

$$\begin{aligned} \mathbf{X}(\mathbf{x}, \mathbf{v}) &= \mathbf{x} - \mathbf{r} & v_{\parallel} &= \mathbf{v} \cdot \mathbf{b}(\mathbf{x}) \\ \mu(\mathbf{x}, \mathbf{v}) &= \frac{mv_{\perp}^2(\mathbf{x})}{2B(\mathbf{x})} & \theta(\mathbf{x}, \mathbf{v}) &= \arccos \left(\frac{1}{v_{\perp}} (\mathbf{b}(\mathbf{x}) \times \mathbf{v}) \cdot \hat{\mathbf{e}}_1 \right) , \end{aligned} \quad (3.9)$$

where the guiding center follows the magnetic field with the parallel velocity v_{\parallel} . The gyromotion is described together with the magnetic moment μ , the guiding center \mathbf{X} and the gyro phase θ which gives a parameter set of six quantities $\{\mathbf{X}, v_{\parallel}, \mu, \theta\}$. Vector $\mathbf{b}(\mathbf{x})$ is the unit vector in the direction of the equilibrium magnetic field and $\mathbf{r} = \rho(\mathbf{x}, \mathbf{v})\mathbf{a}(\mathbf{x}, \mathbf{v})$ is the vector pointing from the guiding center to the particles position, which is defined by the unit vector $\mathbf{a}(\mathbf{x}, \mathbf{v})$ and its length is the Lamor radius $\rho(\mathbf{x}, \mathbf{v})$. The unit vector $\mathbf{a}(\mathbf{x}, \mathbf{v})$ can be expressed in a local orthonormal basis as the function of the gyroangle θ

$$\mathbf{a}(\theta) = \hat{\mathbf{e}}_1 \cos \theta + \hat{\mathbf{e}}_2 \sin \theta . \quad (3.10)$$

The vectors \mathbf{b} , $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ form a local Cartesian coordinate system at the guiding center position.

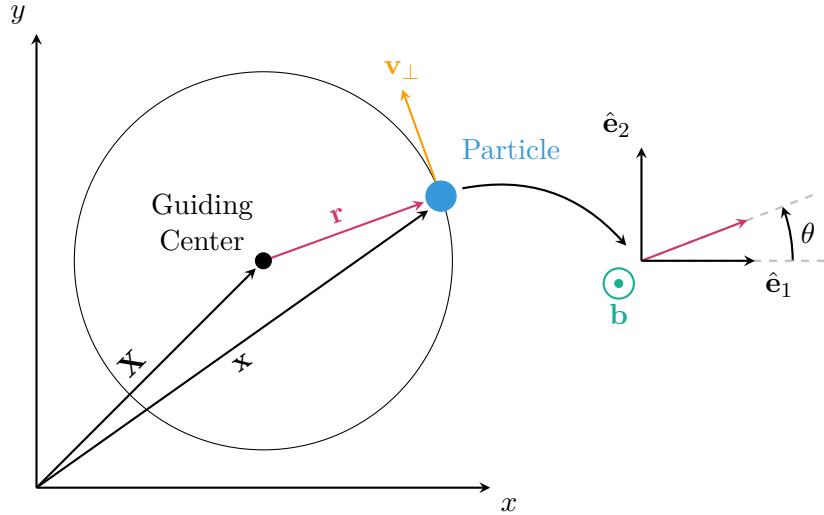


Figure 3.1: Sketch of guiding center coordinates where the charged particle performs a circular motion around the guiding center.¹²

3 Derivation of Gyrokinetic Equation

To transform the fundamental one-form into the guiding center coordinates the following relation will be used

$$\Gamma_\eta = \gamma_\nu \frac{dz^\nu}{dZ^\eta} , \quad (3.11)$$

where Γ_η is a component of the guiding center fundamental one-form. To calculate the new coordinates the transformation [Eq. (3.9)] have to be inverted to provide the old coordinates as function of the new one $z(Z)$. Here, the direct transformation is clearly uniquely determined if the magnetic field is known at the particle position. However, the inverse transformation is not uniquely due to the dependence of the Larmor radius ρ on magnetic field at the particle position \mathbf{x} . Applying Taylor expansion around the guiding center \mathbf{X} for the Larmor radius ρ yields $\rho(\mathbf{x}) \approx \rho(\mathbf{X})$. Note, that terms of order ρ^2 , which leads to second order terms in ρ_\star , will get neglected due to the gyrokinetic ordering. With result of the taylor expansion the particle position \mathbf{x} can be expressed with the guiding center coordinates as

$$\mathbf{x}(\mathbf{X}, \theta) \approx \mathbf{X} + \rho(\mathbf{X})\mathbf{a}(\theta) . \quad (3.12)$$

The particle velocity \mathbf{v} is the sum of the velocity along the magnetic field v_{\parallel} , the gyration velocity \mathbf{v}_{\perp} and the drift velocity, which will be neglected because the particle drifts can be described by the motion of the guiding center. So to summarize the velocity \mathbf{v} in the guiding center frame can be expressed as

$$\mathbf{v} = v_{\parallel}\mathbf{b}(\mathbf{x}) + \mathbf{v}_{\perp} = v_{\parallel}\mathbf{b}(\mathbf{x}) + \rho(\mathbf{x})\dot{\mathbf{a}}(\theta) . \quad (3.13)$$

Applying Taylor expansion again around the guiding center \mathbf{X} the following expression can be obtained

$$\mathbf{v}(\mathbf{X}, v_{\parallel}, \mu, \theta) \approx v_{\parallel} [\mathbf{b}(\mathbf{X}) + \partial_{\mathbf{X}}\mathbf{b}(\mathbf{X}) \cdot \mathbf{a}(\theta)\rho(\mathbf{X}, \mu)] + \rho(\mathbf{X}, \mu)\dot{\mathbf{a}}(\theta) . \quad (3.14)$$

Now, the transformation [Eq. (3.11)] can be applied to express the fundamental one-form in the new coordinates. For that the components of the fundamental one-form of the particle phase space γ_ν and the equation for \mathbf{v} in guiding center coordinates [Eq. (3.14)] will be inserted. After that, the gyroaveraging operator \mathcal{G} will be used, which is defined as the integral over the gyrophase θ

$$\mathcal{G}\{G(\mathbf{x})\} = \bar{G}(\mathbf{X}) = \frac{1}{2\pi} \int_0^{2\pi} d\theta G(\mathbf{X} + \mathbf{r}(\theta)) , \quad (3.15)$$

with an example field $G(\mathbf{x})$. Due to the definition of the vector \mathbf{a} [Eq. (3.10)] the first order terms in \mathbf{a} and $\dot{\mathbf{a}}$ disappear under gyroaveraging. Following all the previous steps, one can obtain the fundamental one-form in guiding center coordinates

$$\begin{aligned} \bar{\Gamma} = & (mv_{\parallel}\mathbf{b}(\mathbf{X}) + m\mathbf{u}_0 + Ze\mathbf{A}(\mathbf{X})) \cdot d\mathbf{X} + \frac{2\mu m}{Ze} d\theta \\ & - \left(\frac{1}{2}mv_{\parallel}^2 - \frac{1}{2}mu_0^2 + Ze\Phi(\mathbf{X}) + \mu B(\mathbf{X}) \right) dt . \end{aligned} \quad (3.16)$$

3 Derivation of Gyrokinetic Equation

Note that as a consequence of the Lagrangian being independent of the gyrophase θ , the magnetic moment μ (the associated conjugated coordinate pair of θ) becomes an invariant of the motion ($\dot{\mu} = 0$).

As in Chapter 3.2.1 perturbations [Eq. (3.7)] will get introduced to the guiding center Lagrangian. The transformation of the equilibrium part is already performed above, so only the perturbation part with the perturbed Lagrangian in the particle phase space γ_1 has to be transformed to the guiding center phase space. The transformation is analogous to the calculation before, the key difference is that the fluctuations quantities vary on a small length scale and Taylor expansion around the guiding center \mathbf{X} can not be applied advantageous. After inserting the components of the perturbed Lagrangian γ_1 into the transformation [Eq. (3.11)] and neglecting terms of order ρ^2 , due to gyrokinetic ordering, the fundamental one form in the guiding center coordinates can be expressed as

$$\begin{aligned}\Gamma &= \bar{\Gamma}_0 + \Gamma_1 \\ \bar{\Gamma}_0 &= (mv_{\parallel}\mathbf{b}(\mathbf{X}) + m\mathbf{u}_0 + Ze\mathbf{A}_0(\mathbf{X})) \cdot d\mathbf{X} + \frac{2\mu m}{Ze}d\theta \\ &\quad - \left(\frac{1}{2}mv_{\parallel}^2 - \frac{1}{2}mu_0^2 + Ze\Phi_0(\mathbf{X}) + \mu B_0(\mathbf{X}) \right) dt \\ \Gamma_1 &= Ze\mathbf{A}_1(\mathbf{x}) \cdot d\mathbf{X} + \frac{Z}{|Z|} \frac{1}{v_{\perp}} \mathbf{A}_1(\mathbf{x}) \cdot \mathbf{a}(\theta) d\mu + \frac{Z}{|Z|} \frac{2\mu}{v_{\perp}} \mathbf{A}_1(\mathbf{x}) \cdot \frac{d\mathbf{a}(\theta)}{d\theta} d\theta - Ze\Phi_1(\mathbf{x}) dt.\end{aligned}\tag{3.17}$$

3.2.3 Lagrangian in Gyrocenter Phase Space

The transformation of the guiding center Lagrangian Γ into the Lagrangian in gyrocenter phase space $\bar{\Gamma}$ aims to remove the gyroangle θ dependence introduced by the fluctuations of the guiding center lagrangian. To distinguish between the guiding center and the gyrocenter coordinates all quantities associated with the gyrocenter are getting an overbar, i.e. $\bar{\Gamma}$. The new set of gyrocenter coordinates are given by $\{\bar{\mathbf{X}}, \bar{v}_{\parallel}, \bar{\mu}\}$. Since the derivation of fundamental one-form in gyrocenter phase space $\bar{\Gamma}$ uses the Lie transform perturbation method, which is beyond the scopes of this thesis, the reader is referred to the Refs. 8 and 23 for more details. The Lagrangian in the gyrocenter phase space can be expressed as

$$\begin{aligned} \bar{\Gamma} &= \bar{\Gamma}_0 + \bar{\Gamma}_1 \\ &= (mv_{\parallel}\mathbf{b}_0(\mathbf{X}) + m\mathbf{u}_0 + Ze(\mathbf{A}_0(\mathbf{X}) + \bar{\mathbf{A}}_1(\mathbf{X}))) \cdot d\mathbf{X} + \frac{2\mu m}{Ze} d\theta \\ &\quad - \left(\frac{1}{2}m(v_{\parallel}^2 - u_0^2) + Ze(\Phi_0(\mathbf{X}) + \bar{\Phi}_1(\mathbf{X})) + \mu(B_0(\mathbf{X}) + \bar{B}_{1\parallel}(\mathbf{X})) \right) dt, \end{aligned} \quad (3.18)$$

where B_0 is the equilibrium magnetic field and $\bar{B}_{1\parallel}$ is the magnetic field introduced by the vector potential $\bar{\mathbf{A}}_1$. Note, that the perturbations of the scalar and vector potential will be separated into an oscillating and a gyroaveraged part which will be expressed as

$$\Phi_1 = \tilde{\Phi}_1 + \bar{\Phi}_1 \quad \mathbf{A}_1 = \tilde{\mathbf{A}}_1 + \bar{\mathbf{A}}_1, \quad (3.19)$$

although the oscillating parts will be added to the gauge function of the Lie transformation and will not be included in the Lagrangian of the gyrocenter phase space. The quantity $\bar{B}_{1\parallel}$ is the shorter notation of following gyroaveraged quantity defined as

$$\mathcal{G}\{Ze\mathbf{A}_1(\mathbf{x}) \cdot \mathbf{v}_{\perp}(\mathbf{X}, \mu, \theta)\} = \mu\bar{B}_{1\parallel}(\mathbf{X}). \quad (3.20)$$

3.3 Gyrokinetic Equation

3.3.1 Vlasov Equation

Because of the large number of particles in the fusion plasma a prediction on the basis of Newton-Maxwell dynamics results in an impossible task for simulation, but this problem can be solved with a statistical approach. For that the distribution function $f(\mathbf{x}, \mathbf{v}, t)$ in the particle phase space $\{\mathbf{x}, \mathbf{v}\}$ will be considered. Because collisions are happening at much smaller frequencies than the characteristic frequencies connected to turbulence, the collisionless model is often preferred⁹ which results through evolution of the particle density distribution function in the *Vlasov equation*

$$\frac{\partial f}{\partial t} + \dot{\mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{x}} + \dot{\mathbf{v}} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0 . \quad (3.21)$$

In the gyrocenter phase space $\{\mathbf{X}, v_{\parallel}, \mu\}$, the overbar introduced in Chapter 3.2.3 gets dropped for simplicity for all quantities, the Vlasov equation with the gyrocenter distribution function F takes the following form

$$\frac{\partial F}{\partial t} + \dot{\mathbf{X}} \cdot \frac{\partial F}{\partial \mathbf{X}} + \dot{v}_{\parallel} \cdot \frac{\partial F}{\partial v_{\parallel}} = 0 , \quad (3.22)$$

where the gyrophase θ is still an ignorable coordinate and the time derivative of the magnetic moment μ is zero, because the magnetic moment μ is an exact invariant. In Equation (3.22) the terms of the time derivative of the gyrocenter $\dot{\mathbf{X}}$ and the parallel velocity \dot{v}_{\parallel} have to be expressed through the gyrocenter Lagrangian with the Euler-Lagrange equation. The Euler-Lagrange equation can be written as

$$\left(\frac{\partial \gamma_j}{\partial z^i} - \frac{\partial \gamma_i}{\partial z^j} \right) \frac{dz^j}{dt} = \frac{\partial H}{\partial z^i} + \frac{\partial \gamma_i}{\partial t} . \quad (3.23)$$

Inserting Equation (3.18) into the Euler-Lagrange equation and apply multiple calculations detailed in Ref. 23 the equations of motion can be obtained as

$$\begin{aligned} \dot{\mathbf{X}} &= \mathbf{b}_0 v_{\parallel} + \mathbf{v}_{\chi} + \mathbf{v}_D & \dot{v}_{\parallel} &= \frac{\dot{\mathbf{X}}}{mv_{\parallel}} \cdot \left(Ze\bar{\mathbf{E}} - \mu \nabla (B_0 + \bar{B}_{1\parallel}) + \underbrace{\frac{1}{2} m \nabla u_0^2}_{mR\Omega^2 \nabla R} \right) \\ \dot{\mu} &= 0 & \dot{\theta} &= \omega_c - \frac{Ze}{m} \partial_{\mu} \left(Ze\bar{\mathbf{A}}_1 \cdot \dot{\mathbf{X}} - Ze\bar{\Phi}_1 - \mu \bar{B}_{1\parallel} \right) , \end{aligned} \quad (3.24)$$

with the drift velocity \mathbf{v}_{χ} defined as the sum of the streaming velocity perpendicular to the perturbed magnetic field $\mathbf{v}_{\bar{B}_{1\perp}}$, the $\mathbf{E} \times \mathbf{B}$ drift in the total electric field $\mathbf{v}_{\bar{E}}$ and the grad- B drift of the parallel perturbed magnetic field $\mathbf{v}_{\nabla \bar{B}_{1\parallel}}$. The drift velocity

3 Derivation of Gyrokinetic Equation

\mathbf{v}_D containing the sum of the curvature drift \mathbf{v}_C , the grad- B drift of the equilibrium magnetic field $\mathbf{v}_{\nabla B_0}$ and the drifts due to the Coriolis force \mathbf{v}_{C_0} and centrifugal force \mathbf{v}_{Ce} . Note that, the term containing u_0^2 got replaced with $u_0^2 = R^2\Omega^2$ from Equation (2.12). The quantity χ can be expressed as

$$\chi = \underbrace{(\Phi_0 + \bar{\Phi}_1)}_{\bar{\Phi}} - v_{\parallel} \bar{A}_{1\parallel} + \frac{\mu}{Ze} \bar{B}_{1\parallel} \quad (3.25)$$

which results the drift velocity \mathbf{v}_χ

$$\mathbf{v}_\chi = \frac{\mathbf{b}_0 \times \nabla \chi}{B_0} = \mathbf{v}_{\bar{E}} + \mathbf{v}_{\bar{B}_{1\perp}} + \mathbf{v}_{\nabla \bar{B}_{1\parallel}}, \quad (3.26)$$

with $\mathbf{b}_0 \times (\nabla A_{1\parallel}) \approx \nabla \times (A_{1\parallel} \mathbf{b}_0) = \mathbf{B}_{1\perp}$. The total electric field $\bar{\mathbf{E}}$ is defined as

$$\bar{\mathbf{E}} = -\nabla \bar{\Phi} - \partial_t \bar{\mathbf{A}}_1 = -\nabla \bar{\Phi} - \partial_t (\mathbf{b}_0 \cdot \bar{A}_{1\parallel} + \bar{\mathbf{A}}_{1\perp}). \quad (3.27)$$

Due to normalization assumptions in gyrokinetics¹⁸ is the time derivative of the vector potential $\partial_t \bar{\mathbf{A}}_1$ one order smaller as the gradient of the electrostatic potential $\nabla \bar{\Phi}$ and the contribution of the vector potential will be neglected in the $\mathbf{v}_{\bar{E}}$ velocity term.

3 Derivation of Gyrokinetic Equation

3.3.2 The delta-*f* Approximation

The delta-*f* approximation separates the density distribution function F into an equilibrium part F_0 and perturbation part F_1 , i.e. $F = F_0 + F_1$. Applying the delta-*f* approximation on the gyrocenter Vlasov equation leads to

$$\frac{\partial F_1}{\partial t} + \dot{\mathbf{X}} \cdot \nabla F_1 + \dot{v}_{\parallel} \cdot \frac{\partial F_1}{\partial v_{\parallel}} = \underbrace{-\dot{\mathbf{X}} \cdot \nabla F_0 - \dot{v}_{\parallel} \frac{\partial F_0}{\partial v_{\parallel}}}_S, \quad (3.28)$$

with the source term S . Substituting the equations for $\dot{\mathbf{X}}$ and \dot{v}_{\parallel} from Equation (3.24) into the delta-*f* approximated Vlasov equation results in

$$\frac{\partial F_1}{\partial t} + \dot{\mathbf{X}} \cdot \nabla F_1 - \frac{\mathbf{b}_0}{m} \cdot (Ze\nabla\Phi_0 + \mu\nabla B_0 - mR\Omega^2\nabla R) \cdot \frac{\partial F_1}{\partial v_{\parallel}} = S. \quad (3.29)$$

Note that only the terms of order ρ_{\star} has to be kept in $\dot{v}_{\parallel}\partial_{v_{\parallel}}F_1$, which results in neglecting the drift velocities \mathbf{v}_{χ} and \mathbf{v}_D and the contribution of $\bar{B}_{1\parallel}$ and $\bar{\Phi}_1$, since these terms are after calculation of order ρ_{\star}^2 .

The equilibrium distribution function F_0 is assumed to be a Maxwellian which includes a finite equilibrium electric field Φ_0 to balance the centrifugal force (in the co-rotating frame) due toroidal rotation of the plasma.

In the rotating frame the included energy term can be written as

$$\mathcal{E} = Ze\langle\Phi_0\rangle - \frac{1}{2}m\omega_{\varphi}^2(R^2 - R_0^2), \quad (3.30)$$

where $\langle \cdot \rangle$ denotes flux-surface averaging, ω_{φ} the plasma rotation frequency [Eq. (2.13)], R the local major radius and R_0 is an integration constant which can be chosen, i.e. major radius of the tokamak or flux surface average of the major radius. The Maxwellian is given by the following expression

$$F_0 = F_M(\mathbf{X}, v_{\parallel}, \mu) = \frac{n_{R_0}}{(2\pi T/m)^{3/2}} \exp\left(-\frac{\frac{1}{2}m(v_{\parallel} - u_{\parallel})^2 + \mu B_0 + \mathcal{E}}{T}\right), \quad (3.31)$$

where n_{R_0} is the particle density at the position $R = R_0$ and is related to equilibrium particle density through the relation $n_0 = n_{R_0} \exp(-\mathcal{E}/T)$.²⁰ Furthermore, u_{\parallel} is the rotation speed of the plasma in the rotating frame parallel to the magnetic field [Eq. (2.14)]. The Maxwellian can be separated in

$$F_M(\mathbf{X}, v_{\parallel}, \mu) = F_M(v_{\parallel})F_M(\mu)e^{-\mathcal{E}/T}, \quad (3.32)$$

$$F_M(v_{\parallel}) = \frac{n_{R_0}}{(2\pi T/m)^{3/2}} \exp\left(-\frac{\frac{1}{2}m(v_{\parallel} - u_{\parallel})^2}{T}\right) \quad F_M(\mu) = \exp\left(-\frac{\mu B_0}{T}\right) \quad (3.33)$$

3 Derivation of Gyrokinetic Equation

and the derivatives of the Maxwellian can be expressed as

$$\begin{aligned}\nabla F_M &= \left[\frac{\nabla n_{R_0}}{n_{R_0}} + \left(\frac{\frac{1}{2}mv_{\parallel}^2 + \mu B_0 + \mathcal{E}}{T} - \frac{3}{2} \right) \frac{\nabla T}{T} - \frac{\mu B_0}{T} \frac{\nabla B_0}{B_0} \right. \\ &\quad \left. + \left(\frac{mv_{\parallel}RB_t}{BT} + m\omega(R^2 - R_0^2) \right) \nabla \omega_{\varphi} \right] F_M \\ \partial_{v_{\parallel}} F_M &= -\frac{mv_{\parallel}}{T} F_M \\ \partial_{\mu} F_M &= -\frac{B_0}{T} F_M ,\end{aligned}\tag{3.34}$$

where the $\nabla \omega_{\varphi}$ terms are the result of the derivatives of the parallel rotation velocity u_{\parallel} and rotation energy \mathcal{E} evaluated at zero rotation speed locally in the co-rotating frame.²⁰ It can be shown with Equations (3.24) and (3.34) that the ∇B_0 term in $-\dot{\mathbf{X}} \cdot \nabla F_M$ cancels with $(\dot{\mathbf{X}}/mv_{\parallel})\mu\nabla B_0\partial_{v_{\parallel}}F_M$ for purely torodial rotation. Finally, the source term can than be written as

$$\begin{aligned}S &= -(\mathbf{v}_{\chi} + \mathbf{v}_D) \cdot \tilde{\nabla} F_M - \frac{Zev_{\parallel}}{T} \partial_t \bar{A}_{1\parallel} F_M \\ &\quad - \frac{F_M}{T} (v_{\parallel} \mathbf{b}_0 + \mathbf{v}_D + \mathbf{v}_{\bar{B}_{1\perp}}) \cdot (Ze\nabla\bar{\Phi} + \mu\nabla\bar{B}_{1\parallel}) ,\end{aligned}\tag{3.35}$$

where $\tilde{\nabla}$ referres to only the ∇n_{R_0} , ∇T and $\nabla \omega_{\varphi}$ terms of ∇F_M .

3.4 Maxwell's Equations

To obtain a closed system the Vlasov equation gets combined with the Maxwell equations to calculate the pertubated electromagnetic fields. As usual in fusion plasma the Coulomb law gets replaced by the quasi neutrality condition which implies that any deviation from neutrality can only happen on small length scales within the Debye length and on a timescale much shorter than of the fluctuations. Due to non-relativistic timescale of the turbulence the displacement current in Ampere's law gets also neglected. Taking everything into account the Maxwell's equations can be written as

$$\begin{aligned} \sum_s Z_s e n_s &= 0 & \nabla \times \mathbf{E}_1 &= -\frac{\partial \mathbf{B}_1}{\partial t} \\ \nabla \cdot \mathbf{B}_1 &= 0 & \nabla \times \mathbf{B}_1 &= \mu_0 \sum_s \mathbf{j}_s , \end{aligned} \quad (3.36)$$

where the index s refers to the species of particles, i.e. proton or electron and \sum_s means that all species will be taken into account. For simplicity of the derivation the "s" index gets droped if not explicitly needed. The Maxwell's equation contain densities n and currents \mathbf{j} of the particles which can be expressed through the moments of the particle phase space distribution function f as follows

$$\begin{aligned} n(\mathbf{x}) &= \int d\mathbf{v} f(\mathbf{x}, \mathbf{v}) \\ j_{\parallel} &= Ze \int d\mathbf{v} v_{\parallel} f(\mathbf{x}, \mathbf{v}) \\ \mathbf{j}_{\perp} &= Ze \int d\mathbf{v} \mathbf{v}_{\perp} f(\mathbf{x}, \mathbf{v}) . \end{aligned} \quad (3.37)$$

3.5 Pull Back Operation into the Guiding Center Phase Space

Since the Vlasov equations [Eq. (3.29)] describes the evolution of the distribution function in the gyrocenter phase space F , the particle moments will be expressed with the guiding center phase space distribution function F^{gc} and which itself will be described through the gyrocenter distribution function F by performing a pull back from F to the guiding center phase space. A schematic about the general idea can be seen in Figure 3.2. The pull back will be performed with the pull back operator \mathcal{P} which results in

$$F^{\text{gc}} = \mathcal{P}\{F\} = F - \underbrace{\frac{F_M}{T} (Ze\tilde{\Phi}_1 - \mu\bar{B}_{1\parallel})}_{\text{Correction Term}}, \quad (3.38)$$

where $\tilde{\Phi}_1$ denotes to the oscillating part of the perturbation Φ_1 . Here, the correction term contain the fluctuations of the electro-magnetic fields and describes physically the polarization and magnetization effects of the fluctuations on the gyro orbit.⁴



Figure 3.2: Idea of gyrokinetic Maxwell's equations: The particle density $n(\mathbf{x})$, the current density $\mathbf{j}(\mathbf{x})$ and the gyrocenter distribution function F are expressed in the guiding center phase space.

The particle density n and currents \mathbf{j} of one species can be expressed with the guiding center distribution function F^{gc} as

$$\begin{aligned} n &= \int d\mathbf{v} f(\mathbf{x}, \mathbf{v}) = \frac{B_0}{m} \int d\mathbf{X} dv_{\parallel} d\theta d\mu \delta(\mathbf{X} + \mathbf{r} - \mathbf{x}) F^{\text{gc}} \\ j_{\parallel} &= Ze \int d\mathbf{v} v_{\parallel} f(\mathbf{x}, \mathbf{v}) = \frac{ZeB_0}{m} \int d\mathbf{X} dv_{\parallel} d\theta d\mu \delta(\mathbf{X} + \mathbf{r} - \mathbf{x}) v_{\parallel} F^{\text{gc}} \\ \mathbf{j}_{\perp} &= Ze \int d\mathbf{v} \mathbf{v}_{\perp} f(\mathbf{x}, \mathbf{v}) = \frac{ZeB_0}{m} \int d\mathbf{X} dv_{\parallel} d\theta d\mu \delta(\mathbf{X} + \mathbf{r} - \mathbf{x}) \mathbf{v}_{\perp} F^{\text{gc}}, \end{aligned} \quad (3.39)$$

where B_0/m is the Jacobian of the guiding center coordinates. The delta function δ appears due to the change of coordinates and guarantees that the spatial region taken into account in the integral remains unchanged during the coordinate transformation. Physically, the delta function δ expresses that all particles which have a Larmor orbit crossing a given point \mathbf{x} in the real space contribute to the particle density [Fig. 3.3].

3 Derivation of Gyrokinetic Equation

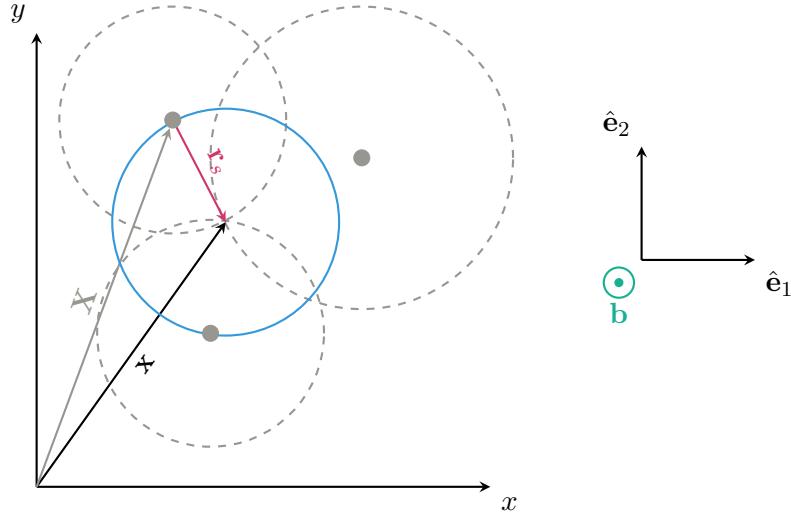


Figure 3.3: Connection between density of particles $n_s(\mathbf{x})$ and density of guiding centers: Gyro orbits with different guiding center \mathbf{X} (gray dashed circles) can cross in position \mathbf{x} , such that the respective gyrating particles add to the particle density n_s there. For a fixed Larmor radius $\rho_s = |\mathbf{r}_s| = |\rho_s \mathbf{a}|$ (red) the particle density n_s at \mathbf{x} is obtained by collecting the contributions of all guiding centers on a circle with radius ρ_s centered at position \mathbf{x} (blue circle).

3 Derivation of Gyrokinetic Equation

3.6 Gyrooperator \mathcal{G}

As stated in Chapter 3.2.2 the gyrooperator \mathcal{G} averages over the gyrophase θ which is mostly used in the derivation of the field equation and is defined as

$$\mathcal{G}\{G(\mathbf{x})\} = \bar{G}(\mathbf{X}) = \frac{1}{2\pi} \int_0^{2\pi} d\theta G(\mathbf{X} + \mathbf{r}(\theta)) . \quad (3.40)$$

To derive the field equations a second kind of gyrooperator will be introduced as

$$\mathcal{G}^\dagger\{G(\mathbf{X})\} = \langle G \rangle(\mathbf{x}) = \frac{1}{2\pi} \int_0^{2\pi} d\mathbf{X} d\theta \delta(\mathbf{X} + \mathbf{r}(\theta) - \mathbf{x}) G(\mathbf{X}) , \quad (3.41)$$

where \mathcal{G}^\dagger the hermitian conjugate of \mathcal{G} ²⁴ and the delta function δ originates from the pull back operation from Chapter 3.5.¹⁵ Furthermore, the double gyroaverage operator is defined as

$$\mathcal{G}^\dagger\{\mathcal{G}\{G(\mathbf{x})\}\} = \langle \bar{G} \rangle(\mathbf{x}) = \frac{1}{(2\pi)^2} \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' G(\mathbf{X} - \mathbf{r}(\theta) + \mathbf{r}(\theta')) , \quad (3.42)$$

which performs a gyroaverage of the field value at all guiding center positions \mathbf{X} with particle position \mathbf{x} in their trajectory.¹⁴

In the case of local simulations the gyrooperators \mathcal{G} and \mathcal{G}^\dagger simplifies to

$$\begin{aligned} \bar{\mathbf{G}}(\mathbf{x}) &= J_0(\lambda) \mathbf{G}(\mathbf{X}) \\ \langle \mathbf{G}(\mathbf{X}) \rangle &= J_0(\lambda) \mathbf{G}(\mathbf{x}) \end{aligned} \quad (3.43)$$

with J_0 as the zeroth order Bessel function. Note, that $\bar{B}_{1\parallel}(\mathbf{X}) = -I_1(\lambda)\mu B_{1\parallel}(\mathbf{X})$, where I_1 is the modified first order Bessel function of first kind defined as $I_1(\lambda) = 2/\lambda J_1(\lambda)$. To obtain Equation (3.43) the process of gyroaveraging will be performed in the Fourier space as follows

$$\begin{aligned} \bar{\mathbf{G}}(\mathbf{x}) &= \bar{\mathbf{G}}(\mathbf{X} + \mathbf{r}) = \mathcal{G} \left\{ \int d\mathbf{k} \hat{\mathbf{G}}(\mathbf{k}) e^{i\mathbf{k}\cdot(\mathbf{X}+\mathbf{r})} \right\} \\ &= \frac{1}{2\pi} \int_0^{2\pi} d\theta \int d\mathbf{k} \hat{\mathbf{G}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{X}} e^{ik_\perp \rho \cos \theta} \\ &= \int d\mathbf{k} \hat{\mathbf{G}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{X}} \underbrace{\frac{1}{2\pi} \int_0^{2\pi} d\theta e^{ik_\perp \rho \cos \theta}}_{J_0(\rho k_\perp)} \\ &= \int d\mathbf{k} J_0(\rho k_\perp) \hat{\mathbf{G}}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{X}} = J_0(\lambda) \mathbf{G}(\mathbf{x}) , \end{aligned} \quad (3.44)$$

where the wave vector \mathbf{k} is defined as $\mathbf{k} = \hat{e}_1 k_\perp$. The argument λ is given by $i\rho\nabla_\perp$ which is the inverse Fourier transformed expression of ρk_\perp . The same routine can be applied

3 Derivation of Gyrokinetic Equation

for the gyrooperator \mathcal{G}^\dagger with the same result. In general the Bessel function is defined as⁸

$$\begin{aligned} J_n(z) &= \left(\frac{z}{2}\right)^n \underbrace{\sum_{\nu=0}^{\infty} \frac{((-1/4z^2)^\nu)}{\nu!(1+\nu)!}}_{I_n(z)}, \\ J_n(z) &= \frac{i^{-n}}{\pi} \int_0^\pi d\theta e^{iz \cos \theta} \cos(n\theta). \end{aligned} \quad (3.45)$$

Integrals with Bessel Function

In the upcoming section there will be often integrals which contain the zeroth Besselfunction $J_0(\lambda)$ and the modified Besselfunctions $I_1(\lambda)$. In general this types of integrals have the form

$$\int dv_{\parallel} d\mu v_{\parallel}^m \mu^n J_0^{2-n}(\lambda) I_1^n(\lambda) F_M, \quad (3.46)$$

wuth the natural number $n, m = \{0, 1, 2\}$. Equation (3.46) can be seperated together with the Maxwellian [Eq. (3.33)] to

$$e^{-\mathcal{E}/T} \int dv_{\parallel} v_{\parallel}^m F_M(v_{\parallel}) \int d\mu \mu^n J_0^{2-n}(\lambda) I_1^n(\lambda) F_M(\mu). \quad (3.47)$$

The first integral appears in the following types

$$\begin{aligned} 1) \int dv_{\parallel} F_M(v_{\parallel}) &= \frac{n_{R_0} m}{2\pi T} \\ 2) \int dv_{\parallel} v_{\parallel} F_M(v_{\parallel}) &= 0 \quad (\text{Due to symmetry}) \\ 3) \int dv_{\parallel} v_{\parallel}^2 F_M(v_{\parallel}) &= \frac{n_{R_0}}{2\pi} \end{aligned} \quad (3.48)$$

and the last integral occurs in three types

$$\begin{aligned} 1) \int d\mu J_0^2(\lambda) F_M(\mu) &= \frac{T}{B_0} \Gamma_0(b) \\ 2) \int d\mu \mu J_0(\lambda) I_1(\lambda) F_M(\mu) &= \frac{T^2}{B_0^2} (\Gamma_0(b) - \Gamma_1(b)) \\ 3) \int d\mu \mu^2 I_1^2(\lambda) F_M(\mu) &= \frac{T^3}{B_0^3} 2(\Gamma_0(b) - \Gamma_1(b)), \end{aligned} \quad (3.49)$$

with the notation $\Gamma_n(b) = I_n(b)e^{-b}$ with the modified Bessel function I_n [Eq. (3.45)] and $b = -\rho_{\text{th}}^2 \nabla^2_{\perp}$. ρ_{th} referres the thermal Larmor radius [Eq. (2.4)].²³

3 Derivation of Gyrokinetic Equation

3.7 Normalization

To implement the field equations into the local version of GKW, one has to normalize the quantities in the equations. The reference values are indicated by the index "ref", the dimensionless normalized by "N" and the relative dimensionless values by the index "R". This section is based on Ref 6, 23 and 18.

A reference mass m_{ref} , density n_{ref} , temperature T_{ref} , magnetic field B_{ref} and major radius R_{ref} is chosen. With these quantities the reference thermal velocity $v_{\text{th,ref}}$ and reference thermal Larmor radius $\rho_{\text{th,ref}}$ gets defines as

$$T_{\text{ref}} = \frac{1}{2} m_{\text{ref}} v_{\text{th,ref}}^2 \quad \rho_{\text{th,ref}} = \frac{m_{\text{ref}} v_{\text{th,ref}}}{e B_{\text{ref}}} = \frac{2 T_{\text{ref}}}{e B_{\text{ref}} v_{\text{th,ref}}} \quad \rho_{\star} = \frac{\rho_{\text{th,ref}}}{R_{\text{ref}}}$$

and for convenience reason the small parameter ρ_{\star} got redefined.

- **Relative Quantities:**

$$m = m_{\text{ref}} m_R \quad n_{R_0} = n_{\text{ref}} n_R \quad T = T_{\text{ref}} T_R \quad v_{\text{th}} = v_{\text{th,ref}} v_{\text{th,R}}$$

- **Normalized Quantities:**

$$\begin{aligned} R &= R_{\text{ref}} R_N & B_0 &= B_{\text{ref}} B_N & \mu &= \frac{2 T_{\text{ref}} T_R}{B_{\text{ref}}} \mu_N & v_{\parallel} &= v_{\text{th}} v_{\parallel N} \\ k &= \frac{k_N}{\rho_{\star}} & k_{\parallel} &= \frac{k_{\parallel N}}{\rho_{\star}} & k_{\perp} &= \frac{k_{\perp N}}{\rho_{\text{th,ref}}} \\ \beta &= \beta_{\text{ref}} \beta_N & \beta_{\text{ref}} &= \frac{2 \mu_0 n_{\text{ref}} T_{\text{ref}}}{B_{\text{ref}}^2} \end{aligned}$$

- **Fluctuating Fields:**

$$\begin{aligned} \Phi_1 &= \rho_{\star} \frac{T_{\text{ref}}}{e} \Phi_{1N} & B_{1\parallel} &= \rho_{\star} B_{\text{ref}} B_{1\parallel N} \\ A_{1\parallel} &= B_{\text{ref}} R_{\text{ref}} \rho_{\star}^2 A_{1\parallel N} & E_{1\parallel} &= \frac{2 T_{\text{ref}}}{e} \frac{1}{R_{\text{ref}}} \rho_{\star} E_{1\parallel N} \end{aligned}$$

- **Time, Frequency and Centrifugal Energy:**

$$t = \frac{R_{\text{ref}}}{v_{\text{th,ref}}} t_N \quad \Omega = \frac{v_{\text{th,ref}}}{R_{\text{ref}}} \Omega_N \quad \mathcal{E} = T_{\text{ref}} \mathcal{E}_N$$

- **Distrubution Function and Vlasov Equation:**

$$F = \rho_{\star} \frac{n_{R_0}}{v_{\text{th}}^3} F_N \quad F_M = \frac{n_{R_0}}{v_{\text{th}}^3} F_{MN} \quad \mathcal{V} = \rho_{\star} \frac{v_{\text{th,ref}}}{R_{\text{ref}}} \frac{n_{R_0}}{v_{\text{th}}^3} \mathcal{V}_N$$

- **Gradients:**

$$\nabla_{\perp} = \frac{1}{R_{\text{ref}}} \nabla_{\perp N} \quad \nabla_{\parallel} = \frac{1}{R_{\text{ref}}} \nabla_{\parallel N} .$$

3.8 Field Equations

3.8.1 Coulomb's Law - Perturbed Electrostatic Potential Φ_1

To evaluate the perturbed electrostatic potential Φ_1 in the gyrocenter phase space Equation (3.38) gets inserted into the equation for the particle density $n(\mathbf{x})$ which results in

$$n_1(\mathbf{x}) = \frac{B_0}{m} \int d\mathbf{X} dv_{\parallel} d\theta d\mu \delta(\mathbf{X} + \mathbf{r} - \mathbf{x}) \left(F_1 - \frac{F_M}{T} (Ze\tilde{\Phi}_1 - \mu\bar{B}_{1\parallel}) \right) \\ = \bar{n}_1(\mathbf{x}) + n_P(\mathbf{x}), \quad (3.50)$$

with the density of the gyrocenter $\bar{n}_1(\mathbf{x})$ and the variations on the gyro orbit to the particle density $n_P(\mathbf{x})$ which describes the polarization effects of the fluctuating fields on the gyro orbit⁴. The gyrocenter density can be simplified with the gyrooperator \mathcal{G}^\dagger [Eq. (3.41)] to

$$\bar{n}_1(\mathbf{x}) = \frac{B_0}{m} \int d\mathbf{X} dv_{\parallel} d\theta d\mu \delta(\mathbf{X} + \mathbf{r} - \mathbf{x}) F_1 = \frac{2\pi B_0}{m} \int dv_{\parallel} d\mu \langle F_1 \rangle. \quad (3.51)$$

The polarization density n_P is given by

$$n_P(\mathbf{x}) = -\frac{2\pi B_0}{m} \int dv_{\parallel} d\mu \frac{F_M}{T} (Ze(\Phi_1(\mathbf{x}) - \langle \bar{\Phi}_1 \rangle(\mathbf{x})) - \mu \langle \bar{B}_{1\parallel} \rangle). \quad (3.52)$$

To derive the term for the polarization density n_P the oscillating Part of the electro static potential got replaced with $\tilde{\Phi}_1(\mathbf{X} + \mathbf{r}) = \Phi_1(\mathbf{X} + \mathbf{r}) - \bar{\Phi}_1(\mathbf{X})$ and the gyrooperator \mathcal{G}^\dagger were used. Taking everything into account and insert it into the quasineutrality equation $\sum_s Z_s e n_{1,s} = 0$ the field equation for the perturbated electrostatic potential Φ_1 is given by

$$\sum_s \frac{Z_s^2 e^2}{m_s} \int dv_{\parallel} d\mu \frac{F_{M,s}}{T_s} (\Phi_1(\mathbf{x}) - \langle \bar{\Phi}_1 \rangle(\mathbf{x})) = \\ \sum_s \frac{Z_s e}{m_s} \int dv_{\parallel} d\mu \langle F_{1,s} \rangle + \frac{F_{M,s}}{T_s} \mu \langle \bar{B}_{1\parallel} \rangle. \quad (3.53)$$

With the use of the local gyrooperator \mathcal{G} the density of the gyrocenter \bar{n}_1 and the polarization density n_P can be expressed as

$$\bar{n}_1(\mathbf{x}) = \frac{2\pi B_0}{m} \int dv_{\parallel} d\mu J_0(\lambda) F_1(\mathbf{x}, v_{\parallel}, \mu), \quad (3.54)$$

$$n_P(\mathbf{x}) = \frac{Zen_{R_0}(\mathbf{x})}{T} e^{-\mathcal{E}/T} (\Gamma_0(b) - 1) \Phi_1(\mathbf{x}) \\ + n_{R_0}(\mathbf{x}) e^{-\mathcal{E}/T} (\Gamma_0(b) - \Gamma_1(b)) \frac{B_{1\parallel}(\mathbf{x})}{B_0}, \quad (3.55)$$

3 Derivation of Gyrokinetic Equation

where n_{R_0} is the background density at position R_0 . To derive the term for the polarization density n_P the integrals mentioned in Chapter 3.6 were performed. Then, the field equation for the perturbated electrostatic potential Φ_1 in the local simulation is given by

$$\begin{aligned} \sum_s \frac{Z_s^2 e^2}{T_s} n_{R_0,s} e^{-\mathcal{E}_s/T_s} (1 - \Gamma_0(b_s)) \Phi_1(\mathbf{x}) = \\ \sum_s Z_s e \left(\bar{n}_{1,s} + n_{R_0,s} e^{-\mathcal{E}_s/T_s} (\Gamma_0(b_s) - \Gamma_1(b_s)) \frac{B_{1\parallel}(\mathbf{x})}{B_0} \right) \end{aligned} \quad (3.56)$$

and takes the following form after performing the Fourier-transform and applying the normalizing expressions [Ch. 3.7]

$$\begin{aligned} \sum_s Z_s n_{R,s} e^{-\mathcal{E}_{N,s}/T_{R,s}} (1 - \Gamma_0(b_s)) \frac{Z_s}{T_{R,s}} \hat{\Phi}_{1N} = \\ \sum_s Z_s n_{R,s} 2\pi B_N \int dv_{\parallel N} d\mu_N J_0(k_{\perp} \rho_s) \hat{F}_{1N,s} \\ + \sum_s Z_s n_{R,s} e^{-\mathcal{E}_{N,s}/T_{R,s}} (\Gamma_0(b_s) - \Gamma_1(b_s)) \frac{\hat{B}_{1\parallel N}}{B_N} \end{aligned} \quad (3.57)$$

3 Derivation of Gyrokinetic Equation

3.8.2 Plasma Compression - Perturbated Parallel Magnetic Field $B_{1\parallel}$

The perpendicular component of Ampere's law can be written as

$$\nabla^2 \mathbf{A}_{1\perp} = (\nabla \times B_{1\parallel})_\perp = \begin{pmatrix} \partial_y B_{1\parallel} - \partial_z B_{1y} \\ \partial_z B_{1x} - \partial_x B_{1\parallel} \end{pmatrix} = \mu_0 \mathbf{j}_{1\perp}, \quad (3.58)$$

where z is the direction of the equilibrium magnetic field B_0 and the Coulomb gauge $\nabla \cdot \mathbf{A}_1 = 0$ was used. The parallel gradients of the perturbed magnetic field can be neglected since they are one order smaller than the perpendicular ones, which results in

$$\begin{pmatrix} \partial_y B_{1\parallel} \\ -\partial_x B_{1\parallel} \end{pmatrix} = \nabla_\perp B_{1\parallel} \times \mathbf{b} = \mu_0 \mathbf{j}_{1\perp}. \quad (3.59)$$

After performing the pull back operation the perpendicular current $\mathbf{j}_{1\perp}$ is given by

$$\mathbf{j}_{1\perp} = \frac{ZeB_0}{m} \int d\mathbf{X} dv_\parallel d\theta d\mu \delta(\mathbf{X} + \mathbf{r} - \mathbf{x}) \mathbf{v}_\perp \left(F_1 - \frac{F_M}{T} (Ze\tilde{\Phi}_1 - \mu\bar{B}_{1\parallel}) \right). \quad (3.60)$$

Inserting the perturbed perpendicular current $\mathbf{j}_{1\perp}$ into Ampere's law and apply the same method as in Ref 23 results in the field equation for the perturbed parallel magnetic field $B_{1\parallel}$ which can be expressed as

$$\begin{aligned} & \left(1 + \sum_s \beta_s (\Gamma_0(b_s) - \Gamma_1(b_s)) e^{-\mathcal{E}_s/T_s} \right) B_{1\parallel} = \\ & - \sum_s \frac{2\pi\mu_0 B_0}{m_s} \int dv_\parallel d\mu \mu I_n(\lambda_s) F_{1,s} \\ & - \sum_s (\Gamma_0(b_s) - \Gamma_1(b_s)) e^{-\mathcal{E}_s/T_s} \frac{Z_s e \mu_0 n_{R0,s}}{B_0} \Phi_1, \end{aligned} \quad (3.61)$$

with β_s as the plasma beta of a given species. Fourier's transformation and normalization yields

$$\begin{aligned} & \left(1 + \sum_s \frac{T_{R,s} n_{N,s}}{B_N^2} \beta_{ref} (\Gamma_0(b_s) - \Gamma_1(b_s)) e^{-\mathcal{E}_{N,s}/T_{R,s}} \right) \hat{B}_{1\parallel N} = \\ & - \sum_s \beta_{ref} 2\pi B_N T_{R,s} n_{R,s} \int dv_{\parallel N} d\mu_N \mu_N I_1(k_\perp \rho_s) \hat{F}_{1N,s} \\ & - \sum_s \beta_{ref} (\Gamma_0(b_s) - \Gamma_1(b_s)) e^{-\mathcal{E}_{N,s}/T_{R,s}} \frac{Z_s n_{R,s}}{2B_N} \hat{\Phi}_{1N}. \end{aligned} \quad (3.62)$$

3 Derivation of Gyrokinetic Equation

3.8.3 Ampere's Law - Plasma Induction $A_{1\parallel}$

To express the parallel perturbation of the vector potential $A_{1\parallel}$, i.e. the plasma induction, the method is analogous to Chapter 3.8.1. The parallel component of Ampere's law can be expressed as

$$\nabla^2 A_{1\parallel} = -\mu_0 j_{1\parallel} = -\mu_0 \sum_s j_{1\parallel,s} . \quad (3.63)$$

Performing the pull back again the parallel perturbation of the current density $j_{1\parallel}$ is given by

$$\begin{aligned} j_{1\parallel} &= \frac{ZeB_0}{m} \int d\mathbf{X} dv_{\parallel} d\theta d\mu \delta(\mathbf{X} + \mathbf{r} - \mathbf{x}) v_{\parallel} \left(F_1 - \frac{F_M}{T} (Ze\tilde{\Phi}_1 - \mu\bar{B}_{1\parallel}) \right) \\ &= \frac{ZeB_0}{m} \int d\mathbf{X} dv_{\parallel} d\theta d\mu \delta(\mathbf{X} + \mathbf{r} - \mathbf{x}) v_{\parallel} F_1 \\ &= \frac{2\pi ZeB_0}{m} \int dv_{\parallel} d\mu \langle v_{\parallel} F_1 \rangle , \end{aligned} \quad (3.64)$$

although the term $v_{\parallel} F_M$ vanishes during the integration along v_{\parallel} , due to symmetry of the Maxwellian F_M . Inserting Equation (3.64) into Ampere's law yields the field equation for the plasma induction as follows

$$\nabla^2 A_{1\parallel} = - \sum_s \frac{2\pi Z_s e \mu_0 B_0}{m_s} \int dv_{\parallel} d\mu \langle v_{\parallel} F_{1,s} \rangle \quad (3.65)$$

and after the use of definition of the local gyrooperator

$$\nabla^2 A_{1\parallel} = - \sum_s \frac{2\pi Z_s e \mu_0 B_0}{m_s} \int dv_{\parallel} d\mu v_{\parallel} J_0(\lambda_s) F_{1,s}(\mathbf{x}, v_{\parallel}, \mu) . \quad (3.66)$$

Fourier transformation and normalization yields

$$k_{\perp N}^2 \hat{A}_{1\parallel N} = 2\pi B_N \beta_{\text{ref}} \sum_s Z_s n_{R,s} v_{\text{thR},s} \int dv_{\parallel N} d\mu_N v_{\parallel N} J_0(k_{\perp} \rho_s) \hat{F}_{1N,s} . \quad (3.67)$$

3 Derivation of Gyrokinetic Equation

3.8.4 Cancellation Problem

Local and global Gyrokinetic simulations are suffering from numerical problems, mainly from saturation of the heat fluxes at a high level of transport (nonzonal transition (NZT)) and the cancellation problem⁵. The cancellation problem is examined in codes which uses the particle-in-cell or the Eulerian methods⁷. In the following chapter the origin of the cancellation problem will be discussed.

As in Chapter 3.3.2 discussed appears in the Source term [Eq. (3.35)] the time derivative of the parallel perturbed vector potential $\partial_t \bar{A}_{1\parallel}$. This $A_{1\parallel}$ -Term and the $\partial t F_1$ term in the gyrokinetic equation are computationally difficult to handle, as it is not immediately clear how to evaluate the terms by using a simple Runge-Kutta scheme. To avoid further complications a modified distribution function g gets introduced

$$g = F_1 + \frac{Zev_{\parallel}}{T} \bar{A}_{1\parallel} F_M . \quad (3.68)$$

Substituting the modified distribution function g into Equation (3.29) results in

$$\frac{\partial g}{\partial t} + \mathbf{v}_x \cdot \nabla g + (v_{\parallel} \mathbf{b}_0 + \mathbf{v}_D) \cdot \nabla F_1 - \frac{\mathbf{b}_0}{m} \cdot (Ze\nabla\Phi_0 + \mu\nabla B_0 - mR\Omega^2\nabla R) \frac{\partial F_1}{\partial v_{\parallel}} = S \quad (3.69)$$

with the source term S defined as

$$S = -(\mathbf{v}_x + \mathbf{v}_D) \cdot \tilde{\nabla} F_M - \frac{F_M}{T} (v_{\parallel} \mathbf{b}_0 + \mathbf{v}_D) \cdot (Ze\nabla\bar{\Phi} + \mu\nabla\bar{B}_{1\parallel}) . \quad (3.70)$$

The advantage of this substitution is clear, because now only one time derivative appears in the gyrokinetic equation. Due to the substitution the distribution function F_1 in the field equations has to be replaced by with the modified distribution g as

$$F_1 = g - \frac{Zev_{\parallel}}{T} \bar{A}_{1\parallel} F_M . \quad (3.71)$$

For the equations of perturbed electrostatic potential Φ_1 and parallel magnetic field $B_{1\parallel}$ the replacement is trivial since the integral $\int dv_{\parallel} v_{\parallel} F_M = 0$ (due to symmetry) results in the elimination of the $A_{1\parallel}$ -Term in both equation leaving the modified distribution g in the integral. For the field equation for $A_{1\parallel}$ the substitution has a different effect. By replacing F_1 with g the new normalized Fourier transformed field equation for $A_{1\parallel}$ is given by

$$\begin{aligned} & \left(k_{\perp N}^2 + \beta_{ref} \sum_s \frac{Z_s^2 n_{R,s}}{m_{R,s}} \Gamma_0(b_s) e^{-\mathcal{E}_{N,s}/T_{R,s}} \right) \hat{A}_{1\parallel N} = \\ & 2\pi B_N \beta_{ref} \sum_s Z_s n_{R,s} v_{thR,s} \int dv_{\parallel N} d\mu_N v_{\parallel N} J_0(k_{\perp} \rho_s) \hat{g}_{N,s} . \end{aligned} \quad (3.72)$$

3 Derivation of Gyrokinetic Equation

Comparing Eq. (3.67) and Eq. (3.72) one will notice the additional term in the brackets of Eq. (3.72). This term is the so-called *skin term*¹⁶ and has no physical meaning and only appears because of the substitution of the distribution function. To understand the cancellation problem completely, one has to consider the "hidden" $A_{1\parallel}$ term in the distribution g . This $A_{1\parallel}$ term has to be cancelled exactly with the skin term. But due to the different numerical representation of the skin term (analytically) and the integral over the modified distribution function (numerically), the cancellation is inexact and leads to the cancellation problem. The error of the cancellation scales with $\beta_{\text{ref}}/k_{\perp N}^2$, making simulations with high plasma beta und small k_\perp very challenging.¹⁷ To mitigate the cancellation problem the gamma function $\Gamma(b)$ gets also calculated with the same numerical scheme as the integral over the distribution, but this approach has to be performed carefully and could still lead to errors.

3 Derivation of Gyrokinetic Equation

3.8.5 Faraday's Law - Inductive Electric Field $E_{1\parallel}$

In this Chapter the $\partial_t \bar{A}_{1\parallel}$ term will get handled differnly with the consideration of electromagnetic fields. This section follows the work of Paul Crandall in chapter 5 of his Dissertation⁶.

First the Vlasov equation gets written down in the F_1 framework with the source term [Eq. (3.29) & (3.35)] and gets simplified into

$$\frac{\partial F_1}{\partial t} + \frac{Zev_{\parallel}}{T} \partial_t \bar{A}_{1\parallel} F_M = \mathcal{V}, \quad (3.73)$$

where \mathcal{V} represents all terms of the Vlasov Equation which excludes the time derivative of plasma induction $\partial_t \bar{A}_{1\parallel}$. The equation of the $\bar{A}_{1\parallel}$ is already established in Chapter 3.8.3 but for this derivation a recall will be made. The equation for $\bar{A}_{1\parallel}$ is given by

$$\nabla^2 A_{1\parallel} = -\mu_0 j_{1\parallel} = -\sum_s \frac{2\pi Z_s e \mu_0 B_0}{m_s} \int dv_{\parallel} d\mu \langle v_{\parallel} F_{1,s} \rangle. \quad (3.74)$$

Now the Faraday law will be considered

$$E_{1\parallel} = -\frac{\partial A_{1\parallel}}{\partial t}. \quad (3.75)$$

Taking the time derivative of Equation (3.74) results directly into the Faraday law and so in the field equation for the induced electric field $E_{1\parallel}$ which can be expressed as

$$\nabla^2 E_{1\parallel} - \sum_s \frac{2\pi Z_s e \mu_0 B_0}{m_s} \int dv_{\parallel} d\mu \langle v_{\parallel} \frac{\partial F_{1,s}}{\partial t} \rangle = 0. \quad (3.76)$$

In Equation (3.76) the time derivative of the gyrocenter distrubution function has to be further simplified. For that, the gyrokinetic equation shall be rewritten as

$$\frac{\partial F_1}{\partial t} = \mathcal{V} + \frac{Zev_{\parallel}}{T} \bar{E}_{1\parallel} F_M. \quad (3.77)$$

Plugging Equation (3.77) into Equation (3.76), one can derive the equation for the induced electric field

$$\begin{aligned} \left(\nabla^2 - \sum_s \frac{2\pi(Z_s e)^2 \mu_0 B_0}{T_s m_s} \int dv_{\parallel} d\mu \mathcal{G}^\dagger v_{\parallel}^2 F_{Ms} \mathcal{G} \right) E_{1\parallel} = \\ \sum_s \frac{2\pi Z_s e \mu_0 B_0}{m_s} \int dv_{\parallel} d\mu \mathcal{G}^\dagger \{v_{\parallel} \mathcal{V}_s\} = \mu_0 \frac{\partial j_{1\parallel}}{\partial t}, \end{aligned} \quad (3.78)$$

although the relation of $\bar{E}_{1\parallel} = \mathcal{G} \{E_{1\parallel}\}$ and the definition of \mathcal{G}^\dagger was used to simplyfied the integral on the right-hand side. To complete the derivation of this section the delta- f

3 Derivation of Gyrokinetic Equation

Vlasov equation will be recalled with the induced electric field $E_{1\parallel}$ in the source term. The delta- f Vlasov equation is given by

$$\frac{\partial F_1}{\partial t} + \dot{\mathbf{X}} \cdot \nabla F_1 - \frac{\mathbf{b}_0}{m} \cdot (Ze\nabla\Phi_0 + \mu\nabla B_0 - mR\Omega^2\nabla R) \cdot \frac{\partial F_1}{\partial v_{\parallel}} = S , \quad (3.79)$$

with the source term

$$S = -(\mathbf{v}_{\chi} + \mathbf{v}_D) \cdot \tilde{\nabla}F_M + \frac{Zev_{\parallel}}{T} \bar{E}_{1\parallel} F_M \\ - \frac{F_M}{T} (v_{\parallel} \mathbf{b}_0 + \mathbf{v}_D + \mathbf{v}_{\bar{B}_{1\perp}}) \cdot (Ze\nabla\bar{\Phi} + \mu\nabla\bar{B}_{1\parallel}) . \quad (3.80)$$

Using the local definition of the gyrooperator into Equation (3.76), one can derive the equation for the included electric field

$$\left(\nabla^2 - \sum_s \frac{2\pi(Z_s e)^2 \mu_0 B_0}{T_s m_s} \int dv_{\parallel} d\mu v_{\parallel}^2 J_0^2(\lambda_s) F_{Ms} \right) E_{1\parallel} = \\ \sum_s \frac{2\pi Z_s e \mu_0 B_0}{m_s} \int dv_{\parallel} d\mu v_{\parallel} J_0(\lambda_s) \mathcal{V}_s = \mu_0 \frac{\partial j_{1\parallel}}{\partial t} . \quad (3.81)$$

The integral itself can be more simplified with performing the integral over v_{\parallel} and μ

$$I = \frac{2\pi(Ze)^2 \mu_0 B_0}{T_m} \int dv_{\parallel} d\mu v_{\parallel}^2 J_0^2(\lambda) F_M \\ = \frac{2\pi(Ze)^2 \mu_0 B_0}{T_m} e^{-\mathcal{E}/T} \underbrace{\int dv_{\parallel} v_{\parallel}^2 F_M(v_{\parallel})}_{n_{R_0}/2\pi} \underbrace{\int d\mu J_0^2(\lambda) F_M(\mu)}_{T/B_0 \Gamma_0(b)} \\ = \frac{(Ze)^2 \mu_0 n_{R_0}}{m} \Gamma_0(b) e^{-\mathcal{E}/T} , \quad (3.82)$$

where the separation of the Maxwellian was used [Eq. (3.33)]. Finally, the field equation for the induced electric field can be written as

$$\left(\nabla^2 - \sum_s \frac{Z_s^2 e^2 \mu_0 n_{R_0,s}}{m_s} \Gamma_0(b_s) e^{-\mathcal{E}_s/T_s} \right) E_{1\parallel} = \\ \sum_s \frac{2\pi Z_s e \mu_0 B_0}{m_s} \int dv_{\parallel} d\mu v_{\parallel} J_0(\lambda_s) \mathcal{V}_s = \mu_0 \frac{\partial j_{1\parallel}}{\partial t} . \quad (3.83)$$

After performing the Fourier transform and normalize Equation (3.83) the final field equation for the induced electric field is given by

$$\left(k_{\perp N}^2 + \beta_{ref} \sum_s \frac{Z_s^2 n_{R,s}}{m_{R,s}} \Gamma_0(b_s) e^{-\mathcal{E}_{N,s}/T_{R,s}} \right) \hat{E}_{1\parallel N} = \\ - 2\pi B_N \beta_{ref} \sum_s Z_s n_{R,s} v_{thR,s} \int dv_{\parallel N} d\mu_N v_{\parallel N} J_0(k_{\perp} \rho_s) \hat{\mathcal{V}}_{N,s} . \quad (3.84)$$

CHAPTER

Mitigation of Cancellation Problem in Local Gyrokinetic Simulations

4

4.1 Gyrokinetic Workshop (**gkw**)

The gyrokinetic simulations of this thesis will be performed with the nonlinear local (flux-tube) version of the Gyrokinetic Workshop (GKW). In this version GKW solves the gyrokinetic equation for the modified distribution function g [Ch. 3.8.4] and uses the field aligned Hamada coordinates¹⁰ with the coordinate set $\{s, \psi, \zeta, v_{\parallel}, \mu\}$. Here, s parameterizes the length along the field lines, ψ is the radial coordinate (normalized by ρ_{th}) and ζ represents the approximately the binormal coordinate. To read more about the code GKW the reader is referred to Ref. 18 or to the manual¹⁹ in the bitbucket repository.

The simulations itself were performed on the **emil**¹ cluster and the cluster **festus**² of the University Bayreuth or on one of the **btppx** machines of the TPV Chair.

4.2 Improvements

At the beginning, it is necessary to talk about the applied code improvements which were done to ensure a valid implementation of the Faraday Law into GKW.

(1) In diagnostics part of GKW several subroutines and functions relies on the definition of the variable `requirements` from the module `diagnostics`. The variable `requirements` is a matrix which communicates which type of data or ghost cell needs to be provided to the diagnostic. In the previous implementation the number of columns were hard coded the field identifier for the gyroaveraged parallel magnetic field $\bar{B}_{1\parallel}$ with `BPAR_GA_FIELD`. This type of problem was found on multiple occasions throughout the code most noteable in `dist.f90` with `derivs_in_lin_terms` and the token array in `diagnos_generic.f90`. Furthermore, it was found that in the code more inconvenece sturctures were established. For example slicing the `requirements` matrix was performend by `PHI_FIELD:BPAR_FIELD` and `PHI_GA_FIELD:BPAR_GA_FIELD`, the start of the tokens for moments with `BPAR_GA_FIELD` and the definition for the distribution function to the number 4. To prevent any errors with future modifications, for example an additional new field, new variabels and scheme gets introduced in `global.f90` as

- `MIN_FIELD` as the smallest number of field identifier,
- `MIN_GA_FIELD` as the smallest number of gyroaverages field identifier
- `MAX_FIELD` as the greatest number of field identifier,
- `MAX_GA_FIELD` as the greatest number of gyroaverages field identifier,
- `DISTRIBUTION` should always have the greatest number and
- `MAX_IDX_FIELD` is the greatest number, i.e. `DISTRIBUTION`.

These changes allowed to the implementation `EPAR_FIELD` and `EPAR_GA_FIELD` as

- `EPAR_FIELD = 4,`
- `EPAR_GA_FIELD = 8`

as well as a new scheme for slicing gets introduced

- `PHI_FIELD:BPAR_FIELD` replaced by `MIN_FIELD:MAX_FIELD`,
- `PHI_GA_FIELD:BPAR_GA_FIELD` replaced by `MIN_GA_FIELD:MAX_GA_FIELD`.

Note that, with the new variables the slicing gets performend from `PHI_FIELD` to `EPAR_FIELD` and `PHI_GA_FIELD` to `EPAR_GA_FIELD`, since the field indentifier for inductive electric field $E_{1\parallel}$ is the greatest number in both cases. The field identifier for the distribution functionen F_1 changed to the number 9 and the size of the arrays or matrix is defined by `MAX_IDX_FIELD`. It is advicable to make sure that the field identifier for the distrubution function is always the greatest number. Further changes were performed in the whole code to ensure the new scheme. The changed code sequence in `global.f90` is listed below

4 Mitigation of Cancellation Problem in Local Gyrokinetic Simulations

```

!-----
!> Please note following notation to prevent errors in diagnostics,
!> check input (through requirements) and creation of ghost cells.
!>
!> - MIN_FIELD = smallest number of field identifier
!> - MIN_GA_FIELD = smallest number of gyroaverages field identifier
!> - MAX_FIELD = greatest number of field identifier
!> - MAX_GA_FIELD = greatest number of gyroaverages field identifier
!>
!> - DISTRIBUTION should always have the greatest number
!>
!> - MAX_IDX_FIELD = the greatest number, i.e. DISTRIBUTION
!-----

integer, parameter, public :: PHI_FIELD = 1, APAR_FIELD = 2
integer, parameter, public :: BPAR_FIELD = 3, EPAR_FIELD = 4
integer, parameter, public :: PHI_GA_FIELD = 5, APAR_GA_FIELD = 6
integer, parameter, public :: BPAR_GA_FIELD = 7, EPAR_GA_FIELD = 8
integer, parameter, public :: DISTRIBUTION = 9
integer, parameter, public :: EVERY_FIELD = 0

integer, parameter, public :: MIN_FIELD      = PHI_FIELD
integer, parameter, public :: MAX_FIELD      = EPAR_FIELD
integer, parameter, public :: MIN_GA_FIELD   = PHI_GA_FIELD
integer, parameter, public :: MAX_GA_FIELD   = EPAR_GA_FIELD
integer, parameter, public :: MAX_IDX_FIELD = DISTRIBUTION

```

- (2) Since the calculation of $E_{1\parallel}$ needs the right-hand side of the Vlasov equation \mathcal{V} and the regular fields perform the calculation with the distribution function F_1 a separation between additional and regular field equations were done. For that purpose, new variables get introduced in `dist.f90` that follows the existing notation

- `nregular_fields_start` as the start of the solutions of the regular fields, i.e. Φ , $A_{1\parallel}$ and $B_{1\parallel}$, in `fdisi`,
- `nadditional_fields_start` as the start of the solutions of the additional fields, i.e. $E_{1\parallel}$, in `fdisi`,
- `nadditional_fields_end` as the end of the solutions of the regular fields in `fdisi`.

Here, `nregular_fields_start` replaces the variable `n_phi_start` to improve the code to a more general naming scheme. Note that, the declaration of the new variables have a specific place in the code and should not be changed. So, if someone wants to add a new regular field the definition of the number of elements in `fdisi` and ghost cells should be put above the declaration from `nregular_fields_start` and `nregular_fields_end` the same goes for additional fields.

4 Mitigation of Cancellation Problem in Local Gyrokinetic Simulations

- (3) The size of the field matrix `poisson_int` and `mat_field_diag` in the module `matdat` was implemented to be big with the size of `ntot` which is the number of grid points for the whole array `fdisi`. To add a more natural way to define the size of the regular field matrices the new variable `nelem_regular_fields` gets defined as `nelem_regular_fields = nf * (1 * number_of_fields)` in `dist.f90`. Here, `nf = nsp*nx*nmu*npar*nmod` stands for the number of grid points for the distribution function F_1 and `number_of_fields` is an integer which gets incremented for every activated field calculation. In general, `nelem_regular_fields` is always less than `ntot` which could improve the runtime of the code, since allocating of the field matrixes takes less time than before. The declaration of `nelem_regular_fields` is in the same code block as `nregular_fields_start` and `nregular_fields_end` and should not be changed, since it relies heavily on the parameter `number_of_fields`.
- (4) In the subroutine `calculate_fields` in `fields.f90` the division of the diagonal parts of the regular fields was performed by a loop starting at `nregular_fields_start` and ends at the size of `mat_field_diag`. Since it was very unintuitive to start a loop not at one, further investigations were done, and it was found that in `linear_terms.f90` a unity block of the size of `nf` was added in the front of the first element of `mat_field_diag`. This performed action was removed and the loop in `calculate_fields` adjusted to loop from the first element to the last element of `mat_field_diag`.
- (5) The subroutine `g2f_correct` from the module `linear_terms` gets renamed. Since it calculates the $A_{1\parallel}$ correction it was more convenient to call the subroutine `apar_correct`. Associated switches in the module control got renamed as well.
- (6) Overall at multiple occasions the code syntax got corrected as well as some minor mistakes.

4.3 $E_{1\parallel}$ Field Equation

Since the code is capable of parallelization the inserted code is written to use the OpenMPI library. This will from now on uncomment. Further subtle changes or import statements will be uncommented since most of them are just following the established structure for $A_{1\parallel}$ and add them for $E_{1\parallel}$ into the code as well. The reader is referred to the branch in bibucket¹³ for more in depth documentation on the process and implementation.

Feature Switch

For the calculation for $E_{1\parallel}$ a new switch has to be added into `control.f90` which relates to the parameter `nlepar`. By default, `nlepar` is set off. Since, the implementation of $E_{1\parallel}$ is electromagnetic it is necessary to activate the switch for $A_{1\parallel}$ as well, i.e. `nlapar`. All upcoming changes in the code can only be accessed if `nlepar` is set to `.true.` which gets implemented with an if statement.

Identifiers `iepar` and `iepar_ga`

In the next step the identifiers for the $E_{1\parallel}$ and $\bar{E}_{1\parallel}$ will be defined in `dist.f90`. These identifiers makes it possible to access only $E_{1\parallel}$ and $\bar{E}_{1\parallel}$ in the solution `fdisi` with the use of the index function from the module `index_function`. The definition of the identifiers follows the already established scheme with the name `iepar` and `iepar_ga` although

- `iepar = 4 + IS_3D_FIELD = 12` with `IS_3D_FIELD = 2**3 = 8` and
- `iepar_ga = 4 + IS_GYROAVG_FIELD = 36` with `IS_GYROAVG_FIELD = 2**5 = 32`.

It was checked, if the new identifiers are not used from the existing implementation.

Definition of Grid Points and Ghost Cells

To continue the implementation, the grid points of $E_{1\parallel}$ gets defined as well as the ghost cells of $E_{1\parallel}$.

```

! if the inductive electric field is kept increase the size
if (nlepar) then
    number_of_fields = number_of_fields + 1
    n_epar           = nf + nfields
    nelem_epar       = nx*ns*nmod
    nfields          = nfields + nelem_epar

```

4 Mitigation of Cancellation Problem in Local Gyrokinetic Simulations

```

ioffset_epar      = ghost_size_s
ioffset_epar2     = ghost_size_x
ghost_size_s      = ghost_size_s      + nx*nmod*ghost_points_s
ghost_size_x      = ghost_size_x      + nmod*ns*ghost_points_x
ghost_size_x_phi = ghost_size_x_phi + nmod*ns*ghost_points_x
endif

```

Here, `number_of_fields` refers to the total number of fields that gets calculated, `n_epar` to the start of the grid points for $E_{1\parallel}$ in `fdisi`, `nelem_epar` for the amount of grid points and `nfields` for all grid points for the fields calculation, i.e. additional and regular fields. Additionally, the maximum number of grid points for $E_{1\parallel}$ (`nf`) gets added to the total number of grid points `ntot` with `ntot = ntot + nf`, where `nf` equals the number of grid points for the distribution function F_1 . The same definition will be also made for the gyroaveraged field $\bar{E}_{1\parallel}$ as listed below. The general scheme is the same as for $E_{1\parallel}$.

```

if ((nlepar).and.(.not. spectral_radius)) then
    n_epar_ga        = nf + nfields
    nelem_epar_ga   = nmod*nx*ns*nmu*nsp
    nfields         = nfields + nelem_epar_ga
    ! the offset of epar_ga within the ghost block
    ioffset_epar_ga = ghost_size_s
    ioffset_epar_ga2 = ghost_size_x
    ! increase the ghost size: derivatives in phi_ga
    ghost_size_s     = ghost_size_s      + nmod*nx*nmu*nsp*ghost_points_s
    ghost_size_x     = ghost_size_x      + nmod*ns*nmu*nsp*ghost_points_x
    ghost_size_x_pga = ghost_size_x_pga + nmod*ns*nmu*nsp*ghost_points_x
    ghost_size_x2_pga = ghost_size_x2_pga + nmod*ns*nmu*nsp*ghost_points_xf
endif

```

It is necessary to mention that the implementation of the grid points for $E_{1\parallel}$ results in an compilation error of GKW. The error is strongly connected to the variable `field_id` which is used to create an quick look up array for the fields in the function `get_field_index_lookup_array` in `diagnos_growth_freq` and `diagnos_timetrace` as well as in a code section in the subroutine `normalise_init` in the module `normalise`. The main problem in the function `get_field_index_lookup_array` is, that the size of the array `field_id` gets allocated with the variable `number_of_fields` from module `dist`. The incrementation of `number_of_fields` by $E_{1\parallel}$ results in an GKW error called "bad identifier". Adding $E_{1\parallel}$ into the function solves the issue. In the subroutine `normalise_init` the code for $E_{1\parallel}$ and $\bar{E}_{1\parallel}$ has to be added as well as the allocated size of `field_id` increased by two. At the end in the subroutine `calc_phase` from the module `diagnos_growth_freq` the size of the array `field_id` has also be incremented by one to prevent any bad identifiers.

Calculation of $E_{1\parallel}$

Before the implementation for the calculation of $E_{1\parallel}$ gets discussed it is necessary to talk about the scheme of the calculation. As stated in Equation (3.84) is the right-hand side of the Vlasov equation \mathcal{V} necessary for the calculation of $E_{1\parallel}$. Now, if one takes a closer look at Equations (3.68) and (3.73) it can be derived that

$$\frac{\partial g}{\partial t} = \mathcal{V}, \quad (4.1)$$

where g is the modified distribution. Since GKW has already implemented $\partial_t g$ as `rhs` (right-hand side) and the Vlasov equation with the modified distribution function [Eq (3.69)] has only one additional term containing g , which is nonlinear, i.e. ignored in linear cases, `rhs` will be used as the right-hand side of the Vlasov equation in the calculation of $E_{1\parallel}$. In the next step the numerical scheme for every Runge Kutta step ($i \rightarrow i + 1$) has to be considered. Figure 4.1 gives an illustration of the scheme.

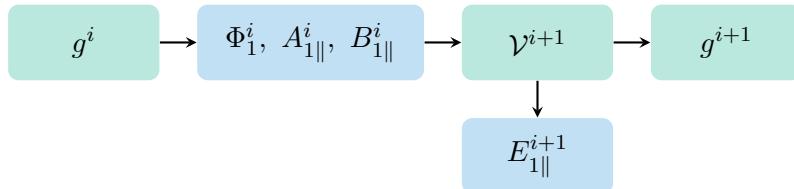


Figure 4.1: Numerical scheme used to calculate the induced eletric field $E_{1\parallel}$ in GKW.

As visible, the order of calculations is important in the numerical scheme, i.e. first calculate the right-hand side of the Vlasov equation \mathcal{V} and afterwards the induced electric field $E_{1\parallel}$. To achieve this structure the calculation of $E_{1\parallel}$ gets performed in `calculate_rhs` from the module `exp_integration`. For that, the following code lines get added in `calculate_rhs`

```

if (nlepar) then
  if (perf_measure) then
    call perfswitch('Calc Additional Fields',2)
    perffields=.true.
  end if

  ! Calculate the additional fields with rhs, i.e. epar,
  ! from rhs into fdis (fdis = mat * rhs) and apply the rhs
  ! correction with rhs = rhs + A * gafepar
  ! Note that rhs gets saved as rhs * deltatime which gets
  ! corrected in calculate_additional_fields
  if (spectral_radius) then
    call calculate_additional_fields(fdis, rhs, deltatime)
  endif

  if (perf_measure) then
  
```

4 Mitigation of Cancellation Problem in Local Gyrokinetic Simulations

```

call perfswitch('Copy fdis vector to tmp',2)
perffields=.false.
end if

```

To ensure that the right Runge Kutta step for the calculation of $E_{1\parallel}$ is used, the following lines have to be adjusted in the subroutine `advance_large_step_explicit` in the same module.

```

! at the end of the timestep, make the electromagnetic field
! consistent with the distribution function
if (spectral_radius) then
    ! Do not change the order because of the calculation scheme
    ! f_i -> Phi_i, Apar_i, Bpar_i -> RHS_i+1 -> Epar_i+1 -> f_i+1
    ! Since calculate_fields gets called in calculate_rhs it does not
    ! have to be called before.
    if (nlepar) then
        call calculate_rhs(fdisi, rhsk(:,1))
    else
        call calculate_fields(fdisi)
    endif
else

```

Note that, before the implementation of $E_{1\parallel}$ feature, `calculate_fields` gets called for local simulations, but since the calculation of $E_{1\parallel}$ requires the calculation of the right-hand side \mathcal{V} , which calculates the regular fields already in the subroutine `calculate_rhs`, the call for `calculate_fields` can be neglected if $E_{1\parallel}$ gets calculated.

The calculation of $E_{1\parallel}$ itself gets performed by `calculate_additional_fields` from the module `fields`. The subroutine is listed below

```

!-----
!> This routine calculates the integrated quantities of the perturbed
!> distribution function with the use of the right hand side. It is
!> used to calculate the induced electric field.
!>
!> It is an optimized routine, meaning that explicit assumptions are
!> made on the memory layout of the code.
!>
!> calls MPI_ALL_REDUCE to sum and distribute the integrals to all
!> processors.
!>
!> Note that, rhs gets saved as rhs * dtim * c1 in calculate_rhs which
!> has to be corrected with rhs/(dtim*c1). For that, the argument
!> DTIM = dtim*c1 = deltatime should be used.
!-----
subroutine calculate_additional_fields(fdis, rhs, DTIM, DPART) ! OPTIMISED
    ← routine

    use dist,           only : nsolc
    use dist,           only : nadditional_fields_start, nadditional_fields_end
    use matdat,         only : mat_add_field_int, mat_add_field_diag
    use control,        only : nlepar, nlapar
    use general,        only : gkw_warn

```

4 Mitigation of Cancellation Problem in Local Gyrokinetic Simulations

```

use mpiinterface,      only : number_of_processors, mpiallreduce_sum_inplace
use mpicomms,         only : COMM_S_EQ
use components,       only : tearingmode
use perform,          only : perfon, perfoff, perffields, perf_measure
use perform,          only : perfswitch
use matrix_format,    only : usmv

complex, optional, intent(inout) :: rhs(nsolc)
complex, intent(inout) :: fdis(nsolc)

integer :: i
integer :: ierr

! small timestep to divide rhs * dtimes to correct to only rhs
complex, intent(in) :: DTIM
real, optional, intent(in) :: DPART

! warning #1 here an assumption of the position of phi in
! the solution is made. I.e. the index function is not
! used. This is of course faster.
!APS: This could be replaced with another appropriate integer denoting the
!APS: index of the first field element.

!Initialise to zero (required if fields are not solved: tearingmodes,
!→ tracers)
!$omp parallel do schedule(static)
zero : do i = nadditional_fields_start, nadditional_fields_end
    fdis(i) = (0.E0,0.E0)
end do zero
!$omp end parallel do

! no point in calculating the potential if it is to
! be zero. (important also for the testcases that do not use phi)
if ((.not. nlepar .and. .not. nlapar)) then
    return
end if

if (perffields .and. perf_measure) then
    call perfon('add fields - int mat multiply',2)
end if

! first calculate the contribution of rhs
call usmv(cmplx(1.0,0.0),mat_add_field_int,rhs/DTIM,fdis,ierr)

if (perffields .and. perf_measure) call perfswitch('add fields - MPI
!→ allreduce',2)

if (number_of_processors > 1) then
    ! finish the species sum and the velspace integral
    call
    → mpiallreduce_sum_inplace(fdis(nadditional_fields_start:nadditional_fields_end),&
        nadditional_fields_end-nadditional_fields_start+1,COMM=COMM_S_EQ)
end if

```

4 Mitigation of Cancellation Problem in Local Gyrokinetic Simulations

```

!
if (perffields .and. perf_measure) call perfswitch('add fields - n4 renorm
    ↪ diag',2)

! Then divide by A: division by fields diagonal part, all elements
! are independent. Since we know that mat_add_field_diag has a NO unity
! block in the distribution part of the matrix, one has to start from i =
    ↪ 1.
normalise : do i = 1, mat_add_field_diag%nmat
    fdis(mat_add_field_diag%ii(i)) =
        ↪ fdis(mat_add_field_diag%ii(i))*mat_add_field_diag%mat(i)
end do normalise

tearingmod : if(tearingmode) then
    call gkw_warn('Tearing modes are not implemented for additional fields
        ↪ yet')
end if tearingmod

if (perffields .and. perf_measure) call perfoff(2)

end subroutine calculate_additional_fields

```

This subroutine has the same structure as `calculate_fields` with two major differences. No tearing modes are implemented yet and if tearing modes are activated, a warning gets printed to the console. `rhs` and `DTIM` as additional argument for the subroutine. The use of `rhs` was already discussed at the beginning of this section but the variable `rhs`, which gets communicated in `calculate_rhs` is not already the right right-hand side needed for the calculation of $E_{1\parallel}$. `rhs` is a product of `rhs` and the time step `deltatime`. This is due to the Runge Kutta scheme for the calculation of the distribution function g . To ensure that the "pure" right-hand side \mathcal{V} will be used for $E_{1\parallel}$ the argument `DTIM` in `calculate_additional_fields` is used to divide `rhs` by `DTIM`, which is `deltatime` from `calculate_rhs`.

In GKW the calculation of the $E_{1\parallel}$ field gets performed in two steps:

- (1) Matrix vector multiplication of the matrix containing the integral part of $E_{1\parallel}$ with the "pure" right-hand side array `rhs/DTIM`

$$\text{fdisi} = \text{mat_add_field_int} * (\text{rhs} / \text{DTIM})$$
- (2) Division of the matrix containing the diagonal part of $E_{1\parallel}$

$$\text{fdisi} = \text{fdisi} / \text{mat_add_field_diag}.$$

Both operations gets saved in the solution array `fdisi`. The integral part (`faraday_int`) and the diagonal part (`faraday_dia`) gets extracted from Equation (3.84) and defined as

$$\begin{aligned}
 \text{faraday_int} &= -2\pi B_N \beta_{\text{ref}} \sum_s Z_s n_{R,s} v_{\text{thR},s} \int dv_{\parallel N} d\mu_N v_{\parallel N} J_0(k_{\perp} \rho_s) \\
 \text{faraday_dia} &= \left(k_{\perp N}^2 + \beta_{\text{ref}} \sum_s \frac{Z_s^2 n_{R,s}}{m_{R,s}} \Gamma_0(b_s) e^{-\mathcal{E}_{N,s}/T_{R,s}} \right). \tag{4.2}
 \end{aligned}$$

4 Mitigation of Cancellation Problem in Local Gyrokinetic Simulations

`faraday_int` and `faraday_dia` both are defined in the module `linear_terms` in two separate subroutines given below

```

!> Adds the part of the Faraday's law that is related with
!> the integral over the right hand side of the vlasov equation.
!-----

subroutine faraday_int

use structures,      only : matrix_element
use control,        only : nlepar, spectral_radius
use grid,           only : nx,ns,nmu,nvpar,nsp,nmod
use components,    only : de, signz, vthrat, veta
use matdat,         only : add_element, set_indx
use geom,           only : bn
use velocitygrid, only : vpgr, intvp, intmu
use dist,           only : ifdis, iepar
use functions,     only : besselj0_gkw

real                :: bes
integer              :: ix, i, j, k, is, imod, ierr
type (matrix_element) :: elem

! non spectral case is done elsewhere (in gyro_average)
if (.not. spectral_radius) return

! Identifier for the term
elem%term = 'faraday_int'

! Type of iih and jjh
elem%itype = iepar
elem%itloc = ifdis
elem%ideriv = 0

if (.not. nlepar) return

do imod = 1, nmod

  if (all(apply_on_imod == 0)) then
    if(.not. lfaraday) cycle
  else
    if (any(apply_on_imod == imod)) then
      if(.not. lfaraday) cycle
    else
      if(.not.linear_term_switch_default) cycle
    endif
  endif

  do ix = 1, nx ; do i = 1, ns ; do j = 1, nmu ; do k = 1, nvpar ; do is =
  ↵ 1, nsp

    call set_indx(elem,imod,ix,i,j,k,is)

    ! Bessel function for gyro-average

```

4 Mitigation of Cancellation Problem in Local Gyrokinetic Simulations

```

bes = besselj0_gkw(imod,ix,i,j,is)

elem%val = -signz(is)*de(ix,is)*veta(ix)*intvp(i,j,k,is)*intmu(j)
    &
    *vthrat(is)*bn(ix,i)*vpgr(i,j,k,is)*bes

call add_element(elem,ierr)

!-----> Add the diagonal part of Faraday's law
!-----

subroutine faraday_dia

use structures,      only : matrix_element
use control,         only : nlepar, spectral_radius
use grid,            only : nx,ns,nsp,nmod,nmu,nvpar
use mpicomms,        only : COMM_S_EQ
use dist,             only : fmaxwl, iepar
use components,      only : de, signz, mas, veta
use matdat,          only : add_element, set_indx
use geom,             only : bn
use mode,             only : krloc
use rotation,         only : cfen
use velocitygrid,    only : intvp, intmu, vpgr
use functions,        only : gamma_gkw, besselj0_gkw
use mpiinterface,     only : mpiallreduce_sum_inplace

real    :: gamma, gamma_num, b, dum
integer :: imod, ix, i, j, k, is, idum, ierr
complex :: mat_elem
type (matrix_element) :: elem

! non spectral case is done elsewhere (in gyro_average - integral term)
if (.not. spectral_radius) return

if (.not. nlepar) return

! identifier of the term
elem%term = 'faraday_dia'
elem%itype = iepar
elem%itloc = iepar
elem%ideriv = 0

! set the dummy
idum = 0

do imod = 1, nmod

    if (all(apply_on_imod == 0)) then
        if(.not. lfaraday) cycle
    else
        if (any(apply_on_imod == imod)) then
            if(.not. lfaraday) cycle
        else

```

4 Mitigation of Cancellation Problem in Local Gyrokinetic Simulations

```

    if(.not.linear_term_switch_default) cycle
  endif
endif

do ix = 1, nx ; do i = 1, ns

  ! reference Epar
  call set_indx(elem,imod,ix,i,idum,idum,idum)

  ! The nabla^2 term
  mat_elem = - krloc(imod,ix,i)**2

  ! calculate the Maxwell correction
  dum = 0.
  do is = 1, nsp

    ! The gamma function of the species
    gamma = gamma_gkw(imod,ix,i,is)*exp(-cfen(i,is))

    ! numerical calculation of the gamma function
    !(includes the strong rotation correction implicitly in the
    ↪ maxwellian)
    gamma_num = 0.
    do j = 1, nmj ; do k = 1, nvpar
      b = besselj0_gkw(imod,ix,i,j,is)
      gamma_num = gamma_num +
    ↪ 2.E0*bn(ix,i)*intmu(j)*intvp(i,j,k,is)*b**2* &
      & vpgr(i,j,k,is)**2*fmaxwl(ix,i,j,k,is)
    ! The implementation below was used until Feb 2023
    ! It is a bit more stable numerically but inaccurate at default
    ↪ vpmax
      ! Leads to large errors at high beta and low kthrho
      ! gamma_num = gamma_num +
    ↪ bn(ix,i)*intmu(j)*intvp(i,j,k,is)*b**2*fmaxwl(ix,i,j,k,is)
      end do ; end do

    dum = dum - veta(ix)*signz(is)**2*de(ix,is)*gamma_num / mas(is)

  end do !nsp

  ! MPI sum of the Maxwell correction, over species, mu and vpar.
  call mpi_allreduce_sum_inplace(dum,1,COMM_S_EQ)

  ! put the inverse element into the matrix. This saves some cycles
  ! later because multiplication is faster than division.
  elem%val = -1.0/(mat_elem + dum)
  call add_element(elem,ierr)

end do ; end do

end do

end subroutine faraday_dia

```

4 Mitigation of Cancellation Problem in Local Gyrokinetic Simulations

Note that, the 2π factor in `faraday_int` is already included in the array `intmu`¹⁹ and will be ignored. Additionally, in `faraday_dia` a new calculation for the $\Gamma_0(b)$ [Eq (3.49)] was introduced from Yann Camenen in February 2023. The integral part will get added to the matrix `mat_add_field_int` and the diagonal part in `mat_add_field_diag`. Both matrices are defined in `matdat`. The size of the matrices gets defined with `nelem_additional_fields` in `dist` with the same method as `nelem_regular_fields`. It is important to mention, that the declaration has to be after the definition of grid points and ghost cells for the additional fields. Also, during the optimizition of the number of elements for `mat_add_field_int` and `mat_add_field_diag` it was found that the `nelem_additional_fields` can not be further reduced. It seems like the field matrices for the additional fields has to account the number of elements for regular field matrices as well. To investigate the matrices, they got added to the diagnostic module `diagnos_matrix` which handles the output.

Diagnostic for $E_{1\parallel}$

To extract $E_{1\parallel}$ from `fdisi` the subroutine `get_epar` in `dist.f90` is needed and is given in greater detail below

```
!+-----+
!> This routine copies the potential from the distribution into the eparc
!> array. If nlepar = .false. eparc is set to zero
!-----
subroutine get_epar(fdis,eparc)
  use general, only : gkw_abort
  use control, only : nlepar
  use grid, only : nmod, nx, ns
  use index_function, only : indx
  complex, dimension(:), intent(inout) :: fdis
  complex, dimension(nmod,1-ghost_points_x:nx+ghost_points_x,&
    & 1-ghost_points_s:ns+ghost_points_s), intent(out) :: eparc
  integer :: ix, i, imod, ghost_s, ghost_x

  if (first_call) then
    call gkw_abort('get_epar: you can not call this before dist_init')
  end if

  if (nlepar) then

    ! copy epar
    if(size(fdis) == nsolc) then
      ! without ghostcells
      ghost_s = 0
      ghost_x = 0
    else
      ! with ghostcells
      ghost_s = ghost_points_s
      ghost_x = ghost_points_x
    end if
  end if
```

4 Mitigation of Cancellation Problem in Local Gyrokinetic Simulations

```

do i = 1-ghost_s, ns+ghost_s
  do ix = 1-ghost_x, nx+ghost_x
    do imod = 1, nmod
      eparc(imod,ix,i) = fdis(indx(iepar,imod,ix,i))
    end do
  end do
end do
else
  ! epar is not solved for and therefore set to zero
  eparc = (0.E0,0.E0)
end if

end subroutine get_epar

```

In general the given subroutine checks if any parallelization was done and copies all real and complex entries of $E_{1\parallel}$ into the argument `eparc` by looping over all grid points for s , ψ and ζ coordinate. If `nlepar` is set to `.false.`, a zero data set will be returned.

The subroutine `get_epar` gets called in `diagnostic` in the subroutine `fill_buffer` where the new defined field identifier `EPAR_FIELD` from module `global` is used.

```

if (nlepar) then
  if(requirements(EPAR_FIELD,X_GHOSTCELLS) .or. &
   & requirements(EPAR_FIELD,S_GHOSTCELLS)) then
    call get_epar(fdisi,epar)
  else if(requirements(EPAR_FIELD,LOCAL_DATA)) then
    call get_epar(fdisi,epar)
  end if
end if

```

Here, the calculated values for $E_{1\parallel}$ gets copied from `fdisi` (or `fdisi_tmp` for parallelization) to the variable `epar` from the module `dist`. In GKW, each diagnostics computes and outputs a distinct physical quantity.¹⁹. In `diagnos_mode_struct` the code gets adapted for the implementation of $E_{1\parallel}$ and the new field identifier `EPAR_FIELD` by increasing the size of the arrays `mode_struct_lus`, `mode_struct_ids`, `mode_struct_names`, `local_write` and `global_write` by one element. It is worth mentioning that from now on the variables `epar` and `eperp` are renamed to `ene_par` and `ene_perp`. After loading the data for $E_{1\parallel}$ from the variable `epar` in a loop for each species N_s , grid points in binormal direction N_{mod} , radial direction N_x and in field direction N_s into the variable `epad`. Then, `epad` gets normalised by rotating relative to the maximum electrostatic potential Φ_1 in the complex plane for each mode is $1 + 0i$ and saved in parallel data structure. For that, the end of the loop with variable `i1` will be increased from 8 to 9 and the parallel data structure got adjusted for two additional columns for $E_{1\parallel}$, i.e. for real and imaginary values. Note that if `nlepar` is set to `.false.` the variable `epad` returns an zero array. Due to this behaviour the testcases

- `adiabat_collisions_momcon_ap`,
- `chease_cf_modebox`,
- `collisions_please_dont_break_me` and

4 Mitigation of Cancellation Problem in Local Gyrokinetic Simulations

- `zonal_flow_sixth_order_FD`

have to adjusted to contain the two zero columns from $E_{1\parallel}$ in the file `parallel.dat`

Additionally, for $\bar{E}_{1\parallel}$ the function `get_averaged_epar` in the module `codefields` gets defined and works as `get_epar`. Here, the function differs between local and global simulations. As defined in Chapter ?? is the gyroaverage of a quantity in a local simulation the product with the zeroth order Besselfunction J_0 and the quantity itself, i.e. $\bar{E}_{1\parallel} = J_0 E_{1\parallel}$, which gets calculated in `get_averaged_epar`. In case of a global simulation the subroutine will return the calculated $\bar{E}_{1\parallel}$ from `fdisi` and as in `get_epar` if `nlepar` is `.false.`. a zero data set will be returned. The function `get_averaged_epar` is listed below for a closer look.

```
!-----  
!> This function gives the value of the gyro-averaged vector potential for a  
!> given grid point. If nlepar = .false. a zero is returned  
!-----  
function get_averaged_epar(imod, ix, i, j, is, fdis)  
  
    use control,           only : nlepar, spectral_radius  
    use index_function, only : indx  
    use functions,        only : besselj0_gkw  
    use dist,             only : iepar, iepar_ga  
  
    integer, intent(in) :: imod, ix, i, j, is  
    complex, intent(in) :: fdis(:)  
    complex :: get_averaged_epar  
  
    if (nlepar) then  
        if (spectral_radius) then  
            get_averaged_epar = besselj0_gkw(imod, ix, i, j, is) * &  
                & fdis(indx(iepar, imod, ix, i))  
        else  
            get_averaged_epar = fdis(indx(iepar_ga, imod, ix, i, j, is))  
        endif  
    else  
        get_averaged_epar = (0.E0, 0.E0)  
    endif  
  
end function get_averaged_epar
```

Benchmark of $E_{1\parallel}$ Field Equation - Linear β Scan

To benchmark the implementation of Faraday's law with $E_{1\parallel}$ multiple linear simulations for different plasma beta β gets performed. The goal is to extract the linear growth rate γ and frequency ω to compare the result for $E_{1\parallel}$ with $A_{1\parallel}$. General a given quantity saved in `parallel.dat` gets saved in two columns containing real and imaginary values. The rows are sorted by the species N_{sp} , bidnormal grid points N_{mod} , radial grid points N_x and grid points along the field line N_s and therefore $N_{sp}N_{mod}N_xN_s$ rows. Data stored in `parallel.dat` is defined as

$$\hat{L}(s, t) = \exp(\gamma t + i\omega t) \hat{L}(s), \quad (4.3)$$

where $\hat{L}(s, t)$ is the calculated data and $\hat{L}(s)$ is the stored values.¹⁹ Note that, $\hat{L}(s, t)$ and $\hat{L}(s)$ are complex. With Equation (4.3) one can derive the relation between $\hat{E}_{1\parallel}$ and $\hat{A}_{1\parallel}$ with Faraday's law [Eq. (3.75)] as

$$\begin{aligned} \hat{E}_{1\parallel}(s, t) &= -\partial_t \hat{A}_{1\parallel}(s, t) \\ \exp(\gamma t + i\omega t) \hat{E}_{1\parallel}(s) &= -\partial_t \left[\exp(\gamma t + i\omega t) \hat{A}_{1\parallel}(s) \right] \\ \hat{E}_{1\parallel}(s) &= -(\gamma + i\omega) \hat{A}_{1\parallel}(s) \\ \hat{E}_{1\parallel}^R(s) + i\hat{E}_{1\parallel}^I(s) &= -(\gamma + i\omega) \left(\hat{A}_{1\parallel}^R(s) + i\hat{A}_{1\parallel}^I(s) \right) \\ \Rightarrow \boxed{\begin{aligned} \hat{E}_{1\parallel}^R(s) &= -\gamma \hat{A}_{1\parallel}^R(s) + \omega \hat{A}_{1\parallel}^I(s) \\ \hat{E}_{1\parallel}^I(s) &= -\omega \hat{A}_{1\parallel}^R(s) - \gamma \hat{A}_{1\parallel}^I(s) \end{aligned}} \end{aligned} \quad (4.4)$$

Simulation Setup

For the plasma beta following values are investigated

$$\beta \in [0.0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.1, 1.2, 1.4, 1.6] \% . \quad (4.5)$$

The values will be set with the parameter `beta` in the input file. As base input file the cyclone benchmark case provided by the GKW Team was used. The input file can be found in the GKW repository under `doc/input/cyclone`. Here, the input parameter were adjust for linear β scan and are displayed in Table 4.1. Additionally following parameter were set

- `NON_LINEAR = .false.` to enable linear simulations,
- `nlepar = .true.` to enable the calculatation of $E_{1\parallel}$,
- `io_format = 'hdf5` output data to `hdf5` format,
- `gama_tol = 1 · 10⁻⁵` defines tolerance for linear growth rate γ ,
- `adiabatic_electrons = .false.` to enable kinetic electrons,
- `mode_box = .false.` deactivates 2D mode grid and
- $k_\zeta\rho = 0.3$ defines “poloidal” wave vector which corresponds roughly to the position of the maximum of the nonlinear transport spectrum in GKW.

DTIM	NTIME	NAVERAGE	N_{mod}	N_x	N_s	$N_{v\parallel}$	N_μ	N_{sp}	nperiod
0.01	2000	100	1	1	288	64	16	2	5

Table 4.1: Adjusted input parameter for linear β scan: **Variables**

Since the simulation perfroms well on the cluster it was shown that parallelization with OpenMPI works. The result of the linear β scan can be seen in Figure 4.2. The obtained data shows good agreement with Ref. 21 although due to the different normalization scheme, i.e. GENE normalises with speed of sound and GKW normalise by the thermal velocity v_{th} , the values are $\sqrt{2}$ times smaller. Furthermore, one would notice by comparing the beta scan in Ref. 21 and Figure 4.2 that the transition from ion temperature gradient mode (ITG) to trapped electron mode (TEM) and from (TEM) to kinetic ballooning mode (KBM) are located at different values of plasma beta β . This behaviour is due to different values for the wave vector $k_\zeta\rho$ and the resolution of parallel velocity grid $N_{v\parallel}$.

4 Mitigation of Cancellation Problem in Local Gyrokinetic Simulations

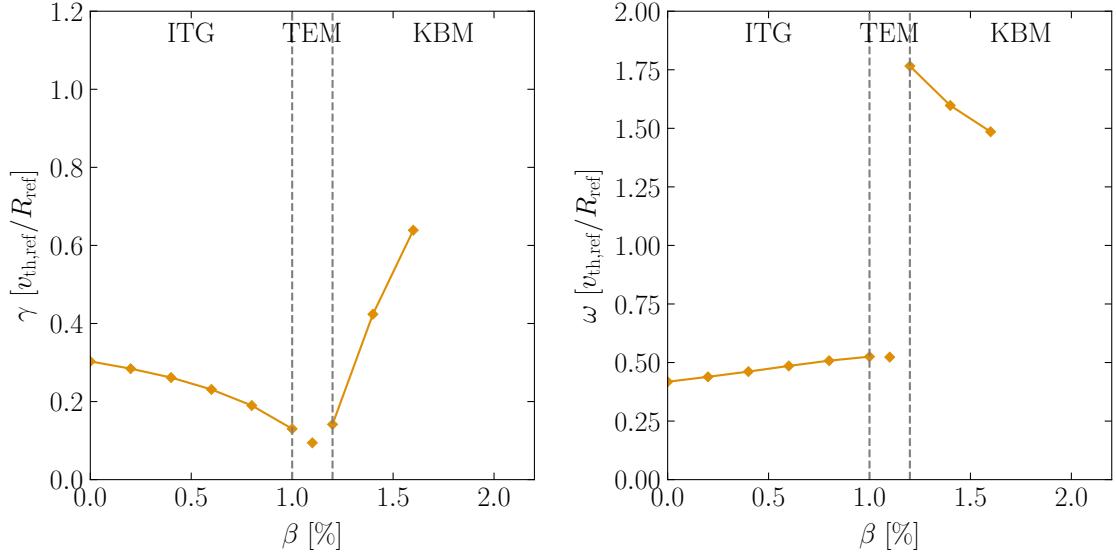


Figure 4.2: Growth rate γ and frequency ω for different plasma beta β . Here, (ITG) stands for ion temperature gradient modes, (TEM) for trapping electron modes and (KBM) for kinetic ballooning modes.

The comparision of the induced electric field $E_{1\parallel}$ and plasma induction $A_{1\parallel}$ yields that the implementation of the $E_{1\parallel}$ field equation was successful. An overall comparison is shown in Figure 4.3 and for a more detailed look for different plasma beta values the reader is referred to Appendix 6.1.1.

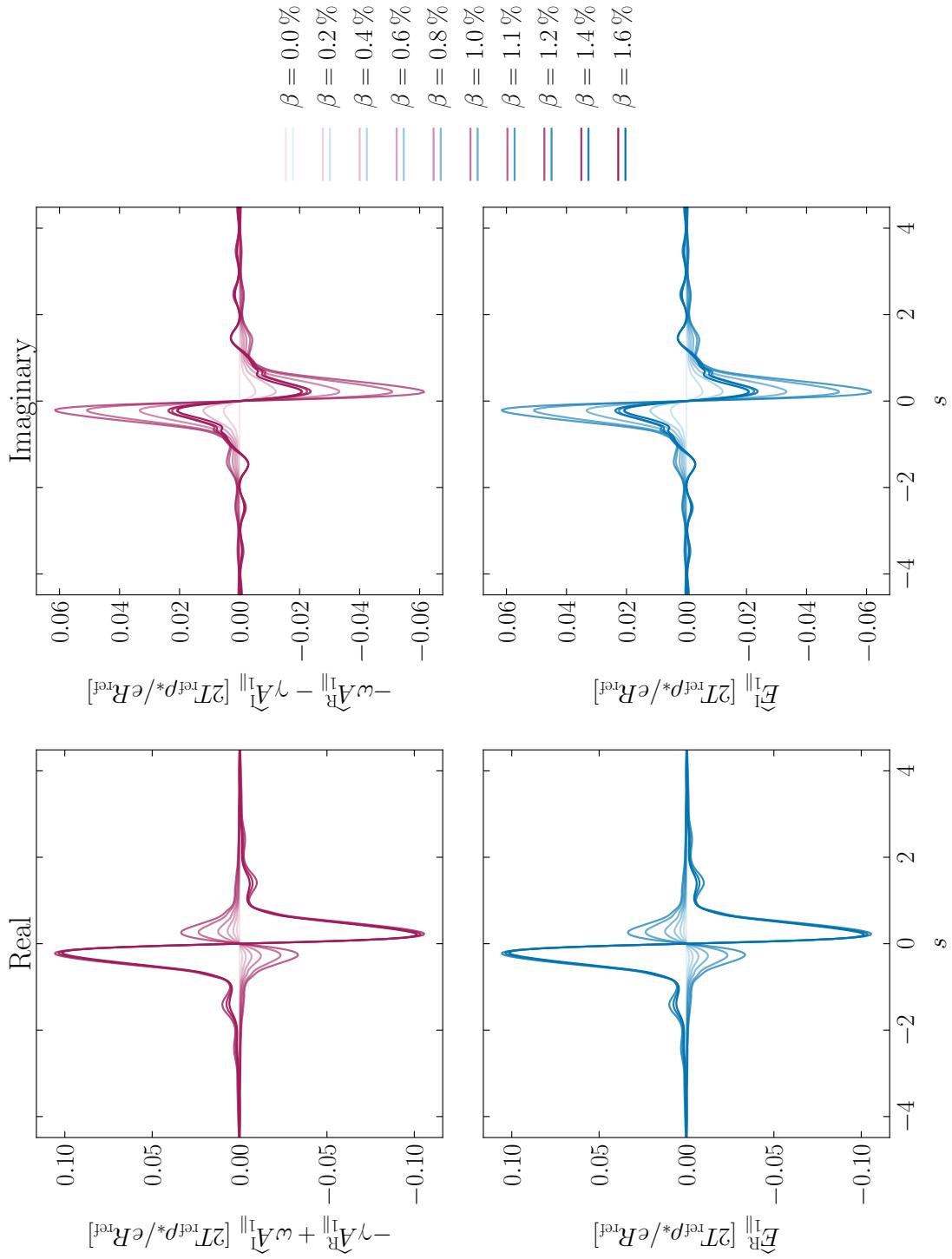


Figure 4.3: Comparision between real and imaginary part of the induced electric field $E_{1\parallel}$ and plasma induction $A_{1\parallel}$ for different plasma beta values for the g-verison of GKW.

4.4 *f* Version of gkw

As stated in Chapter 3.8.4 the current version of GKW implements the gyrokinetic equation with the use of the modified distribution function g . To preserve the established structure of the code a new numerical scheme gets introduced. To recall the right-hand side of the Vlasov equatiuon \mathcal{V} is definded as

$$\frac{\partial g}{\partial t} = \mathcal{V} \quad (4.6)$$

which is implemented numerical with Runge Kutta (one timestep ($i \rightarrow i + 1$)) as

$$g^{i+1} = g^i + \Delta t \cdot (\mathcal{V}^{i+1}) . \quad (4.7)$$

To transform the established scheme to the distribution function F_1 , Equation 3.77 has to be considered and is given by

$$\frac{\partial F_1}{\partial t} = \mathcal{V} + \frac{Zev_{\parallel}}{T} J_0 E_{1\parallel} F_M , \quad (4.8)$$

where $J_0 E_{1\parallel} = \bar{E}_{1\parallel}$ in the local simulation is and can be written normalised and fourier transformed as

$$\frac{\partial \hat{F}_{1N,s}}{\partial t_N} = \hat{\mathcal{V}}_N + \frac{2Zv_{thR}v_{\parallel N}}{T_R} J_0 \hat{E}_{1\parallel N} F_{MN} . \quad (4.9)$$

Here, the numerical Runge Kutta scheme can be expressed as

$$F_1^{i+1} = F_1^i + \Delta t \cdot \left(\mathcal{V}^{i+1} + \frac{Zev_{\parallel}}{T} J_0 E_{1\parallel}^{i+1} \right) . \quad (4.10)$$

Note that the established numerical scheme using the modified distribution g can easily transform to the scheme using the distribution F_1 by applying the so called $E_{1\parallel}$ -Correction to the RHS. The overall new numerical scheme can be seen in Figure 4.4.

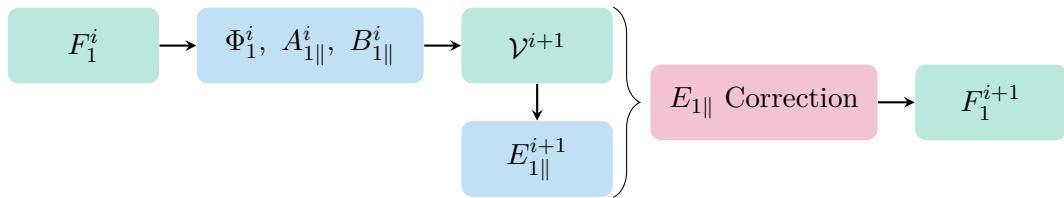


Figure 4.4: Numerical scheme used to calculate the distribution function F_1 in GKW.

4 Mitigation of Cancellation Problem in Local Gyrokinetic Simulations

To implement the $E_{1\parallel}$ -Correction a closer look on the already existing $A_{1\parallel}$ -Correction will be made. The $A_{1\parallel}$ -Correction is used to calculate the distribution F_1 from the modified distribution g , which is necessary since most of the diagnostics need the distribution F_1 for calculation. To recall g is defined as

$$g = F_1 + \frac{Zev_{\parallel}}{T} J_0 A_{1\parallel} F_M . \quad (4.11)$$

To get the gyrocenter distribution function F_1 one has to subtract the $A_{1\parallel}$ -Term, i.e. the $A_{1\parallel}$ -Correction, from the modified distribution g . This gets realised with the use of the matrix `matg2f` which gets defined in the module `matdat`. The elements are set in the subroutine `g2f_correct` in the module `linear_terms` and the calculation for g to F_1 gets either performed with a loop over all elements of array `fdisi` or in the function `ge_f_from_g`. Implementing the $E_{1\parallel}$ -Correction follows the same structure. First the matrix `matrhs4f` (RHS for F_1) gets introduced in the module `matdat` with `nelem_rhs4f` number of elements defined in the module `dist`. The elements itself are set in the subroutine `epar_correct` the main difference to `g2f_correct` is the swap of minus sign and the use of $E_{1\parallel}$ instead of $A_{1\parallel}$. The RHS than gets correct right after the calculation of the $E_{1\parallel}$ field in the subroutine `calculate_rhs` in the module `exp_integration`. Here, the loop method was used because it was found that the calculation of the new RHS needs 50% longer with the `usmv` subroutine. The code sequence is given below. Note that the $E_{1\parallel}$ -Correction has to be multiplied the time step `deltatime` to have the correct expression for the Runge Kutta scheme.

```

! Apply E|| correction to rhs
! Note that rhs = rhs * dtim so the E|| corrections also has to
! be multiplied with dtim
do i = 1, matrhs4f%nmat
    rhs(i) = rhs(i) + deltatime*matrhs4f%mat(i)*fdis(matrhs4f%jj(i))
end do

```

Now that the $E_{1\parallel}$ -Correction is implemented it is important to clarify that the distribution array `fdisi` stores no longer the modified distribution function g instead the gyrocenter distribution F_1 . For that purpose from now on if `nlepar` is set to `.true.` GKW gets executed in the **f-version**. Otherwise, if `nlepar` is set to `.false.` the so called **g-version** of GKW will be used. If `nlepar` is set to `.true.`, GKW switches to the f-version of the code. Additionally, a warning messages will be printed, that the user now uses the f-version of GKW. If needed, a new switch called `fversion` could be implemented to turn all necessary switches on. Until now `nlepar` will be used for that purpose. In the subroutine `control_initt` one could add additional switches for the f-version of GKW into the given code sequence below.

```

! Set switches for f version of gkw
if (nlepar) then
    nlapar = .true.
    call gkw_warn('You use the f version of gkw. nlapar will be set to
    ↪ .true.')
end if

```

4 Mitigation of Cancellation Problem in Local Gyrokinetic Simulations

Furthermore, it is necessary to prevent any code sequence which was meant for the modified distribution function, i.e. the application of the $A_{1\parallel}$ -Correction. For that any code sequence using the $A_{1\parallel}$ -Correction will get deactivated if `nlepar` is set to `.true.`. Since most sequences define a new temporary distribution array `fdis_tmp` and the upcoming code uses this variable it was convenient to just save `fdisi` in `fdis_tmp` for the f-version of GKW. Additionally, the function `get_f_from_g` gets modified to return just `fdisi` and the definition of the matrix `matg2f` gets suppressed as well to save disk space during execution.

As last step the field equation of the perturbed vector potential $A_{1\parallel}$ has to be adjusted, since it is the only field equation which changes significantly through the substitution of the modified distribution g [Ch. 3.8.4]. For that, an if statement for the f-version gets introduced into the subroutine `ampere_dia` in the module `linear_terms` which deactivates the skin term if `nlepar` is set to `.true.`. As already mentioned the other field equation are still implemented right for `fdisi` as distribution F_1 .

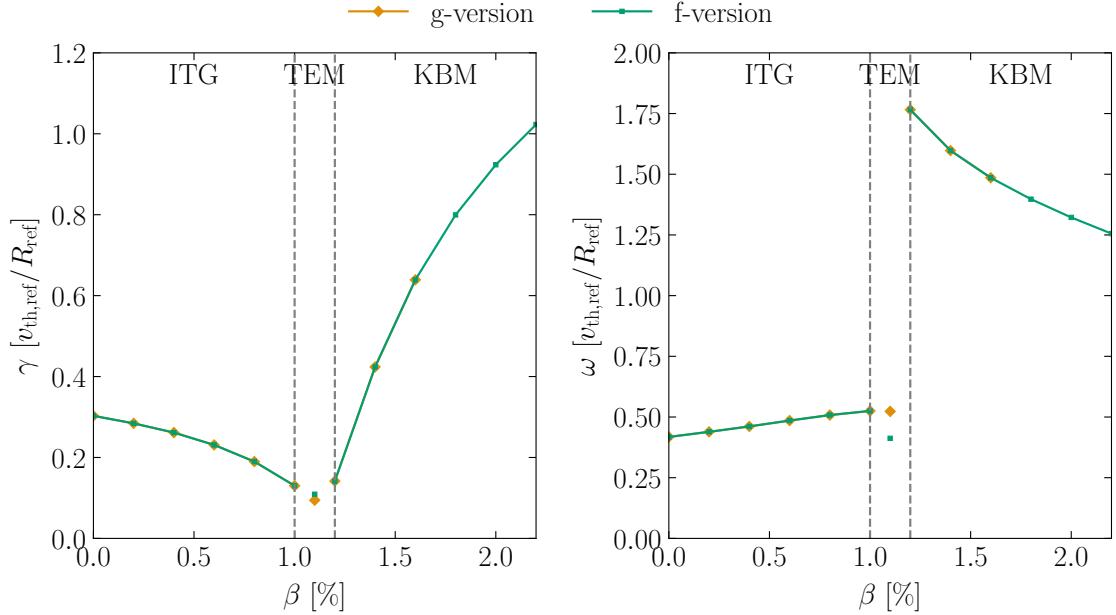
Benchmark of f Version of gkw - Linear β Scan


Figure 4.5: Growth rate γ and frequency ω for different plasma beta β . Here, (ITG) stands for ion temperature gradient modes, (TEM) for trapping electron modes and (KBM) for kinetic ballooning modes.

Again the same benchmark as in Chapter 4.3 will be performed with the same simulation setup. The only difference is that the plasma beta was varied between

$$\beta \in [0.0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.1, 1.2, 1.4, 1.6, 1.8, 2.0, 2.2]\% . \quad (4.12)$$

The result of the linear beta scan of the f-version yields great agreement with the one of the g-version [Fig. 4.5] only for $\beta = 1.1\%$ the values are significant different. This can be explained with the run time of both simulations since the simulation for the g-version run for 5438 normalised timesteps, where the f-version only run for 2000 normalised timesteps. The other simulations run the same noramlized timesteps as the g-version. To verify, that the f-version of GKW calculates the field equation for the plasma induction $A_{1\parallel}$ right, since the field equation got changed (compare Eq. (3.67) and (3.72)), the plasma induction gets compared again with the induced electric field $E_{1\parallel}$ [Fig. 4.6]. Here, the comparision again shows that the field equations are implemented successfully. The only difference exists for $\beta = 1.1\%$, which is also the result of the much shorter run time of the simulation. For a more detailed look for different plasma beta values the reader is referred to Appendix 6.1.2.

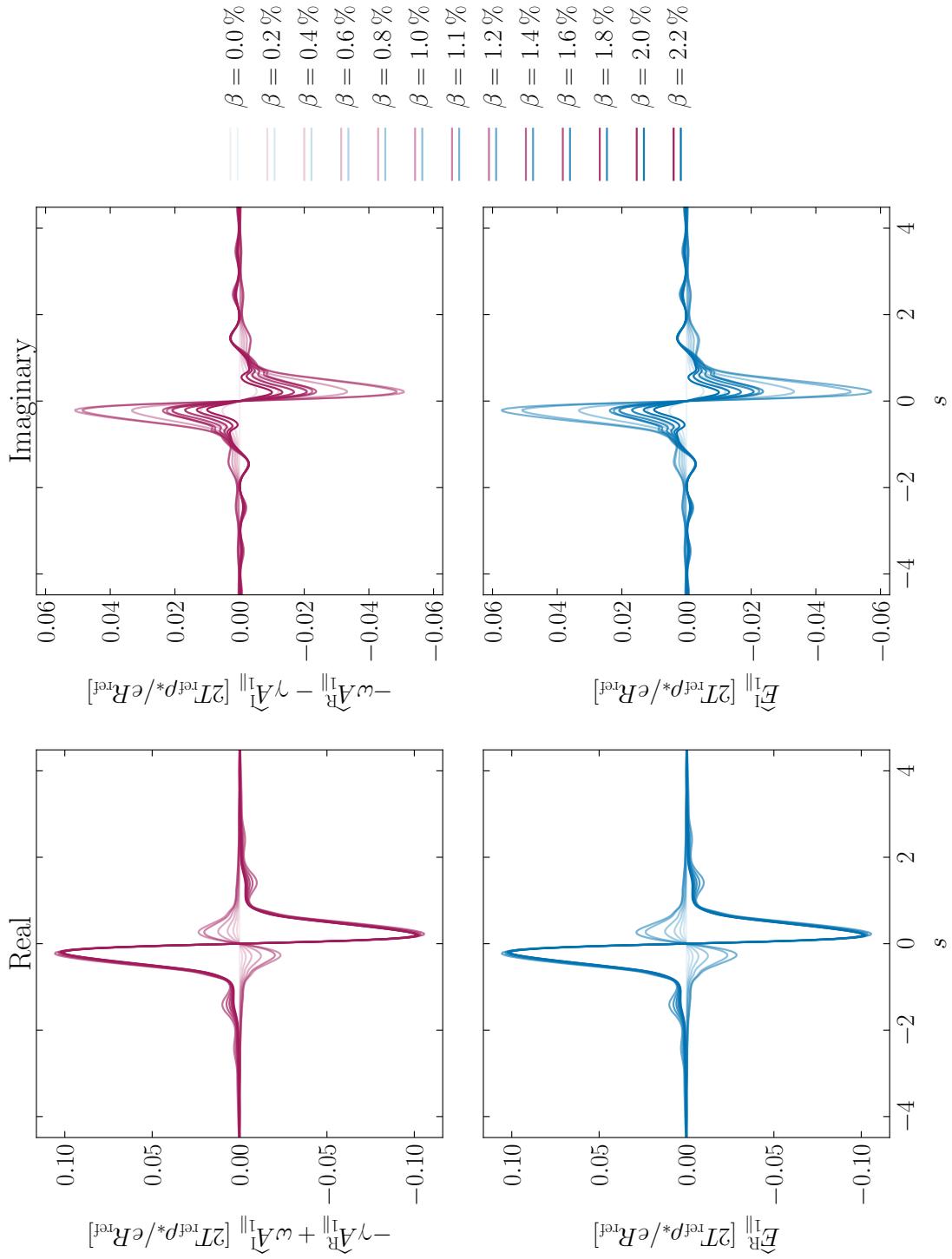


Figure 4.6: Comparision between real and imaginary part of the induced electric field $E_{1\parallel}$ and plasma Induction $A_{1\parallel}$ for different plasma beta values for the f-version of GKW.

Mitigation of Cancellation Problem in linear Simulations

One of the main goals of this thesis is to mitigate the cancellation problem [Ch. 3.8.4] in local simulations. The error of problem scales with $\sim \beta/k_{\perp}^2$ ¹⁶. Because of that, it was investigated if the new f-version improves code run time for lower perpendicular wave vectors k_{\perp} . First, a k_{\perp} scan was performed with the g-version of GKW and cyclone base case for $\beta = 0.8\%$ to find the transition between stable simulations and unstable ones. Here, it was found that the simulation get unstable for $k_{\perp} = 0.0271/\rho_{\text{th,ref}}$. Unfortunately, the f-version gets also unstable for the exact same value of k_{\perp} . To further investigate the behaviour of the plasma induction by changing `naverage` to 1.

AUG Testcases

4.5 Nonlinear Terms

Until now only the linear terms of the gyrokinetic equations got discussed. To complete the transformation to the f-version GKW the nonlinear terms has to be considered as well. Because of the substitution of the modified distribution function g the nonlinear terms

$$\mathbf{v}_\chi \cdot \nabla F_1 - \frac{F_M}{T} \mathbf{v}_{\bar{B}_{1\perp}} \cdot (Ze\nabla\bar{\Phi} + \mu\nabla\bar{B}_{1\parallel})$$

got replaced with $\mathbf{v}_\chi \cdot \nabla g$. To implement the nonlinear terms one has simple to insert the definition of the modified distribution into the already implemented nonlinear term. This results in the following relation

$$\begin{aligned} \mathbf{v}_\chi \cdot \nabla g &= \mathbf{v}_\chi \cdot \nabla \left(F_1 + \frac{Zev_{\parallel}}{T} \bar{A}_{1\parallel} F_M \right) \\ &= \mathbf{v}_\chi \cdot \nabla F_1 + \mathbf{v}_\chi \cdot \nabla \left(\frac{Zev_{\parallel}}{T} \bar{A}_{1\parallel} F_M \right) \\ &= \mathbf{v}_\chi \cdot \nabla F_1 + \frac{Zev_F}{T} \mathbf{v}_\chi \cdot \nabla (v_{\parallel} \bar{A}_{1\parallel}) \\ &= \mathbf{v}_\chi \cdot \nabla F_1 + \frac{Zev_F}{T} \mathbf{v}_{\bar{B}_{1\perp}} \cdot \left(\nabla\bar{\Phi} - \nabla(v_{\parallel} \bar{A}_{1\parallel}) + \frac{\mu}{Ze} \nabla\bar{B}_{1\parallel} \right) \\ &= \mathbf{v}_\chi \cdot \nabla F_1 + \frac{F_M}{T} \mathbf{v}_{\bar{B}_{1\perp}} \cdot (Ze\nabla\bar{\Phi} + \mu\nabla\bar{B}_{1\parallel}), \end{aligned} \quad (4.13)$$

although the term with $\mathbf{v}_\chi \cdot \nabla F_M$ gets neglected due to order of ρ_\star^2 . Since the introduction of the f-version of GKW the distribution array `fdisi` is already F_1 , so only the $A_{1\parallel}$ term has to be added to `fdisi`, i.e. apply the $A_{1\parallel}$ correction. For that a new matrix `matf2g` gets introduced which basically is the same as `matg2f` only the minus signs of the elements are swap. For convienence, the elements for `matf2g` will be set in subroutine `apar_correct`. The $A_{1\parallel}$ correction itself gets performed via loop over the elements of `fdisi` in the subroutine `add_non_linear_terms` and saved in the temporary distribution array `fdis_ttmp`. Then, the array `fdis_ttmp` will be used to calculate the nonlinear terms. It is important to insure the usage of `fdis_ttmp` only since in the f-version the distribution F_1 gets used for further calculation and should not be changed. To advance errors `fdis_ttmp` will be written as well, if the f-version is not used and is simply the array `fdisi`. The above implementation is valid if `nlepar` and `non_linear` is set to `.true.`.

CHAPTER

Conclusion

5

5 Conclusion

In this thesis the cancellation problem gets investigated and one technique to mitigate it was implemented in the local (flux-tube) version of GKW, the Faraday law. In process of the replacement GKW was transformed from the so called g-version to the new f-version, which enables new possibilities to calculate diagnostics.

Experimental Ending !!!

During the benchmark of the f-version of GKW it was found that the cancellation problem could not be mitigated. Further researches about the mitigation technique shows that the mitigation only appears in the global nonlinear cases. Which mechanism are relevant will be from now one part of the upcoming research.

CHAPTER

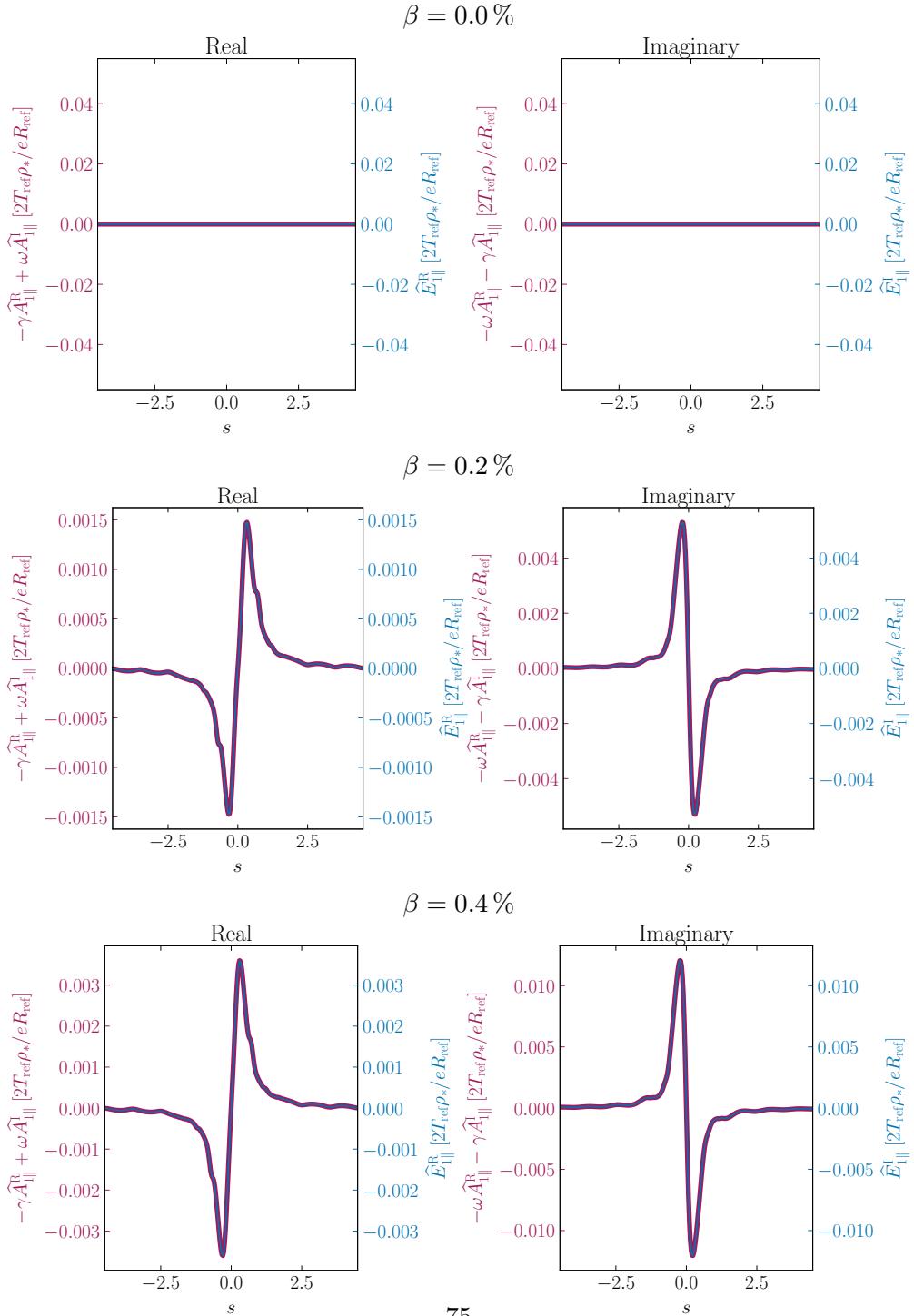
Appendix 6

6 Appendix

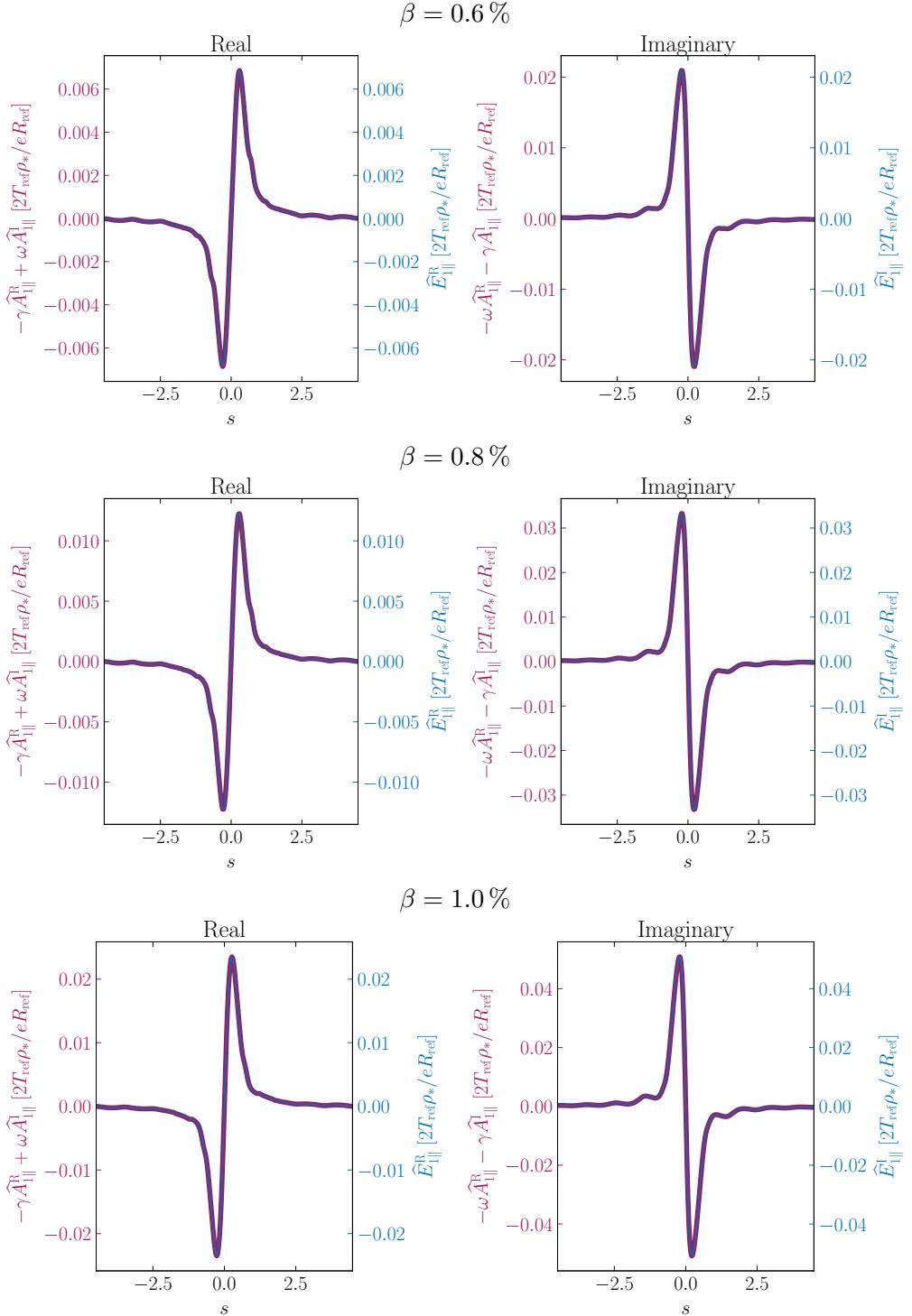
6 Appendix

6.1 Comparision between $E_{1\parallel}$ and $A_{1\parallel}$ for various plasma beta

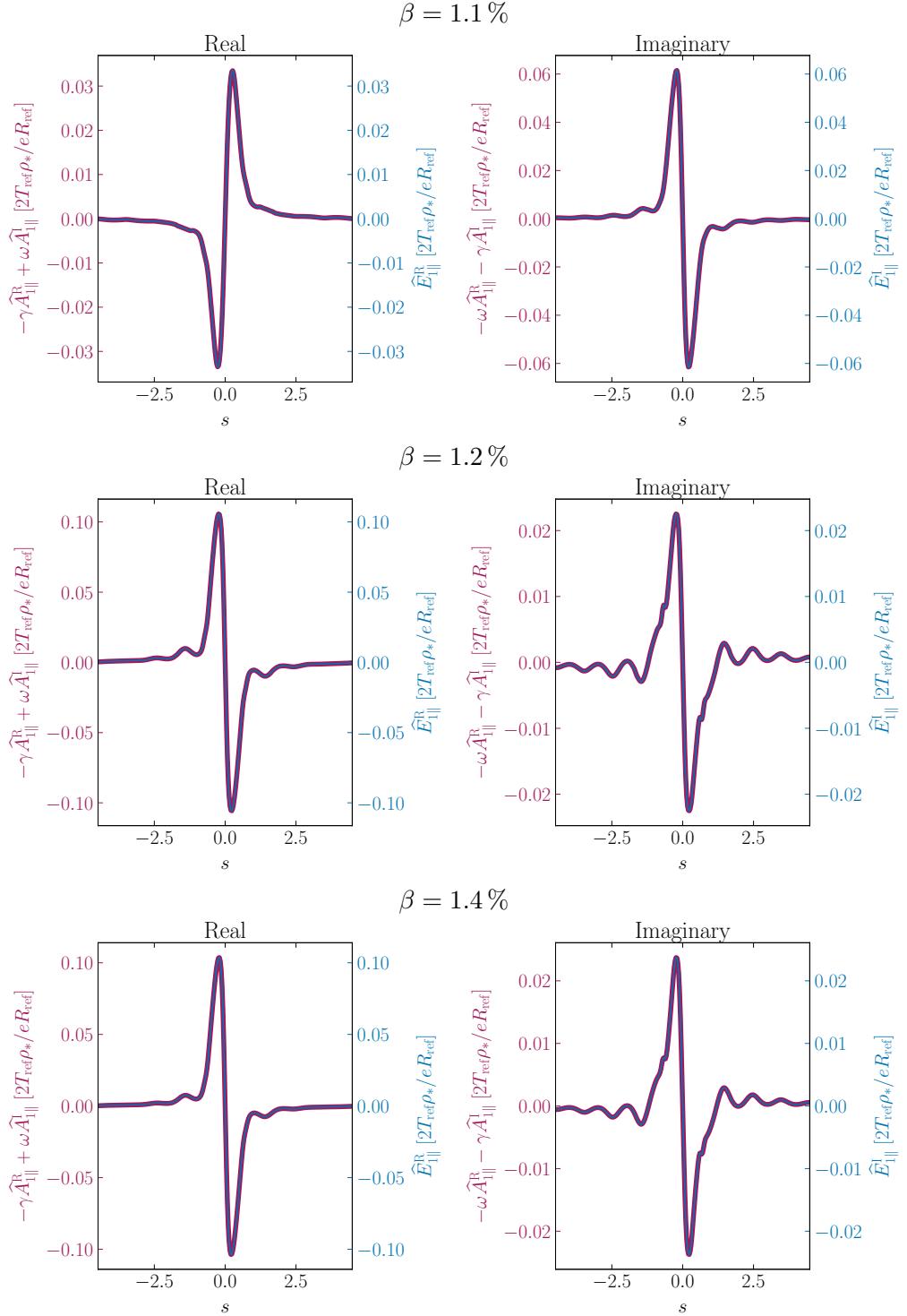
6.1.1 For the g-version of GKW



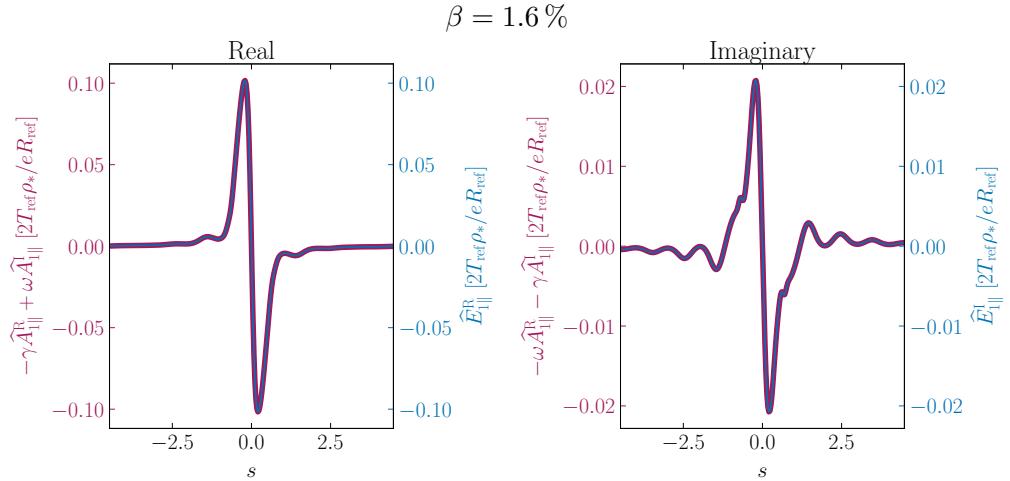
6 Appendix



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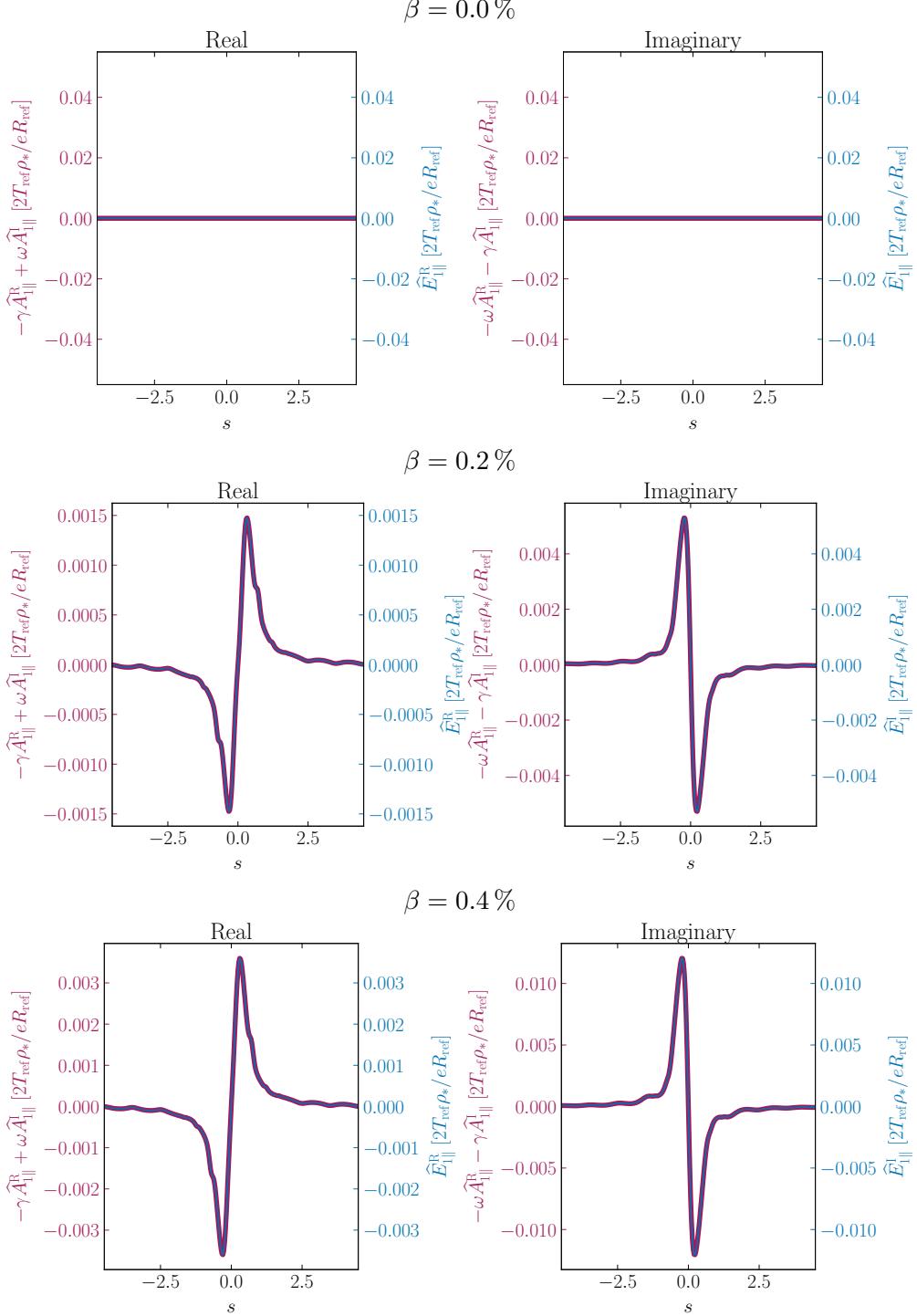


6 Appendix

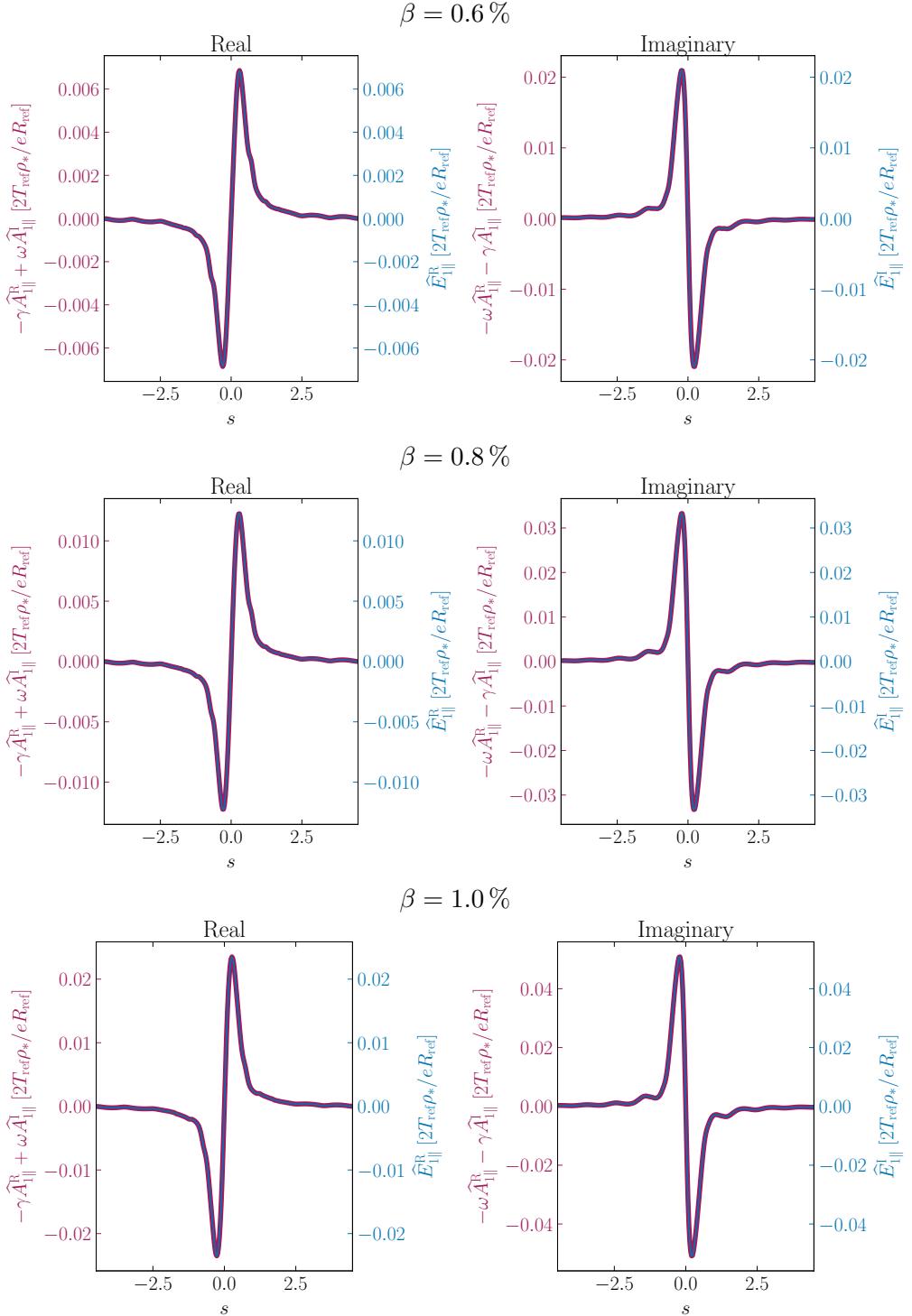


6 Appendix

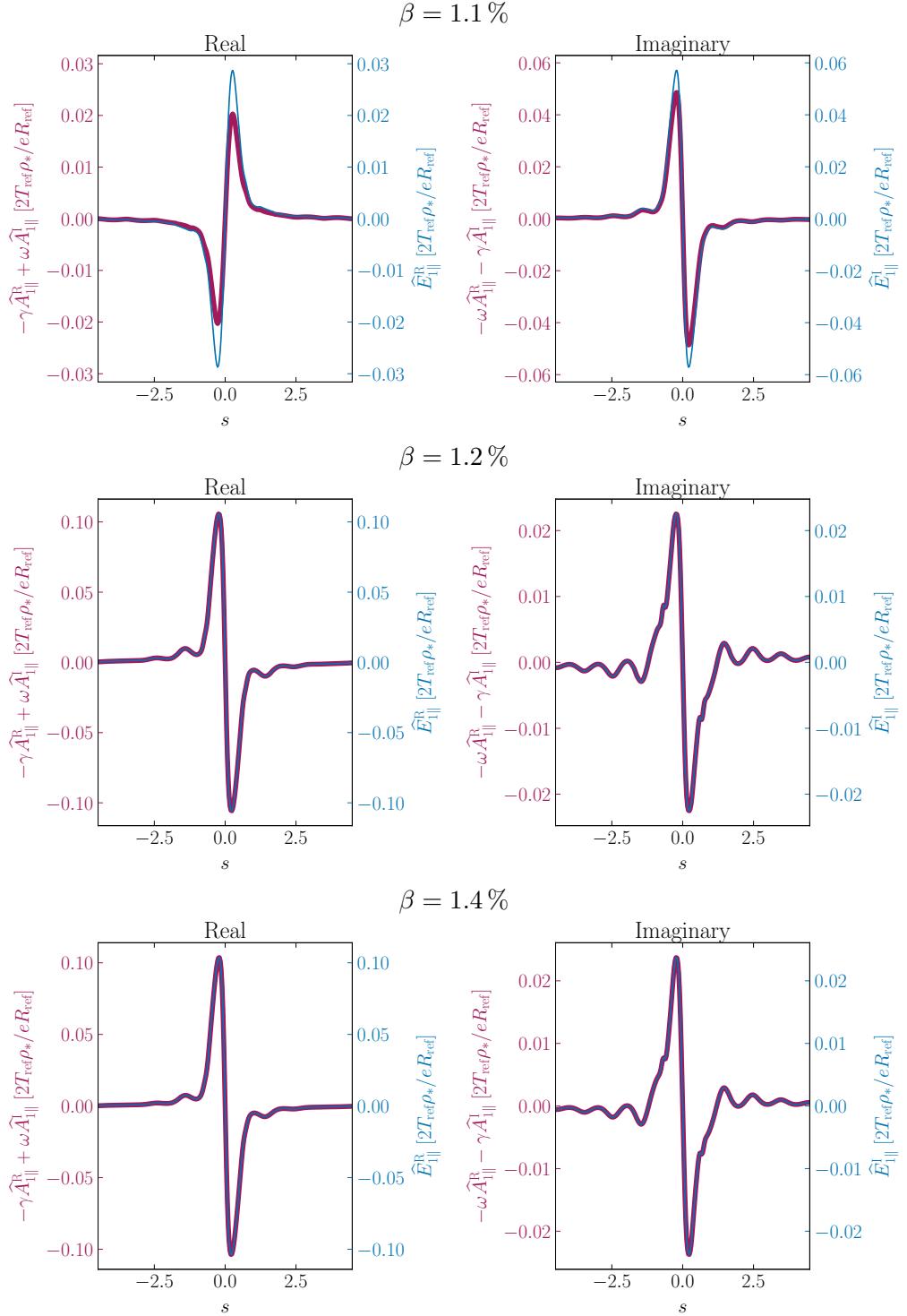
6.1.2 For the f-version of gkw



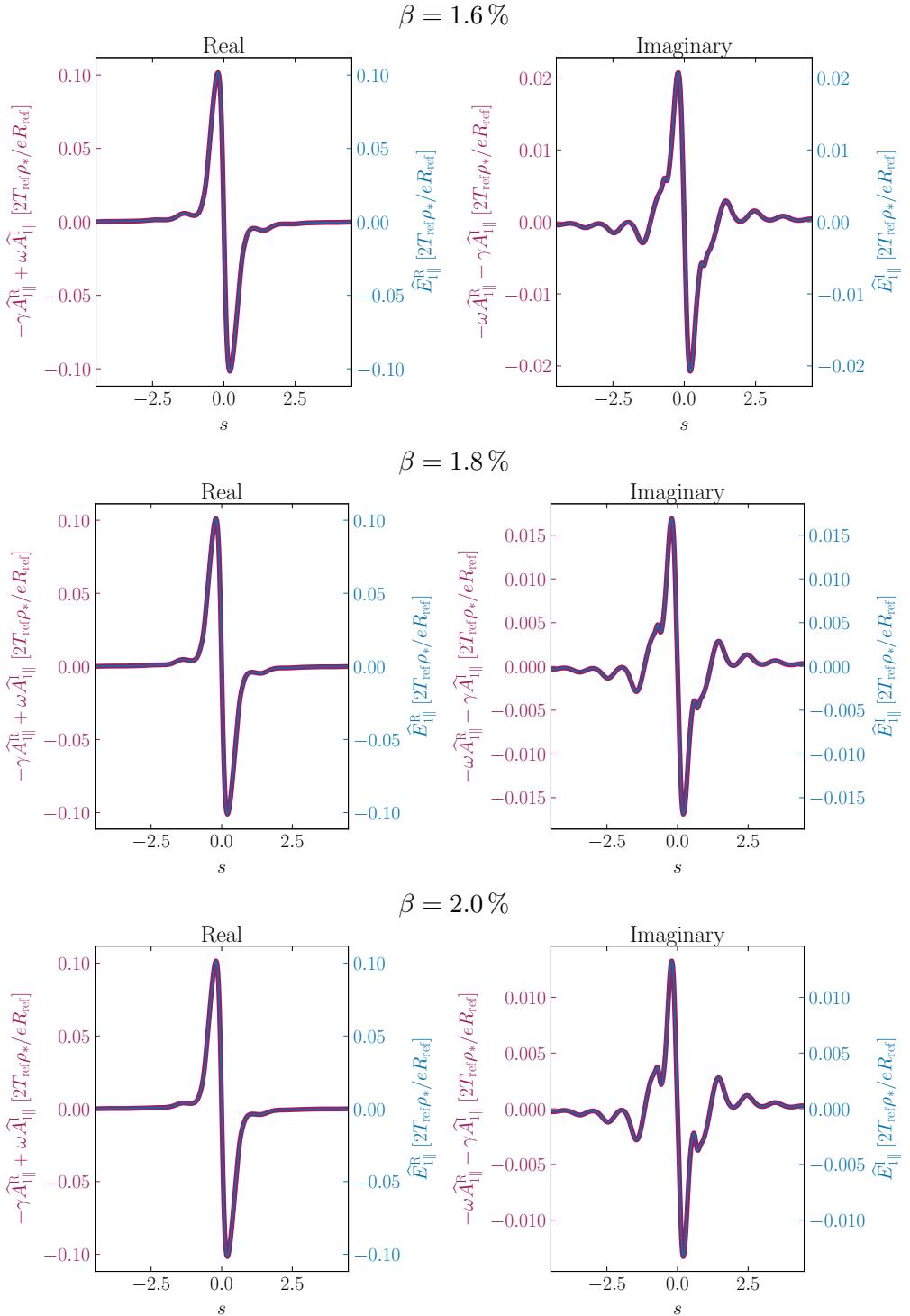
6 Appendix



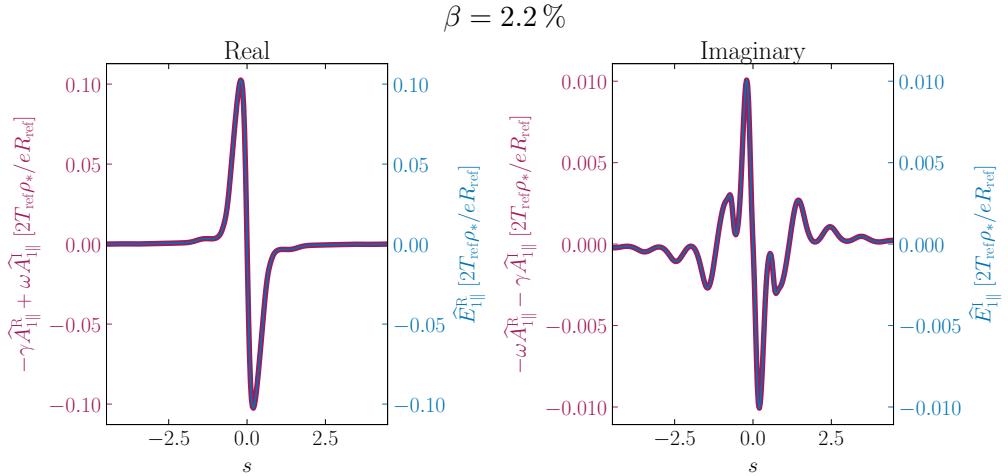
6 Appendix



6 Appendix



6 Appendix



7

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Eidesstattliche Erklärung

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Manuel Lippert