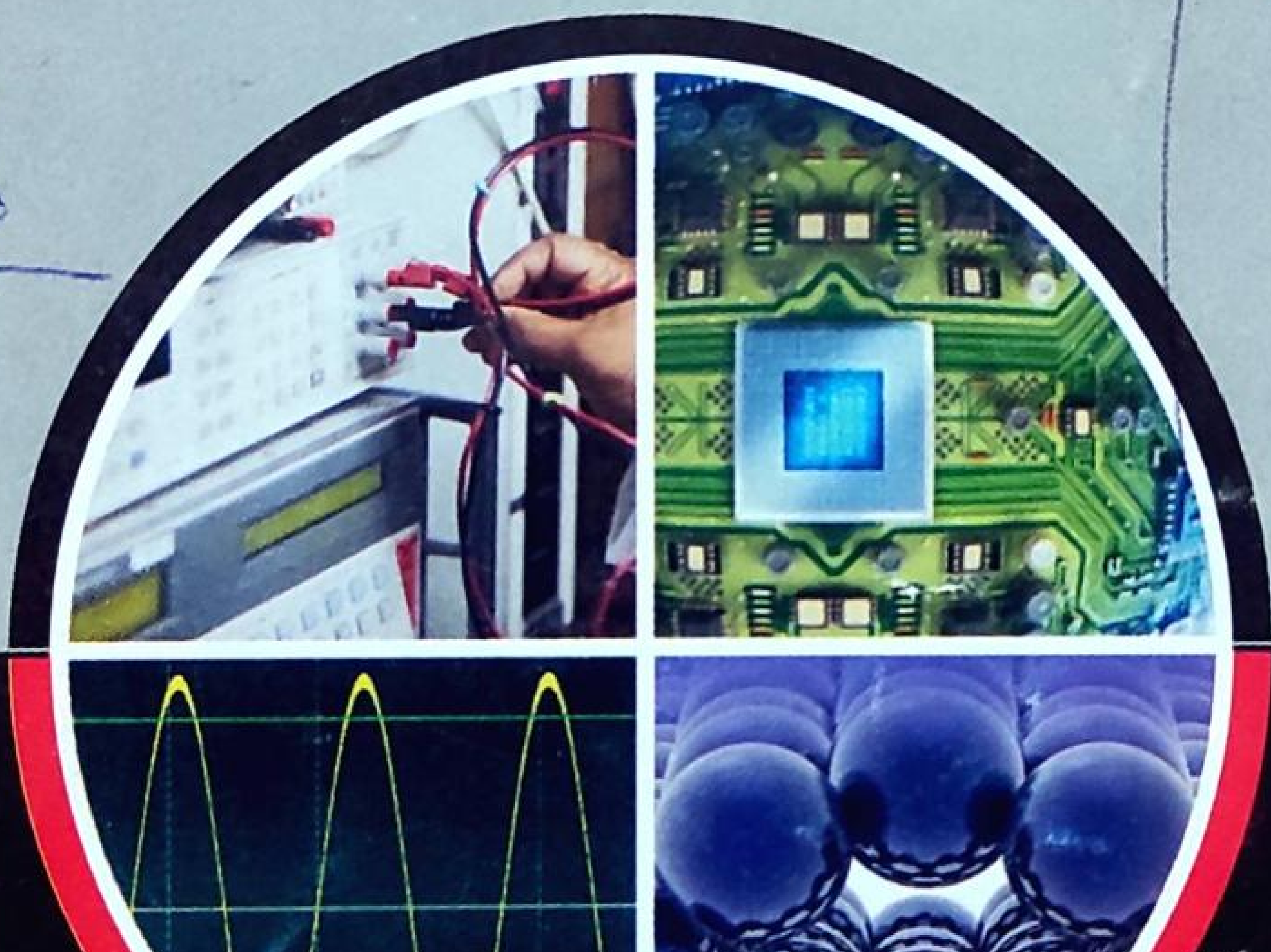
Quantum Series ENGINEERING

QUANTUM Series

Semester - 3 **Electrical & Electronics Engineering**

Basic Signals & Systems



- Topic-wise coverage of entire syllabus in Question-Answer form.
- Short Questions (2 Marks)

Session 2019-20 Odd Semester

Includes solution of following AKTU Question Papers
2014-15 • 2015-16 • 2016-17 • 2017-18 • 2018-19

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CONTENTS

KEE 303 : BASIC SIGNALS & SYSTEMS

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1 UNIT
Introduction to Continuous Time Signals and Systems (Important Formulas)

1. Energy signal, $E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$

2. Power signal, $P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$

2 UNIT
Fourier Transform Analysis (Important Formulas)

1. **Trigonometric Fourier series :**

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

where, $\omega_0 = \frac{2\pi}{T_0}$

2. **Exponential Fourier series :**

$$x(t) = C_0 + \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} C_n \, e^{jn\omega_0 t}$$

where, $C_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jn\omega_0 t} \, dt$ $C_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \, dt$

3. **Fourier transform**

$$x(j\omega) = \int_{-\infty}^{\infty} x(t) \, e^{-j\omega t} . \, dt \text{ for all } \omega$$

5. **DTFS (Discrete time fourier series)**

$$x[n] = \sum_{k=0}^{N_0-1} D_k e^{jk\Omega_0 n}$$

$$D_k = \frac{1}{N_0} \sum_{k=0}^{N_0-1} x[n] e^{-jk\Omega_0 n}$$

6. **DTFT (Discrete time fourier transform)**

$$F[x[n]] = X(\Omega) = \sum_{n=-\infty}^{\infty} x[n] \, e^{-j\Omega n}$$

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3 UNIT Laplace Transform Analysis (Important Formulas)

1. Laplace transform

$$F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$$

2. Convolution theorem : If $f(t)$ and $g(t)$ are two functions,

then $$f(t) * g(t) = \int_0^\infty f(\tau)\, g(t-\tau)\, d\tau$$

3. Initial value theorem

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

4. Final value theorem

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

4 UNIT State Variable Analysis (Important Formulas)

1. Transfer matrix, $\dfrac{Y(s)}{U(s)} = C\,[sI - A]^{-1} B$

$$= \frac{C \operatorname{adj}[sI - A]}{|sI - A|}$$

2. Matrix form

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

3. Diagonal matrix, $\Lambda = P^{-1}AP$

4. State transition matrix

$$\phi(t) = L^{-1}[sI - A]^{-1} = L^{-1}[\phi(s)]$$

where $\phi(s) = [sI - A]^{-1}$

5. Total solution,

$$x(t) = x_c(t) + x_p(t)$$

where $x_c(t) = \phi(t)\, x(0)$

and complementary solution

$$x_p(t) = \int_{t_0}^{t} \phi(t-\tau)\, Bu(\tau)\, d\tau$$

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5
UNIT

Z-Transform Analysis (Important Formulas)

1. Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, Z^{-n}$$

2. Inverse Z-transform

$$x[n] = Z^{-1}\,[X(z)]$$

and
$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} [x[n]\, r^{-n}]\, e^{-j\omega n}$$

$$x[n] = \frac{1}{2zj} \oint_c X(z)\, z^{n-1}\, dz$$

3. $$\log_e (1-x) = \left[-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots\right]$$

4. $$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1\,[k] x_2\,[n-k]$$

5. Initial value theorem :

$$x(0) = \lim_{n \to 0} x[n] = \lim_{z \to \infty} X(z)$$

6. Final value theorem :

$$x(\infty) = \lim_{n \to \infty} x[n] = \lim_{z \to 1} (z-1) X(z)$$

1
UNIT

Introduction to Continuous Time Signals and Systems

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PART-1

Introduction to Continuous Time and Discrete Time Signals, Classification of Signals with Their Mathematical Representation and Characteristics, Transformation of Independent Variable.

CONCEPT OUTLINE : PART-1

- **Signals :** It is a physical quantity that varies with time, space or any other independent variable or variables. It gives the information about behaviour or nature of the phenomenon.
- **Basic continuous time signals :**

i. Unit step
ii. Unit ramp
iii. Unit impulse

- **Relation between step, ramp and impulse signals :**

i. $\frac{d}{dt}$ (ramp signal) = step signal = $\int_0^{\infty}$ (impulse signal) dt

ii. $\frac{d}{dt}$ (step signal) = impulse signal

- **The basic operations done on signals are :**

1. Time shifting
2. Time reversal
3. Time scaling
4. Amplitude scaling
5. Signal addition
6. Signal multiplication

- **Signal modeling :** The representation of a signal by mathematical expression in known as signal modeling.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.1. **Define and give the classification of continuous-time signals and discrete-time signals.**

Answer

Continuous-time signals :

1. The signals that are defined for every instant of time are known as continuous-time signals.

2. Continuous-time signals are also called analog signals.
3. For continuous-time signals, the independent variable is time. They are denoted by $x(t)$. They are continuous in amplitude as well as in time.
4. Fig. 1.1.1(a) and (b) shows the graphical representation of continuous-time signals.

Discrete-time signals :

1. The signals that are defined only at discrete instants of time are known as discrete-time signals.
2. The discrete-time signals are continuous in amplitude but discrete in time.

Classification of continuous and discrete time signals :

1. Deterministic and random signals
2. Periodic and non-periodic signals
3. Energy and power signals
4. Causal and non-causal signals
5. Even and odd signals

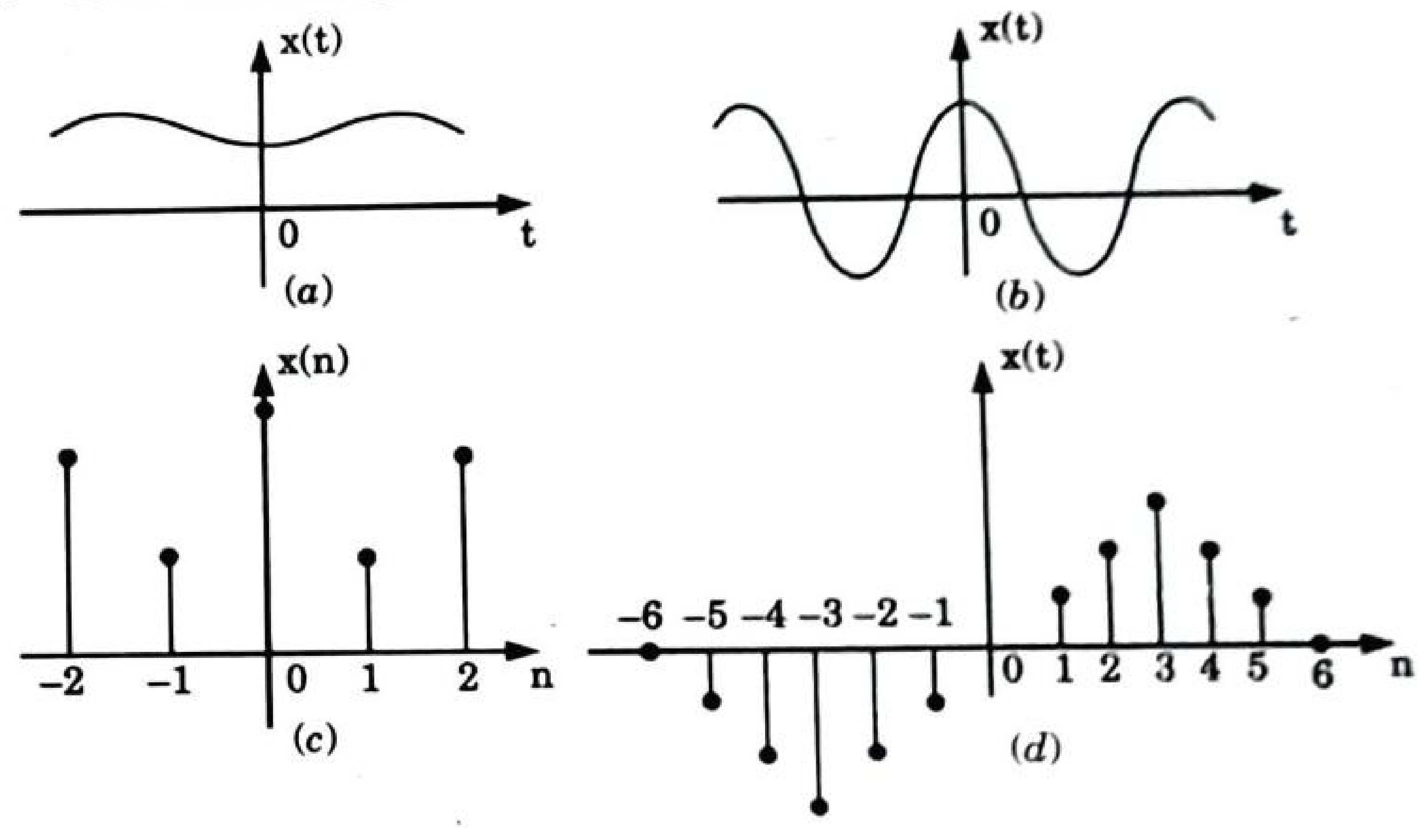

Fig. 1.1.1. (a) and (b) Continuous-time signals, (c) and (d) Discrete-time signals.

Que 1.2. **Define various elementary continuous time signals. Indicate them graphically.** **AKTU 2014-15, Marks 05**

Answer

1. **Unit step :** Signals which start at time t = 0 and have magnitude of unity are called unit step signals.
   They are represented by a unit step function $u(t)$.

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They are defined mathematically as :

$$u(t) = \begin{cases} 1; & t \geq 0 \\ 0; & t < 0 \end{cases}$$

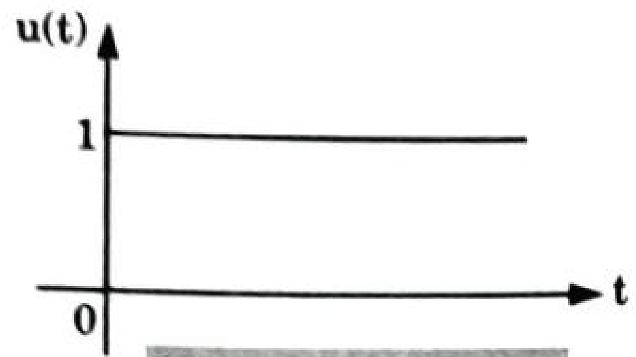

Fig. 1.2.1. Unit step.

2. **Unit ramp :** Signals which start from zero and are linear in nature with a constant slope m are called unit ramp signals.
   They are represented by a unit ramp function $r(t)$.
   They are defined mathematically as :

$$r(t) = \begin{cases} mt; & t \geq 0 \\ 0; & t < 0 \end{cases}$$

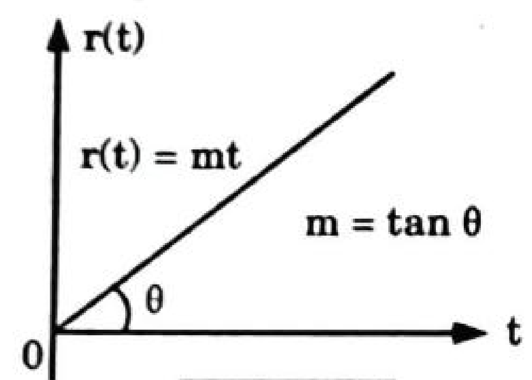

Fig. 1.2.2.

3. **Unit impulse :** Signals which act for very small time but have large amplitude are called unit impulse functions.
   They are represented by $\delta(t)$.
   They are defined mathematically as,

$$\delta(t) = \begin{cases} 0; & t \neq 0 \\ 1; & t = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

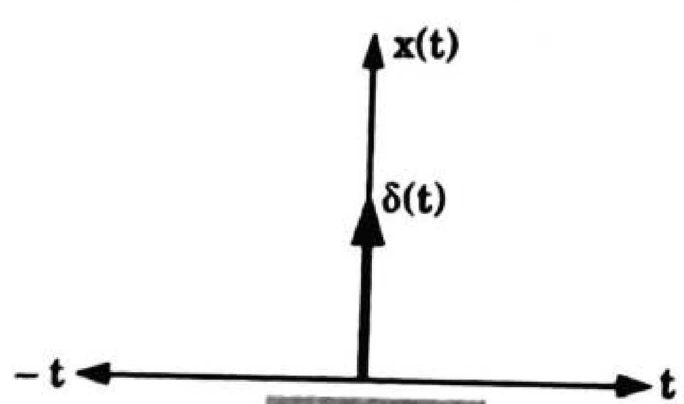

Fig. 1.2.3.

Que 1.3. **Explain the gate, impulse and ramp signal used in basic system analysis.**

AKTU 2011-12, Marks 05

Answer

1. **Gate signal :** A rectangular pulse as shown in Fig. 1.3.1 is called a unit gate function and is defined as :

$$x(t) = \text{rect}\left(\frac{t}{\tau}\right) = \Pi\left(\frac{t}{\tau}\right) = \begin{cases} 1 & ; \text{ for } |t| \leq \tau/2 \\ 0 & ; \text{ otherwise} \end{cases}$$

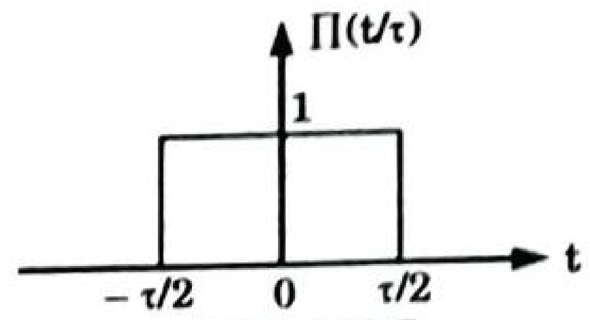

Fig. 1.3.1.

2. **Unit ramp :** Refer Q. 1.2, Page 1–3D, Unit-1.
3. **Unit impulse :** Refer Q. 1.2, Page 1–3D, Unit-1.

Que 1.4. **What is unit parabolic function ?**

Answer

The continuous-time unit parabolic function $p(t)$, also called acceleration signal starts at $t = 0$, and is defined as :

$$p(t) = \begin{cases} \dfrac{t^2}{2} & ; \text{ for } t \geq 0 \\ 0 & ; \text{ for } t < 0 \end{cases}$$

or $$p(t) = \frac{t^2}{2} u(t)$$

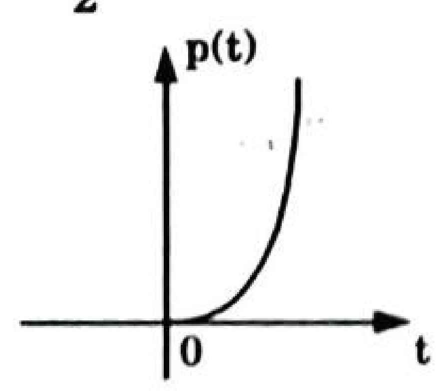

Fig. 1.4.1. Unit parabolic signal.

Que 1.5. **What is the significance of an impulse function ? What does the impulse response mean ? Explain.**

AKTU 2013-14, Marks 05

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Answer

Importance of impulse function :

1. By applying impulse signal to a system one can get the impulse response of the system. From impulse response, it is possible to get the transfer function of the system.
2. For a linear time-invariant system, if the area under the impulse response curve is finite, then the system is said to be stable.
3. From the impulse response of the system, one can easily get the step response and ramp response by integrating it once and twice respectively.
4. Impulse signal is easy to generate and apply to any system.

Impulse response : The impulse response function of a dynamic system is its output when presented with a brief input signal, called an impulse. It is the reaction of any dynamic system in response to some external change.

Que 1.6. **What are deterministic and random signals ?**

Answer

Deterministic signal :

1. A signal exhibiting no uncertainty of its magnitude and phase at any given instant of time is called deterministic signal.
2. A deterministic signal has a regular pattern and can be completely represented by mathematical equation at any time. Its amplitude and phase at any time instant can be predicted in advance.

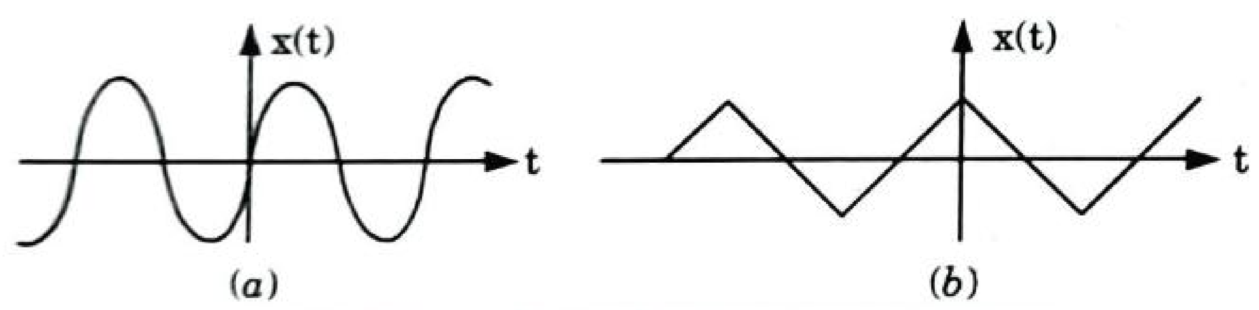

Fig. 1.6.1. (*a*) and (*b*) Deterministic signals.

Random signal :

1. A signal characterized by uncertainty about its occurrence is called a random signal.
2. A random signal cannot be represented by any mathematical equation.
3. The pattern of such a signal is quite irregular. Its amplitude and phase at any time instant cannot be predicted in advance.

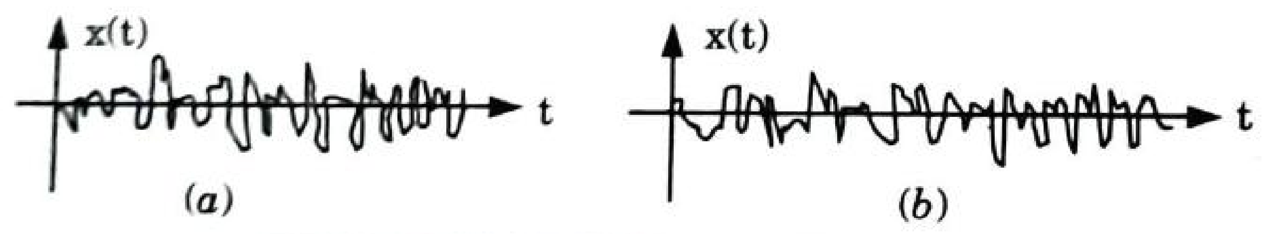

Fig. 1.6.2. (*a*) and (*b*) Random signals.

Que 1.7. **Define periodic and non-periodic signals.**

Answer

1. Signal which has a definite pattern and which repeats itself at regular intervals of time is called a periodic signal, which does not repeat at regular intervals of time is called a non-periodic or aperiodic signal.
2. Mathematically, a continuous-time signal $x(t)$ is called periodic if and only if

$$x(t + T) = x(t) \text{ for all } t, \textit{ i.e.}, \text{ for } -\infty < t < \infty$$

where t denotes time and T is a constant representing the period.

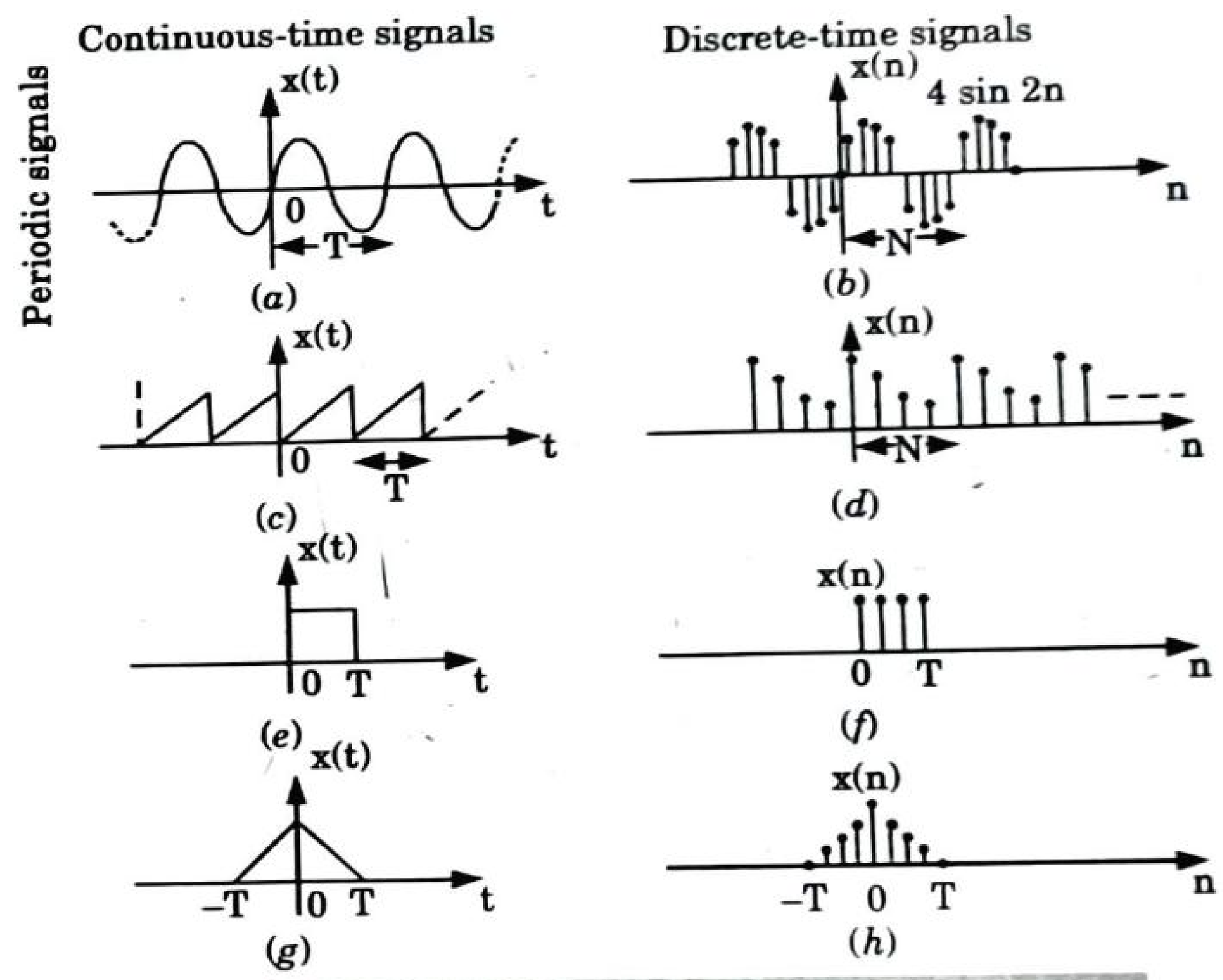

Fig. 1.7.1. (*a*), (*b*), (*c*) and (*d*) periodic signals; (*e*), (*f*), (*g*) and (*h*) non-periodic signals.

Que 1.8. **Explain what are power and energy signals ? Explain their relationship with periodicity.** **AKTU 2012-13, Marks 05**

Answer

Energy and Power signals : Signal having finite energy is called energy signal.

$$E_\infty = \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

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Signal with finite and non-zero power is a power signal.

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt$$

A signal cannot be both energy and power signal at the same time.

Relationship with periodicity :

1. Consider $v(t)$ to be the voltage across a resistor R producing a current $i(t)$. The instantaneous power $p(t)$ per ohm is defined as :

$$p(t) = v(t)\, i(t) = \frac{v(t)\, v(t)}{R} = \frac{v^2(t)}{R}$$
$$= i(t)R\, i(t) = i^2(t)\, R$$

2. Total energy E and average power P on a per-ohm basis are :

$$E = \int_{-\infty}^{\infty} i^2(t)\, dt \text{ joules}$$

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} i^2(t)\, dt \text{ watts}$$

3. For an arbitrary continuous-time signal $x(t)$, the normalized energy content E of $x(t)$ is defined as :

$$E = \lim_{T\to\infty} \int_{-T}^{T} x^2(t)\, dt$$

4. The normalized average power P of $x(t)$ is defined as :

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)\, dt$$

5. Similarly, for a discrete-time signal $x(n)$, the normalized energy content E of $x(n)$ is defined as :

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

6. The normalized average power P of $x(n)$ is defined as :

$$P = \lim_{n\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

7. $x(t)$ (or $x(n)$) is said to be an energy signal if and only if $0 < E < \infty$, and so $P = 0$. Non-periodic signals are example of energy signals.
8. $x(t)$ (or $x(n)$) is said to be power signal if and only if $0 < P < \infty$, thus implying that $E = \infty$. Periodic signals are example of power signal.

Que 1.9. **Prove the following :**

a. The power of the energy signal is zero over infinite time.
b. The energy of the power signal is infinite over infinite time.

Answer

a. Proof : The power of the energy signal is zero over infinite time :

1. Let $x(t)$ be an energy signal, *i.e.*,

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt \text{ is finite.}$$

2. Power of the signal

$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt$$
$$= \lim_{T\to\infty} \frac{1}{2T} \left[\lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\, dt \right] = \lim_{T\to\infty} \frac{1}{2T} \left[\int_{-\infty}^{\infty} |x(t)|^2\, dt \right]$$
$$= \lim_{T\to\infty} \frac{1}{2T} [E] \left[\text{since } E = \int_{-\infty}^{\infty} |x(t)|^2\, dt \right]$$
$$= 0 \times E = 0 \left[\text{since } \lim_{T\to\infty} \frac{1}{2T} = 0 \right]$$

3. Thus, the power of the energy signal is zero over infinite time.

b. Proof : The energy of the power signal is infinite over infinite time :

1. Let $x(t)$ be a power signal, *i.e.*,

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt \text{ is finite.}$$

2. Energy of the signal

$$E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

3. Let us change the limits of integration as $-T$ to T and take limit $T \to \infty$.
4. This will not change the meaning of E,

i.e.,
$$E = \lim_{T\to\infty} \int_{-T}^{T} |x(t)|^2\, dt = \lim_{T\to\infty} \left[2T \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt \right]$$
$$= \lim_{T\to\infty} 2T \left[\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt \right]$$
$$= \lim_{T\to\infty} 2TP = \infty \left[\text{since } P = \frac{1}{2T} \int_{-T}^{T} |x(t)|^2\, dt \right]$$

5. Thus, the energy of the power signal is infinite over infinite time.

Que 1.10. **Explain briefly :**

i. Causal and non-causal signals.
ii. Even and odd signals.

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Answer

i. Causal and non-causal signals.

1. Continuous-time signal $x(t)$ is said to be causal if $x(t) = 0$ for $t < 0$.
2. A continuous-time signal $x(t)$ is said to be non-causal if $x(t) = 0$ for $t > 0$.

ii. Even (symmetric) and odd (non-symmetric) signal :

1. A continuous-time signal $x(t)$ is said to be an even (symmetric) signal if it satisfies the condition

$$x(t) = x(-t) \text{ for all } t$$

2. A continuous-time signal $x(t)$ is said to be an odd (non-symmetric) signal if it satisfies the condition

$$x(-t) = -x(t) \text{ for all } t$$

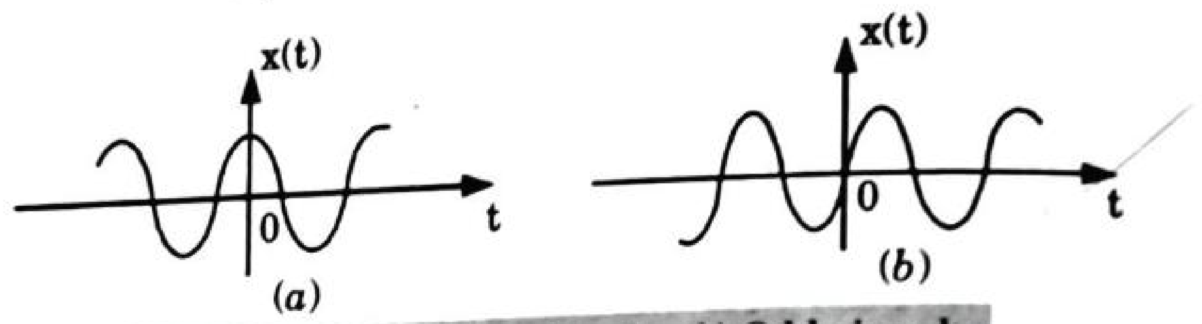

Fig. 1.10.1. (*a*) Even signals, (*b*) Odd signals.

Que 1.11. What do you understand by transformation of the independent variable ?

Answer

Transformation of independent variable :

During the processing of signal several manipulations involving independent variable are required. It means that for analysis of signals, transformation of independent variable is required.

The basic operations on signals are as follows :

i. Time shifting :

1. Mathematically, the time shifting of a continuous-time signal $x(t)$ can be represented by

$$y(t) = x(t - T) \qquad \text{...(1.11.1)}$$

2. The time shifting of a signal may result in time delay or time advance.
3. In eq. (1.11.1) if T is positive the sifting is to the right and then the shifting delays the signal, and if T is negative the shift is to the left and then the shifting advances the signal.
4. An arbitrary signal $x(t)$, its delayed version and advanced version are shown in Fig. 1.11.1(*a*), (*b*) and (*c*).

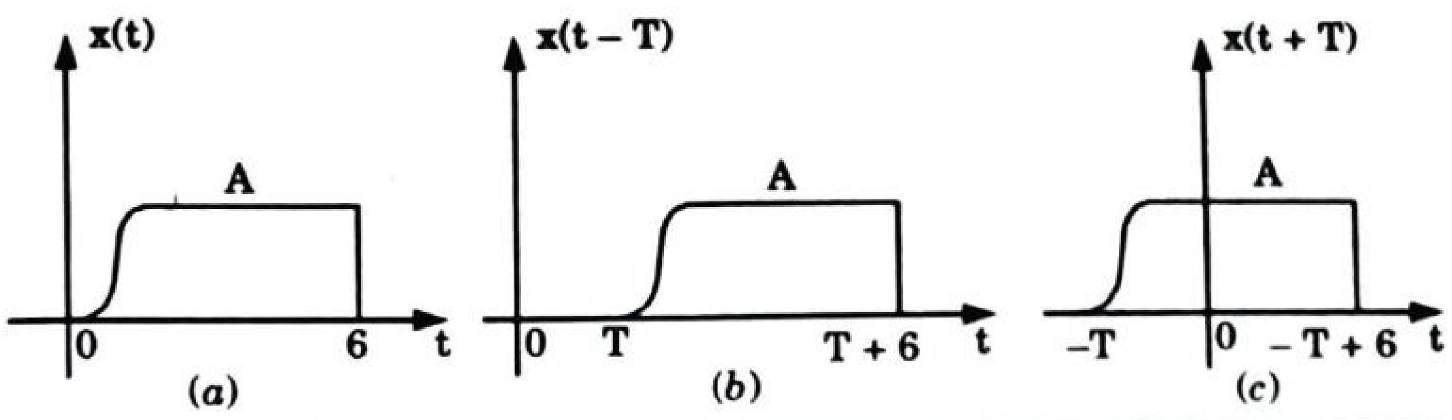

Fig. 1.11.1. (*a*) Signal, (*b*) Its delayed version, (c) Its time advanced version.

ii. Time reversal :

1. Time reversal, also called time folding of a signal $x(t)$ can be obtained by folding the signal about $t = 0$. This operation is very useful in convolution. It is denoted by $x(-t)$.
2. It is obtained by replacing the independent variable t by $(-t)$. Folding is also called as the reflection of the signal about the time origin $t = 0$.
3. Fig. 1.11.2(*a*) shows an arbitrary signal $x(t)$, and Fig. 1.11.2(*b*) shows its reflection $x(-t)$.

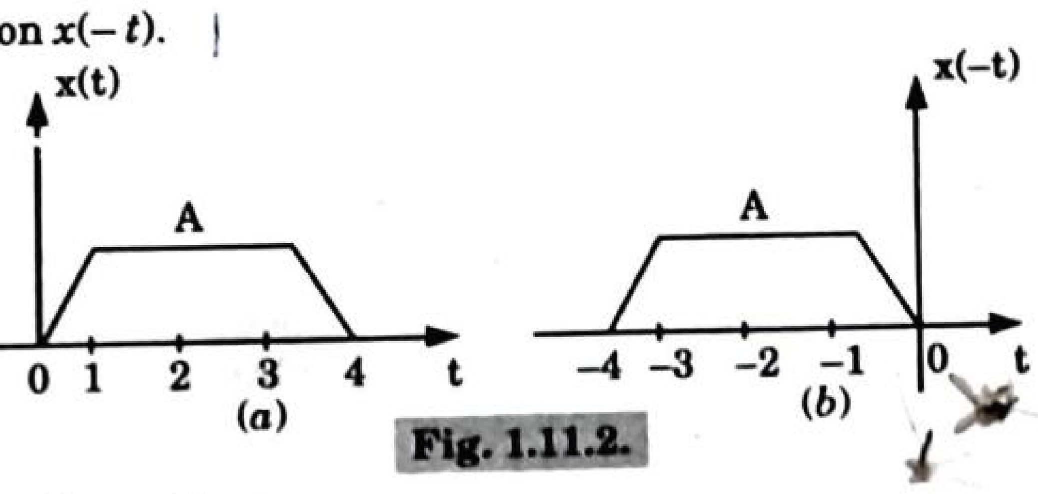

Fig. 1.11.2.

iii. Amplitude scaling :

1. The amplitude scaling of a continuous-time signal $x(t)$ can be represented by

$$y(t) = Ax(t)$$

where A is a constant.

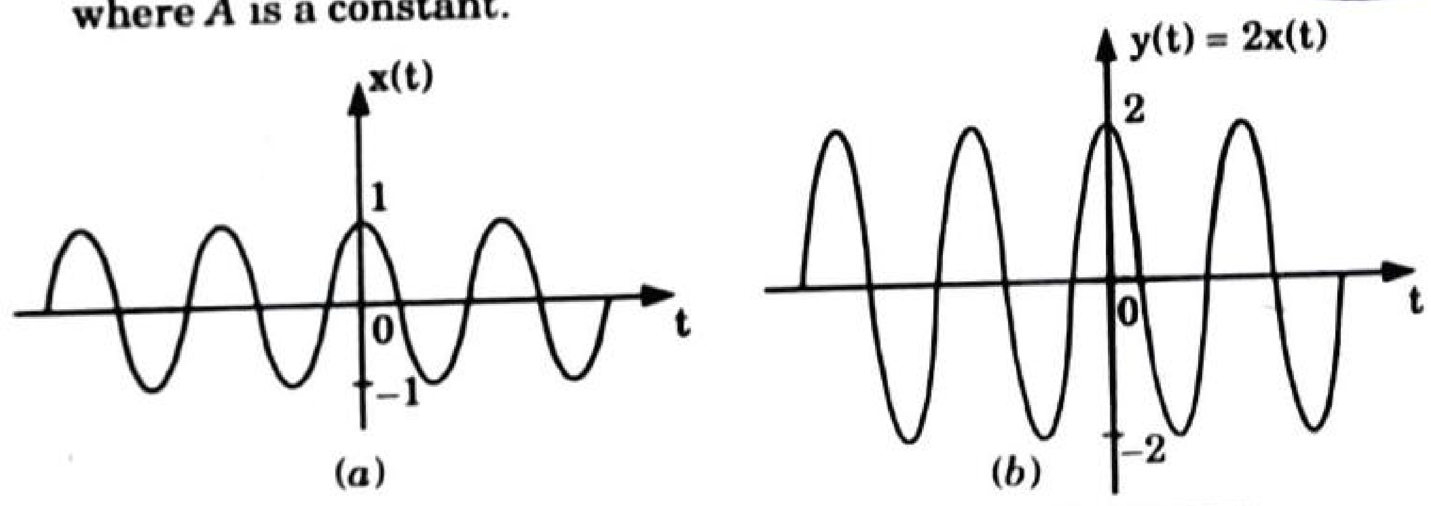

Fig. 1.11.3. Plots of (*a*) $x(t) = \cos \omega t$, (*b*) $y(t) = 2x(t)$.

iv. Time scaling :

1. Time scaling may be time expansion or time compression.
2. The time scaling of a signal $x(t)$ can be accomplished by replacing t by at in it. Mathematically, it can be expressed as :

$$y(t) = x(at)$$

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3. If $a > 1$, it results in time compression by a factor a and if $a < 1$, it results in time expansion by a factor a because with that transformation a point at 'at' in signal $x(t)$ becomes a point at 't' in $y(t)$.

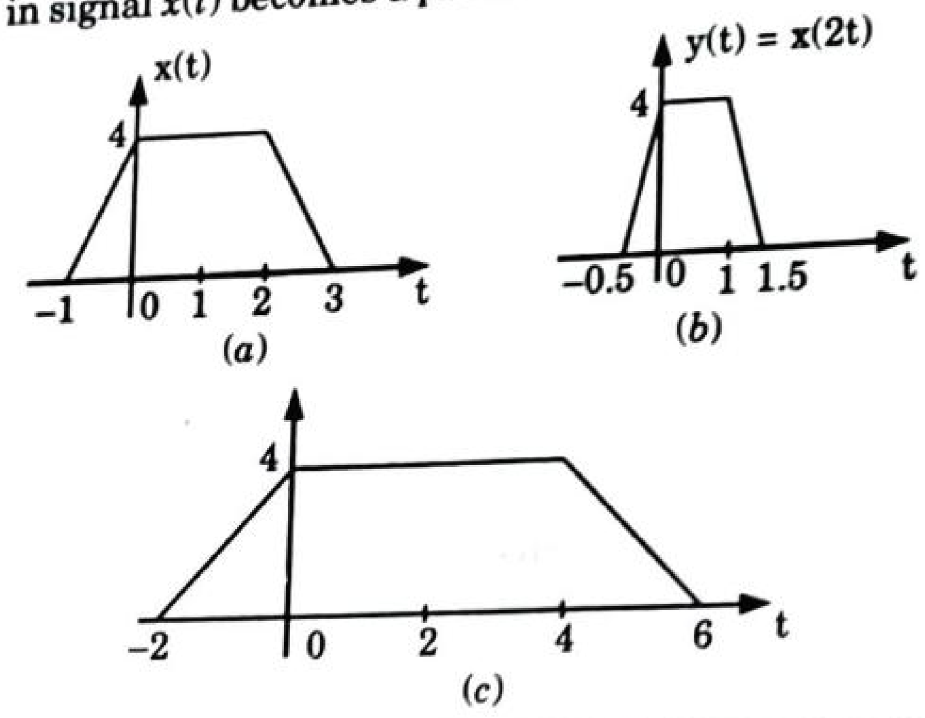

Fig. 1.11.4. *(a)* Original signal, *(b)* Compressed signal, *(c)* Enlarged signal.

v. Signal addition :

1. The sum of two continuous-time signals $x_1(t)$ and $x_2(t)$ can be obtained by adding their values at every instant of time.
2. Similarly, the subtraction of one continuous-time signal $x_2(t)$ from another signal $x_1(t)$ can be obtained by subtracting the value of $x_2(t)$ from that of $x_1(t)$ at every instant.
3. Consider two signals $x_1(t)$ and $x_2(t)$ shown in Fig. 1.11.5*(a)* and *(b)*.

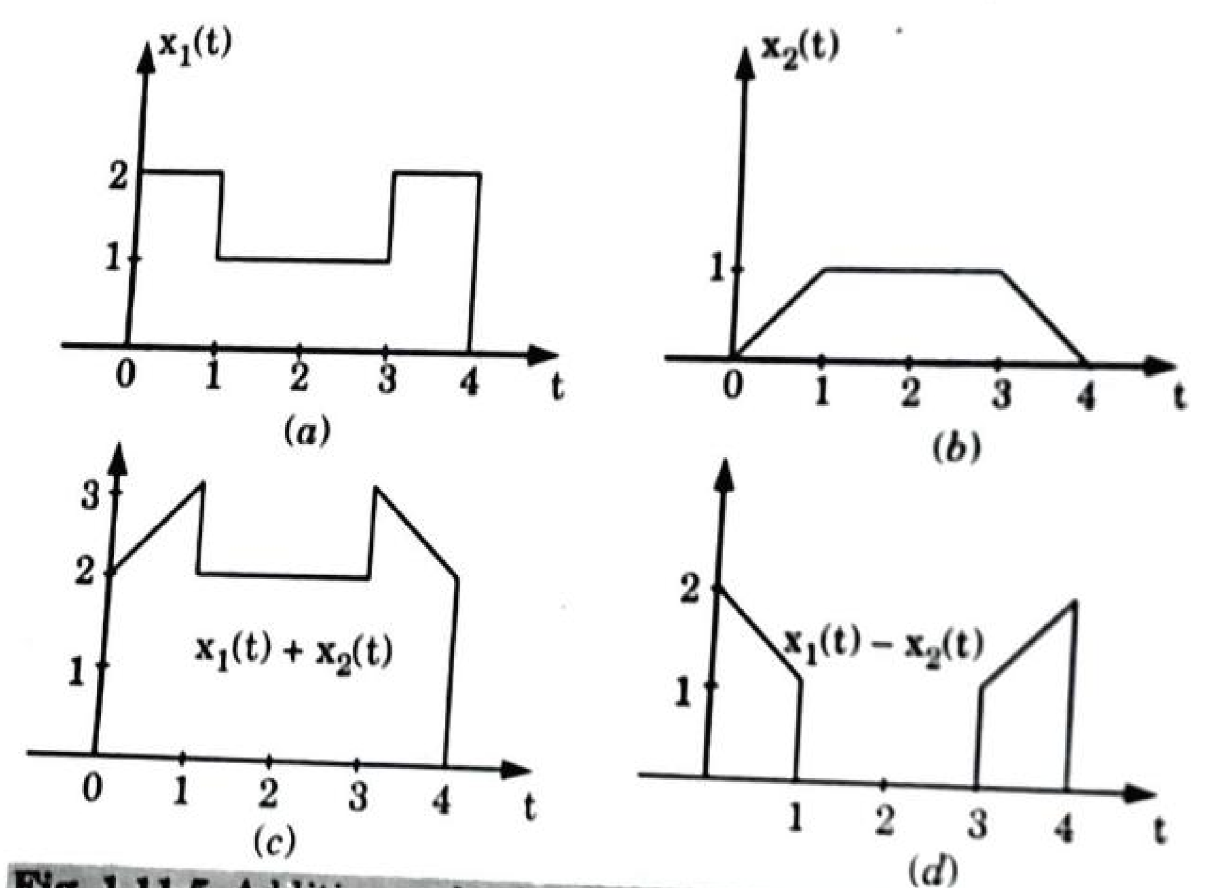

Fig. 1.11.5. Addition and subtraction of continuous-time signals.

4. Fig. 1.11.5*(c)* is the addition of $x_1(t)$ and $x_2(t)$ and Fig. 1.11.5*(d)* is the substraction of $x_2(t)$ from $x_1(t)$.

vi. Signal multiplication :

1. The multiplication of two continuous-time signals can be performed by multiplying their values at every instant.
2. Two continuous-time signals $x_1(t)$ and $x_2(t)$ shown in Fig. 1.11.6*(a)* and *(b)* are multiplied to obtain $x_1(t)\, x_2(t)$ shown in Fig. 1.11.6(c).

For $0 \le t \le 1$ $\quad x_1(t) = 2$ and $x_2(t) = 1$

Hence $\quad x_1(t)\, x_2(t) = 2 \times 1 = 2$

For $1 \le t \le 2$ $\quad x_1(t) = 1$ and $x_2(t) = 1 + (t - 1)$

Hence $\quad x_1(t)\, x_2(t) = (1)\,[1 + (t - 1)] = 1 + (t - 1)$

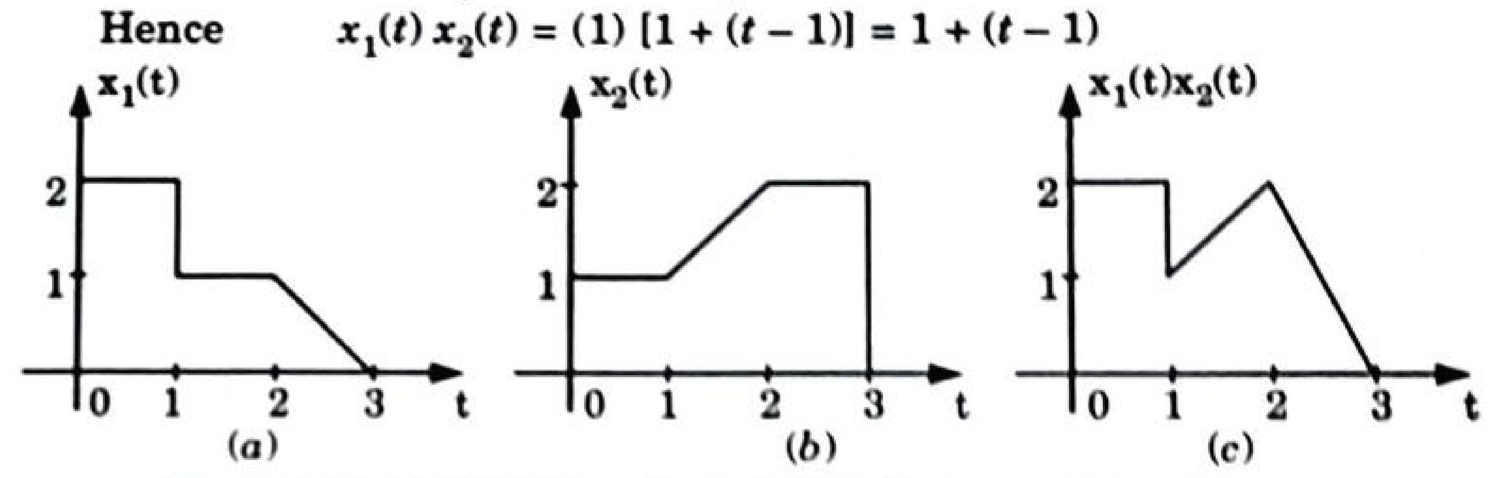

Fig. 1.11.6. Multiplication of continuous-time signals.

Que 1.12. **Synthesize a triangular wave, given in Fig. 1.12.1 in terms of ramp and step signals.**

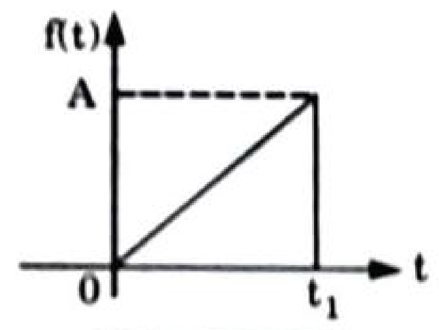

Fig. 1.12.1.

AKTU 2011-12, Marks 05

Answer

1. Signal shown in Fig. 1.12.2 starting from $t = 0$ with a slope A/t_1 is ramp signal $(A/t_1)\, r(t)$.

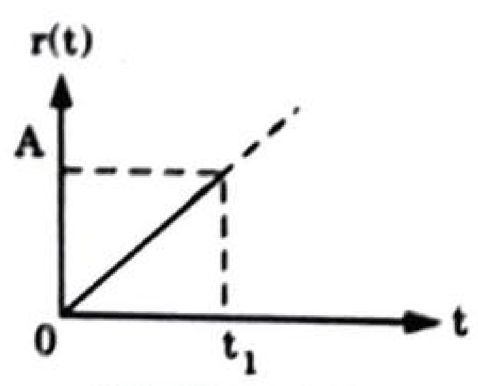

Fig. 1.12.2.

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2. At $t = t_1$, signal terminates. So, we have to multiply it with unit step signal $u(-t + t_1)$.

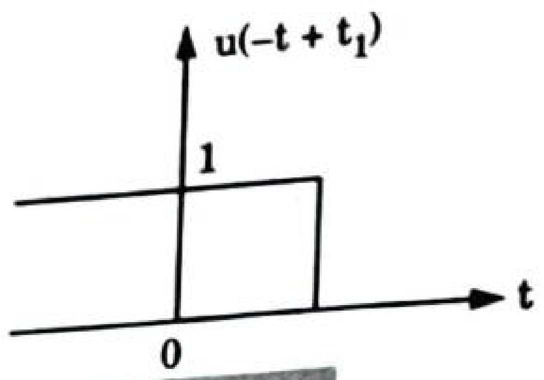

Fig. 1.12.3.

3. Multiplying both signals, we get $f(t) = r(t)\, u(-t + t_1)$

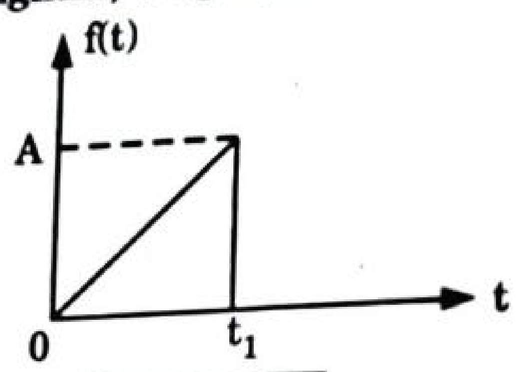

Fig. 1.12.4.

Que 1.13. **Sketch the waveform from the expression :**

$$v(t) = u(t) + \sum_{k=1}^{\infty} (-1)^k 3u(t-k)$$

AKTU 2011-12, Marks 06

Answer

$$v(t) = u(t) + \sum_{k=1}^{\infty} (-1)^k\, 3u(t-k)$$

$$= u(t) - 3u(t-1) + 3u(t-2) - 3u(t-3) + \ldots\ldots$$

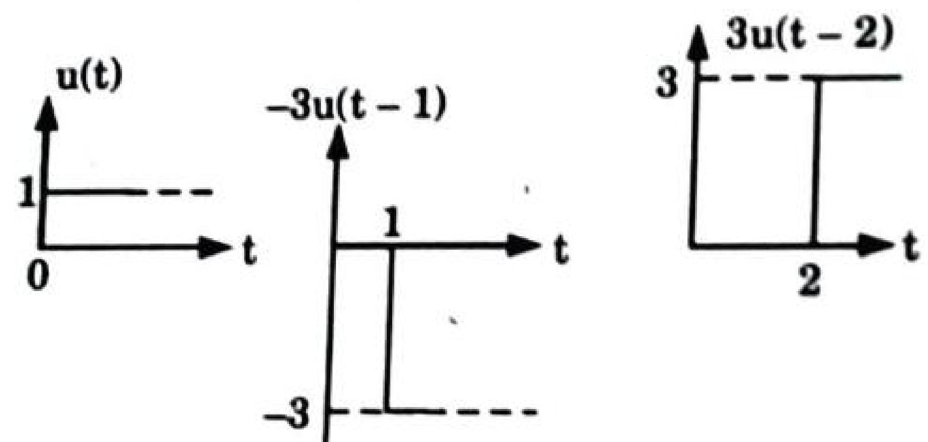

Fig. 1.13.1.

At $t = 0$, $v(t) = 1$
At $t = 1$, $v(t) = 1 - 3 = -2$
At $t = 2$, $v(t) = -2 + 3 = 1$
At $t = 3$, $v(t) = 1 - 3 = -2$
....

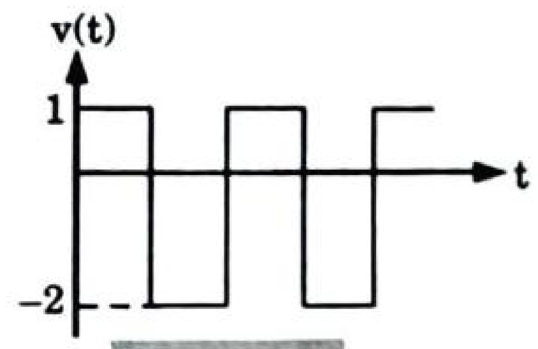

Fig. 1.13.2.

Que 1.14. **A signal $x(t)$ is given by the Fig. 1.14.1. Draw and explain the signal $\phi(t) = x\left(\frac{t}{2} + 6\right)$.**

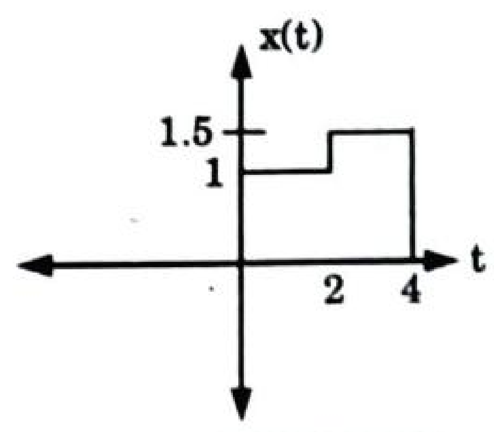

Fig. 1.14.1.

AKTU 2012-13, Marks 05

Answer

1. Here $x(t + 6)$ is shifted by 6 in Fig. 1.14.2.

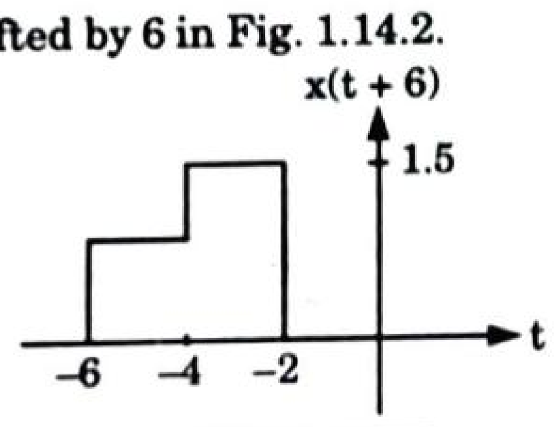

Fig. 1.14.2.

2. Then $x(t + 6)$ function is expanded by 2 as shown in Fig. 1.14.3.

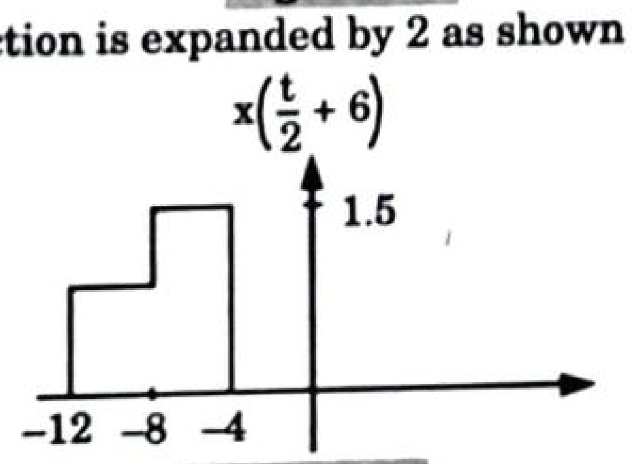

Fig. 1.14.3.

3. ∴ Fig. 1.14.3 is the required waveform.

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Que 1.15. Sketch the signals

$y(t) = r(t+2) - r(t) + r(t-2)$

$y(t) = u(t) + 5u(t-1) - 2u(t-2)$

AKTU 2014-15, Marks 3.5

Answer

i. $y(t) = r(t+2) - r(t) + r(t-2)$

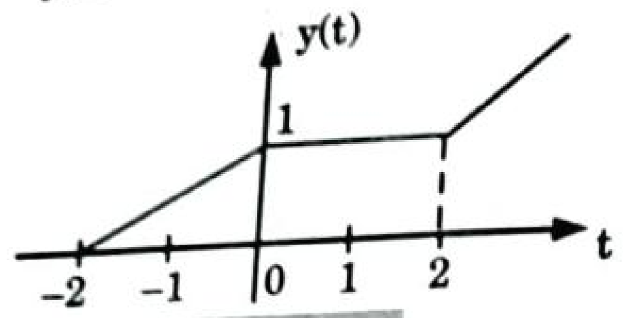

Fig. 1.15.1.

ii. $y(t) = u(t) + 5u(t-1) - 2u(t-2)$

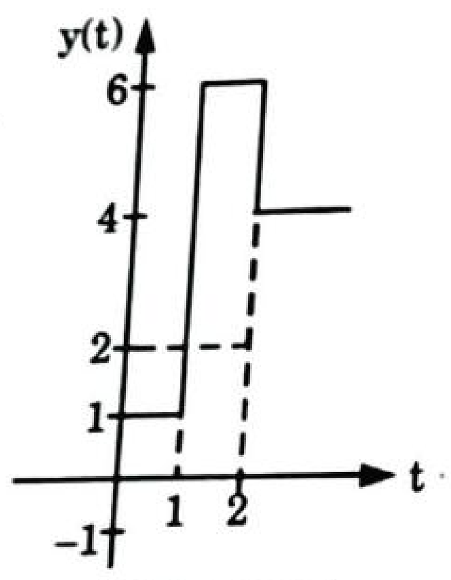

Fig. 1.15.2.

Que 1.16. Find the even and odd components of the signal shown in Fig. 1.16.1.

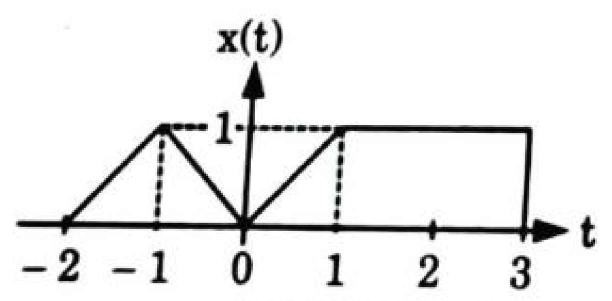

Fig. 1.16.1.

AKTU 2015-16, Marks 05

Answer

$$x(t) = r(t+2) - 2r(t+1) + 2r(t) - r(t-1) - u(t-3)$$

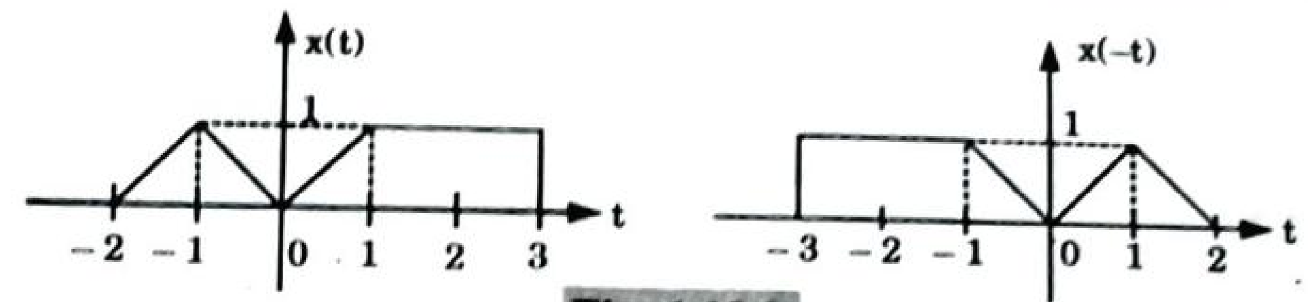

Fig. 1.16.2.

1. **Even component :**

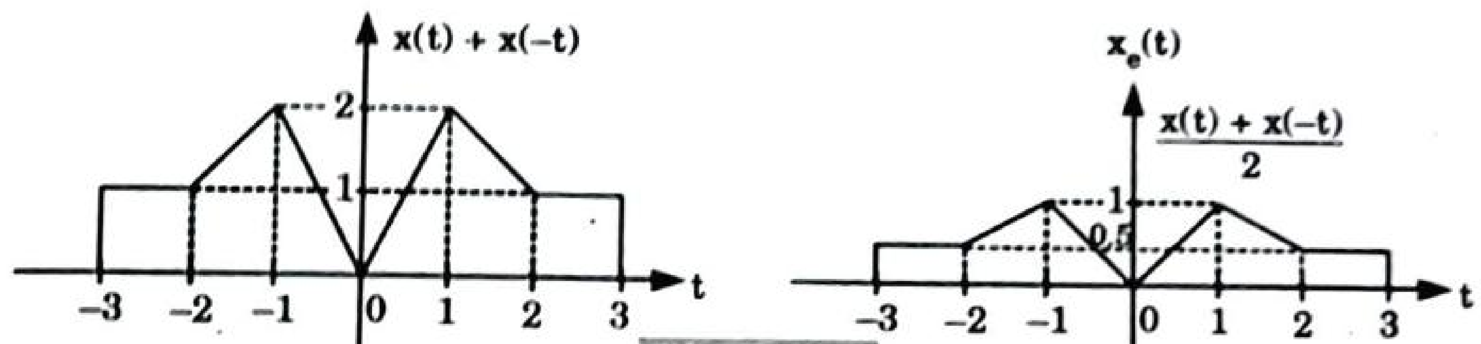

Fig. 1.16.3.

$$x_e(t) = \frac{1}{2}u(t+3) + \frac{1}{2}r(t+2) - \frac{3}{2}r(t+1) + 2r(t) - \frac{3}{2}r(t-1) + \frac{1}{2}r(t-2) - u(t-3)$$

2. **Odd component :**

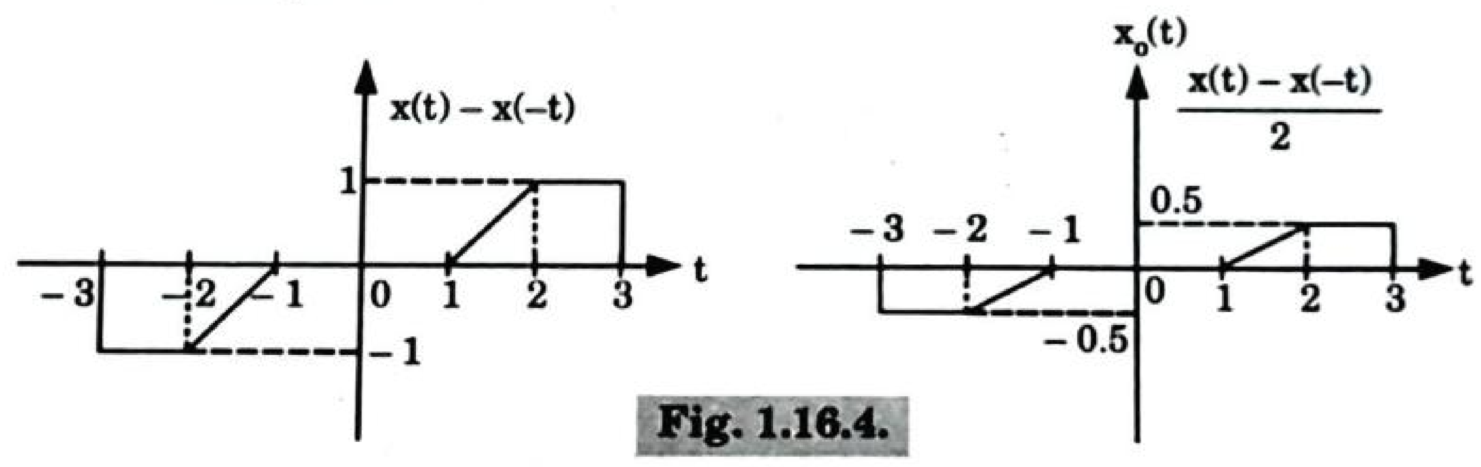

Fig. 1.16.4.

$$x_o(t) = -\frac{1}{2}u(t+3) + \frac{1}{2}r(t+2) - \frac{1}{2}r(t+1) + \frac{1}{2}r(t-1) - \frac{1}{2}r(t-2) - \frac{1}{2}u(t-3)$$

Que 1.17. What are periodic signals ? Find the fundamental period of $X(n) = e^{j(2\pi/3)n} + e^{j(3\pi/4)n}$.

AKTU 2013-14, Marks 05

Answer

Periodic signals : Refer Q. 1.7, Page 1–7D, Unit-1.

Numerical :

1. $$x(n) = \cos\frac{2\pi}{3}n + j\sin\frac{2\pi}{3}n + \cos\frac{3\pi}{4}n + j\sin\frac{3\pi}{4}n$$

Note that there are two different frequencies.

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2. $\therefore$ $2\pi f_1 n = \frac{2\pi}{3} n \Rightarrow f_1 \frac{k_1}{N_1} = \frac{1}{3} \Rightarrow N_1 = 3$

and $2\pi f_2 n = \frac{3\pi}{4} n \Rightarrow f_2 \frac{k_2}{N_2} = \frac{3}{8} \Rightarrow N_2 = 8$

3. Since $\frac{N_1}{N_2} = \frac{3}{8}$ is the ratio of two integers (*i.e.* rational), the signal is periodic.
4. The fundamental period is LCM of 3 and 8 *i.e.* $N = 24$.

Que 1.18. **The waveform is shown in Fig. 1.18.1. Write an equation for this waveform $v(t)$ using step functions.**

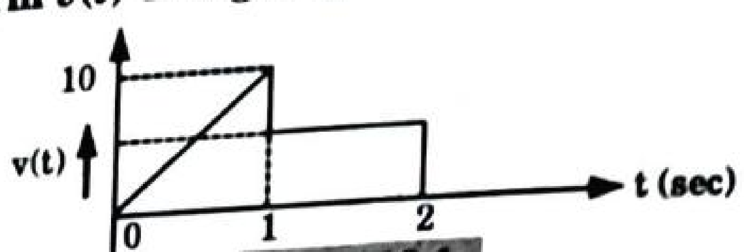

Fig. 1.18.1.

Answer

$$v(t) = 10[r(t) - r(t-1)] - 5u(t-1) - 5u(t-2)$$

PART-2

Introduction to Various Type of System, Basic System Properties.

CONCEPT OUTLINE : PART-2

- **System :** It is a physical device used to generate a response or output signal for a given input signal.
- **Classification of systems :**

i. Linear and non-linear systems.
ii. Time-invariant and time-variant systems.
iii. Instantaneous and dynamic systems.
iv. Causal and non-causal systems.
v. Continuous-time and discrete-time systems.
vi. Analog and digital systems.
vii. Invertible and non-invertible systems.
viii. Stable and astable systems

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.19. **Explain system and its classification.**

Answer

1. A system is represented by a block diagram as shown in Fig. 1.19.1.

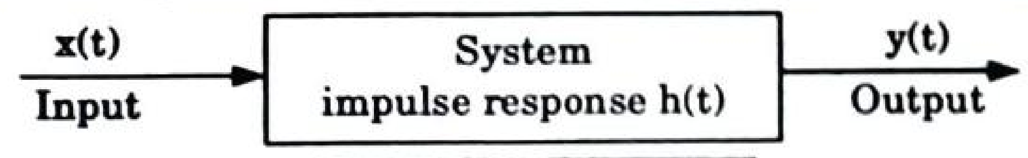

Fig. 1.19.1. A system.

2. An arrow entering the box is the input signal (also called excitation, source or driving function) and an arrow leaving the box is an output signal (also called response).
3. Generally, the input is denoted by $x(t)$ and the output is denoted by $y(t)$.
4. Mathematically, $y(t) = T[x(t)]$

which represents that $x(t)$ is transformed to $y(t)$. In other words $y(t)$ is the transformed version of $x(t)$.

Classification :

Continuous-time system : A continuous-time system is one which transforms continuous-time input signals to continuous-time output signals.

Discrete-time system : A discrete-time system is one which transforms discrete-time input signals to discrete-time output signals.

Que 1.20. **Discuss various types of systems.**

Answer

Static and dynamic system :

1. A system is said to be static or memoryless if the response is due to present input alone, *i.e.*, for a static or memoryless system, the output at any instant t (or n) depends only on the input applied at that instant t (or n) but not on the past or future values of input.

Example, $y(t) = x(t)$
$y(t) = x^2(t)$

2. In contrast, a system is said to be dynamic or memory system if the response depends upon past or future inputs.

Example, $y(t) = x(t-1)$
$y(t) = x(t) + x(t+2)$

Causal and non-causal system :

1. A system is said to be causal (non-anticipative) if the output of the system at any time t depends only on the present and past values of the input but not on future inputs. Causal system are real time system.

Example, $y(t) = x(t-2) + 2x(t)$
$y(t) = tx(t)$

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2. A system is said to be non-causal (anticipative) if the output of the system at any time t depends on future inputs. They do not exist in real time.

Example, $y(t) = x(t + 2) + 2x(t)$

$y(t) = x^2(t) + tx(t + 1)$

Linear and non-linear system :

1. A system which obeys the principle of superposition and principle of homogeneity is called a linear system.
2. A system which does not obey the principle of superposition and homogeneity is called a non-linear system.

Time-invariant and time-varying system :

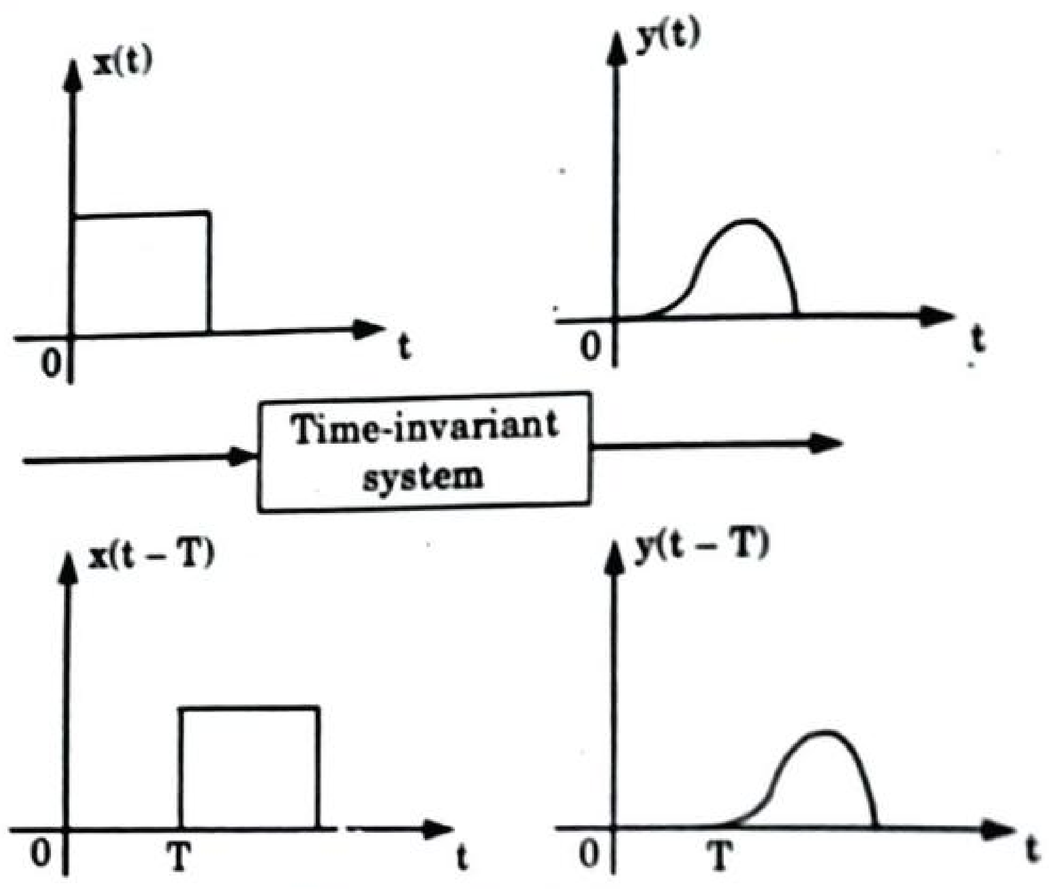

Fig. 1.20.1. Time-invariant system.

1. Time-invariance is the property of a system which makes the behaviour of the system independent of time. This means that the behaviour of the system does not depend on the time at which the input is applied.

   If $x(t) \rightarrow y(t)$

   then $x(t - T) \rightarrow y(t - T)$
2. A system not satisfying these requirements is called a time-varying system.

Que 1.21. **Explain the concepts of linearity and time-invariance.**

AKTU 2013-14, Marks 05

OR

Explain the concepts of stability and time-invariance taking suitable examples.

AKTU 2012-13, Marks 05

Answer

Linearity : Systems having proportional output to their input satisfying additive property as well as homogeneity property (also called scaling property), are called linear systems.

i. **Additive Property :**

$$A \rightarrow B$$
$$C \rightarrow D$$
$$A + C \rightarrow B + D$$

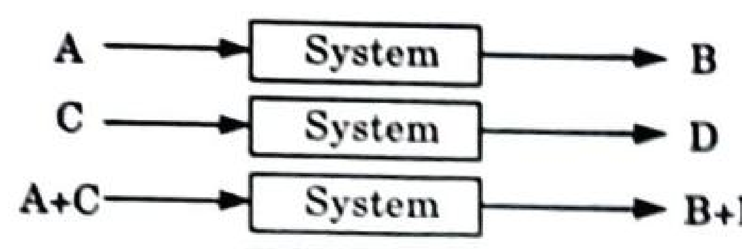

Fig. 1.21.1.

If inputs A and C are acting separately on a system then the output of the system is B and D respectively. But if both A and C act simultaneously then output is $B + D$, this system is said to be additive.

ii. **Homogeneity Property :**

$$A \rightarrow B$$
$$kA \rightarrow kB$$

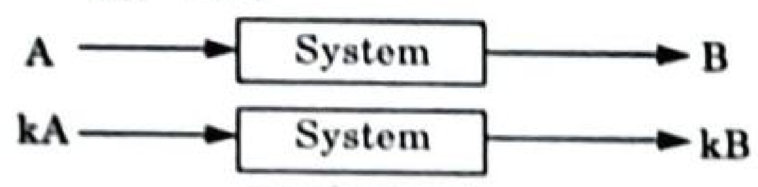

Fig. 1.21.2.

If the input A is increased k times, the output also increases k times, this property is called homogeneity.

Linearity = Additive + Homogeneity

$$k_1A + k_2C \rightarrow k_1B + k_2D$$

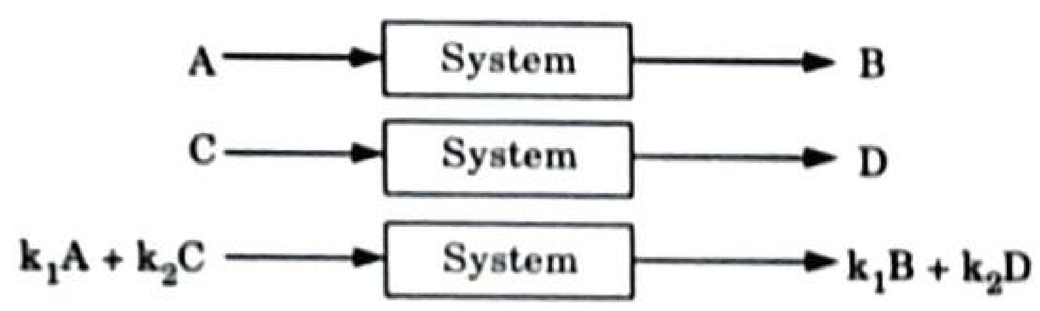

Fig. 1.21.3.

Time invariance : Refer Q. 1.20, Page 1–19D, Unit-1.

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Stability :

1. A system with bounded-input and bounded-output is called stable if for any bounded input x defined by

$$|x| \le k_1$$

the corresponding output y is also bounded defined by

$$|y| \le k_2$$

where, k_1 and k_2 are finite real constants.

2. For example :

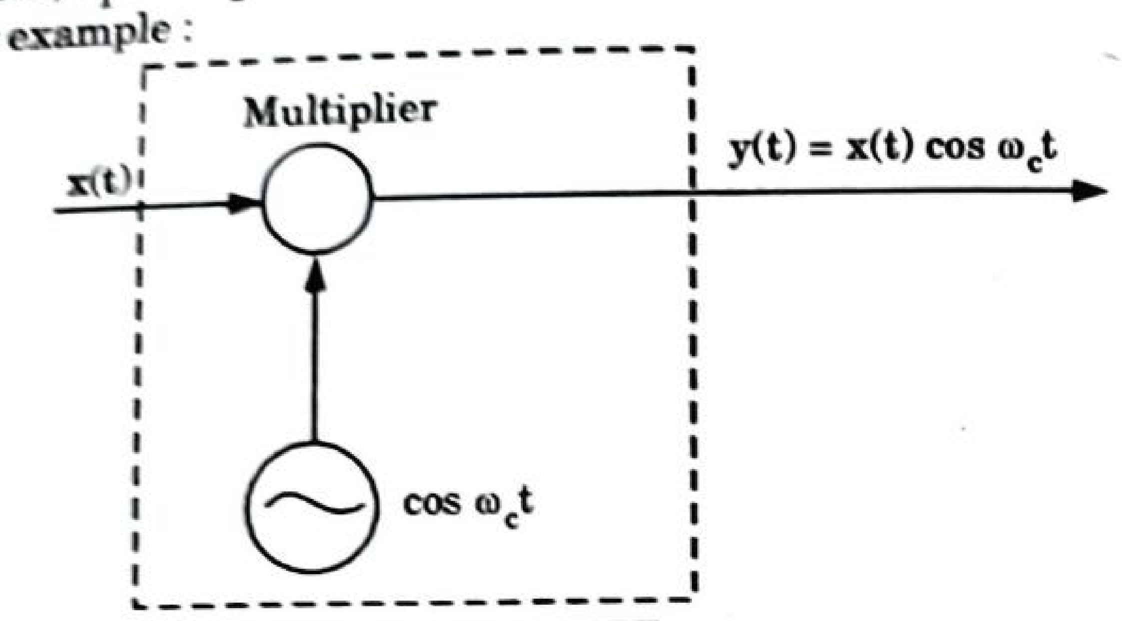

Fig. 1.21.4.

3. In Fig. 1.21.4,

$\because$ $|\cos \omega_c t| \le 1$, we have $|y(t)| = |x(t) \cos \omega_c t| \le |x(t)|$

4. Thus, if the input $x(t)$ is bounded then output $y(t)$ is also bounded and the system is BIBO (bounded-input bounded-output) stable.

Que 1.22. **What is LTI system ? Check the linearity of the system $y(t)$,**

$$\frac{dy(t)}{dt} + y(t) = 5x(t).$$

AKTU 2015-16, Marks 7.5

Answer

LTI system : A system which is both linear and time-invariant is called a linear time invariant (LTI) system. That is, a system for which the principle of superposition and the principle of homogeneity are valid and in addition, the input/output characteristics do not change with time is called an LTI system.

Numerical :

Given : $\frac{dy(t)}{dt} + y(t) = 5x(t)$

To Find : Linearity of system.

1. Let an input $x_1(t)$ produces an output $y_1(t)$.

$$\frac{dy_1(t)}{dt} + y_1(t) = 5x_1(t) \quad \text{...(1.22.1)}$$

2. Similarly, if input $x_2(t)$ produces output $y_2(t)$, then

$$\frac{dy_2(t)}{dt} + y_2(t) = 5x_2(t) \quad \text{...(1.22.2)}$$

3. Linear combination of eq. (1.22.1) and (1.22.2),

$$\frac{ady_1(t)}{dt} + ay_1(t) + \frac{bdy_2(t)}{dt} + by_2(t) = ax_1(t) + bx_2(t)$$

$$\frac{d}{dt}\,[ay_1(t) + by_2(t)] + [ay_1(t) + by_2(t)] = ax_1(t) + bx_2(t)$$

4. This equation follows superposition theorem. Hence system is linear.

Que 1.23. **What are causal systems ? Check the causality of the following system :**

i. $y(n) = x(-n)$

ii. $y(t) = x(t) \cos(t + 1)$

where x is the input and y is the output.

AKTU 2013-14, Marks 05

Answer

Causal system : Refer Q. 1.20, Page 1–19D, Unit-1.

Numerical :

i. Given : $y(n) = x(-n)$

For $n = -2$ $\quad y(-2) = x(2)$

For $n = 0$ $\quad y(0) = x(0)$

For $n = 2$ $\quad y(2) = x(-2)$

For negative values of n, the output depends on the future values of input. Therefore, the system is non-causal.

ii. Given : $y(t) = x(t) \cos(t + 1)$

$$y(\pi) = x(\pi) \cos(\pi + 1)$$

$$y(2\pi) = x(2\pi) \cos(2\pi + 1)$$

As the output depends on the input that occurs later, the system is non-causal.

Que 1.24. **Check whether the following properties hold good for the system $y(t) = atx(t) + bt^2x(t - 2)$:**

i. Static or dynamic

ii. Linear or non-linear

iii. Causal or non-causal

iv. Time-variant or invariant

AKTU 2014-15, Marks 06

Answer

Given : $y(t) = atx(t) + bt^2x(t - 2)$

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i. Static or dynamic :

1. Put $t = 0$

$y(0) = 0$

2. Put $t = 1$

$$y(1) = \underbrace{a\,x(1)}_{\text{Present input}} + \underbrace{b\,x(-1)}_{\text{Past input}}$$

Since system depends on past input means this system has memory, so it is dynamic system.

ii. Linear or non-linear :

1. Let $x_1(t)$ produce an output $y_1(t)$

$\therefore \quad y_1(t) = a\,t\,x_1(t) + bt^2 x_1(t-2)$

2. Let $x_2(t)$ produce an output $y_2(t)$

$\therefore \quad y_2(t) = a\,t\,x_2(t) + bt^2 x_2(t-2)$

3. The weighted sum of output is

$$py_1(t) + qy_2(t) = p[at\,x_1(t) + bt^2 x_1(t-2)] + q[atx_2(t) + bt^2 x_2(t-2)]$$

4. The output due to weighted sum of inputs

$$y_3(t) = px_1(t) + qx_2(t)$$

5. Since superposition principle is not satisfied. Therefore the given system is non-linear.

iii. Causal or non-causal :

1. For $t = 1$, $y(1) = \underbrace{ax(1)}_{\text{Present input}} + \underbrace{bx(-1)}_{\text{Past input}}$

2. For $t = 2$, $y(2) = \underbrace{2ax(2)}_{\text{Present input}} + \underbrace{4bx(0)}_{\text{Past input}}$

3. For $t = -1$, $y(-1) = \underbrace{-ax(-1)}_{\text{Present input}} + \underbrace{bx(-3)}_{\text{Past input}}$

4. Since system is not dependent on future value, so it is a causal system.

iv. Time-variant or invariant :

1. $y(t) = a\,t\,x(t) + bt^2 x(t-2)$

2. The output delay due to input delay by T sec is

$$y(t, T) = y(t)\big|_{x(t) = x(t-T)}$$
$$= a\,t\,x(t-T) + bt^2\,x(t-T-2)$$

3. The output delayed by T sec

$$y(t-T) = y(t)\big|_{t = t-T}$$
$$= a\,(t-T)\,x(t-T) + b\,(t-T)^2\,x(t-T-2)$$

4. $\because \quad y(t, T) \neq y(t-T)$

5. Therefore, the system is time-variant.

PART-3

Analogous System : Linear Mechanical Elements, Force-Voltage and Force-Current Analogy, Modeling of Mechanical and Electro-Mechanical Systems : Analysis of First and Second Order Linear Systems by Classical Method.

CONCEPT OUTLINE : PART-3

- **D'Alembert's Principle :** Sum of external forces and forces resisting motion of a body in any given direction is zero.

  i.e. $f + f_m + f_D + f_K = 0$

  where $f \rightarrow$ External force

  $f_m \rightarrow$ Inertial force

  $f_D \rightarrow$ Damping force

  $f_K \rightarrow$ Spring force

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.25. Explain the force-voltage and force-current analogies.

AKTU 2012-13, Marks 05

OR

Explain the force-current analogy taking a suitable example.

AKTU 2013-14, Marks 05

Answer

Force-voltage analogies :

1. Each junction in a mechanical system corresponds to a closed loop which consists of electrical excitation sources and passive elements analogous to the mechanical driving sources and passive elements connected to the junction.
2. All points on a rigid mass are considered the same junction.

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Table 1.25.1. Force-Voltage Analogy

Mechanical system	Electrical system
Force, f	Voltage, e
Velocity, v	Current, i
Displacement, x	Charge, q
Mass, M	Inductance, L
Damping coefficient, D	Resistance, R
Compliance, $1/K$ (stiffness, K)	Capacitance, C

Example :

1. Consider the mechanical system shown in Fig. 1.25.1. The differential equation for this mechanical system is written as,

$$M\frac{d^2x}{dt^2} + D\frac{dx}{dt} + kx = f(t) \qquad ...(1.25.1)$$

2. Consider the electrical circuit shown in Fig. 1.25.2. The following differential equation is written for this electrical circuit

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int idt = e(t) \qquad ...(1.25.2)$$

3. If $\frac{dx}{dt}$ is replaced by v, the velocity, eq. (1.25.1) could be written as

$$M\frac{dv}{dt} + Dv + k\int vdt = f(t) \qquad ...(1.25.3)$$

4. Comparing eq. (1.25.2) and (1.25.3), we see that the differential equations for the two system are identical.
5. The above analogy is called force-voltage analogy and it is called the loop system.

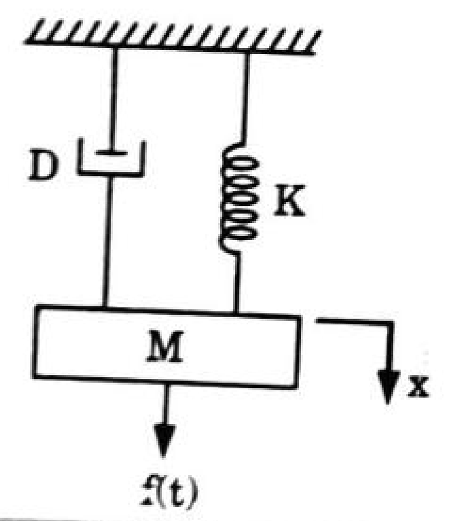

Fig. 1.25.1. Mechanical system.

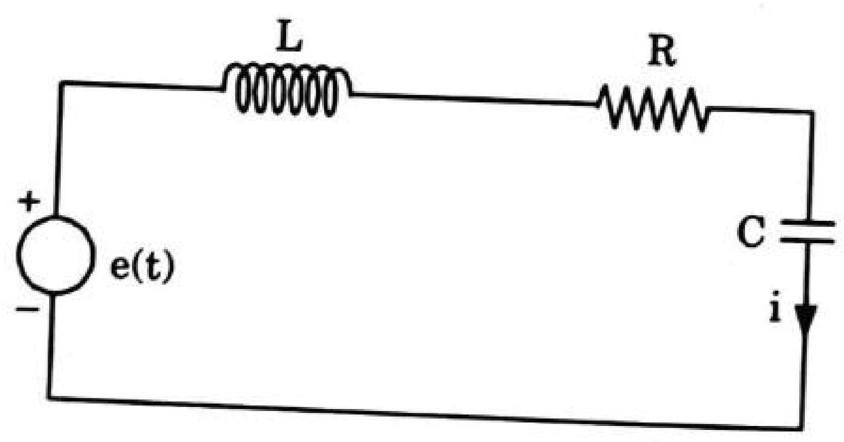

Fig. 1.25.2. Force-voltage analogous circuit.

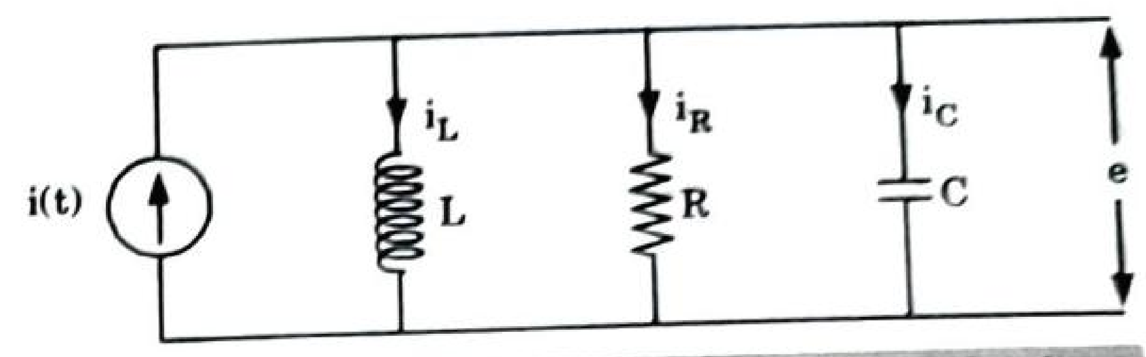

Fig. 1.25.3. Force-current analogous circuit.

Force-current analogies :

1. Each junction in a mechanical system corresponds to a node which joins electrical excitation sources to passive elements analogous to the mechanical driving sources and to passive elements connected to the junction.
2. All points on a rigid mass are considered the same junction.
3. One terminal of the capacitance analogous to a mass is always connected to the ground.

Table 1.25.2. Force-Current Analogy

Mechanical system	Electrical system
Force, f	Current, i
Velocity, v	Voltage, e
Displacement, x	Flux linkage, ϕ
Mass, M	Capacitance, C
Damping coefficient, D	Conductance, G
Stiffness, K (compliance, $1/K$)	Inductance, L

Example :

1. Consider Fig. 1.25.3. For this electrical network, the differential equation is written as

$$\frac{1}{L}\int edt + \frac{e}{R} + C\frac{de}{dt} = i(t) \qquad ...(1.25.4)$$

2. Comparing eq. (1.25.3) and (1.25.4), we see that the differential equations for the mechanical and electrical system are identical.
3. The above analogy is called force-current analogy and it is called the nodal system.

Que 1.26. **What do you understand by analogous systems ? Also mention the F-V and F-I analogy in analogous system.**

AKTU 2014-15, Marks 3.5

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Answer

Analogous system : If there exists a similarity between the equilirium equation of electrical and mechanical system, it is possible :

a. To draw an electrical system which will behave exactly similar to the given mechanical system.

b. To draw a mechanical system which will behave exactly similar to the given electrical system.

c. Both the above systems are called analogous system of each other.

F-V and F-I analogy : Refer Q. 1.25, Page 1–25D, Unit-1.

Que 1.27. **Explain D' Alembert's principle.**

Answer

1. It states that sum of external forces and forces resisting motion of a body in any given direction is zero.

$$f + f_M + f_D + f_K = 0$$

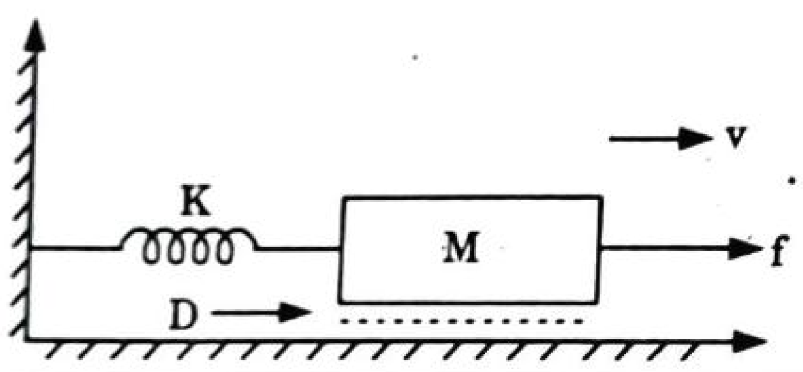

Fig. 1.27.1. Mechanical system in one direction.

2. Inertial force, $f_M = -M\dfrac{dv}{dt}$
3. Damping force, $f_D = -Dv$
4. Spring force, $f_K = -K\left[\int_0^t v\,dt + x(0)\right]$
5. Now, $M\dfrac{dv}{dt} + Dv + K\left[\int_0^t v\,dt + x(0)\right] = f$

Que 1.28. **Draw the force-current analogy of the mechanical system given in Fig. 1.28.1.**

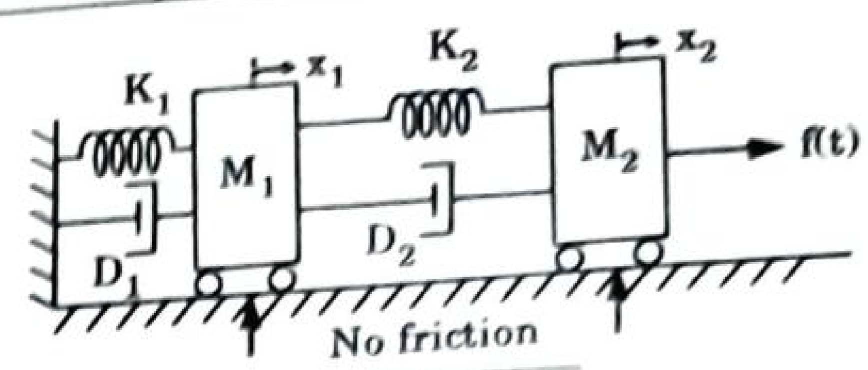

Fig. 1.28.1.

AKTU 2011-12, Marks 06

AKTU 2015-16, Marks 10

Answer

1. Differential equation is written as,

$$M_2\frac{d^2x_2}{dt^2} + D_2\frac{dx_2}{dt} - D_2\frac{dx_1}{dt} + K_2(x_2 - x_1) = f(t)$$

$$M_1\frac{d^2x_1}{dt^2} + D_1\frac{dx_1}{dt} - D_2\frac{dx_2}{dt} + K_2(x_1 - x_2) + K_1x_1 + D_2\frac{dx_1}{dt} = 0$$

2. Differential equation based on force-current analogy,

$$C_2\frac{de_2}{dt} + \frac{e_2}{R_2} - \frac{e_1}{R_2} + \frac{1}{L_2}\int(e_2 - e_1)\,dt = i(t)$$

$$C_1\frac{de_1}{dt} + \frac{e_1}{R_1} + \frac{e_1}{R_2} - \frac{e_2}{R_2} + \frac{1}{L_2}\int(e_1 - e_2)\,dt + \frac{1}{L_1}\int e_1\,dt = 0$$

3. Analogous electrical circuit based on force-current analogy is :

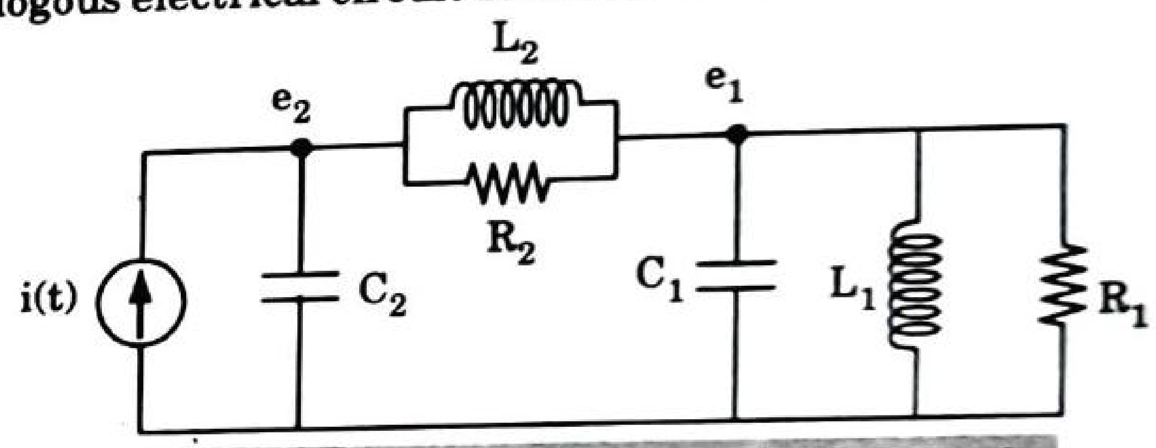

Fig. 1.28.2. Force-current analogous circuit.

Que 1.29. **Determine $\dfrac{X(s)}{F(s)}$ of the given system shown in Fig. 1.29.1.**

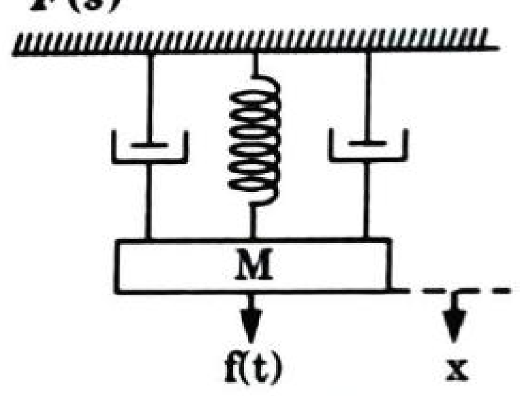

Fig. 1.29.1.

AKTU 2011-12, Marks 05

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Answer

Let D_1 and D_2 be the damping coefficient and K be spring constant.

1. Inertial force, $f_M = \dfrac{-M dv(t)}{dt}$
2. Damping force, $f_D = -(D_1 + D_2)\, v(t)$
3. Spring force, $f_K = -K\left[\int_0^t v(t)\, dt + x(0)\right]$
4. Using D' Alembert's principle

$$f(t) + f_M + f_D + f_K = 0$$

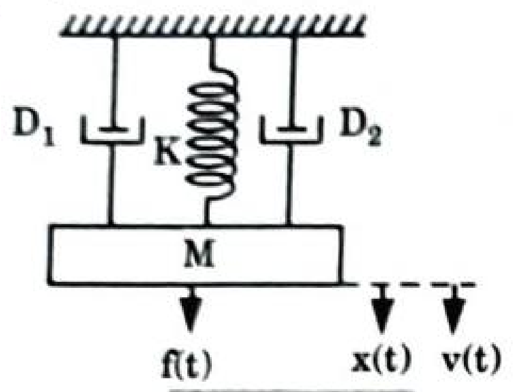

Fig. 1.29.2.

$$f(t) = \frac{M dv(t)}{dt} + (D_1 + D_2)\, v(t) + K\left[\int_0^t v(t)\, dt + x(0)\right]$$

$$f(t) = \frac{Md^2x}{dt^2} + (D_1 + D_2)\frac{dx}{dt} + Kx \qquad \text{...(1.29.1)}$$

5. Taking Laplace transform of eq. (1.29.1), we get

$$F(s) = Ms^2 X(s) + (D_1 + D_2)\, sX(s) + KX(s)$$
$$F(s) = (Ms^2 + (D_1 + D_2)s + K)\, X(s)$$
$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + (D_1 + D_2)s + K}$$

Que 1.30. **Draw the mechanical equivalent of the system shown in Fig. 1.30.1. Obtain the electrical analog system using the force-current analogy.**

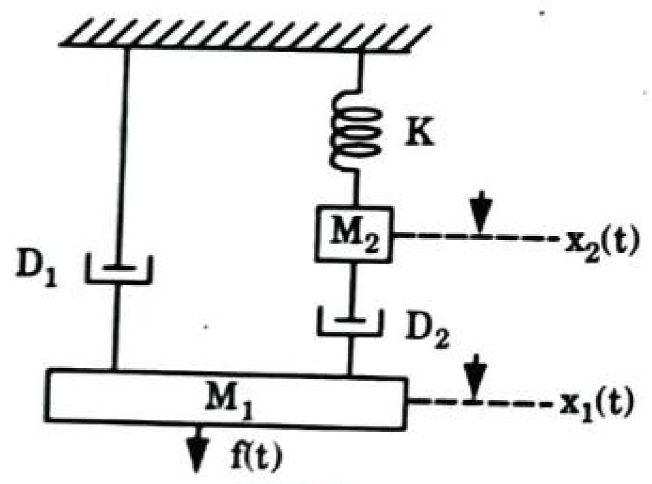

Fig. 1.30.1.

AKTU 2012-13, Marks 05

Answer

1. Mechanical equivalent of system :

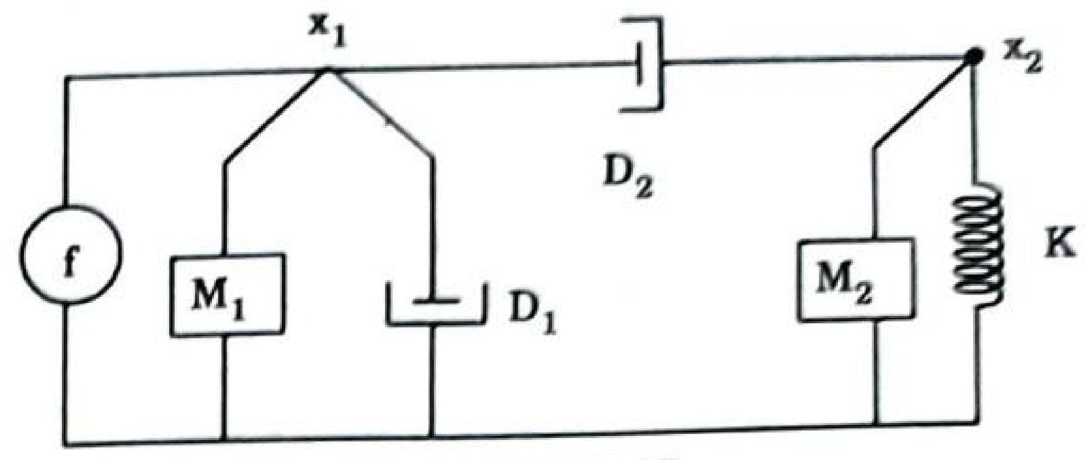

Fig. 1.30.2.

2. Differential equation,

$$f(t) = M_1\frac{d^2x_1}{dt^2} + D_1\frac{dx_1}{dt} + D_2\frac{dx_1}{dt} - D_2\frac{dx_2}{dt} \qquad \text{...(1.30.1)}$$

$$0 = M_2\frac{d^2x_2}{dt^2} + D_2\frac{dx_2}{dt} - D_2\frac{dx_1}{dt} + Kx_2 \qquad \text{...(1.30.2)}$$

3. Converting eq. (1.30.1) and (1.30.2) in KCL form using force-current analogy, eq. (1.30.1) becomes,

$$i(t) = C_1\frac{de_1(t)}{dt} + \frac{e_1(t)}{R_1} + \frac{e_1(t)}{R_2} - \frac{e_2(t)}{R_2} \qquad \text{...(1.30.3)}$$

4. Eq. (1.30.2) becomes,

$$0 = C_2\frac{de_2(t)}{dt} + \frac{e_2(t)}{R_2} - \frac{e_1(t)}{R_2} + \frac{1}{L}\int e_2\, dt \qquad \text{...(1.30.4)}$$

5. Using eq. (1.30.3) and (1.30.4), equivalent electrical analogous circuit is drawn as,

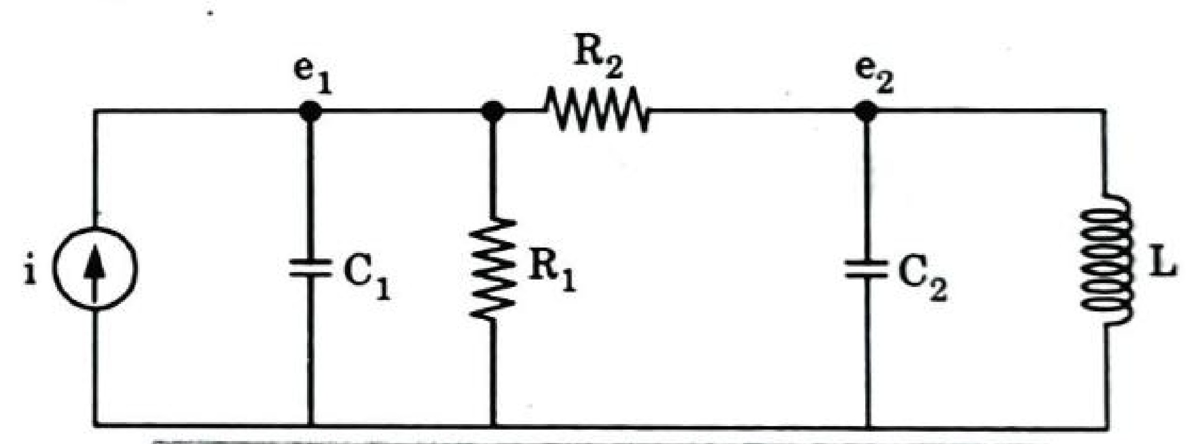

Fig. 1.30.3. Force-current analogous circuit.

Que 1.31. **Obtain the F-V and F-I analogous system of the mechanical system shown in Fig. 1.31.1.**

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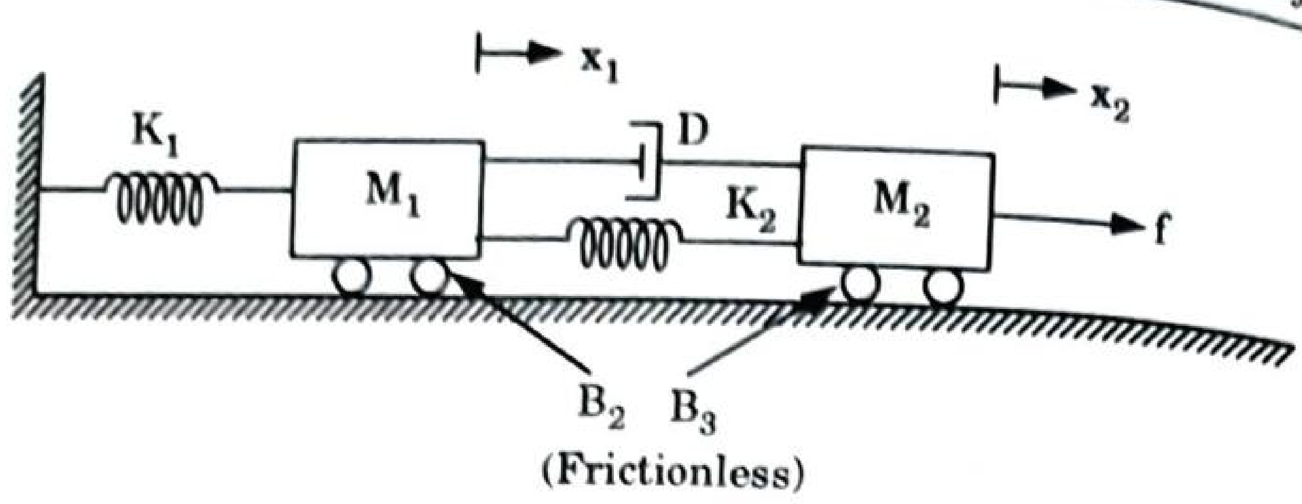

Fig. 1.31.1.

AKTU 2014-15, Marks 06

Answer

1. Differential equations of mechanical system are

$$M_1\frac{d^2x_1}{dt^2}+K_1x_1+D\frac{dx_1}{dt}-D\frac{dx_2}{dt}+K_2(x_1-x_2)=0$$

$$M_2\frac{d^2x_2}{dt^2}+K_2x_2+D\frac{dx_2}{dt}-D\frac{dx_1}{dt}-Kx_1=f(t)$$

2. Differential equation based on F-V analogy are,

$$L_1\frac{di_1}{dt}+R(i_1-i_2)+\frac{1}{C_1}\int i_1dt+\frac{1}{C_2}\int(i_1-i_2)\,dt=0$$

$$L_2\frac{di_2}{dt}+R(i_2-i_1)+\frac{1}{C_2}\int(i_2-i_1)\,dt=e(t)$$

3. Hence F-V analogous system is

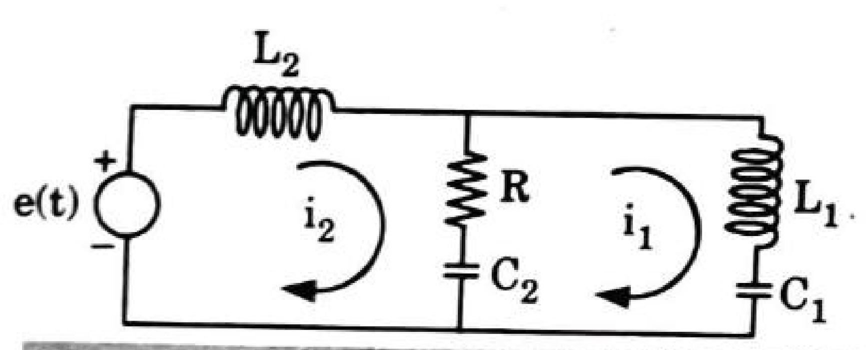

Fig. 1.31.2. Force-voltage analogous circuit.

4. Differential equation based on F-I analogy are

$$C_1\frac{de_1}{dt}+\frac{e_1-e_2}{R}+\frac{1}{L_1}\int e_1\,dt+\frac{1}{L_2}\int(e_1-e_2)\,dt=0$$

$$C_2\frac{de_2}{dt}+\frac{e_2-e_1}{R}+\frac{1}{L_2}\int(e_2-e_1)\,dt=i(t)$$

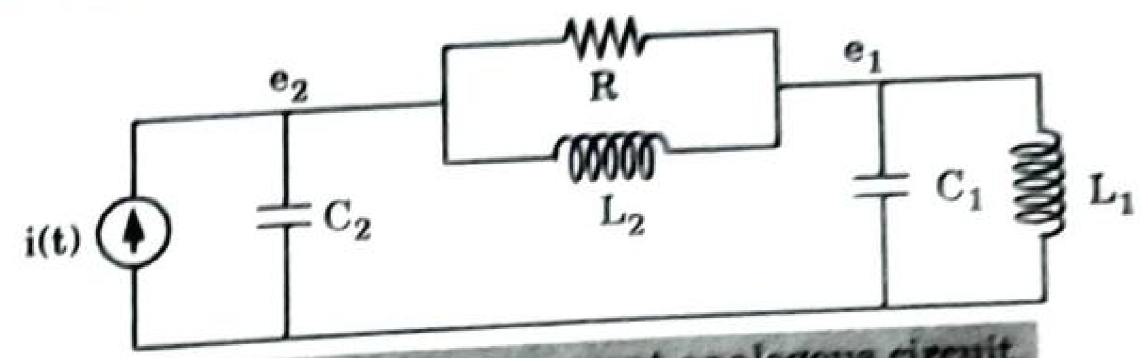

Fig. 1.31.3. Force-current analogous circuit.

Que 1.32. **Discuss first order and second order linear systems by classical method.**

Answer

First order systems :

1. First order homogeneous differential equation,

$$\frac{dy(t)}{dt}+Py(t)=0 \qquad P=\text{constant}$$

$$\frac{dy(t)}{y(t)}=-Pdt$$

Integrating,

$$\ln y(t)=-Pt+k_1$$
$$k_1=\ln k$$
$$\ln y(t)=\ln e^{-Pt}+\ln k$$
$$\ln y(t)=\ln(ke^{-Pt})$$
$$y(t)=ke^{-Pt}$$
$$k=\text{constant}$$

2. First order non-homogenous differential equation,

$$\frac{dy(t)}{dt}+Py(t)=Q$$

P = constant

Q = function of independent variable, or a constant

$$e^{Pt}\frac{dy(t)}{dt}+e^{Pt}Py(t)=Q\,e^{Pt}$$

$$\frac{d}{dt}\left[y(t)\cdot e^{Pt}\right]=Q\,e^{Pt} \qquad [d(x.y)=xdy+ydx]$$

$$y(t)\,e^{Pt}=\int Q\,e^{Pt}dt+k$$

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$$y(t) = e^{-Pt}\int Q\, e^{Pt} dt + k\, e^{-Pt}$$

If Q = constant,

$$y(t) = e^{-Pt} Q \frac{e^{Pt}}{P} + k\, e^{-Pt}$$

$$y(t) = \frac{Q}{P} + k\, e^{-Pt}$$

Second order systems :

Second order differential equations,

$$\frac{Ad^2y(t)}{dt^2} + B\frac{dy(t)}{dt} + Cy(t) = 0$$

General solution : $y(t) = k_1 e^{P_1 t} + k_2 e^{P_2 t}$

k_1, k_2 = constants

P_1, P_2 = roots of quadratic equation

$AP^2 + BP + C = 0$

$$P_1, P_2 = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC}$$

VERY IMPORTANT QUESTIONS

Following questions are very important. These questions may be asked in your SESSIONALS as well as UNIVERSITY EXAMINATION.

Q. 1. Define various elementary continuous time signals, gate signal and parabolic signal.

Ans. **Elementary continuous time signals :** Refer Q. 1.2, Unit-1.
Gate signal : Refer Q. 1.3, Unit-1.
Parabolic signal : Refer Q. 1.4, Unit-1.

Q. 2. Explain what are power and energy signals ? Explain their relationship with periodicity.

Ans. Refer Q. 1.8, Unit-1.

Q. 3. Discuss time-variance and time-invariance.

Ans. Refer Q. 1.20, Unit-1.

Q. 4. What are causal and non-causal system ?

Ans. Refer Q. 1.20, Unit-1.

Q. 5. Explain the concept of linearity and stability.

Ans. Refer Q. 1.21, Unit-1.

Q. 6. Explain the force-voltage and force-current analogies.

Ans. Refer Q. 1.25, Unit-1.

Q. 7. Draw the force-current analogy of the mechanical system given in Fig. 1.7.1.

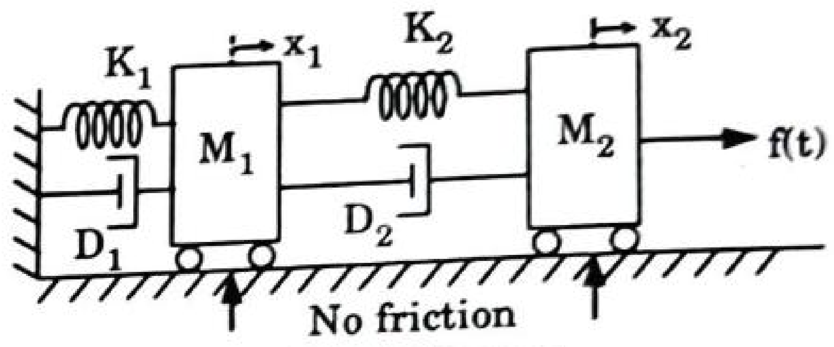

Fig. 1.7.1.

Ans. Refer Q. 1.28, Unit-1.

Q. 8. Find the even and odd components of the signal shown in Fig. 1.8.1.

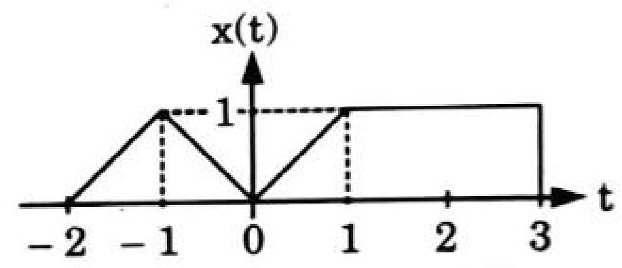

Fig. 1.8.1.

Ans. Refer Q. 1.16, Unit-1.

Q. 9. Check whether the following properties hold good for the system $y(t) = atx(t) + bt^2x(t-2)$:

i. **Static or dynamic**
ii. **Linear or non-linear**
iii. **Causal or non-causal**
iv. **Time-variant or invariant**

Ans. Refer Q. 1.24, Unit-1.

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Q. 10. Draw the mechanical equivalent of the system shown in Fig. 1.10.1. Obtain the electrical analog system using the force-current analogy.

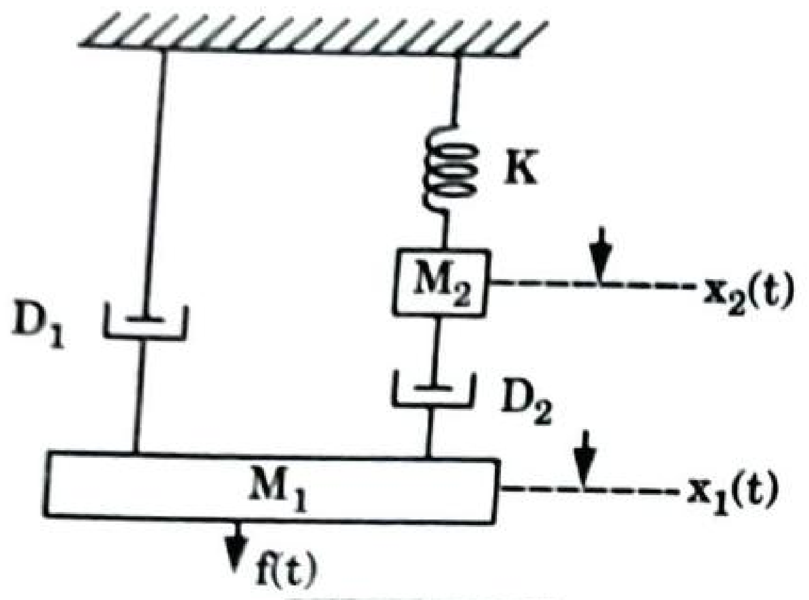

Fig. 1.10.1.

Ans. Refer Q. 1.30, Unit-1.

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2 UNIT

Fourier Transform Analysis

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PART- 1

Exponential form and Compact Trigonometric form of Fourier Series, Fourier Symmetry.

CONCEPT OUTLINE : PART- 1

- **Fourier series :** The representation of signals over a certain interval of time in terms of the linear combination of orthogonal functions is called Fourier series. It is only applicable for periodic signals.
- Three important Fourier series forms are :

1. Trigonometric form
2. Harmonic form
3. Exponential form

- **Dirichlet's condition :** The conditions under which a periodic signal can be represented by a Fourier series are known as Dirichlet's conditions. They are as follows :

1. The function $x(t)$ must be a single valued function.
2. The function $x(t)$ has only a finite number of maxima and minima.
3. The function $x(t)$ has a finite number of discontinuities.
4. The function $x(t)$ is absolutely integrable over one period, that is $\int_0^T |x(t)|\,dt < \infty$.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.1. **Briefly explain the three forms of Fourier series.**

Answer

Three forms of Fourier series are :

i. **Trigonometric Fourier series :** The trigonometric Fourier series representation of a periodic signal $x(t)$ with fundamental period T_0 is given by :

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 t + b_n \sin n\omega_0 t)$$

where, $\omega_0 = \frac{2\pi}{T_0}$

a_0, a_k and b_k are the Fourier coefficients given by :

$$a_0 = \frac{1}{T_0}\int_{T_0} x(t)\,dt$$

$$a_n = \frac{2}{T_0}\int_{T_0} x(t)\cos n\omega_0 t\,dt$$

$$b_n = \frac{2}{T_0}\int_{T_0} x(t)\sin n\omega_0 t\,dt$$

ii. **Harmonic Fourier series :** Another form of Fourier series representation of a real periodic signal $x(t)$ with fundamental period T_0 is :

$$x(t) = c_0 + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n),$$

where, $\omega_0 = \frac{2\pi}{T_0}$

c_0 = DC component

$c_n \cos(k\omega_0 t - \theta_n)$ = n^{th} harmonic component of $x(t)$

$$c_0 = \frac{a_0}{2},\quad c_n = \sqrt{a_n^2 + b_n^2},\quad \theta_n = \tan^{-1}\frac{b_n}{a_n}$$

iii. **Exponential Fourier series :**

1. The exponential Fourier series is the most widely used form of Fourier series.
2. In this, the function $x(t)$ is expressed as a weighted sum of the complex exponential functions.

$$x(t) = C_0 + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} C_n e^{jn\omega_0 t}$$

where, $C_n = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)e^{-jn\omega_0 t}\,dt$

$$C_0 = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt$$

Que 2.2. **Give the derivation of exponential form of Fourier series.**

Answer

1. The set of complex exponential functions

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$\{e^{jn\omega_0 t}, n = 0, \pm 1, \pm 2, \ldots\}$

forms a closed orthogonal set over an interval $(t_0, t_0 + T)$ where $T = (2\pi/\omega_0)$ for any value of t_0, and therefore it can be used as a Fourier series.

2. Using Euler's identity, we can write

$$A_n \cos(n\omega_0 t + \theta_n) = A_n\left[\frac{e^{j(n\omega_0 t+\theta_n)} + e^{-j(n\omega_0 t+\theta_n)}}{2}\right]$$

3. Substituting this in the definition of the cosine Fourier representation,

$$x(t) = A_0 + \sum_{n=1}^{\infty}\frac{A_n}{2}[e^{j(n\omega_0 t+\theta_n)} + e^{-j(n\omega_0 t+\theta_n)}]$$

$$= A_0 + \sum_{n=1}^{\infty}\frac{A_n}{2}[e^{jn\omega_0 t}e^{j\theta_n} + e^{-jn\omega_0 t}e^{-j\theta_n}]$$

$$= A_0 + \sum_{n=1}^{\infty}\left(\frac{A_n}{2}e^{jn\omega_0 t}e^{j\theta_n}\right) + \sum_{n=1}^{\infty}\left(\frac{A_n}{2}e^{-jn\omega_0 t}e^{-j\theta_n}\right)$$

$$= A_0 + \sum_{n=1}^{\infty}\left(\frac{A_n}{2}e^{j\theta_n}\right)e^{jn\omega_0 t} + \sum_{n=1}^{\infty}\left(\frac{A_n}{2}e^{j(-\theta_n)}\right)e^{-jn\omega_0 t} \quad \ldots(2.2.1)$$

4. Let $n = -k$ in the second summation of the eq. (2.2.1), we have

$$x(t) = A_0 + \sum_{n=1}^{\infty}\left(\frac{A_n}{2}e^{j\theta_n}\right)e^{jn\omega_0 t} + \sum_{k=-1}^{-\infty}\left(\frac{A_k}{2}e^{j\theta_k}\right)e^{jk\omega_0 t} \quad \ldots(2.2.2)$$

5. On comparing eq. (2.2.1) and (2.2.2) for $x(t)$, we get

$$A_n = A_k\,;\,(-\theta_n) = \theta_k \qquad n > 0, k < 0$$

6. Let us define

$$C_0 = A_0\,;\, C_n = \frac{A_n}{2}e^{j\theta_n},\ n > 0$$

7. $\therefore$

$$x(t) = A_0 + \sum_{n=1}^{\infty}\frac{A_n}{2}e^{j\theta_n}e^{jn\omega_0 t} + \sum_{n=-1}^{-\infty}\frac{A_n}{2}e^{j\theta_n}e^{jn\omega_0 t}$$

$\therefore$

$$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

where

$$C_n = \frac{1}{T}\int_{T_0} x(t)\,e^{-jn\omega_0 t}$$

Que 2.3. **Discuss waveform symmetries.**

OR

Define odd and even function. Also find Fourier coefficient for odd and even function.

AKTU 2011-12, Marks 06

Answer

1. **Even symmetry :**
   i. A function $f(x)$ is said to have even symmetry if $f(x) = f(-x)$.
   ii. Even function shows even symmetry.
   iii. Even nature is preserved on addition of a constant.
   iv. Sum of even functions remains even.

Example :

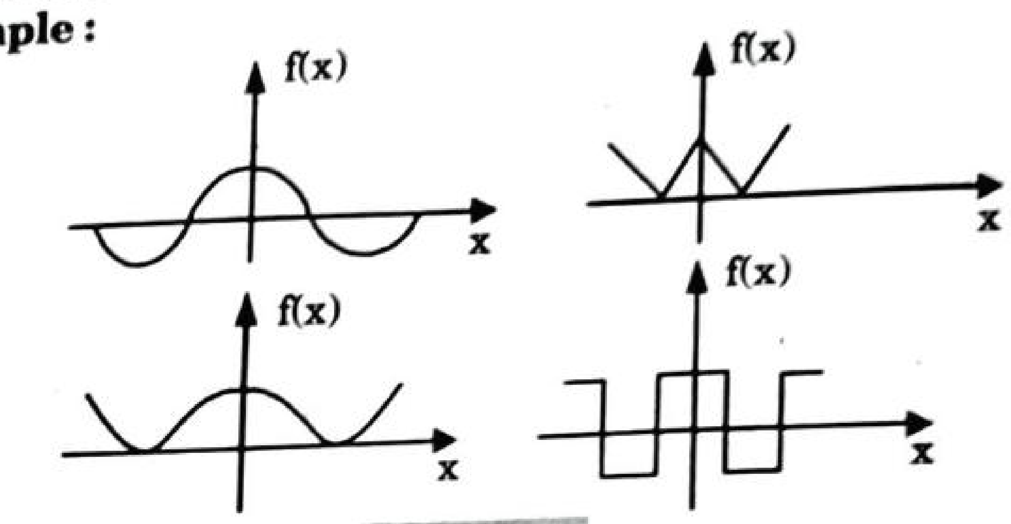

Fig. 2.3.1.

Fourier constants for even function :

$$a_0 = \frac{2}{T}\int_0^{T/2} f(t)\,dt$$

$$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t\,dt$$

$$b_n = 0$$

2. **Odd symmetry :**
   i. A function $f(x)$ is said to have odd symmetry, if $f(x) = -f(-x)$.
   ii. If $f(x) = -f(-x)$ function is an odd function.
   iii. Addition of a constant removes odd nature of the function.
   iv. Sum of odd functions remains odd.

Example :

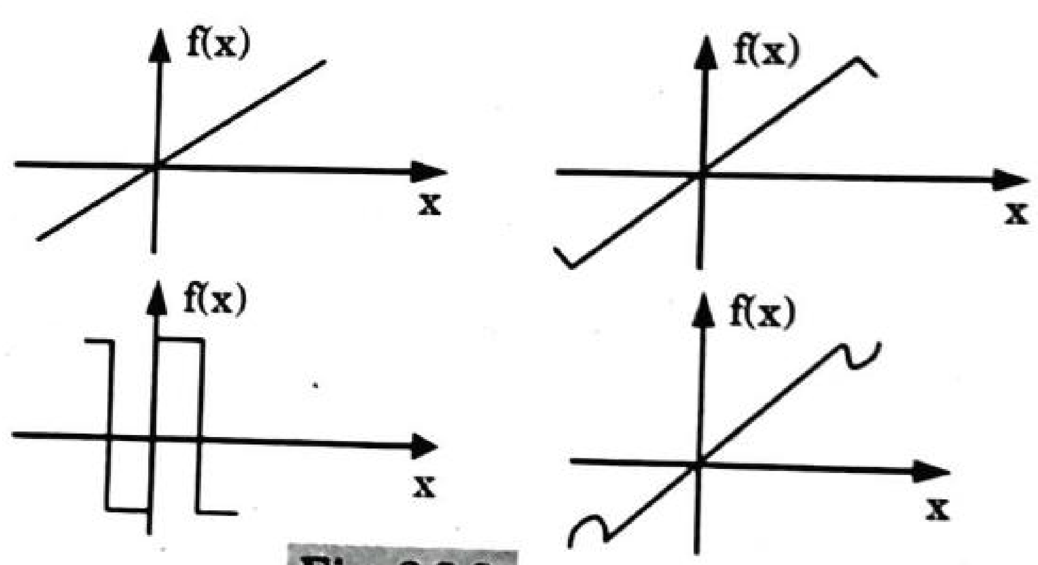

Fig. 2.3.2.

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Fourier constants for odd function :

$$a_0 = 0$$
$$a_n = 0$$
$$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega t\, dt$$

3. **Half wave symmetry :**
A periodic function is said to have half wave symmetry if,
$$f(x) = -f(x \pm T/2)$$

Example :

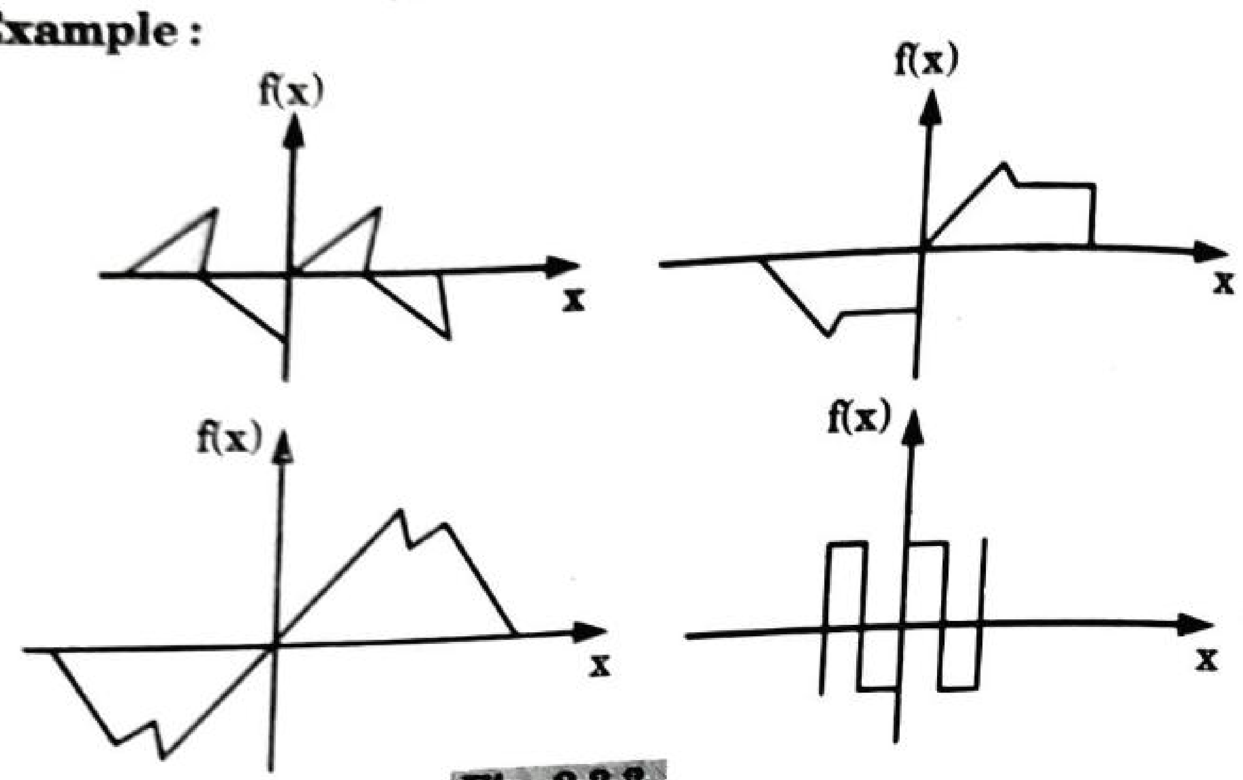

Fig. 2.3.3.

Fourier constants for half wave symmetry :

***n* odd**	***n* even**
$a_0 = 0$	$a_0 = 0$
$a_n = \frac{4}{T}\int_0^{T/2} f(t)\cos n\omega t\, dt$	$a_n = 0$
$b_n = \frac{4}{T}\int_0^{T/2} f(t)\sin n\omega t\, dt$	$b_n = 0$

4. **Quarter wave symmetry :**
If signal has following both properties, it is said to have quarter wave symmetry :
i. It is half wave symmetric.
ii. It has symmetry (odd or even) about the quarter period point.

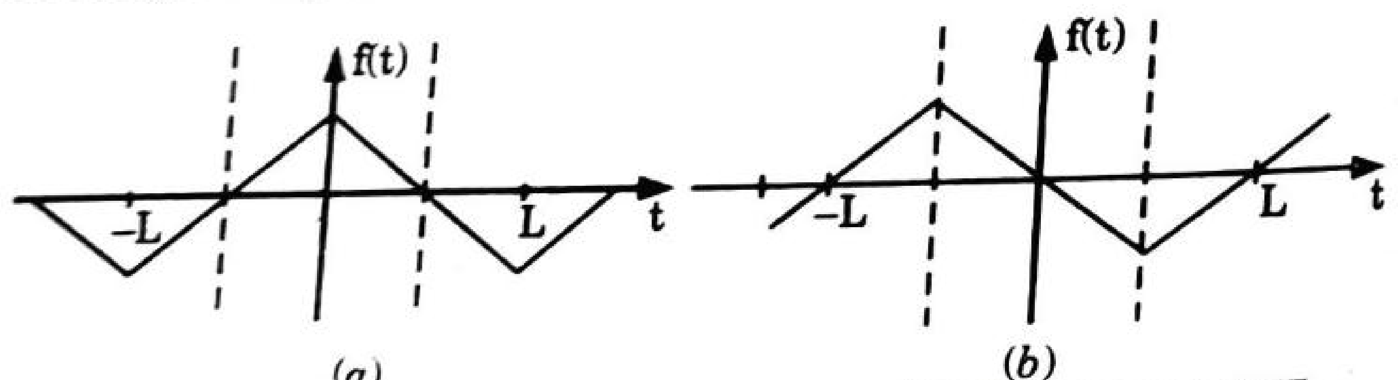

Fig. 2.3.4. (*a*) Even signal with Quarter wave symmetry. (*b*) Odd signal with Quarter wave symmetry.

Que 2.4. **Give the properties of continuous-time Fourier series.**

Answer

1. **Linearity property :**
The linearity property states that, if
$$x_1(t) \xleftrightarrow{FS} C_n \text{ and } x_2(t) \xleftrightarrow{FS} D_n$$
then $$Ax_1(t) + Bx_2(t) \xleftrightarrow{FS} AC_n + BD_n$$

2. **Time shifting property :**
The time shifting property states that, if
$$x(t) \xleftrightarrow{FS} C_n$$
then $$x(t - t_0) \xleftrightarrow{FS} e^{-jn\omega_0 t_0} C_n$$

3. **Time reversal property :**
The time reversal property states that, if
$$x(t) \xleftrightarrow{FS} C_n$$
then $$x(-t) \xleftrightarrow{FS} C_{-n}$$

4. **Time scaling property :**
The time scaling property states that, if
$$x(t) \xleftrightarrow{FS} C_n$$
then $$x(\alpha t) \xleftrightarrow{FS} C_n \text{ with } \omega_0 \to \alpha\omega_0$$

5. **Time differentiation property :**
The time differentiation property states that, if
$$x(t) \xleftrightarrow{FS} C_n$$
then $$\frac{dx(t)}{dt} \xleftrightarrow{FS} jn\omega_0 C_n$$

6. **Time integration property :**
The time integration property states that, if
$$x(t) \xleftrightarrow{FS} C_n$$
then $$\int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{FS} \frac{C_n}{jn\omega_0} \quad (\text{if } C_0 = 0)$$

Que 2.5. **State and prove Parseval's relation.**

Answer

Statement :

If $x_1(t) \xleftrightarrow{FS} C_n$ [for complex $x_1(t)$]

and $x_2(t) \xleftrightarrow{FS} D_n$ [for complex $x_2(t)$]

then, the Parseval's relation states that

$$\frac{1}{T}\int_{t_0}^{t_0+T} x_1(t)\,x_2^*(t)\,dt = \sum_{n=-\infty}^{\infty} C_n D_n^* \qquad \text{[for complex } x_1(t) \text{ and } x_2(t)]$$

and Parseval's identity states that

$$\frac{1}{T}\int_{t_0}^{t_0+T} |x(t)|^2\,dt = \sum_{n=-\infty}^{\infty} |C_n|^2 \qquad \text{if } x_1(t) = x_2(t) = x(t)$$

Proof :

Parseval's relation :

$$\text{LHS} = \frac{1}{T}\int_{t_0}^{t_0+T} x_1(t)x_2^*(t)\,dt = \frac{1}{T}\int_{t_0}^{t_0+T}\left(\sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\right) x_2^*(t)\,dt$$

Interchanging the order of integration and summation in the RHS, we have

$$\frac{1}{T}\int_{t_0}^{t_0+T} x_1(t)x_2^*(t)\,dt = \sum_{n=-\infty}^{\infty} C_n\left[\frac{1}{T}\int_{t_0}^{t_0+T} x_2^*(t)\,e^{jn\omega_0 t}\,dt\right]$$

$$= \sum_{n=-\infty}^{\infty} C_n\left[\frac{1}{T}\int_{t_0}^{t_0+T} x_2(t)\,e^{-jn\omega_0 t}\,dt\right]^* \sum_{n=-\infty}^{\infty} C_n[D_n]^*$$

$$\therefore \frac{1}{T}\int_{t_0}^{t_0+T} x_1(t)\,x_2^*(t)\,dt = \sum_{n=-\infty}^{\infty} C_n D_n^*$$

Que 2.6. **Explain the three forms of Fourier series. Derive the exponential form of Fourier series. Find the exponential form of Fourier series for a triangular wave of maximum value 1 and time period 2 seconds.**

AKTU 2012-13, Marks 10

Answer

Three forms of Fourier series : Refer Q. 2.1, Page 2–2D, Unit-2.

Derivation of exponential form of Fourier series : Refer Q. 2.2, Page 2–3D, Unit-2.

Fourier series of triangular waveform :

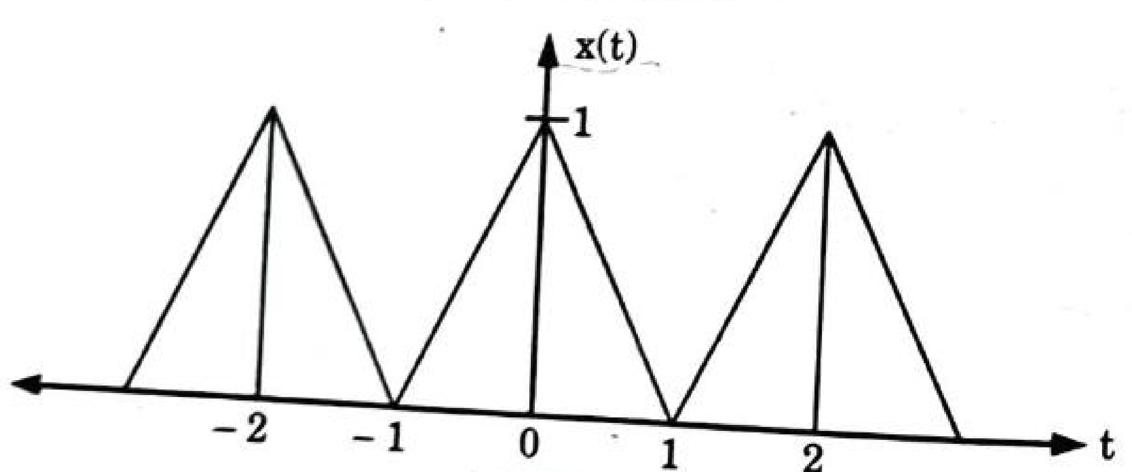

Fig. 2.6.1.

1. $$x(t) = \begin{cases} t+1 \ ; & -1 < t < 0 \\ -t+1 \ ; & 0 < t < 1 \end{cases}$$

2. $$A_0 = C_0 = \frac{1}{2}$$

3. $$C_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{2}\left(\int_{-1}^{0}(t+1)e^{-jn\omega_0 t}\,dt + \int_0^1 -(t-1)e^{-jn\omega_0 t}\,dt\right)$$

$$= \frac{1}{2}\left(\int_{-1}^{0} te^{-jn\omega_0 t}\,dt + \int_{-1}^{0} e^{-jn\omega_0 t}\,dt + \int_0^1 -te^{-jn\omega_0 t}\,dt + \int_0^1 e^{-jn\omega_0 t}\,dt\right)$$

$$= \frac{1}{2}\left(\int_{-1}^{0} te^{-jn\omega_0 t}dt - \int_0^1 te^{-jn\omega_0 t}dt\right) + \frac{1}{2}\int_{-1}^{1} e^{-jn\omega_0 t}\,dt$$

$$= \frac{1}{2}\left\{\left[\frac{e^{-jn\omega_0 t}(-jn\omega_0 t-1)}{(-jn\omega_0)^2}\right]_{-1}^{0} - \left[\frac{e^{-jn\omega_0 t}(-jn\omega_0 t-1)}{(-j\omega_n)^2}\right]_0^1\right\} + 0$$

$$\left[\because \int te^{-nt}\,dt = \frac{e^{-nt}(-nt-1)}{(-n)^2}\right]$$

$$= \frac{1}{2}\left[\frac{-1}{-(n\omega_0)^2} + \frac{e^{jn\omega_0}(jn\omega_0-1)}{(n\omega_0)^2}\right] - \frac{1}{2}\left[\frac{e^{-jn\omega_0}(-j\omega n_0-1)}{-(n\omega_0)^2} - \frac{1}{(n\omega_0)^2}\right]$$

$$= \frac{1}{2}\left[\frac{2}{(n\omega_0)^2} + jn\omega_0\left(\frac{e^{jn\omega_0}-e^{-jn\omega_0}}{(n\omega_0)^2}\right) - \left(\frac{e^{jn\omega_0}+e^{-jn\omega_0}}{(n\omega_0)^2}\right)\right]$$

$$= \frac{1}{2}\left[\frac{2}{(n\omega_0)^2} + \frac{(2j\times jn\omega_0)}{(n\omega_0)^2}\sin n\omega_0 - 2\cos n\omega_0\right]$$

$$= \frac{1-\cos n\omega_0}{(n\omega_0)^2}$$

$$\therefore \quad C_n = \begin{cases} 0 \ ; & \text{for even } n \\ \dfrac{2}{(n\omega_0)^2} \ ; & \text{for odd } n \end{cases}$$

4. $$x(t) = \frac{1}{2} + \sum_{n=\text{odd}}^{\infty} \frac{2}{(n\omega_0)^2} e^{jn\omega_0 t}$$

where, $\omega_0 = \dfrac{2\pi}{T} = \dfrac{2\pi}{2} = \pi$

Que 2.7. **Determine the Fourier series for the sawtooth waveform of unity magnitude.**

AKTU 2011-12, Marks 05

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Answer

1. The waveform is periodic and continuous for $0 < \omega t < 2\pi$. The Dirichlet's conditions are satisfied. The analytical form of the function $f(t)$ is given by

$$f(\omega t) = \frac{1}{2\pi}\omega t \; ; \; 0 < \omega t < 2\pi$$

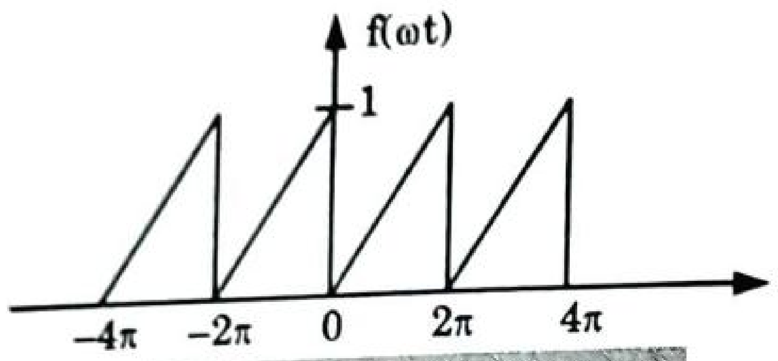

Fig. 2.7.1. Sawtooth wave.

2. $$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\omega t)\, d(\omega t) = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\omega t}{2\pi}\right) d(\omega t) = 0.5$$

3. $$a_n = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{1}{2\pi}\omega t\right)\cos n(\omega t)\, d(\omega t)$$

$$= \frac{1}{2\pi^2}\left[\frac{\omega t}{n}\sin(\omega t) + \frac{\cos n(\omega t)}{n^2}\right]_0^{2\pi} = 0$$

4. $$b_n = \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2\pi}(\omega t)\,\sin n(\omega t)\, d(\omega t)$$

$$= \frac{1}{2\pi^2}\left[-\frac{\omega t}{n}\cos n\,\omega t + \frac{1}{n^2}\sin n\,\omega t\right]_0^{2\pi} = -\frac{1}{n\pi}$$

5. Thus, the Fourier series is

$$f(t) = 0.5 - \sum_{n=1}^{\infty}\frac{1}{n\pi}\sin n\,\omega t$$

Que 2.8. **Find the Fourier series of the function given in Fig. 2.8.1 and is represented by :**

$$f(t) = \begin{cases} 0 \; ; & \text{for } \; 0 \le t \le T/2 \\ A \; ; & \text{for } \; T/2 \le t < T \end{cases}$$

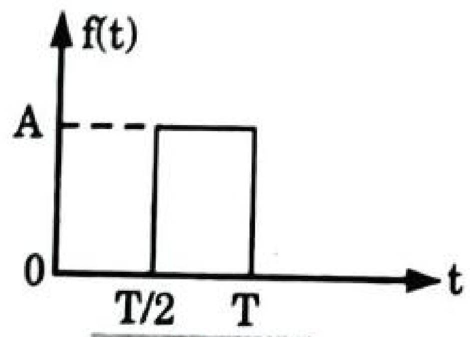

Fig. 2.8.1.

AKTU 2011-12, Marks 05

Answer

1. Fourier series expression

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos\omega_0 t + \sum_{n=1}^{\infty} b_n \sin\omega_0 t \quad ...(2.8.1)$$

2. $$a_0 = \frac{2}{T}\left[\int_0^{T/2} 0\, dt + \int_{T/2}^{T} A\, dt\right]$$

$$a_0 = \frac{2}{T}[At]_{T/2}^{T} = \frac{2A}{T}\left(T - \frac{T}{2}\right) = \frac{2A}{T}\times\frac{T}{2} = A \quad ...(2.8.2)$$

3. $$a_n = \frac{2}{T}\left[\int_0^{T/2} 0\cos n\omega_0 t\, dt + \int_{T/2}^{T} A\cos n\,\omega_0 t\, dt\right]$$

$$a_n = \frac{2}{T}\left[\frac{A\sin n\omega_0 t}{n\omega_0}\right]_{T/2}^{T}$$

$$a_n = \frac{2A}{n\omega_0 T}[\sin n\omega_0 t]_{T/2}^{T}$$

$$a_n = \frac{2A}{n\omega_0 T}\left[\sin n\omega_0 T - \sin n\omega_0\frac{T}{2}\right]$$

$$a_n = \frac{2A}{n\omega_0 T}\left[2\cos\left(\frac{n\omega_0 T + n\omega_0\frac{T}{2}}{2}\right)\sin\left(\frac{n\omega_0 T - n\omega_0\frac{T}{2}}{2}\right)\right]$$

$$a_n = \frac{4A}{n\omega_0 T}\left[\cos\left(\frac{3n\omega_0 T}{4}\right)\sin\left(\frac{n\omega_0 T}{4}\right)\right] \quad ...(2.8.3)$$

4. $$b_n = \frac{2}{T}\left[\int_0^{T/2} 0\sin n\omega_0 t\, dt + \int_{T/2}^{T} A\sin n\omega_0 t\, dt\right]$$

$$= \frac{2}{T}\left[\int_{T/2}^{T} A\sin n\omega_0 t\, dt\right]$$

$$= \frac{2A}{T}\left[\frac{-\cos(n\,\omega_0 t)}{n\omega_0}\right]_{T/2}^{T}$$

$$= \frac{2A}{n\omega_0 T}\left[-\cos(n\omega_0 T) + \cos\left(n\omega_0\frac{T}{2}\right)\right]$$

$$= \frac{4A}{n\omega_0 T}\sin\left(\frac{3n_0\omega T}{4}\right)\sin\left(\frac{n\omega_0 T}{4}\right) \quad ...(2.8.4)$$

5. Now putting eq. (2.8.2), (2.8.3) and (2.8.4) in (2.8.1), we get

$$f(t) = \frac{A}{2} + \sum_{n=1}^{\infty}\frac{4A}{n\omega_0 T}\cos\left(\frac{3n\omega_0 T}{4}\right)\sin\left(\frac{n\omega_0 T}{4}\right)\cos n\omega_0 t +$$

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$$\frac{4A}{n\omega_0 T} \sin\left(\frac{3n_0\omega T}{4}\right) \sin\left(\frac{n\omega_0 T}{4}\right) \sin n\omega_0 t$$

6. $$f(t) = \frac{A}{2} + \frac{4A}{\omega_0 T} \sum_{n=1}^{\infty} \frac{1}{n} \left[\cos\left(\frac{3n\omega_0 T}{4}\right) \sin\left(\frac{n\omega_0 T}{4}\right) \cos n\omega_0 t + \sin\left(\frac{3n_0\omega T}{4}\right) \sin\left(\frac{n\omega_0 T}{4}\right) \sin n\omega_0 t \right]$$

Que 2.9. **Find the exponential Fourier series representation of the infinite square wave shown in Fig. 2.9.1.**

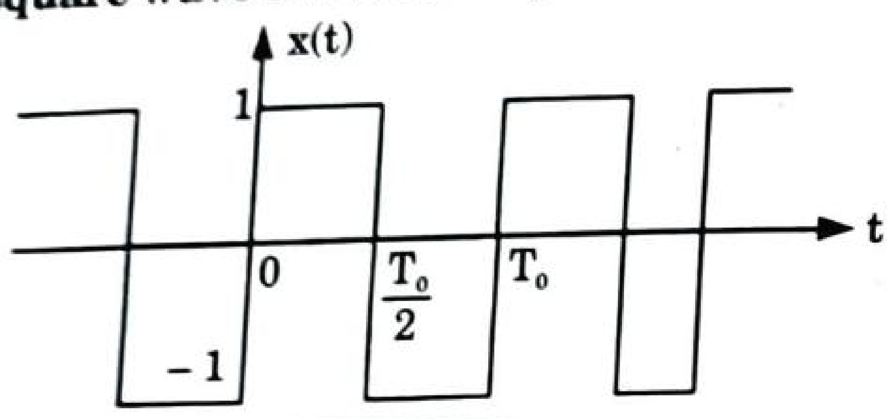

Fig. 2.9.1.

AKTU 2013-14, Marks 10

Answer

1. $$x(t) = \begin{cases} 1 & ; \text{ for } 0 \le t \le T_0/2 \\ -1 & ; \text{ for } \frac{T_0}{2} \le t < T_0 \end{cases}$$

2. $$C_0 = \frac{1}{T_0}\int_0^{T/2} 1\, dt - \int_{T_0/2}^{T_0} 1\, dt = 0$$

3. $$C_n = \frac{1}{T}\int_0^{T_0} x(t) e^{-jn\omega_0 t}\, dt$$

$$= \frac{1}{T_0}\left[\int_0^{T_0/2} 1 e^{-jn\omega_0 t} + \int_{T_0/2}^{T_0} (-1) e^{-jn\omega_0 t} dt\right]$$

$$= \frac{1}{T_0}\left\{\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_0^{T_0/2} - \left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{T_0/2}^{T_0}\right\}$$

$$= \frac{e^{-jn\omega_0 T_0/2} - e^0}{-jn\omega_0 T_0} - \frac{e^{-jn\omega_0 T_0} - e^{-jn\omega_0 T_0/2}}{-jn\omega_0 T_0} \quad \text{...(2.9.1)}$$

4. Since $\omega_0 = \frac{2\pi}{T_0}$, $\omega_0 T_0 = 2\pi$, eq. (2.9.1) will be

$$C_n = \frac{e^{-jn\pi} - 1}{-jn2\pi} - \frac{e^{-jn2\pi} - e^{-jn\pi}}{-jn2\pi} = \frac{e^{-jn\pi} - 1 - e^{-jn2\pi} + e^{-jn\pi}}{-jn2\pi}$$

$e^{-jn2\pi} = \cos(2\pi n) - j\sin(2\pi n) = 1$ for all n.

5. Hence $$C_n = \frac{2e^{-jn\pi} - 2}{-jn2\pi} = \frac{1 - e^{-jn\pi}}{jn\pi}$$

6. $1 - e^{-jn\pi} = 1 - [\cos n\pi - j\sin n\pi] = 1 - \cos n\pi$, since $\sin n\pi = 0$

$$= \begin{cases} 2 & ; \text{ for } n = 1,3,5,\ldots\text{odd} \\ 0 & ; \text{ for } n = 0,2,4,6,\ldots\text{even} \end{cases}$$

7. $$C_n = \begin{cases} \frac{2}{jn\pi} & ; \text{ for } n = 1,3,5,\ldots\text{odd} \\ 0 & ; \text{ for } n = 0,2,4,6,\ldots\text{even} \end{cases}$$

8. Hence, exponential Fourier series is

$$x(t) = \sum_{n=-\infty}^{\infty} X(n) e^{jn\omega_0 t} = \sum_{n=1,3,5,\ldots} \frac{2}{jn\pi} e^{jn\omega_0 t}$$

Que 2.10. **Explain the trigonometric and exponential forms of Fourier series representation of periodic signals. Find the trigonometric Fourier series for the periodic signal shown in Fig. 2.10.1.**

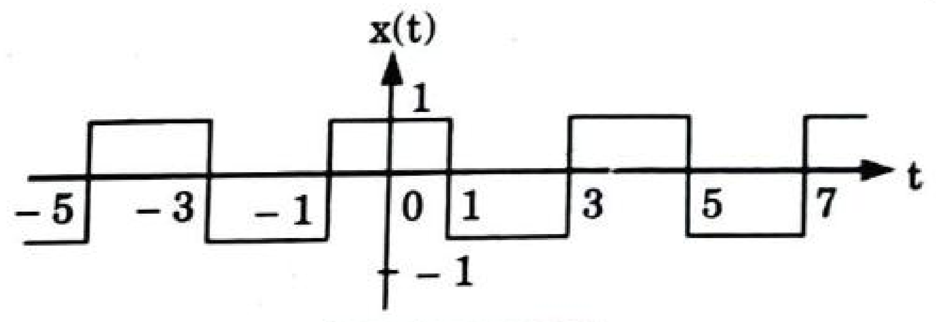

Fig. 2.10.1.

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Answer

Trigonometric and exponential forms : Refer Q. 2.1, Page 2–2D, Unit-2.

Numerical :

Assumption : Shift the signal upward by 1 unit.

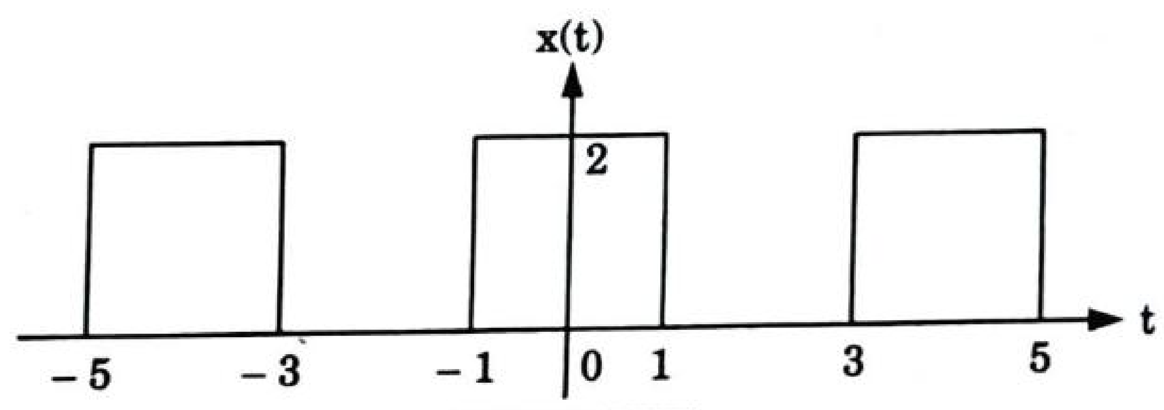

Fig. 2.10.2.

1. $$x(t) = \begin{cases} 2 & ; \; -1 < t < 1 \\ 0 & ; \; 1 < t < 3 \end{cases}$$

2. Fundamental time period = 4

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$$\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \frac{\pi}{2}$$

3. $$a_0 = \frac{2}{T}\int_0^{T/2} x(t)\,dt = \frac{2}{4}\left(\int_0^1 2dt + \int_1^2 0\,dt\right) = 1$$

4. $$a_n = \frac{4}{T}\int_0^{T/2} x(t)\cos n\omega_0 t\,dt$$

$$= \frac{4}{T}\left[\int_0^1 2\cos n\omega_0 t\,dt + \int_1^2 0\cos n\omega_0 t\,dt\right]$$

$$= 2\int_0^1 \cos n\omega_0 t\,dt + 0 = 2\left[\frac{\sin n\omega_0 t}{n\omega_0}\right]_0^1$$

$$= 2\left[\frac{\sin n\omega_0 - \sin n\omega_0 .0}{n\omega_0}\right] = \frac{2\sin n\omega_0}{n\omega_0}$$

Putting $\omega_0 = \dfrac{\pi}{2}$

$$= \frac{2\sin\frac{n\pi}{2}}{\frac{n\pi}{2}} = \frac{4}{n\pi}\sin\frac{n\pi}{2}$$

$$= \begin{cases} 0 & ;\ n \text{ even} \\ \dfrac{4(-1)^n}{n\pi} & ;\ n \text{ odd} \end{cases}$$

5. Since, given signal is even,

$$b_n = 0$$

6. $$x(t) = 1 + \sum_{n=\text{odd}}^{\infty} \frac{4(-1)^n}{n\pi}\cos n\frac{\pi}{2}t$$

Que 2.11. **Obtain the trigonometric Fourier series for the waveform shown in Fig. 2.11.1.**

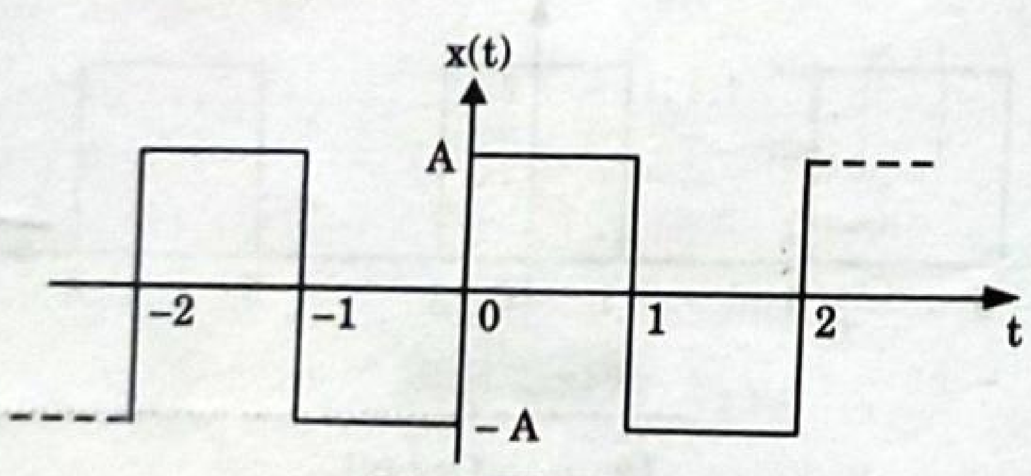

Fig. 2.11.1.

AKTU 2014-15, Marks 06

Answer

1. The periodic waveform shown in Fig. 2.11.1 with a period $T = 2$ can be expressed as

$$x(t) = \begin{cases} A & ;\ 0 \le t \le 1 \\ -A & ;\ 1 \le t \le 2 \end{cases}$$

2. Let $t_0 = 0$

$\therefore$ $t_0 + T = 2$

and fundamental frequency $\omega_0 = \dfrac{2\pi}{T} = \dfrac{2\pi}{2} = \pi$

3. The waveform has odd symmetry because $x(t) = -x(-t)$.
So, $a_0 = 0$, $a_n = 0$

4. $$b_n = \frac{4}{T}\int_{t_0}^{t_0+T/2} x(t)\sin n\omega_0 t\,dt$$

$$b_n = \frac{4}{T}\int_0^{T/2} x(t)\sin n\omega_0 t\,dt = \frac{4}{2}\int_0^1 A\sin n\pi t\,dt$$

$$= 2A\left[\frac{-\cos n\pi t}{\pi n}\right]_0^1 = -\frac{2A}{n\pi}\left[\frac{\cos n\pi - \cos 0}{1}\right]$$

$$= -\frac{2A}{n\pi}[(-1)^n - 1] = \frac{2A}{n\pi}[1-(-1)^n]$$

$$b_n = \begin{cases} \dfrac{4A}{n\pi} & ;\ \text{for odd } n \\ 0 & ;\ \text{for even } n \end{cases}$$

5. The trigonometric Fourier series is

$$x(t) = a_0 + \sum_{n=1}^{\infty}(a_n\cos n\omega_0 t + b_n\sin n\omega_0 t) = \sum_{n=\text{odd}}^{\infty} b_n\sin n\omega_0 t$$

$$x(t) = \sum_{n=\text{odd}}^{\infty}\frac{4A}{n\pi}\sin n\pi t = \frac{4A}{\pi}\sum_{n=\text{odd}}^{\infty}\frac{1}{n}\sin n\pi t$$

$$= \frac{4A}{\pi}\sin\pi t + \frac{4A}{3\pi}\sin 3\pi t + \frac{4A}{5\pi}\sin 5\pi t + \ldots$$

PART-2

Fourier Transform : Properties, Applications to Network Analysis.

CONCEPT OUTLINE : PART-2

Fourier transform : It is a transformation technique which transforms signals from the continuous time domain to the corresponding frequency domain and vice-versa, and which applies for both periodic as well as aperiodic signals.

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$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \text{ for all } \omega \quad ...(1)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega \text{ for all } t \quad ...(2)$$

Eq. (1) transforms the time function $x(t)$ to frequency function $X(j\omega)$ and so it is called Fourier transform.

Eq. (2) converts the frequency function to time function and hence it is called inverse Fourier transform.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.12. State and derive the following four properties of Fourier transform :

i. Duality
ii. Time shifting
iii. Frequency shifting
iv. Scaling

AKTU 2012-13, Marks 10

OR

State and prove duality property of Fourier transform.

AKTU 2015-16, Marks 05

Answer

i. **Duality :** In spectrum analysis, the duality between the time and the frequency is exhibited. The duality property states that

if $\quad x(t) \xleftrightarrow{FT} X(\omega)$

then $\quad X(t) \xleftrightarrow{FT} 2\pi\, x(-\omega)$

Proof : By definition,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$\therefore \quad 2\pi x(t) = \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$\text{or} \quad 2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega$$

Interchanging t and ω, we have

$$2\pi\, x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt = F[X(t)]$$

$$\therefore \quad F[X(t)] = 2\pi\, x(-\omega)$$

$i.e.\quad X(t) \xleftrightarrow{FT} 2\pi\, x(-\omega)$

For even functions,

$$x(-\omega) = x(\omega)$$

$$X(t) \xleftrightarrow{FT} 2\pi\, x(\omega)$$

ii. **Time shifting :** The time shifting property states that if a signal $x(t)$ is shifted by t_0 sec, the spectrum is modified by a linear phase shift of slope $-\omega t_0$, *i.e.*,

if $\quad x(t) \xleftrightarrow{FT} X(\omega)$

then, $\quad x(t - t_0) \xleftrightarrow{FT} e^{-j\omega t_0}\, X(j\omega)$

Proof : By definition,

$$F[x(t - t_0)] = \int_{-\infty}^{\infty} x(t - t_0)\, e^{j\omega t}\, dt$$

Let $\quad t - t_0 = p$

$\therefore \quad t = p + t_0$ and $dt = dp$

$$\therefore \quad F[x(t - t_0)] = \int_{-\infty}^{\infty} x(p)\, e^{-j\omega(p + t_0)}\, dp$$

$$= e^{-j\omega t_0}\int_{-\infty}^{\infty} x(p)\, e^{-j\omega p}\, dp$$

$$= e^{-j\omega t_0} X(\omega)$$

$$\therefore \quad x(t - t_0) \xleftrightarrow{FT} e^{-j\omega t_0} X(\omega)$$

Similarly, $\quad x(t + t_0) \xleftrightarrow{FT} e^{j\omega t_0} X(\omega)$

iii. **Frequency shifting :** Frequency shifting property states that the multiplication of a time domain signal $x(t)$ by $e^{j\omega_0 t}$ results in the frequency spectrum shifted by ω_0, *i.e.*,

if $\quad x(t) \xleftrightarrow{FT} X(\omega)$

then, $\quad e^{j\omega_0 t} x(t) \xleftrightarrow{FT} X(\omega - \omega_0)$

Proof : By definition,

$$F[e^{j\omega_0 t} x(t)] = \int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t) e^{-j(\omega - \omega_0)t}\, dt$$

$$= X(\omega - \omega_0)$$

$$e^{j\omega_0 t} x(t) \xleftrightarrow{FT} X(\omega - \omega_0)$$

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$$e^{-j\omega_0 t}x(t) \xleftrightarrow{FT} X(\omega + \omega_0)$$

iv. Scaling :

If $x(t) \xleftrightarrow{FT} X(\omega)$

then, $x(at) \xleftrightarrow{FT} \dfrac{1}{|a|} X\left(\dfrac{\omega}{a}\right)$

Proof : By definition,

$$F[x(at)] = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

Let $at = p$

$t = \dfrac{p}{a}$ and $dt = \dfrac{dp}{a}$

$\therefore$ $$F[x(at)] = \int_{-\infty}^{\infty} x(p)\, e^{-j\omega(p/a)} \frac{dp}{a}$$

$\therefore$ $$= \frac{1}{a}\int_{-\infty}^{\infty} x(p)\, e^{-j(\omega/a)p}\, dp$$

Case 1 : When $a > 0$,

$$F[x(at)] = \frac{1}{a}\int_{-\infty}^{\infty} x(p)\, e^{-j(\omega/a)p}\, dp = \frac{1}{a} X\left(\frac{\omega}{a}\right)$$

Case 2 : When $a < 0$,

$$F[at] = \frac{1}{a}\int_{\infty}^{-\infty} x(p)\, e^{j(\omega/a)p}\, dp = \frac{1}{a}\int_{-\infty}^{\infty} x(p)\, e^{-j[-(\omega/a)]p}\, dp$$

$$= -\frac{1}{a} X\left(-\frac{\omega}{a}\right)$$

$$= \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

$$x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Que 2.13. State and prove :

i. **Linearity**
ii. **Time differentiation**
iii. **Time integration**

Answer

i. **Linearity :**

If $x_1(t) \xleftrightarrow{FT} X_1(j\omega)$

$x_2(t) \xleftrightarrow{FT} X_2(j\omega)$

then

$$[A\, x_1(t) + B\, x_2(t)] \xleftrightarrow{FT} [A\, X_1(j\omega) + B\, X_2(j\omega)]$$

Proof :

1. Let $x(t) = A\, x_1(t) + B\, x_2(t)$
2. $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} [A\, x_1(t) + B\, x_2(t)]\, e^{-j\omega t}\, dt$
$$= A\int_{-\infty}^{\infty} x_1(t) e^{-j\omega t} + B\int_{-\infty}^{\infty} x_2(t) e^{-j\omega t}\, dt$$
3. $X(j\omega) = A\, X_1(j\omega) + B\, X_2(j\omega)$

ii. Time differentiation :

If $x(t) \xleftrightarrow{FT} X(j\omega)$

then

$$\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega X(j\omega)$$

Proof :

1. $$F[x(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$$
2. $$F\left[\frac{dx(t)}{dt}\right] = \frac{j\omega}{2\pi}\int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega = j\omega\, X(j\omega)$$
3. $$\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega\, X(j\omega)$$
4. In general
$$F\left[\frac{d^n\, x(t)}{dt^n}\right] = (j\omega)^n X(j\omega)$$

iii. Time integration :

If $F[x(t)] = X(j\omega)$

then

$$F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = \frac{1}{j\omega} X(j\omega) + \pi X(0)\, \delta(\omega)$$

Proof :

1. If $X(0) = 0$, this property can be easily proved by using integration by parts as in the case of the differentiation property.
2. By definition,
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$
3. Replacing t by a dummy variable τ, we have
$$x(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega\tau}\, d\omega$$
4. Integrating both sides over $-\infty$ to t, we have

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$$\int_{-\infty}^{t} x(\tau)\, d\tau = \int_{-\infty}^{t}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega\tau}\, d\tau\right] d\omega$$

5. Interchanging the order of integration, we have

$$\int_{-\infty}^{t} x(\tau)\, d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left(\int_{-\infty}^{t} e^{j\omega\tau}\, d\tau\right) d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left[\frac{e^{j\omega\tau}}{j\omega}\right]_{-\infty}^{t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\frac{X(\omega)}{j\omega}\right] e^{j\omega t}\, d\omega = F^{-1}\left[\frac{X(\omega)}{j\omega}\right]$$

$$\therefore \quad F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = \frac{1}{j\omega} X(\omega)$$

$$\therefore \quad \int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{FT} \frac{1}{j\omega} X(\omega)$$

6. If $X(0) \neq 0$ then $x(t)$ is not an energy function, and the Fourier transform of $\int_{-\infty}^{t} x(\tau)\, d\tau$ includes an impulse function, that is

$$F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = \frac{1}{j\omega} X(\omega) + \pi X(0)\, \delta(\omega)$$

Que 2.14. **State and prove convolution and Parseval's theorem.**

Answer

Convolution theorem :

Let
$$y(t) = x(t) * h(t)$$
$$F[y(t)] = Y(j\omega) = X(j\omega)\, H(j\omega)$$

Proof :

1. $$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

2. $$F[y(t)] = Y(j\omega) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau\right] e^{-j\omega t}\, dt$$

3. Interchanging the order of integration, we get

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau)\left[\int_{-\infty}^{\infty} h(t-\tau)\, e^{-j\omega t}\, dt\right] d\tau$$

4. By time shifting property, the term inside the bracket becomes $e^{-j\omega\tau} H(j\omega)$.

$$Y(j\omega) = \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, H(j\omega)\, d\tau$$

$$= H(j\omega)\int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$Y(j\omega) = H(j\omega)\, X(j\omega)$$

Parseval's theorem :

If $x_1(t) \xleftrightarrow{FT} X_1(\omega)$ and $x_2(t) \xleftrightarrow{FT} X_2(\omega)$

then, the Parseval's theorem states that

$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\, X_2^*(\omega)\, d\omega$$

for complex $x_1(t)$ and $x_2(t)$

Proof :

Parseval's relation :

1. $$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \int_{-\infty}^{\infty}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\, e^{j\omega t}\, d\omega\right\} x_2^*(t)\, dt$$

2. Interchanging the order of integration, we have

$$\int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\left\{\int_{-\infty}^{\infty} x_2^*(t)\, e^{j\omega t}\, dt\right\} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\left\{\int_{-\infty}^{\infty} x_2(t)\, e^{-j\omega t}\, dt\right\}^* d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\, \{X_2(\omega)\}^*\, d\omega = \text{RHS}$$

3. $$\therefore \quad \int_{-\infty}^{\infty} x_1(t)\, x_2^*(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(\omega)\, X_2^*(\omega)\, d\omega$$

Que 2.15. **Derive the Fourier transforms of the following functions :**

i. Unit impulse function

ii. rect (t)

iii. $e^{-2|t|}$

iv. sin 2t

AKTU 2012-13, Marks 10

Answer

i. **Unit impulse :**

Given, $x(t) = \delta(t)$,

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$$\delta(t) = \begin{cases} 1 & ; \text{ for } t = 0 \\ 0 & ; \text{ for } t \neq 0 \end{cases}$$

Then $$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt$$

$$= e^{-j\omega t}\big|_{t=0} = 1$$

$\therefore$ $$F[\delta(t)] = 1$$

ii. rect (t):

$$f(t) = \begin{cases} A & ; \; 0 < t < T \\ 0 & ; \; \text{otherwise} \end{cases}$$

$$F[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt = \int_0^T A e^{-j\omega t}\, dt$$

$$= A\left[\frac{e^{-j\omega T} - 1}{-j\omega}\right] = A\left[\frac{1 - e^{-j\omega t}}{j\omega}\right]$$

$$= A\, e^{-j\omega\left(\frac{T}{2}\right)}\left[\frac{e^{-j\omega\left(\frac{T}{2}\right)} - e^{-j\omega\left(\frac{T}{2}\right)}}{j\omega}\right]$$

$$= \frac{2A}{\omega} e^{-j\omega\left(\frac{T}{2}\right)}\left[\frac{e^{j\omega\left(\frac{T}{2}\right)} - e^{-j\omega\frac{T}{2}}}{2j}\right]$$

$$= \frac{2A}{\omega} e^{-j\omega\left(\frac{T}{2}\right)} \sin\frac{\omega T}{2} = AT e^{-j\omega\frac{T}{2}}\left[\frac{\sin\left(\frac{\omega T}{2}\right)}{\left(\frac{\omega T}{2}\right)}\right]$$

$$F(\omega) = AT e^{-j\omega(T/2)} \operatorname{sinc} \frac{\omega t}{2}$$

iii. $e^{-2|t|}$:

$$F(\omega) = \int_{-\infty}^{\infty} e^{-2|t|} e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{(2-j\omega)t}\, dt + \int_0^{\infty} e^{-(2+j\omega)t}\, dt$$

$$= \left[\frac{e^{+(2-j\omega)t}}{(2-j\omega)}\right]_{-\infty}^{0} + \left[\frac{e^{-(2+j\omega)t}}{-(2+j\omega)}\right]_0^{\infty}$$

$$= \left[\frac{1-0}{(2-j\omega)}\right] + \left[\frac{0-1}{-(2+j\omega)}\right] = \frac{2+2}{(2)^2+\omega^2}$$

$$|F(\omega)| = \frac{4}{2^2+\omega^2}$$

iv. $\sin 2t$:

$$F\{\sin 2t\} = F(\omega) = \int_{-\infty}^{\infty} \sin 2t\, e^{-j\omega t} dt$$

$$= \frac{1}{j2}\int_{-\infty}^{\infty} (e^{j2t} - e^{-j2t})\, e^{-j\omega t}\, dt$$

$$= \frac{1}{j2} F\{e^{j2t} - e^{-j2t}\}$$

$$= \frac{1}{j2}\,[2\pi\delta(\omega-2) - 2\pi\delta(\omega+2)]$$

$$F(\omega) = \frac{\pi}{j}\,[\delta(\omega-2) - \delta(\omega+2)]$$

Que 2.16. **Find the Fourier transform of rectangular function shown in Fig. 2.16.1.**

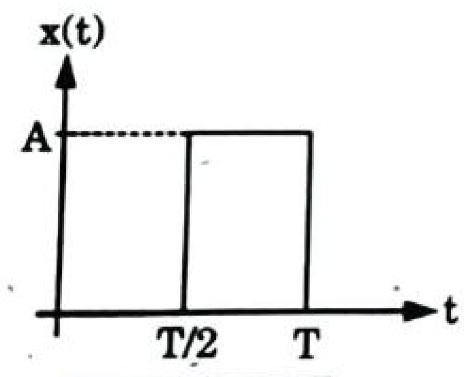

Fig. 2.16.1.

AKTU 2015-16, Marks 05

Answer

1. $$x(t) = \begin{cases} A & ; \; T/2 < t < T \\ 0 & ; \; 0 < t < T/2 \end{cases}$$

2. $$X(\omega) = \int_{T/2}^{T} A e^{-j\omega t}\, dt = \frac{-A}{j\omega}\,[e^{-j\omega t}]_{T/2}^{T}$$

$$= \frac{-A}{j\omega}\left[e^{-j\omega T} - e^{-j\omega T/2}\right]$$

$$= \frac{A}{j\omega} e^{-j\omega T/2} - \frac{A}{j\omega} e^{-j\omega T}$$

$$= \frac{2A}{\omega} e^{-j3\omega T/4}\left[\frac{e^{j\omega T/4} - e^{-j\omega T/4}}{2j}\right]$$

$$= \frac{2AT}{4} e^{-j3\omega T/4}\left[\frac{\sin \omega T/4}{\omega T/4}\right]$$

$$= \frac{AT}{2} e^{-j3\omega T/4} \operatorname{sinc} \omega T/4$$

Que 2.17. **Find the Fourier transform of $e^{2t}u(-t)$ along with amplitude.**

AKTU 2014-15, Marks 3.5

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Answer

Given : $x(t) = e^{2t}\, u(-t)$

To Find : Fourier transform, $X(\omega)$.

1.
$$X(\omega) = F[e^{2t}\, u(-t)]$$
$$= \int_{-\infty}^{\infty} e^{2t}u(-t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{0} e^{2t}\, e^{-j\omega t} dt$$
$$= \int_{-\infty}^{0} e^{(2-j\omega)t} dt = \int_{0}^{\infty} e^{-(2-j\omega)t}\, dt$$
$$= \left[\frac{e^{-(2-j\omega)t}}{-(2-j\omega)}\right]_0^{\infty} = \frac{1}{2-j\omega}$$

Que 2.18. **Find the Fourier transform of the following signals :**

i. $\cos \omega_c t\, u(t)$

ii. $\sin \omega_c t\, u(t)$

AKTU 2013-14, Marks 10

Answer

i. $\cos \omega_c t\, u(t)$ **:**

$$X(\omega) = F[\cos \omega_c(t)] = \int_0^{\infty} \cos \omega_c t e^{-j\omega t} dt$$
$$= \int_0^{\infty} \left[\frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}\right] e^{-j\omega t} dt$$
$$= \int_0^{\infty} \frac{e^{-j(\omega-\omega_c)t}}{2} dt + \int_0^{\infty} e^{-j(\omega+\omega_c)t}\, dt$$
$$= \frac{1}{2}\left[\frac{e^{-j(\omega-\omega_c)t}}{-j(\omega-\omega_c)} + \frac{e^{-j(\omega+\omega_c)t}}{-j(\omega+\omega_c)}\right]_0^{\infty}$$
$$= \frac{1}{2}\left[\frac{e^0}{j(\omega-\omega_c)} + \frac{e^0}{j(\omega+\omega_c)}\right] = \frac{1}{2j}\left[\frac{\omega+\omega_c+\omega-\omega_c}{\omega^2-\omega_c^2}\right]$$
$$\therefore \quad X(\omega) = \frac{\omega}{j(\omega^2-\omega_c^2)}$$

ii. $\sin \omega_c t\, u(t)$ **:**

$$X(\omega) = F[\sin \omega_c t\; u(t)] = \int_0^{\infty} \sin \omega_c t\; e^{-j\omega t} dt$$
$$= \int_0^{\infty} \left(\frac{e^{+j\omega_c t} - e^{-j\omega_c t}}{2j}\right) e^{-j\omega t} dt$$
$$= \frac{1}{2j}\int_0^{\infty} e^{-j(\omega-\omega_c)t} dt - \frac{1}{2j}\int_0^{\infty} e^{-j(\omega+\omega_c)t} dt$$

$$= \frac{1}{2j}\left[\frac{e^{-j(\omega-\omega_c)t}}{-j(\omega-\omega_c)}\right]_0^{\infty} - \frac{1}{2j}\left[\frac{e^{-j(\omega+\omega_c)t}}{-j(\omega+\omega_c)}\right]_0^{\infty}$$
$$= \frac{1}{2}\left[\frac{-1}{(\omega-\omega_c)} + \frac{1}{(\omega+\omega_c)}\right]$$
$$= \frac{-\omega_c}{(\omega^2-\omega_c^2)}$$

Que 2.19. **Write the Fourier transform of step and impulse signals for system analysis.**

Answer

i. **Unit step :**

Given : $x(t) = u(t) = \begin{cases} 1 & ; \; t \ge 0 \\ 0 & ; \; \text{otherwise} \end{cases}$

Since the unit step function is not absolutely integrable, we cannot directly find its Fourier transform. So, expressing the unit step function in terms of signum function as

$$u(t) = \frac{1}{2} + \frac{1}{2}\text{sgn}(t)$$

Fig. 2.19.1 represents $\frac{1}{2}$ sgn(t) and $u(t)$.

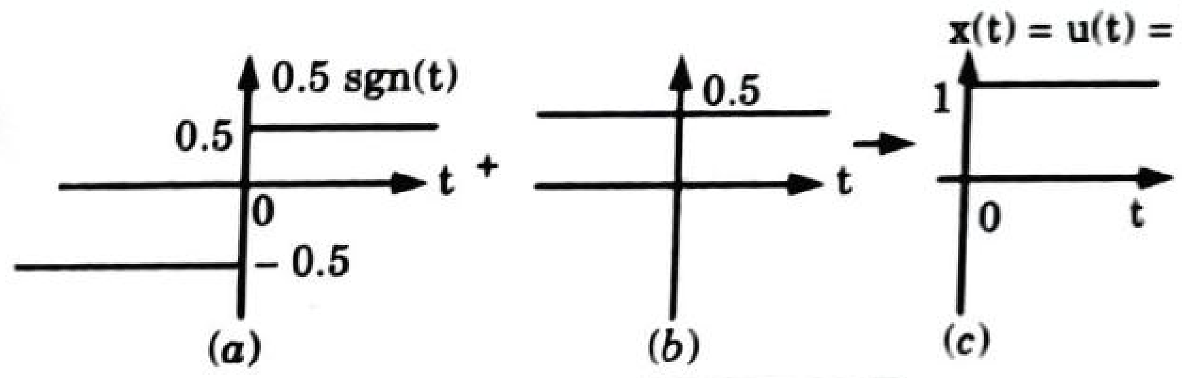

Fig. 2.19.1.

$$x(t) = u(t) = \frac{1}{2}\;[1 + \text{sgn}(t)]$$

Now taking Fourier transform on both sides,

$$X(\omega) = F[u(t)] = F\left[\frac{1}{2}\right] + \frac{1}{2} F\,[\text{sgn}(t)]$$

Now, $\quad F\left[\frac{1}{2}\right] = \pi\,\delta(\omega) \qquad [\because \; F[1] \rightarrow 2\pi\delta(\omega)]$

$$F\left[\frac{1}{2}\text{sgn}\,(t)\right] = \frac{1}{2}\,\frac{2}{j\omega} = \frac{1}{j\omega}$$

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$$\therefore \quad F[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}$$

Similarly, $$F[u(-t)] = \pi\delta(\omega) - \frac{1}{j\omega}$$

ii. **Unit impulse :** Refer Q. 2.15, Page 2–21D, Unit-2.

Que 2.20. **Determine Fourier transform of the following useful functions :**

i. $u(t)\,e^{-at}$ ii. e^{-at}

iii. A iv. $g(t) = \begin{cases} A & ; |t| < T/2 \\ 0 & ; |t| > T/2 \end{cases}$

v. $\text{sgn}(t) = \begin{cases} 1 & ; t > 0 \\ 1 & ; t < 0 \end{cases}$ vi. $\cos \omega_0 t$

Answer

i.

1. $$f(t) = u(t)\,e^{-at}$$

2. $$F(\omega) = \int_{-\infty}^{\infty} u(t)\,e^{-at} e^{-j\omega t} dt = \int_{0}^{\infty} 1 e^{-(a+j\omega)t} dt$$

3. $$F(\omega) = \frac{1}{a + j\omega}$$

ii.

1. $$f(t) = e^{-at}$$

2. $$F(\omega) = \int_{-\infty}^{\infty} e^{-a|t|}\, e^{j\omega t} dt = \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt + \int_{\infty}^{0} e^{-(a+j\omega)t}\, dt$$

$$= \frac{2a}{a^2 + \omega^2}$$

iii.

1. $$f(t) = A$$

2. $$F[f(t)] = \int_{-\infty}^{\infty} A\, e^{-j\omega t}\, dt$$

3. $$F[f(t)] = \lim_{\tau \to \infty} A\tau \left[\frac{\sin\left(\frac{\omega\tau}{2}\right)}{\frac{\omega\tau}{2}} \right] = \lim_{\tau\to\infty} 2\pi A\tau$$

4. Since $$\lim_{x \to \infty} \frac{\sin x}{x} = 2\pi$$

and $$\lim_{\tau\to\infty} (A\tau) = A\delta(\omega)$$

5. Hence, $$F[f(t)] = 2\pi A\, \delta(\omega)$$

iv.

1. Fourier transform of gate function

$$G(\omega) = \int_{-\infty}^{\infty} g(t)\; e^{-j\omega t} dt$$

2. $$G(\omega) = \int_{-\tau/2}^{\tau/2} A\, e^{-j\omega t}\, dt$$

$$= \frac{A}{j\omega}(e^{j\omega\tau/2} - e^{-j\omega\tau/2}) = \frac{2A}{\omega} \sin(\omega\tau/2)$$

$$= A\tau\, \frac{\sin(\omega\,\tau/2)}{(\omega\tau/2)} = A\tau \sin c\left(\frac{\omega\tau}{2}\right)$$

v.

1. This function is not absolutely integrable. So we cannot directly find its Fourier transform.
2. Therefore, let us consider the function $e^{-a|t|}\,\text{sgn}(t)$ and substitute the limit $a \to 0$ to obtain the above sgn(t).
3. $$x(t) = \text{sgn}(t) = \lim_{a\to 0}\; e^{-a|t|}\; \text{sgn}(t)$$

$$= \lim_{a\to 0}\, [e^{-at}\, u(t) - e^{at}\, u(-t)]$$

4. $$\therefore \quad X(\omega) = F[\text{sgn}(t)] = \int_{-\infty}^{\infty} \left[\lim_{a\to 0} [e^{-at}\, u(t) - e^{at}\, u(-t)] \right] e^{-j\omega t}\, dt$$

$$= \lim_{a\to 0} \left[\int_{-\infty}^{\infty} e^{-at}\, e^{-j\omega t}\, u(t)\, dt - \int_{-\infty}^{\infty} e^{at}\, e^{-j\omega t}\, u(-t)\, dt \right]$$

$$= \lim_{a\to 0} \left[\int_{0}^{\infty} e^{-(a+j\omega)t}\, dt - \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt \right]$$

$$= \lim_{a\to 0} \left[\int_{0}^{\infty} e^{-(a+j\omega)t}\, dt - \int_{0}^{\infty} e^{-(a-j\omega)t}\, dt \right]$$

$$= \lim_{a\to 0} \left\{ \left[\frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right]_0^{\infty} - \left[\frac{e^{-(a-j\omega)t}}{-(a-j\omega)} \right]_0^{\infty} \right\}$$

$$= \lim_{a\to 0} \left[\frac{e^{-\infty} - e^0}{-(a+j\omega)} - \frac{e^{-\infty} - e^0}{-(a-j\omega)} \right]$$

$$= \lim_{a\to 0} \left[\frac{1}{a+j\omega} - \frac{1}{a-j\omega} \right] = \frac{1}{j\omega} - \frac{1}{-j\omega} = \frac{2}{j\omega}$$

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vi.

Given : $x(t) = \cos \omega_0 t$

$$X(\omega) = F[x(t)] = F[\cos \omega_0 t] = F\left[\frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t})\right]$$

$$= \frac{1}{2}[F(e^{j\omega_0 t}) + F(e^{-j\omega_0 t})]$$

$$= \frac{1}{2}[2\pi\delta(\omega - \omega_0) + 2\pi\delta(\omega + \omega_0)]$$

$$= \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$$

Que 2.21. **Use fourier transform to find the output voltage V_0 in the Fig. 2.21.1.**

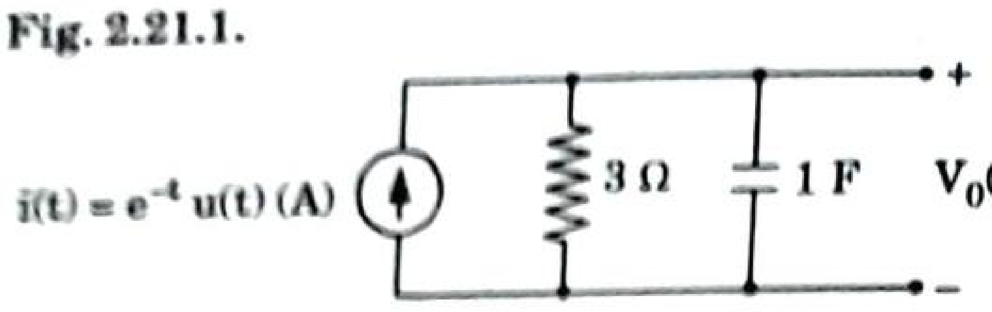

Fig. 2.21.1.

Answer

1. By KCL, $I(j\omega) = \dfrac{V_0(j\omega)}{3} + \dfrac{V_0(j\omega)}{1/j\omega} = V_0(j\omega)\left[\dfrac{1+3j\omega}{3}\right]$

2. Here, $I(j\omega) = \dfrac{1}{1+j\omega}$

3. $\therefore$ $\dfrac{1}{1+j\omega} = V_0(j\omega)\left[\dfrac{1+3j\omega}{3}\right]$

$$V_0(j\omega) = \frac{3}{(1+j\omega)(1+j3\omega)} = \left[\frac{\frac{3}{2}}{\frac{1}{3}+j\omega} - \frac{\frac{3}{2}}{1+j\omega}\right]$$

4. Taking inverse Fourier transform $V_0(t) = \dfrac{3}{2}e^{-t/3} - \dfrac{3}{2}e^{-t}$

Que 2.22. **A voltage $v = 100 \sin 314t + 25 \sin (942\,t + 60°)$ volts is applied to a circuit consisting of a resistance of 10 Ω inductance of 10 mH and a capacitance of 50 μF connected in series.**
Find the :
i. rms values of the applied voltage and current that will flow.
ii. Power consumed in the circuit.

AKTU 2013-14, Marks 10

Answer

Given : $v = 100 \sin 314t + 25 \sin (942t + 60°)\text{ V} = v_1 + v_2$
$R = 10\,\Omega \quad L = 10\text{ mH} \quad C = 50\,\mu\text{F}$

To Find :
i. rms voltage, V_{rms}.
ii. rms current, I_{rms}.
iii. Power consumed in circuit, P.

i. $v = 100 \sin 314t + 25 \sin (942t + 60°)$

$$\therefore \quad V_{rms} = \sqrt{\frac{100^2 + 25^2}{2}} = 72.89\text{ V}$$

ii.
1. Calculation of current at $\omega_1 = 314$ rad/s

$$X_{C1} = \frac{1}{\omega_1 C} = \frac{1}{314 \times 50 \times 10^{-6}} = 63.7\,\Omega$$
$$X_{L1} = \omega_1 L = 314 \times 10 \times 10^{-3} = 3.14\,\Omega$$
$$Z_1 = R + j(X_{L1} - X_{C1}) = 10 + j(3.14 - 63.7)$$
$$= 10 - j\,60.56 = 61.38 \angle -80.62°\,\Omega$$

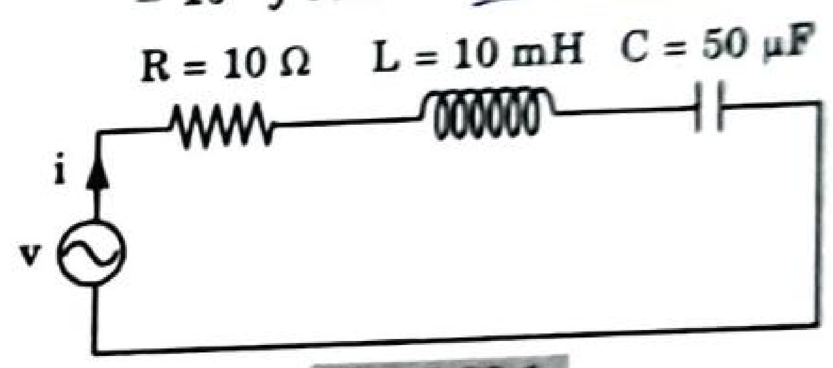

Fig. 2.22.1.

$$i_1 = \frac{v_1}{Z_1} = \frac{100 \angle 0°}{61.38 \angle -80.62°} = 1.63 \angle 80.62°\text{ A}$$

2. Calculation of current i_2 at $\omega_2 = 942$ rad/s

$$X_{C2} = \frac{1}{\omega_2 C} = \frac{1}{942 \times 50 \times 10^{-6}} = 21.23\,\Omega$$
$$X_{L2} = \omega_2 L = 914 \times 10 \times 10^{-3} = 9.14\,\Omega$$
$$Z_2 = R + j(X_{L2} - X_{C2})$$
$$= 10 + j(9.14 - 21.23)$$
$$= 10 - j\,12.09 = 15.69 \angle -50.40°\,\Omega$$
$$i_2 = \frac{v_2}{Z_2} = \frac{25 \angle 60°}{15.69 \angle -50.40°} = 1.60 \angle 110.4°\text{ A}$$

3. $\therefore$ $i = i_1 + i_2 = 1.63 \sin (314t + 80.62°) + 1.60 \sin (942t + 110.4°)$ A

4. $$I_{rms} = \sqrt{\frac{(1.63)^2 + (1.60)^2}{2}} = 1.62\text{ A}$$

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iii. Total consumed power

$$P = \left(\frac{100}{\sqrt{2}} \times \frac{1.63}{\sqrt{2}}\right) \cos 80.62° + \left(\frac{25}{\sqrt{2}} \times \frac{1.60}{\sqrt{2}}\right) \cos 50.4°$$
$$= 13.28 + 12.75$$
$$= 26 \text{ W}$$

PART-3

Definition of DTFS, and DTFT, Sampling Theorem.

CONCEPT OUTLINE : PART-3

- **DTFS (Discrete time fourier series) :** For a discrete-time signal $x[n]$, DTFS of $x[n]$ is defined as

$$x[n] = \sum_{k=0}^{N_0-1} D_k e^{jk\Omega_0 n} \quad \text{...(1)}$$

$$D_k = \frac{1}{N_0} \sum_{k=0}^{N_0-1} x[n] e^{-jk\Omega_0 n} \quad \text{...(2)}$$

Eq. (1) and (2) are called discrete time fourier series pairs.

- **DTFT (Discrete time fourier transform) :** For a discrete-time signal $x[n]$,

$$F(x[n]) = X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

- **Condition for existence of DTFT :** For existence of DTFT, sequence should be absolutely summable *i.e.*,

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

- **Sampling theorem :** A band limited signal of finite energy which has no frequency component higher than f_m can be completely described and recovered back if the sampling frequency is twice the highest frequency of the given signal.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.23. **Define DTFS. What are the different properties of Discrete time Fourier series ?**

Answer

DTFS (Discrete time fourier series) : For a discrete-time signal $x[n]$, DTFS of $x[n]$ is defined as,

$$x[n] = \sum_{k=0}^{N_0-1} D_k e^{jk\Omega_0 n} \quad \text{...(2.23.1)}$$

$$D_k = \frac{1}{N_0} \sum_{k=0}^{N_0-1} x[n] e^{-jk\Omega_0 n} \quad \text{...(2.23.2)}$$

Eq. (2.23.1) and (2.23.2) are called discrete time fourier series pairs.

Properties of discrete time fourier series :

1. **Linearity property :**
   Let $x_1[n]$ and $x_2[n]$ be two periodic signals with fundamental period N_0. According to linearity property the linear combinations of these two signals is also periodic with the same fundamental frequency N_0.

$$Ax_1[n] + Bx_2[n] \xleftrightarrow{\text{DTFS}} AD_{k1} + BD_{k2}$$

2. **Time shifting property :**
   If $x[n]$ is time shifted by n_0, then the periodicity of $x[n - n_0]$ is same as $x[n]$.

$$x[n - n_0] \xleftrightarrow{\text{DTFS}} e^{-jk\Omega_0 n_0} D_k$$

3. **Time reversal property :**
   If $x[n]$ with fundamental period N_0, is the time reversal, the fundamental period is not changed but the Fourier coefficient changes its sign.

$$x[-n] \xleftrightarrow{\text{DTFS}} D_{-k}$$

4. **Multiplication property :**
   If $x_1[n]$ and $x_2[n]$ are two DT signals with Fourier series coefficients as D_l and D_q, then the Fourier series coefficient of $Z[n] = x_1[n] \times x_2[n]$ is

$$D_k = \sum_{q=0}^{N_0-1} D_l D_{k-q}$$

5. **Conjugation property :**
   According to this property, the discrete time Fourier coefficient of $x^*[n]$ conjugate and time reversal of that of $x[n]$,

$$x^*[n] \xleftrightarrow{\text{DTFS}} D_{-k}^*$$

Que 2.24. **Find the discrete Fourier series representation of**

$$x[n] = \cos 0.2\pi n$$

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Answer

1. $x[n] = \cos 0.2\pi n$

$\Omega_0 = 0.2\pi$

$N_0 = 2\pi/\Omega_0 = 10$

2. $\cos 0.2\pi n = [e^{j2\pi n} + e^{-j2\pi n}]/2$ ...(2.24.1)

Also, $\cos 0.2\pi n = [e^{j\Omega_0 n} + e^{-j\Omega_0 n}]/2$ ...(2.24.2)

3. Comparing eq. (2.24.1) and (2.24.2), we get

$D_1 = 1/2$

$D_{-1} = D_{-1+10} = D_9 = 1/2$

4. $\cos[0.2\pi n] = [e^{j\Omega_0 n} + e^{-j9\Omega_0 n}]/2$

5. $\cos[0.2\pi n] = [e^{j.2\pi n} + e^{j1.8\pi n}]/2$

Que 2.25. **What do you understand by DTFT ? What is the condition for existence of DTFT ?**

Answer

DTFT : It stands for discrete-time Fourier transform which gives the description of $x[n]$ in frequency domain. It is given by

$$F[x[n]] = X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

Condition for existence of DTFT : Since the DTFT is given as

$$X(\Omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}$$

Now, we know that

$$|e^{-j\Omega n}| = 1$$

The $X(\Omega)$ can exist only if x[n] is absolutely summable.

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

Que 2.26. **What are the different properties of DTFT ?**

Answer

The DTFT has following properties :

1. **Linearity :** The DTFT is linear in nature.

Mathematically,

if $x_1[n] \xleftrightarrow{\text{DTFT}} X_1(\Omega)$

and $x_2[n] \xleftrightarrow{\text{DTFT}} X_2(\Omega)$

then, $a_1x_1[n] + a_2x_2[n] \xleftrightarrow{\text{DTFT}} a_1X_1(\Omega) + a_2X_2(\Omega)$

2. **Periodicity :** The DTFT is always periodic with period 2π.

Mathematically,

if $x[n] \xleftrightarrow{\text{DTFT}} X(\Omega)$

then, $X(\Omega + 2\pi k) = X(\Omega)$

3. **Shifting property :** This property can be discussed in two ways :

a. Time-shifting

b. Frequency-shifting

a. **Time-shifting :** This property states that if a discrete-time signal is shifted in the time-domain by an amount of 'n_0', a factor '$e^{-j\Omega n_0}$' is multiplied with $X(\Omega)$.

Mathematically,

if $x[n] \xleftrightarrow{\text{DTFT}} X(\Omega)$

then, $x[n - n_0] \xleftrightarrow{\text{DTFT}} e^{-j\Omega n_0} X(\Omega)$

b. **Frequency-shifting :** This property states that the multiplication by a factor $e^{j\Omega_0 n}$ in time domain sequence $x[n]$ results shift in frequency in $X(\Omega)$ by an amount of Ω_0.

Mathematically,

if $x[n] \xleftrightarrow{\text{DTFT}} X(\Omega)$

then, $e^{j\Omega_0 n} x[n] \xleftrightarrow{\text{DTFT}} X(\Omega - \Omega_o)$

4. **Time-reversal :** This property states that if a discrete-time signal is folded about the origin in time, its magnitude spectrum remains unchanged while the phase spectrum of this signal undergoes just change in sign.

Mathematically,

if $x[n] \xleftrightarrow{\text{DTFT}} X(\Omega)$

then, $x[-n] \xleftrightarrow{\text{DTFT}} X(-\Omega)$

5. **Scaling :** Mathematically,

if $x[n] \xleftrightarrow{\text{DTFT}} X(\Omega)$

then, $y[n] = x[kn] \xleftrightarrow{\text{DTFT}} Y(\Omega) = X\left(\dfrac{\Omega}{k}\right)$

6. **Differentiation in frequency-domain :** Mathematically,

if $x[n] \xleftrightarrow{\text{DTFT}} X(\Omega)$

then, $nx[n] \xleftrightarrow{\text{DTFT}} j\dfrac{dX(\Omega)}{d\Omega}$

7. **Conjugation and conjugate symmetry :** Mathematically,

if $x[n] \xleftrightarrow{\text{DTFT}} X(\Omega)$

then, $x^*[n] \xleftrightarrow{\text{DTFT}} X^*(\Omega)$

8. **Convolution :** Mathematically,

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if $x_1[n] \xleftrightarrow{\text{DTFT}} X_1(\Omega)$

and $x_2[n] \xleftrightarrow{\text{DTFT}} X_2(\Omega)$

then, $y[n] = x_1[n] * x_2[n] \xleftrightarrow{\text{DTFT}} Y(\Omega) = X_1(\Omega) . X_2(\Omega)$

Que 2.27. **Prove the periodicity property and convolution property of DTFT.**

AKTU 2016-17, Marks 10

Answer

1. **Convolution property :**
   Let $h(n)$ be the impulse response of discrete time system. Then frequency response of system

$$h[n] \xleftrightarrow{\text{DTFT}} H(\Omega)$$

Input, $x[n] \xleftrightarrow{\text{DTFT}} X(\Omega)$

and, output $y[n] \xleftrightarrow{\text{DTFT}} Y(\Omega)$

Then, $Y(\Omega) = X(\Omega)\, H(\Omega)$

Proof :

$\because$ $y[n] = x[n] * h[n]$

$$Y(\Omega) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} h[k]\, x[n-k]\right]e^{-j\Omega n}$$

Put $n - k = m$, for fixed k, m goes from $-\infty$ to ∞.

$$Y(\Omega) = \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h[k]\, x[m]\, e^{-j\Omega(k+m)}$$

$$= \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} h[k]\, x[m]\, e^{-j\Omega k}\, e^{-j\Omega m}$$

$$= \sum_{m=-\infty}^{\infty} x[m]\, e^{-j\Omega m}\left[\sum_{k=-\infty}^{\infty} h[k]e^{-j\Omega k}\right]$$

$$= X(\Omega)\, H(\Omega)$$

2. **Periodicity property :**
   DTFT is always periodic in Ω with period 2π.

$$X(\Omega + 2\pi) = X(\Omega)$$

Proof : $X(\Omega + 2\pi) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(\Omega+2\pi)n}$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n}\, e^{-j2\pi n}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\Omega n} \times 1 = X(\Omega)$$

Que 2.28. **Determine the DTFT of the following signals :**

i. $x[n] = \left(\frac{1}{2}\right)^n u[n-2]$

ii. $x[n] = 2^n (u[n] - u[n-6])$

Answer

i. Given, $x[n] = \left(\frac{1}{2}\right)^n u[n-2]$

By using the time shifting property, we have

$$\left(\frac{1}{2}\right)^{n-2} u[n-2] \xleftrightarrow{\text{DTFT}} e^{-j2\Omega} \text{ DTFT of } \left[\left(\frac{1}{2}\right)^n u[n]\right] = \frac{e^{-j2\Omega}}{1 - \frac{1}{2}e^{-j\Omega}}$$

Therefore, the DTFT of the given signal is

$$\left(\frac{1}{2}\right)^n u[n-2] \xleftrightarrow{\text{DTFT}} \frac{\frac{1}{4}e^{-j2\Omega}}{1 - \frac{1}{2}e^{-j\Omega}}$$

ii. Given, $x[n] = 2^n[u[n] - u[n-6]]$

$$(\Omega) = \sum_{0}^{\infty} 2^n e^{-j\Omega n} - \sum_{6}^{\infty} 2^n e^{-j\Omega n}$$

$$= \frac{1}{1-2e^{-j\Omega}} - \frac{2^6 e^{-j6\Omega}}{1-2e^{-j\Omega}} = \frac{1-2^6\, e^{-j6\Omega}}{1-2e^{-j\Omega}}$$

Que 2.29. **State and prove the sampling theorem and also explain its applications.**

Answer

Sampling theorem :

A band limited signal of finite energy which has no frequency component higher than f_m can be completely described and recovered back if the sampling frequency is twice the highest frequency of the given signal.

Proof :

1. Consider an arbitrary signal $g(t)$ of finite energy, which is specified for all the time.

$$g(t) = x(t)\delta_T(t) \quad ...(2.29.1)$$

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) \quad ...(2.29.2)$$

2. $\delta_{T_s}(t)$ is called sampling function, T_s as the sampling period and $\omega_s = \frac{2\pi}{T_s}$ is the fundamental sampling frequency.

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3. This is also called radian frequency and is related to cyclic frequency as

$$\omega_s = \frac{2\pi}{T_s} = 2\pi f_s$$

where $f_s = \frac{1}{T_s}$ is the cyclic frequency.

4. The sampled signal $g(t)$ shown in Fig. 2.29.1(a) consists of impulses spaced every T_s seconds, which is the sampling interval. $\delta_T(t)$ can be expressed as a trigonometric Fourier series as

$$\delta_T(t) = \frac{1}{T_s}[1 + 2\cos\omega_s t + 2\cos 2\omega_s t + 2\cos 3\omega_s t + ...] \quad ...(2.29.3)$$

$$g(t) = x(t)\delta_T(t)$$

$$= \frac{1}{T_s}[x(t) + 2x(t)\cos\omega_s t + 2x(t)\cos 2\omega_s t + ...] \quad ...(2.29.4)$$

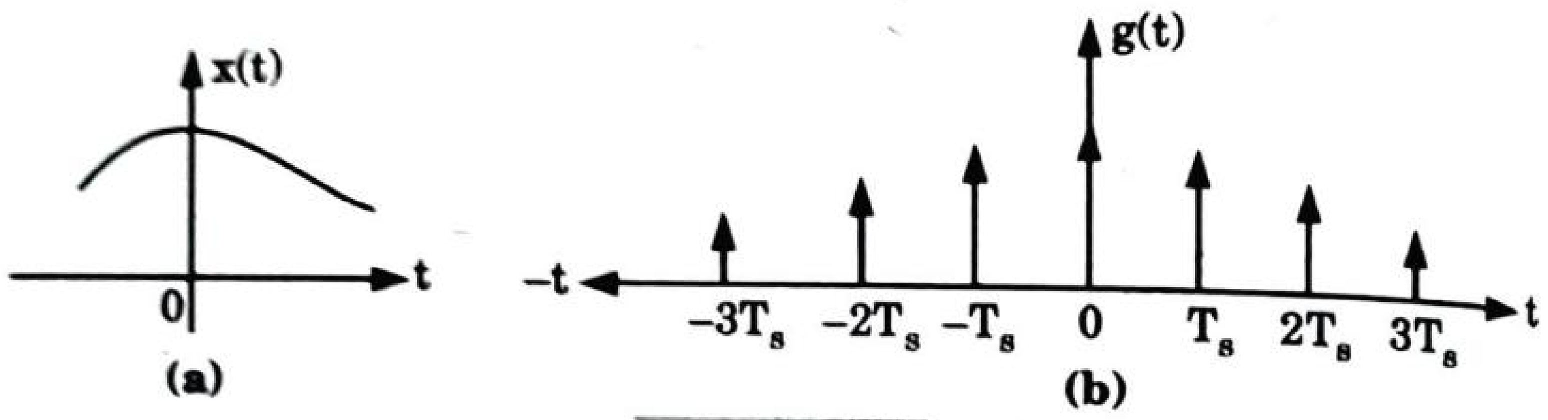


Fig. 2.29.1.

5. From Fourier transform of continuous time signal, the following equations are written :

$$x(t) \xleftrightarrow{\text{DTFT}} X(\omega)$$

$$2x(t)\cos\omega_s t \xleftrightarrow{\text{DTFT}} X(\omega - \omega_s) + X(\omega + \omega_s)$$

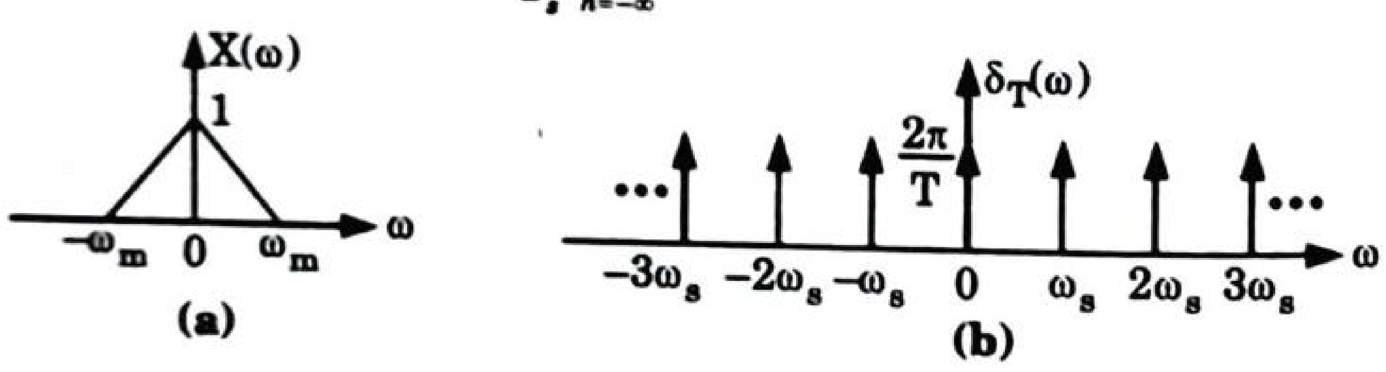
$$2x(t)\cos 2\omega_s t \xleftrightarrow{\text{DTFT}} X(\omega - 2\omega_s) + X(\omega + 2\omega_s)$$

$$g(t) \xleftrightarrow{\text{DTFT}} G(\omega) \quad ...(2.29.5)$$

6. Substituting eq. (2.29.5) in eq. (2.29.4), we get

$$G(\omega) = \frac{1}{T_s}[X(\omega) + X(\omega - \omega_s) + X(\omega + \omega_s) + X(\omega - 2\omega_s) + X(\omega - 2\omega_s) + ...] \quad ...(2.29.6)$$

$$G(\omega) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$



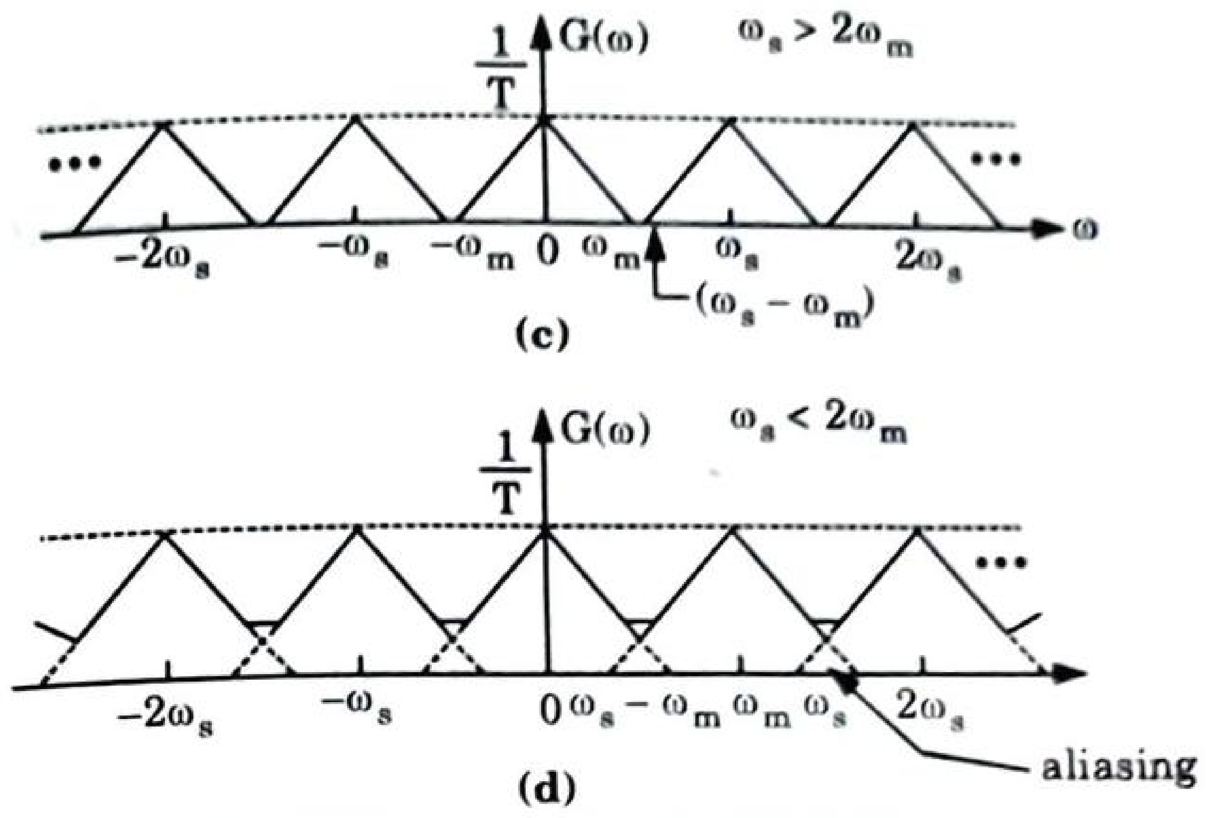


Fig. 2.29.2. Continuous signal, impulse train sampled signal spectrum.

7. To reconstruct the continuous time signal $x(t)$ from sampled signal $g(t)$, we should be able to recover $X(\omega)$ from $G(\omega)$. This recovery is possible if there in no overlap between successive $G(\omega)$.

8. From Fig. 2.29.2(c), this is possible if $\omega_s > 2\omega_n$ or

$$f_s > 2f_m \quad ...(2.29.7)$$

9. The minimum sampling rate $f_s = 2f_m$ is called the Nyquist rate of $x(t)$ and the corresponding time interval $T_s = \frac{1}{f_s} = \frac{1}{2f_m}$ is called Nyquist interval of $x(t)$.

10. Samples of a signal taken at its Nyquist rate are the Nyquist samples.

Applications :

The sampling theorem is used in analysis, processing and transmission of signals.

VERY IMPORTANT QUESTIONS

Following questions are very important. These questions may be asked in your SESSIONALS as well as UNIVERSITY EXAMINATION.

Q. 1. Briefly explain the three forms of Fourier series.

Ans. Refer Q. 2.1, Unit-2.

Q. 2. Give the derivation of exponential form of Fourier series.

Ans. Refer Q. 2.2, Unit-2.

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3 UNIT

Laplace Transform Analysis

PART-1

Review of Laplace Transform, Properties of Laplace Transform, Initial and Final Value Theorems.

CONCEPT OUTLINE : PART-1

- **Bilateral Laplace Transform :**

$$F(s) = \int_{-\infty}^{\infty} f(t)\, e^{-st}\, dt$$

$$f(t) = \frac{1}{j2\pi}\int_{\sigma-j\omega}^{\sigma+j\omega} F(s)\, e^{st}\, ds$$

- **Unilateral Laplace Transform :**

$$F(s) = \int_{0}^{\infty} f(t)\, e^{-st}\, dt$$

$$f(t) = \frac{1}{j2\pi}\int_{\sigma-j\omega}^{\sigma+j\omega} F(s)\, e^{st}\, ds$$

- **Initial Value Theorem :**

$$f(0^-) = \lim_{t\to 0} f(t) = \lim_{s\to\infty} [sF(s)]$$

- **Final Value Theorem :**

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} [sF(s)]$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.1. **What do you understand by "Laplace Transform" ? Also mention its advantages and disadvantages. Enlist applications of Laplace transform.**

Answer

Laplace Transform :

1. Laplace transform of a time function $f(t)$ is define as :

$$F(s) = L[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-st}dt \quad ...(3.1.1)$$

where, s is a complex variable and is equal to $\sigma + j\omega$.

2. The Laplace Transform as define in eq. (3.1.1) with $-\infty$ as the lower limit for the integral is called the 'two-sided' or 'bilateral' Laplace transform.

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3. If the lower limit is changed to '0', we get the 'one-sided' or 'unilateral' Laplace transform.
4. Hence, we define the one-sided Laplace transform as :

$$F(s) = L[f(t)] = \int_0^\infty f(t)e^{-st}dt$$

The two-sided Laplace transform is useful to get a better insight into the system.

Advantages :

1. Laplace transform represents continuous time signals in terms of complex exponentials, *i.e.*, e^{-st}. Hence Laplace transform can be used to analyze the signals or functions which are not absolutely integrable.
2. Continuous time system are also analyzed more effectively using Laplace transforms. Laplace transform can be applied to the analysis of unstable systems also.

Disadvantages :

1. The integral representation of Laplace domain is complicated.
2. Time functions varying in negative values of times, prior to zero, may be by-passed by introducing some artifice to simulate or approximate effects prior to zero.
3. Unsuitable for the purpose of data processing in random vibrations.

Applications of Laplace Transform :

1. It is a technique mainly utilized in engineering purposes for system modeling in which a large differential equation must be solved.
2. It can also be used to solve differential equations and is used extensively in electrical engineering.
3. It is used in electrical circuits for the analysis of linear time-invariant systems.

Que 3.2. **Define Laplace transform and write its properties.**

AKTU 2014-15, Marks 3.5

Answer

Laplace transform : Refer Q. 3.1, Page 3–2D, Unit-3.

Properties of Laplace transform :

i. **Multiplication by a Constant :**

Let $\qquad F(s) \rightarrow$ Laplace Transform of $f(t)$

$\qquad k \rightarrow$ multiplication factor

$$L[k\, f(t)] = kF(s)$$

ii. **Sum and Difference :**

Let $\qquad F_1(s) \rightarrow$ Laplace transform of $f_1(t)$

$\qquad F_2(s) \rightarrow$ Laplace transform of $f_2(t)$

$$L[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$$

iii. **Time Shifting :**

If $\qquad F(s) \rightarrow$ Laplace transform of $f(t)$

Then for $\qquad t_0 \geq 0$

$$L[f(t - t_0)] = F(s)\ e^{-st_0}$$

iv. **Frequency Shifting :**

If $\qquad F(s) \rightarrow$ Laplace transform of $f(t)$

Then $L[f(t)\ e^{s_0 t}] = F(s - s_0)$

v. **Time Differentiation Property :**

If $\qquad F(s) \rightarrow$ Laplace transform of $f(t)$

Then $$L\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0^-)$$

and $$L\left\{\frac{d^n f(t)}{dt^n}\right\} = s^n F(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - .... - f^{n-1}(0^-)$$

vi. **Time Integration Property :**

If $\qquad F(s) \rightarrow$ Laplace transform of $f(t)$

$$L\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{F(s)}{s}$$

and $$\int_{-\infty}^t f(\tau)\, d\tau = \frac{F(s)}{s} + \frac{\int_{-\infty}^0 f(\tau)\, d\tau}{s}$$

vii. **Frequency Differentiation :**

$$L\{t\, f(t)\} = -\frac{d\, F(s)}{s}$$

viii. **Frequency Integration :**

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\, ds$$

ix. **Scaling :**

$$L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$$

Que 3.3. **State and prove initial and final value theorems of Laplace transform.**

AKTU 2015-16, Marks 10

Answer

i. **Initial value theorem :** The initial value theorem states that if $F(s)$ is the Laplace transform of $f(t)$ then,

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$$f(0^+) = \lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

Proof :

1. Consider the Laplace transform of the real differentiation,

$$L\left[\frac{df(t)}{dt}\right] = s\,F(s) - f(0^-)$$

2. Taking limit as $s \to \infty$ on both the sides

$$\lim_{s \to \infty} L\left[\frac{df(t)}{dt}\right] = \lim_{s \to \infty}\left[s\,F(s) - F(0^-)\right] \quad ...(3.3.1)$$

3. Consider the left hand side of eq. (3.3.1)

$$\lim_{s \to \infty} L\left[\frac{df(t)}{dt}\right] = \lim_{s \to \infty} \int_{0^-}^{\infty} e^{-st}\left[\frac{df(t)}{dt}\right]dt \quad \text{(by definition of Laplace)}$$

$$\therefore \quad \lim_{s \to \infty} L\left[\frac{df(t)}{dt}\right] = 0 \quad \text{as } \lim_{s \to \infty} e^{-st} \text{ is zero} \quad ...(3.3.2)$$

4. Putting eq. (3.3.2) in eq. (3.3.1), we get

$$0 = \lim_{s \to \infty}\left[s\,F(s) - f(0^-)\right]$$

i.e. $\quad f(0^-) = \lim_{s \to \infty} s\,F(s)$

But as the function $f(t)$ is continuous, $f(0^-) = f(0^+)$ *i.e.* the initial value of $f(t)$.

$$\therefore \quad f(0^+) = \lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s)$$

ii. **Final value Theorem :** The final value theorem states that if $F(s)$ is the Laplace transform of $f(t)$ then the final value of the function $f(t)$ is given by,

$$f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Proof :

1. Consider the Laplace transform of the real differentiation,

$$L\left[\frac{df(t)}{dt}\right] = s\,F(s) - f(0^-)$$

2. Taking limit as $s \to 0$ on both the sides

$$\lim_{s \to 0} L\left[\frac{df(t)}{dt}\right] = \lim_{s \to 0}\left[s\,F(s) - f(0^-)\right] \quad ...(3.3.3)$$

3. Consider the left hand side of eq. (3.3.3),

$$\lim_{s \to 0} L\left[\frac{df(t)}{dt}\right] = \lim_{s \to 0} \int_{0^-}^{\infty} \frac{df(t)}{dt} e^{-st}\,.\,dt = \int_{0^-}^{\infty} \frac{df(t)}{dt}.dt \text{ as } \lim_{s \to 0} e^{-st} = 1$$

$$= [f(t)]_{0^-}^{\infty} = \lim_{t \to \infty} f(t) - f(0^-) \quad ...(3.3.4)$$

4. Putting eq. (3.3.4) in eq. (3.3.3), we get

$$\lim_{t \to \infty} f(t) - f(0^-) = \lim_{s \to 0} s\,F(s) - f(0^-)$$

$$\therefore \quad \lim_{t \to \infty} f(t) = \lim_{s \to 0} s\,F(s)$$

Que 3.4. **Write the Laplace transforms of :**

i. **Unit impulse**
ii. **Unit step**
iii. **Unit ramp and**
iv. **Parabolic functions.**
v. **Find the Laplace transform of the truncated ramp function as shown in Fig. 3.4.1.**

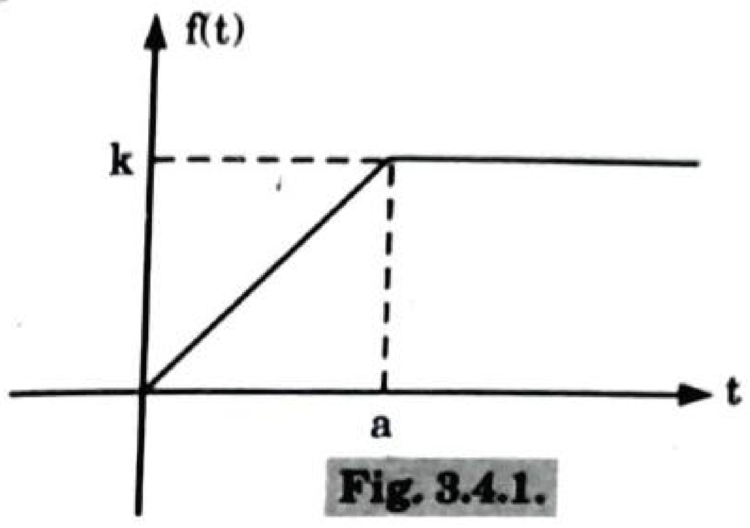

Fig. 3.4.1.

Answer

i. **Unit impulse : $\delta(t)$**

$$f(t) = \delta(t)$$

$$L\{f(t)\} = F(s) = \int_0^{\infty} \delta(t)\, e^{-st}\, dt$$

$$F(s) = e^{-st}\big|_{t=0} = 1 \quad [\because \delta(t) = 1 \text{ only } t = 0]$$

ii. **Unit Step : $u(t)$**

$$f(t) = u(t)$$

$$L\{f(t)\} = F(s) = \int_0^{\infty} u(t)\, e^{-st}\, dt$$

$$= \int_0^{\infty} e^{-st}\, dt \quad [\because u(t) = 1 \text{ for } t > 0]$$

$$= \left.\frac{e^{-st}}{-s}\right|_{t=0}^{t=\infty}$$

$$F(s) = \frac{1}{s}$$

iii. **Unit ramp : $t\,u(t)$**

$$f(t) = r(t) = t\,u(t)$$

$$F(s) = L\{t\,u(t)\} = \int_0^{\infty} t\, e^{-st}\, dt = \left[t\left.\frac{e^{-st}}{-s}\right|_0^{\infty} - \int_0^{\infty} 1.\frac{e^{-st}}{-s}\, dt\right]$$

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$$= [0-0] + \frac{1}{s}\int_0^\infty e^{-st}\,dt = -\frac{1}{s^2}e^{-st}\Big|_0^\infty = \frac{1}{s^2}$$

iv. **Parabolic function :** $kt^2 u(t)$

$$f(t) = kt^2\,u(t)$$

$$F(s) = L\{kt^2\,u(t)\} = \int_0^\infty k\,t^2 e^{-st}\,dt$$

$$= k\left[t^2\frac{e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty 2t\frac{e^{-st}}{-s}\,dt\right]$$

$$= k[0-0] + \frac{2k}{s}\int_0^\infty t\,e^{-st}\,dt$$

$$= \frac{2k}{s}\left[\frac{t\,e^{-st}}{-s}\Big|_0^\infty - \int_0^\infty 1\frac{e^{-st}}{-s}\,dt\right]$$

$$= \frac{2k}{s}[0-0] + \frac{2k}{s^2}\int_0^\infty e^{-st}\,dt$$

$$= -\frac{2k}{s^2}\frac{e^{-st}}{s}\Big|_0^\infty = \frac{2k}{s^3}$$

v. Given Fig. 3.4.1 can be constructed using following functions :

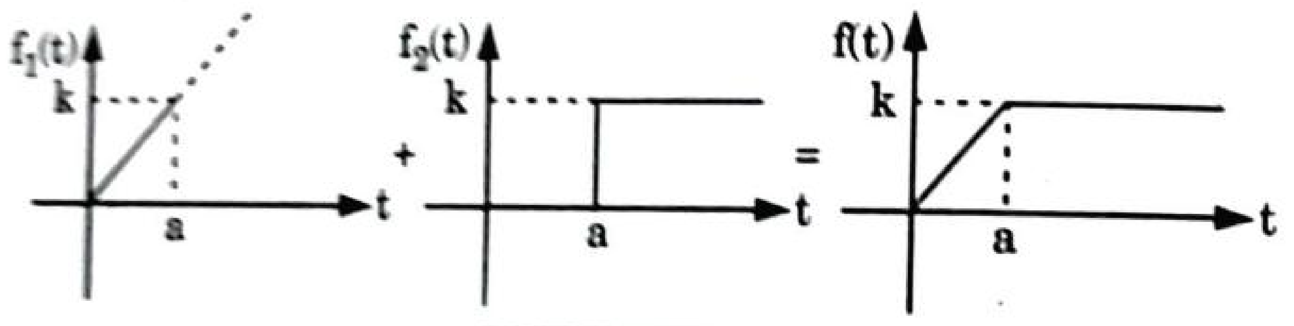

Fig. 3.4.2.

$$f_1(t) = \frac{kt}{a}[u(t) - u(t-a)]$$

$$f_2(t) = k\,u(t-a)$$

$$f(t) = f_1(t) + f_2(t)$$

$$= \frac{kt}{a}[u(t) - u(t-a)] + k\,u(t-a)$$

$$= \frac{kt}{a}u(t) + \frac{k}{a}(a-t)\,u(t-a)$$

$$= \frac{k}{a}t\,u(t) - \frac{k}{a}(t-a)\,u(t-a)$$

Laplace transform,

$$F(s) = L\{f(t)\} = \frac{k}{a}\cdot\frac{1}{s^2} - \frac{k}{a}\frac{e^{-as}}{s^2} = \frac{k}{as^2}\left[1 - e^{-as}\right]$$

Que 3.5. **Find Laplace Transform of following functions :**

i. e^{at}

ii. $u(t)$

iii. $\sin \omega t$

iv. $\cos \omega t$

v. $\sinh at$

vi. $\cosh at$

vii. $e^{-at}\sin \omega t$

viii. $e^{-at}\cos \omega t$

ix. $e^{-at}\cosh bt$

x. $e^{-at}\sinh bt$

xi. t^n

xii. $Ae^{-at}\sin(bt+\theta)$

OR

Calculate the Laplace transform for the function $f(t) = e^{-at}\sinh bt$.

AKTU 2016-17, Marks 10

Answer

i.
$$L\{e^{at}\} = \int_0^\infty e^{at}e^{-st}dt = \int_0^\infty e^{-(s-a)t}dt = \frac{1}{s-a}$$

ii.
$$L\{u(t)\} = \int_0^\infty u(t)\,e^{-st}dt = \int_0^\infty e^{-st}dt = \frac{1}{s}$$

iii.
$$L\{\sin \omega t\} = \frac{1}{2j}L\left\{e^{j\omega t} - e^{-j\omega t}\right\}$$
$$= \frac{1}{2j}\left(\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right) = \frac{\omega}{s^2+\omega^2}$$

iv.
$$L\{\cos \omega t\} = \frac{1}{2}L\left\{e^{j\omega t} + e^{-j\omega t}\right\}$$
$$= \frac{1}{2}\left(\frac{1}{s-j\omega} + \frac{1}{s+j\omega}\right) = \frac{s}{s^2+\omega^2}$$

v.
$$L\{\sinh at\} = \frac{1}{2}L\left[e^{at} - e^{-at}\right] = \frac{1}{2}\left[L\{e^{at}\} - L\{e^{-at}\}\right]$$
$$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = \frac{a}{s^2-a^2}$$

vi.
$$L\{\cosh at\} = \frac{1}{2}L\left[e^{at} + e^{-at}\right] = \frac{1}{2}\left[L\left\{e^{at}\right\} + L\left\{e^{-at}\right\}\right]$$
$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right] = \frac{s}{s^2-a^2}$$

vii.
$$L\left\{e^{-at}\sin \omega t\right\} = L\left[e^{-at}\left\{\frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})\right\}\right]$$
$$= L\left[\frac{1}{2j}\left[e^{-(a-j\omega)t} - e^{-(a+j\omega)t}\right]\right]$$

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$$= \frac{1}{2j}\left[\frac{1}{(s+a)-j\omega} - \frac{1}{(s+a)+j\omega}\right] = \frac{\omega}{(s+a)^2+\omega^2}$$

viii. $$L\{e^{-at}\cos\omega t\} = \frac{1}{2}L\left[\{e^{-(a-j\omega)t} + e^{-(a+j\omega)t}\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s+a)-j\omega} + \frac{1}{(s+a)+j\omega}\right] = \frac{s+a}{(s+a)^2+\omega^2}$$

ix. $$L\{e^{-at}\cosh bt\} = L\left[e^{-at}\left(\frac{e^{bt}+e^{-bt}}{2}\right)\right]$$

$$= \frac{1}{2}L\left[e^{(b-a)t} + e^{-(b+a)t}\right] = \frac{1}{2}L\left[e^{-(a-b)t} + e^{-(a+b)t}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+(a-b)} + \frac{1}{s+(a+b)}\right] = \frac{s+a}{(s+a)^2-b^2}$$

x. $$L\{e^{-at}\sinh bt\} = L\left[e^{-at}\left(\frac{e^{bt}-e^{-bt}}{2}\right)\right] = \frac{1}{2}L\left[e^{-(a-b)t} - e^{-(a+b)t}\right]$$

$$= \frac{1}{2}\left[\frac{1}{s+(a-b)} - \frac{1}{s+(a+b)}\right] = \frac{b}{(s+a)^2-b^2}$$

xi. $$L[t^n] = \int_0^\infty (t^n)\, e^{-st}\, dt$$

Integrating by parts

Let
$$u = t^n$$
$$dv = e^{-st}\, dt$$
$$du = nt^{n-1}$$
$$v = \int_0^\infty e^{-st}\, dt = \frac{-e^{-st}}{s}$$

$$L[t^n] = \int_0^\infty u\, dv = uv\Big|_0^\infty - \int_0^\infty v\, du$$

$$= -\frac{t^n}{s}\left[e^{-st}\right]_0^\infty + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt$$

$$= \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt = \frac{n}{s}L\left[t^{n-1}\right] = \frac{n}{s}\left(\frac{n-1}{s}\right)L\left[t^{n-2}\right]$$

$$= \frac{n}{s}\times\frac{(n-1)}{s}\times\frac{(n-2)}{s}\ldots\ldots\frac{1}{s}L\left[t^0\right]$$

$$= \frac{\lfloor n}{s^n}L[u(t)] = \frac{\lfloor n}{s^{n+1}}$$

xii. $L[Ae^{-at}\sin(bt+\theta)]$

As, $$L[A\sin(bt+\theta)] = A\,L\left[\sin bt\cos\theta + \sin\theta\cos bt\right]$$

$$= A\left[\frac{b\cos\theta}{s^2+b^2} + \frac{s\sin\theta}{s^2+b^2}\right]$$

So, $$L\left[Ae^{-at}\sin(bt+\theta)\right] = A\frac{b\cos\theta}{(s+a)^2+b^2} + A\frac{(s+a)\sin\theta}{(s+a)^2+b^2}$$

$$= A\left[\frac{b\cos\theta + (s+a)\sin\theta}{(s+a)^2+b^2}\right]$$

Que 3.6. **Find $L[t^2 \sin \omega t]$ using the following relation :**

$$L[t\,f(t)] = -\frac{d}{ds}F(s).$$

AKTU 2011-12, Marks 05

Answer

1. We know, $L\{tf(t)\} = \frac{-d}{ds}F(s)$
2. $$L\{t^2\sin\omega t\} = (-1)^2\frac{d^2}{ds^2}L\{\sin\omega t\}$$

$$= \frac{d^2}{ds^2}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{2\omega\,(3s^2-\omega^2)}{(s^2+\omega^2)^3}$$

Que 3.7. **Find the Laplace transforms of :**

i. $\sin\omega t$

ii. $e^{-at}\sin\omega t$

iii. te^{-at}

AKTU 2013-14, Marks 10

Answer

i. $\sin\omega t$ **:** Refer Q. 3.5, Page 3–8D, Unit-3.

ii. $e^{-at}\sin\omega t$ **:** Refer Q. 3.5, Page 3–8D, Unit-3.

iii. te^{-at}**:**

$$L\{t\} = \frac{1}{s^2}$$

$$L\{te^{-at}\} = \frac{1}{(s+a)^2} \qquad \text{(using shifting property)}$$

Que 3.8. **State and prove Initial value and Final value theorems. Find the final value of the function $f(t) = 2 + e^{-3t}\cos 2t$.**

AKTU 2013-14, Marks 10

Answer

Initial and final value theorems : Refer Q. 3.3, Page 3–4D, Unit-3.

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Numerical :

Given : $f(t) = 2 + e^{-3t} \cos 2t$

To find : Laplace transform, $F(s)$

1. Using Laplace transform,

$$F(s) = \frac{2}{s} + \frac{s+3}{(s+3)^2 + 4}$$

2. Using final value theorem, we get

$$f(\infty) = \lim_{s \to 0} sF(s) = \lim_{s \to 0} s\left[\frac{2}{s} + \frac{s+3}{(s+3)^2 + 4}\right]$$

$$= \lim_{s \to 0} s \cdot \frac{2}{s} + \lim_{s \to 0} s \cdot \frac{s+3}{(s+3)^2 + 4}$$

$$= 2 + 0 = 2$$

Que 3.9. **Find the Laplace transform of the following signals :**

i. $x(t) = te^{-t} u(t)$

ii. $x(t) = te^{-2t} \sin 2t\, u(t)$

AKTU 2016-17, Marks 10

Answer

i. 1. $x(t) = te^{-t} u(t)$

2. $\because$ $L[u(t)] = \frac{1}{s}$

3. $L[e^{-t} u(t)] = \frac{1}{s+1}$

4. $L[te^{-t} u(t)] = (-1)^1 \frac{d}{ds}\left(\frac{1}{s+1}\right)$

5. $\therefore$ $L[te^{-t} u(t)] = \frac{1}{(s+1)^2}$

ii. 1. $x(t) = te^{-2t} \sin 2t\, u(t)$

2. $\because$ $L[\sin 2t\, u(t)] = \frac{2}{s^2 + 4}$

3. $L[e^{-2t} \sin 2t\, u(t)] = \frac{2}{(s+2)^2 + 4}$

4. $L[te^{-2t} \sin 2t\, u(t)] = (-1)^1 \frac{d}{ds}\left(\frac{2}{(s+2)^2 + 4}\right)$

$$= -1\left[\frac{-2(2s+4)}{[(s+2)^2 + 4]^2}\right] = \frac{2(2s+4)}{(s^2 + 4 + 4s + 4)^2}$$

5. $\therefore$ $L[te^{-2t} \sin 2t\, u(t)] = \frac{4(s+2)}{(s^2 + 4s + 8)^2}$

PART-2

Inverse Laplace Transform, Convolution Theorem, Impulse Response.

CONCEPT OUTLINE : PART-2

- **Inverse Laplace transform :** Given a function $G(s)$, a function $g(t)$ such that $L[g(t)] = G(s)$ is called Inverse Laplace transform of $G(s)$.
  In this event, we write
  $$g(t) = L^{-1}[G(s)]$$
  For example $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$
  and $L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$
- **Convolution Theorem :**
  Convolution theorem states that if there are two functions given by $f_1(t)$ and $f_2(t)$, which are zero for $t < 0$ and their Laplace transforms are
  $$L[f_1(t)] = F_1(s) \quad \text{and} \quad L[f_2(t)] = F_2(s)$$
  then $L^{-1}[F_1(s) . F_2(s)] = f_1(t) * f_2(t)$
- **Impulse response :** The impulse response of LTI continuous system, is defined as the response of the system to a unit impulse as input when the system is in ground state.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.10. **What is inverse Laplace transform ?**

Answer

It is used to convert frequency-domain signal $F(s)$ to the time-domain signal $f(t)$. It is given as,

$$f(t) = L^{-1}[F(s)]$$

$$f(t) = \frac{1}{2\pi} \int_{\sigma - j\omega}^{\sigma + j\omega} F(s)e^{st} ds$$

Que 3.11. **State and prove convolution theorem.**

Answer

Convolution theorem states that if there are two functions given by $f_1(t)$ and $f_2(t)$, which are zero for $t < 0$ and their Laplace transforms are

$$L\{f_1(t)\} = F_1(s) \quad \text{and} \quad L\{f_2(t)\} = F_2(s)$$

then,

$$L^{-1}\{F_1(s).\, F_2(s)\} = f_1(t) * f_2(t)$$

Proof :

1. Let $F(s) = L\{f(t)\} = \int_0^{+\infty} e^{-st} f(t)dt$ [Assuming $f_1(t) = f(t)$]

and $G(s) = L\{g(t)\} = \int_0^{+\infty} e^{-st} g(t)\, dt$ [Assuming $f_2(t) = g(t)$]

be the two Laplace transforms.

2. Lets convolution of $f(t)$, $g(t)$ is denoted as :

$$f(t) * g(t) = \int_0^{+\infty} f(\tau)\, g(t-\tau)\, d\tau$$

3. Thus the Laplace transform for the convolution would be,

$$L\{f(t) * g(t)\} = \int_0^{+\infty} e^{-st} \left(\int_0^{+\infty} f(\tau)\, g(t-\tau)\, d\tau \right) dt$$

$$= \int_0^{+\infty}\int_0^{+\infty} e^{-st} f(\tau)\, g(t-\tau)\, d\tau\, dt$$

4. Let $t - \tau = p$, then $dt = dp$

5. $\therefore$ $L\{f(t) * g(t)\} = \int_0^{+\infty} f(\tau) \int_0^{+\infty} g(p)\, e^{-s(\tau+p)}\, d\tau\, dp$

$$= \int_0^{+\infty} f(\tau)\, e^{-s\tau}\, d\tau \int_0^{+\infty} g(p)\, e^{-sp}\, dp = F(s).\, G(s)$$

6. $\therefore$ $L\{f(t) * g(t)\} = L\{f(t)\}.\, L\{g(t)\} = F(s)\, G(s)$

Que 3.12. **Find Laplace inverse of the function :** $\left(\dfrac{s+4}{2s^2+5s+3}\right)$

AKTU 2011-12, Marks 06

Answer

1. $F(s) = \dfrac{s+4}{2s^2+5s+3} = \dfrac{s+4}{(s+1)(2s+3)} = \dfrac{A}{s+1} + \dfrac{B}{(2s+3)}$

2. $\therefore$ $A = (s+1)F(s)\Big|_{s=-1} = \dfrac{s+4}{2s+3}\Big|_{s=-1} = 3$

$$B = (2s+3)F(s)\Big|_{s=\frac{-3}{2}} = \frac{s+4}{s+1}\Big|_{s=\frac{-3}{2}} = -5$$

3. $F(s) = \dfrac{3}{s+1} - \dfrac{5}{2s+3}$

4. Taking inverse Laplace transform on both sides, we get

$$f(t) = \left(3e^{-t} - 5/2e^{\frac{-3}{2}t}\right)u(t).$$

Que 3.13. **State and prove the convolution theorem. Find the inverse Laplace transform of the following function using the convolution theorem.**

$$F(s) = \frac{1}{s(s^2+2s+4)}$$

AKTU 2012-13, Marks 10

Answer

Convolution theorem : Refer Q. 3.11, Page 3–12D, Unit-3.

Numerical :

1. $F(s) = \dfrac{1}{s(s^2+2s+4)}$

2. $F_1(s) = \dfrac{1}{s}$

$f_1(t) = u(t)$

3. $F_2(s) = \dfrac{1}{s^2+2s+4} = \dfrac{1}{(s-\alpha)(s-\beta)}$

where, $\alpha = -1 + \sqrt{3}i$

$$\beta = -1 - \sqrt{3}i = \frac{1}{\alpha-\beta}\left[\frac{1}{s-\alpha} - \frac{1}{s-\beta}\right]$$

4. $f_2(t) = \dfrac{1}{2\sqrt{3}i}\left[e^{(-1+\sqrt{3}i)t} - e^{-(1+\sqrt{3}i)t}\right]$

5. $f(t) = \int_0^t f_1(\tau)\, f_2(t-\tau)d\tau = \int_0^t u(t)\dfrac{1}{2\sqrt{3}i}\left[e^{\alpha(t-\tau)} - e^{\beta(t-\tau)}\right]d\tau$

$$= \frac{1}{2\sqrt{3}i} \times e^{\alpha t}\int_0^t e^{-\alpha\tau}\, d\tau - \frac{1}{2\sqrt{3}i} \times e^{\beta t}\int_0^t e^{-\beta\tau}d\tau$$

$$= \frac{e^{\alpha t}}{2\sqrt{3}i}\left[\frac{e^{-\alpha\tau}}{-\alpha}\right]_0^t - \frac{e^{\beta t}}{2\sqrt{3}i}\left[\frac{e^{-\beta\tau}}{-\beta}\right]_0^t$$

$$= \frac{e^{\alpha t}}{2\sqrt{3}i}\left[\frac{e^{-\alpha t}-1}{-\alpha}\right] - \frac{e^{\beta t}}{2\sqrt{3}i}\left[\frac{e^{-\beta t}-1}{-\beta}\right]$$

$$= \left[\frac{e^{\alpha t}-1}{2\sqrt{3}\alpha i}\right] - \left[\frac{e^{\beta t}-1}{2\sqrt{3}\beta i}\right] = \frac{1}{2\sqrt{3}i}\left[\frac{\beta e^{\alpha t} - \beta - \alpha e^{-\beta t} + \alpha}{\alpha\beta}\right]$$

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$$= \frac{(-1-\sqrt{3}i)\,e^{(-1+\sqrt{3}i)t} - (-1+\sqrt{3}i)\,e^{(-1-\sqrt{3}i)t} + 2\sqrt{3}i}{8\sqrt{3}i}$$

Que 3.14. **State convolution property of Laplace transform. Also find the inverse Laplace transform of the function using it.**

$$X(s) = \frac{1}{s^2\,(s+1)}$$

AKTU 2014-15, Marks 06

Answer

Convolution theorem : Refer Q. 3.11, Page 3–12D, Unit-3.

Numerical :

1. Given, $X(s) = \dfrac{1}{s^2\,(s+1)}$
2. Let $X(s) = X_1(s)\,X_2(s)$

   where $X_1(s) = \dfrac{1}{s^2}$ and $X_2(s) = \dfrac{1}{s+1}$
3. $$x_1(t) = L^{-1}\left[\frac{1}{s^2}\right] = tu(t)$$
   $$x_2(t) = L^{-1}\left[\frac{1}{s+1}\right] = e^{-t}\,u(t)$$
4. Using convolution property of Laplace transform, we have

$$L^{-1}[X(s)] = L^{-1}[X_1(s)\,X_2(s)] = L^{-1}\left[\frac{1}{s^2\,(s+1)}\right] = x_1(t) * x_2(t)$$

$$= \int_0^t x_1(\tau)\,x_2(t-\tau)\,d\tau = \int_0^t \tau\, e^{-(t-\tau)}\,d\tau$$

$$= e^{-t}\int_0^t \tau e^{\tau}\,d\tau = e^{-t}\left[\tau e^{\tau}\Big|_0^t - \int_0^t e^{\tau}d\tau\right]$$

$$= e^{-t}\left[te^t - e^{\tau}\Big|_0^t\right] = e^{-t}\left[te^t - e^t + 1\right] = t - 1 + e^{-t}$$

5. $$L^{-1}\left[\frac{1}{s^2\,(s+1)}\right] = [(t-1) + e^{-t}]\,u(t)$$

Que 3.15. **Find the impulse response and step response of the following system.**

$$H(s) = 5/(s^2 + 5s + 6)$$

AKTU 2016-17, Marks 10

Answer

Given : $H(s) = 5/(s^2 + 5s + 6)$

To find :

i. Impulse response.

ii. Step response.

i. **For impulse response :**

1. $$x(t) = \delta(t)$$
   $$X(s) = 1$$
   $$Y(s) = H(s)\,X(s) = \frac{5}{(s+2)(s+3)}$$
2. Using partial fraction,
   $$Y(s) = 5\left[\frac{1}{s+2} - \frac{1}{s+3}\right]$$
3. Taking inverse Laplace transform,
   $$y(t) = 5\,e^{-2t}\,u(t) - 5\,e^{-3t}\,u(t)$$

ii. **For step response :**

1. Input signal, $x(t) = u(t)$

   $\therefore$ $X(s) = \dfrac{1}{s}$
2. Then, $Y(s) = H(s)\,X(s)$
   $$Y(s) = \frac{5}{s(s^2+5s+6)} = \frac{5}{s(s+2)(s+3)}$$
3. Using partial fraction,
   $$Y(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$$
   $$A = sY(s)\big|_{s=0} = \frac{5}{(s+2)(s+3)}\bigg|_{s=0} = \frac{5}{6}$$
   $$B = (s+2)\,Y(s)\big|_{s=-2} = \frac{5}{s(s+3)}\bigg|_{s=-2} = -\frac{5}{2}$$
   $$C = (s+3)\,Y(s)\big|_{s=-3} = \frac{5}{s(s+2)}\bigg|_{s=-3} = \frac{5}{3}$$
   $$Y(s) = \frac{5}{6}\frac{1}{s} - \frac{5}{2}\frac{1}{(s+2)} + \frac{5}{3}\frac{1}{(s+3)}$$
4. Taking inverse Laplace transform,
   $$y(t) = \frac{5}{6}u(t) - \frac{5}{2}e^{-2t}u(t) + \frac{5}{3}e^{-3t}u(t)$$

PART-3

Application of Laplace Transform to Analysis of Networks, Waveform Synthesis and Laplace Transform of Complex Waveforms.

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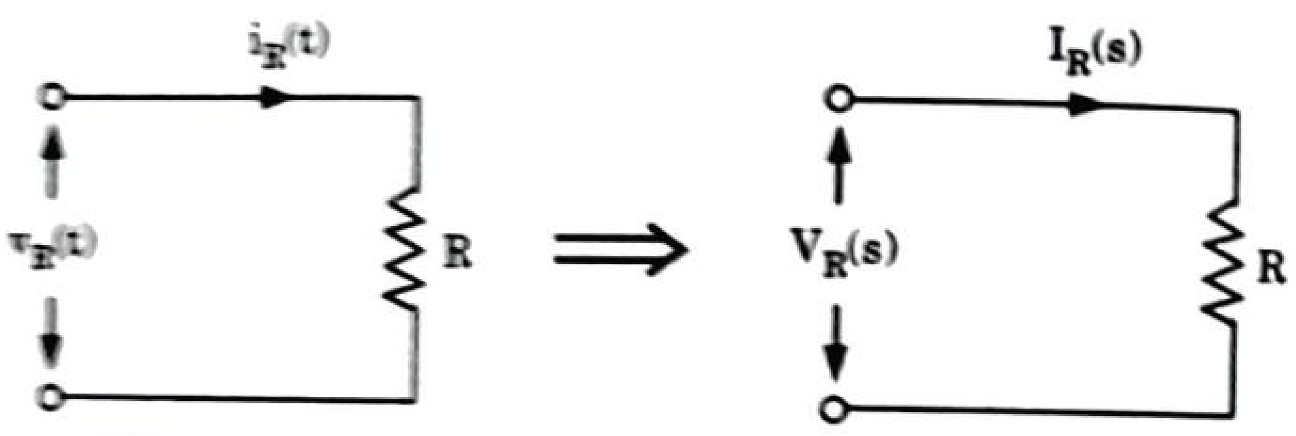
CONCEPT OUTLINE : PART-3

Circuit Components Representation in Laplace Form :

Voltage Source :

Current Source :

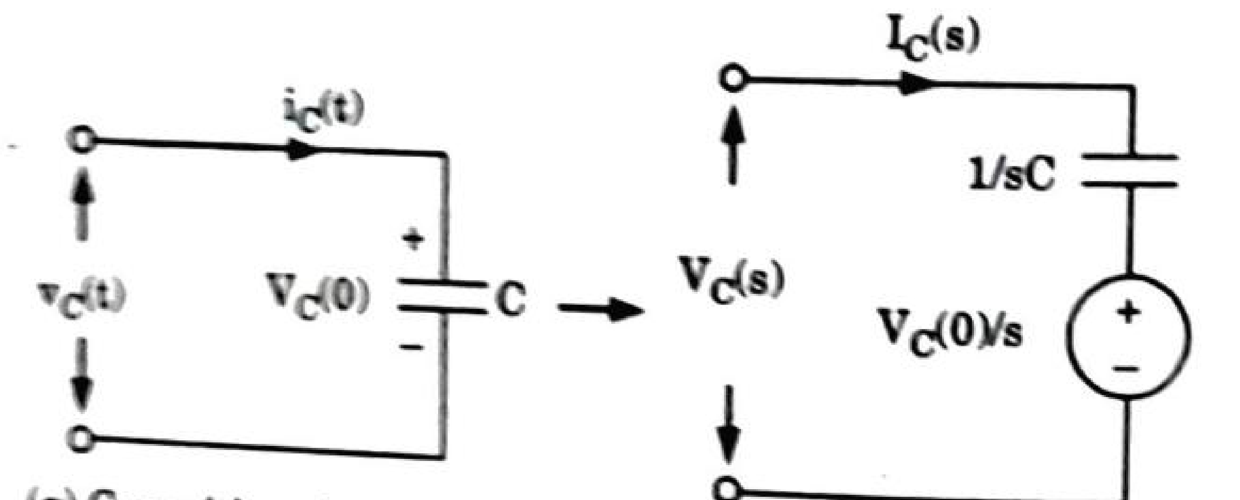
Resistor :



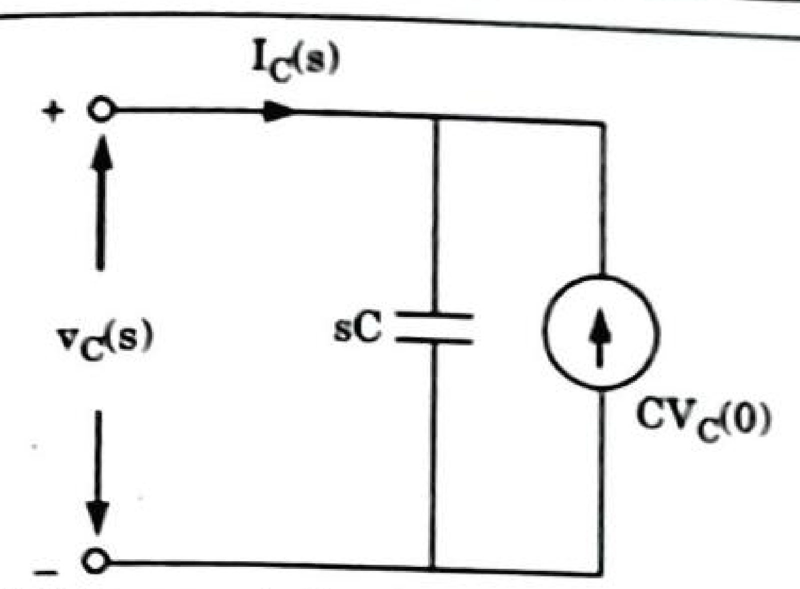
KVL : $V_R(t) = Ri_R(t) \Rightarrow V_R(s) = RI_R(s)$

KCL : $i_R(t) = (1/R)\, V_R(t) \Rightarrow I_R(s) = (1/R)\, V_R(s)$

Capacitor :



(*a*) Capacitive element (*b*) Representation in series form



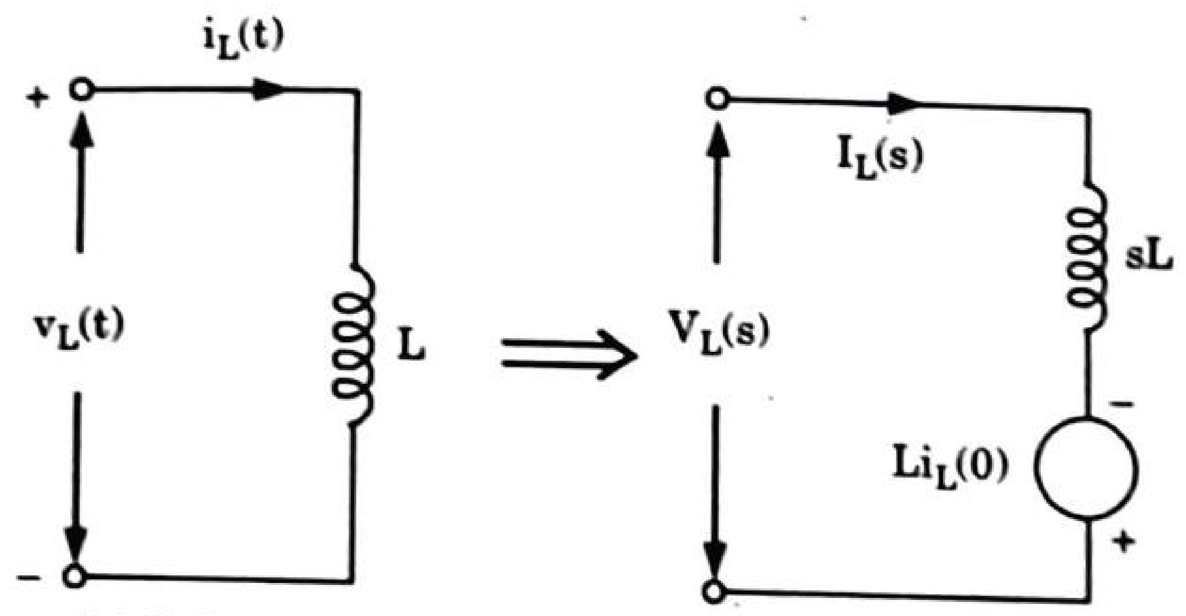
(*c*) Representation in parallel form

KVL : $v_C(t) = \frac{1}{C}\int_0^t i_C(t)dt + V_C(0) \Rightarrow V_C(s)$

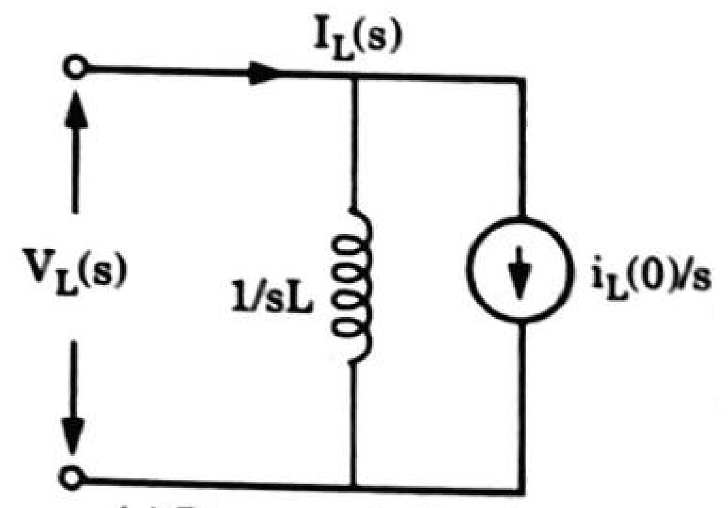
$$= \frac{1}{sC} I_C(s) + \frac{V_C(0)}{s}$$

KCL : $i_C(t) = C\frac{dV_C}{dt} \Rightarrow I_C(s) = sCV_C(s) - CV_C(0)$

Inductor :



(*a*) Inductive element (*b*) Representation in series form



(*c*) Representation in parallel form

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KVL :	$v(t) = \dfrac{L\,di_L}{dt} \Rightarrow V_L(s) = sLI_L(s) - Li_L(0)$
KCL :	$i_L(t) = \dfrac{1}{L}\int_0^t V_L dt + i_L(0) \Rightarrow I_L(s) = \dfrac{V_L(s)}{sL} + \dfrac{i_L(0)}{s}$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.16. **Consider a series RL circuit shown in Fig. 3.16.1. The switch is closed at time $t = 0$, find the current $i(t)$ using Laplace transform.**

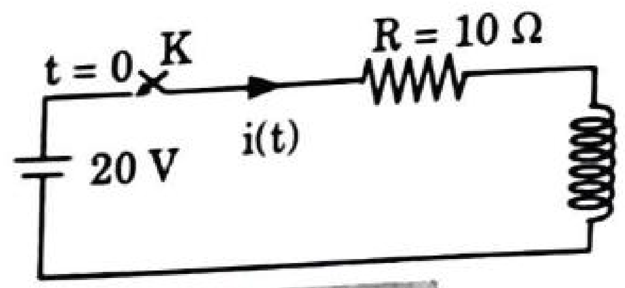

Fig. 3.16.1.

AKTU 2011-12, Marks 05

Answer

1. Initial condition, $i(0) = 0$
2. Using KVL,

$$R\,i(t) + 5\frac{di}{dt} = 20$$

3. Taking Laplace transform,

$$10I(s) + 5[sI(s) - i(0)] = \frac{20}{s}$$

$$10I(s) + 5sI(s) = \frac{20}{s} \qquad (\because i(0) = 0)$$

$$I(s) = \frac{20/s}{10 + 5s} = \frac{20}{s(5s+10)} = \frac{4}{s(s+2)}$$

4. Using partial fraction,

$$I(s) = \frac{2}{s} - \frac{2}{s+2}$$

5. Taking inverse Laplace,

$$i(t) = L^{-1}[I(s)] = L^{-1}\left[\frac{2}{s} - \frac{2}{s+2}\right]$$

$$i(t) = 2(1 - e^{-2t})$$

Que 3.17. **An R-L-C series circuit is as shown in Fig. 3.17.1. The switch is moved from position 1 to 2 at $t = 0$ after it remained in position 1 for a long time. The initial current at ($t = 0^-$) in the inductor is 2 A and the voltage across the capacitor at that instant is 4 volts. Find the expression for the inductor current $i(t)$ for $t > 0$.**

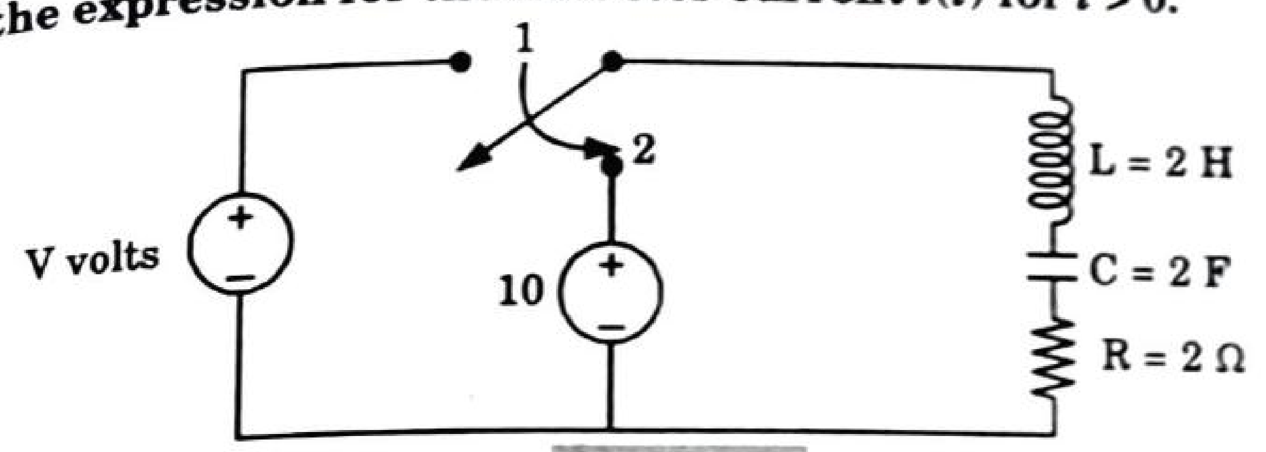

Fig. 3.17.1.

AKTU 2012-13, Marks 05

Answer

1. For switch position 2,

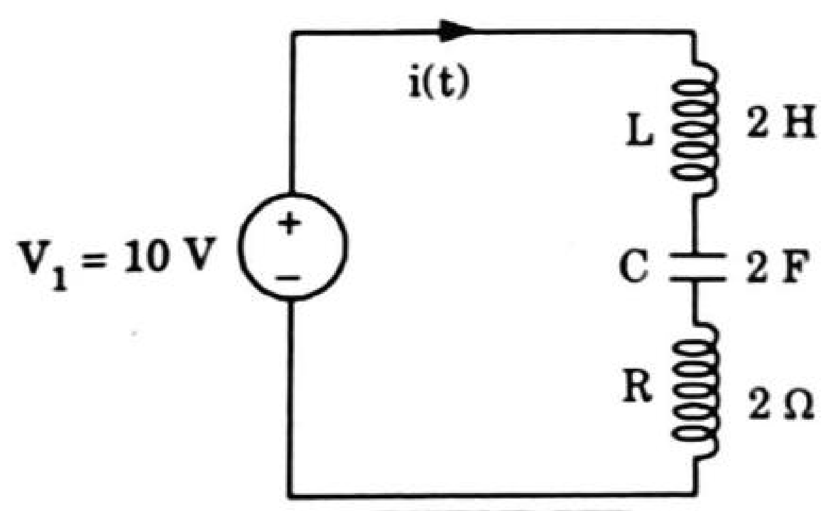

Fig. 3.17.2.

2. Using KVL

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_0^t i\,dt + V_C(0^-) = V_1$$

3. Taking Laplace transform,

$$RI(s) + L\,sI(s) - Li(0^-) + \frac{1}{C}\frac{I(s)}{s} + \frac{V_C(0^-)}{s} = \frac{V_1}{s}$$

$$\left[R + Ls + \frac{1}{Cs}\right]I(s) - Li(0^-) + \frac{V_C(0^-)}{s} = \frac{V_1}{s}$$

$$\left[2 + 2s + \frac{1}{2s}\right]I(s) = \frac{10}{s} + 4 - \frac{4}{s}$$

$$I(s) = \frac{8s + 12}{4\left(s + \frac{1}{2}\right)^2}$$

4. ∴ $$i(t) = L^{-1}[I(s)] = \frac{1}{4}L^{-1}\left[\frac{8s+12}{\left(s+\frac{1}{2}\right)^2}\right] = 2L^{-1}\left[\frac{s+\frac{3}{2}}{\left(s+\frac{1}{2}\right)^2}\right]$$

$$= 2L^{-1}\left[\frac{s+\frac{1}{2}+1}{\left(s+\frac{1}{2}\right)^2}\right] = 2L^{-1}\left[\frac{s+\frac{1}{2}}{\left(s+\frac{1}{2}\right)^2} + \frac{1}{\left(s+\frac{1}{2}\right)^2}\right]$$

$$= 2L^{-1}\left[\frac{1}{\left(s+\frac{1}{2}\right)} + \frac{1}{\left(s+\frac{1}{2}\right)^2}\right] = 2e^{-\frac{1}{2}t} + 2te^{-\frac{1}{2}t}\ \text{A}$$

Que 3.18. **i. State and prove the initial and final value theorems.**
ii. A pulse of width one second and magnitude one volt is applied across a series R-L circuit with $R = 1$ ohm and $L = 1$ henry. Find the current $i(t)$ flowing in the circuit as a function of time. Use Laplace transform method.

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Answer

i. **Initial and final value theorem :** Refer Q. 3.3, Page 3–4D, Unit-3.

ii. **Given :** $R = 1\,\Omega$, $L = 1$ H, $v = \begin{cases} 1 & ; \quad 0 \le t \le 1 \\ 0 & ; \quad \text{otherwise} \end{cases}$

To find : Current, $i(t)$.

1. Applying KVL,
$$R\,i(t) + L\frac{di(t)}{dt} = v(t)$$
2. Taking Laplace transform,

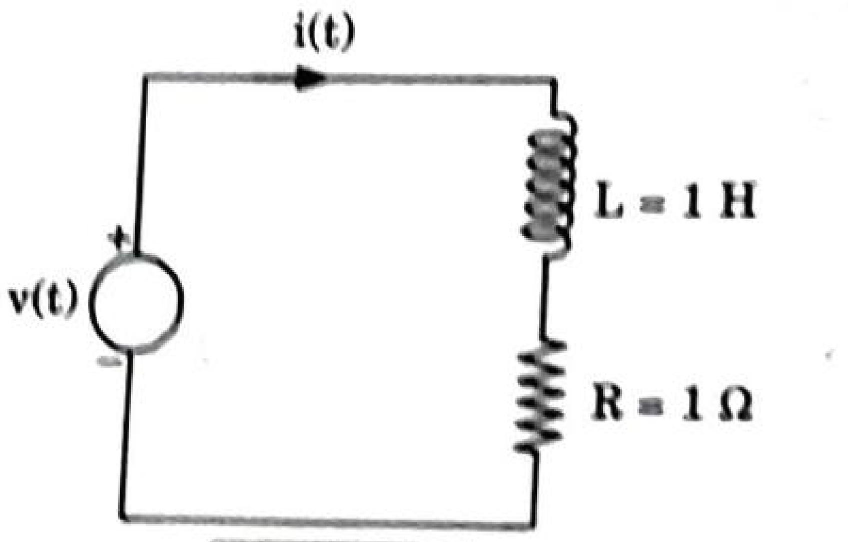

Fig. 3.18.1.

$$RI(s) + Ls\,I(s) = L[v(t)]$$
(Assuming initial condition zero)

3. $$V(s) = \int_0^1 1.e^{-st}dt = \int_0^1 e^{-st}dt$$
$$= \left[\frac{e^{-st}}{-s}\right]_0^1 = \left[\frac{e^{-s}-e^{0}}{-s}\right] = \frac{1}{s} - \frac{e^{-s}}{s}$$
4. $$I(s)[1+s] = \frac{1}{s} - \frac{e^{-s}}{s}$$
$$I(s) = \frac{1}{s(s+1)} - \frac{e^{-s}}{s+1} = \frac{1}{s} - \frac{1}{s+1} - \frac{e^{-s}}{s} + \frac{e^{-s}}{s+1}$$
5. Taking inverse Laplace transform,
$$i(t) = u(t) - e^{-t}\,u(t) - u(t-1) + e^{-s(t-1)}\,u(t-1)$$

Que 3.19. **In the circuit shown in Fig. 3.19.1, the input is 0.4 $u(t)$. Find $v_C(t)$ with the switch closed at $t = 0$ assuming zero initial conditions.**

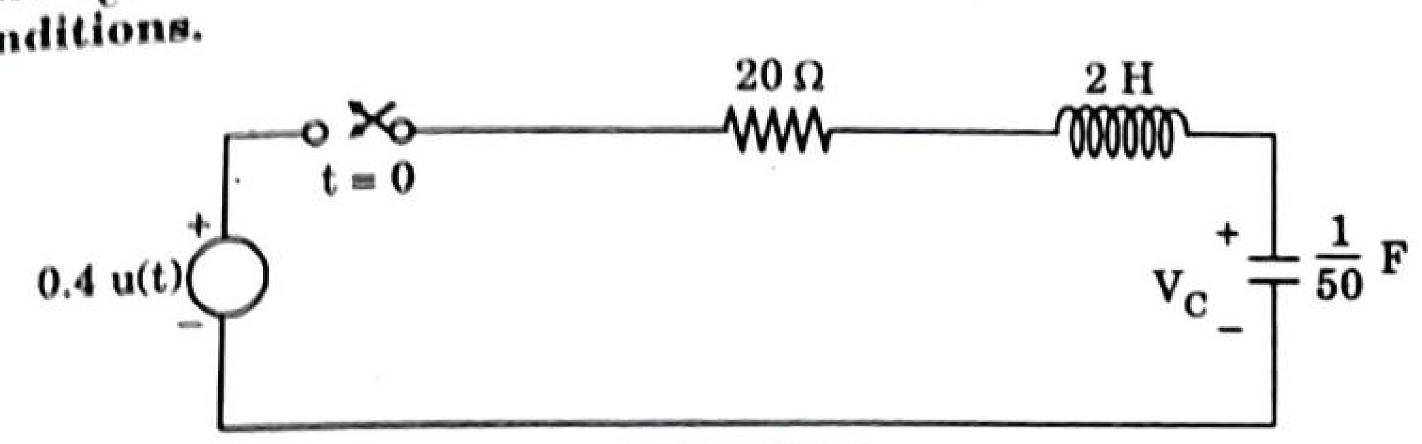

Fig. 3.19.1.

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Answer

1. As the initial conditions are zero, the s-domain network of the given circuit is shown in the Fig. 3.19.2.

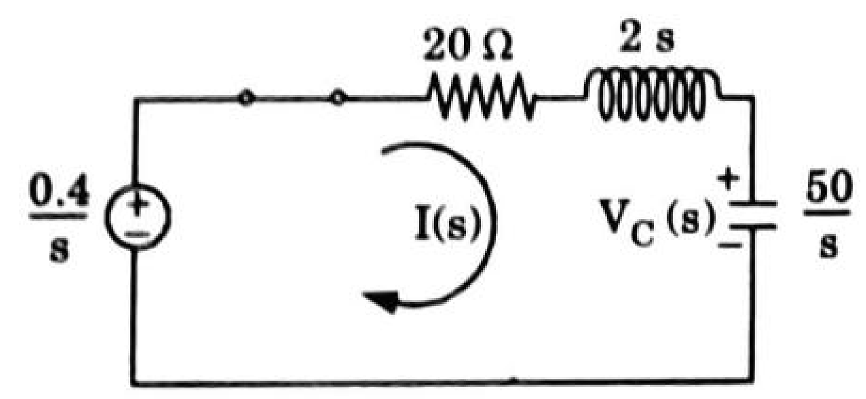

Fig. 3.19.2.

2. Applying KVL to the loop,
$$I(s) = \frac{\left(\frac{0.4}{s}\right)}{20+2s+\frac{50}{s}} = \frac{0.4}{2s^2+20s+50}$$
$$= \frac{0.2}{s^2+10s+25}$$

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4. $\therefore \quad V_C(s) = \dfrac{50}{s} \times I(s) = \dfrac{50}{s} \times \dfrac{0.2}{s^2+10s+25} = \dfrac{10}{s(s^2+10s+25)}$

$\therefore \quad V_C(s) = \dfrac{10}{s(s+5)^2} = \dfrac{A}{s} + \dfrac{B}{s+5} + \dfrac{C}{(s+5)^2}$

$\therefore A(s+5)^2 + Bs(s+5) + Cs = 10$

5. Comparing, $\quad A = 0.4, \quad B = -0.4, \quad C = -2$

$\therefore \quad V_C(s) = \dfrac{0.4}{s} - \dfrac{0.4}{s+5} - \dfrac{2}{(s+5)^2}$

6. $\therefore \quad v_C(t) = L^{-1}[V_C(s)] = 0.4\,u(t) - 0.4e^{-5t} - t\,e^{-5t}$ V

Que 3.20. **Find the expression for the currents $i_1(t)$ and $i_2(t)$ in Fig. 3.20.1 using Laplace transform method.**

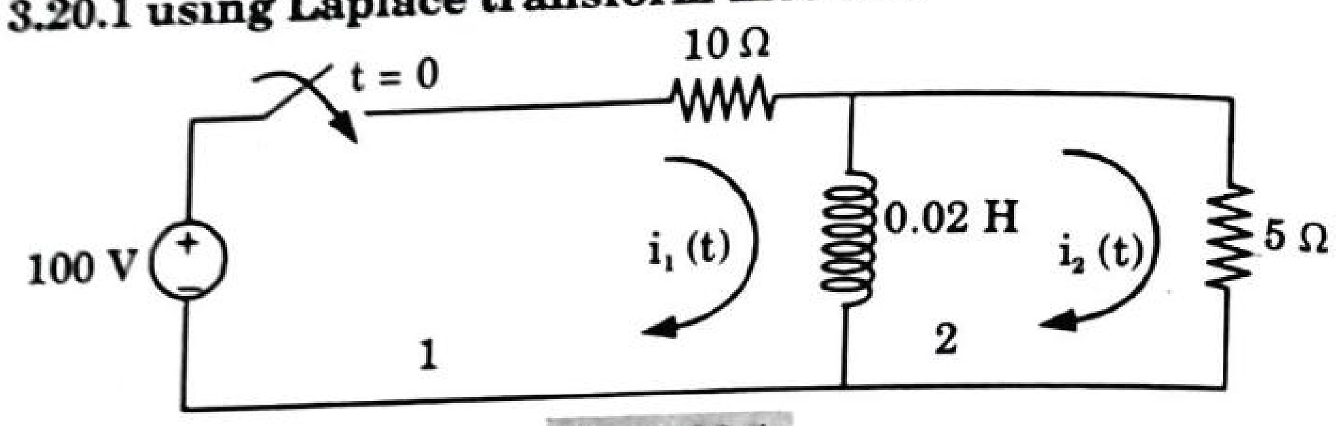

Fig. 3.20.1.

The switch is closed at $t = 0$ and the circuit is initially a relaxed one.

AKTU 2013-14, Marks 10

Answer

1. The initial current through the inductor is zero as the switch was open before $t = 0$.

$\therefore \quad i_L(0^-) = 0$ A

2. Applying KVL to the two loops we get,

Loop 1, $\quad -10i_1 - 0.02\dfrac{di_1}{dt} + 0.02\dfrac{di_2}{dt} + 100 = 0$

$$10i_1 + 0.02\frac{di_1}{dt} - 0.02\frac{di_2}{dt} - 100 = 0 \qquad ...(3.20.1)$$

Loop 2, $\quad 5i_2 - 0.02\dfrac{di_1}{dt} + 0.02\dfrac{di_2}{dt} = 0$

i.e. $$5i_2 - 0.02\frac{di_1}{dt} + 0.02\frac{di_2}{dt} = 0 \qquad ...(3.20.2)$$

3. Taking Laplace transform of eq. (3 20.1) and (3.20.2),

$$10\,I_1(s) + 0.02\,sI_1(s) - 0.02\,i_1(0^-) - 0.02\,sI_2(s) + 0.02\,i_2(0^-) = \frac{100}{s}$$

$$\therefore I_1(s)\,[10 + 0.02\,s] - 0.02\,s\,I_2(s) - 0.02\,[i_1(0^-) - i_2(0^-)] = \frac{100}{s}$$

but $\quad i_1(0^-) = i_2(0^-) = 0$

$\therefore \quad I_1(s)\,[10 + 0.02\,s] - 0.02\,s\,I_2(s) = \dfrac{100}{s} \qquad ...(3.20.3)$

4. $5\,I_2(s) - 0.02\,sI_1(s) + 0.02\,i_1(0^-)$

$\qquad + 0.02\,sI_2(s) - 0.02\,i_2(0^-) = 0$

$\therefore -0.02\,sI_1(s) + I_2(s)\,[5 + 0.02s] + 0.02\,[i_1(0^-) - i_2(0^-)] = 0$

but $\quad i_1(0^-) = i_2(0^-) = 0$

$\therefore \quad -0.02\,s\,I_1(s) + I_2(s)[5 + 0.02\,s] = 0 \qquad ...(3.20.4)$

5. Applying Cramer's rule to the eq. (3.20.3) and (3.20.4),

$$D = \begin{vmatrix} 10+0.02s & -0.02s \\ -0.02s & 5+0.02s \end{vmatrix} = 50 + 0.3s$$

$$D_1 = \begin{vmatrix} \dfrac{100}{s} & -0.02s \\ 0 & 5+0.02s \end{vmatrix} = \frac{100}{s}(5 + 0.02s)$$

$$D_2 = \begin{vmatrix} 10+0.02s & \dfrac{100}{s} \\ -0.02s & 0 \end{vmatrix} = \frac{100}{s} \times 0.02s = 2$$

6. $\therefore \quad I_1(s) = \dfrac{D_1}{D} = \dfrac{100(5+0.02s)}{s(50+0.3s)} = \dfrac{6.67(s+250)}{s(s+166.67)}$

and $\quad I_2(s) = \dfrac{D_2}{D} = \dfrac{2}{50+0.3s} = \dfrac{6.67}{(s+166.67)}$

7. $\quad I_1(s) = \dfrac{A}{s} + \dfrac{B}{(s+166.67)}$

$$A = \left.\frac{6.67(s+250)}{(s+166.67)}\right|_{s=0} = 10$$

$$B = \left.\frac{6.67(s+250)}{s}\right|_{s=-166.67} = -3.33$$

8. $\quad I_1(s) = \dfrac{10}{s} + \dfrac{(-3.33)}{(s+166.67)}$

9. Taking inverse Laplace transform of $I_1(s)$ and $I_2(s)$,

$\quad i_1(t) = 10 - 3.33\,e^{-166.67t}$ A

and $\quad i_2(t) = 6.67\,e^{-166.67t}$ A

Que 3.21. **In the circuit shown in Fig. 3.21.1, determine the current $i(t)$ when the switch is at position 2. The switch S is moved from position 1 to position 2 at $t = 0$. Initially the switch has been at position 1 for a long time.**

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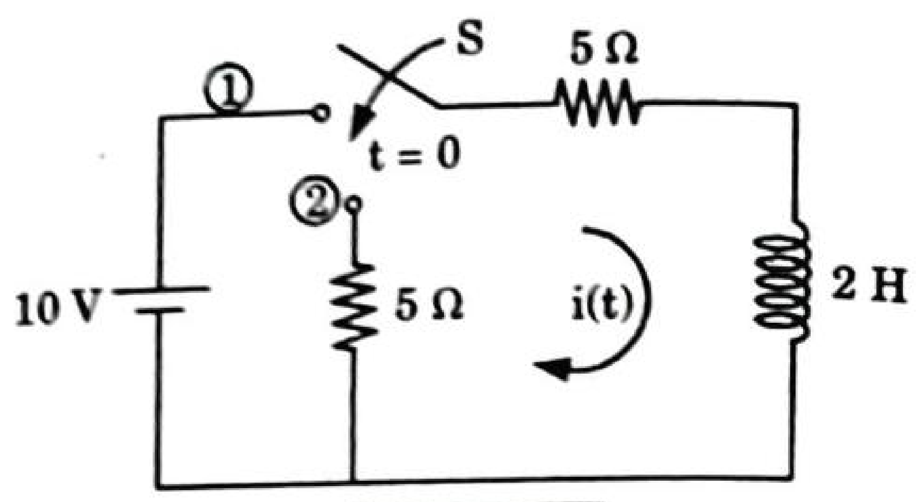

Fig. 3.21.1.

AKTU 2014-15, Marks 06

Answer

1. When switch S is in position 1,
   Steady state current

$$i(0) = \frac{10}{5} = 2 \text{ A}$$

2. When switch is at position 2, for $t > 0$, we have

$$2\frac{di(t)}{dt} + 5i(t) + 5i(t) = 0$$

$$2\frac{di(t)}{dt} + 10i(t) = 0$$

$$\frac{di(t)}{dt} + 5i(t) = 0$$

3. Taking Laplace transform, we obtain
   $s\,I(s) - i(0) + 5\,I(s) = 0$
   $s\,I(s) - 2 + 5\,I(s) = 0$

$$I(s) = \frac{2}{s+5}$$

4. Taking inverse Laplace transform, we get
   $i(t) = 2e^{-5t}\,u(t)$

Que 3.22. **Moved to position 2, find the current $i(t)$ in the circuit.**

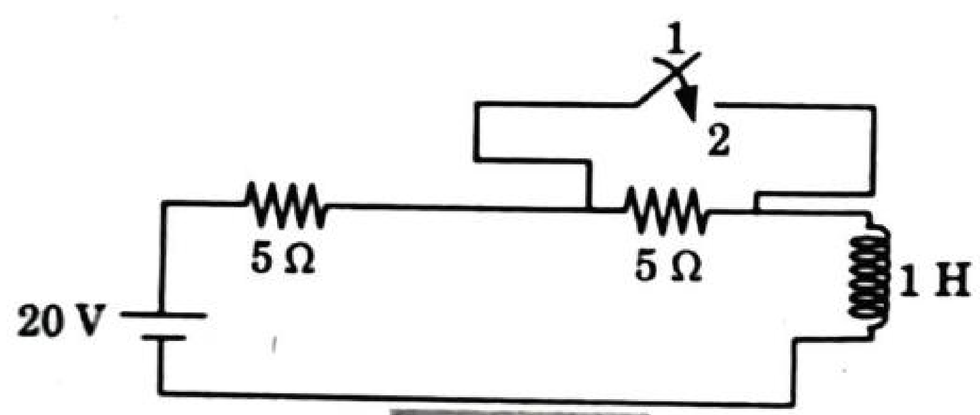

Fig. 3.22.1.

AKTU 2015-16, Marks 05

Answer

1. When switch is at position 1, circuit achieves steady state and inductor behaves as short circuit.

$$\therefore \quad i(0) = \frac{20}{10} = 2 \text{ A}$$

2. When switch at position 2,

$$5i + \frac{di}{dt} = 20$$

3. Taking Laplace transform,

$$5i(s) + sI(s) - i(0) = \frac{20}{s}$$

$$(5 + s)\,I(s) - 2 = \frac{20}{s}$$

$$(5 + s)\,I(s) = \frac{20}{s} + 2 = \frac{20 + 2s}{s}$$

$$I(s) = \frac{20 + 2s}{s(s+5)} = \frac{4}{s} - \frac{2}{s+5}$$

4. Taking inverse Laplace transform,
   $i(t) = 4 - 2e^{-5t}$ A

Que 3.23. **How differential equation is solved ?**

Answer

1. When the initial conditions are specified for the given differential equation, they have to be accounted for when LT is taken to convert the differential equation to algebraic equation. Thus

$$L\left[\frac{dy}{dt}\right] = sY(s) - y(0^-)$$

$$L\left[\frac{d^2y}{dt^2}\right] = s^2Y(s) - sy(0^-) - \dot{y}(0-)$$

$$L\left[\frac{d^3y}{dt^3}\right] = s^3Y(s) - s^2y(0^-) - s\dot{y}(0-) - \ddot{y}(0^-)$$

The initial conditions $y(0^-)$, $\dot{y}(0^-)$ and $\ddot{y}(0^-)$ are meant that the system initial conditions are given just before the input is applied to the system. The initial condition $y(0^+)$ indicates that the initial condition is given to the system after the input is applied which is not realistic. Unless otherwise mentioned, $y(0^-)$ means $y(0)$ (and $y(0)$ is not $y(0^+)$).

2. The zero initial conditions explained in step 1 are applicable to the input also. Thus

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$$\frac{dx}{dt} = sX(s) - x(0^-)$$

3. The initial conditions for an input multiplied by $u(t)$ implies that the signals are zero prior to $t = 0$.
4. The solution of the differential equation contains two components :
   a. The first component is the response due to the initial conditions only where the input is assumed to be absent. The response is called the zero input response.
   b. The second component is the response due to the input alone and the initial conditions here are assumed to be zero. Such response is called zero state response.
5. The total response = zero state response + zero input response.
6. If one is interested to find out the zero initial conditions for verification of the results, only the zero input response has to be considered and not the total response. The total response satisfies the initial conditions at $t = 0^-$

Que 3.24. **Using Laplace transform, solve differential equation :**

$$2\ddot{x} + 7\dot{x} + 6x = 0$$

where $x(0) = 0,\ \dot{x}(0) = 1$ **AKTU 2011-12, Marks 05**

OR

Discuss the important properties and applications of Laplace transform. Using Laplace transform solve the following differential equation

$$2\ddot{x}(t) + 7\dot{x}(t) + 6x(t) = 0;\ x(0) = 0,\ \dot{x}(0) = 1$$

AKTU 2015-16, Marks 10

Answer

Properties of Laplace transform : Refer Q. 3.2, Page 3–3D, Unit-3.
Applications of Laplace transform : Refer Q. 3.1, Page 3–2D, Unit-3.
Numerical :

Given : $2\ddot{x} + 7\dot{x} + 6x = 0,\ x(0) = 0,\ \dot{x}(0) = 1$
To find : Output, $x(t)$.

1. Taking Laplace transform,

$$2s^2X(s) - 2sx(0) - \frac{2dx(0)}{dt} + 7sX(s) - 7x(0) + 6X(s) = 0$$

2. $$X(s)\,(2s^2 + 7s + 6) = (2s + 7)x(0) + \frac{2dx(0)}{dt}$$

$$X(s) = \frac{2}{2s^2 + 7s + 6}$$

3. $$x(t) = L^{-1}[X(s)]$$

4. $$= L^{-1}\left[\frac{2}{(2s+3)(s+2)}\right]$$

$$= 2L^{-1}\left[\frac{1}{s+\frac{3}{2}} - \frac{1}{s+2}\right]$$

$$= 2[e^{-1.5t} - e^{-2t}]$$

Que 3.25. **A system is described by the differential equation**

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t)$$

For the initial conditions, $\frac{dy(0)}{dt} = 2$ **and** $y(0) = 1$ **and input** $x(t) = u(t)$, **find the free and forced response of the system.**

AKTU 2014-15, Marks 06

Answer

Given : $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t),\ \frac{dy(0)}{dt} = 2,\ y(0) = 1,\ x(t) = u(t)$
To find : Response of system, $x(t)$.

1. Taking Laplace transform

$$\left[s^2 Y(s) - sy(0) - \frac{dy(0)}{dt}\right] + 5\ [sY(s) - y(0)] + 6Y(s) = X(s)$$

$$s^2 Y(s) - (s + 5)\,y(0) - \frac{dy(0)}{dt} + 5s\,Y(s) + 6Y(s) = X(s)$$

$$(s^2 + 5s + 6)\,Y(s) = X(s) + (s + 5)\,y(0) + \frac{dy(0)}{dt}$$

$$Y(s) = \frac{(s+5)\,y(0) + \left[\frac{dy(0)}{dt}\right]}{s^2 + 5s + 6} + \frac{X(s)}{s^2 + 5s + 6}$$

2. The first term on right side of response depends only on the initial condition and has no effect of input so it is called natural or free response while second term is called forced response.
3. Substituting the initial conditions,

$$\frac{dy(0)}{dt} = 2,\ y(0) = 1 \text{ and } X(s) = \frac{1}{s}$$

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$$Y(s) = \frac{(s+5)+2}{s^2+5s+6} + \frac{\frac{1}{s}}{s^2+5s+6}$$

$$= \frac{s+7}{s^2+5s+6} + \frac{1}{s(s^2+5s+6)}$$

$$= \frac{s+7}{(s+2)(s+3)} + \frac{1}{s(s+2)(s+3)}$$

$$= \frac{5}{s+2} - \frac{4}{s+3} + \frac{1}{6}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2} + \frac{1}{3}\frac{1}{s+3}$$

4. Taking inverse Laplace transform on both side

$$y(t) = \underbrace{5e^{-2t}u(t) - 4e^{-3t}u(t)}_{\text{Free response}} + \underbrace{\frac{1}{6}u(t) - \frac{1}{2}e^{-2t}u(t) + \frac{1}{3}e^{-3t}u(t)}_{\text{Forced response}}$$

Total response

Que 3.26. **A system has impulse response $h(t) = e^{-2t}u(t)$. Find its system function and the output if the input to the system is $x(t) = e^{-t}u(t)$.**

AKTU 2016-17, Marks 10

Answer

Given : $h(t) = e^{-2t}u(t)$, $x(t) = e^{-t}u(t)$

To find :

i. System function, $T(s)$.

ii. Output, $y(t)$.

i. 1. Taking Laplace transform,

$$H(s) = \frac{1}{(s+2)}$$

2. Let, $x(t) = \delta(t)$

$\therefore$ $X(s) = 1$

System function,

$$T(s) = \frac{H(s)}{X(s)} = \frac{1}{(s+2)}$$

ii. 1. Again, $x(t) = e^{-t}u(t)$

Taking Laplace transform,

$$X(s) = \frac{1}{s+1}$$

2. Output, $Y(s) = T(s)\,X(s)$

$$Y(s) = \frac{1}{(s+1)(s+2)}$$

$$Y(s) = \frac{1}{(s+1)} - \frac{1}{(s+2)}$$

3. Taking inverse Laplace transform,

$$y(t) = e^{-t}u(t) - e^{-2t}u(t)$$

Que 3.27. **Using Laplace transform solve the following differential equation. $d^2y(t)/dt^2 + 5dy(t)/dt + 4y(t) = x(t)$, if $x(t) = e^{-2t}u(t)$ and $y(0^-) = -2$, $dy(0^-)/dt = -1$, and find auto-correlation of sequence $x(n) = (-1, 1, -1)$.**

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Answer

Given : $\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4y(t) = x(t)$, $x(t) = e^{-2t}u(t)$, $y(0^-) = -2$, $\frac{dy(0^-)}{dt} = -1$, $x(n) = (-1, 1, -1)$

To find :

i. Output, $y(t)$.

ii. Auto-correlation.

i. 1. Taking Laplace transform on both sides,

$$[s^2Y(s) - s\,y(0^-) - \dot{y}(0^-)] + 5[s\,Y(s) - y(0^-)] + 4Y(s) = X(s)$$

2. Putting initial conditions,

$$[s^2Y(s) + 2s + 1] + 5[s\,Y(s) + 2] + 4Y(s) = L\{e^{-2t}u(t)\}$$

$$(s^2+5s+4)Y(s) = \frac{1}{(s+2)} - 2s - 1 - 10$$

$$(s^2+5s+4)Y(s) = \frac{1}{(s+2)} - (2s+11)$$

$$Y(s) = \frac{1-(2s^2+4s+11s+22)}{(s+2)(s+1)(s+4)}$$

$$Y(s) = \frac{-(2s^2+15s+21)}{(s+2)(s+1)(s+4)}$$

3. Using partial fraction,

$$Y(s) = \frac{A}{(s+1)} + \frac{B}{(s+2)} + \frac{C}{(s+4)}$$

$$A = (s+1)Y(s)\Big|_{s=-1} = \frac{-(2s^2+15s+21)}{(s+2)(s+4)}\Bigg|_{s=-1} = -4$$

$$B = (s+2)Y(s)\Big|_{s=-2} = \frac{-(2s^2+15s+21)}{(s+1)(s+3)}\Bigg|_{s=-2} = -1$$

$$C = (s+4)Y(s)\Big|_{s=-4} = \frac{-(2s^2+15s+21)}{(s+1)(s+2)}\Bigg|_{s=-3} = \frac{7}{6}$$

$$Y(s) = -\frac{4}{(s+1)} - \frac{1}{(s+2)} + \frac{7}{6}\frac{1}{(s+4)}$$

4. Taking inverse Laplace transform,

$$y(t) = -4e^{-t}\,u(t) - e^{-2t}\,u(t) + 7/6e^{-4t}\,u(t)$$

ii. Auto correlation :

$$r_{xx}(k) = \sum_{n=0}^{2} x(n)x(n-k)$$

$$r_{xx}(-2) = (-1)(-1) + (1)(0) + 0 = 1$$
$$r_{xx}(-1) = (-1)(1) + (1)(-1) + 0 = -2$$
$$r_{xx}(0) = (-1)(-1) + (1)(1) + (-1)(-1) = 3$$
$$r_{xx}(1) = (-1)(0) + (1)(-1) + (-1)(1) = -2$$
$$r_{xx}(2) = (-1)(0) + (1)(0) + (-1)(-1) = 1$$
$$\{r_{xx}(k)\} = (1, -2, 3, -2, 1)$$

Que 3.28. **Find the Laplace transform of the following waveforms shown in Fig. 3.28.1(*a*) and 3.28.1(*b*).**

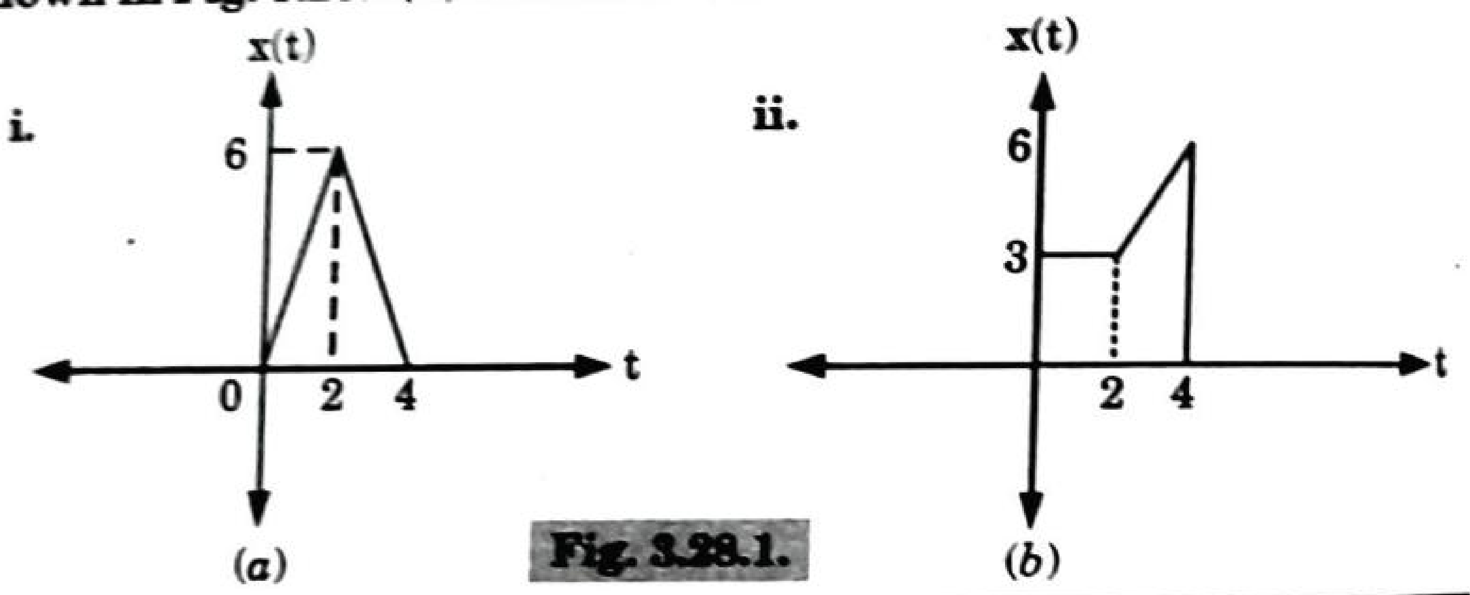

Fig. 3.28.1.

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Answer

i. 1. Here, $x_1(t) = 3t \quad , \ 0 \le t \le 2$

$x_2(t) = -3t + c, \quad 2 \le t \le 4$

2. When $t = 2, x_1(t) = 6$

Hence, $6 = -3 \times 2 + c$

or $c = 12$

3. $x_2(t) = -3t + 12, \ 2 \le t \le 4$

4. $$X_1(s) = \int_0^2 3te^{-st}\,dt = 3\int_0^2 te^{-st}\,dt$$

$$= 3\left[\frac{te^{-st}}{-s}\Bigg|_0^2 - \int_0^2 \frac{e^{-st}}{-s}\,dt\right]$$

$$= 3\left(\left[\frac{2e^{-2s}}{-s} - 0\right] + \left[\frac{-e^{-st}}{s^2}\Bigg|_0^2\right]\right)$$

$$= 3\left[\frac{-2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{1}{s^2}\right] = \frac{3}{s^2} - \left(\frac{6}{s} + \frac{3}{s^2}\right)e^{-2s}$$

5. $$X_2(s) = \int_2^4 (12-3t)\,e^{-st}\,dt = \int_2^4 12e^{-st}\,dt - 3\int_2^4 te^{-st}\,dt$$

$$= 12\left[\frac{e^{-st}}{-s}\right]_2^4 - 3\left[\frac{te^{-st}}{-s}\Bigg|_2^4 - \int_2^4 \frac{e^{-st}}{-s}\,dt\right]$$

$$= 12\left[\frac{e^{-4s}}{-s} + \frac{e^{-2s}}{s}\right] - 3\left[\frac{4e^{-4s}}{-s} + \frac{2e^{-2s}}{s} - \frac{e^{-st}}{s^2}\Bigg|_2^4\right]$$

$$= 12\left[\frac{e^{-2s}}{s} - \frac{e^{-4s}}{s}\right] - 3\left[\frac{2e^{-2s}}{s} - \frac{4e^{-4s}}{s} - \frac{e^{-4s}}{s^2} + \frac{e^{-2s}}{s^2}\right]$$

$$= \frac{12e^{-2s}}{s} - \frac{12e^{-4s}}{s} - \frac{6e^{-2s}}{s} + \frac{12e^{-4s}}{s} + \frac{3e^{-4s}}{s^2} - \frac{3e^{-2s}}{s^2}$$

$$= \frac{6e^{-2s}}{s} + \frac{3e^{-4s}}{s^2} - \frac{3e^{-2s}}{s^2}$$

6. $\therefore$ $$X(s) = X_1(s) + X_2(s)$$

$$= \frac{3}{s^2} - \left(\frac{6}{s} + \frac{3}{s^2}\right)e^{-2s} + \frac{6e^{-2s}}{s} + \frac{3e^{-4s}}{s^2} - \frac{3e^{-2s}}{s^2}$$

$$= \frac{3}{s^2} - \frac{6}{s^2}e^{-2s} + \frac{3}{s^2}e^{-4s}$$

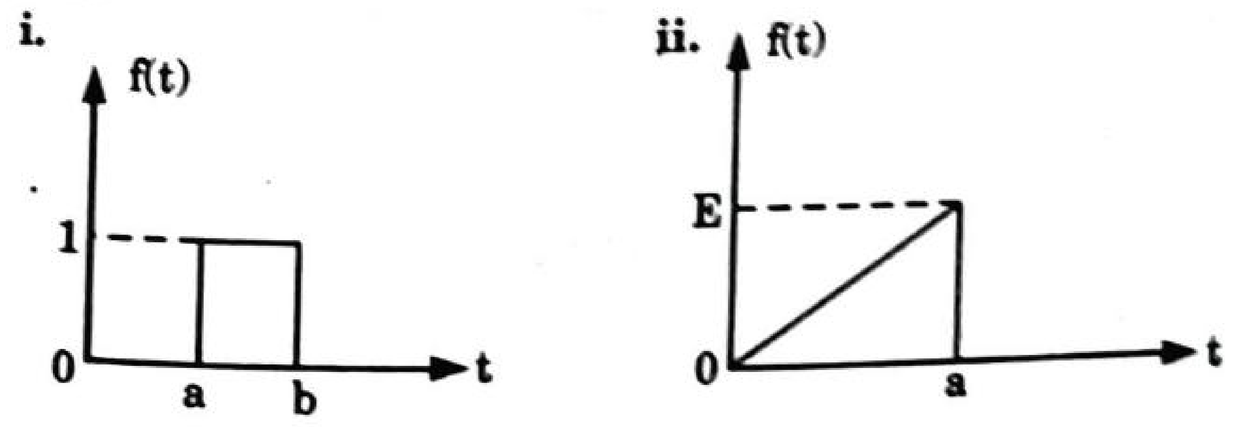
ii. 1. Signal can be written as,

$$x(t) = 3u(t) + 1.5t\,u(t-2) - 1.5t\,u(t-4) - 6u(t-4)$$

2. Taking Laplace transform on both sides,

$$X(s) = \frac{3}{s} + \frac{1.5e^{-2s}}{s^2} - \frac{1.5e^{-4s}}{s^2} - \frac{6}{s}e^{-4s}$$

Que 3.29. **Obtain Laplace transform of the following graphs :**



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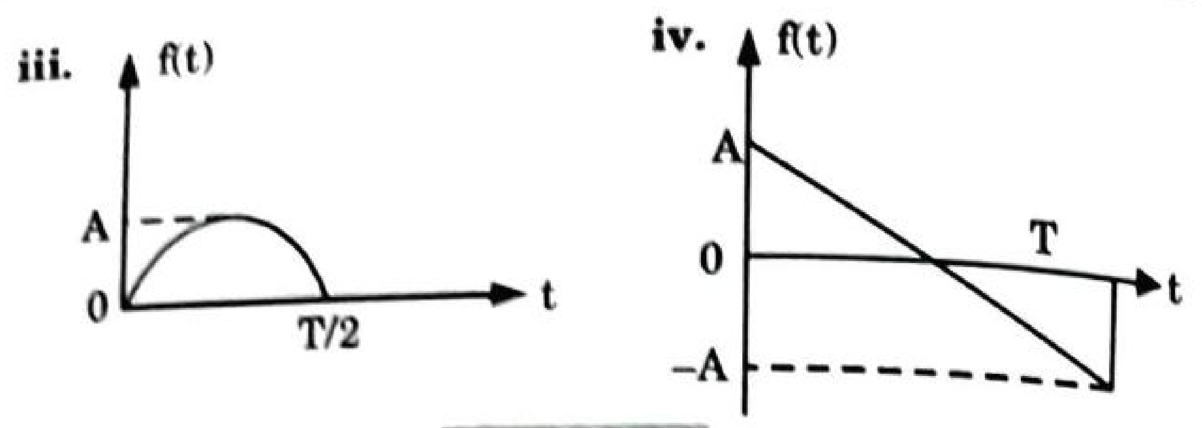

Fig. 3.29.1.

Answer

i. Function can be written as :

$$f(t) = u(t-a) - u(t-b)$$

$$L[f(t)] = L\,[u(t-a) - u(t-b)]$$

$$= \frac{1}{s}e^{-as} - \frac{1}{s}e^{-bs} = \frac{1}{s}[e^{-as} - e^{-bs}]$$

ii. The sawtooth waveform is expressed as :

$$f(t) = \frac{E}{a}t \quad , \quad 0 \le t \le a$$

$$= 0 \quad , \quad \text{otherwise}$$

Taking Laplace transform for the time function $f(t)$,

$$F(s) = \int_0^a \frac{E}{a} t\, e^{-st}\, dt = \frac{E}{a}\int_0^a t\, e^{-st} dt$$

Integrating by parts,

$$F(s) = \frac{E}{a}\left[t\left(\frac{e^{-st}}{-s}\right)\Bigg|_0^a + \int_0^a \frac{1}{s} e^{-st}\, dt \right]$$

$$= \frac{E}{a}\left[\frac{-a}{s}s^{-as} + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)\Bigg|_0^a\right]$$

$$= \frac{E}{a}\left[-\frac{ae^{-as}}{s} - \frac{e^{-as}}{s^2} + \frac{1}{s^2}\right]$$

$$F(s) = \frac{E}{as^2} - \left(\frac{E}{s} + \frac{E}{as^2}\right)e^{-as}$$

iii.

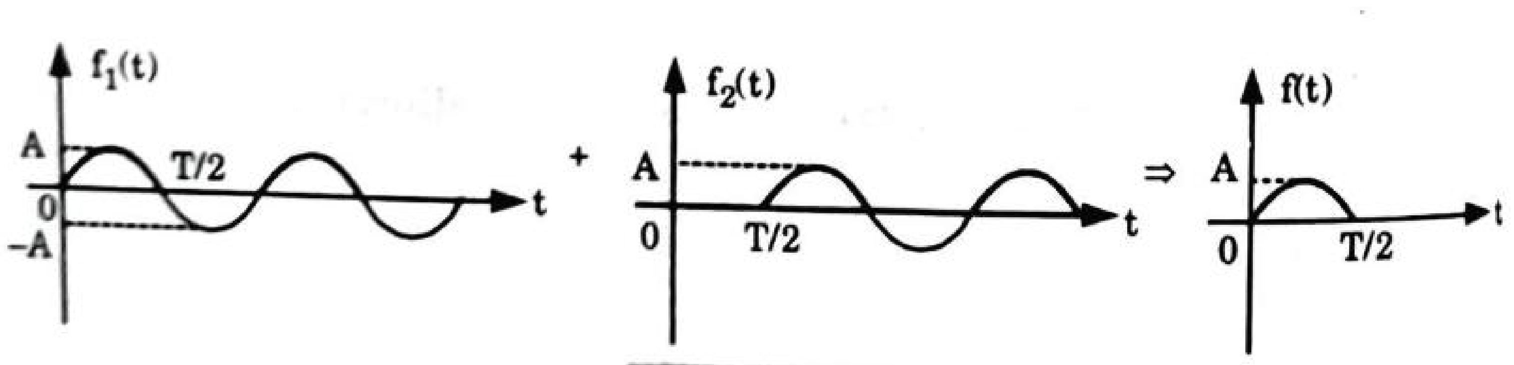

Fig. 3.29.2.

Function $f(t)$ can be defined as

$$f(t) = A\sin\left(\frac{2\pi}{T}t\right)u(t) + A\sin\frac{2\pi}{T}\left(t - \frac{T}{2}\right)u\left(t-\frac{T}{2}\right)$$

$$L\{f(t)\} = AL\left\{\sin\left(\frac{2\pi}{T}t\right)u(t)\right\} + AL\left\{\sin\frac{2\pi}{T}\left(t-\frac{T}{2}\right)u\left(t-\frac{T}{2}\right)\right\}$$

$$= \frac{A(2\pi/T)}{s^2 + (2\pi/T)^2} + \frac{A(2\pi/T)}{s^2+(2\pi/T)^2}e^{-\frac{T}{2}s}$$

$$= \left[1 + e^{\frac{-T}{2}s}\right]\frac{A(2\pi/T)}{s^2 + (2\pi/T)^2}$$

iv. Using gate function, we can write

$$f(t) = \frac{-2A}{T}\left(t - \frac{T}{2}\right)[u(t) - u(t-T)]$$

$$= \frac{-2A}{T}\left(t-\frac{T}{2}\right)u(t) + \frac{2A}{T}[(t-T) + T/2]u(t-T)$$

$$= \frac{-2A}{T}\left(t-\frac{T}{2}\right)u(t) + \left[\frac{2A}{T}(t-T) + A\right]u(t-T)$$

$$L\{f(t)\} = \frac{-2A}{T}\left[\frac{1}{s^2} - \frac{T}{2s}\right] + \left[\frac{2A}{Ts^2} + \frac{A}{s}\right]e^{-sT}$$

$$= \frac{2A}{Ts}\left[\frac{T}{2}(1 + e^{-Ts}) - \frac{1}{s}(1 - e^{-Ts})\right]$$

Que 3.30. **Evaluate Laplace transform of following periodic waveforms :**

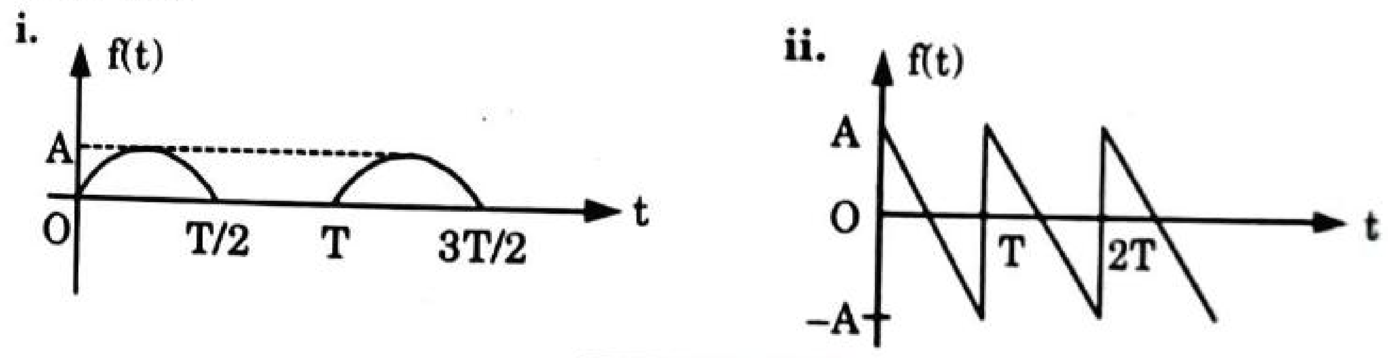

Fig. 3.30.1.

Answer

i. Laplace transform of single half-sine wave is

$$F_1(s) = \frac{A\,(2\pi/T)}{s^2 + (2\pi/T)^2}\left(1 + e^{-\frac{T}{2}s}\right)$$

So,

$$F(s) = \frac{1}{1 - e^{-Ts}}F_1(s) = \frac{A\left(\frac{2\pi}{T}\right)\left(1 + e^{-\frac{T}{2}s}\right)}{\left(1 - e^{-Ts}\right)\left(s^2 + \left(\frac{2\pi}{T}\right)^2\right)}$$

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$$= \frac{\left(1+e^{-\frac{T}{2}s}\right) A(2\pi / T)}{\left(1+e^{-\frac{T}{2}s}\right)\left(1-e^{-\frac{T}{2}s}\right)\left(s^2+(2\pi / T)^2\right)}$$

$$= \frac{1}{\left(1+e^{-\frac{T}{2}s}\right)} \times \frac{A(2\pi / T)}{(s^2+(2\pi / T)^2)}$$

ii. Laplace transform of the following waveform for the first period,

$$F_1(s) = \frac{2A}{Ts}\left[\frac{T}{2}(1+e^{-Ts}) - \frac{1}{s}(1-e^{-Ts})\right]$$

Laplace transform of signal with period T

$$F(s) = L\{f(t)\} = \frac{1}{(1-e^{-Ts})}F_1(s)$$

$$= \frac{2A}{Ts}\left[\frac{T}{2}\left(\frac{1+e^{-Ts}}{1-e^{-Ts}}\right) - \frac{1}{s}\right] = \frac{2A}{Ts}\left[\frac{T}{2}\coth\frac{Ts}{2} - \frac{1}{s}\right]$$

Que 3.31. **Determine the Laplace transform of the non-sinusoidal waveform in Fig. 3.31.1.**

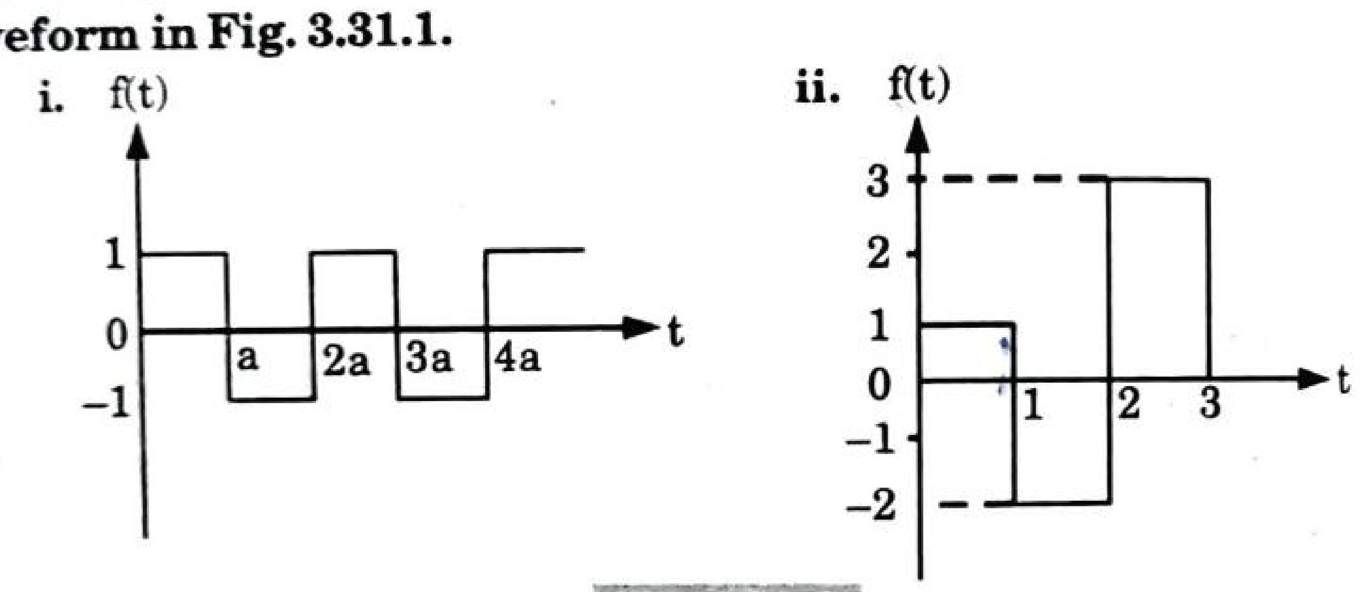

Fig. 3.31.1.

Answer

i. The mathematical description of the periodic wave with period 2a is written as

$$f_1(t) = 1\ ;\ 0 \le t \le a$$
$$= -1\ ;\ a \le t \le 2a$$

Let $F_1(s)$ be the LT of $f_1(t)$ for the time $0 \le t \le 2a$.

$$F_1(s) = \frac{1}{s}(1-e^{-as})^2$$

$$F(s) = \frac{F_1(s)}{[1-e^{-2as}]}$$

$$F(s) = \frac{(1-e^{as})^2}{s(1-e^{-2as})}$$

ii. Function can be written as

$$f(t) = u(t) - 3u(t-1) + 5u(t-2) - 3u(t-3)$$

$$L\{f(t)\} = \frac{1}{s} - \frac{3e^{-s}}{s} + \frac{5e^{-2s}}{s} - \frac{3e^{-3s}}{s}$$

$$F(s) = \frac{1}{s}[1-3e^{-s}+5e^{-2s}-3e^{-3s}]$$

Que 3.32. **Find the Laplace transform of the waveform shown in Fig. 3.32.1. It is to be noted that $v(t) = 0$ for $t > 2T$ and $t < 0$.**

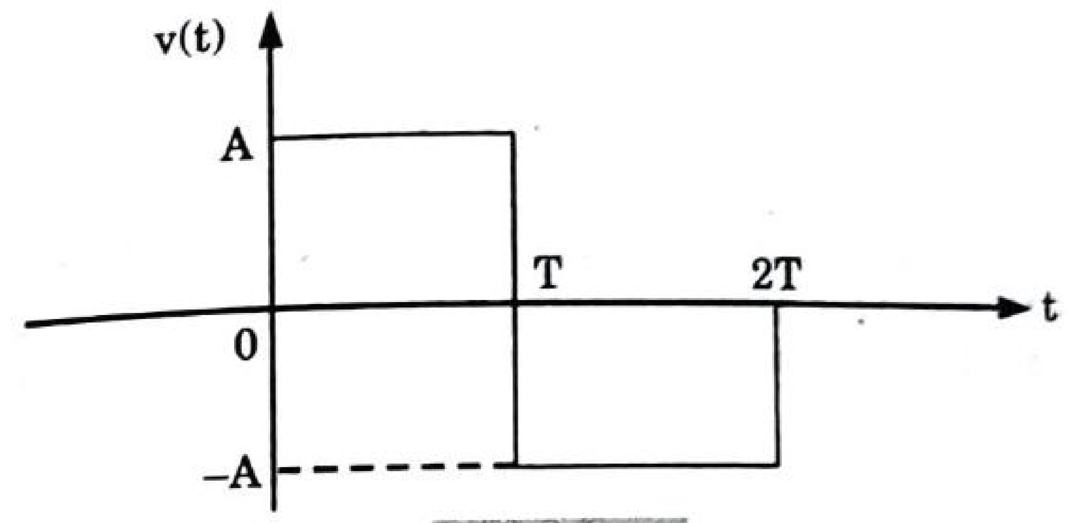

Fig. 3.32.1.

Answer

$$v(t) = A[u(t) - 2\,u(t-T) + u(t-2T)]$$

$$L\{v(t)\} = A\left[\frac{1}{s} - \frac{2e^{-Ts}}{s} + \frac{1}{s}e^{-2Ts}\right]$$

$$= \frac{A}{s}[1-2e^{-sT}+e^{-2Ts}] = \frac{A}{s}(1-e^{-Ts})^2$$

VERY IMPORTANT QUESTIONS

Following questions are very important. These questions may be asked in your SESSIONALS as well as UNIVERSITY EXAMINATION.

Q. 1. Define Laplace transform and write its properties.

Ans. Refer Q. 3.2, Unit-3.

Q. 2. State and prove initial and final value theorems of Laplace transform.

Ans. Refer Q. 3.3, Unit-3.

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Q. 3. Find Laplace Transform of following functions :

i. e^{at}
ii. $u(t)$
iii. $\sin \omega t$
iv. $\cos \omega t$
v. $\sinh at$
vi. $\cosh at$
vii. $e^{-at} \sin \omega t$
viii. $e^{-at} \cos \omega t$
ix. $e^{-at} \cosh bt$
x. $e^{-at} \sinh bt$
xi. t^n
xii. $Ae^{-at} \sin (bt + \theta)$

Ans. Refer Q. 3.5, Unit-3.

Q. 4. Find the Laplace transform of the following signals :

i. $x(t) = te^{-t} u(t)$
ii. $x(t) = te^{-2t} \sin 2t\, u(t)$

Ans. Refer Q. 3.9, Unit-3.

Q. 5. State and prove the Convolution theorem. Find the inverse Laplace transform of the following function using the Convolution theorem.

$$F(s) = \frac{1}{s(s^2 + 2s + 4)}$$

Ans. Refer Q. 3.13, Unit-3.

Q. 6. Find Laplace inverse of the function : $\left(\dfrac{s+4}{2s^2 + 5s + 3}\right)$

Ans. Refer Q. 3.12, Unit-3.

Q. 7. An R-L-C series circuit is as shown in Fig. 3.7.1. The switch is moved from position 1 to 2 at $t = 0$ after it remained in position 1 for a long time. The initial current at ($t = 0^-$) in the inductor is 2 A and the voltage across the capacitor at that instant is 4 volts.
Find the expression for the inductor current $i(t)$ for $t > 0$.

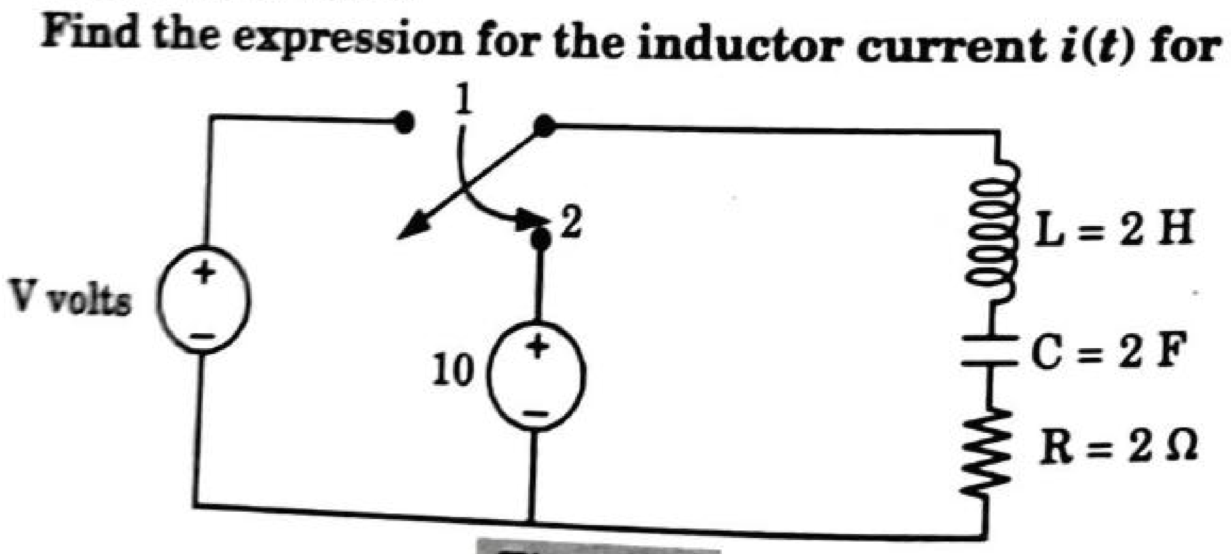

Fig. 3.7.1.

Ans. Refer Q.3.17, Unit-3.

Q. 8. A pulse of width one second and magnitude one volt is applied across a series R-L circuit with $R = 1$ ohm and $L = 1$ henry. Find the current $i(t)$ flowing in the circuit as a function of time. Use Laplace transform method.

Ans. Refer Q. 3.18, Unit-3.

Q. 9. A system is described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t)$$

For the initial conditions, $\dfrac{dy(0)}{dt} = 2$ and $y(0) = 1$ and input $x(t) = u(t)$, find the free and forced response of the system.

Ans. Refer Q. 3.25, Unit-3.

Q. 10. Find the impulse response and step response of the following system.
$H(s) = 5/(s^2 + 5s + 6)$

Ans. Refer Q. 3.15, Unit-3.

Q.11. Evaluate Laplace transform of following periodic waveforms.

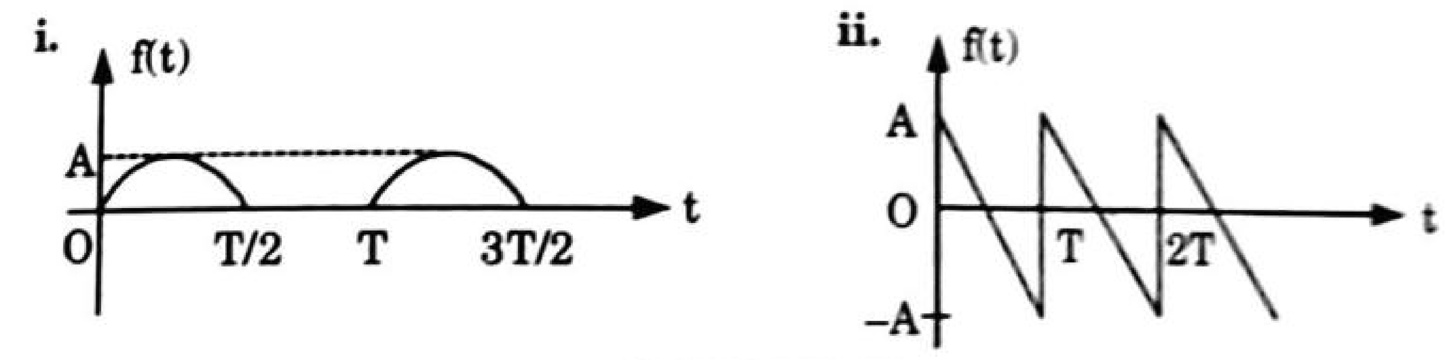

Fig. 3.11.1.

Ans. Refer Q. 3.30, Unit-3.

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4 UNIT State Variable Analysis

PART-1

Introduction, State Space Representation of Linear System Transfer Function and State Variables.

CONCEPT OUTLINE : PART-1

- An n^{th} order differential equation is not generally suitable for computer solution; it is the best to obtain a set of n first-order differential equation, using a set of auxiliary variables called state variables and this approach is called State Variable Analysis.
- **Transfer Matrix :**

  Transfer matrix = $\dfrac{Y(s)}{U(s)} = C[sI - A]^{-1} B$
- **Transfer Function :** Transfer function is a ratio of two polynomials of s. For a single input single output system, Y and U are scalars. Hence we get the transfer matrix which becomes scalar as

  Transfer function = $C[sI - A]^{-1} B = \dfrac{C \text{ adj}[(sI - A)]}{|sI - A|}$
- **Diagonalization :** Transformation of a matrix into a diagonal matrix so that the diagonal elements are represented by eigen values is called diagonalization.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.1. Define state, state variables, state vector.

Answer

State : State of a system is the minimum amount of information needed along with initial conditions at $t = t_0$ and input excitation so that future response of system can be completely described at any time $t > t_0$.

State Variables : A set of at least n variables $x_1(t), x_2(t) \ldots x_n(t)$, are needed to completely describe how a system will behave in future, along with initial state and input excitation. These minimal set of variables which can determine the state of a system are known as state variables.

State Vector : The set of state variables are the components of state vector $x(t)$

$$x(t) = [x_1(t), x_2(t), \ldots. x_n(t)]^T$$

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Quantum - Family

Que 4.2. **Discuss state space representation.**

Answer

1. In a state variable system, state variables are represented by $x_1(t)$, $x_2(t)$ ...
2. Input variables by $u_1(t)$, $u_2(t)$, ...
3. Output variables by $y_1(t)$, $y_2(t)$ ...
   where,

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ \vdots \\ x_n(t) \end{bmatrix};\; u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ \vdots \\ u_n(t) \end{bmatrix};\; y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ \vdots \\ y_n(t) \end{bmatrix}$$

4. So, dynamics of a system can be represented by n^{th} order differential equation

$$\frac{d^n y(t)}{dt^n} + a_1 \frac{d^{n-1}y(t)}{dt^{n-1}} + a_2 \frac{d^{n-2}y(t)}{dt^{n-2}} \ldots + a_{n-1}\frac{dy(t)}{dt} + a_n y(t) = u(t) \quad \ldots(4.2.1)$$

5. The knowledge of initial conditions $y(0), \frac{dy(0)}{dt}, \ldots, \frac{d^{n-1}y(0)}{dt^{n-1}}$, along with the input $u(t)$ for $t \geq 0$, completely determines the future behavior of the system.
6. We take $y(t), \frac{dy(t)}{dt}, \ldots, \frac{d^{n-1}y(t)}{dt^{n-1}}$ as a set of n-state variables.
7. Let us define

$$x_1 = y$$
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\ldots\ldots$$
$$\dot{x}_{n-1} = x_n$$

8. Now from eq. (4.2.1)

$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 \ldots - a_1 x_n + u$$

9. We can write this set of equations in matrix form as,

$$\dot{x} = Ax + Bu$$

10. Output :

$$y = Cx$$

   where,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix};$$

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix};\; B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

11. For n^{th} order single-input-single-output system :

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$A \rightarrow$ System matrix of order $(n \times n)$
$B \rightarrow$ Input coupling matrix of order $(n \times 1)$
$C \rightarrow$ Output coupling matrix of order $(1 \times n)$
$x \rightarrow$ State vector of order $(n \times 1)$
$u \rightarrow$ Scalar input of order (1×1)
$y \rightarrow$ Scalar output of order (1×1)

12. For multivariable system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix};\; x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix};\; u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}$$

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix}$$

Que 4.3. **What is Transfer Function ? Also derive the expression for transfer function of a state model.**

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Answer

Transfer Function : Transfer function is a ratio of two polynomials of s. For a single input single output system, Y and U are scalars. Hence we get the transfer matrix which becomes scalar as

$$\text{Transfer function} = C[sI - A]^{-1} B = \frac{C\,\text{adj}[(sI - A)]}{|sI - A|}$$

Derivation :

1. Let us consider a vector matrix differential equation

$$\dot{x} = Ax + Bu$$

and output, $y = Cx$

2. Now, taking Laplace transform with zero initial conditions

$$sX(s) = AX(s) + BU(s)$$
$$X(s) = [sI - A]^{-1} BU(s)$$

and

$$Y(s) = CX(s)$$
$$Y(s) = C\,[sI - A]^{-1} BU(s)$$

3. For a single-input-single-output system, Y and U are scalars.
4. Now transfer matrix can be given as

$$\text{Transfer matrix} = \frac{Y(s)}{U(s)} = C[sI - A]^{-1} B$$

5. Transfer function $= C[sI - A]^{-1} B = \dfrac{C\,\text{adj}([sI - A])B}{\det\,[sI - A]}$
6. Denominator part *i.e.* $|sI - A|$ is called the characteristic equation.

$$|sI - A| = 0$$

7. n^{th} degree characteristic equation $|sI - A| = 0$ has n roots or eigen values.

Que 4.4. **Explain diagonalization.**

Answer

1. Transformation of a matrix into a diagonal matrix so that the diagonal elements are represented by eigen values is called diagonalization.
2. Matrix A of order $n \times n$ with distinct eigen values is given by :

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & & -a_1 \end{bmatrix}$$

3. The transformation $x = PZ$, where P is the similarity transformation matrix

$$P = \begin{bmatrix} 1 & 1 & \cdots\cdots & 1 \\ s_1 & s_2 & \cdots\cdots & s_n \\ s_1^2 & s_2^2 & \cdots\cdots & s_n^2 \\ \vdots & & & \\ s_1^{n-1} & s_2^{n-1} & & s_n^{n-1} \end{bmatrix}$$

$s_1, s_2 \ldots\ldots s_n \rightarrow n$ distinct eigen values.

4. This will transform the matrix A into a diagonal matrix as

$$\Lambda = P^{-1} AP = \begin{bmatrix} s_1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & s_2 & 0 & & \cdots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & & \cdots & s_n \end{bmatrix}$$

where, $\Lambda \rightarrow$ Diagonal matrix.

Que 4.5. **Obtain the state space representation of the system** $\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$

where,

$y \rightarrow$ **output**
$u \rightarrow$ **input**

Answer

1. Let $x_1 = y$
2. Then $x_2 = \dot{y}\,;\ \dot{x}_1 = x_2$

$x_3 = \ddot{y}\,;\ \dot{x}_2 = x_3$

3. Now, $\dot{x}_3 = -6x_3 - 11x_2 - 6x_1 + 6u$
4. In matrix form,

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} u$$

$$[y] = [1 \;\; 0 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Que 4.6. **Obtain the state variable representation of the system described by the following differential equations :**

i. $\dddot{y} + 4\ddot{y} + 5\dot{y} + 2y = u$

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ii. $$\frac{d^3x}{dt^3} + \frac{3d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = u_1(t) + 3u_2(t) + 4u_3(t)$$
and outputs,

$$y_1 = 4\frac{dx}{dt} + 3u_1$$

$$y_1 = \frac{d^2x}{dt^2} + 4u_2 + u_3$$

Answer

i. Given : $\dddot{y} + 4\ddot{y} + 5\dot{y} + 2y = u$
To Find : State variable representation.

1. Let
$$x_1 = y$$
2. Then
$$x_2 = \dot{y} \quad \dot{x}_1 = x_2$$
$$x_3 = \ddot{y} \quad \dot{x}_2 = x_3$$
3. $\dot{x}_3 + 4x_3 + 5x_2 + 2x_1 = u$
$$\dot{x}_3 = -4x_3 - 5x_2 - 2x_1 + u$$
4. In matrix form $\quad \dot{x} = Ax + Bu$
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = Cx$$
$$[y] = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

ii. Given : $$\frac{d^3x}{dt^3} + \frac{3d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = u_1(t) + 3u_2(t) + 4u_3(t)$$
$$y_1(t) = 4\frac{dx}{dt} + 3u_1$$
$$y_2(t) = \frac{d^2x}{dt^2} + 4u_2 + u_3$$
To Find : State space representation.

1. Let
$$x_1 = x$$
$$\dot{x}_1 = \dot{x} = x_2$$
$$\ddot{x}_1 = \dot{x}_2 = x_3$$
2. Then,
$$\dot{x}_3 + 3x_3 + 4x_2 + 4x_1 = u_1 + 3u_2 + 4u_3$$

$$\dot{x}_3 = -3x_3 - 4x_2 - 4x_1 + u_1 + 3u_2 + 4u_3$$
3. State space matrix
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -4 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$
4. Output,
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

Que 4.7. **Derive the expression for the transfer function of a State Model. Find the transfer function of the system given by :**
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
AKTU 2013-14, Marks 10

Answer

Transfer function of state model : Refer Q. 4.3, Page 4–4D, Unit-4.
Numerical :
1. From the given model,
$$A = \begin{bmatrix} 0 & 3 \\ -2 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$$
2. $$\text{T.F} = C\,[s\,I - A]^{-1}\,B = \frac{C\,\text{adj}[s\,I - A]B}{|s\,I - A|}$$
3. $$[s\,I - A] = \begin{bmatrix} s & -3 \\ 2 & s+5 \end{bmatrix}, \text{ adj } [s\,I - A] = \begin{bmatrix} s+5 & 3 \\ -2 & s \end{bmatrix}$$
4. $$|s\,I - A| = s^2 + 5\,s + 6 = (s+2)\,(s+3)$$
5. $$\text{T. F} = \frac{\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} s+5 & 3 \\ -2 & s \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}}{(s+2)(s+3)} = \frac{\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & s+8 \\ s & s-2 \end{bmatrix}}{(s+2)(s+3)}$$
6. $\therefore$
$$\text{T. F} = \begin{bmatrix} \dfrac{s+6}{(s+2)(s+3)} & \dfrac{3s+14}{(s+2)(s+3)} \\ \dfrac{3}{(s+2)(s+3)} & \dfrac{s+8}{(s+2)(s+3)} \end{bmatrix}$$

Que 4.8. **Find the transfer function from state variable representation. Also determine the roots.**

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$$\ddot{y} + 5\dot{y} + 6y = u$$

Answer

Given : $\ddot{y} + 5\dot{y} + 6y = u$

To Find : i. Transfer function.

ii. Roots *i.e.*, eigen values.

1. Let $x_1 = y$ and $x_2 = \dot{x}_1$

$$x_2 = \dot{y}$$

$$\dot{x}_2 + 5x_2 + 6x_1 = u$$

$$\dot{x}_2 = -5x_2 - 6x_1 + u$$

2. State space matrix,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}[u]$$

$$y = Cx$$

$$[y] = [1 \quad 0]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

3. Transfer function $= C[sI - A]^{-1}B = \dfrac{C \text{ adj}[sI - A]B}{|sI - A|}$

$$= \frac{[1 \quad 0]\begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\begin{vmatrix} s & -1 \\ 6 & s+5 \end{vmatrix}} = \frac{1}{s^2 + 5s + 6}$$

Transfer function $= \dfrac{1}{s^2 + 5s + 6}$

ii. Characteristic equation,

$$|sI - A| = \begin{vmatrix} s & -1 \\ 6 & s+5 \end{vmatrix} = s^2 + 5s + 6 = 0$$

Eigen values are –2 and –3.

Que 4.9. **Transform the system into diagonal form of representation, $\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y = 6u$.**

Answer

1. Let $x_1 = y$ Then $x_2 = \dot{x}_1$

Let $x_2 = \dot{y}$ Then $x_3 = \dot{x}_2$

and $x_3 = \ddot{y}$

$$\dot{x}_3 = -6x_3 - 11x_2 - 6x_1 + 6u$$

2.

3. State space matrix,

$$\dot{x} = Ax + Bu$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix}u$$

and the output, $y = Cx$

$$y = [1\ 0\ 0]\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

4. Characteristic equation,

$$|sI - A| = \det\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 6 & 11 & s+6 \end{bmatrix} = s^3 + 6s^2 + 11s + 6 = 0$$

5. Eigen values are,

$$s_1 = -1,\ s_2 = -2,\ s_3 = -3$$

6. The transformation matrix,

$$P = \begin{bmatrix} 1 & 1 & 1 \\ s_1 & s_2 & s_3 \\ s_1^2 & s_2^2 & s_3^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$P^{-1} = \frac{-1}{2}\begin{bmatrix} -6 & -5 & -1 \\ 6 & 8 & 2 \\ -2 & -3 & -1 \end{bmatrix}$$

7. Hence, diagonal matrix Λ

$$\Lambda = P^{-1}AP$$

$$= \frac{-1}{2}\begin{bmatrix} -6 & -5 & -1 \\ 6 & 8 & 2 \\ -2 & -3 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

8. $$\hat{B} = P^{-1}B = \frac{-1}{2}\begin{bmatrix} -6 & -5 & -1 \\ 6 & 8 & 2 \\ -2 & -3 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ -6 \\ 3 \end{bmatrix}$$

9. $$\hat{C} = CP = [1 \quad 0 \quad 0]\begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} = [1 \quad 1 \quad 1]$$

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10. Vector matrix differential equation of transformed system :

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\\ \dot{z}_3\end{bmatrix} = \begin{bmatrix}-1 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -3\end{bmatrix}\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix} + \begin{bmatrix}3\\ -6\\ 3\end{bmatrix}u$$

11. Output :

$$y = [1\ 1\ 1]\begin{bmatrix}z_1\\ z_2\\ z_3\end{bmatrix}$$

Que 4.10. **Derive the state equation of a system having transfer function as follows :**

$Y(s)/U(s) = 8/s(s+2)(s+3)$

Use

i. Cascade and

ii. Parallel decomposition.

AKTU 2016-17, Marks 10

Answer

i. **Cascade form :**

1. $$\frac{Y(s)}{U(s)} = \frac{8}{s(s+2)(s+3)}$$

$$= \frac{1}{s}\frac{1}{(s+2)}\frac{1}{(s+3)}8 = H_1(s)\ H_2(s)\ H_3(s)$$

2. Let
$$\dot{q}_1 = x(t)$$
$$\dot{q}_2 = q_1 - 2q_2$$
$$\dot{q}_3 = q_2 - 3q_3$$
$$y(t) = 8q_3$$

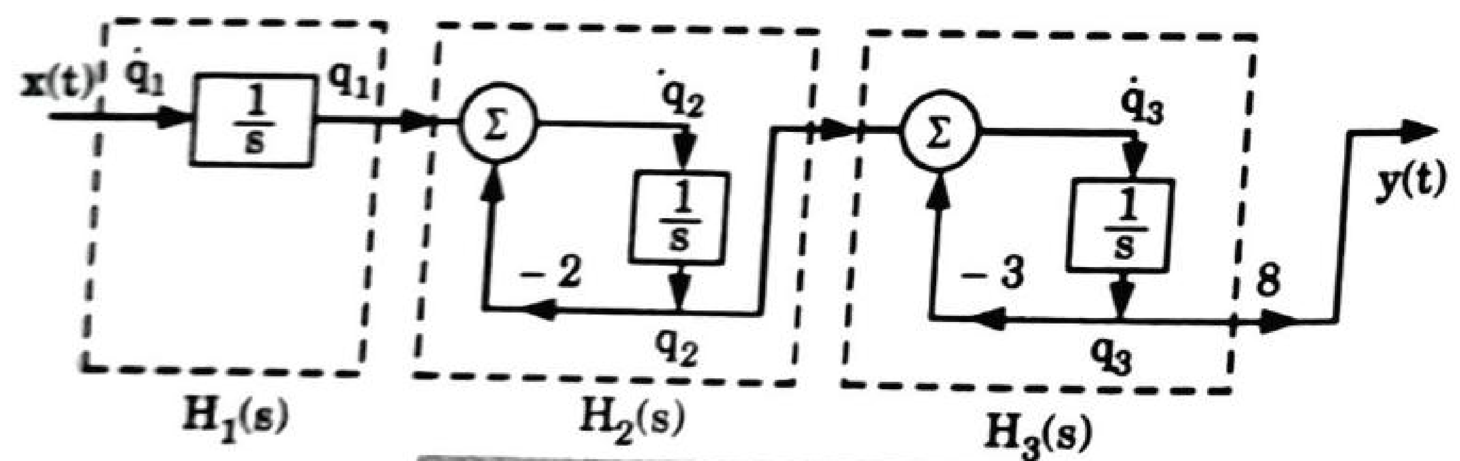

Fig. 4.10.1. Cascade form.

3. State equations are given as :

$$\dot{q}(t) = \begin{bmatrix}0 & 0 & 0\\ 1 & -2 & 0\\ 0 & 1 & -3\end{bmatrix}q(t) + \begin{bmatrix}1\\ 0\\ 0\end{bmatrix}x(t)$$

$$y(t) = [0\ 0\ 8]q(t)$$

ii. **Parallel decomposition :**

1. $$\frac{Y(s)}{U(s)} = \frac{8}{s(s+2)(s+3)}$$

2. Using partial fraction,

$$T(s) = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$A = sT(s)|_{s=0}$$

$$= \frac{8}{(s+2)(s+3)}\bigg|_{s=0} = \frac{4}{3}$$

$$B = (s+2)\,T(s)|_{s=-2} = \frac{8}{s(s+3)}\bigg|_{s=-2} = -4$$

$$C = (s+3)\,T(s)|_{s=-3} = \frac{8}{s(s+2)}\bigg|_{s=-3} = \frac{8}{3}$$

3. $$T(s) = \frac{4}{3s} - \frac{4}{(s+2)} + \frac{8}{3}\frac{1}{(s+3)}$$

4. Let
$$\dot{q}_1 = x(t) = 0q_1 + 0q_2 + 0q_3 + x(t)$$
$$\dot{q}_2 = -2q_2 + x(t) = 0q_1 - 2q_2 + 0q_3 + x(t)$$
$$\dot{q}_3 = 0q_1 + 0q_2 - 3q_3 + x(t)$$
$$y(t) = \frac{4}{3}q_1 - 4q_2 + \frac{8}{3}q_3$$

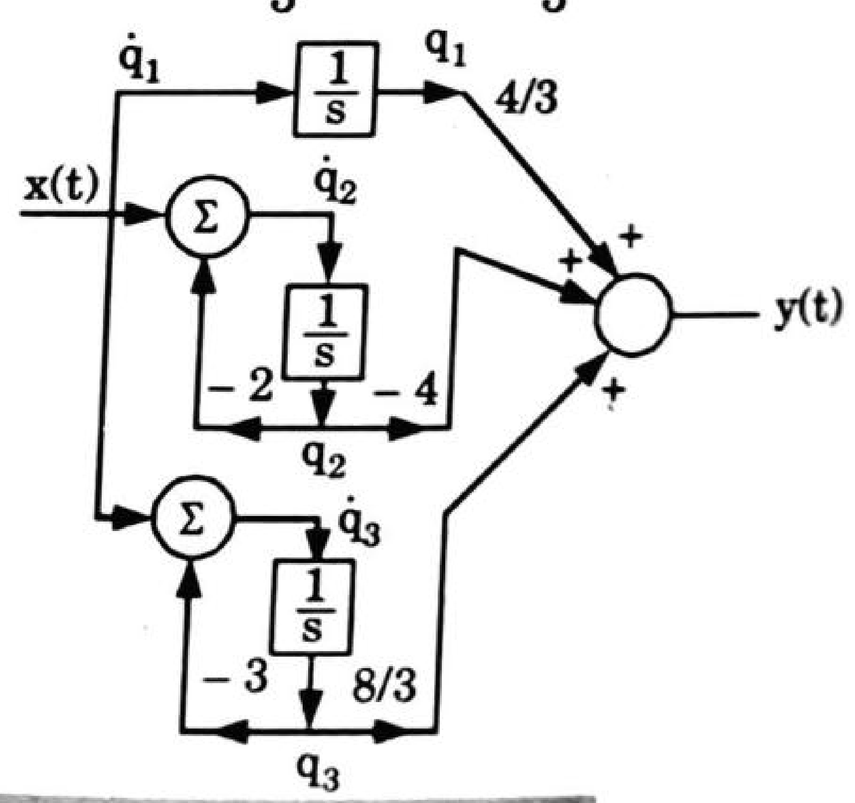

Fig. 4.10.2. Parallel form.

5. The state equations are given as :

$$\dot{q}(t) = \begin{bmatrix}0 & 0 & 0\\ 0 & -2 & 0\\ 0 & 0 & -3\end{bmatrix}q(t) + \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}x(t)$$

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$$y(t) = \begin{bmatrix} \frac{4}{3} & -4 & \frac{8}{3} \end{bmatrix} q(t)$$

Que 4.11. **Obtain the state model of the electrical circuit shown in Fig. 4.11.1.**

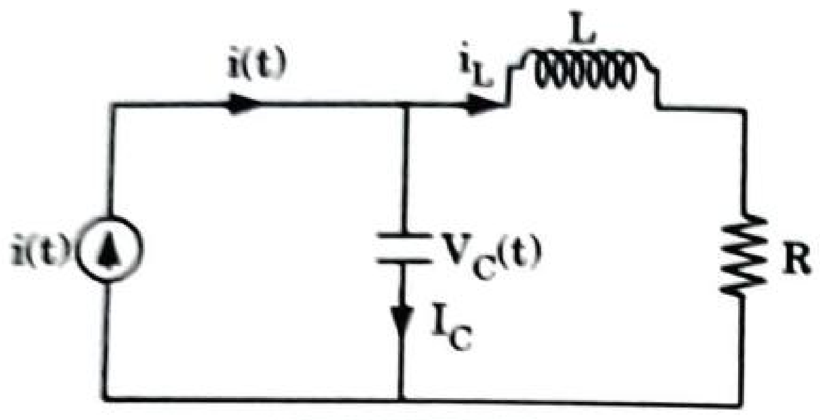

Fig. 4.11.1.

AKTU 2015-16, Marks 05

Answer

1. Applying KCL,

$$i = i_C + i_L$$

$$i = \frac{CdV_C}{dt} + i_L$$

$$\frac{dV_C}{dt} = \frac{-1}{C} i_L + \left(\frac{1}{C}\right) i \quad ...(4.11.1)$$

2. Applying KVL,

$$V_C = \frac{Ldi_L}{dt} + Ri_L$$

$$\frac{di_L}{dt} = \left(\frac{1}{L}\right) V_C + \left(-\frac{R}{L}\right) i_L \quad ...(4.11.2)$$

3. Using eq. (4.11.1) and (4.11.2), state model of electrical circuit is,

$$\begin{bmatrix} \dot{V}_c \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} 0 & \frac{-1}{C} \\ \frac{1}{L} & \frac{-R}{L} \end{bmatrix} \begin{bmatrix} V_C \\ i_L \end{bmatrix} + \begin{bmatrix} 1/C \\ 0 \end{bmatrix} i$$

4. Output, $V_R = Ri_L = [0 \quad R] \begin{bmatrix} V_C \\ i_L \end{bmatrix}$

Que 4.12. **Write the state variable formulation of the parallel RLC network.**

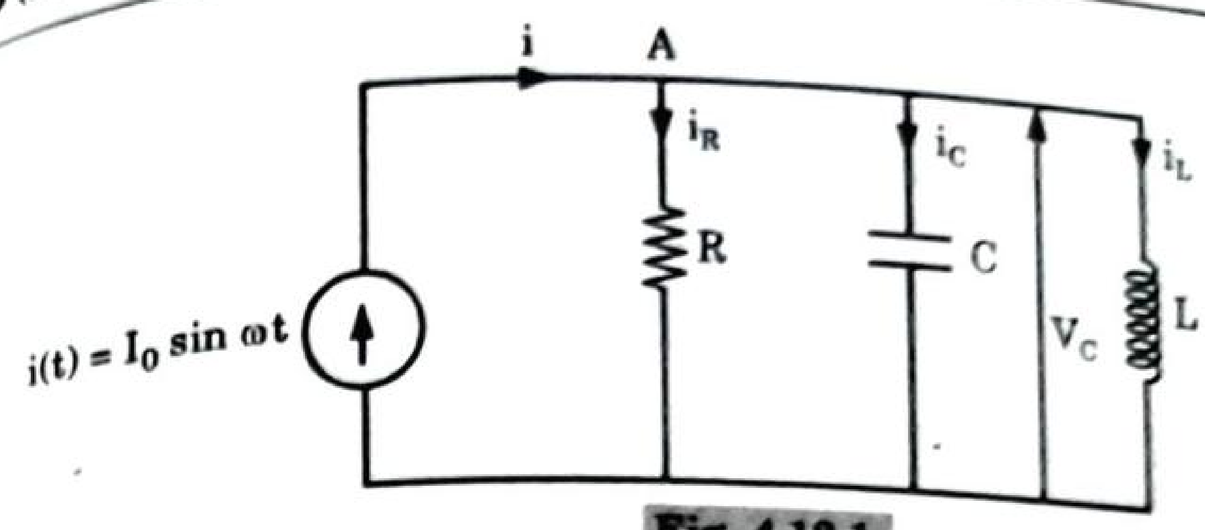

Fig. 4.12.1.

Answer

1. KCL at node A,

$$i_R + i_C + i_L = I_0 \sin \omega t$$

$$\frac{V_C(t)}{R} + \frac{1}{L}\int V_C(t)\, dt + \frac{CdV_C(t)}{dt} = I_0 \sin \omega t$$

2. Differentiating $w.r.t.$ t and dividing by C,

$$\frac{1}{RC}\frac{dV_C(t)}{dt} + \frac{1}{LC} V_C(t) + \frac{d^2V_C(t)}{dt^2} = \frac{I_0\omega}{C}\cos \omega t$$

$$\frac{d^2V_C(t)}{dt^2} + \frac{1}{RC}\frac{dV_C(t)}{dt} + \frac{1}{LC} V_C(t) = \frac{I_0\omega}{C}\cos \omega t \quad ...(4.12.1)$$

3. Let, $V_C(t) = x_1(t)$

$$\dot{x}_1(t) = x_2(t)$$

Eq. (4.12.1) becomes,

$$\dot{x}_2(t) = -\frac{1}{RC} x_2(t) - \frac{1}{LC} x_1(t) + \frac{I_0\omega}{C}\cos \omega t$$

4. State space representation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \frac{-1}{LC} & \frac{-1}{RC} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{I_0\omega}{C}\cos \omega t \end{bmatrix}$$

5. Output

$$V_C = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

PART-2

State Transition Matrix, Solution of State Equation for Homogeneous and Non-Homogeneous System, Application of State Variable Technique to the Analysis of Linear Systems.

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CONCEPT OUTLINE : PART-2

- **State Transition Matrix :**

$$\phi(t) = e^{At} = \sum_{r=0}^{\infty} \frac{A^r t^r}{r!}$$

$$\phi(t) = L^{-1}[sI - A]^{-1} = L^{-1}[\phi(s)]$$

- **Homogeneous and non-homogeneous system :** The state equation of a linear time-invariant system is given by

$$\dot{x}(t) = Ax(t) + Bu(t)$$

Case 1 : If A is a constant matrix and $u(t)$ is a zero vector, *i.e.*, no control forces are applied to the system, then this equation represents a homogeneous linear time-invariant system.

Case 2 : If A is a constant matrix and $u(t)$ is non-zero vector, *i.e.*, control forces are applied to the system, then this equation represents a non-homogeneous linear time-invariant system.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.13. What is the State Transition Matrix ? Derive its expression. Enlist its properties with proofs.

AKTU 2013-14, Marks 10

OR

Define state transition matrix. Explain the properties of state transition matrix.

AKTU 2011-12, Marks 06

OR

What do you mean by STM ? Also mention its properties.

AKTU 2014-15, Marks 3.5

OR

What is state transition matrix ? List the important properties of state transition matrix.

AKTU 2015-16, Marks 05

Answer

State transition matrix : The matrix $\phi(t) = \exp(At)$ is an $n \times n$ matrix and it helps in transition from initial state $X(0)$ to any other state $x(t)$ for $t > 0$, hence $\phi(t)$ is called state transition matrix.

Derivation :

1. Consider homogeneous equation, with $u(t) = 0$

$$\dot{x} = Ax$$

Laplace transform,

$$sX(s) - x(0) = A\,X(s)$$

$$(sI - A)\,X(s) = x(0)$$

$I \rightarrow$ Identity matrix

$s \rightarrow$ Scalar Laplace operator

2. Multiplying both sides by $[sI - A]^{-1}$

$$X(s) = [sI - A]^{-1} x(0)$$

$$X(s) = \phi(s)\,x(0)$$

where,

$$\phi(s) = [sI - A]^{-1}$$

$$\phi(s) = \frac{1}{s}\left[I - \frac{A}{s}\right]^{-1} = \frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \ldots$$

3. Taking inverse Laplace transform of $X(s)$

$$x(t) = L^{-1}[X(s)] = L^{-1}[sI - A]^{-1} x(0)$$

$$= L^{-1}\left[\frac{I}{s} + \frac{A}{s^2} + \frac{A^2}{s^3} + \ldots\right] x(0)$$

$$= \left[I + At + \frac{A^2 t^2}{\underline{|2}} + \ldots\right] x(0)$$

$$x(t) = e^{At}\,x(0) = \phi(t)\,x(0)$$

4. State transition matrix,

$$\phi(t) = L^{-1}[sI - A]^{-1} = L^{-1}[\phi(s)]$$

5. So, state transition matrix (STM) can be given as,

$$\phi(t) = e^{At} = I + At + \frac{1}{\underline{|2}}A^2t^2 + \frac{1}{\underline{|3}}A^3t^3 + \ldots$$

Properties of STM :

1. $\phi(0) = e^{A0} = I$

Proof : $\phi(0) = e^{A \times 0} = I$

2. $\phi^{-1}(t) = \phi(-t)$

Proof : $\phi^{-1}(t) = \dfrac{1}{\phi(t)} = \dfrac{1}{e^{At}} = e^{-At} = \phi(-t)$

3. $\phi(t_2 - t_1)\,\phi(t_1 - t_0) = \phi(t_2 - t_0)$

Proof : $\phi(t_2 - t_1)\phi(t_1 - t_0) = e^{A(t2 - t1)} \cdot e^{A(t1 - t0)} = e^{A(t2 - t1 + t1 - t0)} = e^{A(t2 - t0)}$

$= \phi(t_2 - t_0)$

4. $[\phi(t)]^k = \phi(kt)$

Proof : $\phi(t)^k = \phi(t).\phi(t) \ldots k$ times $= e^{At} \cdot e^{At} \ldots k$ times $= e^{Akt} = \phi(kt)$

5. $\phi(t_1 + t_2) = \phi(t_1)\,\phi(t_2) = \phi(t_2)\,\phi(t_1)$

Proof : $\phi(t_1 + t_2) = e^{A(t1 + t2)} = e^{At1} \cdot e^{At2} = e^{At2} \cdot e^{At1} = \phi(t_1) \cdot \phi(t_2) = \phi(t_2) \cdot \phi(t_1)$

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Que 4.14. **What are homogeneous and non-homogeneous systems ? Derive the solution of the two systems in terms of the state variables.**

AKTU 2012-13, Marks 10

Answer

Homogeneous system : If in a state model of a system, the matrix A is a constant matrix and input control forces zero, then the equation takes the form, $\dot{X}(t) = A\,X(t)$, such an equation is called homogeneous equation and the system is called homogeneous system.

Non-homogeneous system : If in a state model of a system, if A is a constant matrix and input control forces are applied to the system, then the system is called non-homogeneous system.

Solution of homogeneous state equation :

1. $\dot{x} = A(x)$ $\{u(t) = 0$ for homogeneous equation$\}$
2. Taking Laplace transform
   $$sX(s) - x(0) = A\,X(s)$$
   Hence, $[sI - A]\,X(s) = x(0)$
   where I is identity matrix and s is the scalar Laplace operator.
3. Premultiplying both side by $[sI - A]^{-1}$
   $$X(s) = [sI - A]^{-1}\,x(0) = \phi(s)\,x(0)$$
4. Taking the inverse Laplace transform of $X(s)$
   $$x(t) = L^{-1}\left[\frac{1}{s} + \frac{A}{s^2} + \frac{A_2}{s^3} + \ldots\right]x(0)$$
   $$x(t) = \phi(t)\,x(0)$$
   $\phi(t) = e^{At}$ is $(n \times n)$ matrix and is called State Transition Matrix.
   $\phi(t)$ is unique solution of
   $$\frac{d\phi(t)}{dt} = A\,\phi(t)\,,\ \phi(0) = I$$

Solution of non-homogeneous equation :

1. Consider state equation
   $$\dot{x}(t) = Ax(t) + Bu(t)$$
   $$U(s) = L[u(t)]$$
   $$X(s) = L[x(t)]$$
2. Taking Laplace transform,
   $$sX(s) - x(0) = AX(s) + BU(s)$$
   $$[sI - A]\,X(s) = x(0) + BU(s)$$
3. Premultiplying by $[sI - A]^{-1}$
   $$X(s) = [sI - A]^{-1}\,x(0) + [sI - A]^{-1}\,BU(s)$$
4. Taking inverse Laplace transform
   $$x(t) = L^{-1}\{X(s)\}$$
   $$= L^{-1}[sI - A]^{-1}\,x(0) + L^{-1}[sI - A]^{-1}\,BU(s)$$
   $$x(t) = \phi(t)x(0) + \int_0^t \phi(t-\tau)\,Bu(\tau)\,d\tau$$
   $$= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

5. If the initial time is other than zero, say t_0 then
   $$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$
   $$= \phi(t-t_0)\,x(t_0) + \int_{t_0}^t \phi(t-\tau)\,Bu(\tau)\,d\tau$$
6. $$x(t) = x_c(t) + x_p(t)$$
   where $x_c(t) = \phi(t)\,x(0)$
   is the complementary solution of the state vector
   and $x_p(t) = \int_{t_0}^t \phi(t-\tau)\,Bu(\tau)\,d\tau$
   is a particular solution of vector.

Que 4.15. **Obtain the state transition matrix of the following system :**

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} X \text{ with } X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

AKTU 2011-12, Marks 05

Answer

Given : $A = \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}$, $X(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

To Find : State transition matrix, $X(t)$.

1. $$[sI - A] = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 6 & s+5 \end{bmatrix}$$
2. $$\text{Adj}\,[sI - A] = \begin{bmatrix} s+5 & -6 \\ 1 & s \end{bmatrix}^T = \begin{bmatrix} s+5 & 1 \\ -6 & s \end{bmatrix}$$
3. $$|sI - A| = s^2 + 5s + 6 = (s+2)(s+3)$$
4. $$[sI - A]^{-1} = \frac{\text{Adj}\,[sI - A]}{|sI - A|}\begin{bmatrix} \dfrac{s+5}{(s+2)(s+3)} & \dfrac{1}{(s+2)(s+3)} \\ \dfrac{-6}{(s+2)(s+3)} & \dfrac{s}{(s+2)(s+3)} \end{bmatrix} = \phi(s)$$

5. $$e^{At} = L^{-1}[sI - A]^{-1} = L^{-1}\begin{bmatrix} \frac{3}{s+2} - \frac{2}{s+3} & \frac{1}{s+2} - \frac{1}{s+3} \\ \frac{-6}{s+2} + \frac{6}{s+3} & \frac{-2}{s+2} + \frac{3}{s+3} \end{bmatrix}$$

6. $\therefore$ $$e^{At} = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6e^{-2t} + 6e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}$$

7. $\therefore$ $$X(t) = e^{At}X(0) = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} & e^{-2t} - e^{-3t} \\ -6e^{-2t} + 6e^{-3t} & -2e^{-2t} + 3e^{-3t} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

8. $\therefore$ $$X(t) = \begin{bmatrix} 3e^{-2t} - 2e^{-3t} \\ -6e^{-2t} + 6e^{-3t} \end{bmatrix}$$

Que 4.16. **What is the state transition matrix ? What are its properties ? Find the state transition matrix for a system matrix** $A = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}$.

AKTU 2012-13, Marks 10

Answer

State transition matrix and properties : Refer Q. 4.13, Page 4–15D, Unit-4.

Numerical :

Given : $A = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}$

To Find : State transition matrix, $\phi(t)$.

1. $$[sI - A] = \begin{bmatrix} s & 1 \\ -2 & s+3 \end{bmatrix}$$

2. $$[sI - A]^{-1} = \frac{\text{adj}[sI - A]}{\det[sI - A]} = \frac{1}{s^2 + 3s + 2}\begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix}$$

3. $$\phi(t) = L^{-1}[sI - A]^{-1}$$
$$= \begin{bmatrix} L^{-1}\left(\frac{s+3}{(s+1)(s+2)}\right) & L^{-1}\left(\frac{-1}{(s+1)(s+2)}\right) \\ L^{-1}\left(\frac{2}{(s+1)(s+2)}\right) & L^{-1}\left(\frac{s}{(s+1)(s+2)}\right) \end{bmatrix}$$
$$= \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix}$$

4. $$\phi_{11}(t) = L^{-1}\left(\frac{s+3}{(s+1)(s+2)}\right) = L^{-1}\left[\frac{2}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right]$$
$$= 2e^{-t} - e^{-2t}$$

5. $$\phi_{12}(t) = -1\left[L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s+2}\right)\right]$$
$$= -e^{-t} + e^{-2t}$$

6. $$\phi_{21}(t) = 2\left[L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s+2}\right)\right] = 2(e^{-t} - e^{-2t})$$

7. $$\phi_{22}(t) = L^{-1}\left(\frac{2}{s+2}\right) - L^{-1}\left(\frac{1}{s+1}\right) = 2e^{-2t} - e^{-t}$$

8. State transition matrix, $$\phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & -e^{-t} + e^{-2t} \\ 2(e^{-t} - e^{-2t}) & 2e^{-2t} - e^{-t} \end{bmatrix}$$

Que 4.17. **Obtain the response of the system :**

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}X + \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}U(t), X(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and $$Y(t) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}X$$

to the following input $U(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} u(t) \\ e^{-3t}u(t) \end{bmatrix}$

where, $u(t)$ **is a unit step function.**

AKTU 2012-13, Marks 10

AKTU 2013-14, Marks 10

AKTU 2014-15, Marks 06

Answer

1. $$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

2. $$[sI - A] = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

3. $$[sI - A]^{-1} = \frac{1}{s(s+3)+2}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} = \frac{1}{s^2 + 3s + 2}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$
$$= \frac{1}{(s+1)(s+2)}\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

4. $$\phi(t) = L^{-1}[sI - A]^{-1}$$

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$$= L^{-1}\begin{bmatrix} \dfrac{s+3}{(s+1)(s+2)} & \dfrac{1}{(s+1)(s+2)} \\ \dfrac{-2}{(s+1)(s+2)} & \dfrac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$= L^{-1}\begin{bmatrix} \dfrac{2}{s+1}-\dfrac{1}{s+2} & \dfrac{1}{s+1}-\dfrac{1}{s+2} \\ \dfrac{-2}{s+1}+\dfrac{2}{s+2} & \dfrac{-1}{s+1}+\dfrac{2}{s+2} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}$$

5. $$x(t) = \phi(t)\,x(0) + \int_0^t \phi(t-\tau)\,BU(\tau)\,d\tau$$

6. $$\phi(t)\,x(0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

7. $$\int_0^t \phi(t-\tau)\,BU(\tau)\,d\tau = \int_0^t \begin{bmatrix} 2e^{-(t-\tau)}-e^{-2(t-\tau)} & e^{-(t-\tau)}-e^{-2(t-\tau)} \\ -2e^{-(t-\tau)}+2e^{-2(t-\tau)} & -e^{-(t-\tau)}+2e^{-2(t-\tau)} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u(\tau) \\ e^{-3\tau}\,u(\tau) \end{bmatrix} d\tau$$

$$= \int_0^t \begin{bmatrix} 4e^{-(t-\tau)}-2e^{-2(t-\tau)} & 3e^{-(t-\tau)}-2e^{-2(t-\tau)} \\ -4e^{-(t-\tau)}+4e^{-2(t-\tau)} & -3e^{-(t-\tau)}+4e^{-2(t-\tau)} \end{bmatrix}\begin{bmatrix} u(\tau) \\ e^{-3\tau}\,u(\tau) \end{bmatrix} d\tau$$

$$= \int_0^t \begin{bmatrix} 4e^{-(t-\tau)}-2e^{-2(t-\tau)}+3e^{-(t-\tau+3\tau)}-2e^{-2(t-\tau)-3\tau} \\ -4e^{-(t-\tau)}+4e^{-2(t-\tau)}-3e^{-(t-\tau+3\tau)}+4e^{-2t+2\tau-3\tau} \end{bmatrix} d\tau$$

$$= \int_0^t \begin{bmatrix} 4e^{-(t-\tau)}-2e^{-2(t-\tau)}+3e^{-(t+2\tau)}-2e^{-(2t+\tau)} \\ -4e^{-(t-\tau)}+4e^{-2(t-\tau)}-3e^{-(t+2\tau)}+4e^{-(2t+\tau)} \end{bmatrix} d\tau$$

8. $$x_1(t) = \int_0^t 4e^{-(t-\tau)}\,d\tau - \int_0^t 2e^{-2(t-\tau)}\,d\tau + \int_0^t 3e^{-(t+2\tau)}\,d\tau - \int_0^t 2e^{-(2t+\tau)}\,d\tau$$

$$= 4\int_0^t e^{-t}e^{\tau}\,d\tau - \int_0^t 2e^{-2t}e^{2\tau}\,d\tau + \int_0^t 3e^{-t}e^{-2\tau}\,d\tau - \int_0^t 2e^{-2t}e^{-\tau}\,d\tau$$

$$= 4e^{-t}\left[e^{\tau}\right]_0^t - 2e^{-2t}\left[\frac{e^{2\tau}}{2}\right]_0^t + 3e^{-t}\left[\frac{e^{-2\tau}}{-2}\right]_0^t - 2e^{-2t}\left[\frac{e^{-\tau}}{-1}\right]_0^t$$

$$= 4e^{-t}\left[e^{t}-1\right] - e^{-2t}\left[e^{2t}-1\right] - \frac{3}{2}e^{-t}\left[e^{-2t}-1\right] + 2e^{-2t}\left[e^{-t}-1\right]$$

$$= 4 - 4e^{-t} - 1 + e^{-2t} - \frac{3}{2}e^{-3t} + \frac{3}{2}e^{-t} + 2e^{-3t} - 2e^{-2t}$$

$$= 3 - \frac{5}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}$$

9. $$x_2(t) = \int_0^t -4e^{-(t-\tau)}\,d\tau + \int_0^t 4e^{-2(t-\tau)}\,d\tau - \int_0^t 3e^{-(t+2\tau)}\,d\tau \quad \int_0^t 4e^{-(2t+\tau)}\,d\tau$$

$$= -4\int_0^t e^{-t}e^{\tau}\,d\tau + 4\int_0^t e^{-2t}e^{2\tau}\,d\tau - 3\int_0^t e^{-t}e^{-2\tau}\,d\tau + 4\int_0^t e^{-2t}e^{-\tau}\,d\tau$$

$$= -4e^{-t}\left[e^{\tau}\right]_0^t + 4e^{-2t}\left[\frac{e^{2\tau}}{2}\right]_0^t - 3e^{-t}\left[\frac{e^{-2\tau}}{-2}\right]_0^t - 4e^{-2t}\left[e^{-\tau}\right]_0^t$$

$$= -4e^{-t}\left[e^{t}-1\right] + 2e^{-2t}\left[e^{2t}-1\right] + \frac{3e^{-t}}{2}\left[e^{-2t}-1\right] - 4e^{-2t}\left[e^{-t}-1\right]$$

$$= -4 + 4e^{-t} + 2 - 2e^{-2t} + \frac{3}{2}e^{-3t} - \frac{3}{2}e^{-t} - 4e^{-3t} + 4e^{-2t}$$

$$= -2 + \frac{5}{2}e^{-t} + 2e^{-2t} - \frac{5}{2}e^{-3t}$$

10. $\therefore$ $$X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 3-\frac{5}{2}e^{-t}-e^{-2t}+\frac{1}{2}e^{-3t} \\ -2+\frac{5}{2}e^{-t}+2e^{-2t}-\frac{5}{2}e^{-3t} \end{bmatrix}$$

11. $\therefore$ Output response,

$$Y(t) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 5-\frac{5}{2}e^{-t}-e^{-2t}+\frac{1}{2}e^{-3t} \\ -2+\frac{5}{2}e^{-t}+2e^{-2t}-\frac{5}{2}e^{-3t} \end{bmatrix}$$

12. $\therefore$ $$Y(t) = \begin{bmatrix} 3-\frac{5}{2}e^{-t}-e^{-2t}+\frac{1}{2}e^{-3t} \\ 3+e^{-2t}-2e^{-3t} \end{bmatrix}$$

Que 4.18. **Obtain the state transition matrix of a system given by**

$$A = \begin{bmatrix} -1/2 & -5/2 \\ 1/2 & -7/2 \end{bmatrix}$$

AKTU 2015-16, Marks 05

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Answer

Given : $A = \begin{bmatrix} -1/2 & -5/2 \\ 1/2 & -7/2 \end{bmatrix}$

To Find : State transition matrix, $\phi(t)$.

1. $$\phi(t) = L^{-1}[sI - A]^{-1}$$

2. $$[sI - A] = \begin{bmatrix} s+1/2 & 5/2 \\ -1/2 & s+7/2 \end{bmatrix}$$

3. Co-factor of $[sI - A] = \begin{bmatrix} s+7/2 & 1/2 \\ -5/2 & s+1/2 \end{bmatrix}$

4. $$\text{adj}\,[sI - A] = \begin{bmatrix} s+7/2 & -5/2 \\ 1/2 & s+1/2 \end{bmatrix}$$

5. $$|sI - A| = \left(s+\frac{1}{2}\right)\left(s+\frac{7}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{-5}{2}\right)$$
$$= s^2 + \frac{7s}{2} + \frac{s}{2} + \frac{7}{4} + \frac{5}{4} = s^2 + \frac{8s}{2} + \frac{12}{4}$$
$$= s^2 + 4s + 3$$

6. $$[sI - A]^{-1} = \frac{\begin{bmatrix} s+7/2 & -5/2 \\ 1/2 & s+1/2 \end{bmatrix}}{s^2 + 4s + 3}$$
$$= \begin{bmatrix} \dfrac{s+7/2}{(s+1)(s+3)} & \dfrac{-5/2}{(s+1)(s+3)} \\ \dfrac{1/2}{(s+1)(s+3)} & \dfrac{s+1/2}{(s+1)(s+3)} \end{bmatrix}$$

7. $$\phi(t) = L^{-1}[sI - A]^{-1}$$
$$= L^{-1}\begin{bmatrix} \dfrac{s+7/2}{(s+1)(s+3)} & \dfrac{-5/2}{(s+1)(s+3)} \\ \dfrac{1/2}{(s+1)(s+3)} & \dfrac{s+1/2}{(s+1)(s+3)} \end{bmatrix}$$
$$= \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix}$$

8. $$\phi_{11}(t) = L^{-1}\left[\frac{s+7/2}{(s+1)(s+3)}\right]$$
$$= L^{-1}\left[\frac{A}{s+1} + \frac{B}{s+3}\right]$$
$$\phi_{11}(t) = L^{-1}\left[\frac{5/4}{s+1} - \frac{1/4}{s+3}\right] = \frac{5}{4}e^{-t} - \frac{1}{4}e^{-3t}$$

9. $$\phi_{12}(t) = L^{-1}\left[\frac{-5/2}{(s+1)(s+3)}\right]$$
$$= \left[\frac{-5/4}{s+1} + \frac{5/4}{s+3}\right] = \frac{-5}{4}e^{-t} + \frac{5}{4}e^{-3t}$$

10. $$\phi_{21}(t) = L^{-1}\left[\frac{1/2}{(s+1)(s+3)}\right]$$
$$= L^{-1}\left[\frac{1/4}{s+1} - \frac{1/4}{s+3}\right] = \frac{1}{4}e^{-t} - \frac{1}{4}e^{-3t}$$

11. $$\phi_{22}(t) = L^{-1}\left[\frac{s+1/2}{(s+1)(s+3)}\right] = L^{-1}\left[\frac{-1/4}{s+1} + \frac{5/4}{s+3}\right]$$
$$= -\frac{1}{4}e^{-t} + \frac{5}{4}e^{-3t}$$

12. ∴ $$\phi(t) = \begin{bmatrix} \frac{5}{4}e^{-t} - \frac{1}{4}e^{-3t} & \frac{-5}{4}e^{-t} + \frac{5}{4}e^{-3t} \\ \frac{1}{4}e^{-t} - \frac{1}{4}e^{-3t} & \frac{-1}{4}e^{-t} + \frac{5}{4}e^{-3t} \end{bmatrix}$$

Que 4.19. **Define controllability and observability in state variable analysis with suitable example.**

AKTU 2011-12, Marks 05

OR

What do you mean by controllability and observability in state variable analysis of system ?

AKTU 2015-16, Marks 05

Answer

Controllability :

1. A system is said to be state controllable if there exists an input that brings the system from an arbitrary initial state $x(0)$ to the zero state in a finite time.
2. In order to be able to command the given dynamic system under control input, the system must be controllable.
3. The state equation is,
$$\dot{x} = [A]\,[x] + [B]\,u(t)$$
4. This condition is satisfied if the controllability matrix
$$[Q_c] = [B]\,[AB]\,[A^2B]\,......\,[A^{n-1}B] \text{ be of rank } n.$$
5. For output controllability, the matrix,
$$[M] = [CB]\,[CAB]\,[CA^2B]\,......\,[CA^{n-1}B] \text{ be of rank } n.$$

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Observability :

1. A system is said to be observable, if the information to estimate the initial state $x(0)$ is made available by observing the output response with no input for a specified time interval.
2. In order to see what is going on inside the system under observation, the system must be observable.
3. The above situation is satisfied if the matrix

$$[0] = [C^T]\ [A^TC^T] \ \ldots\ldots\ [(A^T)^{n-1}C^T]$$

is of the rank 'n' and 'T' stands for transpose.

4. The necessary and sufficient conditions for complete observability is that no pole zero cancellation exists in $G(s)$.

Example :

1. $$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}u$$

2. $$y = [20\ \ 9\ \ 1]\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

3. $$A = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -6\end{bmatrix}, [B] = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}, [C] = [20\ \ 9\ \ 1]$$

4. The controllability matrix

$$C_M = [B\ \ AB\ \ A^2B] = \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & -6\\ 1 & -6 & 25\end{bmatrix}$$

5. The rank of this matrix is 3 and hence the system is completely state controllable.
6. The observability matrix

$$O_M = [C^T\ \ A^TC^T\ \ (A^T)^2C^T]$$

$$= \begin{bmatrix}20 & -6 & -18\\ 9 & 9 & -39\\ 1 & 3 & -9\end{bmatrix}$$

7. The rank of this matrix is 3 and hence the system is completely observable.

VERY IMPORTANT QUESTIONS

Following questions are very important. These questions may be asked in your SESSIONALS as well as UNIVERSITY EXAMINATION.

Q. 1. Define state, state variables, state vector.

Ans. Refer Q. 4.1, Unit-4.

Q. 2. Derive the expression for the transfer function of a State Model. Find the transfer function of the system given by :

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 3\\ -2 & -5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}2 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Ans. Refer Q. 4.7, Unit-4.

Q. 3. Derive the state equation of a system having transfer function as follows :

$Y(s)/U(s) = 8/s(s+2)(s+3)$

Use

i. **Cascade and**

ii. **Parallel decomposition.**

Ans. Refer Q. 4.10, Unit-4.

Q. 4. What do you mean by STM ? Also mention its properties.

Ans. Refer Q. 4.13, Unit-4.

Q. 5. What are homogeneous and non-homogeneous systems ? Derive the solution of the two systems in terms of the state variables.

Ans. Refer Q. 4.14, Unit-4.

Q. 6. Obtain the state transition matrix of the following system :

$$\dot{X} = \begin{bmatrix}0 & 1\\ -6 & -5\end{bmatrix}X \text{ with } X(0) = \begin{bmatrix}1\\ 0\end{bmatrix}$$

Ans. Refer Q. 4.15, Unit-4.

Q. 7. Obtain the response of the system :

$$\dot{X} = \begin{bmatrix}0 & 1\\ -2 & -3\end{bmatrix}X + \begin{bmatrix}2 & 1\\ 0 & 1\end{bmatrix}U(t), X(0) = \begin{bmatrix}0\\ 0\end{bmatrix}$$

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and $$Y(t) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} X$$

to the following input $U(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} u(t) \\ e^{-3t}u(t) \end{bmatrix}$

where, $u(t)$ is a unit step function.

Ans. Refer Q. 4.17, Unit-4.

Q. 8. Define controllability and observability in state variable analysis with suitable example.

Ans. Refer Q. 4.19, Unit-4.

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5 UNIT Z-Transform Analysis

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PART-1

Concept of Z-Transform, Properties of Z-Transform, Z-Transform of Common Functions.

CONCEPT OUTLINE : PART-1

Z-transform : It is the discrete counterpart of Laplace transform. Z-transform converts difference equations of discrete time system to algebraic equations which simplifies the discrete time system analysis.

ROC : ROC stands for region of convergence. The values of z in the complex z-plane for which the sum in the Z-transform equation converges is called the region of convergence.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.1. **Discuss concept of Z-transform and ROC.**

Answer

Z-transform :

1. Z-transform is the counterpart of Laplace transform for discrete-time systems.
2. It converts difference equations into algebraic equations.

$$\text{Differential equation} \xrightarrow[\text{transform}]{Z} \text{Algebraic equation}$$

Fig. 5.1.1.

3. Z-transform provides a method for the analysis of discrete-time systems in the frequency domain, and is more efficient than time domain analysis.
4. Consider the discrete time sequence

$$x[nT] \text{ or } x[n]$$

5. Z-transform is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}$$

$$x[n] \rightleftarrows X(z)$$

6. Z-transform is an infinite power series, it exists only for those values of z for which this series converges.

Region of Convergence :

The set of values of z or equivalently the set of points to z-plane, for which $X(z)$ converges is called the region of convergence (ROC) of $X(z)$. If there is no value of z(*i.e.*, no point in the z-plane) for which $X(z)$ converges, then the sequence $x[n]$ is said to be having no Z-transform.

Advantages of Z-transform :

1. Z-transform converts the difference equations of a discrete-time system into linear algebraic equations so that the analysis becomes easy and simple.
2. Convolution in the time domain converted into multiplication in z-domain.
3. Z-transform exists for most of the signals for which discrete-time Fourier transform (DTFT) does not exist.

Limitation :

Frequency domain response cannot be achieved and cannot be plotted.

Que 5.2. **Enlist the properties of the ROC in Z-transform.**

AKTU 2012-13, Marks 05

Answer

Properties of ROC :

1. The ROC is a ring or disk in the z-plane centered at the origin.
2. The ROC cannot contain any poles.
3. If $x[n]$ is an infinite duration causal sequence, the ROC is $|z| > \alpha$, *i.e.*, it is the exterior of a circle of radius α.
4. If $x[n]$ is an infinite duration anti-causal sequence, the ROC is $|z| < \beta$, *i.e.*, it is the interior of a circle of radius β.
5. If $x[n]$ is a finite duration anti-causal sequence (left-sided sequence), the ROC is entire z-plane except at $z = \infty$.
6. If $x[n]$ is a finite duration two-sided sequence, the ROC is entire z-plane except at $z = 0$ and $z = \infty$.
7. If $x[n]$ is an infinite duration, two-sided sequence, the ROC consists of a ring in the z-plane (ROC ; $\alpha < |z| < \beta$) bounded on the interior and exterior by a pole, not containing any poles.
8. The ROC of an LTI stable system contains the unit circle.
9. The ROC must be a connected region. If $X(z)$ is rational, then its ROC is bounded by poles or extends up to infinity.
10. $x[n] = \delta[n]$ is the only signal whose ROC is entire z-plane.

Que 5.3. **Show that for a stable system, the ROC of a system function includes the unit circle.**

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Answer

1. Necessary and sufficient condition for a causal linear time invariant system to be BIBO stable is :

$$= \sum_{n=0}^{\infty} |h[n]| < \infty$$

2. The system function of a causal LTI system is :

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

3. The magnitude of the Z-transform is :

$$|H(z)| = \sum_{n=0}^{\infty} |h[n] z^{-n}| \le \sum_{n=0}^{\infty} |h[n]|\,|z^{-n}|$$

4. The evaluation of $|H(z)|$ on unit circle yields

$$|H(z)| \le \sum_{n=0}^{\infty} |h[n]| < \infty \qquad (\because \ |z| = 1 \text{ for unit circle})$$

Therefore, for a stable system, the ROC of a system function includes the unit circle.

Que 5.4. Define the properties of Z-transform.

AKTU 2012-13, Marks 05

Answer

Properties of Z-transform :

1. **Linearity :**

If $x_1[n] \xleftrightarrow{ZT} X_1(z)$

$x_2[n] \xleftrightarrow{ZT} X_2(z)$

then $x[n] = a_1 x_1[n] + a_2 x_2[n] \xleftrightarrow{ZT} X(z) = a_1 X_1(z) + a_2 X_2(z)$

2. **Multiplication by a constant :**

If $x[n] \xleftrightarrow{ZT} X(z)$; ROC $= R$

then $ax[n] \xleftrightarrow{ZT} aX(z)$; ROC $= R$

where $a \to$ constant

3. **Time shifting :**

If $x[n] \xleftrightarrow{ZT} X(z)$

then, $x[n-k] \xleftrightarrow{ZT} z^{-k} X(z)$

4. **Scaling in Z-transform :**

$x[n] \xleftrightarrow{ZT} X(z)$; ROC : $r_1 < z < r_2$

$a^n x[n] \xleftrightarrow{ZT} X(a^{-1}z)$; ROC : $|a| r_1 < z < |a| r_2$

5. **Time reversal :**

$x[n] \xleftrightarrow{ZT} X(z)$; ROC $= R$

$x[-n] \xleftrightarrow{ZT} X(z^{-1})$; ROC $= \dfrac{1}{r_2} < |z| < \dfrac{1}{r_1}$

6. **Time expansion :**

$x[n] \xleftrightarrow{ZT} X(z)$; ROC $= R$

$x_k[n] \xleftrightarrow{ZT} X(z^k)$; ROC $= R^{1/k}$

7. **Conjugation :**

$x[n] \xleftrightarrow{ZT} X(z)$; ROC $= R$

$x^*[n] \xleftrightarrow{ZT} X^*(z^*)$; ROC $= R$

8. **Initial value theorem :**

If limit exists, then

$$x[0] = \lim_{n\to 0} x[n] = \lim_{z\to\infty} X(z)$$

9. **Final value theorem :**

$$x[\infty] = \lim_{n\to\infty} x[n] = \lim_{z\to 1} (z-1) X(z)$$

10. **Differentiation in the z-domain :**

If $x[n] \xleftrightarrow{ZT} X(z)$

then $nx[n] \xleftrightarrow{ZT} \dfrac{-z\, dX(z)}{dz}$

11. **Convolution of two sequences :**

If $x_1[n] \xleftrightarrow{ZT} X_1(z)$

$x_2[n] \xleftrightarrow{ZT} X_2(z)$

then, $x[n] = x_1[n] * x_2[n] \xleftrightarrow{ZT} X(z) = X_1(z)\, X_2(z)$

ROC : intersection of $X_1(z)$ and $X_2(z)$

Que 5.5. Derive the relationship between Laplace and Z-transforms.

Answer

1. Let $x(t)$ be a continuous signal.
2. The discrete version of the signal $x(t)$ is $x^*(t)$.
3. $x^*(t)$ is obtained by sampling $x(t)$ with a sampling period of T second, *i.e.*, $x^*(t)$ is obtained by multiplying $x(t)$ with a sequence of impulses T second apart.

$$\therefore \quad x^*(t) = \sum_{n=0}^{\infty} x[nT]\, \delta[t - nT]$$

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4. The Laplace transform of $x^*(t)$ is given by

$$L(x^*(t)) = L\left[\sum_{n=0}^{\infty} x[nT]\,\delta\,[t-nT]\right]$$

$$= \sum_{n=0}^{\infty} x[nT]\,L\,[\delta[t-nT]] = \sum_{n=0}^{\infty} x[nT]\,e^{-nTs}$$

5. The Z-transform of signal $x[nT]$ is given by

$$Z(x^*(t)) = Z(x[nT]) = \sum_{n=0}^{\infty} x[nT]z^{-n}$$

6. So the relation between Laplace transform and Z-transform is :

$$L(x^*(t)) = Z(x^*(t)) \text{ with } z = e^{sT}$$

Que 5.6. **What is the difference between the Z-transform and the Laplace transform ? Explain.**

AKTU 2012-13, Marks 05

Answer

S. No.	Laplace transform	Z-transform
1.	It is used to analyse LTI continuous-time systems.	It is used to analyse LTI discrete-time systems.
2.	It converts differential equations which are in time domain into algebraic equations in s-domain.	It converts difference equations which are in time domain into algebraic equations in z-domain.
3.	It is a simple and systematic method and the complete solution can be obtained in one step and also the initial conditions can be introduced in the beginning of the process itself.	It is also a simple and systematic method and the complete solution can be obtained in one step and also the initial conditions can be introduced in the beginning of the process itself.
4.	Laplace transform may be one-sided (unilateral) or two-sided (bilateral).	Z-transform also may be one-sided (unilateral) or two-sided (bilateral).
5.	The range of values of s for which $X(s)$ converges is called ROC of $X(s)$.	The range of values of z for which $X(z)$ converges is called ROC of $X(z)$.
6.	The ROC of $X(s)$ consists of strips parallel to $j\omega$ axis in s-plane.	The ROC of $X(z)$ consists of a ring or disc in z-plane centered at the origin.

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7.	If the real part of s, *i.e.* $\sigma = 0$, then the Laplace transform becomes the continuous-time Fourier transform.	If the magnitude of z, *i.e.* $\lvert z \rvert = 1$, then the Z-transform becomes DTFT.
8.	Convolution in time domain is equal to multiplication in s-domain.	Convolution in time domain is equal to multiplication in z-domain.

Que 5.7. **a. Prove the convolution theorem of Z-transform.**
b. Find the Z-transform of cos $\omega_0 n u[n]$.

AKTU 2015-16, Marks 10

Answer

a. **Convolution Theorem :** The convolution property of Z-transform states that the Z-transform of the convolution of two signals is equal to the multiplication of their Z-transforms, *i.e.*

if $\quad x_1[n] \xleftrightarrow{ZT} X_1(z)$, with ROC $= R_1$

and $\quad x_2[n] \xleftrightarrow{ZT} X_2(z)$ with ROC $= R_2$

then $\quad x_1[n] * x_2[n] \xleftrightarrow{ZT} X_1(z)\,X_2(z)$, with ROC $= R_1 \cap R_2$

Proof :

1. We know that

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]\,x_2[n-k]$$

2. Let $\quad x[n] = x_1[n] * x_2[n]$

3. We have

$$Z(x[n]) = X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

4. $$Z(x_1[n] * x_2[n]) = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1[k]\,x_2[n-k]\right]z^{-n}$$

$$= \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x_1[k]\,x_2[n-k]\,z^{-(n-k)}z^{-k}$$

5. Interchanging the order of summations,

$$X(z) = \sum_{k=-\infty}^{\infty} x_1[k]\,z^{-k}\sum_{n=-\infty}^{\infty} x_2[n-k]\,z^{-(n-k)}$$

6. Replacing $(n-k)$ by p in the second summation, we get

$$X(z) = \sum_{k=-\infty}^{\infty} x_1[k]\,z^{-k}\sum_{p=-\infty}^{\infty} x_2[p]\,z^{-p}$$

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$$= X_1(z)\, X_2(z)$$

7. $x_1[n] * x_2[n] \xleftrightarrow{ZT} X_1(z)\, X_2(z)$; ROC $= R_1 \cap R_2$

b. Z-transform of $\cos \omega_0 n u[n]$:

1. $$x[n] = (\cos \omega_0 n)\, u[n]$$
$$= \frac{1}{2} e^{j\omega_0 n} u[n] + \frac{1}{2} e^{-j\omega_0 n} u[n]$$

2. $$X(z) = \frac{1}{2} Z\left\{e^{j\omega_0 n} u[n]\right\} + \frac{1}{2} Z\left\{e^{-j\omega_0 n} u[n]\right\}$$

3. Let
$$e^{j\omega_0 n}\, u[n] \xleftrightarrow{ZT} \frac{1}{1 - e^{j\omega_0} z^{-1}} \qquad \text{ROC} : |z| > 1$$
and
$$e^{-j\omega_0 n}\, u[n] \xleftrightarrow{ZT} \frac{1}{1 - e^{-j\omega_0} z^{-1}} \qquad \text{ROC} : |z| > 1$$

4. $\therefore$
$$X(z) = \frac{1}{2}\frac{1}{(1 - e^{-j\omega_0} z^{-1})} + \frac{1}{2}\frac{1}{(1 - e^{-j\omega_0} z^{-1})} \quad \text{ROC} : |z| > 1$$
$$X(z) = \frac{1}{2}\left[\frac{z(z - e^{-j\omega_0}) + z(z - e^{j\omega_0})}{(z - e^{j\omega_0})(z - e^{-j\omega_0})}\right]$$
$$= \frac{1}{2}\left[\frac{z\{2z - (e^{-j\omega_0} + e^{j\omega_0})\}}{z^2 - z(e^{j\omega_0} + e^{-j\omega_0}) + 1}\right] = \frac{z(z - \cos \omega_0)}{z^2 - 2z \cos \omega_0 + 1}$$
$$X(z) = \frac{1 - z^{-1} \cos \omega_0}{1 - 2z^{-1} \cos \omega_0 + z^{-2}} \qquad \text{ROC} : |z| > 1$$

Que 5.8. **Find the Z-transform of the following :**

i. $x[n] = a^n\, u[n]$

ii. $x[n] = -b^n\, u[-n-1]$

AKTU 2011-12, Marks 05

OR

Define ROC of Z-transform. Determine Z-transform of $x_1(t) = a^n u[n]$ and $x_2[n] = -a^n u[-n-1]$ and also indicate their region of convergence.

AKTU 2014-15, Marks 06

OR

Find the Z-transform of following functions :

i. $x[n] = a^n u[n]$

ii. $x[n] = -b^n u[-n-1]$

AKTU 2015-16, Marks 10

Answer

Region of Convergence : Refer Q. 5.1, Page 5–2D, Unit-5.

i. **Given :** $x[n] = a^n\, u[n]$
To Find : Z-transform of $x[n]$, $X(z)$.

1. $$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

2. If $|az^{-1}| < 1$
$$|z| > |a|$$
then,
$$X(z) = \frac{1}{1 - az^{-1}}$$

3. Region of convergence (ROC), is the exterior of circle having radius $|a|$.

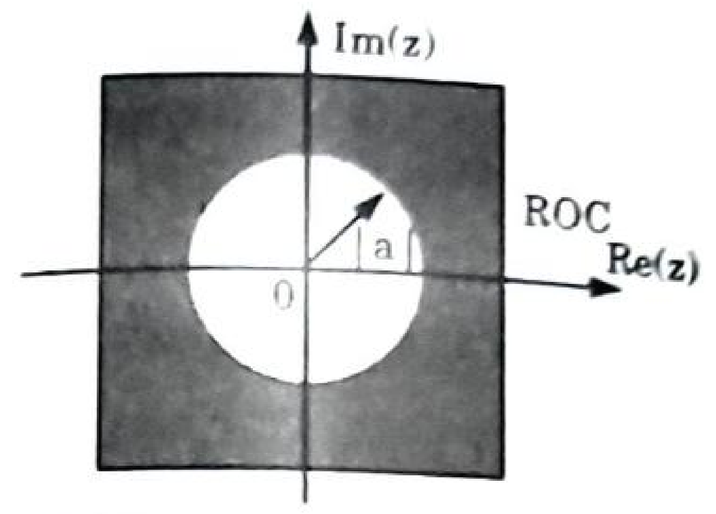

Fig. 5.8.1. ROC of $X[z]$.

ii. **Given :** $x[n] = -b^n\, u[-n-1]$.
To Find : Z-transform of $x[n]$, $X(z)$.

1. $$X(z) = -\sum_{n=-\infty}^{-1} (b^n)\, z^{-n} = -\sum_{n=1}^{\infty} (b^{-n})\, z^n = 1 - \sum_{n=0}^{\infty} (b^{-1}z)^n$$

2. $$X(z) = 1 - \frac{1}{1 - b^{-1}z} = \frac{1}{1 - bz^{-1}}$$

3. Since, $|b^{-1} z| < 1$
or $|z| < |b|$
ROC is interior of circle having radius $|b|$.

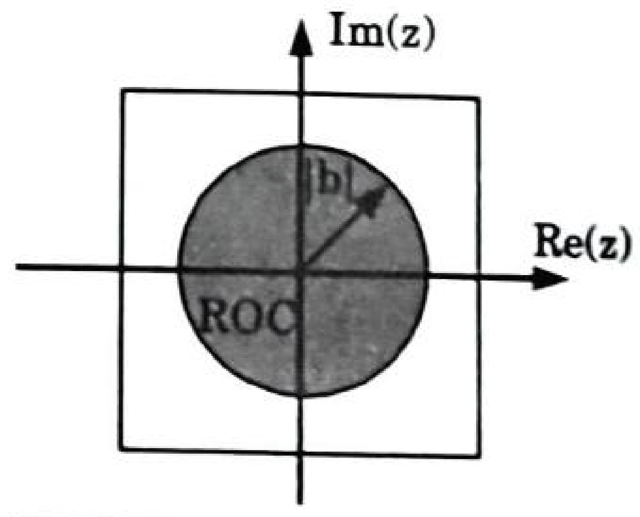

Fig. 5.8.2. ROC of $X(z)$.

Que 5.9. **Find the Z-transform of the following sequences :**

i. $Y_1[n] = \{2, 0, 3, 6, 8\}$
$\uparrow$

ii. $u[n]$.

AKTU 2012-13, Marks 05

Answer

i. **Given :** $y_1[n] = \{2, 0, 3, \underset{\uparrow}{6}, 8\}$.

To Find : $y(z)$.

$$y_1(z) = 2z^3 + 3z + 6 + 8z^{-1}$$

ii. **Given :** $u[n]$.

To Find : $X(z)$.

1. $$x[n] = 1; \quad \text{for } n \geq 0$$
$$= 0; \quad \text{for } n < 0$$

2. $$\therefore \; X(z) = Z(x[n]) = Z(u[n]) = \sum_{n=0}^{\infty} u[n] z^{-n}$$
$$= \sum_{n=0}^{\infty} 1 z^{-n} = 1 + z^{-1} + z^{-2} + z^{-3} + \ldots$$
$$= \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

3. The above series converges if $|z^{-1}| < 1$, *i.e.* ROC is $|z| > 1$. So, the ROC is the exterior of the unit circle in the z-plane.

$$u[n] \xleftrightarrow{ZT} \frac{z}{z-1} = \frac{1}{1-z^{-1}}; \; \text{ROC} = |z| > 1$$

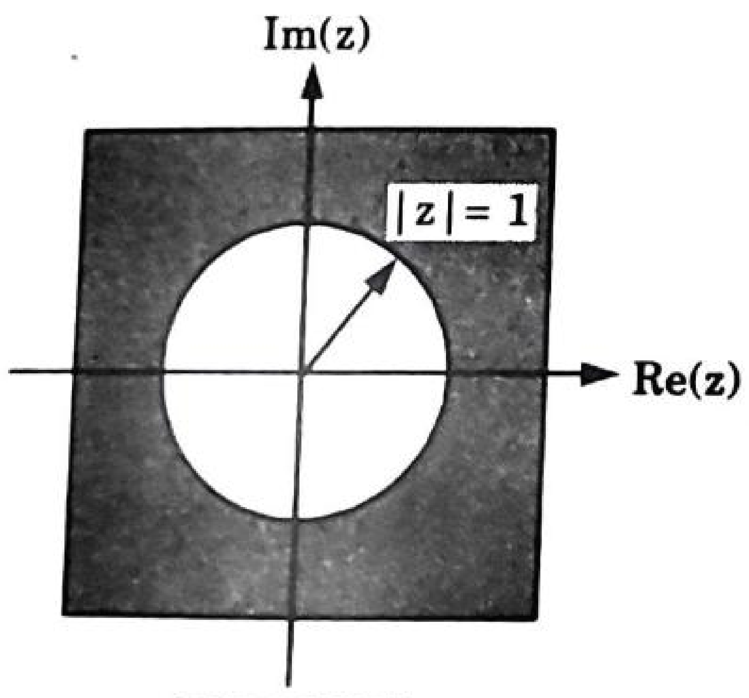

Fig. 5.9.1.

Que 5.10. **Find the Z-transform of $x[n] = na^n u[n]$.**

AKTU 2012-13, Marks 05

Answer

Given : $x[n] = na^n u[n]$

To Find : $X(z)$.

1. $$Z(x[n]) = \frac{1}{1-az^{-1}} \qquad \text{ROC} : |z| > |a|$$

2. Now, $$Z(nx[n]) = \frac{-z \, dX(z)}{dz} = \frac{az^{-1}}{(1-az^{-1})^2} \qquad \text{ROC} : |z| > |a|$$

Que 5.11. **Find the Z-transform of the signal $x[n] = n2^n u[n]$. Also find the ROC.**

AKTU 2016-17, Marks 10

Answer

Given : $x[n] = n2^n u[n]$

To find : i. Z-transform of $x[n]$, $X(z)$.

ii. ROC.

i. From the definition of Z-transform,

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=0}^{\infty} n \, 2^n z^{-n} = \sum_{n=0}^{\infty} n \, (2z^{-1})^n$$
$$= 0 + 1(2z^{-1}) + 2(2z^{-1})^2 + 3(2z^{-1})^3 + \ldots\ldots$$
$$= 2z^{-1} (1 + 2\,(2z^{-1}) + 3\,(2z^{-1})^2 + \ldots\ldots)$$

$$\therefore \quad X(z) = \frac{2z^{-1}}{(1-2z^{-1})^2}$$

ii. For ROC, $|1 - 2z^{-1}| > 0$

$$|2z^{-1}| < 1$$
$$|z^{-1}| < \frac{1}{2}$$
$$\therefore \quad |z| > 2$$

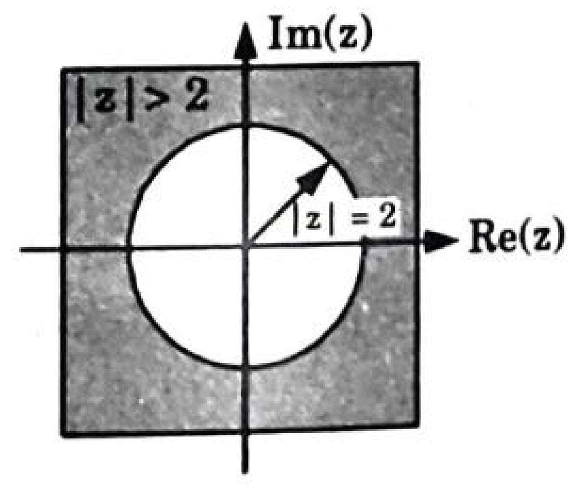

Fig. 5.11.1.

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Que 5.12. Find the Z-transform of the following sequences :

i. $x_1[n] = \{1, 2, 3, 4, 5, 0, 7\}$

ii. $x_2[n] = \{1, 2, 3, \underset{\uparrow}{4}, 5, 0, 7\}$

iii. $\delta[n]$

AKTU 2013-14, Marks 10

Answer

i. **Given :** $x_1[n] = \{1, 2, 3, 4, 5, 0, 7\}$
To Find : $X_1(z)$.

1. $x_1(0) = 1, x_1(1) = 2, x_1(2) = 3,$
$x_1(3) = 4, x_1(4) = 5, x_1(5) = 0, x_1(6) = 7$

2. By definition, $X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$

$$X_1(z) = \sum_{n=0}^{6} x_1[n]z^{-n}$$

3. Putting for $x_1[n], X_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4} + 0z^{-5} + 7z^{-6}$

$$X_1(z) = 1 + \frac{2}{z} + \frac{2}{z^2} + \frac{4}{z^3} + \frac{5}{z^4} + \frac{7}{z^6}$$

ii. **Given :** $x_2[n] = \{1, 2, 3, \underset{\uparrow}{4}, 5, 0, 7\}$
To Find : $X_2(z)$

1. $x_2(0) = 4, x_2(1) = 5, x_2(2) = 0, x_2(3) = 7$, and
$x_2(-1) = 3, x_2(-2) = 2, x_2(-3) = 1$

2. $$X_2(z) = \sum_{n=-3}^{3} x_2[n]z^{-n}$$

3. Putting for $x_2[n], X_2(z) = 1 \cdot z^3 + 2 \cdot z^2 + 3z^1 + 4z^0 + 5z^{-1} + 0z^{-2} + 7z^{-3}$

$$= z^3 + 2z^2 + 3z + 4 + \frac{5}{z} + \frac{7}{z^3}$$

iii. **Given :** $x[n] = \delta[n]$
To Find : $X(z)$.

1. $$x[n] = 1; \quad \text{for } n = 0$$
$$= 0; \quad \text{for } n \neq 0$$

2. $X(z) = Z(x[n]) = Z(\delta[n]) = \sum_{n=0}^{\infty} \delta[n]z^{-n} = 1$ (for all z)

i.e., the ROC is the entire z-plane.

$$\delta[n] \xleftrightarrow{ZT} 1 \qquad \text{for } n \leq -z$$

Que 5.13. Find the Z-transform of the following :

i. $n\left[\frac{1}{3}\right]^{n+3} u(n+3)$

ii. $n^2 u[n]$

AKTU 2013-14, Marks 10

Answer

i. **Given :** $x[n] = n\left[\frac{1}{3}\right]^{n+3} u[n+3]$
To Find : $X(z)$.

1. $$\left(\frac{1}{3}\right)^n u[n] \xleftrightarrow{ZT} \frac{1}{1-\frac{1}{3}z^{-1}}$$

2. By time shifting, $\left(\frac{1}{3}\right)^{n+3} u[n+3] \xleftrightarrow{ZT} \frac{z^3}{1-\frac{1}{3}z^{-1}}$

3. By differentiation in z-domain, $n x_1[n] \xleftrightarrow{ZT} -z\frac{d}{dz}X_1(z)$

$$\therefore \quad n\left(\frac{1}{3}\right)^{n+3} u[n+3] \xleftrightarrow{ZT} -z\frac{d}{dz}\left(\frac{z^3}{1-\frac{1}{3}z^{-1}}\right)$$

4. $\therefore$ $$X(z) = -z\frac{d}{dz}\left(\frac{z^3}{1-\frac{1}{3}z^{-1}}\right) = -z\frac{d}{dz}\left(\frac{z^4}{z-\frac{1}{3}}\right)$$

$$= -z.\frac{\left(z-\frac{1}{3}\right)4z^3 - z^4(1)}{\left(z-\frac{1}{3}\right)^2} = \frac{z^4(4/3-3z)}{(z-1/3)^2}$$

ii. **Given :** $x[n] = n^2 u[n]$
To Find : $X(z)$.

1. $$u[n] \xleftrightarrow{ZT} \frac{1}{1-z^{-1}}$$

2. By differentiation in z-domain, $nu[n] \xleftrightarrow{ZT} -z\frac{d}{dz}\left(\frac{1}{1-z^{-1}}\right)$

3. Applying differentiation again,

$$n.[nu[n]] = -z\frac{d}{dz}\left[-z\frac{d}{dz}\left(\frac{1}{1-z^{-1}}\right)\right]$$

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4. ∴
$$X(z) = -z\frac{d}{dz}\left[-z\frac{d}{dz}\left(\frac{1}{1-z^{-1}}\right)\right] = -z\frac{d}{dz}\left[-z\frac{d}{dz}\left(\frac{z}{z-1}\right)\right]$$
$$= -z\frac{d}{dz}\left[-z.\frac{(z-1)\times 1 - z}{(z-1)^2}\right] = -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$
$$= -z\left[\frac{(z-1)^2\times 1 - z\times 2(z-1)\times 1}{(z-1)^4}\right] = \frac{z^3-z}{(z-1)^4}$$
$$= \frac{z(z+1)}{(z-1)^3}$$

Que 5.14. **Find the convolution of sequences,**
$x_1[n] = (1/4)^n u[n]$ and $x_2[n] = (1/5)^{n-2} u[n-2]$ using :

i. Convolution in Z-transform

ii. Time Domain Method.

AKTU 2016-17, Marks 10

Answer

Given : $x_1[n] = \left(\frac{1}{4}\right)^n u[n]$, $x_2[n] = \left(\frac{1}{5}\right)^{n-2} u[n-2]$

To Find : i. $X(z)$ using convolution in Z-transform.
ii. $X(z)$ using time domain method.

i. **Convolution in Z-transform :**

1. Z-transform of $x_1[n]$ is given as,
$$Z(x_1[n]) = \frac{z}{z-1/4}$$

2. Again, $$x_2[n] = \left(\frac{1}{5}\right)^{n-2} u[n-2]$$

3. ∵ $$Z\left[\left(\frac{1}{5}\right)^n u[n]\right] = \frac{z}{z-1/5}$$
Using shifting property,
$$Z\left[\left(\frac{1}{5}\right)^{n-2} u[n-2]\right] = z^{-2}\frac{z}{z-1/5}$$
$$X_2(z) = \frac{z^{-1}}{z-1/5} = \frac{1}{z(z-1/5)}$$

4. Using convolution theorem,
$$X(z) = X_1(z)\,X_2(z)$$
$$= \frac{z}{(z-1/4)}\,\frac{1}{z(z-1/5)}$$

$$X(z) = \frac{1}{(z-1/4)(z-1/5)}$$

6. ∴
$$x[n] = 20\left\{Z^{-1}\left[\frac{1}{(z-1/4)} - \frac{1}{(z-1/5)}\right]\right\}$$
$$= 20\left[\left(\frac{1}{4}\right)^{n-1} u[n-1] - \left(\frac{1}{5}\right)^{n-1} u[n-1]\right]$$

ii. **Time domain method :**
$$x[n] = x_1[n] * x_2[n] = \sum_{k=0}^{\infty} n_1[k]\,x_2[n-k]$$
$$= \sum_{k=0}^{n-2}\left(\frac{1}{4}\right)^k u[k]\left(\frac{1}{5}\right)^{n-2-k} u[n-2-k]$$
$$= \sum_{k=0}^{n-2}\left(\frac{1}{4}\right)^k\left(\frac{1}{5}\right)^{n-2-k}$$
$$= \sum_{k=0}^{n-2}\left(\frac{1}{4}\right)^k\left(\frac{1}{5}\right)^n\left(\frac{1}{5}\right)^{-2}\left(\frac{1}{5}\right)^{-k}$$
$$= 25\left(\frac{1}{5}\right)^n\sum_{k=0}^{n-2}\left(\frac{5}{4}\right)^k$$
$$= 25\left(\frac{1}{5}\right)^n\left[\frac{1-\left(\frac{5}{4}\right)^{n-1}}{1-\left(\frac{5}{4}\right)}\right]$$
$$= -100\left(\frac{1}{5}\right)^n\left[1-\left(\frac{5}{4}\right)^{n-1}\right]$$
$$= -20\left[\left(\frac{1}{5}\right)^{n-1} u[n-1] - \left(\frac{1}{5}\right)^{n-1}\left(\frac{5}{4}\right)^{n-1} u[n-1]\right]$$
$$= -20\left[\left(\frac{1}{5}\right)^{n-1} u[n-1] - \left(\frac{1}{4}\right)^{n-1} u[n-1]\right]$$
$$= -20\left[\left(\frac{1}{4}\right)^{n-1} u[n-1] - \left(\frac{1}{5}\right)^{n-1} u[n-1]\right]$$

Que 5.15. **Find the Z-transform of the following signals :**

i. $x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$

ii. $x[n] = a^n u[n]$.
Also discuss the region of convergence (ROC) for each signal.

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Answer

i. **Given :** $x[n] = 7\left(\frac{1}{3}\right)^n u[n] - 6\left(\frac{1}{2}\right)^n u[n]$

To Find : $X(z)$.

To Draw : ROC.

1.
$$X(z) = \sum_{n=0}^{\infty}\left[7\left(\frac{1}{3}\right)^n - 6\left(\frac{1}{2}\right)^n\right] z^{-n}$$
$$= \sum_{n=0}^{\infty} 7\left(\frac{1}{3}\right)^n z^{-n} - \sum_{n=0}^{\infty} 6\left(\frac{1}{2}\right)^n z^{-n}$$
$$= 7\left[1 + \frac{1}{3}z^{-1} + \left(\frac{1}{3}\right)^2 z^{-2} + \left(\frac{1}{3}\right)^3 z^{-3} \ldots\right]$$
$$- 6\left[1 + \frac{1}{2}z^{-1} + \left(\frac{1}{2}\right)^2 z^{-2} + \left(\frac{1}{2}\right)^3 z^{-3} \ldots\right]$$

2. Using concept of sum of infinite power series
$$X(z) = 7\frac{1}{1 - \frac{1}{3}z^{-1}} - 6\frac{1}{1 - \frac{1}{2}z^{-1}}$$
$$= \frac{7 - \frac{7}{2}z^{-1} - 6 + \frac{6}{3}z^{-1}}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)}$$
$$X(z) = \frac{1 - \frac{3}{2}z^{-1}}{\underbrace{\left(1 - \frac{1}{3}z^{-1}\right)}_{F_1(z)}\underbrace{\left(1 - \frac{1}{2}z^{-1}\right)}_{F_2(z)}}$$

3. The ROC of $F_1(z)$ is $\left|\frac{1}{3}z^{-1}\right| < 1$ and $F_2(z)$ is $\left|\frac{1}{2}z^{-1}\right| < 1$
$$\left|\frac{1}{3}z^{-1}\right| < 1 \Rightarrow |z| > 1/3$$
$$\left|\frac{1}{2}z^{-1}\right| < 1 \Rightarrow |z| > \frac{1}{2}$$

4. Thus overall ROC of $X(z)$ is $|z| > 1/3$

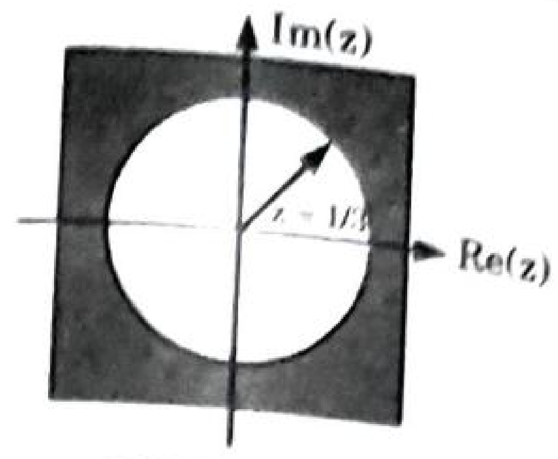

Fig. 5.15.1.

ii. Refer Q. 5.8, Page 5–8D, Unit-5.

Que 5.16. **Determine the Z-transform of the signal**

$$x[n] = \begin{cases} a^n, & \text{for } n \geq 0 \\ -b^n, & \text{for } n \leq -1 \end{cases}$$

OR

$$x[n] = a^n u[n] + b^n u[-n-1] \qquad \text{also, } |a| < |b|$$

Answer

1.
$$X(z) = \sum_{n=-\infty}^{-1} -b^n z^{-n} + \sum_{n=0}^{\infty} a^n z^{-n}$$
$$= \frac{z}{z-b} + \frac{z}{z-a} = \frac{z(2z-a-b)}{(z-a)(z-b)}$$

2. ROC : $|a| < |z| < |b|$

3. Pole zero plot and region of convergence,

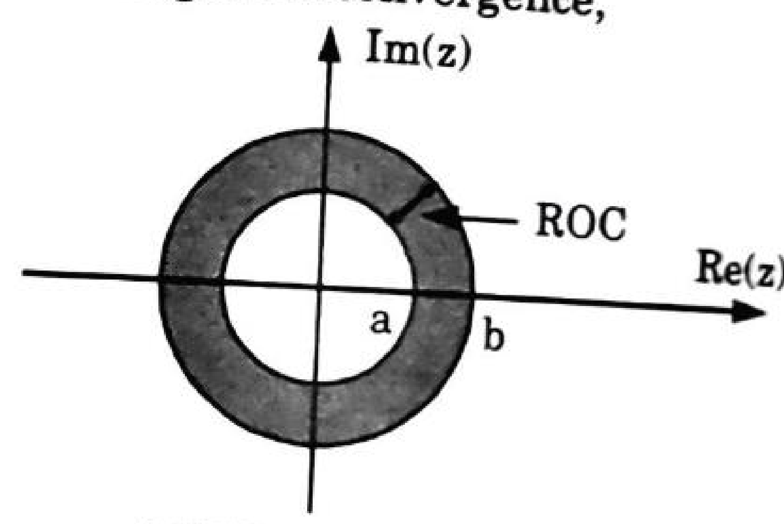

Fig. 5.16.1.

Que 5.17. **Compute the convolution $x[n]$ of the signals**

$$x_1[n] = \{1, -2, 1\}$$
$$x_2[n] = \begin{cases} 1; & 0 \leq n \leq 5 \\ 0; & \text{otherwise} \end{cases}$$

Answer

1.

2.
$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$
$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

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3. $$X(z) = X_1(z)\,X_2(z) = 1 - z^{-1} - z^{-6} + z^{-7}$$

(Using convolution property)

4. $$x[n] = \{\underset{\uparrow}{1}, -1, 0, 0, 0, 0, -1, 1\}$$

Que 5.18. **Determine the Z-transform of the following finite duration signals :**

i. $x_1[n] = \{1, 2, 5, 7, 0, 1\}$

ii. $x_2[n] = \{1, 2, \underset{\uparrow}{5}, 7, 0, 1\}$

iii. $x_3[n] = \{0, 0, 1, 2, 5, 7, 0, 1\}$

iv. $x_4[n] = \{2, 4, \underset{\uparrow}{5}, 7, 0, 1\}$

v. $x_5[n] = \delta[n]$

vi. $x_6[n] = \delta[n - k],\ k > 0$

vii. $x_7[n] = \delta[n + k],\ k > 0$

Answer

i. $$X_1(z) = 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

ROC : entire z-plane except $z = 0$.

ii. $$X_2(z) = z^2 + 2z + 5 + 7z^{-1} + z^{-2}$$

ROC : entire z-plane except $z = 0$ and $z = \infty$.

iii. $$X_3(z) = z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$$

ROC : entire z-plane except $z = 0$.

iv. $$X_4(z) = 2z^2 + 4z + 5 + 7z^{-1} + z^{-3}$$

ROC : entire z-plane except $z = 0$ and $z = \infty$.

v. $$X_5(z) = 1 \quad i.e.,\ \delta[n] \xleftrightarrow{ZT} 1$$

ROC : entire z-plane.

vi. $$X_6(z) = z^{-k} \quad i.e.,\ \delta[n-k] \xleftrightarrow{ZT} z^{-k}$$

ROC : entire z-plane except $z = 0$.

vii. $$X_7(z) = z^{k} \quad i.e.,\ \delta[n+k] \xleftrightarrow{ZT} z^{k}$$

ROC : entire z-plane except $z = \infty$.

PART-2

Inverse Z-transform, Initial and Final Value Theorems.

CONCEPT OUTLINE : PART-2

- The process of finding the time domain signal $x[n]$ from its Z-transform $X(z)$ is called the inverse Z-transform which is denoted as :

$$x[n] = Z^{-1}(X(z))$$

We have $$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} [x[n]r^{-n}]\,e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi j}\oint_c X(z)\,z^{n-1}\,dz$$

where the symbol $\oint_c$ denotes integration around the circle of radius $|z| = r$ in counter clockwise direction.

This is the Direct method of finding inverse Z-transform of $X(z)$.

- Basically there are four methods to find inverse Z-transform :
  a. Power series method or long division method
  b. Partial fraction expansion method
  c. Complex inversion integral method (Residue method)
  d. Convolution integral method.
- **Initial value theorem :** $x[0] = \lim_{n\to 0} x[n] = \lim_{z\to\infty} X(z)$
- **Final value theorem :** $x[\infty] = \lim_{n\to\infty} x[n] = \lim_{z\to 1} (z-1)X(z)$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.19. **State and explain initial and final value theorem using Z-transform analysis.**

AKTU 2011-12, Marks 05

Answer

A. **Initial value theorem :** The initial value theorem of Z-transform states that, for a causal signal $x[n]$

if $$x[n] \xleftrightarrow{ZT} X(z)$$

then $$\lim_{n\to 0} x[n] = x(0) = \lim_{z\to\infty} X(z)$$

Proof :

1. For a causal signal

$$Z(x[n]) = X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = x(0) + x(1)\,z^{-1} + x(2)z^{-2} + ...$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + ...$$

2. Taking the limit $z \to \infty$ on both sides, we have

$$\lim_{z\to\infty} X(z) = \lim_{n\to\infty}\left[x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \frac{x(3)}{z^3} + ...\right]$$

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$$= x(0) + 0 + 0 + \ldots = x(0)$$

$$x(0) = \lim_{z\to\infty} X(z)$$

3. This theorem helps us to find the initial value of $x[n]$ from $X(z)$ without taking its inverse Z-transform.

B. Final value Theorem : The final value theorem of Z-transform states that, for a causal signal

if $\quad x[n] \xleftrightarrow{ZT} X(z)$

and if $X(z)$ has no poles outside the unit circle, and it has no double or higher order poles on the unit circle centered at the origin of the z-plane, then

$$\lim_{n\to\infty} x[n] = x(\infty) = \lim_{z\to 1} (z-1)X(z)$$

Proof :

1. For a causal signal

$$Z(x[n]) = X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

2. $$Z(x[n+1]) = zX(z) - zx(0) = \sum_{n=0}^{\infty} x[n+1]z^{-n}$$

3. $$\therefore \; Z(x[n+1]) - Z(x[n]) = zX(z) - zx(0) - X(z)$$

$$= \sum_{n=0}^{\infty} x[n+1]z^{-n} - \sum_{n=0}^{\infty} x[n]z^{-n}$$

i.e., $$(z-1)X(z) - zx(0) = \sum_{n=0}^{\infty}[x[n+1] - x[n]]z^{-n}$$

$$(z-1)X(z) - zx(0) = [x(1) - x(0)]z^{-0} + [x(2) - x(1)]z^{-1} + [x(3) - x(2)]z^{-2} + \ldots$$

4. Taking limit as $z \to 1$ on both sides, we have

$$\lim_{z\to 1}(z-1)X(z) - x(0) = [x(1) - x(0) + x(2) - x(1) + x(3) - x(2) + \ldots + x(\infty) - x(\infty - 1)]$$

$$= x(\infty) - x(0)$$

$$x(\infty) = \lim_{z\to 1}(z-1)X(z)$$

5. This theorem enables us to find the steady-state value of $x[n]$, *i.e.* $x(\infty)$ without taking the inverse Z-transform of $X(z)$.

Que 5.20. **Determine the inverse Z-transform of the following functions :**

i. $X(z) = (z-1)/(z^2 - 4z + 4)$

ii. $X(z) = z^2/(z^2 - 5/4z + 3/8)$

AKTU 2016-17, Marks 10

Answer

i. Given : $X(z) = (z-1)/(z^2 - 4z + 4)$

To Find : $x[n]$.

$$X(z) = \frac{(z-1)}{(z-2)^2}$$

1. $$\frac{X(z)}{z} = \frac{(z-1)}{z(z-2)^2}$$

$$\frac{X(z)}{z} = \frac{A}{z} + \frac{B}{(z-2)} + \frac{C}{(z-2)^2}$$

2. $$A = z\frac{X(z)}{z}\bigg|_{z=0}$$

3. $$= \frac{z-1}{(z-2)^2}\bigg|_{z=0} = \frac{-1}{4}$$

4. $$B = \frac{1}{1!}\frac{d}{dz}\left[(z-2)^2\frac{X(z)}{z}\right]_{z=2}$$

$$= \frac{d}{dz}\left[\frac{(z-1)}{z}\right]_{z=2} = \left[\frac{1}{z^2}\right]_{z=2} = \frac{1}{4}$$

5. $$C = (z-2)^2\frac{X(z)}{z}\bigg|_{z=2} = \frac{z-1}{z}\bigg|_{z=2} = \frac{1}{2}$$

6. $$\frac{X(z)}{z} = \left(\frac{-1}{4}\right)\frac{1}{z} + \left(\frac{1}{4}\right)\frac{1}{(z-2)} + \left(\frac{1}{2}\right)\frac{1}{(z-2)^2}$$

$$X(z) = \left(\frac{-1}{4}\right) + \left(\frac{1}{4}\right)\frac{z}{(z-2)} + \left(\frac{1}{2}\right)\frac{z}{(z-2)^2}$$

7. Taking inverse Z-transform,

$$x[n] = \left(\frac{-1}{4}\right)\delta(n) + \frac{1}{4}2^n u(n) + \frac{1}{2}n\,2^n u(n)$$

ii. Given : $X(z) = \dfrac{z^2}{\left(z^2 - \frac{5}{4}z + \frac{3}{8}\right)}$

To Find : $x[n]$.

1. $$\frac{X(z)}{z} = \frac{8z}{8z^2 - 10z + 3}$$

$$\frac{X(z)}{z} = \frac{8z}{(2z-1)(4z-3)}$$

2. Using partial fraction,

$$\frac{X(z)}{z} = \frac{A}{(2z-1)} + \frac{B}{(4z-3)}$$

3. $$A = (2z-1)\frac{X(z)}{z}\bigg|_{z=1/2} = \frac{8z}{(4z-3)}\bigg|_{z=1/2} = -4$$

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4.
$$B = (4z-3)\left.\frac{X(z)}{z}\right|_{z=3/4}$$
$$= \left.\frac{8z}{(2z-1)}\right|_{z=3/4} = 12$$

5.
$$\frac{X(z)}{z} = \frac{-4}{(2z-1)} + \frac{12}{(4z-3)}$$

6. $\therefore$
$$X(z) = \frac{-4z}{2(z-1/2)} + \frac{12z}{4(z-3/4)}$$

7. Taking inverse Z-transform,
$$x[n] = -2\left(\frac{1}{2}\right)^n u(n) + 3\left(\frac{3}{4}\right)^n u(n)$$

Que 5.21. **Find the inverse Z-transform of following :**

i.
$$X(z) = \frac{1-\frac{1}{3}z^{-1}}{(1-z^{-1})(1+2z^{-1})}; \quad \text{ROC} : |z| > 2$$

ii.
$$X(z) = \log_{10}(1 + az^{-1}); \quad \text{ROC} : |z| > |a|$$

AKTU 2015-16, Marks 05

Answer

i. **Given :** $X(z) = \dfrac{1-\frac{1}{3}z^{-1}}{(1-z^{-1})(1+2z^{-1})}$; ROC : $|z| > 2$

To Find : $x[n]$.

1.
$$X(z) = \frac{1-\dfrac{1}{3z}}{\left(1-\dfrac{1}{z}\right)\left(1+\dfrac{2}{z}\right)}$$
$$= \frac{\dfrac{3z-1}{3z}}{\left(\dfrac{z-1}{z}\right)\left(\dfrac{z+2}{z}\right)}$$
$$= \frac{z(3z-1)}{3(z-1)(z+2)}$$

2. Using partial fraction,
$$\frac{X(z)}{z} = \frac{1}{3}\left[\frac{(3z-1)}{(z-1)(z+2)}\right]$$

$$\frac{X(z)}{z} = \frac{1}{3}\left[\frac{A}{z-1} + \frac{B}{z+2}\right]$$
$$A = \left.\frac{3z-1}{z+2}\right|_{z=1} = \frac{2}{3}$$
$$B = \left.\frac{3z-1}{z-1}\right|_{z=-2} = \frac{7}{3}$$

3.
$$\frac{X(z)}{z} = \frac{1}{3}\left[\frac{2/3}{z-1} + \frac{7/3}{z+2}\right]$$
$$X(z) = \frac{1}{3}\left[\frac{2/3\,z}{z-1} + \frac{7/3\,z}{z+2}\right]$$
$$= \frac{1}{3}\left[\frac{2/3}{1-z^{-1}} + \frac{7/3}{1+2z^{-1}}\right]$$

4. For $|z| > 2$,

$\therefore$
$$x[n] = \frac{1}{3}[2/3\,u(n) + 7/3\,(-2)^n\,u(n)]$$

ii. **Given :** $X(z) = \log_{10}(1 + az^{-1})$; ROC : $|z| > |a|$.

To Find : $x[n]$.

1.
$$X(z) = \log_{10}(1 + az^{-1})$$
$$= \frac{\log_e(1 + az^{-1})}{\log_e(10)}$$
$$= \frac{1}{\log_e(10)}\left[az^{-1} - \frac{(az^{-1})^2}{2} + \frac{(az^{-1})^3}{3} - \frac{(az^{-1})^4}{4} + \frac{(az^{-1})^5}{5} - \dots\right]$$
$$= -\sum_{n=1}^{\infty}\frac{(-az^{-1})^n}{n\log_e(10)} = -\sum_{n=1}^{\infty}\frac{(-a)^n}{n\log_e(10)}z^{-n}$$

2. Therefore, the inverse Z-transform is
$$x[n] = \left[0, \frac{a}{\log_e(10)}, -\frac{a^2}{2\log_e(10)}, \frac{a^3}{3\log_e(10)}, -\frac{a^4}{4\log_e(10)}, \dots\right]$$
$$= -\frac{(-a)^n}{n\log_e(10)}u(n-1)$$

Que 5.22. **Find the inverse Z-transform of the following function :**

$$X(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}}$$

for ROC (*i*) $|z| > 1$, (*ii*) $|z| < 0.5$ and (*iii*) $0.5 < |z| < 1$. Draw the various ROCs.

AKTU 2012-13, Marks 05

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Answer

Given : $X(z) = \dfrac{1}{1 - 1.5z^{-1} + 0.5z^{-2}}$

To Find :
i. $x[n]$ for ROC $|z| > 1$.
ii. $x[n]$ for ROC $|z| < 0.5$.
iii. $x[n]$ for ROC $0.5 < |z| < 1$.

$$X(z) = \frac{1}{(1 - z^{-1})\left(1 - \frac{1}{2}z^{-1}\right)}$$

Using partial fraction,

$$X(z) = \frac{2}{(1 - z^{-1})} - \frac{1}{\left(1 - \frac{1}{2}z^{-1}\right)}$$

i. For ROC $|z| > 1$, both signals must be causal.

Hence, $x[n] = (2 - (0.5)^n)\, u[n]$

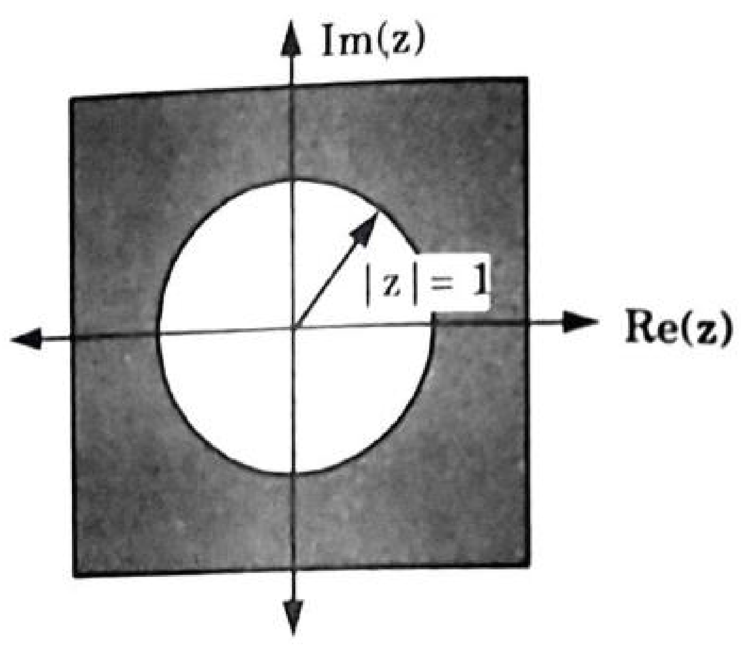

Fig. 5.22.1.

ii. For ROC $|z| < 0.5$, both signals must be anti-causal.

$\therefore$ $x[n] = \left[2 + \left(\frac{1}{2}\right)^n\right] u(-n-1)$

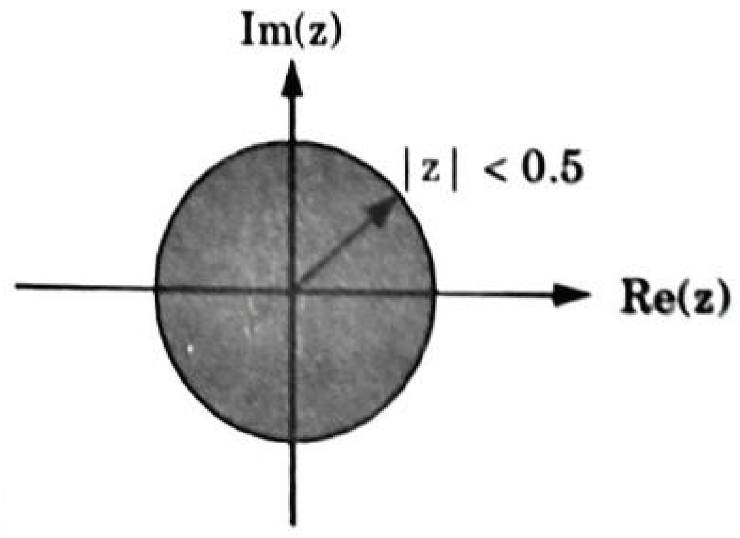

Fig. 5.22.2.

iii. For ROC $0.5 < |z| < 1$, $\dfrac{1}{\left(1 - \frac{1}{2}z^{-1}\right)}$ is causal and $\dfrac{1}{(1 - z^{-1})}$ is anti-causal.

Hence $x[n] = -2u[-n-1] - \left(\frac{1}{2}\right)^n u[n]$

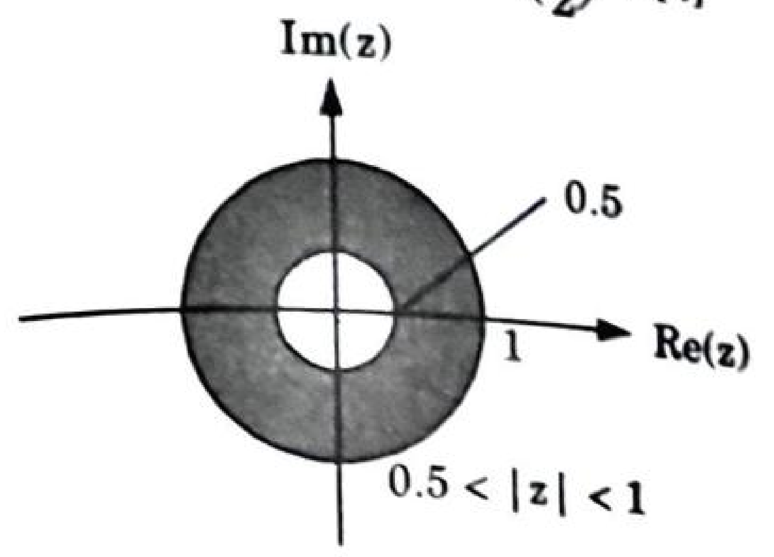

Fig. 5.22.3.

Que 5.23. **Find the inverse Z-transform of the following function :**
$X(z) = 1/(1 + z^{-1})^2 (1 - z^{-1})$; **ROC :** $|z| > 1$

AKTU 2016-17, Marks 10

Answer

Given : $X(z) = \dfrac{1}{(1 + z^{-1})^2 (1 - z^{-1})}$

To Find : $x[n]$

1. $$X(z) = \frac{z^3}{(z+1)^2 (z-1)}$$

$$\frac{X(z)}{z} = \frac{z^2}{(z+1)^2 (z-1)}$$

2. Using partial fraction,

$$\frac{X(z)}{z} = \frac{A}{(z+1)} + \frac{B}{(z+1)^2} + \frac{C}{(z-1)}$$

$$A = \frac{1}{1!}\frac{d}{dz}\left[(z+1)^2 \frac{X(z)}{z}\right]_{z=-1}$$

$$= \frac{d}{dz}\left[\frac{z^2}{z-1}\right]_{z=-1}$$

$$= \left[\frac{-z^2}{(z-1)^2} + \frac{2z}{z-1}\right]_{z=-1} = \frac{3}{4}$$

$$B = (z+1)^2 \left.\frac{X(z)}{z}\right|_{z=-1}$$

$$= \left.\frac{z^2}{(z-1)}\right|_{z=-1} = -\frac{1}{2}$$

$$C = (z-1)\left.\frac{X(z)}{z}\right|_{z=1}$$

$$= \left.\frac{z^2}{(z+1)^2}\right|_{z=1} = \frac{1}{4}$$

3. $$\frac{X(z)}{z} = \frac{3/4}{(z+1)} - \frac{1/2}{(z+1)^2} + \frac{1/4}{(z-1)}$$

$$X(z) = \frac{3}{4}\left(\frac{z}{z+1}\right) - \frac{1}{2}\frac{z}{(z+1)^2} + \frac{1}{4}\frac{z}{(z-1)}$$

4. Taking inverse Z-transform,

$$x[n] = \left[\frac{3}{4}(-1)^n - \frac{1}{2}n(-1)^n + \frac{1}{4}(1)^n\right]u(n)$$

Que 5.24. Find inverse Z-transform of the following function :

$$F(z) = \frac{1}{2(z+0.5)(z-1)}$$

AKTU 2011-12, Marks 05

Answer

Given : $F(z) = \frac{1}{2(z+0.5)(z-1)}$

To Find : $f[n]$.

1. $$\frac{F(z)}{z} = \frac{1}{2z(z+0.5)(z-1)}$$

2. Using partial fraction,

$$\frac{F(z)}{z} = \frac{A}{z} + \frac{B}{(z+0.5)} + \frac{C}{(z-1)}$$

3. $$A = \left.\frac{F(z)}{z}\right|_{z=0} = \frac{1}{2(z+0.5)(z-1)} = -1$$

4. $$B = (z+0.5)\left.\frac{F(z)}{z}\right|_{z=-0.5} = \frac{1}{2z(z+1)} = \frac{2}{3}$$

5. $$C = (z-1)\left.\frac{F(z)}{z}\right|_{z=1} = \frac{1}{2z(z+0.5)} = \frac{1}{3}$$

$$\frac{F(z)}{z} = \frac{-1}{z} + \frac{\frac{2}{3}}{(z+0.5)} + \frac{\frac{1}{3}z}{z-1}$$

6. $$F(z) = -1 + \frac{\frac{2}{3}z}{z+0.5} + \frac{\frac{1}{3}z}{z-1}$$

7. Taking inverse Z-transform,

$$f[n] = -\delta[n] + \frac{2}{3}(-0.5)^n u(n) + \frac{1}{3}(1)^n u(n)$$

Que 5.25. Determine the inverse Z-transform of $X(z) = \frac{z}{(3z^2 - 4z + 1)}$ in region of convergence

i. $|z|>1$ ii. $|z|<\frac{1}{3}$ iii. $\frac{1}{3}<|z|<1$

AKTU 2014-15, Marks 06

Answer

Given : $X(z) = \frac{z}{(3z^2 - 4z + 1)}$

To Find : i. $x[n]$ for $|z| > 1$.
ii. $x[n]$ for $|z| < 1/3$.
iii. $x[n]$ for $1/3 < |z| < 1$.

$$X(z) = \frac{z}{(3z-1)(z-1)}$$

$$= \frac{1}{3}\frac{z}{(z-1)\left(z-\frac{1}{3}\right)}$$

$$\frac{X(z)}{z} = \frac{1}{3}\frac{1}{(z-1)\left(z-\frac{1}{3}\right)}$$

By partial fraction,

$$\frac{X(z)}{z} = \frac{1}{3}\left[\frac{3}{2}\frac{1}{z-1} - \frac{3}{2}\frac{1}{z-\frac{1}{3}}\right] = \frac{1}{2}\left[\frac{1}{z-1} - \frac{1}{z-\frac{1}{3}}\right]$$

$$X(z) = \frac{1}{2}\left[\frac{z}{z-1} - \frac{z}{z-\frac{1}{3}}\right]$$

i. For $|z|>1$,

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$$x[n] = \frac{1}{2}\left[u[n] - \left(\frac{1}{3}\right)^n u[n]\right]$$

ii. For $|z| < \frac{1}{3}$,

$$x[n] = \frac{1}{2}\left[-u[-n-1] + \left(\frac{1}{3}\right)^n u[-n-1]\right]$$

iii. For $\frac{1}{3} < |z| < 1$,

$$x[n] = \frac{1}{2}\left[-u[-n-1] + \left(\frac{1}{3}\right)^n u[n]\right]$$

Que 5.26. **Find inverse Z-transform of $X(z) = \ln(1/(1 - \alpha^{-1} z))$.**

AKTU 2016-17, Marks 10

Answer

Given : $X(z) = \ln\left(\frac{1}{1-a^{-1}z}\right)$

To Find : $x[n]$.

Assuming ROC : $|z| < |a|$.

1. $$X(z) = \ln\left(\frac{1}{1-a^{-1}z}\right) = -\ln(1-a^{-1}z)$$

$$= -\left[-a^{-1}z - \frac{(a^{-1}z)^2}{2} - \frac{(a^{-1}z)^3}{3} - \frac{(a^{-1}z)^4}{4} - \dots\right]$$

2. $\therefore$ $$X(z) = \dots + \frac{1}{4a^4}z^4 + \frac{1}{3a^3}z^3 + \frac{1}{2a^2}z^2 + \frac{1}{a}z = \sum_{n=-\infty}^{-1}\left(\frac{a^n}{-n}\right)z^{-n}$$

3. Hence $$x[n] = -\frac{a^n}{n} \text{ for } n < 0$$

$$x[n] = \left(-\frac{a^n}{n}\right)u[-n-1]$$

Que 5.27. **Find $x[n]$ if**

$$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

i. $|z| > \frac{1}{3}$, ii. $\frac{1}{4} < |z| < \frac{1}{3}$, iii. $|z| < \frac{1}{4}$

Answer

Given : $$X(z) = \frac{3 - \frac{5}{6}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

To Find : i. $x[n]$ for $|z| > 1/3$.
ii. $x[n]$ for $1/4 < |z| < 1/3$.
iii. $x[n]$ for $|z| < 1/4$.

1. Using partial fraction,

$$X(z) = \frac{1}{1-\frac{1}{4}z^{-1}} + \frac{2}{1-\frac{1}{3}z^{-1}}$$

$$x[n] = x_1[n] + x_2[n]$$

2. Let,

3. $$x_1[n] = z^{-1}\left\{\frac{1}{1-\frac{1}{4}z^{-1}}\right\} = \left(\frac{1}{4}\right)^n u[n]; \qquad |z| > \frac{1}{4}$$

4. $$x_2[n] = z^{-1}\left\{\frac{2}{1-\frac{1}{3}z^{-1}}\right\} = 2\left(\frac{1}{3}\right)^n u[n]; \qquad |z| > \frac{1}{3}$$

5. $\therefore$ $$x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n];$$

i. $$x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]; \qquad |z| > \frac{1}{3}$$

ii. $$x[n] = \left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[-n-1]; \qquad \frac{1}{4} < |z| < \frac{1}{3}$$

iii. $$x[n] = -\left(\frac{1}{4}\right)^n u[-n-1] - 2\left(\frac{1}{3}\right)^n u[-n-1]; \qquad |z| < \frac{1}{4}$$

PART-3

Applications to Solution of Difference Equations.

CONCEPT OUTLINE : PART-3

- **Solution of Difference Equations Using Z-Transforms :**
1. To solve the difference equation, first it is converted into algebraic equation by taking its Z-transform.
2. The solution is obtained in Z-domain and the time domain solution is obtained by taking its inverse Z-transform.

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3. The response of the system due to input alone when the initial conditions are neglected is called the forced response of the system.
4. The response of the system due to initial conditions alone when the input is neglected is called the free or natural response of the system.
5. The response due to input and initial conditions considered simultaneously is called the total response of the system.
6. When input is a unit impulse input, the response is called the impulse response of the system and when the input is a unit step input, the response is called the step response of the system.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.28. **Solve the following difference equation**

$y[n] - 3y[n-1] - 4y[n-2] = 0, n \geq 0$

given $y(-1) = 5$ and $y(-2) = 0$.

AKTU 2013-14, Marks 10

Answer

Given : $y[n] - 3y[n-1] - 4y[n-2] = 0, y[-1] = 5, y[-2] = 0$
To Find : $y[n]$.

1. Taking Z transform of the given difference

$$Y(z) - 3[z^{-1}Y(z) + y(-1)] - 4[z^{-2}Y(z) + y(-1)z^{-1} + y(-2)] = 0 \quad ...(5.28.1)$$

2. Putting the initial conditions in eq. (5.28.1),

$$Y(z) - 3[z^{-1}Y(z) + 5] - 4[z^{-2}Y(z) + 5z^{-1} + 0] = 0$$

$$Y(z)[1 - 3z^{-1} - 4z^{-2}] - 20z^{-1} - 15 = 0$$

3. $$Y(z) = \frac{15 + 20z^{-1}}{1 - 3z^{-1} - 4z^{-2}} = \frac{z(15z+20)}{z^2 - 3z - 4}$$

4. $$\frac{Y(z)}{z} = \frac{15z+20}{z^2 - 3z - 4} = \frac{15z+20}{(z-1)(z-4)} = -\frac{1}{z+1} + \frac{16}{z-4}$$

5. $\therefore$ $$Y(z) = -\frac{z}{z+1} + \frac{16z}{z-4} = -\frac{1}{1+z^{-1}} + \frac{16}{1-4z^{-1}}$$

6. Taking inverse Z-transform,

$$y[n] = [-(-1)^n + 16(4)^n]u[n]$$

Que 5.29. **For the discrete system described by the difference equation $y[n] = 0.6y[n-1] - 0.08y[n-2] + x[n]$, determine :**

i. **The unit impulse response sequence, $h[n]$.**

ii. **The step response.**

AKTU 2016-17, Marks 10

Answer

Given : $y[n] = 0.6\,y[n-1] - 0.08\,y[n-2] + x[n]$
To Find : i. Unit impulse response, $h[n]$.
ii. Step response, $y[n]$.

$$y[n] - 0.6\,y[n-1] + 0.08\,y[n-2] = x[n]$$

1. Taking Z-transform on both sides,
2. $Y(z) - 0.6\,z^{-1}\,Y(z) + 0.08\,z^{-2}\,Y(z) = X(z)$

$Y(z)\,(1 - 0.6\,z^{-1} + 0.08\,z^{-2}) = X(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{(1 - 0.6z^{-1} + 0.08z^{-2})}$$

3. $\therefore$ $$H(z) = \frac{z^2}{(100z^2 - 60z + 8)}$$

i. **The unit impulse response sequence, $h[n]$:**

1. For unit impulse input,

$$x[n] = \delta[n]$$
$$X(z) = 1$$

2. Then impulse response,

$$Y(z) = H(z)\;X(z) = \frac{z^2}{(100z^2 - 60z + 8)}$$

3. $$\frac{Y(z)}{z} = \frac{z}{(5z-2)(20z-4)} = \frac{1}{4}\left[\frac{z}{(5z-2)(5z-1)}\right]$$

$$A = (5z-2)\left.\frac{Y(z)}{z}\right|_{z=\frac{2}{5}}$$

$$= \frac{1}{4}\left.\frac{z}{(5z-1)}\right|_{z=\frac{2}{5}} = \frac{1}{10}$$

$$B = (5z-1)\left.\frac{Y(z)}{z}\right|_{z=\frac{1}{5}}$$

$$= \frac{1}{4}\left.\frac{z}{(5z-2)}\right|_{z=\frac{1}{5}} = -\frac{1}{20}$$

4. $$\frac{Y(z)}{z} = \frac{1}{10}\frac{1}{(5z-2)} - \frac{1}{20}\frac{1}{(5z-1)}$$

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$$Y(z) = \frac{1}{10}\frac{z}{(5z-2)} - \frac{1}{20}\frac{z}{(5z-1)}$$

5.
$$Y(z) = \frac{1}{50}\frac{z}{(z-2/5)} - \frac{1}{100}\frac{z}{(z-1/5)}$$

6. Taking inverse Z-transform,

$$y[n] = h[n] = 0.02(0.4)^n u[n] - 0.01\,(0.2)^n u[n]$$

ii. The step response :

1. For step input,

$$x[n] = u[n]$$

2.
$$X(z) = \frac{z}{z-1}$$

3. $\therefore$
$$Y(z) = H(z)\,X(z)$$
$$= \frac{z^2}{(100z^2 - 60z + 8)}\,\frac{z}{(z-1)}$$

4.
$$\frac{Y(z)}{z} = \frac{z^2}{4\,(5z-2)(5z-1)(z-1)}$$

5. Using partial fraction,

$$\frac{Y(z)}{z} = \frac{A}{(5z-2)} + \frac{B}{(5z-1)} + \frac{C}{(z-1)}$$

$$A = (5z-2)\left.\frac{Y(z)}{z}\right|_{z=\frac{2}{5}}$$
$$= \left.\frac{z^2}{4(5z-1)(z-1)}\right|_{z=\frac{2}{5}} = -\frac{1}{15}$$

$$B = (5z-1)\left.\frac{Y(z)}{z}\right|_{z=\frac{1}{5}}$$
$$= \left.\frac{z^2}{4(5z-2)(z-1)}\right|_{z=\frac{1}{5}} = \frac{1}{80}$$

$$C = (z-1)\left.\frac{Y(z)}{z}\right|_{z=1}$$
$$= \left.\frac{z^2}{4(5z-2)(5z-1)}\right|_{z=1}$$
$$= \frac{1}{48}$$

6.
$$\frac{Y(z)}{z} = -\frac{1}{15}\frac{1}{(5z-2)} + \frac{1}{80\,(5z-1)} + \frac{1}{48\,(z-1)}$$

$$= \frac{-z}{75(z-2/5)} + \frac{z}{400(z-1/5)} + \frac{z}{48\,(z-1)}$$

7. Taking inverse Z-transform,

$$y[n] = (-0.013\,(0.4)^n + 0.0025\,(0.2)^n + 0.208\,(1)^n)\,u[n]$$

Que 5.30. **A causal LTI system is described by the difference equation**

$$y[n] = y[n-1] + 2y\,[n-2] + x[n-1]$$

Find the system function, impulse response and step response.

Answer

Given : $y[n] = y[n-1] + 2y\,[n-2] + x[n-1]$

To Find :
i. System function, $H(z)$.
ii. Impulse response, $h[n]$.
iii. Step response, $y[n]$.

i.
$$y[n] - y[n-1] - 2y\,[n-2] = x[n-1]$$

1. Taking Z-transform on both sides, we have

2. $Y(z) - z^{-1}\,Y(z) - 2z^{-2}\,Y(z) = z^{-1}\,X(z)$

$Y(z)\,(1 - z^{-1} - 2\,z^{-2}) = z^{-1}\,X(z)$

3. The transfer function of the system $H(z)$ is :

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}}{1 - z^{-1} - 2z^{-2}} = \frac{z}{z^2 - z - 2} = \frac{z}{(z-2)(z+1)}$$

ii. The impulse response of the system is :

$$h[n] = Z^{-1}(H(z))$$
$$= Z^{-1}\left[\frac{z}{(z-2)(z+1)}\right] = Z^{-1}\left[\frac{A}{z-2} + \frac{B}{z+1}\right]$$
$$= Z^{-1}\left[\frac{2/3}{z-2} + \frac{1/3}{z+1}\right] = Z^{-1}\left[\frac{2/3}{z-2}\right] + Z^{-1}\left[\frac{1/3}{z+1}\right]$$
$$= \frac{2}{3}(2)^{n-1}\,u[n-1] + \frac{1}{3}(-1)^{n-1}\,u[n-1]$$

iii.

1. **For step response** $x[n] = u[n]$

$$X(z) = \frac{z}{z-1}$$

2. Output,
$$Y(z) = H(z)\,X(z) = \frac{z}{(z-2)(z+1)}\cdot\frac{z}{(z-1)}$$

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$$\frac{Y(z)}{z} = \frac{z}{(z-2)(z+1)(z-1)} = \frac{A}{z-2} + \frac{B}{z+1} + \frac{C}{z-1}$$

$$\frac{Y(z)}{z} = \frac{2/3}{z-2} - \frac{1/6}{z+1} - \frac{1/2}{z-1}$$

$$Y(z) = \frac{2}{3}\left[\frac{z}{z-2}\right] - \frac{1}{6}\left[\frac{z}{z+1}\right] - \frac{1}{2}\left[\frac{z}{z-1}\right]$$

3. Taking inverse Z-transform on both sides, we have

$$y[n] = Z^{-1}(Y(z))$$

$$y[n] = \frac{2}{3}(2)^n u[n] - \frac{1}{6}(-1)^n u[n] - \frac{1}{2} u[n]$$

Que 5.31. **Discuss the significance of the difference equation. Solve the following difference equation using the Z-transform method :**
$y[n+2] - 0.1\,y[n+1] - 0.2\,y[n] = r[n+1] + r[n]$
where, $r[n] = U[n]$
$y(0) = 0$, and $y(1) = 0$, $r(0) = 0$

Answer

Significance of difference equation : A difference equation is solved either recursively or by using the unilateral Z-transform. The recursive method however, does not yield a closed form expression of $x[n]$ in terms of n, while the Z-transform method yields a closed form expression.

Given : $y[n+2] - 0.1\,y[n+1] - 0.2\,y[n] = r[n+1] + r[n]$
$r[n] = u[n]$, $c[n] = 0$, $y(1) = 0$, $r(0) = 0$
To Find : $y[n]$.

1. Taking Z-transform,

$$\{z^2 Y(z) - zY(0) - y(1)\} - 0.1\,\{z\,Y(z) - y(0)\} - 0.2\,Y(z) = zR(z) - r(0) + R(z)$$

2. $\{z^2 Y(z)\} - 0.1\,\{z\,Y(z)\} - 0.2\,Y(z) = zR(z) + R(z)$

3. $Y(z)\,\{z^2 - 0.1\,z - 0.2\} = \dfrac{z^2}{z-1} + \dfrac{z}{z-1}$

$$Y(z) = \frac{z}{(z-1)(z^2-0.1z-0.2)} + \frac{z^2}{(z-1)(z^2-0.1z-0.2)}$$

$$\frac{Y(z)}{z} = \frac{10}{(z-1)(10z^2-z-2)} + \frac{10z}{(z-1)(10z^2-z-2)}$$

$$\frac{Y(z)}{z} = \frac{10}{(z-1)(2z-1)(5z+2)} + \frac{10z}{(z-1)(2z-1)(5z+2)}$$

4. Using partial fraction,

$$\frac{Y(z)}{z} = 10\left(\frac{1}{7(z-1)} - \frac{4}{9(2z-1)} + \frac{25}{63(5z+2)}\right) + 10\left(\frac{1}{7(z-1)} - \frac{2}{9(2z-1)} - \frac{10}{63(5z+2)}\right)$$

$$Y(z) = \frac{20}{7}\frac{z}{z-1} - \frac{30}{9}\frac{z}{\left(z-\frac{1}{2}\right)} + \frac{30}{63}\frac{z}{\left(z+\frac{2}{5}\right)}$$

5. Taking inverse Z-transform,

$$y[n] = \left[\frac{20}{7} - \frac{10}{3}\left(\frac{1}{2}\right)^n + \frac{10}{21}\left(-\frac{2}{5}\right)^n\right]u[n]$$

Que 5.32.

i. **Find the response of the system**
$f[n+2] - 3f[n+1] + 2f[n] = \delta[n]$
when all initial conditions are zero.

ii. **Find the initial and final values of the sequence $f[n]$ given $F(z) = 2 + 3z^{-1} + 4z^{-2}$**

Answer

i. **Given :** $f[n+2] - 3f[n+1] + 2f[n] = \delta[n]$
Initial conditions are zero.
To Find : $f[n]$.

1. Taking Z-transform,
$z^2 F(z) - 3z\,F(z) + 2\,F(z) = 1$

$$F(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)}$$

$$\frac{F(z)}{z} = \frac{1}{z(z-1)(z-2)} = \frac{1}{2z} - \frac{1}{2(z-1)} + \frac{1}{2(z-2)}$$

$$\frac{F(z)}{z} = \frac{0.5}{z} - \frac{0.5}{z-1} + \frac{0.5}{z-2}$$

$$F(z) = 0.5 - \frac{0.5}{1-z^{-1}} + \frac{0.5}{1-2z^{-1}}$$

2. Taking inverse Z-transform,

$$f[n] = 0.5\delta[n] - 0.5u[n] + 0.5\,2^n\,u[n]$$

$$f[n] = 0.5\delta[n] - 0.5\,(1-2^n)\,u[n]$$

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ii. Given : $F(z) = 2 + 3z^{-1} + 4z^{-2}$
To Find : i. Initial value, $x(0)$.
ii. Final value, $f(\infty)$.

i. Initial value,
$$f(0) = \lim_{n\to 0} f[n] = \lim_{z\to\infty} F(z)$$
$$= \lim_{z\to\infty}(2 + 3z^{-1} + 4z^{-2}) = 2$$

ii. Final value,
$$f(\infty) = \lim_{n\to\infty} f[n] = \lim_{z\to 1}(1 - z^{-1})F(z)$$
$$= \lim_{z\to 1}(1 - z^{-1})(2 + 3z^{-1} + 4z^{-2})$$
$$= \lim_{z\to 1}(2 + 3z^{-1} + 4z^{-2} - 2z^{-1} - 3z^{-2} - 4z^{-3})$$
$$= 0$$

Que 5.33. **A LTI system is represented by the following difference equation $3y[n] = 5y[n-1] - 7y[n-2] + 4x[n-1]$ for $n \geq 0$. Determine**
i. Impulse response, $h[n]$.
ii. Obtain cascade and parallel form realization for discrete time system.

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Answer

Given : $3y[n] = 5y[n-1] - 7y[n-2] + 4x[n-1]$
To Find : i. Impulse response, $h[n]$.
ii. Cascade and parallel form realization.

Taking Z-transform on both sides,
$$3Y(z) = 5z^{-1}Y(z) - 7z^{-2}Y(z) + 4z^{-1}X(z)$$

Transfer function, $H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{4z^{-1}}{3 - 5z^{-1} + 7z^{-2}}$

i. Impulse response $h[n]$:

1. For impulse response, $X(z) = 1$.
2. $$H(z) = \frac{4z^{-1}}{3 - 5z^{-1} + 7z^{-2}} = \frac{4z}{3z^2 - 5z + 7}$$
3. By long division method,

$$
\begin{array}{l}
3z^2 - 5z + 7 \overline{)\,4z\,(}\;\; \frac{4}{3}z^{-1} + \frac{20}{9}z^{-2} + \frac{16}{27}z^{-3} - \frac{340}{81}z^{-4} \\
\quad \underline{4z - \frac{20}{3} + \frac{28}{3}z^{-1}} \\
\qquad \frac{20}{3} - \frac{28}{3}z^{-1} \\
\qquad \underline{\frac{20}{3} - \frac{100}{9}z^{-1} + \frac{140}{9}z^{-2}} \\
\qquad\quad -\frac{16}{9}z^{-1} - \frac{140}{9}z^{-2} \\
\qquad\quad \underline{-\frac{16}{9}z^{-1} - \frac{80}{27}z^{-2} + \frac{112}{27}z^{-3}} \\
\qquad\qquad -\frac{340}{27}z^{-2} - \frac{112}{27}z^{-3} \\
\qquad\qquad \underline{-\frac{340}{27}z^{-2} \ldots}
\end{array}
$$

$$H(z) = \frac{4}{3}z^{-1} + \frac{20}{9}z^{-2} + \frac{16}{27}z^{-3} - \frac{340}{81}z^{-4} + \ldots$$

4. Taking inverse Z-transform,
$$h[n] = \frac{4}{3}\delta[n-1] + \frac{20}{9}\delta[n-2] + \frac{16}{27}\delta[n-3] - \frac{340}{81}\delta[n-4] \ldots$$

ii.
$$H(z) = \frac{4z}{3z^2 - 5z + 7}$$
$$= \frac{4z^{-1}}{3 - 5z^{-1} + 7z^{-2}}$$

Cascade / parallel form realization :

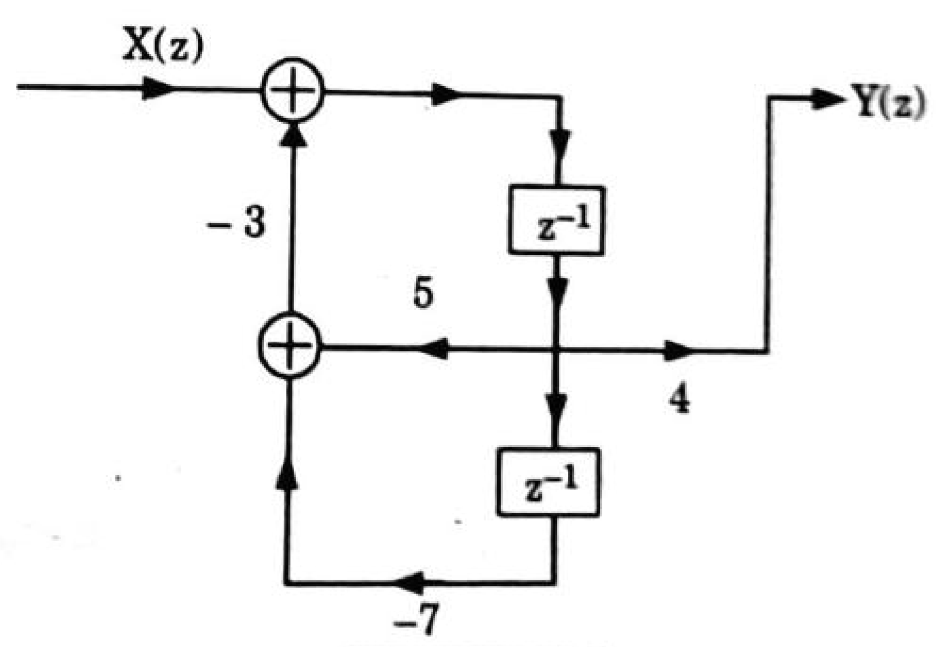

Fig. 5.33.1.

VERY IMPORTANT QUESTIONS

Following questions are very important. These questions may be asked in your SESSIONALS as well as UNIVERSITY EXAMINATION.

Q. 1. Enlist the properties of the ROC in Z-transform.

Ans. Refer Q. 5.2, Unit-5.

Q. 2. Define the properties of Z-transform.

Ans. Refer Q. 5.4, Unit-5.

Q. 3. What is the difference between the Z-transform and the Laplace transform ? Explain.

Ans. Refer Q. 5.6, Unit-5.

Q. 4.

a. Prove the convolution theorem of Z-transform.

b. Find the Z-transform of cos $\omega_0 n u[n]$.

Ans. Refer Q. 5.7, Unit-5.

Q. 5. Find the Z-transform of the following :

i. $x[n] = a^n u[n]$

ii. $x[n] = -b^n u[-n-1]$

Ans. Refer Q. 5.8, Unit-5.

Q. 6. Find the Z-transform of the following sequences :

i. $x_1[n] = \{1, 2, 3, 4, 5, 0, 7\}$

ii. $x_2[n] = \{1, 2, 3, \underset{\uparrow}{4}, 5, 0, 7\}$

iii. $\delta[n]$

Ans. Refer Q. 5.12, Unit-5.

Q. 7. Find the convolution of sequences, $x_1[n] = (1/4)^n u[n]$ & $x_2[n] = (1/5)^{n-2} u[n-2]$ using :

i. Convolution in Z-transform

ii. Time Domain Method.

Ans. Refer Q. 5.14, Unit-5.

Q. 8. State and explain initial and final value theorem using Z-transform analysis.

Ans. Refer Q. 5.19, Unit-5.

Q. 9. Determine the inverse Z-transform of the following functions :

i. $X(z) = (z-1)/(z^2 - 4z + 4)$

ii. $X(z) = z^2 / (z^2 - 5/4z + 3/8)$

Ans. Refer Q. 5.20, Unit-5.

Q. 10. Determine the inverse Z-transform of

$$X(z) = \frac{z}{(3z^2 - 4z + 1)} \text{ in region of convergence}$$

i. $|z| > 1$ ii. $|z| < \frac{1}{3}$ iii. $\frac{1}{3} < |z| < 1$

Ans. Refer Q. 5.25, Unit-5.

Q. 11. Solve the following difference equation $y[n] - 3y[n-1] - 4y[n-2] = 0, n \geq 0$ given $y(-1) = 5$ and $y(-2) = 0$.

Ans. Refer Q. 5.28, Unit-5.

Q. 12. For the discrete system described by the difference equation $y[n] = 0.6y[n-1] - 0.08y[n-2] + x[n]$, determine :

i. The unit impulse response sequence, $h[n]$.

ii. The step response.

Ans. Refer Q. 5.29, Unit-5.

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