Lecture 1

Differential Equations of First Order and First Degree

INTRODUCTION

An equation of the form $F\left(x, y, \frac{dy}{dx}\right) = 0$ in which x is the independent variable and $\frac{dy}{dx}$ appears with first degree is called a first order and first degree differential equation. It can also be written in the form $\frac{dy}{dx} = f(x, y)$ or in the form $Mdx + Ndy = 0$, where M and N are functions of x and y. Generally, it is difficult to solve the first order differential equations and in some cases they may not possess any solution. There are certain standard types of first order, first degree equations. In this chapter we shall discuss the methods of solving them..

VARIABLES SEPARABLE

If the equation is of the form $f_1(x)\,dx = f_2(y)\,dy$ then, its solution, by integration is $\int f_1(x)dx = \int f_2(y)\,dy + C$

where C is an arbitrary constant.

Example 1. Solve $\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$

Solution. The given equation is

$$\frac{dy}{dx} = e^{x-y} + x^2 e^{-y}$$

or $$e^y\,dy = (e^x + x^2)\,dx$$

Integrating, $e^y = e^x + \frac{x^3}{3} + c$, where C is an arbitrary constant is the required solution.

Example 2. Solve $\left(y - x\frac{dy}{dx}\right) = a\left(y^2 + \frac{dy}{dx}\right)$

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Solution. The given equation is

$$y - x\frac{dy}{dx} = ay^2 + a\frac{dy}{dx}$$

or $$(a + x)\frac{dy}{dx} = (y - ay^2)$$

or $$\frac{dy}{y - ay^2} = \frac{dx}{a + x}$$

or $$\frac{dy}{y - ay^2} = \frac{dx}{a + x}$$

or $$\frac{dy}{y(1 - ay)} = \frac{dx}{x + a}$$

or $$\left[\frac{a}{1 - ay} + \frac{1}{y}\right] dy = \frac{dx}{x + a},$$ resolving into partial fractions.

Integrating, $[-\log(1 - ay) + \log y] = \log(x + a) + \log C$

where C is an arbitrary constant

or $$\log\left(\frac{y}{1 - ay}\right) = \log\{C(x + a)\}$$

or $$\frac{y}{1 - ay} = C(x + a)$$

or $y = C(x + a)(1 - ay)$ is the required solution.

Example 2. Solve $(x + y)^2 \dfrac{dy}{dx} = a^2$

Solution. Let $x + y = v$

then from (1) on differentiating with respect to x, we have

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

or $$\frac{dy}{dx} = \left(\frac{dv}{dx} - 1\right)$$

Substituting these values from (1) and (2) in the given equation, we get

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$$v^2\left[\frac{dv}{dx}-1\right]=a^2$$

or $$v^2\frac{dv}{dx}=a^2+v^2$$

or $$\frac{v^2}{a^2+v^2}\,dv=dx$$

or $$\frac{a^2+v^2-a^2}{a^2+v^2}\,dv=dx$$

or $$\left[1-\frac{a}{a^2+v^2}\right]dv=dx$$

Integrating,

$$v-a^2\frac{1}{a}\tan^{-1}\left(\frac{v}{a}\right)=x+c$$

where c is an arbitrary constant

or $$v-a\tan^{-1}(v/a)=x+c$$

or $$(x+y)-a\tan\{(x+y)/a\}=x+c,\ \text{from (1)}$$

or $$y-a\tan^{-1}\{(x+y)/a\}=C$$

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Lecture 2

Linear Differential Equations

A differential equation of the form $\frac{dy}{dx} + Py = Q$

where P and Q are constants or functions of x alone (and not of y) is called a linear differential equation of the first order in y

its integrating factor $= e^{\int P\,dx}$

Multiplying both sides of (1) by this integrating factor (I.F.) and then integrating we get

$y.\, e^{\int P dx} = C + \int Q.\, e^{\int P dx}\, dx$, where C is an arbitrary constant, is the complete solution of (1)

Example Solve $\frac{dy}{dx} + 2y \tan x = \sin x$, given that $y = 0$ when $x = \pi/3$.

Solution. Here $P = 2 \tan x$ and $Q = \sin x$

$\therefore$ Integrating factor $= e^{\int P dx} = e^{\int 2\tan x\, dx}$

$= e^{2\log \sec x}$

$= e^{\log (\sec x)^2} = \sec^2 x$

Multiplying the given equation by $\sec^2 x$, we get

$$\sec^2 x \left(\frac{dy}{dx} + 2y \tan x\right) = \sin x \sec^2 x$$

or $\frac{d}{dx}(y \sec^2 x) = \sec x \tan x$

Integrating both sides with respect to x, we get

$y \sec^2 x = C + \int \sec x \tan x\, dx$, where C is an arbitrary constant

or $$y \sec^2 x = C + \sec x \qquad (1)$$

it is given that when $x = \pi/3$, $y = 0$

$\therefore$ from (1) $0 \times \sec^2 \frac{\pi}{3} = C + \sec \frac{\pi}{3}$

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or $\qquad 0 = C + 2 \qquad \therefore \sec\frac{\pi}{3} = 2$

$\therefore$ from (1) the required solution is

$$y \sec^2 x = -2 + \sec x$$

or $\qquad y = -2\cos^2 x + \cos x$

Example . Solve $(1 + y^2)\,dx + (x - e^{\tan^{-1}y})\,dy = 0$

Solution. The given equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{\tan^{-1}y}}{1+y^2}$$

Therefore the integrating factor $= e^{\int \frac{1}{1+y^2}dy}$

$$= e^{\tan^{-1}y}$$

Multiplying both sides of (1) by the integrating factor and integrating, we have

$$x.\, e^{\tan^{-1}y} = C + \int \frac{e^{\tan^{-1}y}}{1+y^2} \times e^{\tan^{-1}y}\, dy$$

where C is an arbitrary costant

or $\qquad x\, e^{\tan^{-1}y} = C + \int e^{2t}\, dt,$ where $t = \tan^{-1} y$

$$= C + \frac{1}{2} e^{2t}$$

or $\qquad x.\, e^{\tan^{-1}y} = C + \frac{1}{2} e^{2\tan^{-1}y}$

Example . Solve $\frac{dy}{dx} + y\cos x = \frac{1}{2}\sin 2x$

Solution. Here $P = \cos x$ and $Q = \frac{1}{2}\sin 2x = \sin x \cos x$

$\therefore$ Integrating factor $= e^{\int P dx} = e^{\int \cos x\, dx} = e^{\sin x}$

Multiplying the given equation by the integrating factor $e^{\sin x}$ and integrating with respect to x, we get

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$y.\, e^{\sin x} = C + \int e^{\sin x} \sin x \cos x \, dx$, where C is an arbitrary constant.

or $$y.\, e^{\sin x} = C + \int e^{t}\, t\, dt, \qquad \text{where } t = \sin x$$

$$= C + t.\, e^{t} - e^{t}$$

$$= C + e^{\sin x}(\sin x - 1)$$

or $$y.\, e^{\sin x} = C + e^{\sin x}(\sin x - 1)$$

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Lecture 3

Linear Differential Equations with Constant Coefficients and Applications

INTRODUCTION

A differential equation is of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots\dots\dots + a_n y = Q \qquad (1)$$

where $a_1, a_2, \dots\dots a_n$ are constants and Q is a function of x only, is called a linear differential equation of nth order. Such equations are most important in the study of electro-mechanical vibrations and other engineering problems.

The operator $\frac{d}{dx}$ is denoted by D.

$\therefore D^n y + a_1 D^{n-1} y + \dots\dots\dots\dots + a_n y = Q$

or $\qquad f(D)\, y = Q$

where $f(D) = D^n + a_1 D^{n-1} + \dots\dots\dots\dots + a_n$

Solution of the Differential Equation

If the given equation is

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \dots\dots + a_n y = 0 \qquad (1)$$

or $\qquad (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots\dots + a_n)\, y = 0 \qquad (2)$

Let $y = e^{mx}$

$\Rightarrow \qquad D^r y = m^r e^{mx}, 1 \le r \le n$

$\therefore$ Then from equation (2)

$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots\dots\dots + a_n)\, e^{mx} = 0$

$y = e^{mx}$ is a solution of (1), if

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$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots\ldots\ldots\ldots + a_n = 0$

This equation is called the auxiliary equation.

Case 1. When Auxiliary Equation has Distinct and real Roots

Let $m_1, m_2, \ldots\ldots\ldots\ldots\ldots\ldots m_n$ are distinct roots of the auxiliary equation, then the general solution of (1) is $y = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \ldots\ldots\ldots\ldots\ldots + C_n e^{m_n x}$

where $C_1, C_2, \ldots\ldots\ldots\ldots C_n$ are arbitrary constants

Illustration. Solve the differential equation

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 54y = 0$$

Solution. The given equation is

$$(D^2 + 3D - 54)\, y = 0$$

Here auxiliary equation is

$$m^2 + 3m - 54 = 0$$

or $\quad (m + 9)(m - 6) = 0$

$\Rightarrow \quad m = 6, -9$

Hence the general solution of the given differential equation is $y = C_1 e^{6x} + C_2 e^{-9x}$

Case II. When Auxiliary Equation has real and some equal roots.

If the auxiliary equation has two roots equal, say $m_1 = m_2$ and others are distinct say $m_3, m_4, \ldots\ldots\ldots\ldots m_n$. In this Case the general solution of the equation (1) is

$y = (C_1 + C_2 x)\, e^{m_1 x} + C_3 e^{m_3 x} + \ldots\ldots\ldots\ldots + C_n e^{m_n x}$

where $C_1, C_2, C_3 \ldots\ldots\ldots\ldots C_n$ are arbitrary constants

Illustration. Solve the differential equation

$$(D^4 - D^3 - 9D^2 - 11D - 4)\, y = 0$$

Solution. The auxiliary equation of the give equation is

$m^4 - m^3 - 9m^2 - 11m - 4 = 0$

or $\quad (m + 1)^3 (m - 4) = 0$

$\Rightarrow \quad m = -1, -1, -1, 4$

Hence, the required solution is

$$y = (C_1 + C_2 x + C_3 x^2)\, e^{-x} + C_4 e^{4x}$$

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or $$y = (C_1 x^2 + C_2 x + C_3) e^{-x} + C_4 e^{4x}$$

Case III. When the auxiliary equation has imaginary roots

If there are one pair of imaginary roots say $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ i.e. $\alpha \pm i\beta$ say then the required solution is

$e^{x\,(\text{Real part})}$ [C_1 {cos (imaginary part) x} + C_2 {sin (imaginary part) x}]

i.e. $$e^{\alpha x} [C_1 \cos \beta x + C_2 \sin \beta x]$$

or $$y = C_1\, e^{\alpha x} \cos (\beta x + C_2)$$

Illustration. Solve $(D^2 - 2D + 5)\, y = 0$

Solution. Here the auxiliary equation is

$$m^2 - 2m + 5 = 0$$

or $$m = \frac{1}{2}\left[2 \pm \sqrt{(4 - 20)}\right] = 1 \pm 2i$$

$\therefore$ The required solution is

$y = e^x (C_1 \cos 2x + C_2 \sin 2x)$

Particular Integral (P.I.)

when the equation is

$$D^n + a_1 D^{n-1} + \ldots\ldots\ldots + a_n)\, y = Q$$

or $$f(D)\, y = Q$$

The general solution of $f(D)\, y = Q$ is equal to the sum of the general solution of $f(D)\, y = 0$ called complementary function (C.F.) and any particular integral of the equation $f(D)\, y = Q$

$\therefore$ General solution = C.F. + P.I.

A particular integral of the differential equation

$f(D)\, y = Q$ is given by $\dfrac{1}{f(D)}\, Q$

Methods of finding Particular integral

(A)

Case I. P.I., when Q is of the form of e^{ax}, where a is any constant and $f(a) \neq 0$

we know that $D(e^{ax}) = a\, e^{ax}$

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$$D^2(e^{ax}) = a^2 e^{ax}$$

$$D^3(e^{ax}) = a^3 e^{ax}$$

In general $$D^n(e^{ax}) = a^n e^{ax}$$

$$\therefore f(D)(e^{ax}) = f(a)\, e^{ax}$$

or $$\frac{1}{f(D)} f(D)\, e^{ax} = \frac{1}{f(D)} f(a)\, e^{ax}$$

or $$e^{ax} = f(a) \frac{1}{f(D)} e^{ax} \because f(a) \text{ is constant}$$

or $$\frac{1}{f(a)} e^{ax} = \frac{1}{f(D)} e^{ax}$$

Hence P.I. $= \dfrac{1}{f(D)} e^{ax} = \dfrac{1}{f(a)} e^{ax}$, if $f(a) \neq 0$

Case II. P.I., when Q is of the form of e^{ax}, and $f(a) = 0$

Then $\dfrac{1}{f(D)} (e^{ax}) = e^{ax} \dfrac{1}{f(D+a)} 1$, which shows that if e^{ax} is brought to the left from the right of $\dfrac{1}{f(D)}$, then D should be replaced by $(D + a)$

Another method for Exceptional Case

If $f(a) = 0$, then

$$\text{P.I.} = \frac{1}{f(D)} e^{ax} = x \frac{1}{f'(D)} e^{ax}$$

$$= x \frac{e^{ax}}{f'(a)}, \text{ if } f'(a) \neq 0$$

If $f'(a) = 0$, then

$$\text{P.I.} = \frac{1}{f(D)} e^{ax}$$

$$= x^2 \frac{e^{ax}}{f''(a)}, \text{ if } f''(a) \neq 0$$

Example 1. Solve $(D^2 - 2D + 5)\, y = e^{-x}$

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Solution. Here auxiliary equation is $m^2 - 2m + 5 = 0$, whose roots are

$$m = -1 \pm 2i$$

$\therefore$ C.F. $= e^{-x}[C_1 \cos 2x + C_2 \sin 2x]$, where C_1 and C_2 are arbitrary constants

and
$$\text{P.I.} = \frac{1}{D^2 - 2D + 5} e^{-x}$$

$$= \frac{1}{(-1)^2 - 2(-1) + 5} e^{-x} \qquad \because \text{here } a = -1$$

$$= \frac{1}{8} e^{-x}$$

$\therefore$ The required solution is y = C.F. + P.I.

i.e. $y = e^{-x}(C_1 \cos 2x + C_2 \sin 2x) + \frac{1}{8} e^{-x}$

Example 2. Solve $(D - 1)^2 (D^2 + 1)^2 y = e^x$

Solution. Here the auxiliary equation is

$$(m - 1)^2 (m^2 + 1)^2 = 0$$

or
$$m = 1, 1, \pm i, \pm i$$

$\therefore$ C.F. $= (C_1 + C_2 x) e^x + (C_3 x + C_4) \cos x + (C_5 x + C_6) \sin x$

where c's are arbitrary constant

and
$$\text{P.I.} = \frac{1}{(D-1)^2 (D^2+1)^2} e^x$$

$$= \frac{1}{(D-1)^2} \frac{1}{(1^2+1)^2} e^x$$

$$= \frac{1}{(D-1)^2} \frac{1}{(2)^2} e^x$$

$$= \frac{1}{(D-1)^2} \frac{1}{4} e^x$$

$$= e^x \frac{1}{(D+1-1)^2} \frac{1}{4}$$

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$$= e^x \frac{1}{D^2} \frac{1}{4} = \frac{1}{4} e^x \frac{1}{D^2} (1) = \frac{1}{4} e^x \frac{x^2}{2} = \frac{1}{8} x^2 e^x$$

$\therefore$ The required solution is y = C. F + P.I.

or $$y = (C_1 x + C_2) e^x + (C_3 x + C_4) \cos x + (C_5 x + C_6) \sin x + \frac{1}{8} x^2 e^x$$

Example 3. Solve $(D + 2)(D - 1)^3 y = e^x$

Solution. Here the auxiliary equation is

$$(m + 2)(m - 1)^3 = 0$$

or $$m = -2 \text{ and } m = 1 \text{ (thrice)}$$

Therefore C. F = $C_1 e^{-2x} + (C_2 x^2 + C_3 x + C_4) e^x$, where C_1, C_2 and C_3 are constants and

$$\text{P.I.} = \frac{1}{(D + 2)(D - 1)^3} e^x$$

$$= \frac{1}{(1 + 2)(D - 1)^3} e^x$$

$$= \frac{1}{3} \frac{1}{(D - 1)^3} e^x = \frac{1}{3} e^x \frac{1}{\{(D + 1) - 1\}} 1$$

$$= \frac{1}{3} e^x \frac{1}{D^3} 1$$

$$= \frac{1}{3} e^x \left(\frac{1}{6} x^3\right) = \frac{1}{18} x^3 e^x$$

$\therefore$ the required solution is y = C.F + P.I

or $$y = C_1 e^{-2x} + (C_2 x^2 + C_3 x + C_4) e^x + \frac{1}{18} x^3 e^x$$

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Lecture 4(P.I.Continued)

(B) (i) P.I. when Q is of the form sin ax or cos ax and f $(-a^2) \neq 0$

$$\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax, \text{ if } f(-a^2) \neq 0$$

$$\& \ \frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax, \text{ if } f(-a^2) \neq 0$$

(ii) P.I. when Q is of the form sin ax or cos ax and f $(-a^2) = 0$

Example . Solve $(D^2 - 5D + 6)\, y = \sin 3x$

Solution. Its auxiliary equation is $m^2 - 5m + 6 = 0$ which gives

$m = 2, 3$

$\therefore$ C.F. $= C_1 e^{2x} + C_2 e^{3x}$, where C_1 and C_2 are arbitrary constants

and $$\text{P.I.} = \frac{1}{D^2 - 5D + 6} \sin 3x$$

$$= \frac{1}{-3^2 - 5D + 6} \sin 3x, \text{ replacing } D^2 \text{ by } -3^2$$

$$= \frac{1}{-(5D + 3)} \sin 3x = \frac{-1}{(5D + 3)(5D - 3)} (5D - 3) \sin 3x$$

$$= \frac{-1}{(25D^2 - 9)} (5D - 3) \sin 3x$$

$$= \frac{-1}{\{25(-3^2) - 9\}} (5D - 3) \sin 3x$$

$$= \frac{1}{234} [5D(\sin 3x) - 3\sin 3x]$$

$$= \frac{1}{234} [5 \times 3 \cos 3x - 3 \sin 3x]$$

$$= \frac{1}{78} (5 \cos 3x - \sin 3x)$$

Hence the required solution is

$$y = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{78} (5 \cos 3x - \sin 3x)$$

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(C) To find P.I. when Q is of the form x^m

In this case $P.I = \frac{1}{f(D)} x^m$, where m is a positive integer

To evaluate this we take common the lowest degree from f(D), so that the remaining factor reduces to the form [1 + F (D)] or [1 - F (D)]. Now take this factor in the numerator with a negative index and expand it by Binomial theorem in powers of D upto the term D^m, (Since other higher derivatives of x^m will be zero) and operate upon x^m. The following examples will illustrate the method.

Example 12. Solve $(D^3 - D^2 - 6D)\, y = x^2 + 1$ Where $D \equiv \frac{d}{dx}$

Solution. Here the auxiliary equation is $m^3 - m^2 - 6m = 0$

or $\quad m(m^2 - m - 6) = 0$

or $\quad m(m-3)(m+2) = 0$

or $\quad m = 0, 3, -2$

$\therefore C_1 F = C_1 e^{0x} + C_2 e^{3x} + C_3 e^{-2x}$

or $\quad$ C.F. $= C_1 + C_2 e^{3x} + C_3 e^{-2x}$, where C_1, C_2 and C_3 are arbitrary constants.

and $$P.I. = \frac{1}{D^2 - D^2 - 6D}(x^2 + 1)$$

$$= \frac{1}{-6D\left(1 + \frac{1}{6}D - \frac{1}{6}D^2\right)}(x^2 + 1)$$

$$= -\frac{1}{6D}\left[1 - \left(\frac{D}{6} - \frac{D^2}{6}\right) + \left(\frac{D}{6} - \frac{D^2}{6}\right)^2 \ldots\ldots\ldots\ldots\right](x^2 + 1)$$

$$= -\frac{1}{6D}\left[1 + x^2 - \frac{1}{6}D(x^2) + \frac{7}{36}D^2(x^2)\right] \qquad \because D(1) = 0$$

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(E) To show that $\frac{1}{D-a}Q = e^{ax}\int Qe^{-ax}\,dx$, where Q is a function of x.

Proof Let $y = \frac{1}{D-a}Q$

Then $(D-a)\,y = Q$, operating both sides with $D-a$

or $\frac{dy}{dx} - ay = Q$, Which is a linear equation in y whose integrating factor is e^{-ax} its solution is $ye^{-ax} = \int Q\,e^{-ax}\,dx$, neglecting the constant of integration as P.I. is required

or $\quad y = e^{ax}\int Q\,e^{-ax}\,dx$

or $\quad \frac{1}{D-a}Q = e^{ax}\int Q\,e^{-ax}\,dx$

Example : Solve $(D^2 + a^2)\,y = \sec ax$

Solution. The auxiliary equation is $m^2 + a^2 = 0$ or $m = \pm ai$

$\therefore$ C.F. $= C_1 \cos ax + C_2 \sin ax$, where C_1 and C_2 are arbitrary constants

and $\text{P.I} = \frac{1}{D^2+a^2}\sec ax = \frac{1}{2ia}\left[\frac{1}{D-ia} - \frac{1}{D+ia}\right]\sec ax \qquad (1)$

Now $\quad \frac{1}{D-ia}\sec ax = e^{iax}\int \sec ax.\,e^{-iax}\,dx$

$$= e^{iax}\int \sec ax.\,(\cos ax - i\sin ax)\,dx$$

$$= e^{iax}\int (1 - i\tan ax)\,dx$$

$$= e^{iax}\left[x + \left(\frac{i}{a}\right)\log\cos ax\right]$$

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$$\frac{1}{D+ia}\sec ax = e^{-iax}\int \sec ax.\, e^{iax}\, dx$$

$$= e^{-iax}\int \sec ax\,(\cos ax + i\sin ax)\, dx:$$

$$= e^{-iax}\int (1 + i\tan ax)\, dx$$

$$= e^{-iax}\left[x - \left(\frac{i}{a}\right)\log\cos ax\right]$$

$\therefore$ From (1) we, have

$$P.I = \frac{1}{2ia}\left[e^{iax}\left\{x + \frac{i}{a}\log\cos ax\right\} - e^{-iax}\left\{x - \frac{i}{a}\log\cos ax\right\}\right]$$

$$= \left[x\left(\frac{e^{iax} - e^{-iax}}{2ia}\right) + \frac{i}{a}(\log\cos ax).\left(\frac{e^{iax} + e^{-iax}}{2ia}\right)\right]$$

$$= \left(\frac{x}{a}\right)\sin ax + \frac{1}{a^2}\cos ax.\,(\log\cos ax)$$

$\therefore$ The required solution is

$$y = C.F. + P.I.$$

or $\quad y = C_1\cos ax + C_2\sin ax + \left(\frac{x}{a}\right)\sin ax + \left(\frac{1}{a^2}\right)\cos ax.\log\cos ax$

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Lecture 5

Equations Reducible To Linear Equations with Constant Coefficients

INTRODUCTION

Now the shall study two such forms of linear differential equations with variable coefficients which can be reduced to linear differential equations with constant coefficients by suitable substituitions.

1. Cauchy's Homogeneous Linear Equations

A differential equation of the form

$$x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots\ldots + P_n y = X$$

where $P_1, P_2, \ldots\ldots\ldots P_n$ are constants and X is either a function of x or a constant is called Cauchy-Euler homogeneous linear differential equation.

The solution of the above homogenous linear equation may be obtained after transforming it into linear equation with constant coefficients by using the substitution.

By the substitution $x = e^z$ or $z = \log_e x$; $\therefore \frac{dz}{dx} = \frac{1}{x}$

Now $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$

$$\therefore x\frac{dy}{dx} = \frac{dy}{dz} \qquad (1)$$

Again $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right)$

$$= \frac{x\frac{d^2y}{dz^2}\frac{dz}{dx} - \frac{dy}{dz}}{x^2} = \frac{x\frac{d^2y}{dz^2}\frac{1}{x} - \frac{dy}{dz}}{x^2}$$

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$$\therefore x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} \qquad (2)$$

Also $$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left[\frac{1}{x^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)\right]$$

$$= \frac{x^2\left[\frac{d^3y}{dz^3}\frac{dz}{dx} - \frac{d^2y}{dz^2}\frac{dz}{dx}\right] - 2x\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)}{x^4}$$

Substituting $\frac{dz}{dx} = \frac{1}{x}$ and simplifying, we get

$$x^3 \frac{d^3y}{dx^3} = \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dz} \qquad (3)$$

Using $x\frac{d}{dx} \equiv \frac{d}{dz} \equiv D$, in (1), (2) and (3)

we get $$x\frac{dy}{dx} = Dy$$

$$x^2 \frac{d^2y}{dx^2} = D(D-1)y$$

$$x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

In general, we have

$$x^n \frac{d^ny}{dx^n} = D(D-1)(D-2)\ldots\ldots\ldots\ldots(D-n+1)y$$

Using, these results in homogeneous linear equation, it will be transformed into a linear differential equation with constant coefficients.

Example : . Solve $x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} - 3y = x^2 \log x$

Solution. On changing the independent variable by substituting

$$x = e^z \text{ or } z = \log_e x \text{ and } \frac{d}{dz} \equiv D$$

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The differential equation becomes

$$[D(D-1) - D - 3]\, y = ze^{2z}$$

or $$(D^2 - 2D - 3)\, y = ze^{2Z}$$

Now the auxiliary equation is $m^2 - 2m - 3 = 0$

$\Rightarrow$ $$m = 3, -1$$

Hence, the C.F. $= C_1 e^{3Z} + C_2 e^{-Z} = C_1 x^3 + \dfrac{C_2}{x}$

and $$\text{P.I.} = \frac{1}{D^2 - 2D - 3} ze^{2z}$$

$$= e^{2z} \frac{1}{(D+2)^2 - 2(D+2) - 3} z = e^{2z} \frac{1}{D^2 + 4 + 4D - 2D - 4 - 3} z$$

$$= e^{2z} \frac{1}{D^2 + 2D - 3} z$$

$$= e^{2z} \frac{1}{-3\left(1 - \frac{2D}{3} - \frac{D^2}{3}\right)} z$$

$$= \frac{e^{2z}}{-3}\left[1 - \left(\frac{2D}{3} + \frac{D^2}{3}\right)\right]^{-1} z$$

$$= \frac{e^{2z}}{-3}\left[1 + \frac{2D}{3} + \frac{D^2}{3} + \ldots\ldots\right] z$$

$$= \frac{e^{2z}}{-3}\left(z + \frac{2}{3}\right) = e^{2z}\left(-\frac{z}{3} - \frac{2}{9}\right)$$

$$= x^2\left(-\frac{1}{3}\log_e x - \frac{2}{9}\right)$$

Hence solution of the given differential equation is

$$y = C_1 x^3 + \frac{C_2}{x} + x^2\left(-\frac{1}{3}\log_e x - \frac{2}{9}\right)$$

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2. Legendre's linear differential equation (Equation reducible to homogeneous form)

An equation of the form

$$(ax + b)^n \frac{d^n y}{dx^n} + k_1 (ax + b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots + k_n y = X \qquad (1)$$

Where a, b, k_1, k_2,.....k_n are all constants and X is a function of x, is called Legendre's linear equation.

Such equations can be reduced to linear equations with constant coefficients by substituting $ax + b = e^z$ i.e. $z = \log (ax + b)$

Then if $D = \frac{d}{dz}$, $\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = \frac{a}{ax + b} \frac{dy}{dz}$

i.e. $(ax + b) \frac{dy}{dx} = aDy$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{a}{ax + b} \frac{dy}{dz}\right) = \frac{-a^2}{(ax + b)^2} \frac{dy}{dz} + \frac{a}{ax + b} \frac{d}{dz}\left(\frac{dy}{dz}\right) \frac{dz}{dx}$$

$$= \frac{a^2}{(ax + b)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

i.e. $(ax + b)^2 \frac{d^2y}{dx^2} = a^2 D (D - 1) y$

Similarly $(ax + b)^3 \frac{d^3y}{dx^3} = a^3 D (D - 1) (D - 2) y$ and so on.

After making these replacements in (1), there results a linear equation with constant coefficients.

Example Solve $(1 + x)^2 \frac{d^2y}{dx^2} + (1 + x) \frac{dy}{dx} + y = 4 \cos \log (1 + x)$

Solution. put $1 + x = e^z$ and $\frac{d}{dz} = D$

Hence the given differential equation becomes

$[D (D - 1) + D + 1] y = 4 \cos z$

$\therefore$ Auxiliary equation is

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$$D^2 + 1 = 0 \text{ or } D = \pm i$$

$\therefore$ C.F. $= C_1 \cos (z + C_2) = C_1 \cos [\log (1 + x) + C_2]$

and $\text{P.I} = \dfrac{1}{D^2 + 1} 4 \cos z = 4. \dfrac{z}{2} \sin z$

$$= 2z \sin z$$

$$= 2 \log (1 + x) \sin \log (1 + x)$$

Hence the required solution is

$y = C_1 \cos [\log (1 + x) + C_2] + 2 \log (1 + x) \sin \log (1 + x)$

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Lecture 6

METHOD OF VARIATION OF PARAMETERS

Method of variation of parameters enables to find solution of any linear non homogeneous differential equation of second order even (with variable coefficients also) provided its complimentary function is given (known). The particular integral of the non-homogeneous equation is obtained by varying the parameters i.e. by replacing the arbitrary constants in the C.F. by variable functions.

Consider a linear non-homogeneous second order differential equation with variable coefficients

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = X(x) \qquad (1)$$

Suppose the complimentary function of (1) is $= C_1y_1(x) + C_2y_2(x)$ (2)

so that y_1 and y_2 satisfy

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$$

In method of variation of parameters the arbitrary constants C_1 and C_2 in (2) are replaced by two unknown functions u (x) and v (x).

Let us assume particular integral is $= u(x)\,y_1(x) + v(x)\,y_2(x)$ (3)

where $$u = \int \frac{-X\,y_2}{y_1\,y_2' - y_1'\,y_2}\,dx$$

and $$v = \int \frac{X\,y_1}{y_1\,y_2' - y_1'\,y_2}\,dx$$

on putting the values of u and v in (3) we get P.I

Thus, required general solution = C.F + P.I

Example Apply the method of variation of parameters to solve $\frac{d^2y}{dx^2} + y = \tan x$

Solution. The auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = \pm i$

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$\therefore$ C.F. $= C_1 \cos x + C_2 \sin x$ (1)

Here $y_1 = \cos x$, $y_2 = \sin x$

Therefore $y_1 y_2' - y_1' y_2 = \cos^2 x + \sin^2 x = 1$

Let us suppose P.I $= u.y_1 + v\, y_2$ (2)

where

$$u = \int \frac{-Xy_2}{y_1 y_2' - y_1' y_2}\, dx = -\int \frac{\sin x \tan x}{1}\, dx$$

$$= -\int \frac{\sin^2 x}{\cos x}\, dx = -\int \frac{1 - \cos^2 x}{\cos x}\, dx$$

$$= \int (\cos x - \sec x)\, dx$$

$$= \sin x - \log(\sec x + \tan x)$$

& $$v = \int \frac{X\, y_1}{y_1 y_2' - y_1' y_2}\, dx = \int \frac{\tan x \cos x}{1}\, dx$$

$$= \int \sin x\, dx = -\cos x$$

Putting the values of u and v in (2), we get

$$\text{P.I} = u\, y_1 + v\, y_2$$

$$= [\sin x - \log(\sec x + \tan x)] \cos x - \cos x \sin x$$

$$= -\cos x \log(\sec x + \tan x)$$

Therefore, complete solution is

$y = C_1 \cos x + C_2 \sin x - \cos x \log(\sec x + \tan x)$

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Lecture 7

SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS (Solution by Changing dependent and independent variables)

INTRODUCTION

The general form of linear differential equation of the second order may be written as

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad (1)$$

where P, Q and R are functions of x only. There is no general method for the solution of this type of equations. Some particular methods used to solve these equations are, change of independent variables, Variation of parameters and removal of first order derivatives etc. As this kind of differential equations are of great significance in physics, especially in connection with vibrations in mechanics and theory of electric circuit. In addition many profound and beautiful ideas in pure mathematics have grown out to the study of these equations.

Method I: Complete solution is terms of known integral belonging to the complementary function (i.e. part of C.F. is known or one solution is known).

Let u be a part of complementary function of equation (1) and v is remaining solution of differential equation (1)

Then the complete solution of equation (1) is

$$y = u\,v \qquad (2)$$

$$\Rightarrow \qquad \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} \text{ and } \frac{d^2y}{dx^2} = v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}$$

Putting these values in equation (1) then, we get

$$v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2} + P\left(v\frac{du}{dx} + u\frac{dv}{dx}\right) + Quv = R$$

$$\text{or} \qquad v\left[\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right] + u\left[\frac{d^2v}{dx^2} + P\frac{dv}{dx}\right] + 2\frac{du}{dx}\frac{dv}{dx} = R \qquad (3)$$

Since u is a part of C.F. i.e. solution of (1)

$$\therefore \qquad \frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu = 0$$

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Hence equation (3) becomes

$$u\left(\frac{d^2v}{dx^2} + P\frac{dv}{dx}\right) + 2\frac{du}{dx}\frac{dv}{dx} = R$$

or $$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u} \qquad (4)$$

Let $$\frac{dv}{dx} = z, \text{ so that } \frac{d^2v}{dx^2} = \frac{dz}{dx}$$

Equation (4) becomes

$$\frac{dz}{dx} + \left(P + \frac{2}{u}\frac{du}{dx}\right)z = \frac{R}{u},$$

which is linear in z, Hence z can be determined

We obtain v, by integration the relation $\frac{dv}{dx} = z$

$\Rightarrow$ $$v = \int z\,dx + C_1$$

Therefore, the solution of (1) is $y = u\left[\int z\,dx + C_1\right]$

i.e. $$y = uv$$

Remark. Solving by the above method, u determined by inspection of the following rules

(1) If $P + Qx = 0$, then $u = x$

(2) If $1 + P + Q = 0$, then $u = e^x$

(3) If $1 - P + Q = 0$, then $u = e^{-x}$

(4) If $1 + \frac{P}{a} + \frac{Q}{a^2} = 0$, then $u = e^{ax}$

(5) If $2 + 2Px + Qx^2 = 0$, then $u = x^2$

(6) If $m(m-1) + Pmx + Qx^2 = 0$, then $u = x^m$

Example Solve $y'' - 4xy' + (4x^2 - 2)y = 0$ given that $y = e^{x^2}$ is an integral induced in the complementary function.

Solution. The given equation may be written as

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$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 2)\,y = 0$$

Here $P = -4x$, $Q = 4x^2 - 2$, $R = 0$

and $u = e^{x^2}$, so that $\frac{du}{dx} = 2\,xe^{x^2}$

Let $y = uv \Rightarrow y = e^{x^2}\,v$ (1)

we know that

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = 0, \text{ As } R = 0$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + \left(\frac{2}{e^{x^2}}\,2xe^{x^2} - 4x\right)\frac{dv}{dx} = 0$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} = 0$$

$$\Rightarrow \quad \frac{dv}{dx} = C$$

$$\Rightarrow \quad v = C_1x + C_2$$

Hence the complete solution is $y = e^{x^2}\,v$

or $\quad y = e^{x^2}\,(C_1x + C_2)$

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Lecture 8

Method II. Normal form (Removal of first derivative)

Let $$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad (1)$$

putting $y = uv$, we get

$$v\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) + u\left(\frac{d^2v}{dx^2} + P\frac{dv}{dx}\right) + 2\frac{du}{dx}\frac{dv}{dx} = R$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + \left(\frac{2}{u}\frac{du}{dx} + P\right)\frac{dv}{dx} + v\left(\frac{1}{u}\frac{d^2u}{dx^2} + \frac{P}{u}\frac{du}{dx} + Q\right) = \frac{R}{u} \qquad (2)$$

But the first order derivative must be remove

so $$\frac{2}{u}\frac{du}{dx} + P = 0 \Rightarrow \frac{du}{u} = -\frac{1}{2}Pdx$$

$$\Rightarrow \quad \log u = -\int\frac{P}{2}dx$$

$$\Rightarrow \quad u = e^{-\int\frac{P}{2}dx}$$

Since $$\frac{du}{dx} = -\frac{Pu}{2} \quad \Rightarrow \quad \frac{d^2u}{dx^2} = -\frac{1}{2}\left[P\frac{du}{dx} + u\frac{dP}{dx}\right]$$

$$\Rightarrow \quad \frac{d^2u}{dx^2} = -\frac{1}{2}\left[P\left(-\frac{Pu}{2}\right) + u\frac{dP}{dx}\right] = \frac{P^2u}{4} - \frac{u}{2}\frac{dp}{dx}$$

From (2) $$\frac{d^2v}{dx^2} + v\left(\frac{P^2}{4} - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{2} + Q\right) = \frac{R}{u}$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + v\left(Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}\right) = \frac{R}{u}$$

$$\Rightarrow \quad \frac{d^2v}{dx^2} + Iv = \frac{R}{u} \qquad (3)$$

This equation is called normal form of equation (1)

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where $$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$$

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Example : . Solve

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)\,y = -3\,e^{x^2}\sin 2x$$

Solution. Here $P = -4x$, $Q = 4x^2 - 1$, $R = -3e^{x^2}\sin 2x$

so $$I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = 4x^2 - 1 - \frac{1}{2}(-4) - \frac{1}{4}(-4x)^2$$

$$= 4x^2 - 1 + 2 - 4x^2 = 1$$

$$u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int(-4x)\,dx}$$

$$= e^{2\int x\,dx} = e^{x^2}$$

Then substituting these values in the equation

$$\frac{d^2v}{dx^2} + Iv = \frac{R}{u}, \text{ We have}$$

$$\frac{d^2v}{dx^2} + v = \frac{-3e^{x^2}\sin 2x}{e^{x^2}} = -3\sin 2x$$

its $C.F = C_1\cos x + C_2\sin x$

and $$P.I = -\frac{1}{D^2+1}\,2\sin 2x = -3\,\frac{1}{-2^2+1}\sin 2x$$

$$= \sin 2x$$

Thus $v = C_1\cos x + C_2\sin x + \sin 2x$

Therefore required solution is $y = uv$

or $y = e^{x^2}(C_1\cos x + C_2\sin x + \sin 2x)$

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Lecture 9

Method III. Change of independent variable

consider $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ (1)

Let us change the independent variable x to z and z = f(x).

Then $$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} \quad (2)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dz}\frac{dz}{dx}\right) = \frac{dy}{dz}\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2\frac{d^2y}{dx^2} \quad (3)$$

Putting the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1) we get

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2} + P\frac{dy}{dz}\frac{dz}{dx} + Qy = R$$

or $$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \left(P\frac{dz}{dx} + \frac{d^2z}{dx^2}\right)\frac{dy}{dz} + Qy = R$$

or $$\frac{d^2y}{dz^2} + \frac{\left(P\frac{dz}{dx} + \frac{d^2z}{dx^2}\right)}{\left(\frac{dz}{dx}\right)^2}\frac{dy}{dz} + \frac{Q}{\left(\frac{dz}{dx}\right)^2}y = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

$\Rightarrow$ $$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \quad (4)$$

where $$P_1 = \frac{\left(P\frac{dz}{dx} + \frac{d^2z}{dx^2}\right)}{\left(\frac{dz}{dx}\right)^2},\ Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \text{ and } R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2}$$

Equation (4) is solved either by taking $P_1 = 0$ or Q_1 = a constant

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Example Solve by changing the independent variable

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3y = x^5$$

Solution. Given equation is

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2y = x^4 \qquad (1)$$

Here $P = -\frac{1}{x}, Q = 4x^2$ and $R = x^4$

On changing the independent variable x to z, the equation (1) transformed as

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1 \qquad (2)$$

where $$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4x^2}{\left(\frac{dz}{dx}\right)^2} = \text{constant} = 1 \text{ say}$$

or $$\left(\frac{dz}{dx}\right)^2 = 4x^2$$

$$\Rightarrow \quad \frac{dz}{dx} = 2x$$

$$\Rightarrow \quad z = x^2$$

$$\Rightarrow \quad \frac{d^2z}{dx^2} = 2$$

$$P_1 = \frac{P\frac{dz}{dx} + \frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2} = \frac{\left\{2 + \left(\frac{-1}{x}\right)2x\right\}}{4x^2} = 0$$

$$R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{x^4}{4x^2} = \frac{x^2}{4} = \frac{z}{4}$$

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on putting the values of P_1, Q_1 and R_1 in (2), we get

$$\frac{d^2y}{dz^2} + y = \frac{z}{4}$$

or $$(D^2 + 1)\, y = \frac{z}{4}$$

its A.E. is $m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$

$\therefore \quad C.F = C_1 \cos z + C_2 \sin z$

or $\quad C.F = C_1 \cos x^2 + C_2 \sin x^2$

and $$P.I = \frac{1}{D^2 + 1}\, \frac{z}{4} = \frac{1}{4}\,(1 + D^2)^{-1}\, z$$

$$= \frac{1}{4}\,(1 - D^2 + \ldots\ldots\ldots)\, z$$

$$= \frac{z}{4}$$

$$= \frac{x^2}{4}$$

Hence the complete solution is $y = C.F + P.I$

or $$y = C_1 \cos x^2 + C_2 \sin x^2 + \frac{x^2}{4}$$

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Lecture 10

SIMULTANEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS:

In Several applied mathematics problems, there are more than one dependent variables, each of which is a function of one independent variable, usually say time t. The formulation of such problems leads to a system of simultaneous linear differential equation with constant coefficients. Such a system can be solved by the method of elimination. Laplace transform method, using matrices and short cut operator methods.

Example Solve $\frac{dx}{dt} + y = \sin t$, $\frac{dy}{dt} + x = \cos t$ $x(o) = 2, y(o) = 0$

Solution. We have

$$\frac{dx}{dt} + y = \sin t \tag{1}$$

$$\frac{dy}{dt} + x = \cos t \tag{2}$$

Differentiating (1) w.r.t. 't' we have

$$\frac{d^2x}{dt^2} + \frac{dy}{dt} = \cos t \tag{3}$$

Using (2) in (3) we get

$$\frac{d^2x}{dt^2} - x = 0 \quad \Rightarrow \quad (D^2 - 1)x = 0$$

its auxiliary equation is

$m^2 - 1 = 0 \quad \Rightarrow \quad m = \pm 1$

$$\therefore \quad x = C_1 e^t + C_2 e^{-t} \tag{4}$$

$$\Rightarrow \quad \frac{dx}{dt} = C_1 e^t - C_2 e^{-t}$$

putting this value of $\frac{dx}{dt}$ in (1) we get

$$y = \sin t - C_1 e^t + C_2 e^{-t} \tag{5}$$

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Using given conditions

$$\left.\begin{array}{l}\text{from (iv) } C_1 + C_2 = 2 \\ \text{from (v) } -C_1 + C_2 = 0\end{array}\right\} \Rightarrow C_1 = C_2 = 1$$

putting these values of C_1 and C_2 in (4) & (5) we get

$$x = e^t + e^{-t}$$

and $$y = \sin t - e^t + e^{-t}$$

is the required solution

On changing the independent variable x to z, the equation (1) transformed as

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1 \qquad (2)$$

where $$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = \frac{4x^2}{\left(\frac{dz}{dx}\right)^2} = \text{constant} = 1 \text{ say}$$

or $$\left(\frac{dz}{dx}\right)^2 = 4x^2$$

$$\Rightarrow \quad \frac{dz}{dx} = 2x \Rightarrow z = x^2 \Rightarrow \frac{d^2z}{dx^2} = 2$$

$$P_1 = \frac{P\frac{dz}{dx} + \frac{d^2z}{dx^2}}{\left(\frac{dz}{dx}\right)^2} = \frac{\left\{2 + \left(\frac{-1}{x}\right)2x\right\}}{4x^2} = 0 \quad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = \frac{x^4}{4x^2} = \frac{x^2}{4} = \frac{z}{4}$$

on putting the values of P_1, Q_1 and R_1 in (2), we get

$$\frac{d^2y}{dz^2} + y = \frac{z}{4}$$

or $$(D^2 + 1)\, y = \frac{z}{4}$$

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Lecture 11

Series Solutions

INTRODUCTION

In this chapter, we shall describe methods of solving variable coefficient differential equations.

Consider the second order homogeneous linear differential equation

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0, \qquad a_0(x)\neq 0 \qquad \text{(i)}$$

where $a_0(x)$, $a_1(x)$ and $a_2(x)$ are polynomials.

The equation (i) is called a variable coefficient second order homogeneous equation. A number of problems of physics and engineering involve differential equations of the form (i) in which $a_0(x)$, $a_1(x)$ and $a_2(x)$ are polynomials. Solution of such equations can be obtained in terms of infinite series. In this chapter, we shall present methods for determining the solution of (i) in terms of infinite series. The method can be classified into two categories, power series method and generalized power series method (Frobenius method). In the following sections, we shall discuss the applications of these methods in solving special functions. The study of those solutions (and of other "higher" functions not discussed in calculus) is called the theory of special functions. The series solution of certain differential equations give rise to special functions such as Bessel's function, Legendre's Polynomial, Lagurre's polynomial, Hermite's polynomial, chebyshev polynomial, strum - Lioville problem based on the orthogonality of function is also included which shows that Bessel's Legendre's and other equations can be considered from a common point of view. These special functions, have many applications in mathematical physics and engineering.

ORDINARY AND SINGULAR POINTS

Consider the second order linear homogeneous equation

$$a_0(x)\frac{d^2y}{dx^2}+a_1(x)\frac{dy}{dx}+a_2(x)y=0, \qquad a_0(x)\neq 0 \qquad \text{(i)}$$

where $a_0(x)$, $a_1(x)$ and $a_2(x)$ are polynomials. Dividing the equation by $a_0(x)$, we get

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$$\frac{d^2y}{dx^2}+\frac{a_1(x)}{a_0(x)}\frac{dy}{dx}+\frac{a_2(x)}{a_0(x)}y=0 \qquad \text{(ii)}$$

Taking $p(x)=\frac{a_1(x)}{a_0(x)}$, $q(x)=\frac{a_2(x)}{a_0(x)}$. The equation (ii) Can be written as

$$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(y)y=0$$

Equation is called the standard form or normal form of (i). It is also called the canonical form of equation (i). Consider the interval I. let $x_0 \in I$ and $a_1(x)$, $a_2(x)$ be analytic (differentiable) at the point x_0. We define the following.

Ordinary Point

A point $x_0 \in I$ is said to be ordinary point (a regular point) of equation (i) if $a_0(x_0) \neq 0$.

Example $x = 0$ is an ordinary point of the equation

$$(1+x^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}-y=0$$

Singular point A point $x_0 \in I$ is said to be a singular point of the equation (i) if $a_0(x_0) = 0$

Example $x = 0$ is a singular point of the equation

$$x^3\frac{d^2y}{dx^2}+3x\frac{dy}{dx}+6y=0$$

REGULAR AND IRREGULAR SINGULAR POINTS

Consider the normal form of equation (i) i.e.

$$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=0$$

where $p(x)=\frac{a_1(x)}{a_0(x)}$, $q(x)=\frac{a_2(x)}{a_0(x)}$

Let $x_0 \in I$ be a singular point of (i) and $a_1(x)$, $a_2(x)$ be analytic (differentiable) at $x = x_0$.

write above equation as

$$\frac{d^2y}{dx^2}+\frac{p_1(x)}{(x-x_0)}\frac{dy}{dx}+\frac{q_1(x)}{(x-x_0)^2}y=0 \qquad \text{(iii)}$$

where $p_1(x) = (x - x_0)\, p(x)$ and $q_1(x) = (x - x_0)^2\, q(x)$ we define the following.

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Regular singular point

A singular point $x_0 \in I$ of (i) is said to be a regular singular point if the function $p_1(x) = (x - x_0)\, p(x) = \dfrac{(x-x_0)a_1(x)}{a_0(x)}$ and $q_1(x) = (x-x_0)^2 q(x) = \dfrac{(x-x_0)^2 a_2(x)}{a_0(x)}$ have removable discontinuities at x_0 and become analytic when these discontinuities are removed.

Irregular singular point

A singular point of the equation (i) is said to be an irregular singular point of the equation (i) if and only if x_0 is not a regular singular point of (i).

Example 1. Find the regular and singular points of

(i) $$x^3\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 6y = 0$$

(ii) $$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

(iii) $$x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0$$ (a, b are contants)

Solution

(i) we have $a_0(x) = x^3$

$a_0(x) = 0 \quad \Rightarrow \quad x^3 = 0$

or $x = 0$

x_0 is the singular point of the given equation i.e.,

$$x^3\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + 6y = 0$$

All other points are regular (ordinary) points.

(ii) Here we have $a_0(x) = 1 - x^2$

$a_0 = 0 \Rightarrow 1 - x^2 = 0$

or $x = \pm 1$

Hence, $x = -1$ and $x = 1$ are singular points of the equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

while all other points are regular (ordinary) points.

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Lecture 12

SERIES SOLUTION OF THE DIFFERENTIAL EQUATION WHEN X =0 IS AN ORDINARY POINT

There are following steps for solution of equation

Step I: Assume that $y = \sum_{n=0}^{\infty} a_n x^n$ or $y = a_0 + a_1 x + a_2 x^2 + \ldots\ldots\ldots$ (i)

Step II: Substitute $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ obtained by equation (i) in equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

Step III. Equate to zero the coefficients of various powers of x and find a_2, a_3..... in terms of a_0 and a_1.

Step IV. Substitute the value of a_2, a_3.....in relation (i) which will be the required series solution.

EXAMPLE

Solve the following differential equation in series

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 4y = 0$$

Solution The given equation can be written as

$$\frac{d^2y}{dx^2} - \frac{x}{(1-x^2)}\frac{dy}{dx} + \frac{4y}{(1-x^2)} = 0 \qquad \text{(i)}$$

Here $p(x) = -\frac{x}{1-x^2}$, $q(x) = \frac{4}{1-x^2}$

p(x) and q(x) both exist at x =0, so x = 0 is an ordinary point of the equation

$$\text{Let } y = \sum_{n=0}^{\infty} a_n x^n \qquad \text{(ii)}$$

$$\therefore \frac{dy}{dx} = \sum_{n=0}^{\infty} a_n \cdot n x^{n-1} \text{ and } \frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} a_n \cdot n(n-1)x^{n-2}$$

putting these values in given equation, we get

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$$(1-x^2)\sum_{n=0}^{\infty} a_n \cdot n(n-1)x^{n-2} - x\sum_{n=0}^{\infty} a_n \cdot nx^{n-1} + 4\sum_{n=0}^{\infty} a_n \cdot x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n \cdot n(n-1)x^{n-2} - \sum_{n=0}^{\infty} a_n .(n^2-4)x^n = 0$$

Equating to zero the coefficients of various powers of x.

$\therefore$ Coefficient of x^0

$a_2 . 2.1 + 4\, a_0 = 0 \Rightarrow a_2 = -2a_0$

Now equating the coefficient of x^n, we get

$$a_{n+2}(n+2)(n+1) - (n-2)(n+2)\, a_n = 0$$

$$\Rightarrow \qquad a_{n+2} = \frac{n-2}{n+1} a_n$$

putting n = 1, 2, 3, ... etc we get

$$a_3 = -\frac{1}{2} a_1$$

$$a_4 = -\frac{0}{2} a_2 a_2 = 0, a_5 = \frac{1}{4} a_3 = \frac{1}{4}\left(-\frac{1}{2} a_1\right) = \frac{-a_1}{8}$$

$$a_6 = \frac{2}{5} a_4 = \frac{2}{5}.0 = 0 \text{.....} \text{ and so on.}$$

Substituting these values in equation (ii) we get

$$y = a_0 + a_1 x - 2a_0 x^2 + \left(-\frac{1}{2} a_1\right) x^3 + 0x^4 + \left(\frac{-a_1}{8}\right) x^5 + \text{.....}$$

$$= a_0 + a_1 x - 2a_0 x^2 - \frac{a_1}{2} x^3 - \frac{a_4}{8} x^5 + \text{.....}$$

$$\therefore y = a_0(1-2x^2) + a_1 x\left(1 - \frac{x^2}{2} - \frac{x^4}{8} + \text{.....}\right)$$

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Lecture 13

FROBENIUS METHOD (SERIES SOLTUTION WHEN x =0 IS NOT AN ORDIANRY POINT)

If x =0 is not an ordinary point (is regular singular point) then we use the following steps for solution.

Step I. Assume a solution of the form

$$y = \sum_{r=0}^{\infty} a_r x^{m+r}$$

Step II. Find $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ and substitute these values in the given equation.

Step III Equate to zero the coefficients of lowest power of x in m gives a quadratic equation in m known as the identical equation of given differential equation.

Thus we get two values of m. The series solution of given equation depends on nature of the roots of the identical equation. Let $y_1(x)$ and $y_2(x)$ be two nontrivial linearly independent solution of given differential equation.

Then the general solution is

$y(x) = Ay_1(x) + By_2(x)$, where A and B are arbitrary constants

There are following cases to solve the equation

Case I: The roots i.e. m_1, and m_2 are distinct ($m_1 \neq m_2$) and not differing by an integer 1, 2, 3,.....

The complete solution is $y = A\,(y)_{m_1} + B(y)_{m_2}$

Case II: ($m_1 \neq m_2$) and differ by an integer.

There are two cases for solving the equation when $m_1 \neq m_2$ and differ by an integer.

(a) If $m_1 < m_2$ and making some coefficients indeterminate at $m = m_1$, then solution is $y = (y)_{m_1}$.

Similarly if $m_1 > m_2$ then the solution is $y = (y)_{m_2}$.

Remark : The solution at m_1 or m_2 has already contain all those terms which present in the solution at other root, so we obtained solution only m_1 or m_2

(b) Let $m_1 > m_2$ if some of the coefficients of y become infinite when $m = m_2$ then we substitute $a_0 = b_0\,(m - m_2)$ where $b_0 \neq 0$ in the summation complete solution.

$$y = A\,(y)_{m_1} + B\left(\frac{\partial y}{\partial m}\right)_{m_2}$$

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Remark: The value of y at $m = m_2$ after replacing a_0 by $b_0(m - m_2)$ gives only the multiple of $(y)_{m_1}$ and it not be an independent solution.

Case III. When the roots are equal i.e.

$(m_1 = m_2 = k)$

Then the complete solution is

$$y = A(y)_k + B\left(\frac{\partial y}{\partial m}\right)_k$$

$$\text{As } m_1 = m_2$$
$$= k$$

Case I.

EXAMPLE Solve the following equation in power series about x =0.

$$2x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x+1)y = 0$$

Solution The given equation can be written as

$$\frac{d^2y}{dx^2} + \frac{1}{2x}\frac{dy}{dx} - \left(\frac{x+1}{2x^2}\right)y = 0$$

$$p(x) = \frac{1}{2x} \text{ and } Q(x) = -\left(\frac{x+1}{2x^2}\right)$$

Hence x P(x) and $x^2Q(x)$ both exist at the point x =0. So x =0 is a regular point, we assume the solution in the form

$$y = \sum_{r=0}^{\infty} a_r x^{m+r} \qquad \text{(i)}$$

$$\therefore \frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (m+r)x^{m+r-1}$$

$$\text{and } \frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (m+r)(m+r-1)x^{m+r-2}$$

substituting the values of $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in given equation we get

$$2x^2\sum_{r=0}^{\infty} a_r (m+r)(m+r-1)x^{m+r-2} + x\sum_{r=0}^{\infty} a_r (m+r)x^{m+r-1} - (x+1)\sum_{r=0}^{\infty} a_r x^{m+r} = 0$$

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$$\text{or } \sum_{r=0}^{\infty} a_r\left[2(m+r)(m+r-1)+(m+r)-1\right]x^{m+r}-\sum_{r=0}^{\infty} a_r x^{m+r-1}=0$$

Equating to zero the coefficients of lowest power of x, we get

$\Rightarrow \quad a_0[2m(m-1)+m-1]=0$

$\Rightarrow \quad 2m(m-1)+(m-1)=0$

$\Rightarrow \quad (m-1)(2m+1)=0$

$\Rightarrow \quad m=1, -\frac{1}{2}$

Coefficient of next lowest power of x i.e. x^{m+1}

$$a_1[2(m+1)m+(m+1)-1]-a_0=0$$

$$\Rightarrow a_1=\frac{a_0}{2m^2+2m+m}=\frac{a_0}{m(2m+3)}$$

Now equating to zero the coefficient of x^{m+r} by putting r = r-1 in second summation, we get

$a_r\{2(m+r)(m+r-1)+(m+r-1)\}-a_{r-1}=0$

$$\Rightarrow a_r=\frac{a_{r-1}}{(m+r-1)(2m+2r+1)}$$

Now putting r = 1, 2, 3, 4,....., we get

$$a_1=\frac{a_0}{m(2m+3)}$$

$$a_2=\frac{a_1}{(m+1)(2m+5)}, a_3\frac{a_2}{(m+2)(2m+7)}, a_4=\frac{a_3}{(m+3)(2m+9)}$$

For m=1

$$a_1=\frac{a_0}{5}, a_2=\frac{a_1}{14}=\frac{a_0}{70}, a_3=\frac{a_2}{27}=\frac{a_0}{1890}, \ldots\ldots\ldots\ldots$$

Here $y_1=a_0x^{1+0}a_1x^{1+1}+a_2x^{1+2}+a_3x^{1+3}+\ldots\ldots\ldots$ (from (i))

$$\Rightarrow y_1=a_0x+\frac{a_0}{5}x^2+\frac{a_0}{70}x^3+\frac{a_0}{1890}x^4+\ldots\ldots\ldots\ldots$$

$$\Rightarrow y_1=a_0x\left[1+\frac{x}{5}+\frac{x^2}{70}+\frac{x^3}{1890}+\ldots\ldots\ldots\right] \qquad \text{(ii)}$$

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For $m = -\frac{1}{2}$

$$a_1 = \frac{a_0}{-\frac{1}{2}(-1+3)} = \frac{a_0}{-\frac{1}{2}2} = -a_0$$

$$a_2 = -\frac{a_0}{2},\ a_3 = \frac{a_2}{9} = -\frac{a_0}{18}, \ldots\ldots\ldots\ldots \text{ (from (i))}$$

$$\text{or } y_2 = a_0 x^{-\frac{1}{2}} + (-a_0)x^{-\frac{1}{2}+1} - \frac{a_0}{2}x^{-\frac{1}{2}+2} - \frac{a_0}{18}x^{-\frac{1}{2}+3} \ldots\ldots\ldots\ldots$$

$$\text{or } y_2 = a_0 x^{-\frac{1}{2}}\left[1 - x - \frac{x^2}{2} - \frac{x^3}{18} \ldots\ldots\ldots\ldots\right]$$

Hence the required solution is

$$y_1 = Aa_0 x\left(1 + \frac{x}{5} + \frac{x^2}{70} + \frac{x^3}{1890} + \ldots\ldots\ldots\ldots\right) + Ba_0 x^{-\frac{1}{2}}\left(1 - x - \frac{x^2}{2} - \frac{x^3}{18} - \ldots\ldots\ldots\right)$$

$$\text{or } y_1 = c_1 x\left(1 + \frac{x}{5} + \frac{x^2}{70} + \frac{x^3}{1890} + \ldots\ldots\ldots\ldots\right) + c_2 x^{-\frac{1}{2}}\left(1 - x - \frac{x^2}{2} - \frac{x^3}{18} - \ldots\ldots\ldots\right)$$

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Lecture 14

Case II (a) Non logarithm case

EXAMPLE Solve in series the differential equation

$$x\frac{d^2y}{dx^2}+2\frac{dy}{dx}+xy=0$$

Solution: Here $p(x)=\frac{2}{x}$, $Q(x)=1$

At $x=0$, $x\,P(x)$ and $x^2\,Q(x)$ both exist $\therefore$ $x=0$ is a regular singular point

Let us suppose

$$y=\sum_{r=0}^{\infty}a_r r^{m+r} \qquad \text{(i)}$$

$$\therefore \frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(m+r)x^{m+r-1}$$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(m+r)(m+r-1)x^{m+r-2}$$

substituting these values in the given equation we get

$$x\sum_{r=0}^{\infty}a_r(m+r)(m+r-1)x^{m+r-2}+2\sum_{r=0}^{\infty}a_r(m+r)x^{m+r-1}+x\sum_{r=0}^{\infty}a_r x^{m+r}=0$$

$$\Rightarrow \sum_{r=0}^{\infty}a_r(m+r)(m+r+1)x^{m+r-1}+\sum_{r=0}^{\infty}a_r x^{m+r+1}=0$$

Now, equate to zero the lowest power of x i.e. x^{m-1}, we get

$a_0(m+0)(m+1)=0 \Rightarrow m=0,-1$

Here $m_1-m_2=0-(-1)=1$

and $m_1>m_2$

So, obtained the solution for $m=m_2$ i.e. $m_2=-1$.

next equate to zero the coefficient of x^m, then we get

$a_1(m+1)(m+2)=0 \Rightarrow a_1=\frac{0}{0}$ indeterminate form

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and coefficient of x^{m+1}

$$a_2(m+2)(m+3)+a_0=0 \Rightarrow a_2=-\frac{a_0}{(m+2)(m+3)} \Rightarrow a_2=\frac{a_0}{(-1+2)(-1+3)}$$

$$=-\frac{a_0}{1.2}(\text{put } m=-1)=-\frac{a_0}{\lfloor 2}$$

Now equating to zero the coefficient of x^{m+r}, we get

$$a_{r+1}(m+r+1)(m+r+2)+a_{r-1}=0$$

$$\Rightarrow a_{r+1}=-\frac{a_{r-1}}{(m+r+1)(m+r+2)}$$

Putting $m_1=m_2=-1$, we get $a_{r+1}=-\frac{a_{r+1}}{r(r+1)}$

putting $r=2, 3, 4, \ldots\ldots\ldots\ldots$

$$a_3=-\frac{a_1}{2.3}=-\frac{a_1}{\lfloor 3}$$

$$a_4=-\frac{a_2}{3.4}=-\frac{1}{3.4}\left(-\frac{a_0}{1.2}\right)=\frac{1}{4.3.2.1}a_0 \Rightarrow a_4=\frac{a_0}{\lfloor 4}$$

$$a_5=-\frac{a_3}{4.5}=-\frac{1}{4.5}\left(-\frac{a_1}{2.3}\right)=\frac{1}{5.4.3.2.1}a_1=\frac{1}{\lfloor 5}a_1$$

$$a_6=-\frac{a_4}{5.6}=-\frac{1}{5.6}\left(\frac{a_0}{4.3.2.1}\right)=\frac{-1}{6.5.4.3.2.1}a_0 \Rightarrow a_6=-\frac{a_0}{\lfloor 6}$$

and so on.

Hence the required solution is

$$y=(y)_{m_2}=\sum_{r=0}^{\infty}a_r x^{-1+r}=a_0x^{-1}+a_1x^{-1+1}+a_2x^{-1+2}+a_3x^{-1+3}+\ldots\ldots\ldots\ldots \quad \text{(From (i))}$$

$$y=a_0x^{-1}+a_1+\left(-\frac{a_0}{\lfloor 2}\right)x+\left(-\frac{a_1}{\lfloor 3}\right)x^2+\left(\frac{a_0}{\lfloor 4}\right)x^3+\left(\frac{a_1}{\lfloor 5}\right)x^4+\ldots\ldots\ldots\ldots$$

$$y=x^{-1}\left[a_0\left(1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}-\ldots\ldots\ldots\ldots\right)+a_1\left(x-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}-\ldots\ldots\ldots\ldots\right)\right]$$

$$y=x^{-1}[a_0\cos x+a_1\sin x]$$

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For $m = -\frac{1}{2}$

$$a_1 = \frac{a_0}{-\frac{1}{2}(-1+3)} = \frac{a_0}{-\frac{1}{2}2} = -a_0$$

$$a_2 = -\frac{a_0}{2},\ a_3 = \frac{a_2}{9} = -\frac{a_0}{18}, \ldots\ldots\ldots\ldots \text{ (from (i))}$$

$$\text{or } y_2 = a_0 x^{-\frac{1}{2}} + (-a_0)x^{-\frac{1}{2}+1} - \frac{a_0}{2}x^{-\frac{1}{2}+2} - \frac{a_0}{18}x^{-\frac{1}{2}+3} \ldots\ldots\ldots\ldots$$

$$\text{or } y_2 = a_0 x^{-\frac{1}{2}}\left[1 - x - \frac{x^2}{2} - \frac{x^3}{18} \ldots\ldots\ldots\ldots\right]$$

Hence the required solution is

$$y_1 = Aa_0 x\left(1 + \frac{x}{5} + \frac{x^2}{70} + \frac{x^3}{1890} + \ldots\ldots\ldots\ldots\right) + Ba_0 x^{-\frac{1}{2}}\left(1 - x - \frac{x^2}{2} - \frac{x^3}{18} - \ldots\ldots\ldots\right)$$

$$\text{or } y_1 = c_1 x\left(1 + \frac{x}{5} + \frac{x^2}{70} + \frac{x^3}{1890} + \ldots\ldots\ldots\ldots\right) + c_2 x^{-\frac{1}{2}}\left(1 - x - \frac{x^2}{2} - \frac{x^3}{18} - \ldots\ldots\ldots\right)$$

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Lecture 15

Case II (b) Logarithmic case

EXAMPLE

Obtain the general solution of the equation

$$x^2\frac{d^2y}{dx^2}+5x\frac{dy}{dx}+x^2y=0$$

Solution Let us assume the series solution of the given equation as

$$y=\sum_{r=0}^{\infty}a_r x^{m+r} \qquad \text{(i)}$$

Substituting this in the given equation we get

$$\sum_{r=0}^{\infty}(m+r)(m+r-1)a_r x^{m+r}+5\sum_{r=0}^{\infty}(m+r)a_r x^{m+r}+\sum_{r=0}^{\infty}a_r x^{m+r+2}=0$$

$$\text{or } \sum_{r=0}^{\infty}(m+r)(m+r+4)a_r x^{m+r}+\sum_{r=0}^{\infty}a_r x^{m+r+2}=0 \qquad \text{(ii)}$$

Equating to zero the coefficient of the lowest degree term x^m, we get the identical equation as $m(m+4)\,a_0=0$, which gives $m=0,-4$ as $a_0\neq 0$

Equating to zero the coefficient of x^{m+1}, we get $(m+1)(m+5)\,a_1=0$, $\Rightarrow a_1=0$ as $m\neq -1$ and $m\neq -5$. Equating to zero the coefficient of x^{m+r}, we get the recurrence relation as

$(m+r)(m+r+4)\,a_r+a_{r-2}=0$

$$\Rightarrow a_r=-\frac{a_{r-2}}{(m+r)(m+r+4)}$$

putting $r=2,3,4,\ldots\ldots\ldots$

$$a_2=-\frac{a_0}{(m+2)(m+6)},\ a_4=\frac{-a_2}{(m+4)(m+8)}=\frac{(-1)^2a_0}{(m+2)(m+4)(m+6)(m+8)}$$

and so on

Therefore (i) becomes

$$y=a_0x^m\left[1-\frac{x^2}{(m+2)(m+6)}+\frac{x^4}{(m+2)(m+4)(m+6)(m+8)}-\frac{x^6}{(m+2)(m+6)(m+6)(m+8)(m+10)}+\ldots\ldots\right] \qquad \text{(iii)}$$

putting $m=0$, we get one independent solution as

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$$y_1 = a_0\left[1-\frac{x^2}{2.6}+\frac{x^4}{2.4.6.8}-\frac{x^6}{2.4.6.8.10}+\ldots\ldots\ldots\ldots\right] \qquad \text{(iv)}$$

Next putting m=–4 in (iii) the denominator of each term from third onwards vanishes, To overcome this problem we take

$a_0 = b_0\,(m + 4)$ in (iii) and taking limit as $m \to -4$, we get

$$y = \lim_{m\to -4} b_0\,(m+4)x^m\left\{1-\frac{x^2}{(m+2)(m+6)}+\frac{x^4}{(m+2)(m+4)(m+6)(m+8)}-\ldots\ldots\ldots\ldots\right\}$$

$$= b_0x^{-4}\left[0-0+\frac{x^4}{(-2)(2)(4)}-\frac{x^6}{(-2)(2)(4)(6)}+\ldots\ldots\ldots\right]$$

$$= \frac{b_0}{(-2)(2)(4)}\left[1-\frac{x^2}{2.6}+\ldots\ldots\ldots\right]$$

which is nothing but a multiple of y_1 hence not an independent solution.

Here the second independent solution of the given differential equation is taken as

$$y_2 = \left[\frac{\partial}{\partial m}\left\{b_0\,(m+4)\,y_m\right\}_{m=-4}\right]$$

Now

$$y = b_0\,(m+4)\,x^m - \frac{b_0\,(m+4)}{(m+2)(m+6)}x^{m+2}+\frac{b_0\,(m+4)}{(m+2)(m+4)(m+6)(m+8)}x^{m+4}-\ldots\ldots$$

$$y = b_0x^m\left[(m+4)-\frac{(m+4)}{(m+2)(m+6)}x^2+\frac{1}{(m+2)(m+6)(m+8)}x^4-\ldots\ldots\ldots\ldots\right]$$

Differentiating partially the above equation w.r.t. m, we get

$$\frac{\partial y}{\partial m} = y\log x + b_0x^m\left[1+\frac{m^2+8m+20}{\left(m^2+8m+12\right)^2}-\frac{\left(3m^2+32m+76\right)}{\left(m^3+16m^2+75m+96\right)^2}x^4+\ldots\ldots\ldots\ldots\right]$$

$$\therefore y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=-4} = (y)_{m=-4}\log x + b_0x^{-4}\left(1+\frac{x^2}{4}-\frac{x^4}{4}+\ldots\ldots\ldots\right)$$

$$= b_0x^{-4}\log x\left[-\frac{x^4}{16}-\frac{x^6}{16}+\ldots\ldots\ldots\ldots\right]+b_0x^{-4}\left(1+\frac{x^2}{4}-\frac{x^4}{4}+\ldots\ldots\right)$$

Hence the complete solution is given by

$y = A(y_1) + B(y_2)$

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For $m = -\frac{1}{2}$

$$a_1 = \frac{a_0}{-\frac{1}{2}(-1+3)} = \frac{a_0}{-\frac{1}{2}2} = -a_0$$

$$a_2 = -\frac{a_0}{2},\ a_3 = \frac{a_2}{9} = -\frac{a_0}{18}, \ldots\ldots\ldots\ldots \text{ (from (i))}$$

$$\text{or } y_2 = a_0 x^{-\frac{1}{2}} + (-a_0)x^{-\frac{1}{2}+1} - \frac{a_0}{2}x^{-\frac{1}{2}+2} - \frac{a_0}{18}x^{-\frac{1}{2}+3} \ldots\ldots\ldots\ldots$$

$$\text{or } y_2 = a_0 x^{-\frac{1}{2}}\left[1 - x - \frac{x^2}{2} - \frac{x^3}{18} \ldots\ldots\ldots\ldots\right]$$

Hence the required solution is

$$y_1 = Aa_0 x\left(1 + \frac{x}{5} + \frac{x^2}{70} + \frac{x^3}{1890} + \ldots\ldots\ldots\ldots\right) + Ba_0 x^{-\frac{1}{2}}\left(1 - x - \frac{x^2}{2} - \frac{x^3}{18} - \ldots\ldots\ldots\right)$$

$$\text{or } y_1 = c_1 x\left(1 + \frac{x}{5} + \frac{x^2}{70} + \frac{x^3}{1890} + \ldots\ldots\ldots\ldots\right) + c_2 x^{-\frac{1}{2}}\left(1 - x - \frac{x^2}{2} - \frac{x^3}{18} - \ldots\ldots\ldots\right)$$

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Lecture 16

Case III

EXAMPLE Find the series solution of the equation

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+x^2y=0$$

Solution we have

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}+x^2y=0 \qquad \text{(i)}$$

Here x P(x) and x^2 Q(x) exist at the point x = 0. So x =0 is a regular singular point.

$$y=\sum_{r=0}^{\infty}a_r r^{m+r} \qquad \text{(i)}$$

Let $y=\sum_{r=0}^{\infty}a_r x^{m+r}$

$$\therefore \frac{dy}{dx}=\sum_{r=0}^{\infty}a_r(m+r)x^{m+r-1}$$

$$\frac{d^2y}{dx^2}=\sum_{r=0}^{\infty}a_r(m+r)(m+r-1)x^{m+r-2}$$

Substituting these values in equation (i) we get

$$x\sum_{r=0}^{\infty}a_r(m+r)(m+r-1)x^{m+r-2}+\sum_{r=0}^{\infty}a_r(m+r)x^{m+r-1}+x^2\sum_{r=0}^{\infty}a_r x^{m+r}=0$$

$$\Rightarrow \sum_{r=0}^{\infty}a_r(m+r)^2x^{m+r-1}+\sum_{r=0}^{\infty}a_r x^{m+r-2}=0$$

Equating to zero the coefficient of lowest power of x i.e. x^{m-1}, we get the identical equation as

$a_0 m^2=0 \quad \Rightarrow \quad m=0, 0$ as $a_0 \neq 0$

Equating the coefficient of x^m, we get

$a_1(m+1)^2=0 \quad \Rightarrow \quad a_1=0,$ as $m \neq -1$

equating the coefficient of x^{m+1}, we get

$a_2(m+2)^2=0 \quad \Rightarrow \quad a_2=0,$ as $m \neq -2$

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Equating the coefficient of x^{m+2}

$$a_3 (m+3)^2 + a_0 = 0 \quad \Rightarrow \quad a_3 = \frac{-a_0}{(m+3)^2}$$

Next equating the coefficient of x^{m+r+2}, we get $a_{r+3}(m+r+3)^2 + a_r = 0$

$$a_{r+3} = -\frac{a_r}{(m+r+3)^2}$$

putting $r = 1,2,3$....., we get

$$a_4 = -\frac{a_1}{(m+4)^2} \Rightarrow a_4 = 0, a_5 = -\frac{a_2}{(m+5)^2} \Rightarrow a_5 = 0$$

$$a_6 = \frac{-a_3}{(m+6)^2} = \frac{a_0}{(m+3)^2(m+6)^2}$$

$$a_7 = 0, a_8 = 0,\ a_9 = \frac{-a_6}{(m+9)^2} \Rightarrow a_9 = -\frac{a_0}{(m+3)^2(m+6)^2(m+9)^2}$$

and so on.

From equation (ii), we have

$$y_1 = (y)_{m=0} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots\ldots\ldots\ldots$$

$$\Rightarrow y_1 = (y)_{m=0} = a_0 - \frac{a_0}{(3)^2}x^3 + 0 + 0 + \frac{a_0}{(3)^2(6)^2}x^6 + 0 + 0 - \frac{a_0}{(3)^2(6)^2(9)^2}x^9 + \ldots\ldots\ldots$$

$$\Rightarrow y_1 = a_0\left[1 - \frac{x^3}{3^2} + \frac{x^6}{3^2.6^2} - \frac{x^9}{3^2.6^2.9^2} + \ldots\ldots\ldots\right] \qquad \text{(iii)}$$

Again

$$y = a_0 x^m - \frac{a_0}{(m+3)^2}x^{m+3} + \frac{a_0}{(m+3)^2(m+6)^2}x^{m+6} - \ldots\ldots\ldots\ldots \qquad \text{(From (ii))}$$

or $$y = a_0 x^m\left[1 - \frac{x^3}{(m+3)^2} + \frac{x^6}{(m+3)^2(m+6)^2} - \ldots\ldots\ldots\ldots\right]$$

$$\frac{\partial y}{\partial m} = a_0 x^m \log x\left[1 - \frac{x^3}{(m+3)^2} + \frac{x^6}{(m+3)^2(m+6)^2} - \ldots\ldots\ldots\ldots\right]$$

$$+ a_0 x^m\left[0 + \frac{2x^3}{(m+3)^3} - \frac{4m^3 + 54m^2 + 234m + 324}{(m+3)^4(m+6)^4}x^6 + \ldots\ldots\ldots\ldots\right]$$

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$$y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=0} = \log x (y)_{m=0} + a_0\left[\frac{2}{3^3}x^3 - \frac{324}{3^4.6^4}x^6 + \ldots\ldots\ldots\ldots\right]$$

$$y_2 = a_0 \log x\left[1 - \frac{x^3}{3^2} + \frac{x^6}{3^2.6^2} - \frac{x^9}{3^2.6^2.9^2} + \ldots\ldots\ldots\ldots\right] + a_0\left[\frac{2}{3^3}x^3 - \frac{324}{3^4.6^4}x^6 + \ldots\ldots\ldots\ldots\right]$$

Hence the complete solution is

$$y = Ay_1 + B\left(\frac{\partial y}{\partial m}\right)_{m=0}$$

$$\text{or } y = (Aa_0 + Ba_0 \log x)\left(1 - \frac{x^3}{9} + \frac{x^6}{108} - \ldots\ldots\ldots\right) + \frac{2}{3^3}Ba_0x^3\left(1 - \frac{x^3}{24} + \ldots\ldots\ldots\right)$$

$$\text{or } y = (c_1 + c_2 \log x)\left(1 - \frac{x^3}{9} + \frac{x^6}{108} - \ldots\ldots\ldots\right) + \frac{2c_2}{9}x^3\left(1 - \frac{x^3}{24} + \ldots\ldots\ldots\right)$$

where $c = Aa_0$, $c_2 = B\,a_0$

For $m = -\frac{1}{2}$

$$a_1 = \frac{a_0}{-\frac{1}{2}(-1+3)} = \frac{a_0}{-\frac{1}{2}2} = -a_0$$

$$a_2 = -\frac{a_0}{2},\ a_3 = \frac{a_2}{9} = -\frac{a_0}{18}, \ldots\ldots\ldots \text{ (from (i))}$$

$$\text{or } y_2 = a_0x^{-\frac{1}{2}} + (-a_0)x^{-\frac{1}{2}+1} - \frac{a_0}{2}x^{-\frac{1}{2}+2} - \frac{a_0}{18}x^{-\frac{1}{2}+3} \ldots\ldots\ldots\ldots$$

$$\text{or } y_2 = a_0x^{-\frac{1}{2}}\left[1 - x - \frac{x^2}{2} - \frac{x^3}{18}\ldots\ldots\ldots\ldots\right]$$

Hence the required solution is

$$y_1 = Aa_0x\left(1 + \frac{x}{5} + \frac{x^2}{70} + \frac{x^3}{1890} + \ldots\ldots\ldots\ldots\right) + Ba_0x^{-\frac{1}{2}}\left(1 - x - \frac{x^2}{2} - \frac{x^3}{18} - \ldots\ldots\ldots\right)$$

$$\text{or } y_1 = c_1x\left(1 + \frac{x}{5} + \frac{x^2}{70} + \frac{x^3}{1890} + \ldots\ldots\ldots\ldots\right) + c_2x^{-\frac{1}{2}}\left(1 - x - \frac{x^2}{2} - \frac{x^3}{18} - \ldots\ldots\ldots\right)$$

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Lecture 1

Improper Integrals

we defined definite integral of a function f for the case when f is a *bounded function* defined on a *closed interval* $[a, b]$. can we still have a notion of integral even when the above assumptions on f and the domain of integration are not satisfied? We consider a notion of integral, called *improper integral*, in a few cases.

Integrals over infinite intervals

First we consider integrals of functions defined over infinite integrals of the form $[a, \infty)$, $(-\infty, b]$ and $(-\infty, \infty)$. Recall that Rieman integral was defined over intervals of the form $[a, b]$.

Definition (i) Suppose f is defined on $[a, \infty)$ and integrable on $[a, t]$ for all $t > a$. If $\lim_{t\to\infty} \int_a^t f(x)\, dx$ exists, then we define the **improper integral of f over** $[a, \infty)$ as

$$\int_a^\infty f(x)\, dx := \lim_{t\to\infty} \int_a^t f(x)\, dx.$$

(ii) Suppose f is defined on $(-\infty, b]$ and integrable on $[t, b]$ for all $t < b$. If $\lim_{t\to-\infty} \int_t^b f(x)\, dx$ exists, then we define the **improper integral of f over** $(-\infty, b]$ as

$$\int_{-\infty}^b f(x)\, dx = \lim_{t\to-\infty} \int_t^b f(x)\, dx.$$

(iii) Suppose f is defined on $\mathbb{R} := (-\infty, \infty)$ and integrable on $[a, b]$ for every closed and bounded interval $[a, b] \subseteq \mathbb{R}$. If $\int_{-\infty}^c f(x)\, dx$ and $\int_c^\infty f(x)\, dx$ exist for some $c \in \mathbb{R}$, then we define the **improper integral of f over** $(-\infty, \infty)$ as

$$\int_{-\infty}^\infty f(x)\, dx = \int_{-\infty}^c f(x)\, dx + \int_c^\infty f(x)\, dx.$$

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In Definition the following results is used without mentioning:

- Suppose f is defined on $(-\infty, \infty)$ and integrable on $[a, b]$ for every closed and bounded interval $[a, b] \subseteq \mathbb{R}$. If the integrals $\int_{-\infty}^{c} f(x)\, dx$ and $\int_{c}^{\infty} f(x)\, dx$ exist for some $c \in \mathbb{R}$, then they exist for every $c \in \mathbb{R}$, and

$$\int_{-\infty}^{c} f(x)\, dx + \int_{c}^{\infty} f(x)\, dx = \int_{-\infty}^{a} f(x)\, dx + \int_{a}^{\infty} f(x)\, dx.$$

  for every $a \in \mathbb{R}$.

Remark We may observe that the existence of $\lim_{t\to\infty} \int_{-t}^{t} f(x)\, dx$ does not, in general, imply the existence of $\int_{-\infty}^{\infty} f(x)\, dx$. To see this, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x, \quad x \in \mathbb{R}.$$

Then we have $\int_{-t}^{t} f(x)\, dx = 0$ for every $t \in \mathbb{R}$, but the integrals $\int_{-\infty}^{c} f(x)\, dx$ and $\int_{c}^{\infty} f(x)\, dx$ do not exist for any $c \in \mathbb{R}$.

Next we consider integrals of functions defined over infinite integrals of the form (a, ∞) and $(-\infty, b)$.

Definition (i) Suppose f is defined on (a, ∞) and $\int_{t}^{\infty} f(x)dx$ exists for all $t > a$. If $\lim\limits_{t\to a} \displaystyle\int_{t}^{\infty} f(x)\, dx$ exists, then we define the **improper integral of f over** (a, ∞) as

$$\int_{a}^{\infty} f(x)\, dx := \lim_{t\to a} \int_{t}^{\infty} f(x)\, dx.$$

(ii) Suppose f is defined on $(-\infty, b)$ and $\int_{-\infty}^{t} f(x)dx$ exists for all $t < b$. If $\lim\limits_{t\to b} \displaystyle\int_{-\infty}^{t} f(x)\, dx$ exists, then we define the **improper integral of f over** $(-\infty, b)$ as

$$\int_{a}^{\infty} f(x)\, dx := \lim_{t\to b} \int_{-\infty}^{t} f(x)\, dx.$$

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Lecture 2

Improper integrals over finite intervals

Now we consider the case when f is defined on a interval J of finite length, but either the function is not defined at any one of the end points or the function is not bounded on J.

Definition (i) Suppose f is defined on $(a, b]$. If $\int_t^v f(x)\,dx$ exists for every $t \in (a, b)$, and if $\lim_{t\to a} \int_t^b f(x)\,dx$ exists, then we define the **improper integral of f over** $(a, b]$ as

$$\int_a^b f(x)\,dx = \lim_{t\to a} \int_t^b f(x)\,dx.$$

(ii) Suppose f is defined on $[a, b)$. If $\int_a^t f(x)\,dx$ exists for every $t \in (a, b)$, and if $\lim_{t\to b} \int_a^t f(x)\,dx$ exists, then we define the **improper integral of f over** $[a, b)$ as

$$\int_a^b f(x)\,dx = \lim_{t\to b} \int_a^t f(x)\,dx.$$

(iii) Suppose f is defined on $[a, c)$ and $(c, b]$. If $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ exist, then we define the **improper integral of f over** $[a, b]$ as

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Definition If improper integral of a function f over an interval J (of finite or infinite length) exists, then we say that the the **improper integral exists** or **improper integral converges**; otherwise we say that the **improper integral does not exist** or **improper integral diverges**.

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Lecture 3

Examples of Improper Integrals

Example 1 Consider the improper integral $\int_1^\infty \frac{1}{x}\,dx$. Note that

$$\int_1^t \frac{1}{x}\,dx = [\ln x]_1^t = \ln t \to \infty \quad \text{as} \quad t \to \infty.$$

Hence, $\int_1^\infty \frac{1}{x}\,dx$ diverges.

Example 2 Consider the improper integral $\int_1^\infty \frac{1}{x^2}\,dx$ Note that

$$\int_1^t \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^t = 1 - \frac{1}{t} \to 1 \quad \text{as} \quad t \to \infty.$$

Hence, $\int_1^\infty \frac{1}{x^2}\,dx$ converges.

Example 3 For $p \neq 1$, consider the improper integral $\int_1^\infty \frac{1}{x^p}\,dx$. In this case, we have

$$\int_1^t \frac{1}{x^p}\,dx = \left[\frac{x^{-p+1}}{-p+1}\right]_1^t = \frac{t^{-p+1}-1}{-p+1}.$$

Note that,

$$p > 1 \quad \Longrightarrow \quad \frac{t^{-p+1}-1}{-p+1} \to \frac{1}{p-1} \quad \text{as} \quad t \to \infty,$$

and

$$p < 1 \quad \Longrightarrow \quad \frac{t^{-p+1}-1}{-p+1} \to \infty \quad \text{as} \quad t \to \infty,$$

The above observations combined with Example 1 show that

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \begin{cases} \text{converges for } p > 1, \\ \text{diverges for } p \leq 1. \end{cases}$$

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Example 4 (i) We consider the improper integral $\int_0^1 \frac{1}{x}\, dx$: Note that for $0 < \delta < 1$,

$$\int_\delta^1 \frac{1}{x}\, dx = [\log x]_\delta^1 = \log 1 - \log \delta = -\log \delta = \log\left(\frac{1}{\delta}\right) \to \infty \quad \text{as} \quad \delta \to 0.$$

Thus, the integral diverges.

(ii) For $p \neq 1$, consider the improper integral $\int_0^1 \frac{1}{x^p}\, dx$. In this case, we have

$$\int_\delta^1 \frac{1}{x^p}\, dx = \left[\frac{x^{-p+1}}{-p+1}\right]_\delta^1 = \frac{1-\delta^{-p+1}}{-p+1}.$$

Note that,

$$p > 1 \quad \Longrightarrow \quad \frac{\delta^{-p+1}-1}{-p+1} \to \infty \quad \text{as} \quad \delta \to 0,$$

and

$$p < 1 \quad \Longrightarrow \quad \frac{\delta^{-p+1}-1}{-p+1} \to \frac{1}{1-p} \quad \text{as} \quad \delta \to 0.$$

The above observations combined with (i) above give

$$\int_0^1 \frac{1}{x^p}\, dx \quad \begin{cases} \text{converges for } p < 1, \\ \text{diverges for } p \geq 1. \end{cases}$$

Example 5 Let $a < b$ and $\alpha < 1$. Then $\int_a^b \frac{dx}{(b-x)^\alpha}$ converges:

We observe that for $a < t < b$,

$$\int_a^t \frac{dx}{(b-x)^\alpha} = \int_{b-t}^{b-a} \frac{du}{u^\alpha}.$$

Now,

$$\begin{aligned} \lim_{t\to b} \int_a^t \frac{dx}{(b-x)^\alpha} \text{ exists} \quad &\Longleftrightarrow \quad \lim_{t\to b} \int_{b-t}^{b-a} \frac{du}{u^\alpha} \text{ exists} \\ &\Longleftrightarrow \quad \lim_{\varepsilon\to 0} \int_\varepsilon^{b-a} \frac{du}{u^\alpha} \text{ exists} \\ &\Longleftrightarrow \quad \alpha < 1. \end{aligned}$$

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Lecture 4

Integrability by Comparison

We state a result which will be useful in asserting the existence of certain improper integral by comparing it with certain other improper integral.

Suppose J is either an interval of finite or infinite length. Suppose f is defined on J, except possibly at a finite number of point in J. We denote the improper integral of f over J by

$$\int_J f(x)\, dx.$$

We say that the improper integral $\int_J f(x)\, dx$ converges whenever it exists, and diverges if it does not exist.

For example, if $J = [a, b]$, then f may not be defined at a or at b or at some point $c \in (a, b)$, and the corresponding improper integrals, by definition, are

$$\lim_{t\to a}\int_t^b f(x)\, dx, \quad \lim_{t\to b}\int_a^t f(x)\, dx, \quad \lim_{t\to c^-}\int_a^t f(x)\, dx + \lim_{t\to c^+}\int_t^b f(x)\, dx$$

respectively.

Theorem 1 *Suppose f and g are defined on J.*

(i) If $0 \le f(x) \le g(x)$ for all $x \in J$, and $\int_J g(x)\, dx$ exists, then $\int_J f(x)\, dx$ exists.

(ii) If $\int_J |f(x)|\, dx$ exists, then $\int_J f(x)\, dx$ exists.

Example Since

$$\left|\frac{\sin x}{x^p}\right| \le \frac{1}{x^p}, \qquad \left|\frac{\cos x}{x^p}\right| \le \frac{1}{x^p}$$

it follows from Example 3 and Theorem 1(ii) that the improper integrals

$$\int_1^\infty \frac{\sin x}{x^p}\, dx \quad \text{and} \quad \int_1^\infty \frac{\cos x}{x^p}\, dx$$

converge for all $p > 1$.

In fact $\int_1^\infty \frac{\sin x}{x^p}\, dx$ and $\int_1^\infty \frac{\cos x}{x^p}\, dx$ converge for all $p > 0$ as we see in the next example.

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Example 7 Let $p > 0$. Then for $t > 0$,

$$\begin{aligned}\int_1^t \frac{\sin x}{x^p}\,dx &= \left[\frac{1}{x^p}(-\cos x)\right]_1^t - p\int_1^t \frac{1}{x^{p+1}}\cos x\,dx \\ &= \left[\cos 1 - \frac{\cos t}{t^p}\right] - p\int_1^t \frac{\cos x}{x^{p+1}}\,dx.\end{aligned}$$

By the result in Example 4.6, $\int_1^\infty \frac{\cos x}{x^{p+1}}\,dx$ converges for all $p > 0$. Also, $\frac{\cos t}{t^p} \to 0$ as $t \to \infty$. Hence,

$$\int_1^\infty \frac{\sin x}{x^p}\,dx \quad \text{converges for all} \quad p > 0.$$

Similarly, we see that

$$\int_1^\infty \frac{\cos x}{x^p}\,dx \quad \text{converges for all} \quad p > 0.$$

Example 8 Since

$$\left|\frac{\sin x}{x^p}\right| = \left|\frac{\sin x}{x}\right|\frac{1}{x^{p-1}} \le \frac{1}{x^{p-1}}, \qquad \left|\frac{\cos x}{x^p}\right| \le \frac{1}{x^p}$$

it follows from Example 4.4 above and Theorem 4.1(ii)that

$$\int_0^1 \frac{\sin x}{x^p}\,dx \quad \text{converges for all} \quad p < 2,$$

$$\int_0^1 \frac{\cos x}{x^p}\,dx \quad \text{converges for all} \quad p < 1.$$

Example 9 Observe that

$$\frac{\sin x}{x^p} = \frac{\sin x}{x}\frac{1}{x^{p-1}} \ge \frac{\sin 1}{x^{p-1}} \quad \forall x \in (0, 1].$$

Since $\int_0^1 \frac{1}{x^{p-1}}\,dx$ diverges for $p - 1 \ge 1$, i.e., for $p \ge 2$, it follows that

$$\int_0^1 \frac{\sin x}{x^p}\,dx \quad \text{diverges for all} \quad p \ge 2,$$

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Lecture 5

Integrability Using Limits

Now some more results which facilitate the assertion of convergence/divergence of improper integrals, whose proofs follow from the definition of limits.

Theorem 2 *Suppose $f(x) \geq 0$, $g(x) \geq 0$ for all $x \in [a, \infty)$, $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ exists for every $b > a$. Suppose further that $\dfrac{f(x)}{g(x)} \to \ell$ as $x \to \infty$ for some $\ell \geq 0$.*

(i) *If $\ell \neq 0$, then $\int_a^\infty f(x)dx$ converges $\iff$ $\int_a^\infty g(x)dx$ converges.*

(ii) *If $\ell = 0$, then $\int_a^\infty g(x)dx$ converges $\Longrightarrow \int_a^\infty f(x)dx$ converges.*

Further, if $\dfrac{f(x)}{g(x)} \to \infty$ as $x \to \infty$, then

$$\int_a^\infty g(x)dx \text{ converges } \Longrightarrow \int_a^\infty f(x)dx \text{ converges.}$$

Proof. Suppose further that $\dfrac{f(x)}{g(x)} \to \ell$ as $x \to \infty$ for some $\ell \geq 0$.

(i) Suppose $\ell \neq 0$. Then $\ell > 0$, and for $\varepsilon > 0$ with $\ell - \varepsilon > 0$, there exists $x_0 \geq a$ such that

$$\ell - \varepsilon < \frac{f(x)}{g(x)} < \ell + \varepsilon \quad \forall x \geq x_0.$$

Hence

$$(\ell - \varepsilon)g(x) < f(x) < (\ell + \varepsilon)g(x) \quad \forall x \geq x_0.$$

Consequently, $\int_{x_0}^\infty f(x)dx$ converges iff $\int_{x_0}^\infty g(x)dx$ converges. As $\int_a^{x_0} f(x)dx$ and $\int_a^{x_0} g(x)dx$ exist, the result in (i) follows.

(ii) Suppose $\ell = 0$. Then for $\varepsilon > 0$, there exists $x_0 \geq a$ such that

$$\frac{f(x)}{g(x)} < \varepsilon \quad \forall x \geq x_0.$$

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Thus, $f(x) < \varepsilon g(x)$ for all $x \geq x_0$. Hence, convergence of $\int_{x_0}^{\infty} g(x)dx$ implies the convergence of $\int_{x_0}^{\infty} f(x)dx$. From this the result in (ii) follows.

Next, suppose further that $\dfrac{f(x)}{g(x)} \to \infty$ as $x \to \infty$. Then for $M > 0$, there exists $x_0 \geq a$ such that

$$0 \leq \frac{f(x)}{g(x)} \leq M \quad \forall\, x \geq x_0.$$

Hence

$$0 \leq f(x) \leq Mg(x) \quad \forall\, x \geq x_0.$$

Consequently, $\int_{x_0}^{\infty} g(x)dx$ converges implies $\int_{x_0}^{\infty} f(x)dx$ converges. As $\int_a^{x_0} f(x)dx$ and $\int_a^{x_0} g(x)dx$ exist, the proof is over.

Exercise Suppose f and g are non-negative continuous functions on J. Then

$$\int_a^b f(x)dx \text{ exists } \iff \int_a^b g(x)dx \text{ exists}$$

in the following cases:

1. $J = (a, b]$ and $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \ell$ and $\ell > 0$.
2. $J = [a, b)$ and $\lim\limits_{x \to b} \dfrac{f(x)}{g(x)} = \ell$ and $\ell > 0$.
3. $J = [a, \infty)$ and $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = \ell$ and $\ell > 0$.
4. $J = (-\infty, b]$ and $\lim\limits_{x \to -\infty} \dfrac{f(x)}{g(x)} = \ell$ and $\ell > 0$.

In 1-4 above, if $\ell = 0$, then $\displaystyle\int_a^b g(x)dx$ exists $\implies \displaystyle\int_a^b g(x)dx$ exists .

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Lecture 6

Gamma and Beta Functions

Gamma and Beta Functions are certain improper integrals which appear in many applications.

Gamma function

We show that for $x > 0$, the improper integral

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\, dt$$

converges. The function $\Gamma(x)$, $x > 0$, is called the **gamma function.**

Note that for $t^{x-1}e^{-t} \leq t^{x-1}$ for all $t > 0$, and $\int_0^1 t^{x-1}\, dt$ converges for $x > 0$. Hence, by Theorem 1,

$$\int_0^1 t^{x-1}e^{-t}\, dt \quad \text{converges for} \quad x > 0.$$

Also, we observe that $\dfrac{t^{x-1}e^{-t}}{t^{-2}} \to 0$ as $t \to \infty$, and $\int_1^\infty t^{-2}dt$ converges. Hence, by Theorem 2, $\int_1^\infty t^{x-1}e^{-t}\, dt$ converges. Thus,

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\, dt = \int_0^1 t^{x-1}e^{-t}\, dt + \int_1^\infty t^{x-1}e^{-t}\, dt$$

converges for every $x > 0$.

Beta function

We show that for $x > 0$, $y > 0$, the improper integral

$$\beta(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$$

converges. The function $\beta(x, y)$ for $x > 0, y > 0$ is called the **beta function.**

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Clearly, the above integral is *proper* for $x \geq 1$, $y \geq 1$. Hence it is enough to consider the case of $0 < x < 1$, $0 < y < 1$. In this case both the points $t = 0$ and $t = 1$ are problematic. hence, we consider the integrals

$$\int_0^{1/2} t^{x-1}(1-t)^{y-1}\, dt, \qquad \int_{1/2}^{1} t^{x-1}(1-t)^{y-1}\, dt.$$

We note that if $0 < t \leq 1/2$, then $(1-t)^{y-1} \leq 2^{1-y}$ so that $t^{x-1}(1-t)^{y-1} \leq 2^{1-y}t^{x-1}$. Since $\int_0^{1/2} t^{x-1}\, dt$ converges it follows that $\int_0^{1/2} t^{x-1}(1-t)^{y-1}\, dt$ converges. To deal with the second integral, consider the change of variable $u = 1 - t$. Then

$$\int_{1/2}^{1} t^{x-1}(1-t)^{y-1}\, dt = \int_0^{1/2} u^{y-1}(1-u)^{x-1}\, du$$

which converges by the above argument. Hence,

$$\beta(x,y) := \int_0^1 t^{x-1}(1-t)^{1-y}\, dt, \qquad x > 0,\ y > 0$$

converges for every $x > 0$, $y > 0$.

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Lecture 7

BETA AND GAMMA FUNCTIONS

The first and second Eulerian Integrals which are also called "Beta and Gamma functions" respectively are defined as follows:

$$\boxed{\beta\,(m,\,n) \;=\; \int_0^1 x^{m-1}(1-x)^{n-1}\;dx}$$

and

$$\boxed{\Gamma\,n \;=\; \int_0^\infty e^{-x}x^{n-1}dx}$$

$\beta\,(m,\,n)$ is read as "Beta $m,\,n$" and $\Gamma\,n$ is read as "Gamma n". Here the quantities m and n are positive numbers which may or may not be integrals.

Properties of Beta and Gamma Functions

(a) The function $\beta\,(m,\,n)$ is symmetrical w.r.t. $m,\,n$ *i.e.,*

$$\beta\,(m,\,n) \;=\; \beta\,(n,\,m)$$

we have $\quad \beta\,(m,\,n) \;=\; \int_0^1 x^{m-1}(1-x)^{n-1}\;dx$

putting $\quad 1-x \;=\; t \Rightarrow dx = -\,dt$

$$= -\int_1^0 (1-t)^{m-1}\cdot t^{n-1}\,dt = \int_0^1 (1-t)^{m-1}\cdot t^{n-1}\,dt$$

$$= \int_0^1 (1-x)^{m-1}\cdot x^{n-1}\cdot dx = \int_0^1 x^{n-1}(1-x)^{m-1} \quad \left(\text{By } \int_a^b f(t)\,dt = \int_a^b f(x)\,dx\right)$$

$$\Rightarrow \quad \boxed{\beta\,(m,\,n) \;=\; \beta\,(n,\,m)}.$$

(*b*) Evaluation of Beta Function $\beta\,(m,\,n)$

$$\beta\,(m,\,n) \;=\; \frac{\lfloor m-1 \;\lfloor n-1}{\lfloor (m+n-1)} \;=\; \frac{\Gamma\,m\;\Gamma\,n}{\Gamma\,m+n}$$

We have $\quad \beta\,(m,\,n) \;=\; \int_0^1 x^{m-1}\,(1-x)^{n-1}.\,dx$

$$= \left[\frac{x^m}{m}(1-x)^{n-1}\right]_0^1 + \frac{(n-1)}{m}\int_0^1 x^m\,(1-x)^{n-2}\cdot dx \quad \text{(Integrate by part)}$$

$\beta\,(m,\,n) \;=\; \dfrac{(n-1)}{m}\displaystyle\int_0^1 x^m\,(1-x)^{n-2}.\,dx$ Again integrate by parts above,

we get $\quad \beta\,(m,\,n) \;=\; \dfrac{(n-1)}{m}\cdot\dfrac{(n-2)}{m+1}\displaystyle\int_0^1 x^{m+1}\,(1-x)^{n-3}.\,dx$

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Continuing the above process of integrating by parts

$$\beta(m, n) = \frac{(n-1)(n-2)\ldots 2\cdot 1}{m(m+1)\ldots(m+n-2)} \int_0^1 x^{m+n-2} \cdot dx$$

or

$$\boldsymbol{\beta(m, n) = \frac{\lfloor n-1}{m(m+1)\ldots(m+n-2)(m+n-1)}} \qquad \ldots(i)$$

n is a positive integer

In case n is alone a positive integer.

Similarly, if m is positive integer, then $\beta(m, n) = \beta(n, m)$

$$\boldsymbol{\beta(n, m) = \frac{\lfloor m-1}{n(n+1)\ldots(n+m-1)}} \qquad \ldots(ii)$$

In case both m and n are positive integer, then multiplying (i) numerator and denominator by 1·2·3... (m – 1) or (ii) by 1·2·3... (n – 1), then, we get

$$\beta(m, n) = \frac{1\cdot 2\cdot 3\ldots(m-1)\lfloor n-1}{1\cdot 2\cdot 3\ldots(m-1)\cdot m(m+1)\ldots(m+n-1)}$$

or

$$\boxed{\beta(m, n) = \frac{\lfloor m-1 \lfloor n-1}{\lfloor m+n-1} = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}}\,.$$

(c) Evaluation of Gamma Function

$$\Gamma n = (n-1)\,\Gamma n-1 \text{ or } \Gamma n+1 = n\,\Gamma n = \lfloor n$$

we know that $$\Gamma n = \int_0^\infty x^{n-1} . e^{-x} \cdot dx$$

Integrating by parts keeping x^{n-1} as first function

$$\Gamma n = \left[-e^{-x} \cdot x^{n-1}\right]_0^\infty + (n-1) \int_0^\infty x^{n-2} \cdot e^{-x}\, dx$$

$$= \left[-\lim_{x\to\infty} e^{-x} \cdot x^{n-1} + 0\right] + (n-1) \int_0^\infty x^{n-2} . e^{-x}\, dx \qquad \ldots(i)$$

$$\left| \text{As } \lim_{x\to 0} e^{-x} . x^{n-1} = 0 \right.$$

$$\lim_{x\to\infty} e^{-x} \cdot x^{n-1} = \lim_{x\to\infty} \cdot \frac{x^{n-1}}{e^x} = \lim_{x\to\infty} \frac{x^{n-1}}{1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+\ldots+\frac{x^n}{\lfloor n}+\ldots}$$

$$= \lim_{x\to\infty} \frac{1}{\frac{1}{x^{n-1}}+\frac{1}{x^{n-2}}+\frac{1}{\lfloor 2\cdot x^{n-3}}\ldots} = \frac{1}{\infty} = 0$$

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$\therefore$ From (*i*), we get

$$\Gamma n = 0 + (n-1)\int_0^\infty x^{n-2}\cdot e^{-x}\,dx = (n-1)\int_0^\infty x^{(n-1)-1}\cdot e^{-x}\,dx$$

$$\Rightarrow \quad \boxed{\Gamma n = (n-1)\,\Gamma n-1}. \qquad ...(ii)$$

Replace n by $n+1$ in equation (*ii*) then, we get

$$\boxed{\Gamma n+1 = n\,\Gamma n}. \qquad ...(iii)$$

(d) $\Gamma n+1 = \lfloor n$

from (*i*) $\Gamma n = (n-1)(n-2)\ ....\ 3.\ 2.\ 1$

$\Gamma n = \lfloor n-1$

replace n by $(n+1)$, then

$$\boxed{\Gamma n+1 = \lfloor n}.$$

TRANSFORMATIONS OF GAMMA FUNCTION

We have $\Gamma n = \int_0^\infty e^{-x}x^{n-1}\,dx \qquad ...(i)$

(1) Put $x = \lambda y$ or $dx = \lambda\,dy$

Then from (*i*), we get $\Gamma n = \int_0^\infty e^{-\lambda y}(\lambda y)^{n-1}\,\lambda\,dy$

$$= \lambda^n \int_0^\infty e^{-\lambda y}y^{n-1}\,dy$$

or

$$\boxed{\int_0^\infty y^{n-1}\,e^{-\lambda y}\,dy = \frac{\Gamma n}{\lambda^n}} \qquad ...(ii)$$

(2) Put $x^n = z$ in (*i*) then $nx^{n-1}\,dx = dz$ and $x = z^{1/n}$

$\therefore$ From (*i*), we get $\Gamma n = \int_0^\infty e^{-z^{1/n}}\left(\frac{1}{n}\right)dz$

$$\boxed{\int_0^\infty e^{-z^{1/n}}\,dz = n\,\Gamma n = \Gamma(n+1)} \qquad ...(iii)$$

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Lecture 8

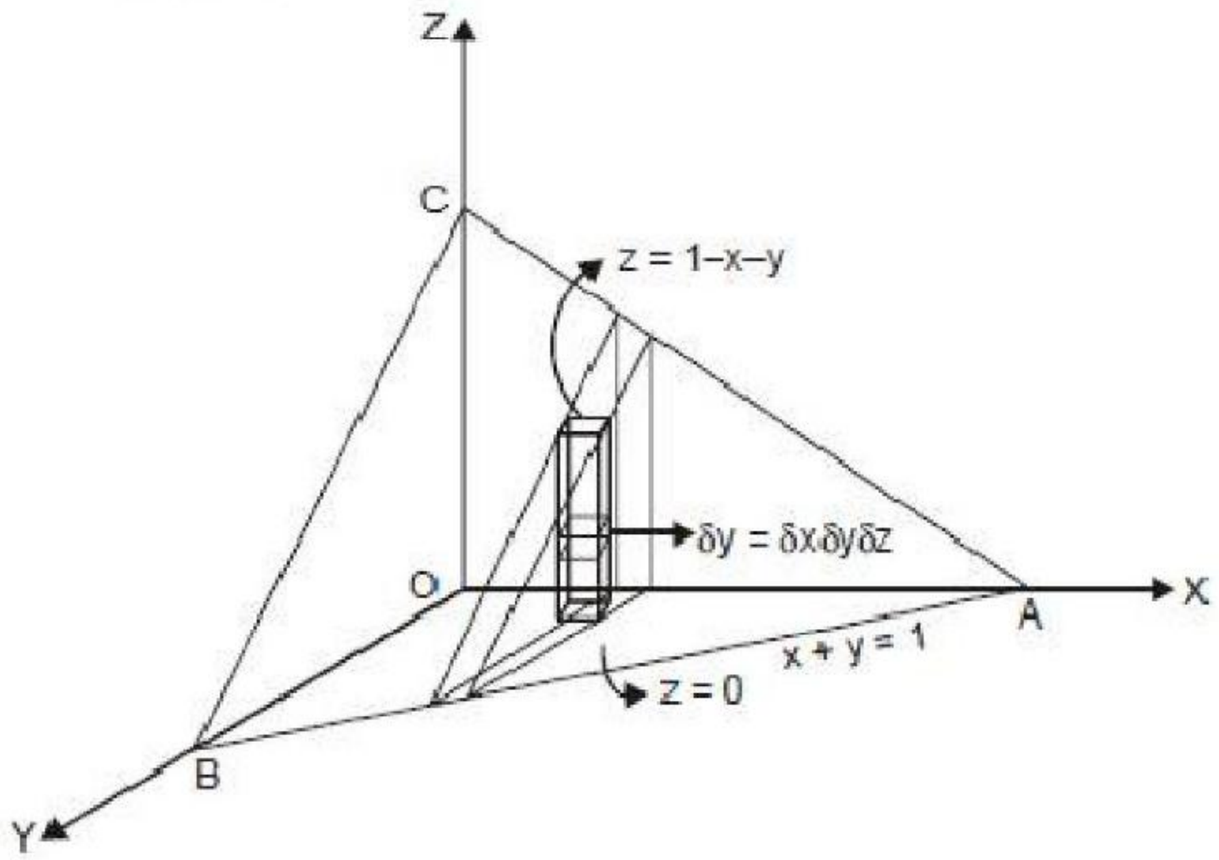
DRITCHLET'S* THEOREM

If V is a region bounded by $x \geq 0$, $y \geq 0$ and $x + y + z \leq 1$, then

$$\iiint_V x^{l-1} y^{m-1} z^{n-1} dx\, dy\, dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(m)}{\Gamma(l+m+n+1)}$$

The given triple integral may be written as

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$$



$$= \int_0^1 \int_0^{1-x} x^{l-1} y^{m-1} \left[\frac{z^n}{n} \right]_0^{1-x-y} dx\, dy$$

$$= \frac{1}{n} \int_0^1 \int_0^{1-x} x^{l-1} y^{m-1} (1-x-y)^n dx\, dy$$

Put $\quad y = (1-x)\,t$, so that $dy = (1-x)\,dt$

Then $\quad I = \dfrac{1}{n} \displaystyle\int_0^1 \int_0^1 x^{l-1} \{(1-x)\,t\}^m \cdot \{(1-x)-(1-x)\,t\}^n (1-x)\, dx\, dt$

$$= \frac{1}{n} \int_0^1 x^{l-1} (1-x)^{(m+n-1)-1} dx \int_0^1 t^{m-1} (1-t)^{(n+1)-1} dt = \frac{1}{n} \beta(l, m+n+1) \cdot \beta(m, n+1)$$

$$= \frac{1}{n} \frac{\Gamma(l) \cdot \Gamma(m+n+1)}{\Gamma(l+m+n+1)} \cdot \frac{\Gamma(m)\,\Gamma(n+1)}{\Gamma(m+n+1)} = \frac{1}{n} \frac{\Gamma(l) \cdot \Gamma(m)\,\Gamma(n+1)}{\Gamma(l+m+n+1)} = \frac{1}{n} \frac{\Gamma(l) \cdot \Gamma(m) \cdot n\,\Gamma(n)}{\Gamma(l+m+n+1)}$$

$$= \frac{\Gamma(l) \cdot \Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)} \quad \text{Hence,} \quad \boxed{\iiint_V x^{l-1} y^{m-1} z^{n-1} dx\, dy\, dz = \frac{\Gamma(l) \cdot \Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}}$$

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Example . Find the value of $\iiint x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz$ where x, y, z are always positive with $\left(\frac{x}{a}\right)^p+\left(\frac{y}{b}\right)^q+\left(\frac{z}{c}\right)^r \le 1.$

Sol. Putting $\left(\frac{x}{a}\right)^p = u,\ \left(\frac{y}{b}\right)^q = v, \left(\frac{z}{c}\right)^r = w$, we have

$$x = au^{1/p},\ y = bv^{1/q},\ z = cw^{1/r}$$

$$\therefore \qquad dx = \left(\frac{a}{p}\right)u^{\left(\frac{1}{p}-1\right)}du, dy = \left(\frac{b}{q}\right)v^{\left(\frac{1}{q}-1\right)}dv, dz = \left(\frac{c}{r}\right)w^{\left(\frac{1}{r}-1\right)}dw$$

$$\therefore \text{ The given integral} = \iiint\left(au^{\frac{1}{p}}\right)^{l-1}\left(bu^{\frac{1}{q}}\right)^{m-1}\left(cw^{\frac{1}{r}}\right)^{n-1}\left(\frac{a}{p}\right)u^{\frac{1}{p}-1}\cdot\frac{b}{q}v^{\frac{1}{q}-1}\cdot\frac{c}{r}w^{\frac{1}{r}-1}du\,dv\,dw$$

$$= \frac{a^l b^m c^n}{pqr}\iiint u^{\left(\frac{l}{p}-1\right)}v^{\frac{m}{q}-1}w^{\left(\frac{n}{r}-1\right)}du\,dv\,dw, \text{ where } u+v+w \le 1$$

$$= \frac{a^l b^m c^n}{pqr}\frac{\Gamma\left(\frac{l}{p}\right)\Gamma\left(\frac{m}{q}\right)\Gamma\left(\frac{n}{r}\right)}{\Gamma\left(\frac{l}{p}+\frac{m}{q}+\frac{n}{r}+1\right)}.$$

Example The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$ meets the axis in A, B and C. Apply Dritchlet's integral to find the volume of the tetrahedran $OABC$. Also find its mass if the density at any point is $kxyz$.

Sol. Let $\frac{x}{a} = u,\ \frac{y}{b} = v,\ \frac{z}{c} = w$, then $u \ge 0,\ v \ge 0,\ w \ge 0$ and $u + v + w \le 1$

Also, $dx = a\,du,\ dy = b\,dv,\ dz = c\,dw.$

$$\text{Volume } OABC = \iiint_D dx\,dy\,dz = \iiint_D abc\,du\,dv\,dw, \text{ where } u+v+w<1$$

$$= abc\iiint_D u^{1-1}v^{1-1}w^{1-1}du\,dr\,dw = abc\frac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1+1)} = \frac{abc}{3!} = \frac{abc}{6}$$

$$\text{Mass} = \iiint_D kxyz\,dx\,dy\,dz = \iiint_{D'} k\,(au)(bv)(cw)\,abc\,du\,dv\,dw$$

$$= ka^2b^2c^2\iiint_{D'}u^{2-1}v^{2-1}w^{2-1}du\,dv\,dw = ka^2b^2c^2\frac{\Gamma(2)\Gamma(2)\Gamma(2)}{\Gamma(2+2+2+1)} = ka^2b^2c^2\frac{1!1!1!}{6!}$$

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Lecture 1

Sequences and Their Limits

A sequence in a set S is a function whose domain is the set $\mathbb{N}$ of natural numbers, and whose range is contained in the set S. In this chapter, we will be concerned with sequences in $\mathbb{R}$ and will discuss what we mean by the convergence of these sequences.

Definition A **sequence of real numbers** (or a **sequence in** $\mathbb{R}$) is a function defined on the set $\mathbb{N} = \{1, 2, \ldots\}$ of natural numbers whose range is contained in the set $\mathbb{R}$ of real numbers.

In other words, a sequence in $\mathbb{R}$ assigns to each natural number $n = 1, 2, \ldots$ a uniquely determined real number. If $X : \mathbb{N} \to \mathbb{R}$ is a sequence, we will usually denote the value of X at n by the symbol x_n rather than using the function notation $X(n)$. The values x_n are also called the **terms** or the **elements** of the sequence. We will denote this sequence by the notations

$$X, \quad (x_n), \quad (x_n : n \in \mathbb{N}).$$

Of course, we will often use other letters, such as $Y = (y_k)$, $Z = (z_i)$, and so on, to denote sequences.

seems evident. For example, we may define the sequence of reciprocals of the even numbers by writing

$$X := \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \cdots\right),$$

though a more satisfactory method is to specify the formula for the general term and write

$$X := \left(\frac{1}{2n} : n \in \mathbb{N}\right)$$

or more simply $X = (1/2n)$.

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Introduction to Infinite Series

In elementary texts, an infinite series is sometimes "defined" to be "an expression of the form"

$$x_1 + x_2 + \cdots + x_n + \cdots. \tag{1}$$

However, this "definition" lacks clarity, since there is *a priori* no particular value that we can attach to this array of symbols, which calls for an *infinite* number of additions to be performed.

Definition If $X := (x_n)$ is a sequence in $\mathbb{R}$, then the **infinite series** (or simply the **series**) **generated by** X is the sequence $S := (s_k)$ defined by

$$\begin{aligned} s_1 &:= x_1 \\ s_2 &:= s_1 + x_2 \quad (= x_1 + x_2) \\ &\cdots \\ s_k &:= s_{k-1} + x_k \quad (= x_1 + x_2 + \cdots + x_k) \\ &\cdots \end{aligned}$$

The numbers x_n are called the **terms** of the series and the numbers s_k are called the **partial sums** of this series. If lim S exists, we say that this series is **convergent** and call this limit the **sum** or the **value** of this series. If this limit does not exist, we say that the series S is **divergent.**

It is convenient to use symbols such as

$$\sum (x_n) \quad \text{or} \quad \sum x_n \quad \text{or} \quad \sum_{n=1}^{\infty} x_n$$

to denote both the infinite series S generated by the sequence $X = (x_n)$ and also to denote the value lim S, in case this limit exists.

Examples Consider the sequence $X := (r^n)_{n=0}^{\infty}$ where $r \in \mathbb{R}$, which generates the **geometric series**:

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \cdots + r^n + \cdots.$$

We will show that if $|r| < 1$, then this series converges to $1/(1-r)$. Indeed, if $s_n := 1 + r + r^2 + \cdots + r^n$ for $n \geq 0$, and if we multiply s_n by r and subtract the result from s_n, we obtain (after some simplification):

$$s_n(1-r) = 1 - r^{n+1}.$$

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Therefore, we have $$s_n - \frac{1}{1-r} = \frac{r^{n+1}}{1-r},$$

from which it follows that $$\left| s_n - \frac{1}{1-r} \right| \leq \frac{|r|^{n+1}}{|1-r|}$$

Since $|r|^{n+1} \to 0$ when $|r| < 1$, it follows that the geometric series converges to $1/(1-r)$ when $|r| < 1$.

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Lecture-2

INFINITE SERIES:

Definition: Let $\{u_n\}$ be a sequence of real numbers.

i.e. $\{u_1, u_2, u_3, u_4, ..., u_n, ...\infty\}$ be a sequence of real numbers.

Then, the expression $u_1 + u_2 + ... + u_n + ...\infty$ [i.e. the sum of the terms of the sequence, which are infinite in number] is called an **infinite series** and is denoted by $\sum_{n=1}^{\infty} u_n$ or more briefly, by $\sum u_n$.

n^{th} Term: The term u_n in an infinite series is called the nth term of the series.

Finite series:

If the number of terms in the series is limited, then the series is called a ***finite series.***

Infinite series:

If the number of terms in the series is infinite, then the series is called an ***infinite series.***

n^{th} Partial sum:

The sum of the first n terms of the series is called its ***nth partial sum*** of $\sum u_n$ and is denoted by S_n.

i.e., $S_n - u_1 + u_2 + ... + u_n$.

Clearly, S_n is a function of n and as n increases indefinitely, then the following three cases arise:

(i) If S_n tends to a finite limit as $n \to \infty \Rightarrow \sum u_n$ is ***convergent.***

(ii) If S_n tends to $+\infty$ or $-\infty$ as $n \to \infty \Rightarrow \sum u_n$ is ***divergent.***

(iii) If S_n does not tends to a unique limit as $n \to \infty \Rightarrow \sum u_n$ is ***oscillatory or non-convergent.***

Remark: Any series which diverge or oscillate are said to be non-convergent series.

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Now let us examine the behavior of the well known series:

Q.No.1.: Examine the behavior of the series: $1+\frac{1}{2}+\frac{1}{2^2}+.....+\frac{1}{2^{n-1}}+\frac{1}{2^n}+.....$.

Sol.: Given series is $1+\frac{1}{2}+\frac{1}{2^2}+.....+\frac{1}{2^{n-1}}+\frac{1}{2^n}+.....$

Here $S_n=1+\frac{1}{2}+\frac{1}{2^2}+.....+n^{th}$ term.

$\left[\because \text{ it is a G.P. whose first term is 1 and common ratio is } 1/2.\ \therefore S_n=\frac{a\left(1-r^n\right)}{1-r}\right]$

$$\rightarrow S_n-\frac{1\left[1-\left(\frac{1}{2}\right)^n\right]}{1-\frac{1}{2}} \rightarrow S_n=2\left(1-\frac{1}{2^n}\right).$$

Taking limit on both sides, we get

$$\lim_{n\to\infty} S_n=\lim_{n\to\infty} 2\left(1-\frac{1}{2^n}\right)=2(1-0)=2.$$

$\Rightarrow$ The given series converges to 2.

Q.No.2.: Examine the behavior of the series: $1+2+3+.....+n+.....\infty$.

Sol.: Given series is $1+2+3+.....+n+.....\infty$.

Here $S_n=1+2+3+.....+n=\frac{n(n+1)}{2}$.

Taking limit on both sides, we get

$$\lim_{n\to\infty} S_n=\lim_{n\to\infty}\frac{n(n+1)}{2}\rightarrow\infty.$$

$\Rightarrow$ The given series is divergence.

Q.No.3.: Examine the behavior of the series: $1-1+1-1+.....=\sum_{n=1}^{\infty}(-1)^{n-1}$.

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Sol.: Given series is $1-1+1-1+\ldots\ldots\ldots = \sum_{n=1}^{\infty}(-1)^{n-1}$.

Here $S_n = 1-1+1-1+\ldots\ldots\ldots$upto n terms $= 0,\ n = 2m$ (even)

$= 1,\ n = 2m+1$ (odd).

Taking limit on both sides, we get

$\lim_{n\to\infty} S_n = 0,$

$= 1.$

Clearly, in this case, S_n does not tend to a unique limit as $n \to \infty$.

Hence, the given series is oscillatory and oscillates between 0 and 1.

Comparison Test:

Statement: Let $\sum u_n$ and $\sum v_n$ be two positive term series, such that from and after some particular term, $\lim_{n\to\infty} \frac{u_n}{v_n} = k$ (a non-zero, finite quantity), then $\sum u_n$ and $\sum v_n$ either both converge or both diverge, or in other words, both the series behave alike.

Remark: To select auxiliary series (or Harmonic series) $\sum v_n = \sum \frac{1}{n^p}$, it should be noted that p = difference in degree of n in denominator and numerator of u_n.

3. p-series Test:

(Behaviour of Auxiliary Series or Harmonic Series)

Statement: Show that the series $\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\ldots\ldots+\frac{1}{n^p}+\ldots\ldots = \sum\frac{1}{n^p}$

(i) converges if $p > 1$ and

(ii) diverges if $p \le 1$.

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Q.No.1. Discuss the behaviour of the series $\sum_{n=1}^{\infty}\frac{n+3}{n^3-n+1}$.

Sol.: Here $u_n = \frac{n+3}{n^3-n+1}$. Take $v_n = \frac{1}{n^2}$.

$$\therefore \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1+\frac{3}{n}}{1-\frac{1}{n^2}+\frac{1}{n^3}} = 1 \text{ (a finite, non-zero number)}.$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum\frac{1}{n^2} \approx \sum\frac{1}{n^p}$ (here $p = 2 > 1$) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent.

Q.No.2.: Discuss the behaviour of the series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}$.

Sol.: Here $u_n = \frac{1}{\sqrt{n+1}+\sqrt{n}}$. Take $v_n = \frac{1}{\sqrt{n}}$.

$$\therefore \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2} \text{ (a finite, non-zero number)}.$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum\frac{1}{n^{1/2}} \approx \sum\frac{1}{n^p}$ (here $p = 1/2 < 1$) is divergent. [by p-series test]

Hence $\sum u_n$ is also divergent.

Q.No.3.: Discuss the behaviour of the series $\sum_{n=0}^{\infty}\frac{2n^3+5}{4n^5+1}$.

Sol.: Here $u_n = \frac{2n^3+5}{4n^5+1}$. Take $v_n = \frac{1}{n^2}$.

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$$\therefore \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{2+\frac{5}{n^3}}{4+\frac{1}{n^5}} = \frac{2}{4} = \frac{1}{2} \text{ (a finite, non-zero number).}$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \frac{1}{n^2}$ (here $p = 2 > 1$) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent.

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Lecture-3

D'Alembert's Ratio Test:

Statement: If $\sum u_n$ be a positive term series s.t. from and after some particular term,

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=k\text{, then }\sum u_n$$

(i) converges if $k<1$ and

(ii) diverges if $k>1$.

This test fails when limit does not exist or equal to 1.

Here $\frac{u_{n+1}}{u_n}$ measures the rate or growth of the terms of the series.

Note: This test was developed by Jean Le-Rond D'Alembert (1717–1783) a French Mathematician.

Proof: Let the series from and after the particular term be

$u_1 + u_2 + \ldots\ldots\ldots\ldots\ldots + u_n + \ldots\ldots\ldots\ldots$

Case I: When $k<1$.

Then by definition of limit, a positive number $\lambda(k<\lambda<1)$ can be found s.t.

$$\frac{u_{n+1}}{u_n}<\lambda\ \forall n$$

$$\Rightarrow \frac{u_2}{u_1}<\lambda,\ \frac{u_3}{u_2}<\lambda,\ \frac{u_4}{u_3}<\lambda,\ldots\ldots\ldots\ldots\ldots\text{, and so on.}$$

Now $S_n = u_1+u_2+\ldots\ldots\ldots\ldots\ldots+u_{n-1}+u_n$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\ldots\ldots\ldots\ldots+\frac{u_n}{u_1}\right)$$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}.\frac{u_2}{u_1}+\ldots\ldots\ldots\ldots\right)$$

$$<u_1\left(1+\lambda+\lambda^2+\ldots\ldots\ldots\ldots\lambda^{n-1}\right)=u_1\frac{1-\lambda^n}{1-\lambda}$$

$\lim_{n\to\infty}S_n<\lim_{n\to\infty}u_1\frac{1-\lambda^n}{1-\lambda}=\frac{u_1}{1-\lambda}$, a finite quantity $(\because \lambda<1)\Rightarrow\sum u_n$ is convergent.

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Case II: When $k > 1$.

Then, by definition of limit, $\frac{u_{n+1}}{u_n} > 1$

$$\Rightarrow \frac{u_2}{u_1} > 1, \frac{u_3}{u_2} > 1, \frac{u_4}{u_3} > 1, \ldots\ldots\ldots\ldots\ldots\ldots, \text{and so on}.$$

Now $S_n = u_1 + u_2 + \ldots\ldots\ldots\ldots\ldots\ldots + u_{n-1} + u_n$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \ldots\ldots\ldots\ldots\ldots + \frac{u_n}{u_1}\right)$$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}.\frac{u_2}{u_1} + \ldots\ldots\ldots\ldots\ldots\ldots\right)$$

$$> u_1(1 + 1 + 1 + \ldots\ldots\ldots\ldots\ldots 1) = nu_1$$

$\lim\limits_{n\to\infty} S_n > \lim\limits_{n\to\infty} nu_1 \to +\infty$, $\sum u_n$ is divergent.

This completes the proof.

Remark: Ratio test fails when k = 1. So when k = 1, in that case comparison test is helpful in determining the behaviour of the series.

Counter example:

For the convergent series, $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$: $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{n^2}{(1+n)^2} = 1$

Also for the divergent series, $\sum\limits_{n=1}^{\infty} \frac{1}{n}$: $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim\limits_{n\to\infty} \frac{n}{(1+n)} = 1$

Thus ratio test cannot be used to distinguish between convergent and divergent series when k = 1

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Q.No.1.: Discuss the behaviour of infinite series $\sum \frac{n!}{n^n}$.

Sol.: Here $u_n = \frac{n!}{n^n}$ and so $u_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)}}$.

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{(n+1)}} \times \frac{n^n}{n!}$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} = \frac{1}{e} < 1. \quad \text{[since e=2.71828...]}$$

Hence $\sum u_n$ is convergent (by D'Alembert's Ratio Test).

Q.No.2.: Discuss the behaviour of infinite series $1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \ldots\ldots$.

Sol.: Here $u_n = \frac{n^p}{n!}$ and so $u_{n+1} = \frac{(n+1)^p}{(n+1)!}$.

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)^p}{(n+1)!} \times \frac{n!}{n^p} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^p . \frac{1}{(n+1)} = (1)^p.0 = 0 < 1.$$

Hence $\sum u_n$ is convergent (by D'Alembert's Ratio Test).

Q.No.3.: Discuss the behaviour of infinite series $\frac{x}{1.2} + \frac{x^2}{3.4} + \frac{x^3}{5.6} + \frac{x^4}{7.8} + \ldots\infty \quad (x > 0)$.

Sol.: Here $u_n = \frac{x^n}{(2n-1)(2n)}$ and so $u_{n+1} = \frac{x^{n+1}}{(2n+1)(2n+2)}$.

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\left(2-\frac{1}{n}\right)(2)}{\left(2+\frac{1}{n}\right)\left(2+\frac{2}{n}\right)} . x = x .$$

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Case 1: The given series is convergent if $x < 1$.

Case 2: The given series is divergent if $x > 1$.

Case 3: When $x = 1$. $u_n = \dfrac{1}{(2n-1)(2n)}$. Take $v_n = \dfrac{1}{n^2}$.

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(2-\frac{1}{n}\right)2} = \frac{1}{4} \text{ (a finite non-zero number)} .$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words, both the series behave alike.

But $\sum v_n = \sum \dfrac{1}{n^2} \approx \sum \dfrac{1}{n^p}$ (here $p = 2 > 1$) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent when $x = 1$.

Hence the given series converges if $x \le 1$ and diverges if $x > 1$.

Q.No.4.: Discuss the behaviour of infinite series $\dfrac{1}{2\sqrt{1}} + \dfrac{x^2}{3\sqrt{2}} + \dfrac{x^4}{4\sqrt{3}} + \ldots\ldots\ldots\infty$.

Sol.: Here $u_n = \dfrac{x^{2n-2}}{(n+1)\sqrt{n}}$ and so $u_{n+1} = \dfrac{x^{2n}}{(n+2)\sqrt{n+1}}$.

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{2n}}{(n+2)\sqrt{n+1}} \times \frac{(n+1)\sqrt{n}}{x^{2n-2}} = \lim_{n\to\infty} \frac{n\left(1+\frac{1}{n}\right)}{n\left(1+\frac{2}{n}\right)} \frac{\sqrt{n}}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}\right)} x^2 = x^2 .$$

Case 1: The given series is convergent if $x^2 < 1$.

Case 2: The given series is divergent if $x^2 > 1$.

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Case 3: When $x^2 = 1$. $u_n = \dfrac{1}{(n+1)\sqrt{n}}$. Take $v_n = \dfrac{1}{n^{3/2}}$.

$$\therefore \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)} = 1 \text{ (a finite non-zero number)}.$$

Thus, by comparison test, both the series $\sum u_n$ and $\sum v_n$ converges or diverges together, or in other words , both the series behave alike.

But $\sum v_n = \sum \dfrac{1}{n^{3/2}} \approx \sum \dfrac{1}{n^p}$ (here $p = 3/2 > 1$) is convergent. [by p-series test]

Hence $\sum u_n$ is also convergent when $x^2 = 1$.

Hence the given series converges if $x^2 \le 1$ and diverges if $x^2 > 1$.

Q.No.5.: Discuss the behaviour of infinite series $\sum \dfrac{x^n}{(2n)!}$, $(x > 0)$.

Sol.: Here $u_n = \dfrac{x^n}{(2n)!}$ and so $u_{n+1} = \dfrac{x^{n+1}}{(2n+2)!}$.

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{x^{n+1}}{(2n+2)!} \times \frac{(2n)!}{x^n} = \lim_{n\to\infty} \frac{x}{(2n+2)(2n+1)} = 0 < 1.$$

Hence $\sum u_n$ is convergent for all x (by D'Alembert's Ratio Test) .

Q.No.6.: Discuss the behaviour of infinite series $\sum \dfrac{n!2^n}{n^n}$.

Sol.: Here $u_n = \dfrac{n!2^n}{n^n}$ and so $u_{n+1} = \dfrac{(n+1)!2^{n+1}}{(n+1)^{n+1}}$.

$$\therefore \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{(n+1)!2^{n+1}}{(n+1)^{n+1}} \times \frac{n^n}{n!2^n} = \lim_{n\to\infty} 2.\frac{n^n}{(n+1)^{n+1}} = \lim_{n\to\infty} \frac{2}{\left(1+\frac{1}{n}\right)^n} = \frac{2}{e} < 1 .$$

Hence $\sum u_n$ is convergent (by D'Alembert's Ratio Test).

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Q.No.7: Discuss the behaviour of infinite series

$$x^2(\log 2)^q + x^3(\log 3)^q + x^4(\log 4)^q + \ldots\ldots \; (x > 0)$$

Sol.: Here $u_n = x^n(\log n)^q$ for $n \geq 2$ and so $u_{n+1} = x^{n+1}[\log(n+1)]^q$.

$$\therefore \frac{u_{n+1}}{u_n} = x\left[\frac{\log(n+1)}{\log n}\right]^q = x\left[\frac{\log n\left(1+\frac{1}{n}\right)}{\log n}\right]^q = x\left[\frac{\log n + \log\left(1+\frac{1}{n}\right)}{\log n}\right]^q$$

$$= x\left[\frac{\log n + \frac{1}{n} - \frac{\left(\frac{1}{n}\right)^2}{2} + \frac{\left(\frac{1}{n}\right)^3}{3} + \ldots\ldots}{\log n}\right]^q = x\left[\frac{\left(\log n + \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \ldots\ldots\right)}{\log n}\right]^q$$

$$\Rightarrow \lim_{n\to\infty}\frac{u_{n+1}}{u_n} = \lim_{n\to\infty} x\left[1 + \frac{1}{n\log n} - \frac{1}{2n^2\log n} + \frac{1}{3n^3\log n} - \ldots\ldots\right]^q = x$$

Case 1: The given series is convergent if $x < 1$.

Case 2: The given series is divergent if $x > 1$.

Case 3: When $x = 1$. $u_n = (\log n)^q \Rightarrow \lim_{n\to\infty}(\log n)^q \to \infty \neq 0$.

Hence given series is divergent if $x = 1$. [by nth term test]

Hence the given series converges if $x < 1$ and diverges if $x \geq 1$.

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Lecture-4

RAABE'S TEST*:

Statement: If $\sum u_n$ be a positive term series s.t. from and after some particular term, $\lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right]=k$ or $\lim_{n\to\infty} n\left[1-\left|\frac{u_{n+1}}{u_n}\right|\right]=k$,

then $\sum u_n$ **(i)** converges if $k>1$ and

(ii) diverges if $k<1$.

Remarks: **(i)** Raabe's test fails when $k=1$.

(ii) This test is very useful when D'Alembert's ratio test fails.

Q.No.1.: Discuss the behaviour of infinite series $\sum \frac{4.7.10.....(3n+1)}{1.2.3.....n}x^n$.

Sol.: Here $\frac{u_n}{u_{n+1}}=\frac{4.7.....(3n+1)}{1.2.....n}x^n\times\frac{1.2....n.(n+1)}{4.7....(3n+1).(3n+4)}\frac{1}{x^{n+1}}=\frac{(n+1)}{(3n+4)}.\frac{1}{x}$

$\Rightarrow \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\frac{1}{3x}$. Hence, by D'Alembert's ratio test

$\sum u_n$ (i) converges if $\frac{1}{3x}>1$ and (ii) diverges if $\frac{1}{3x}<1$.

i.e. $\sum u_n$ (i) converges if $x<\frac{1}{3}$ and (ii) diverges if $x>\frac{1}{3}$.

But this test fails if $x=\frac{1}{3}$. Now if $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}$ does not involve e, then we can apply Raabe's

Now let us try the Raabe's test. When $x=\frac{1}{3}$, then

$$\frac{u_n}{u_{n+1}}=\left[1+\frac{1}{n}\right]\left[1+\frac{4}{3n}\right]^{-1}=\left[1+\frac{1}{n}\right]\left[1-\frac{4}{3n}+\frac{16}{9n^2}.....\right]=1-\frac{1}{3n}+\frac{4}{9n^2}-.....$$

$$\Rightarrow \lim_{n\to\infty} n\left[\frac{u_n}{u_{n+1}}-1\right]=\lim_{n\to\infty}\left[-\frac{1}{3}+\frac{4}{9n}-.....\right]=-\frac{1}{3}<1.$$

$\Rightarrow$ By Raabe's test, the given series diverges when $x=\frac{1}{3}$.

Hence the given series $\sum u_n$ (i) converges if $x<\frac{1}{3}$ and (ii) diverges if $x\geq\frac{1}{3}$.

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Q.No.2.: Discuss the behaviour of infinite series

$$1+\frac{3}{7}x+\frac{3.6}{7.10}x^2+\frac{3.6.9}{7.10.13}x^3+\frac{3.6.9.12}{7.10.13.16}x^4+.....\infty .$$

Sol.: Here (by omitting first term) $u_n=\dfrac{3.6.9.....(3n)}{7.10.13.....(3n+4)}x^n$

and $\therefore u_{n+1}=\dfrac{3.6.9.....(3n)(3n+3)}{7.10.13.....(3n+4)(3n+7)}x^{n+1}$.

Now $\dfrac{u_n}{u_{n+1}}=\dfrac{3.6.9.....(3n)}{7.10.13.....(3n+4)}x^n\times\dfrac{7.10.13.....(3n+4)(3n+7)}{3.6.9.....(3n).(3n+3)}\dfrac{1}{x^{n+1}}=\dfrac{(3n+7)}{(3n+3)}.\dfrac{1}{x}$

$$\Rightarrow \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{\left(1+\frac{7}{3n}\right)}{\left(1+\frac{1}{n}\right)}.\frac{1}{x}=\frac{1}{x}.$$

Hence by D'Alembert's ratio test:

$\sum u_n$ (i) converges if $\dfrac{1}{x}>1$ and (ii) diverges if $\dfrac{1}{x}<1$.

i.e. $\sum u_n$ (i) converges if $x<1$ and (ii) diverges if $x>1$.

But this test fails if $x=1$. Now if $\lim\limits_{n\to\infty}\dfrac{u_n}{u_{n+1}}$ does not involve e, then we can apply Raabe's test or Gauss's test. Now let us try the Raabe's test.

When $x=1$, then

$$\frac{u_n}{u_{n+1}}=\left[1+\frac{7}{3n}\right]\left[1+\frac{1}{n}\right]^{-1}=\left[1+\frac{7}{3n}\right]\left[1-\frac{1}{n}+\frac{1}{n^2}.....\right]=1+\frac{4}{3n}-\frac{4}{3n^2}-.....$$

$$\Rightarrow \lim_{n\to\infty}n\left[\frac{u_n}{u_{n+1}}-1\right]=\lim_{n\to\infty}\left[\frac{4}{3}-\frac{4}{3n}-.....\right]=\frac{4}{3}>1.$$

$\Rightarrow$ By Raabe's test, the given series converges when $x=1$.

Hence the given series $\sum u_n$ (i) converges if $x\le 1$ and (ii) diverges if $x>1$.

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Q.No.3.: Discuss the behaviour of infinite series $\frac{x}{3}+\frac{1.2}{3.5}x^2+\frac{1.2.3}{3.5.7}x^3+.....\infty\ (x>0)$.

Sol.: Here $u_n=\frac{1.2.3....n}{3.5.7....(2n+1)}x^n$ and so $u_{n+1}=\frac{1.2.3....n.(n+1)}{3.5.7....(2n+1)(2n+3)}x^{n+1}$

Now $\frac{u_n}{u_{n+1}}=\frac{1.2.3...n}{3.5.7....(2n+1)}x^n\times\frac{3.5.7....(2n+1)(2n+3)}{1.2.3....n.(n+1)}\frac{1}{x^{n+1}}=\frac{(2n+3)}{(n+1)}\frac{1}{x}=\frac{2\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{n}\right)}\frac{1}{x}$.

$$\therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{2\left(1+\frac{3}{2n}\right)}{\left(1+\frac{1}{n}\right)}\frac{1}{x}=\frac{2}{x}.$$

Hence by D'Alembert's ratio test

$\sum u_n$ (i) converges if $\frac{2}{x}>1$ and (ii) diverges if $\frac{2}{x}<1$.

i.e. $\sum u_n$ (i) converges if $x<2$ and (ii) diverges if $x>2$.

But this test fails if $x=2$. Now if $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}$ does not involve e, then we can apply Raabe's test or Gauss's test. Now let us try the Raabe's test.

When $x=2$ then

$$\frac{u_n}{u_{n+1}}=\left[1+\frac{3}{2n}\right]\left[1+\frac{1}{n}\right]^{-1}=\left[1+\frac{3}{2n}\right]\left[1-\frac{1}{n}+\frac{1}{n^2}.....\right]=1+\frac{1}{2n}-\frac{1}{2n^2}-.....$$

$$\Rightarrow \lim_{n\to\infty}n\left[\frac{u_n}{u_{n+1}}-1\right]=\lim_{n\to\infty}\left[\frac{1}{2}-\frac{1}{2n}-.....\right]=\frac{1}{2}<1.$$

$\Rightarrow$ By Raabe's test, the given series diverges when $x=1$.

Hence the given series $\sum u_n$ (i) converges if $x<2$ and (ii) diverges if $x\geq 2$.

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Lecture 5

Fourier Series

Introduction

Most of the events in nature and many other systems, being periodic in nature. In many engineering problems, especially in the study of periodic phenomena in conduction of heat, electro-dynamics and acoustic, it is necessary to express a function in a series of sines and cosines. Most of the single-valued functions which occur in applied mathematics can be expressed in the form

$$\frac{a_o}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \dots\dots\dots + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \dots\dots$$

where $a_o, a_1, a_2, a_3, \dots\dots, b_1, b_2, b_3, \dots\dots$ are real constants and the series is known as Fourier series. The constants $a_o, a_1, a_2, a_3, \dots\dots, b_1, b_2, b_3, \dots\dots$ are called Fourier's Coefficients of the periodic function.

Euler's Formulae

The Fourier series for the function f(x) in the interval $\alpha<x<\alpha + 2\pi$ is given by

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \text{(i)}$$

Determination of the Fourier's Constants:

To find a_o: assume that (i) can be integrated from x = c to x = c+2π, term by term, so that we have

$$\int_C^{C+2\pi} f(x)\,dx = \frac{a_o}{2}\int_C^{C+2\pi} 1\,dx + \int_C^{C+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)dx + \int_C^{C+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)dx$$

$$= \frac{a_o}{2}(x)_C^{C+2\pi} + 0 + 0$$

$$= \frac{a_o}{2}(C + 2\pi - C) = \pi a_o$$

or $$a_o = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\,dx$$

To find a_n: multiplying each side of (i) by cosnx and integrate with respect to x between the limits x = C to x = C + 2π, so that we have

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$$\int_C^{C+2\pi} f(x)\cos nx\,dx = \frac{a_o}{2}\int_C^{C+2\pi}\cos nx\,dx + \int_C^{C+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\cos nx\,dx + \int_C^{C+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\cos nx\,dx$$

$$= 0 + \int_C^{C+2\pi}\sum_{n=1}^{\infty} a_n \cos^2 nx\,dx + 0$$

$$= a_n \pi$$

or $$a_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\cos nx\,dx$$

To find b_n multiply both side of (i) by sinnx and integrate with respect to x between the limits x = c to x = C+2π, so that we have

$$\int_C^{C+2\pi} f(x)\sin nx\,dx = \frac{a_o}{2}\int_C^{C+2\pi}\sin nx\,dx + \int_C^{C+2\pi}\left(\sum_{n=1}^{\infty} a_n \cos nx\right)\sin nx\,dx + \int_C^{C+2\pi}\left(\sum_{n=1}^{\infty} b_n \sin nx\right)\sin nx\,dx$$

$$= 0 + 0 + \int_C^{C+2\pi}\sum_{n=1}^{\infty} b_n \sin^2 nx\,dx = b_n\pi$$

$$b_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\sin nx\,dx$$

The values $a_o = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\cos nx$$

$$b_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\sin nx\,dx$$

are called Euler's formulae

Euler's Formulae for Different Intervals

Case (i): If C = 0, then the interval for the above series (i) become $0<x<2\pi$ and the Euler's formulae reduce to

$$a_o = \frac{1}{\pi}\int_o^{2\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_o^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_o^{2\pi} f(x)\sin nx\,dx$$

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Example 1. Find a Fourier series to represent $x-x^2$ from $x = -\pi$ to $x = \pi$ and hence deduce

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots$$

Solution. Let $x - x^2 = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ (1)

$$\text{Then } a_o = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x)^2\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi} = -\frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\frac{\sin nx}{n} - (1-2x) - \left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{-4(-1)^n}{n^2} \qquad \because \cos n\pi = (-1)^n$$

$$\therefore a_1 = 4/1^2,\ a_2 = -4/2^2,\ a_3 = 4/3^2,\ a_4 = -4/4^2 \text{ etc}$$

Finally

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x-x^2)\left(-\frac{\cos nx}{n}\right) - (1-2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= -2(-1)^n/n$$

$$\therefore b_1 = 2/1,\ b_2 = -2/2,\ b_3 = 2/3,\ b_4 = -2/4 \text{ etc}$$

Substituting the values of a_o, a_n, b_n in (i) we get

$$x - x^2 = -\frac{\pi^2}{3} + 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2} + \ldots\ldots\right]$$

$$+ 2\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \ldots\ldots\right]$$

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Lecture 6

Example Obtain the Fourier Series expansion for the function

$f(x) = x^2, -\pi<x<\pi$

Hence, deduce that

(a) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots = \frac{\pi^2}{6}$

(b) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\ldots = \frac{\pi^2}{12}$

(c) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\ldots = \frac{\pi^2}{8}$

Solution: We have $f(x) = x^2$

$f(x)$ is an even function, therefore, $f(x)$ contains only cosine terms. Hence $b_n = 0$

Let $f(x) = x^2 = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ (1)

Then we have $a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{1}{3\pi}\left[\pi^3 - (-\pi)^3\right]$$

$$= \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}$$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{2}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + \frac{2\pi}{n^2}\cos n\pi + 0\right]$$

$$= \frac{2}{\pi}\,\frac{2\pi}{n^2}(-1)^n = \frac{4}{n^2}(-1)^n$$

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Substituting in (1), we get

$$\text{i.e. } x^2 = \frac{\pi^2}{3} + 4\left[\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx\right]$$

$$= \frac{\pi^2}{3} + 4\left[\frac{(-1)^1}{1^2}\cos x + \frac{(-1)^2}{2^2}\cos 2x + \frac{(-1)^3}{3^2}\cos 3x + \ldots\ldots\right]$$

$$= \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\ldots\ldots\ldots\right]$$

$$\text{or } x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \ldots\ldots\right]$$

At $x = \pi$ and $x = 0$, the function $f(x)$ is continuous

Putting $x = \pi$, is (2), we get

$$\pi^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos \pi}{1^2} - \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} - \ldots\ldots\right]$$

$$\text{or } \pi^2 = \frac{\pi^2}{3} - 4\left[-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \ldots\ldots\ldots\right]$$

$$\pi^2 - \frac{\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\right]$$

$$\text{or } 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\right] = \frac{2\pi^2}{3}$$

$$\text{i.e. } \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\ldots\ldots = \frac{\pi^2}{6}$$

putting $x = 0$, in (2), we get

$$0 = \frac{\pi^2}{3} - 4\left[\frac{\cos 0}{1^2} - \frac{\cos 0}{2^2} + \frac{\cos 0}{3^2} - \ldots\ldots\right]$$

$$\text{or } 0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\right]$$

$$\text{or } 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots\right] = \frac{\pi^2}{3}$$

$$\text{or } \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots = \frac{\pi^2}{12} \qquad (4)$$

Adding (3) and (4) we get $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$

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Lecture 7

FUNCTIONS HAVING POINT OF DISCONTINUITY

In deriving Euler's formulae for the constants a_o, a_n and b_n we have assumed that f(x) is continuous in the given interval. In some cases f(x) may have a finite number of discontinuities. We can also express such functions as Foruier series. For example, consider a function f(x) defined as follows:

$$f(x) = f_1(x), \qquad c<x<x_o$$
$$= f_2(x), \qquad x_o<x<c+2\pi$$

Where x_o is a point of discontinuities for f(x) is the interval (c, c+2π) and $\lim_{x\to o-} f(x)$ i.e. $f(x_o - o)$ and $\lim_{x\to o+} f(x)$ i.e. $f(x_o + o)$ exist unequal and are finite. We determine the values of a_o, a_n and b_n can be computed as

$$a_o = \frac{1}{\pi}\left[\int_c^{x_o} f_1(x)\,dx + \int_{x_o}^{C+2\pi} f_2(x)\,dx\right]$$

$$a_n = \frac{1}{\pi}\left[\int_c^{x_o} f_1(x)\cos nx\,dx + \int_{x_o}^{c+2\pi} f_2(x)\cos nx\,dx\right]$$

$$\text{and } b_n = \frac{1}{\pi}\left[\int_c^{x_o} f_1(x)\sin nx\,dx + \int_{x_o}^{c+2\pi} f_2(x)\sin nx\,dx\right]$$

At the point of discontinuity i.e. at $x = x_o$ the Fourier series converges to

$$\frac{1}{2}[f(x_o + o) + f(x_o - o)] = \frac{1}{2}(FB + FC)$$

If f(x) satisfies Dirichlet's conditions and f(x) is expressed as

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos nx + \sum_1 b_n \sin nx$$

in the interval [C, C+2π], then f(x) converges to $f(x_o)$ if x_o is a point of continuity of f(x) in the given interval

Thus from above we conclude that

(i) It may be seen from the graph, that at a point of finite discontinuity $x = x_o$ there is a finite jump equal to BC in the value of the function f(x) at $x = x_o$

(ii) A given function f(x) may be defined by different formulae in different regions. Such types of functions are quite common in Fourier series

(iii) At a point of discontinuity the sum of the series is equal to mean of the limits on the right and left.

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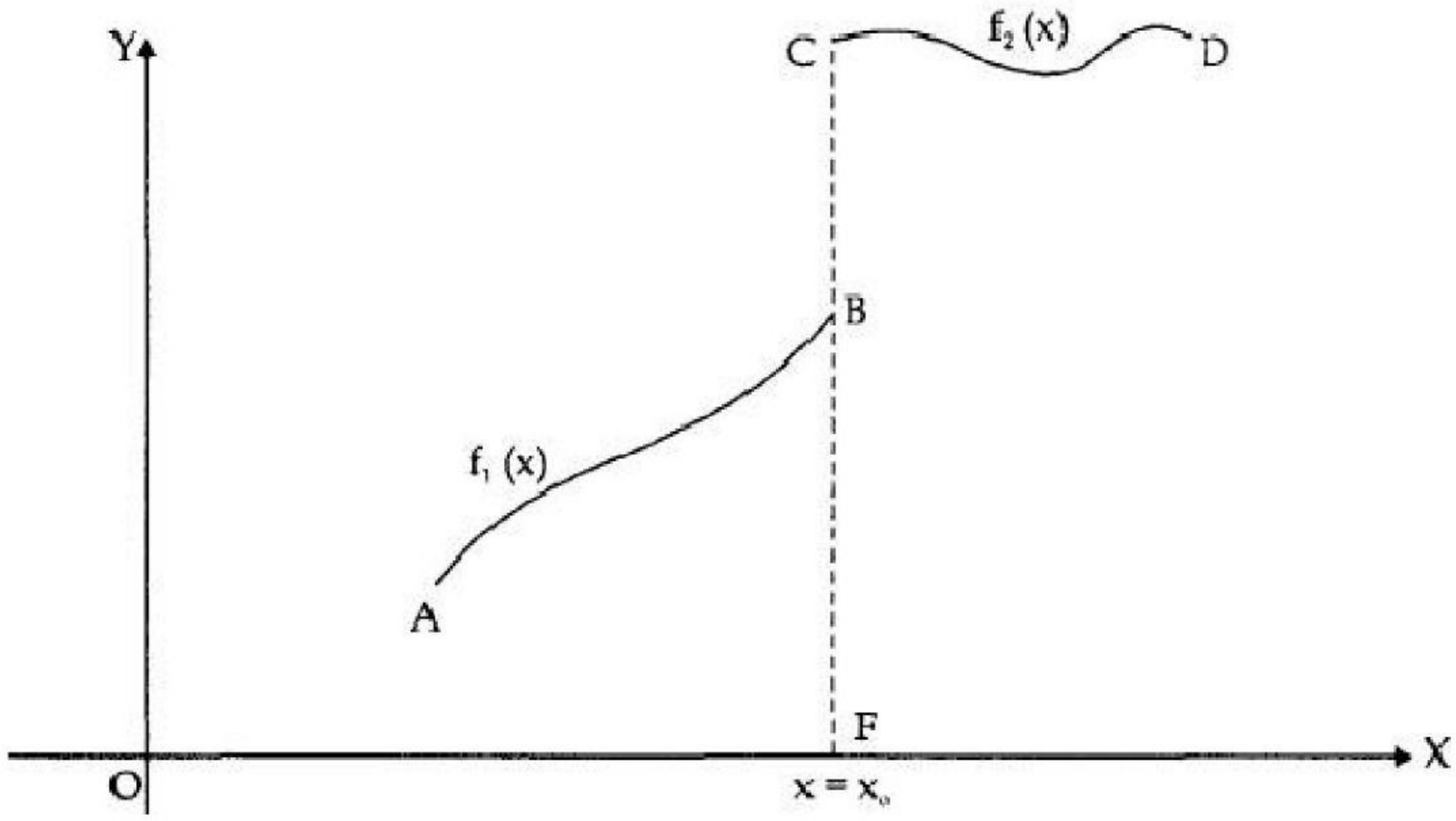

Example Find Fourier series for the function defined by

$$f(x) = \begin{cases} -1 & \text{for } -\pi<x<o \\ 0 & \text{for } x = o \\ 1 & \text{for } 0<x<\pi \end{cases}$$

hence proved that

$$\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}+.....$$

Solution: By Fourier series, we have

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\text{Then } a_o = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{o}(-1)\,dx + \int_{o}^{\pi} 1\,dx\right]$$

$$= \frac{1}{\pi}\left[(-x)_{-\pi}^{o} + (x)_{o}^{\pi}\right]$$

$$a_o = \frac{1}{\pi}[-\pi+\pi] = 0$$

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and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\cos nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}\left[\sin nx\right]_{-\pi}^{0} + \frac{1}{n}\left[\sin nx\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[-0+0\right] = 0$$

Finally, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -\sin nx\,dx + \frac{1}{\pi}\int_{0}^{\pi}\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\frac{1}{n}\left[\cos nx\right]_{-\pi}^{0} - \frac{1}{n}\left[\cos nx\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{n\pi}\left[1-\cos n\pi - \cos n\pi + 1\right] = \frac{1}{n\pi}\left[2-2\cos n\pi\right] = \frac{2}{n\pi}\left[1-(-1)^n\right]$$

$$\Rightarrow b_n = \frac{2}{n\pi}\left[1-(-1)^n\right],\ n = 1, 2, 3, \ldots\ldots$$

If n is even $b_n = 0$ if n is odd $b_n = \frac{4}{n\pi}$ Hence from (1) we have

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx = b_1\sin x + b_2\sin 2x + b_3\sin 3x + b_4\sin 4x + \ldots\ldots$$

$$f(x) = \frac{4}{\pi}\sin x + \frac{4}{3\pi}\sin 3x + \frac{4}{5\pi}\sin 5x + \ldots\ldots \qquad (2)$$

The expansion (2) is required Fourier expansion putting $x = \frac{\pi}{2}$ is (2) we get

$$1 = \frac{4}{\pi}\left[\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \ldots\ldots\right] = \frac{4}{\pi}\left[1 + \frac{1}{3}(-1) + \frac{1}{5}(1) + \frac{1}{7}(-1) + \ldots\right]$$

$$= \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots\right] = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

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Lecture 8

HALF RANGE SERIES

Many a time it is required to obtain a Fourier expansion of a function f(x) for the range (0, C) which is half the period of the Fourier series. As it is immaterial whatever the function may be outside the range $0<x<c$, we extend the function to cover the range $-c<x<c$ so that the new function may be odd or even. The extension of the functions period being made in such a way that their graphs became either symmetrical to the axis of y or symmetrical to origin, and then the expansion contains either only the cosine terms along with a_o or only the sine terms. Thus, we may get the different forms of series for the same functions.

Sine Series: If we have to expand a function f(x) as a sine series in 0 to C, then we expand the function first from –C to C and then the make the reflection at the origin, so that $f(x) = -f(-x)$ then the expanded function became odd and will give the required Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{C}$$

$$\text{Where } b_n = \frac{2}{C} \int_o^C f(x) \sin \frac{n\pi x}{C}\, dx$$

Cosine Series: If it is required to express f(x) as a cosine series in $0<x<C$, then first we expand the function from –C to C so that its reflection became about the axis of y i.e., the graph became symmetrical about the axis of y then the expanded Fourier Cosine series contains

$$f(x) = \frac{a_o}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{C}$$

$$\text{where } a_o = \frac{2}{C} \int_o^C f(x)\, dx$$

$$\text{and } a_n = \frac{2}{C} \int_o^C f(x) \cos \frac{n\pi x}{C}\, dx$$

Example . Expand for $f(x) = k$ for $0<x<2$ in a half range

(i) Sine series (ii) cosine series

Solution. $f(x) = k$ and $C = 2$

(i) $b_n = \frac{2}{C} \int_o^C f(x) \sin \frac{n\pi x}{C}$ in half range (0, C)

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$$= \frac{2}{2}\int_{o}^{C} k \sin\frac{n\pi x}{2}\, dx$$

$$= k\,\frac{2}{n\pi}\left(-\cos\frac{n\pi x}{2}\right)_{o}^{2} = \frac{2k}{n\pi}\left[-\cos n\pi + 1\right]$$

Half range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2}$$

$$k = \sum_{n=1}^{\infty}\frac{2k}{n\pi}\left[1-\cos n\pi\right]\sin\frac{n\pi x}{2} = \frac{2k}{\pi}\sum_{n=1}^{\infty}\left[1-(-1)^n\right]\sin\frac{n\pi x}{2}$$

$$\Rightarrow k = \frac{2k}{\pi}\left[2\sin\frac{\pi x}{2} + 2\sin\frac{3\pi x}{2} + 2\sin\frac{5\pi x}{2} + \ldots\right]$$

(ii) $$a_o = \frac{2}{C}\int_{o}^{C} f(x)\, dx = \frac{2}{2}\int_{o}^{2} k\, dx = k\,(x)_{o}^{2} = 2k$$

$$a_n = \frac{2}{C}\int_{o}^{C} f(x)\cos\frac{n\pi x}{C}\, dx = \frac{2}{2}\int_{o}^{2} k\cos\frac{n\pi x}{2}\, dx$$

$$= k\,\frac{2}{n\pi}\left[\sin\frac{n\pi x}{2}\right]_{o}^{2} = \frac{2k}{n\pi}\sin n\pi$$

$$\Rightarrow a_n = 0$$

Therefore, from

$$f(x) = \frac{a_o}{2} + \sum a_n \cos\frac{n\pi x}{2} \text{ we have}$$

$$f(x) = k = \frac{1}{2}(2k) + \sum_{n=1}^{\infty} 0\cos\frac{n\pi x}{2}$$

$$\Rightarrow f(x) = k$$

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LECTURE -1

Limit of a complex function f(z):

Definition:

A function $w = f(z)$ tends to the **limit** ℓ as z tends to a point z_0 along **any path**, if to each positive arbitrary number $\in$, however small, there corresponds a positive number δ, such that

$|f(z) - \ell| < \epsilon$, whenever $0 < |z - z_0| < \delta$

i.e., $(\ell - \in) < f(z) < (\ell + \in)$, whenever $(z_0 - \delta) < z < (z_0 + \delta)$, $z \neq z_0$

and we write $\underset{z \to z_0}{\text{Lt}} f(z) = \ell$.

Continuity of f(z):

A single valued function $w = f(z)$ is said to be **continuous** at a point $z = z_0$, if $\underset{z \to z_0}{\text{Lt}} f(z) = f(z_0)$.

In other words:

A function f(z) is said to be continuous at a point z_0 if $f(z_0)$ exists, $\underset{z \to z_0}{\text{Lim}} f(z)$ exists and $\underset{z \to z_0}{\text{Lim}} f(z) = f(z_0)$,

i.e., limiting value of f(z) as z approaches z_0 coincides with the value $f(z_0)$.

A function f(z) is said to be continuous in any region R of the z-plane, if it is continuous at every point of the region.

This means that, a function is said to be continuous in a domain if it is continuous at every point of the domain.

A function, which is not continuous at z_0, is known as discontinuous at z_0.

This means that a function, in which $f(z_0)$ does not exist, or $\underset{z \to z_0}{\text{Lim}} f(z)$ does not exist or $\underset{z \to z_0}{\text{Lim}} f(z) \neq f(z_0)$, is known as discontinuous at z_0.

Result 1: If f(z) and g(z) are continuous function in D. Then their sum $f + g$, difference $f - g$, product fg, quotient f/g are all continuous in D. Continuous function of a continuous function is continuous.

Result 2: $f = u + iv$ is continuous if both u and v are continuous.

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Derivative of f(z) (Differentiability) :

Let $w = f(z)$ be a single valued function of the variable $z(= x + iy)$, then the derivative or differential coefficient of $w = f(z)$ is defined as

$$\frac{dw}{dz} = f'(z) = \underset{\delta z \to 0}{Lt} \frac{f(z+\delta z) - f(z)}{\delta z}$$

provided the limit exists and has the **same value** for all the different ways in which $\delta z \to 0$.

In other words:

A function f(z) is said to be differentiable at a point z_0 if the limit

$$f'(z_0) = \underset{\delta z \to 0}{Lim} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z} = \underset{z \to z_0}{Lim} \frac{f(z) - f(z_0)}{z - z_0} \qquad (\text{with } z = z_0 + \delta z)$$

exists.

The limit $f'(z_0)$ is known as the derivative of f(z) at z_0.

The above limit should be the same along any path from z to z_0.

Thus, differentiability of a complex function is a stringent requirement.

Differentiation rules:

Differentiation rules of real calculus are valid in complex differentiation also.

1. $\frac{dc}{dz} = 0$ where c = complex constant
2. $\frac{d}{dz}[f \pm g] = \frac{df}{dz} \pm \frac{dg}{dz}$
3. $\frac{d}{dz}[cf(z)] = c\frac{df}{dz}$
4. $\frac{d}{dz}[f.g] = f\frac{dg}{dz} + \frac{df}{dz}.g$
5. $\frac{d}{dz}\left[\frac{f}{g}\right] = \frac{g\frac{df}{dg} - f\frac{dg}{dz}}{g^2}$
6. a. $\frac{d}{dz}[f(z)]^n = n[f(z)]^{n-1}\frac{df}{dz}$

   b. $\frac{d}{dz}z^n = nz^{n-1}$

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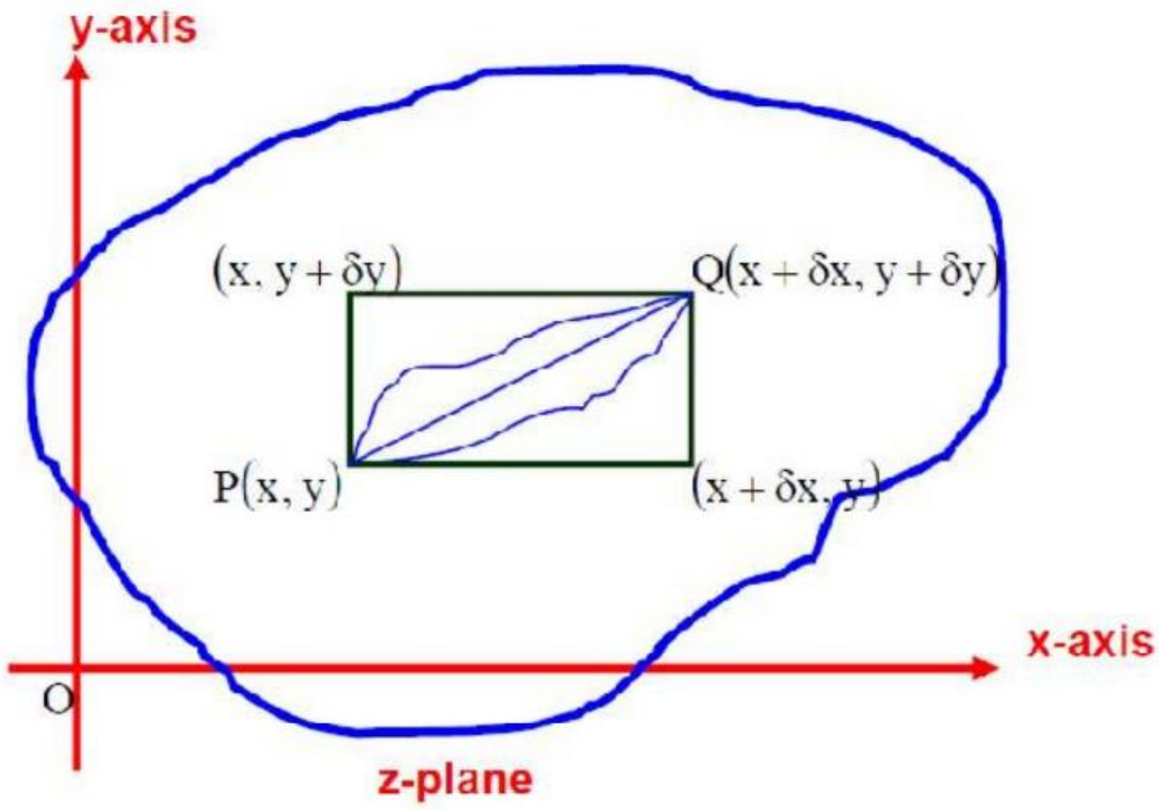
7. Chain rule $\frac{dw}{dz} = \frac{dw}{d\zeta}.\frac{d\zeta}{dz}$ if $w = f(\zeta)$ and $\zeta = g(z)$.



Remarks: Suppose $P(z)$ is fixed and $Q(z+\delta z)$ neighbouring point. The point Q may approach P along any straight or curved path in the given region, i.e., δz may tends to zero in any manner and $\frac{dw}{dz}$ may not exist. Then it becomes a fundamental problem to determine the necessary and sufficient conditions for $\frac{dw}{dz}$ to exist. This fact is settled by the following theorem.

Q.No.1.: Show that $\lim\limits_{z\to 0}\frac{x^2y}{x^4+y^2}$ does not exist even though this function approached the same limit along every straight line through the origin.

Sol.: Path I. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\frac{x^2y}{x^4+y^2} = \lim\limits_{y\to 0}\frac{0}{y^2} = \lim\limits_{y\to 0} 0 = 0$

Path II. $\lim\limits_{\substack{y\to 0\\ x\to 0}}\frac{x^2y}{x^4+y^2} = \lim\limits_{x\to 0}\frac{0}{x^4} = \lim\limits_{x\to 0} 0 = 0$

Path III. Along any straight line through origin.

Let $y = mx$

$$\lim_{\substack{y=mx\\ x\to 0}}\frac{x^2y}{x^4+y^2} = \lim_{x\to 0} = \frac{mx^3}{x^4+m^2x^2} = \lim_{x\to 0}\frac{mx}{x^2+m^2} = 0$$

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Path IV as $y = mx^2$, then

$$\lim_{\substack{y=mx^2 \\ x\to 0}} \frac{x^2 y}{x^4 + y^2} = \lim_{x\to 0} = \frac{m.x^4}{x^4 + m^2 x^4} = \lim_{x\to 0} \frac{m}{1+m^2} = \frac{m}{1+m^2} \neq 0$$

and different for different values of m.

Therefore, the limit does not exist.

Q.No.2.: Determine where the given function is continuous

(a) $\frac{1}{1+z^2}$, (b). $\frac{1}{z-1}$ inside a unit circle.

How about in the complex plane.

Sol.: $\frac{1}{1+z^2}$ is continuous everywhere except where $1+z^2 = 0$, i.e., at $z = \pm i$.

When unit circle is considered, $|z| < 1$, $z = \pm i$ are excluded.

Thus $\frac{1}{1+z^2}$ is continuously inside $|z| = 1$.

Similarly, $\frac{1}{z-1}$ is also continuous inside $|z| = 1$.

If the entire complex plane is considered, then both $\frac{1}{1+z^2}$ and $\frac{1}{z-1}$ are discontinuous, at $z = \pm i$ and $z = 1$ respectively.

Q.No.3.: Is $f(z) = \frac{z}{|z|}$ continuous at origin (defined for $z \neq 0$ and $f(0) = 0$).

Sol.: $\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{x+iy}{\sqrt{x^2+y^2}}$

Path I. $\lim_{\substack{x\to 0 \\ y\to 0}} \frac{x+iy}{\sqrt{x^2+y^2}} = i$ **Path II.** $\lim_{\substack{y\to 0 \\ x\to 0}} \frac{x+iy}{\sqrt{x^2+y^2}} = 1$

Since limit does not exist, so f is discontinuous at $z_0 = 0$.

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LECTUER-2

Analytic functions:

Definition:

If a single-valued function f(z) possesses a unique derivative w.r.t. z at all points of a region R, then f(z) is called an analytic function.

In other words:

A function f(z) is said to be analytic at a point z_0 if f is differentiable not only at z_0 but at every point of some neighbourhood of z_0. An analytic function is also known as "holomorphic function", "regular function", "monogenic function".

A function f(z) is analytic in a domain if it is analytic every point of the domain.

A point at which an analytic function ceases to possess a derivative is called a singular point of the function.

Thus, if u and v are real single-valued functions of x and y such that $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous throughout a region R, then the Cauchy's-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$,

are both necessary and sufficient conditions for the function $f(z) - u + iv$ to be analytic in the region R.

The derivative of f(z) is then given by

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

Thus Cauchy's-Riemann equations (or conditions) are used to determine whether a complex function is analytic or not.

Note:

1. If f is analytic in a domain D, then u, v satisfy C-R equations at all points in D.
2. C-R conditions are **necessary**, but not sufficient conditions.
3. C-R conditions are **sufficient** if the partial derivatives are continuous, i.e., u(x,y), v(x,y) have continuous first partial derivatives and satisfy C-R equations then f = u + iv is analytic.

f analytic $\rightarrow$ C-R conditions – Continuous P.D. $\rightarrow$ Analyticity

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Entire Function:

A function which is analytic everywhere, for all z in the complex plane, is known as entire function.

e.g., Polynomials, rational functions are entire functions, means analytic everywhere.

$|z|^2$ is differentiable only at z = 0. So, it is nowhere analytic.

Thus, analyticity is very stringent condition

Properties of analytic functions:

1. If f(z) and g(z) are analytic, then $f+g, fg, f/g$ are analytic if $g(z) \neq 0$.
2. Analytic function of an analytic function is analytic.
3. An entire function of an entire function is entire.
4. If f is analytic, then it is continuous (analyticity) $\Rightarrow$ differentiability $\Rightarrow$ continuity.
5. Derivative of an analytic function is itself analytic.

   Proof: $f' = u_x + iv_x = U + iV$.

   f is analytic, so $u_x = v_y, u_y = -v_x$

   Differentiating w.r.t. and y, we get

   $u_{xx} = v_{yx}, u_{yy} = -v_{xy}$ or $U_x = V_x$ and $U_y = -V_x$,

   i.e. U, V satisfy CR conditions. Hence f' is analytic.
6. If $f = u + iv$ is analytic, then the family of curves $u(x, y) = c_1$ and $v(x, y) = c_2$ are mutually orthogonal, i.e., $u = c_1$ are orthogonal trajectories of $v = c_2$ and vice-versa.

Proof: By implicit differentiation of $u = c_1$, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x} = 0 \Rightarrow \frac{\partial y}{\partial x} = \frac{ux}{uy}.$$

Similarly $v_x + v_y \dfrac{dy}{dx} = 0 \rightarrow \dfrac{dy}{dx} = \dfrac{-v_x}{v_y}$.

Product of slopes $-\dfrac{ux}{uy}\cdot\left(\dfrac{-v_x}{v_y}\right) = -1$ by CR conditions.

Remarks: The real and imaginary parts of an analytic function are called **conjugate functions**. Thus if, $f(z) = u(x, y) + iv(x, y)$ is an analytic function, then $u(x, y)$ and $v(x, y)$ are conjugate functions. The relation between two conjugate functions is given by C-R equations.

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LECTUER-3

[Necessary and Sufficient conditions for the derivative of the function f(z)]

Theorem:

The necessary and sufficient conditions for the derivative of the function $w = u(x, y) + iv(x, y) = f(z)$ to exist for all values of z in a region R are

(i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in the region R;

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

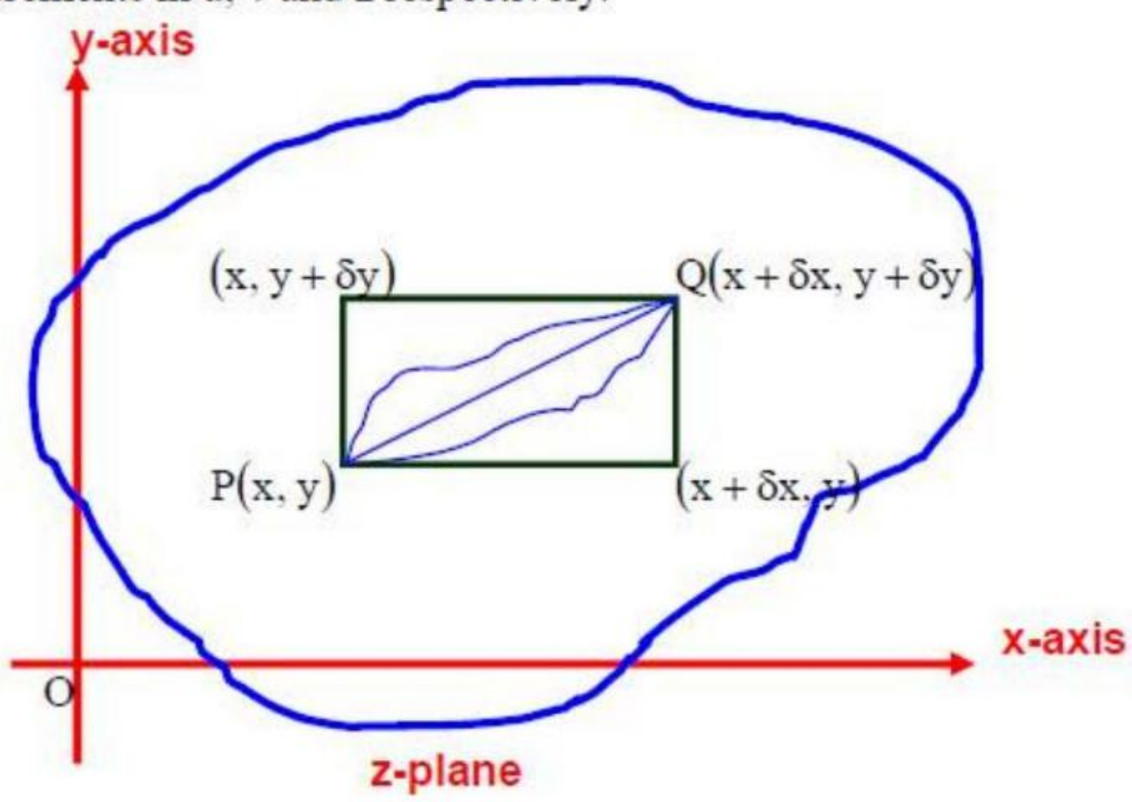
The conditions (ii) are known as **Cauchy-Riemann equations** or briefly **C-R equations.**

Proof: (a) Necessary condition:

Given: f(z) possesses a unique derivative.

To show: C-R equations are satisfied.

Let δx and δy be the increments of x and y respectively. Let δu, δv and δz be the corresponding increments in u, v and z respectively.



Now since f(z) possesses a unique derivative at P(z), then

$$f'(z) = \underset{\delta z \to 0}{Lt} \frac{f(z+\delta z) - f(z)}{\delta z} = \underset{\delta z \to 0}{Lt} \frac{(u+\delta u) + i(v+\delta v) - (u+iv)}{\delta z} = \underset{\delta z \to 0}{Lt} \left(\frac{\delta u}{\delta z} + i \frac{\delta v}{\delta z} \right)$$

Since δz can approach zero in any manner, we can first assume δz to be completely real and then completely imaginary.

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When δz is wholly real, then $\delta y = 0$ and $\delta z = \delta x$.

$$\therefore f'(z) = \underset{\delta x \to 0}{\text{Lt}}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}. \qquad \text{(i)}$$

When δz is wholly imaginary, then $\delta x = 0$ and $\delta z = i\delta y$.

$$\therefore f'(z) = \underset{\delta y \to 0}{\text{Lt}}\left(\frac{\delta u}{i\delta y} + i\frac{\delta v}{i\delta y}\right) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}. \qquad \text{(ii)}$$

Now since f(z) possesses a unique derivative, this means (i) and (ii) are equal.

$$\therefore \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.$$

On equating the real and imaginary parts from both sides, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \qquad \text{(iii)}$$

Thus, the necessary conditions for the existence of the derivative of f(z) is the C-R equations should be satisfied.

(b) Sufficient condition:

Given: (i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in the region R;

i.e. f(z) is a single-valued function possessing partial derivatives

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at each point of the region

(ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

To show: f(z) possesses a unique derivative at every point of the region R.

Now since $f'(z) = \underset{\delta z \to 0}{\text{Lt}} \frac{f(z+\delta z) - f(z)}{\delta z}$.

By Taylor's theorem for function of two variables, we have

$$f(z+\delta z) = u(x+\delta x,\ y+\delta y) + iv(x+\delta x,\ y+\delta y)$$

$$= u(x,\ y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right) + \ldots\ldots i\left[v(x,\ y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right) + \ldots\ldots\right]$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

[Omitting terms beyond the first powers of δx and δy]

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$$\Rightarrow f(z+\delta z)-f(z)=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta x+\left(\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}\right)\delta y$$

$$\rightarrow f(z+\delta z)-f(z)-\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta x+\left(-\frac{\partial v}{\partial x}+i\frac{\partial u}{\partial x}\right)\delta y \qquad \text{[Using C-R equations]}$$

$$=\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta x+\left(i\frac{\partial v}{\partial x}+\frac{\partial u}{\partial x}\right)i\delta y$$

$$-\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)(\delta x+i\delta y)-\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)\delta z.$$

$$\therefore f'(z)=\underset{\delta z\to 0}{\mathrm{Lt}}\frac{f(z+\delta z)-f(z)}{\delta z}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x} \quad \text{or} \quad \frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}.$$

Thus $f'(z)$ exists, because $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial x}$ exist.

This completes the proof.

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LECTUER-4

POLAR FORM OF CAUCHY-RIEMANN EQUATIONS

Show that the polar form of Cauchy-Riemann equations are

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial 0},\quad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial 0}.$$

Also deduce that $\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0$.

Proof: If $(r, 0)$ be the co-ordinates of the point whose cartesian co-ordinates are (x, y), then

$z=x+iy=re^{i\theta}$.

$\therefore u+iv=f(z)=f\left(re^{i\theta}\right)$, where u and v are now expressed in terms of r and θ.

Differentiating it partially w.r.t. r and θ, we get

$$\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}=f'\left(re^{i\theta}\right).e^{i\theta} \qquad \text{and} \qquad \frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \theta}=f'\left(re^{i\theta}\right).ire^{i\theta}=ir\left(\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}\right)$$

Equating real and imaginary parts, we get

$$\frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial 0} \quad \text{(i)} \qquad\qquad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial 0}, \quad \text{(ii)}$$

which is the required polar form of Cauchy-Riemann equations.

Now, differentiating (i) partially w. r. t. r, we get

$$\frac{\partial^2 u}{\partial r^2}=-\frac{1}{r^2}\frac{\partial v}{\partial \theta}+\frac{\partial^2 v}{\partial \theta \partial r}. \quad \text{(iii)}$$

Differentiating (ii) partially w. r. t. θ, we get

$$\frac{\partial^2 u}{\partial \theta^2}=-r\frac{\partial^2 v}{\partial r \partial \theta}. \quad \text{(iv)}$$

Thus using (i), (ii) and (iv), we get

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=\frac{1}{r^2}\frac{\partial v}{\partial \theta}+\frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r}+\frac{1}{r}\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right)+\frac{1}{r^2}\left(-r\frac{\partial^2 v}{\partial r \partial \theta}\right)=0. \left[\because \frac{\partial^2 v}{\partial 0 \partial r}=\frac{\partial^2 v}{\partial r \partial 0}\right]$$

Q-1

$$f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} \quad (z\neq 0),\quad f(0)=0,$$

is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist.

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Sol.: Here $f(z) = \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$ $(z \neq 0)$.

$$\therefore \underset{z\to 0}{\text{Lt}}\, f(z) = \underset{\substack{x\to 0\\ y\to 0}}{\text{Lt}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \underset{y\to 0}{\text{Lt}} \frac{-y^3(1-i)}{x^2} = \underset{y\to 0}{\text{Lt}}\left[-y(1-i)\right] = 0.$$

$$\text{Also } \underset{z\to 0}{\text{Lt}}\, f(z) = \underset{\substack{y\to 0\\ x\to 0}}{\text{Lt}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \underset{x\to 0}{\text{Lt}} \frac{x^3(1+i)}{x^2} = \underset{y\to 0}{\text{Lt}}\left[x(1+i)\right] = 0.$$

Also $f(0) = 0$ (given)

Thus $\underset{z\to 0}{\text{Lt}}\, f(z) = f(0)$ when $x \to 0$ first and then $y \to 0$ and also vice-versa. Now let both x and y tend to zero simultaneously along the path $y = mx$. Then

$$\underset{z\to 0}{\text{Lt}}\, f(z) = \underset{\substack{y\to mx\\ x\to 0}}{\text{Lt}} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = \underset{x\to 0}{\text{Lt}} \frac{x^3(1+i) - m^3x^3(1-i)}{(1+m^2)x^2}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{x\left[1+i-m^3(1-i)\right]}{1+m^2} = 0.$$

Hence $\underset{z\to 0}{\text{Lt}}\, f(z) = f(0)$, in whatever manner $z \to 0$.

$\Rightarrow f(z)$ is continuous at the origin.

Hence f(z) is continuous for all values of z.

$$\text{Now } f(z) = \frac{x^3 - y^3}{x^2+y^2} + i\frac{x^3+y^3}{x^2+y^2} = u(x, y) + iv(x, y)$$

Since $f(0) = 0 \Rightarrow u(0, 0) = 0$, and $v(0, 0) = 0$

$$\therefore \left(\frac{\partial u}{\partial x}\right)_{0,0} = \underset{x\to 0}{\text{Lt}} \frac{u(x, 0) - u(0, 0)}{x} = \underset{x\to 0}{\text{Lt}} \frac{x}{x} = 1$$

$$\left(\frac{\partial u}{\partial y}\right)_{0,0} = \underset{y\to 0}{\text{Lt}} \frac{u(0, y) - u(0, 0)}{y} = \underset{x\to 0}{\text{Lt}} \frac{-y}{y} = -1$$

$$\left(\frac{\partial v}{\partial x}\right)_{0,0} = \underset{x\to 0}{\text{Lt}} \frac{v(x, 0) - v(0, 0)}{x} = \underset{x\to 0}{\text{Lt}} \frac{x}{x} = 1$$

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and $\left(\frac{\partial v}{\partial y}\right)_{0,0} = \underset{y\to 0}{\text{Lt}} \frac{u(0, y)-u(0, 0)}{y} = \underset{x\to 0}{\text{Lt}} \frac{y}{y} = 1$

$\to \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.

Thus, the C-R equations are satisfied at the origin.

But $f'(0) = \underset{z\to 0}{\text{Lt}} \frac{f(z)-f(0)}{z} = \underset{z\to 0}{\text{Lt}} \frac{f(z)}{z} = \underset{z\to 0}{\text{Lt}} \frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$.

If $z \to 0$ along the path $y = mx$, then $f'(0) = \frac{1-m^3+i(1+m^3)}{(1+m^2)(1+im)}$,

which assume different values as m varies. So $f'(z)$ is not unique at (0, 0).

Thus, f(z) is not analytic at the origin even though it is continuous and satisfies the C.R. equations at the origin.

Q-2 Show that

$$f(z) = \begin{cases} \frac{x^2y^3(x+iy)}{x^6+y^{10}}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is not analytic at the origin, even though CR equations are satisfied at the origin.

Sol.: Let $z \neq 0$ and given $f(z) = \frac{x^3y^3}{x^6+y^{10}} + i\frac{x^2y^4}{x^6+y^{10}} = u + iv$ (say).

Here $u = \frac{x^3y^3}{x^6+y^{10}}$, $v = \frac{x^2y^4}{x^6+y^{10}}$.

Now $\left(\frac{\partial u}{\partial x}\right)_{\substack{x=0\\y=0}} = \underset{x\to 0}{\text{Lt}} \frac{u(x,0)-u(0,0)}{x} = \underset{x\to 0}{\text{Lt}} \frac{0-0}{x} = 0$.

$$\left(\frac{\partial u}{\partial y}\right)_{\substack{x=0\\y=0}} = \underset{y\to 0}{\text{Lt}} \frac{u(0,y)-u(0,0)}{y} = \underset{y\to 0}{\text{Lt}} \frac{0-0}{y} = 0.$$

$$\left(\frac{\partial v}{\partial y}\right)_{\substack{x=0\\y=0}} = \underset{y\to 0}{\text{Lt}} \frac{v(0,y)-v(0,0)}{y} = \underset{y\to 0}{\text{Lt}} \frac{0-0}{y} = 0.$$

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$$\left(\frac{\partial v}{\partial x}\right)_{\substack{x=0\\y=0}} = \underset{x\to 0}{\text{Lt}}\frac{v(x,0)-(0,0)}{x} = \underset{x\to 0}{\text{Lt}}\frac{0-0}{x} = 0.$$

$\Rightarrow u_x = v_y,\ \ u_y = -v_x.$

Hence, CR equations are satisfied at the origin.

Further $f'(0) = \underset{z\to 0}{\text{Lt}}\dfrac{f(z)-f(0)}{z} = \underset{z\to 0}{\text{Lt}}\dfrac{\dfrac{x^3y^3}{x^6+y^{10}} + i\dfrac{x^2y^4}{x^6+y^{10}} - 0}{z}.$

Choose the path y = x, then

$$f'(0) = \underset{x\to 0}{\text{Lt}}\left(\frac{x^6}{x^6+x^{10}} + i\frac{x^6}{x^6+x^{10}}\right)\frac{1}{x+ix}$$

$$= \underset{x\to 0}{\text{Lt}}\frac{(1+i)x^6}{x^6(1+x^4)(1+i)x} = \underset{x\to 0}{\text{Lt}}\frac{1}{x(1+x^4)} = \frac{1}{0} = \infty.$$

$\Rightarrow f'(0)$ does **not exist** at origin. $\Rightarrow f(z)$ is not analytic at the origin.

Q-3

Find the derivatives of $f(z) = \dfrac{1}{z^n}$ for $n \neq -1$.

Sol.: Introducing polar coordinates

$$f(z) = \frac{1}{(r\cos\theta + ir\sin\theta)^n} = \frac{1}{r^n}(\cos n\theta - i\sin n\theta)$$

so $u = r^{-n}\cos n\theta,\ v = -r^{-n}\sin n\theta$

$$f'(z) = e^{-i\theta}(u_r + iv_r) = e^{-i\theta}\left(-nr^{-n-1}.\cos n\theta\right) + (-1)(-n)r^{-n-1}.\sin n\theta$$

$$= \frac{-n}{r^{n+1}}.e^{-i\theta}(\cos n\theta - i\sin n\theta) = \frac{-n}{r^{n+1}}.e^{-i\theta}.e^{-in\theta} = \frac{n}{r^{n+1}}e^{-i(n+1)\theta}$$

$$f'(z) = \frac{-n}{r^{n+1}.e^{-(n+1)\theta}} = \frac{-n}{z^{n+1}},\quad n \neq -1.$$

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Q-4

Show that for $f(z) = \dfrac{2xy(x+iy)}{x^2+y^2}$ if $z \neq 0$

$= 0$ if $z = 0$.

The C-R are satisfied at origin but derivative of f(z) at origin does not exist.

Sol.: C-R conditions at origin;

$f(z) = \dfrac{2xy(x+iy)}{x^2+y^2}$, so $u = \dfrac{2x^2y}{x^2+y^2}$, $v = \dfrac{2xy^2}{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0}\frac{v(x,0)\quad v(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0}\frac{v(0,y)-u(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0$$

So C-R conditions are satisfied at $z = 0$

Derivative at $z = 0$:

$$f'(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z-0} = \lim_{z\to 0}\frac{\dfrac{2xy(x+iy)}{x^2+y^2}}{x+iy}$$

$$= \lim_{z\to 0}\frac{2xy}{x^2+y^2} = 0 \quad \text{as} \quad \begin{matrix} x\to 0 & y\to 0 \\ y\to 0 & x\to 0 \end{matrix} \quad \text{and}$$

$$= \lim_{x\to 0}\frac{2xy}{x^2+y^2} = \lim_{x\to 0}\frac{2mx^2}{x^2+m^2x^2} = \lim_{x\to 0}\frac{2m}{1+m^2} = \frac{2m}{1+m^2} \neq 0.$$

Thus derivative of f(z) does not exist at $z = 0$.

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Q-5

If f is analytic show that $f' = (\cos\theta - i\sin\theta)\frac{\partial f}{\partial r}$.

Sol.: $x = r\cos\theta$, $y = r\sin\theta$, $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\frac{y}{x}$.

$$r_x = \frac{1}{2}\frac{2x}{\sqrt{xx^2 + y^2}} = \frac{x}{r} = \cos\theta, \quad r_y = \sin\theta.$$

$$\theta_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}.\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2 + y^2} = \frac{-y}{r^2} = \frac{-\sin\theta}{r}, \quad \theta_y = \frac{\cos\theta}{r}.$$

We know that $f' = u_x + iv_x = u_x - iu_y$

$$f' = (u_r.r_x + u_\theta.\theta_x) - (u_r.r_y + u_\theta.\theta_y)$$

since u is function of r, θ which are functions of x, y.

$$f' = \left(\cos\theta.u_r - \frac{\sin\theta}{r}u\theta\right) - i\left(\sin\theta.u_r + \frac{\cos\theta}{r}.u\theta\right)$$

since f is analytic, C-R conditions are satisfied (in polar coordinates) $u_r = \frac{v_\theta}{r}$, $u_\theta = -rv_r$.

$$f' = \left(\cos\theta. + \frac{\sin\theta}{r}.rv_r\right) - i(u_r\sin\theta - v_r\cos\theta) = (\cos\theta - i\sin\theta)u_r + i(\cos\theta - i\sin\theta)v_r$$

$$= (\cos\theta - i\sin\theta)(u_r + iv_r)$$

$$f' = (\cos\theta - i\sin\theta)\frac{\partial f}{\partial r}, \text{ since } \frac{\partial f}{\partial r} = \frac{\partial}{\partial r}(u + iv) = u_r + iv_r.$$

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LECTUER-5

Harmonic Functions:

Definition: A function f(x, y), is said to be a harmonic function if it satisfies the Laplace's equation, i.e., $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$, i.e., $\nabla^2 f=0$.

If $f(z)=u+iv$ be an analytic function in some region of the z-plane, then the Cauchy-Riemann equations are satisfied.

i. e. $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ (i) and $\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}$ (ii)

Differentiate (i) w. r. t. x and (ii) w. r. t. y, we get

$$\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y} \quad \text{(iii)}$$

$$\frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y\partial x} \quad \text{(iv)}$$

Adding (iii) and (iv) and assuming that $\frac{\partial^2 v}{\partial x\partial y}=\frac{\partial^2 v}{\partial y\partial x}$, we get $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ (v)

Similarly, by differentiating (i) w. r. t. y and (ii) w. r. t. x and subtracting, we obtain

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0. \quad \text{(vi)}$$

Thus, both the functions u and v satisfy the Laplace's equation in two variables. For this reason, they are known as harmonic functions and their theory is called potential theory.

Thus, a function f(x, y), is said to be a harmonic function if it satisfies the Laplace's equation, i.e., $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$, i.e., $\nabla^2 f=0$.

Orthogonal system: Consider the two families of curves

$$u(x, y)=c_1 \quad \text{(i)}$$

$$v(x, y)=c_2 \quad \text{(ii)}$$

Differentiating (i), we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}.\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}=\frac{\frac{\partial v}{\partial y}}{\frac{\partial v}{\partial x}}=m_1 \text{ (say)} \qquad \text{[by C-R equations]}$$

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Similarly (ii) gives $\frac{dy}{dx} = -\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} = m_2$ (say)

$\therefore m_1 m_2 = -1$, i. e. (i) and (ii) form an orthogonal system.

Hence every function $f(z) = u + iv$ defines two families of curves $u(x, y) = c_1$ and $v(x, y) = c_2$, which form an orthogonal system.

Q-1

Show that an analytic function with constant modulus is constant.

or

Show that an analytic function cannot have a constant absolute value without reducing to a constant.

Sol.: Let $f(z) = u + iv$.

Since $|f(z)| = \text{constant} = c$, say, then we have

$\sqrt{u^2 + v^2} = c \Rightarrow u^2 + v^2 = c^2$

$\therefore 2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$ and $2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0$

$\Rightarrow u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0$ and $u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0$. [by C-R equations]

Eliminating $\frac{\partial u}{\partial y}$, we get $(u^2 + v^2)\frac{\partial u}{\partial x} = 0$.

Thus provided $w = u + iv \neq 0$.

Similarly, $\frac{\partial u}{\partial y} = 0$, $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$.

Since, the four partial derivatives of u and v are zero.

$\Rightarrow$ The functions u, v are constant.

$\Rightarrow w = u + iv$ is also constant.

This completes the proof.

Q-2

Is the function $u(x, y) = 2xy + 3xy^2 - 2y^2$ harmonic (i.e., solution of Laplace equation)?

Sol.: $u_x = 2y + 3y^2$, $u_{xx} = 0$, $u_y = 2x + 6xy - 6y^2$, $u_{yy} = 6x - 12y$.

So $u_{xx} + u_{yy} \neq 0$. Therefore u is not harmonic.

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Q-3

Show that $v(x,y) = -\sin x \sinh y$ is harmonic. Find the conjugate harmonic of v

(or find an analytic function $f = u + iv$)

Sol.: Differentiating v partially w.r.t. x and y, we get

$v_x = -\cos x \sinh y, \quad v_{xx} = \sin x \sinh y$

$v_y = -\sin x \cosh y, \quad v_{yy} = -\sin x \sinh y$

Then $v_{xx} + v_{yy} = \sin x \sinh y + (-\sin x \sinh y) = 0$

Therefore v is harmonic.

To find conjugate harmonic u of v:

From C-R conditions $u_x = v_y$ and $u_y = -v_x$

So $u_x = v_y = -\sin x.\cosh y$ (i)

Integrating (i) partially w.r.t., x, we get

$u(x,y) = \cos x.\cosh y + c(y)$ (ii)

Differentiating (ii) partially w.r.t. y and using second C-R condition, $(u_y = -v_x)$, we have

$\cos x.\sinh y + \frac{dc}{dy} = \frac{\partial u}{\partial y} = v_x = \cos x.\sinh y$

Q-4

Find the analytic function $f(z) = u + iv$, given

(i) $u = a(1+\cos\theta)$, **(ii)** $v = \left(r - \frac{1}{r}\right)\sin\theta, \ r \neq 0.$

and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{1}{r}[-a\sin\theta] = \frac{a}{r}\sin\theta.$ (ii)

Integrating (i) w. r. t. θ, we get

$v = f(r) \quad \therefore \frac{\partial v}{\partial r} = f'(r).$ (iii)

From (ii) and (iii), we get

$\frac{\partial v}{\partial r} = \frac{a}{r}\sin\theta = f'(r) \quad \Rightarrow f'(r) = \frac{a}{r}\sin\theta.$

$\therefore f(r) = a\sin\theta \log r + c$ Thus $v = a\sin\theta \log r + c$

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Hence $f(z) = u + iv = a[1+\cos\theta] + i[a\sin\theta\log r + c] = a[1+\cos\theta + i\sin\theta\log r] + ic$,

which is the required analytic function.

(ii): Let $f(z) = u + iv$ be the required analytic function.

Given $v = \left(r - \frac{1}{r}\right)\sin\theta,\ r \neq 0.$

Using C-R equation in polar form i.e. $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

$$\therefore r\frac{\partial u}{\partial r} = r\frac{\partial v}{\partial \theta} = \left(r - \frac{1}{r}\right)\cos\theta \qquad \Rightarrow \frac{\partial u}{\partial r} = \left[1 - \frac{1}{r^2}\right]\cos\theta \qquad \text{(i)}$$

$$\text{and } -\frac{1}{r}\frac{\partial u}{\partial 0} - \frac{\partial v}{\partial r} - -\left[1 + \frac{1}{r^2}\right]\sin 0 \rightarrow \frac{\partial u}{\partial 0} - -\left[r + \frac{1}{r}\right]\sin 0. \qquad \text{(ii)}$$

Integrating (i) w. r. t. r, we get

$$u = \left[r - \frac{1}{r}\right]\cos\theta + f(\theta) \quad \therefore \frac{\partial u}{\partial \theta} = \left[r - \frac{1}{r}\right](-\sin\theta) + f'(\theta). \qquad \text{(iii)}$$

From (ii) and (iii), we get

$$\frac{\partial u}{\partial \theta} = -\left[r + \frac{1}{r}\right]\sin\theta = -\left[r - \frac{1}{r}\right]\sin\theta + f'(\theta) \quad \Rightarrow f'(\theta) = -\frac{2}{r}\sin\theta.$$

$$\therefore f(\theta) = \frac{2}{r}\cos\theta + c$$

Thus $u = \left[r - \frac{1}{r}\right]\cos\theta + \frac{2}{r}\cos\theta + c = \left[r + \frac{1}{r}\right]\cos\theta + c$

Hence $f(z) = u + iv = \left[r + \frac{1}{r}\right]\cos\theta + i\left[r - \frac{1}{r}\right]\sin\theta + c$,

which is the required analytic function.

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LECTUER-6

Conformal Mapping (Transformation):

Suppose two curves C, C_1 in the z-plane intersect at the point P and the corresponding curves C'and C_1' in the w-plane intersect at P'. If the angle of intersection of the curves at P is the **same** as the angle of intersection of the curves at P' in magnitude and sense, then the **transformation** is said to be **conformal** at P.

A transformation, which preserves angles both in magnitude and sense between every pair of curves through a point, is said to conformal at the point. But basically, conformal mapping has two kinds:

1. Conformal mapping of the first kind:

If the conformal mapping preserves angles both in magnitude and sense, then conformal mapping is known as conformal mapping of the first kind.

2. Conformal mapping of the second kind (Isogonal mapping):

If the conformal mapping preserve angles only in magnitude but not in sense, which is reserved, like $w - \bar{z}$, where $\arg \bar{z} = -\arg z$, then conformal mapping is known as conformal mapping of the second kind.

Remarks: Given two mutually orthogonal one-parameter families of curves say $\phi(x, y) = c_1$ and $\psi(x, y) = c_2$, then their image curves in the w-plane $\phi(u, v) = c_3$ and $\psi(u, v) = c_4$ under a conformal mapping are also mutually orthogonal . Thus, conformal mapping preserves the property of mutual orthogonality of system of curves in the plane.

Note: Conformal mapping is used to map complicated regions conformally onto simpler, standard regions such as circular disks, half planes and strips for which the boundary value problems are easier.

Condition under which the transformation $w = f(z)$ is conformal

Condition for Conformality:

Theorem:

A mapping w=f(z) is conformal at each point z_0 where f(z) is analytic and $f'(z_0) \neq 0$.

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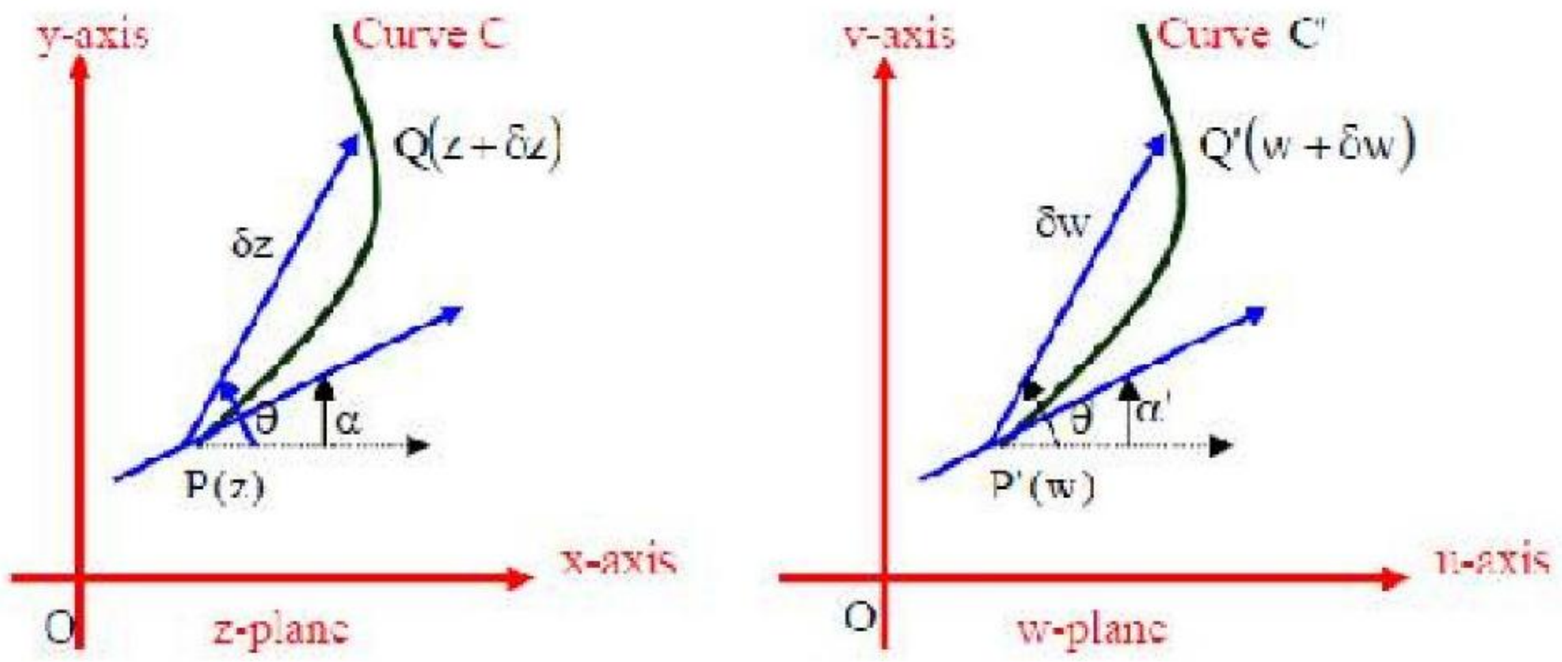
Proof: Given

(i) f(z) is analytic, i.e. f(z) possesses a unique derivative w.r.t. z at all points of a region R

(ii) $f'(z) \neq 0$

Let P(z) be a point in the region R of the z-plane and $P'(w)$ the corresponding point in the region R' of the w-plane. Suppose P(z) moves on a curve C and $P'(w)$ moves on the corresponding curve C'. Let $Q(z-\delta z)$ be a neighbouring point on C and $Q'(w+\delta w)$ be the corresponding neighbouring point on C' so that $\overrightarrow{PQ} - \delta z$ and $\overrightarrow{P'Q'} - \delta w$.



Then, δz is a complex number whose modulus r is the length PQ and amplitude θ is the angle which PQ makes with the x-axis.

$\therefore \delta z = re^{i\theta}$.

Similarly, if the modulus and amplitude of δw be r' and θ', then $\delta w = r'e^{i\theta'}$.

Hence $\dfrac{\delta w}{\delta z} = \dfrac{r'}{r}e^{i(\theta'-\theta)}$.

Let the tangent at P to the curve C makes an angle α with the x-axis and the tangent at P' to C' makes an angle α' with the u-axis, then as $\delta z \to 0$, $\theta \to \alpha$ and $\theta' \to \alpha'$.

$$\therefore f'(z) = \frac{dw}{dz} = \underset{\delta z \to 0}{\mathrm{Lt}} \frac{\delta w}{\delta z} = \underset{\delta z \to 0}{\mathrm{Lt}} \left(\frac{r'}{r}\right) . e^{i(\alpha'-\alpha)}. \qquad \text{[f(z) possesses a unique derivative]} \qquad \text{(i)}$$

Also given $f'(z) \neq 0$, and let us consider $f'(z) = \rho e^{i\phi}$. (ii)

then $\rho = |f'(z)|$ is the modulus and $\phi =$ amplitudeof $f'(z)$.

$\therefore$ From (i), we get $\rho e^{i\phi} = \underset{\delta z \to 0}{\mathrm{Lt}} \left(\dfrac{r'}{r}\right) e^{i(\theta'-\theta)}$.

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$\therefore$ From (i) and (ii), we have $\rho = \underset{\delta z \to 0}{\text{Lt}} \dfrac{r'}{r}$. (iii)

and $\phi = \underset{\delta z \to 0}{\text{Lt}} (\theta' - \theta) = \alpha' - \alpha$. (iv)

Now, let C' be another curve through P in the z-plane and C_1' the corresponding curve through P' in the w-plane. If the tangent at P to C' makes an angle β with the x-axis and tangent at P' to C_1' makes an angle β' with the u-axis, then as in (iv), we get

$$\phi = \beta' - \beta. \qquad \text{(v)}$$

Then, from (iv) and (v), we obtain $\alpha' - \alpha = \beta' - \beta \Rightarrow \beta - \alpha = \beta' - \alpha' = \gamma$.

Thus, the angle between the curves before and after the mapping is preserved in magnitude and sense (direction). Hence, the mapping by the analytic function $w = f(z)$ is conformal at each point where $f'(z) \neq 0$.

Remarks:

(1). Relation (iv), i.e. $\alpha' = \alpha + \phi$ shows that the tangent at P to the curve C is rotated through an angle $\phi = \text{amp}\{f'(z)\}$ under the given transformation.

(2). The relation (iii), i.e. $\rho = \underset{\delta z \to 0}{\text{Lt}} \dfrac{r'}{r}$ shows that in the conformal transformation, the lengths of arcs passing through P any direction are magnified in the ratio $\rho : 1$, where $\rho = |f'(z)|$. Thus, an infinitesimal length in the z-plane is magnified by the factor $|f'(z)|$ in the w-plane and consequently the infinitesimal areas are in the z-plane magnified by the factor $|f'(z)|^2$ in the w-plane in a conformal transformation.

(3). Jacobian of a transformation:

If $w = f(z)$, i.e., $u + iv = f(x + iy)$ is an analytic function which maps a closed region D of the z-plane into a closed region D' of w-plane, then u and v must satisfy C-R equation.

$$\therefore J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{-\partial v}{\partial x} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial u}{\partial x} \end{vmatrix} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = \left|\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right| = |f'(z)|^2.$$

Hence, in a conformal transformation, infinitesimal areas are magnified by the factor $J\left(\dfrac{u, v}{x, y}\right)$. Also the condition of a conformal mapping is $J\left(\dfrac{u, v}{x, y}\right) \neq 0$.

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Conformal Mapping by Elementary Functions:

General Linear Transformation or Linear Transformation

General Linear Transformation or simply linear transformation defined by the function

$$w = f(z) = az + b \qquad (i)$$

($a \neq 0$, and b are arbitrary complex constants) maps conformally the extended complex z-plane onto the extended w-plane, since this function is analytic and $f'(z) = a \neq 0$ for any z. If a = 0, (i) reduces to a constant function.

Special cases of linear transformation are:

i. Identity Transformation:

$$w = z$$

for a = 1, b = 0, which maps a point z onto itself.

ii. Translation:

$$w = z + b$$

for a = 1, which translates (shifts) z through a distance $|b|$ in the direction b.

If $z = x + iy$, $b = b_1 + ib_2$ and $w = u + iv$,

then the transformation becomes $u + iv = x + iy + b_1 + ib_2 \Rightarrow u = x + b_1$ and $v = y + b_2$,

i.e., the point P(x, y) in the z-plane is mapped onto the point $P'(x + b_1, y + b_2)$ in the w-plane. Every point in the z-plane is mapped onto w-plane in the same way. Thus, if the w-plane is superposed on the z-plane, figure is shifted through a distance given by the vector b.

Accordingly, this transformation maps a figure in the z-plane into a figure in the w-plane of the same shape and size. Thus, this transformation is a mere translation of the axis and preserves the shape and size. In particular, this transformation changes circles into circles.

iii. Magnification and Rotation:

$$w = e^{i\theta}.z$$

for $a = e^{i\theta}, b = 0$ which rotates (the radius vector of point) z through a scalar angle θ (counter clockwise if $\theta > 0$ while clockwise if $\theta < 0$)

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Let $w = cz$, where c is complex constant.

Let $c = \rho e^{i\alpha}$, $z = re^{i\theta}$ and $w = Re^{i\phi}$, then the transformation becomes

$Re^{i\phi} = \rho e^{i\alpha} re^{i\theta} = \rho r e^{i(\theta+\alpha)} \rightarrow R = \rho r$ and $\phi = \theta + \alpha$.

Thus, the transformation maps a point $P(r, \theta)$ in the z-plane onto the point $P'(\rho r, \theta + \alpha)$ in the w-plane. Hence this transformation consists of magnification (or contraction) of the radius vector P by $\rho = |c|$ and its rotation through an angle $\alpha = \text{amp}(c)$.

Accordingly, it maps any figure in the z-plane into a geometrically similar figure in the w-plane. In particular, this transformation maps circles into circles.

iv. Stretching (scaling):

$$w = az$$

for 'a' real stretches if $a > 1$ (contracts if $0 < a < 1$) the radius vector by factor 'a'.

Thus, the linear transformation $w = f(z) = az + b$ consists of rotation through angle arg a, scaling by factor $|a|$, followed by translation through vector b. This transformation is used for constructing conformal mapping of 'similar' figures.

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Result: Show that circles are invariant under translation, rotation and stretching.

i.e., Linear transformation preserves circles i.e., a circle in the z-plane under transformation maps to a circle in the w-plane.

Proof: Consider any circle in the z-plane

$$A\left(x^2+y^2\right)+Bx+Cy+D=0 \qquad \text{(i)}$$

From $w=f(z)=az+b$, we get

$$u+iv=w=az+b=a(x+iy)+(b_1+ib_2)\Rightarrow u=ax+b_1,\ \ v=ay+b_2$$

$$\Rightarrow x=\frac{u-b_1}{a},\ \ y=\frac{v-b_2}{a},\ \ a\neq 0 \qquad \text{(ii)}$$

Substituting (ii) in (i), we get

$$A^*\left(u^2+v^2\right)+B^*u+c^*v+D^*=0 \qquad \text{(iii)}$$

which is a circle in the w-plane.

Here

$$A^*=\frac{A}{a^2},\ \ B^*=\frac{B-2Ab_1}{a},\ \ C^*=\frac{C-2Ab_2}{a}$$

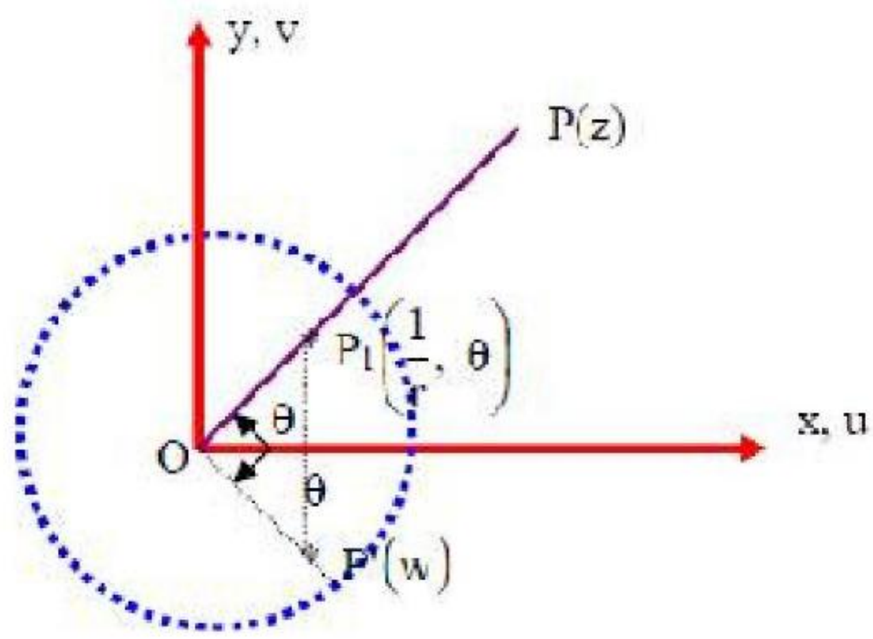
$$D^*=D+A=\left(\frac{b_1^2+b_2^2}{a^2}\right)-\frac{Bb_1}{a}-\frac{Cb_2}{a}.$$

Thus, circles are invariant under translation, rotation and stretching.

Some standard transformations:

Inversion and Reflection: $w=\frac{1}{z}$.

Here, it is convenient to think the w-plane as superposed on z-plane.



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Let $z = re^{i\theta}$ and $w = Re^{i\phi}$, then the transformation $w = \frac{1}{z}$ becomes $Re^{i\phi} = \frac{1}{r}e^{-i\theta}$

$\Rightarrow R = \frac{1}{r}$ and $\phi = -\theta$.

Thus, under the transformation $w = \frac{1}{z}$, a point $P(r, \theta)$ in z-plane is mapped into the point $P_1\left(\frac{1}{r}, -\theta\right)$.

Consider the w-plane superposed on the z-plane. If P is (r, θ) and $P_1 = \left(\frac{1}{r}, -\theta\right)$, then

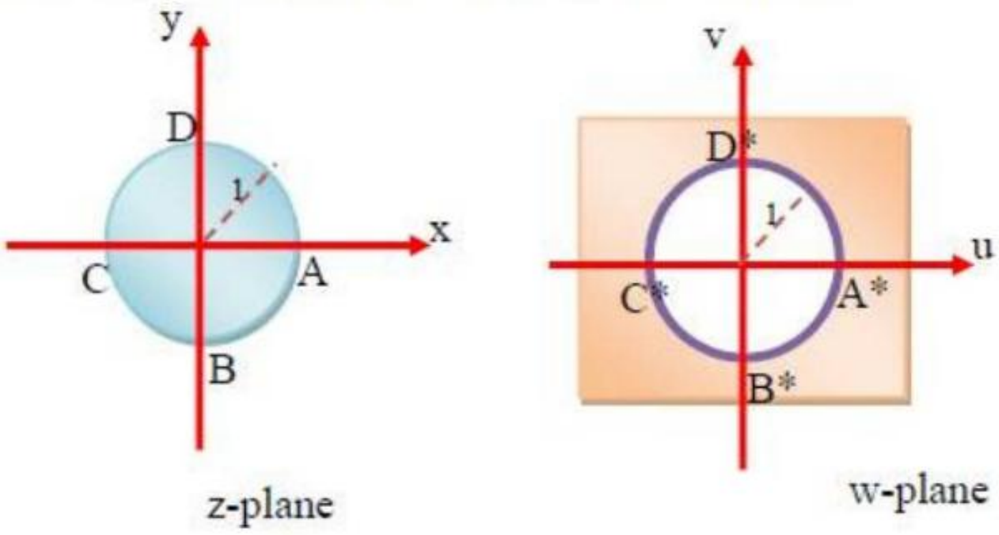
$OP_1 = \frac{1}{r} = \frac{1}{OP} \Rightarrow OP.OP_1 = 1$

$\Rightarrow P_1$ is the inverse of P w.r.t. the unit circle with centre O.

Then the reflection P' of P_1 in the real axis represents $w = \frac{1}{z}$.

Hence, this transformation is an inversion of z w.r.t. the unit circle $|z| = 1$ followed by reflection of the inverse into the real axis.

Clearly, the function $w = \frac{1}{z}$ maps the interior of the unit circle $|z| = 1$ onto the exterior of the unit circle $|w| = 1$ and the exterior of $|z| = 1$ onto the interior of $|w| = 1$.



However, the origin $z = 0$ is mapped to the point $w = \infty$, called the point at infinity.

Result: Prove that circles are invariant under $w = \frac{1}{z}$.

or

Show that the transformation $w = \frac{1}{z}$ always maps circles into circles.

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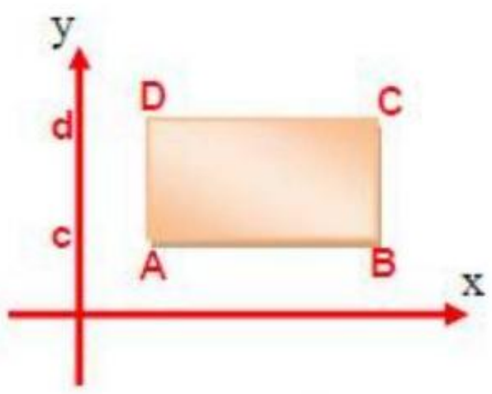
Transformation $w = e^z$:

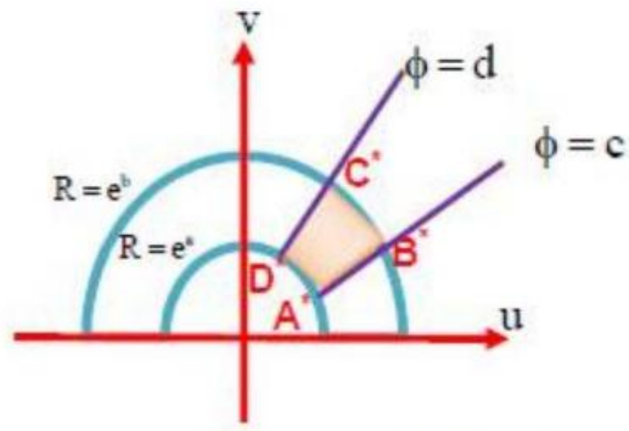
Rewriting

$Re^{i\phi} = w = e^z = e^{x+iy} = e^x.e^{iy}$

Therefore $R = e^x$ and $\phi = y$ (i)

i.e., modulus of w is e^x and argument of w is y. The line x = c = constant maps onto the circle $R = e^c$.





The line y = c maps on to the ray $\phi = c$. Thus, the region $a \le x \le b$, $c \le y \le d$ in the z-plane is mapped to the region A*B*C*D* in the w-plane bounded by the concentric circles $R = e^b$ and $R = e^a$ and by the rays $\phi = c$ and $\phi = d$.

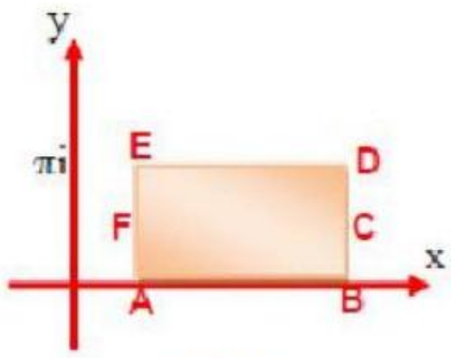
Note 1: Since $e^z \ne 0$, w = 0 is not mapped. Thus, the origin in w-plane is excluded.

Note 2: This mapping is one to one if $d - c < 2\pi$.

Particular case: $c = 0$, $d = \pi$.

Consider the rectangular region in z-plane

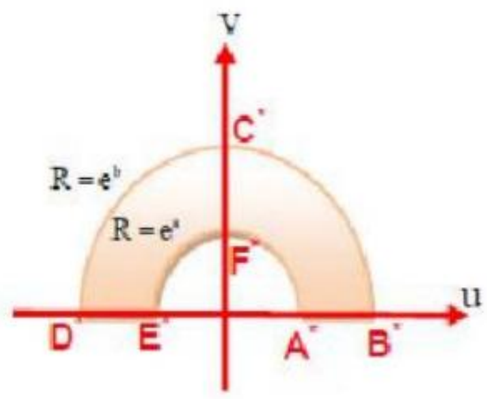
$a \le x \le b$, $0 \le y \le \pi$ (see figure)



z-plane

By (i), $e^a \le R = e^x \le e^b$ and $0 \le \phi = y \le \pi$.

Thus, the rectangular region maps onto the upper half of the annulus ring $e^a < R < e^b$, $0 \le \phi \le \pi$. (see figure)



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LECTUER-7

Bilinear Transformation or Mobius Transformation:

The transformation $w = \frac{az+b}{cz+d}$, (i)

where a, b, c and d are complex constants and $ad - bc \neq 0$ is called as **Bilinear transformation or Mobius transformation or linear fractional transformation.**

In other words, Bilinear transformation is the function w of a complex variable z of the form $w = f(z) = \frac{az+b}{cz+d}$.

Bilinear transformation is conformal for all z:

Differentiating (i) w.r.t., z, we get

$$\frac{dw}{dz} = \frac{ad-bc}{(cz+d)^2}$$

If $ad - bc \neq 0$, then $\frac{dw}{dz} \neq 0$ for any z and therefore Bilinear transformation is conformal for all z, i.e., it maps z-plane conformally onto the w-plane,

Thus, the condition $ad - bc \neq 0$ ensures that $\frac{dw}{dz} \neq 0$, i.e., the transformation is conformal.

If $ad - bc = 0$, then $\frac{dw}{dz} = 0$ for any z. Then every point of the z-plane is critical point and the function is not conformal.

Special cases of Bilinear transformation:

For a choice of constants a, b, c, d, we get special cases of Bilinear transformation as

$w = z + b$ Translation

$w = az$ Rotation

$w = az + b$ Linear transformation

$w = \frac{1}{z}$ inversion in the unit circle.

Thus B.T. can be considered as combination of these transformations.

Inverse of bilinear transformation $w = \frac{az+b}{cz+d}$:

Solving (i) for z, we find that inverse of the Bilinear transformation is

$$z = \frac{-dw+b}{cw-a} \qquad \text{(ii)}$$

which is also a bilinear transformation.

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Remarks:

(1). From (i), we see that each point in the z-plane except $z = -\frac{d}{c}$, maps into a unique point in the w-plane. Similarly, (ii) shows that each point in the w-plane except $w = \frac{a}{c}$, maps into a unique point in the z-plane. Considering the two exceptional points as points at infinity in the two planes, we can say that there is one to one correspondence between all points in the two planes.

From (i), observe that the point $z = -\frac{d}{c}$ corresponds to $w = \infty$, point at infinity in the w-plane. Similarly from (ii), the point $w = \frac{a}{c}$ corresponds to $z = \infty$, point at infinity in the z-plane.

(2). Invariant points of bilinear transformation:

If z maps into itself in the w-plane

(i.e. $w = z$), then $z = \frac{az+b}{cz+d} \Rightarrow z = \frac{az+b}{cw+d} \Rightarrow cz^2 + (d-a)z - b = 0$.

The roots of this equation (say: z_1, z_2) are defined as the invariant or fixed points of the bilinear transformation $z = \frac{az+b}{cz+d}$.

In other words, fixed (or invariant) points of function $w = f(z)$ are points which are mapped onto themselves i.e., $w = f(z) = z$.

Example:

$w = z$ has every point a fixed point

$w = \bar{z}$ infinitely many

$w = \frac{1}{z}$ has two

$w = z + b$ has no fixed point

To obtain the fixed points of $w = \frac{az+b}{cz+d}$, solve

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$z = \dfrac{az+b}{cz+d}$, which is a quadratic in z given by

$$cz^2 - (a-d)z - b - 0. \qquad \text{(i)}$$

Thus, the roots, say α, β of (i) are fixed points of $w = \dfrac{az+b}{cz+d}$.

If two roots of (i) are equal then bilinear transformation is said to be parabolic.

The quadratic with α, β as roots is $z^2 - (\alpha+\beta)z + \alpha\beta = 0$.

For any complex constant γ,

$$z^2 - (\alpha+\beta)z + \gamma z - \gamma z + \alpha\beta = 0$$

$$z(z - (\alpha+\beta-\gamma)) - \gamma z - \alpha\beta$$

$$z = \frac{\gamma z - \alpha\beta}{z - (\alpha+\beta) + \gamma}.$$

Thus, the bilinear transformations, whose fixed points α, β are given by

$$w = \frac{\gamma z - \alpha\beta}{z - (\alpha+\beta) + \gamma}. \qquad \text{(ii)}$$

For various values γ, (ii) gives B.T. with fixed points α, β.

(3). In the linear transformation $w = \dfrac{az+b}{cz+d}$, $ad - bc \neq 0$, dividing the numerator and denominator of the right side by one of the four constants, we observe that there are only three independent constants. Hence, three conditions are required to determine a bilinear transformation. For instance, three distinct points z_1, z_2, z_3 can be mapped into any three specified points w_1, w_2, w_3.

Cross-ratio or anharmonic ratio of four numbers:

The cross-ratio or anharmonic ratio of four numbers z_1, z_2, z_3, z_4 is the linear fraction given by

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)}$$

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Theorem: A bilinear transformation preserves cross-ratio of four points.

or

The cross ratio is invariant under a bilinear transformation.

Let the points z_1, z_2, z_3, z_4 of the z-plane map onto the points w_1, w_2, w_3, w_4 of the w-plane respectively under the bilinear transformation $w = \frac{az+b}{cz+d}$.

This means that w_1, w_2, w_3, w_4 are respectively the images of z_1, z_2, z_3, z_4 under the bilinear transformation $w = \frac{az+b}{cz+d}$.

If these points are finite, then from $w = \frac{az+b}{cz+d}$, we have

$$w_j - w_k = \frac{az_j+b}{cz_j+d} - \frac{az_k+b}{cz_k+d} = \frac{ad-bc}{(cz_j+d)(cz_k+d)}(z_j - z_k)$$

Using this relation for j, k = 1, 2, 3, 4, we get

$$\frac{(w_1-w_2)(w_3-w_4)}{(w_1-w_4)(w_3-w_2)} = \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)}$$

Thus the cross-ratio of four points is invariant under bilinear transformation.

This property is very useful in finding a bilinear transformation.

Determination of Bilinear Transformation:

A bilinear transformation can be uniquely determined by three given conditions. Although four constants a, b, c, d appear in (i), essentially they are three ratio of three of these constants to the fourth one.

To find the unique bilinear transformation which maps three given points z_1, z_2, z_3 onto three distinct images w_1, w_2, w_3, consider w which is the image of a general point z under this transformation. Now as we know the cross-ratio is preserved under bilinear transformation, so the cross-ratio of the four points w_1, w_2, w_3, w must be equal to the cross-ratio of z_1, z_2, z_3, z.

Hence, the unique bilinear transformation that maps three given points z_1, z_2, z_3 onto three given images w_1, w_2, w_3 is given by

$$\frac{(w_1-w_2)(w_3-w)}{(w_1-w)(w_3-w_2)} = \frac{(z_1-z_2)(z_3-z)}{(z_1-z)(z_3-z_2)}$$

Note 1: If one of the points, say: $z_1 \to \infty$, the quotient of those two differences which contain z_1, is replaced by 1 i.e., $\frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_4)(z_3-z_2)} = \frac{(z_3-z_4)}{(z_3-z_2)}$.

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Lecture-1

Line Integrals

The basic theme here is that complex line integrals will mirror much of what we've seen for multivariable calculus line integrals. But, just like working with $e^{i\theta}$ is easier than working with sine and cosine, complex line integrals are easier to work with than their multivariable analogs. At the same time they will give deep insight into the workings of these integrals.

- The complex plane: $z = x + iy$
- The complex differential $dz = dx + idy$
- A curve in the complex plane: $\gamma(t) = x(t) + iy(t)$, defined for $a \leq t \leq b$.
- A complex function: $f(z) = u(x, y) + iv(x, y)$

Note. Line integrals are also called path or contour integrals.

Given the ingredients we define the complex line integral $\int_\gamma f(z)\,dz$ by

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt. \tag{1}$$

You should note that this notation looks just like integrals of a real variable. We don't need the vectors and dot products of line integrals in $\mathbf{R}^2$. Also, make sure you understand that the product $f(\gamma(t))\gamma'(t)$ is just a product of complex numbers.

An alternative notation uses $dz = dx + idy$ to write

$$\int_\gamma f(z)\,dz = \int_\gamma (u + iv)(dx + idy) \tag{2}$$

Let's check that Equations 1 and 2 are the same. Equation 2 is really a multivariable calculus expression, so thinking of $\gamma(t)$ as $(x(t), y(t))$ it becomes

$$\int_\gamma f(z)\,dz = \int_a^b (u(x(t), y(t)) + iv(x(t), y(t))(x'(t) + iy'(t))dt$$

But, $u(x(t), y(t)) + iv(x(t), y(t)) = f(\gamma(t))$ and $x'(t) + iy'(t) = \gamma'(t)$ so the right hand side of this equation is

$$\int_a^b f(\gamma(t))\gamma'(t)\,dt.$$

That is, it is exactly the same as the expression in Equation 1.

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Example 3.1. Compute $\int_\gamma z^2 dz$ along the straight line from 0 to $1+i$.

<u>answer:</u> We parametrize the curve as $\gamma(t) = t(1+i)$ with $0 \le t \le 1$. So $\gamma'(t) = 1+i$. The line integral is

$$\int z^2 dz = \int_0^1 t^2(1+i)^2(1+i)\,dt = \frac{2i(1+i)}{3}.$$

Example 3.2. Compute $\int_\gamma \overline{z}\,dz$ along the straight line from 0 to $1+i$.

<u>answer:</u> We can use the same parametrization as in the previous example. So,

$$\int_\gamma \overline{z} dz = \int_0^1 t(1-i)(1+i)\,dt = 1.$$

Example 3.3. Compute $\int_\gamma z^2\,dz$ along the unit circle.

<u>answer:</u> We parametrize the unit circle by $\gamma(\theta) = e^{i\theta}$, where $0 \le \theta \le 2\pi$. We have $\gamma'(\theta) = ie^{i\theta}$. So, the integral becomes

$$\int_\gamma z^2\,dz = \int_0^{2\pi} e^{2i\theta} ie^{i\theta}\,d\theta = \int_0^{2\pi} ie^{3i\theta}\,d\theta = \frac{e^{i3\theta}}{3}\Big|_0^{2\pi} = 0.$$

Example 3.4. Compute $\int \overline{z} dz$ along the unit circle.

<u>answer:</u> Parametrize C: $\gamma(t) = e^{it}$, with $0 < t < 2\pi$. So, $\gamma'(t) = ie^{it}$. Putting this into the integral gives

$$\int_C \overline{z}\,dz = \int_0^{2\pi} \overline{e^{it}}\, i\, e^{it}\,dt = \int_0^{2\pi} i\,dt = \boxed{2\pi i.}$$

<u>Lecture-2</u>

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Cauchy Integral Theorem

Let f be holomorphic inside Ω be bounded by a closed piecewise-smooth simple curve γ and also at the point of γ. Then

$$\oint_\gamma f(z)\,\mathrm{d}z = 0$$

Proof. Let $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$, so

$$\begin{aligned}\oint_\gamma f(z)\,\mathrm{d}z &= \oint_\gamma (u+iv)(\mathrm{d}x + i\,\mathrm{d}y)\\ &= \oint_\gamma u\,\mathrm{d}x - v\,\mathrm{d}y + i\oint_\gamma v\,\mathrm{d}x + u\,\mathrm{d}y\end{aligned}$$

and by Green's formula

$$= \iint_\Omega (-v'_x - u'_y)\,\mathrm{d}x\,\mathrm{d}y + i\iint_\Omega (u'_x - v'_y)\,\mathrm{d}x\,\mathrm{d}y = 0$$

Deformation

Let γ_1 and γ_2 be two simple, closed, piecewise-smooth curves with γ_2 lying wholly inside γ_1 and suppose f is holomorphic in a domain containing the region between γ_1 and γ_2. Then

$$\oint_{\gamma_1} f(z)\,\mathrm{d}z = \oint_{\gamma_2} f(z)\,\mathrm{d}z$$

Proof.

Cauchy-Goursat implies

$$\oint_{\tilde{\gamma}} f(z)\,\mathrm{d}z = 0, \quad \oint_{\tilde{\tilde{\gamma}}} f(z)\,\mathrm{d}z = 0$$

Then

$$\begin{aligned}0 &= \oint_{\tilde{\gamma}} f(z)\,\mathrm{d}z + \oint_{\tilde{\tilde{\gamma}}} f(z)\,\mathrm{d}z\\ &= \oint_{\gamma_1} f(z)\,\mathrm{d}z - \oint_{\gamma_2} f(z)\,\mathrm{d}z\end{aligned}$$

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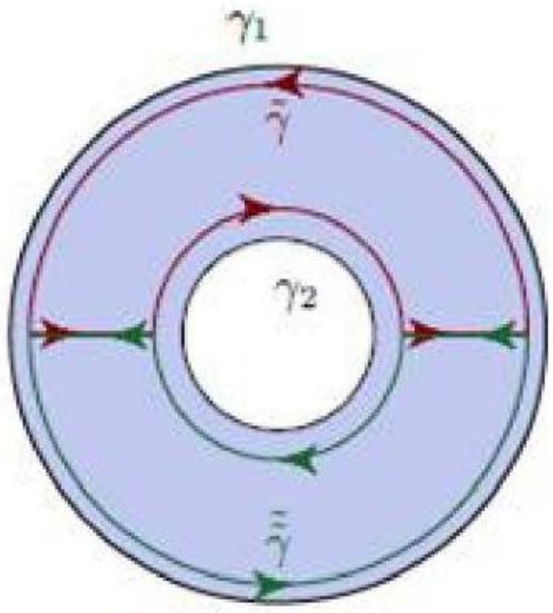
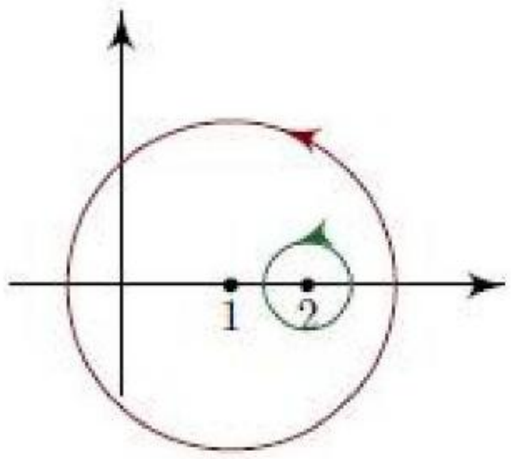


Picture

Example Take $\gamma = \{z \in \mathbb{C} : |z-1| = 2\}$

$$\oint_\gamma \frac{1}{z^2-4}\,\mathrm{d}z = \oint_\gamma \frac{1}{(z-2)(z+2)}\,\mathrm{d}z = \frac{1}{4}\oint_\gamma \left(\frac{1}{z-2} - \frac{1}{z+2}\right)\mathrm{d}z$$



Since $\dfrac{1}{z+2}$ is holomorphic inside γ and on γ, by Cauchy Goursat:

$$\oint_\gamma \frac{1}{z+2}\,\mathrm{d}z = 0$$

Now

$$\oint_\gamma \frac{1}{z-2}\,\mathrm{d}z = \oint_{z:|z-2|=\frac{1}{2}} \frac{1}{z-2}\,\mathrm{d}z$$

Changing variables $z-2=w$

$$= \oint_{|w|=\frac{1}{2}} \frac{1}{w}\,\mathrm{d}w$$

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Parametrising $w = \frac{1}{2}e^{it}$, $t \in [0, 2\pi]$

$$= \int_0^{2\pi} \frac{1}{\frac{1}{2}e^{it}} \cdot \frac{1}{2} i e^{it} \, \mathrm{d}t = 2\pi i$$

Hence

$$\oint_\gamma \frac{1}{z^2 - 4} \, \mathrm{d}z = i\frac{\pi}{2}$$

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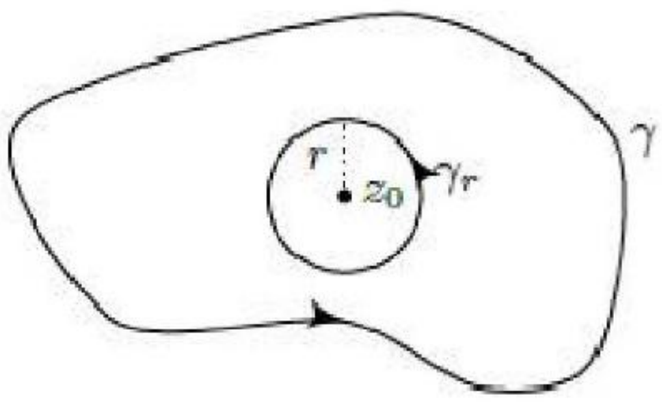
Lecture-3

Cauchy integral Formula

Let f be holomorphic inside and on a simple, closed piecewise-smooth curve γ. Then for any point z_0 interior to γ, we have

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} \, \mathrm{d}z$$



Proof. If z_0 is interior to γ, then for any $r > 0$ such that

$$\gamma_r = \{z : |z - z_0| = r\} \subset \Omega$$

we obtain by using the Deformation Theorem

$$\oint_\gamma \frac{f(z)}{z - z_0} \, \mathrm{d}z = \oint_{\gamma_r} \frac{f(r)}{z - z_0} \, \mathrm{d}z$$

We have

$$\frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(z)}{z - z_0} \, \mathrm{d}z = \frac{1}{2\pi i} f(z_0) \oint_{\gamma_r} \frac{1}{z - z_0} \, \mathrm{d}z \qquad (I_1)$$

$$+ \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0} \, \mathrm{d}z \qquad (I_2)$$

For I_1 we note that

$$\oint_{\gamma_r} \frac{1}{z - z_0} \, \mathrm{d}z = \oint_{z:|z-z_0|=r} \frac{1}{z - z_0} \, \mathrm{d}z$$

Write $z - z_0 = re^{it}$ for $t \in [0, 2\pi]$

$$= \int_0^{2\pi} \frac{1}{re^{it}} \cdot rie^{it} \, \mathrm{d}t = 2\pi i$$

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This implies

$$I_1 = \frac{1}{2\pi i} f(z_0) \oint_{\gamma_r} \frac{1}{z - z_0} \, \mathrm{d}r = f(z_0)$$

For I_2 we use the following fact: f is holomorphic $\implies$ it is continuous at $z_0 \implies \forall \epsilon > 0, \exists \delta \, (=: r) > 0$ such that as soon as $|z - z_0| < \delta$ we have

$$|f(z) - f(z_0)| < \epsilon \qquad (r < \delta)$$

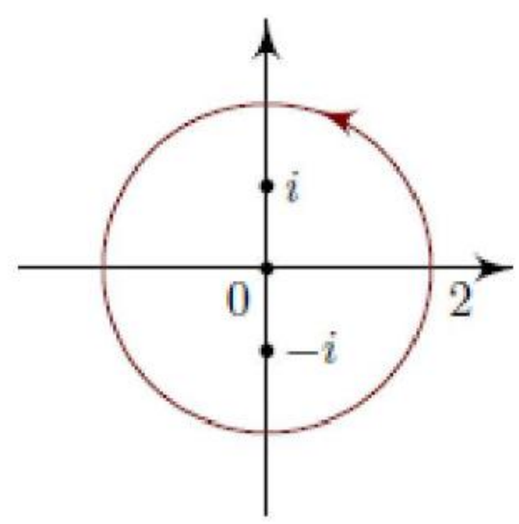
Then

$$\begin{aligned} |I_2| &= \left| \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(z) - f(z_0)}{z - z_0} \, \mathrm{d}z \right| \\ &\leq \frac{1}{2\pi} \max_{z : |z - z_0| = r} \frac{|f(z) - f(z_0)|}{|z - z_0|} \cdot 2\pi r \\ &\leq \frac{\epsilon}{r} \cdot r = \epsilon \end{aligned}$$

So $I_2 = 0$, and the theorem follows.

Example Calculate

$$I = \frac{1}{2\pi i} \oint_{|z|=2} \frac{e^z}{(z - i)(z + i)} \, \mathrm{d}z$$



Using partial fractions, we get

$$I = \frac{1}{2\pi i} \cdot \frac{1}{2i} \oint_{|z|=2} \left(\frac{e^z}{z - i} - \frac{e^z}{z + i} \right) \mathrm{d}z$$

By Cauchy's integral formula:

$$\begin{aligned} \frac{1}{2\pi i} \oint_{|z|=2} \frac{e^z}{z - i} \, \mathrm{d}z &= e^i \\ \frac{1}{2\pi i} \oint_{|z|=2} \frac{e^z}{z + i} \, \mathrm{d}z &= e^{-i} \end{aligned}$$

And so

$$I = \frac{1}{2i}(e^i - e^{-i}) = \sin 1$$

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Generalized Cauchy Formula

Let f be holomorphic in an open set Ω. Then f has infinitely many complex derivatives in Ω. Moreover for a simple, closed piecewise-smooth curve $\gamma \subset \Omega$ and any z lying inside γ we have

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n} f(z) = \frac{n!}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z)^{n+1}} \, \mathrm{d}\eta$$

and so

$$\frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} \to \frac{(n-1)!}{2\pi i} \oint_\gamma f(z) \frac{1}{(\eta - z)^2} \cdot \frac{n}{(\eta - z)^{(n-1)}} \, \mathrm{d}\eta$$

$$= \frac{n!}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z)^{n+1}} \, \mathrm{d}\eta \qquad \blacksquare$$

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Lecture-4

Complex Power Series

Definition. A complex power series is a series of the form

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \dots$$

where $z_0 \in \mathbb{C}$ and a_n are complex constants.

Example

$$\sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!}$$

In this case,

$$\begin{aligned} L &= \lim_{n\to\infty} \left| \frac{(z-z_0)^{n+1}}{(n+1)!} \cdot \frac{n!}{(z-z_0)^n} \right| \\ &= \lim_{n\to\infty} \frac{|z-z_0|}{n+1} = 0 \end{aligned}$$

Therefore $\sum_{n=0}^{\infty} \frac{(z-z_0)^n}{n!}$ converges for all $z \in \mathbb{C}$.

Definition (Radius of convergence)**.** There is a real number $R \in \{[0,\infty]\}$ such that $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges for all $z : |z-z_0| < R$ and diverges for all $z : |z-z_0| > R$ and may or may not converge for $z : |z-z_0| = R$. Such value of R is called the radius of convergence for the power series.

Remark. If $R = 0$ then the power series converges only at $z = z_0$. If $R = \infty$ the power series converges absolutely for all $z \in \mathbb{C}$.

For many power series the radius of convergence can be calculated according to

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \qquad (**)$$

or

$$R = \lim_{n\to\infty} \frac{1}{|a_n|^{1/n}}$$

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provided either limit exists or is equal to infinity. Indeed, let us check (**): we know that the series converges if

$$L = \lim_{n\to\infty} \frac{|a_{n+1}(z-z_0)^{n+1}|}{a_n(z-z_0)^n|} = |z-z_0| \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$$

This implies that if

$$|z-z_0| < \lim_{n\to\infty} |a_n|/|a_{n+1}|$$

then the power series converges.

Example Find R for $\sum_{n=1}^{\infty} n^2(z-z_0)^n$.

$$R = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$$

$\implies \sum_{n=1}^{\infty} n^2(z-z_0)^n$ is absolutely convergent for all $z : |z-z_0| < 1$.

Taylor Expansion

Let f be holomorphic in an open set Ω and let $z_0 \in \Omega$. Then

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \dots$$

valid in all circles $\{z \in \mathbb{C} : |z-z_0| < r\} \subset \Omega$.

Proof. Let $\gamma = \{\eta \in \mathbb{C} : |\eta - z_0| = r\} \subset \Omega$ and let $z : |z-z_0| < r$. We calculate

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i} \oint_\gamma \frac{f(\eta)}{\eta - z}\, \mathrm{d}\eta \\
&= \frac{1}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z_0) - (z - z_0)}\, \mathrm{d}\eta \\
&= \frac{1}{2\pi i} \oint_\gamma \frac{f(\eta)}{\eta - z_0} \cdot \frac{1}{1 \cdot \frac{z-z_0}{\eta - z_0}}\, \mathrm{d}\eta \\
&= \frac{1}{2\pi i} \oint_\gamma \frac{f(\eta)}{\eta - z_0} \left[1 + \frac{z-z_0}{\eta - z_0} + \frac{(z-z_0)^2}{(\eta - z_0)^2} + \dots + \frac{(z-z_0)^n}{(\eta - z_0)^{n-1}} + \frac{(\frac{z-z_0}{\eta - z_0})^n}{1 - \frac{z-z_0}{\eta - z_0}} \right] \mathrm{d}\eta
\end{aligned}$$

We can calculate each integral:

(i) $\frac{1}{2\pi i} \oint_\gamma \frac{f(\eta)}{\eta - z_0} \cdot 1\, \mathrm{d}\eta = f(z_0)$

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(ii) $\frac{1}{2\pi i}\oint_\gamma \frac{f(\eta)\cdot(z-z_0)}{(\eta-z_0)^2}\cdot 1\ \mathrm{d}\eta = \frac{1}{1!}f'(z_0)(z-z_0)$

(iii) ...

And so

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \cdots + \frac{f^{(n-1)}(z_0)}{(n-1)!}(z-z_0)^{n-1} + R_n$$

where

$$R_n = \frac{(z-z_0)^n}{2\pi i}\oint_\gamma \frac{f(\eta)}{(\eta-z)(\eta-z_0)^n}\ \mathrm{d}\eta$$

Let $\rho = |z-z_0|$. We have to show that $R_n \to 0$ as $n\to\infty$.

$$|R_n| \le \frac{\rho^n}{2\pi}\cdot\frac{M}{r^n}\cdot\frac{1}{r-\rho}2\pi r$$

Where $M = \max_{\eta\in\gamma}|f(\eta)|$, $|\eta - z_0| = r$ and $\min_{\eta\in\gamma}|\eta - z| = r-\rho$. We know that $\rho < r \implies \frac{\rho^n}{r^n}\to 0$ as $n\to\infty \implies R_n \to 0$ as $n\to\infty$. ■

Definition. The expansion

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \ldots$$

is called the Taylor series of f about z_0. The special case in which $z_0 = 0$ we have

$$f(z) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}z^n$$

this series is called the Maclaurin series for f.

Examples

(i) $f(z) = e^z; z_0 = 0 \implies f^{(n)}|_{z=0} = 1$. So

$$e^z = \sum_{n=0}^{\infty}\frac{1}{n!}z^m, \quad R = \lim_{n\to\infty}\frac{(n+1)!}{n!} = \infty$$

(ii) $$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, |z|<1 \quad \text{Then } R = 1.$$

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(iii) $\mathrm{Log}(1-z)$. Note that

$$(\mathrm{Log}(1-z))' = -\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n$$

Integrating both sides we get

$$\mathrm{Log}\,(1-z) = -\sum_{n=0}^{\infty} \frac{1}{n+1} z^{n+1} + C = -\sum_{n=1}^{\infty} \frac{1}{n} z^n + C$$

where $C = \mathrm{Log}\,(1-0) = 0$.

(iv) $f(z) = \frac{1}{1+z}$ about $z_0 = i$. Then

$$\begin{aligned}
\frac{1}{1+z} &= \frac{1}{1+i+z-i} \\
&= \frac{1}{1+i} \cdot \frac{1}{1-(-\frac{z-i}{1+i})} \\
&= \frac{1}{1+i} \sum_{n=0}^{\infty} (-1)^n \frac{(z-i)^n}{(1+i)^n} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{1}{(1+i)^{n+1}} (z-i)^n
\end{aligned}$$

where R is defined by the inequality

$$\frac{|z-i|}{1+i} < 1 \text{ or } |z-i| < \sqrt{2}$$

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Lecture-5

Laurent Expansion

Let f be holomorphic in the annulus

$$D = \{z : r < |z - z_0| < R\}, \quad r, R > 0$$

Then $f(z)$ can be expressed in the form $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$, where

$$a_n = \frac{1}{2\pi i} \oint_\gamma \frac{f(\eta)}{(\eta - z_0)^{(n+1)}} \, d\eta$$

and whose γ is any simple, closed, piecewise-smooth curve in D that contains z_0 in its interior.

Examples

(i) Find the Laurent series at $z_0 = 0$ for $f(z) = 1/(z-1)$ for $z : |z| > 1$.

$$\frac{1}{z-1} = \frac{1}{z(1-1/z)} = \frac{1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{k=1}^{\infty} \frac{1}{z^k}$$

This converges for $|z| > 1$.

(ii) Find the Laurent series at $z_0 = 0$ for $f(z) = 1/z(z+2)$ for $0 < |z| < 2$.

$$\begin{aligned}
\frac{1}{z(z+2)} &= \frac{1}{2}\left(\frac{1}{z} - \frac{1}{z+2}\right) \\
&= \frac{1}{2} \cdot \frac{1}{z} - \frac{1}{4(1+z/2)} \\
&= \frac{1}{2} \cdot \frac{1}{z} - \frac{1}{4} \sum_{n=0}^{\infty} (-z/2)^n \\
&= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{z^n}{2^{n+2}} + \frac{1}{2} \cdot \frac{1}{z}
\end{aligned}$$

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Lecture-6

Singularities

Singularity . A point z_0 is called a *singularity* of a complex function f it f is not holomorphic at z_0, but every neighbourhood of z_0 contains at least one point at which f is holomorphic.

Isolated Singularity . A singularity z_0 of a complex function is said to be *isolated* if there exists a neighbourhood of z_0 in which z_0 is the only singularity of f.

Examples (Singularities).

(i) $f(z) = \frac{1}{1-z}, z_0 = 1.$

(ii) $f(z) = e^{1/z^2}, z_0 = 0,$

(iii) $f(z) = \frac{1}{(z+2)^2}, z_0 = -2.$

Definition (Pole of order m, removable and essential singularity). Suppose a holomorphic function f has an isolated singularity at z_0 and $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ is the Laurent expansion of f valid in some annulus $0 < |z - z_0| < R$. Then

(i) If $a_n = 0$ for all $n < 0$, z_0 is called a *removable singularity.*

(ii) If $a_n = 0$ for $n < -m$ where m is a fixed positive integer, but $a_{-m} \neq 0$, z_0 is called a *pole of order m.*

(iii) If $a_n \neq 0$ for infinitely many negative n's, z_0 is called an *essential singularity.*

Examples (i) $f(z) = \frac{\sin z}{z}$, (ii) $f(z) = \frac{1}{z^3(z+2)^2}$, (iii) $f(z) = e^{1/z}$.

Theorem

A function f has a pole of order m at z_0 if and only if it can be written in the form

$$f(z) = \frac{g(z)}{(z-z_0)^m}$$

where g is holomorphic at z_0 and $g(z_0) \neq 0$.

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Proof. If g is holomorphic at z_0 and $g(z_0) \neq 0$ then for some $R > 0$,

$$g(z) = a_0 + a_1(z - z_0) + \dots, \quad |z - z_0| < R$$

where $a_0 = g(z_0) \neq 0$. Then

$$f(z) = \frac{a_0}{(z - z_0)^m} + \frac{a_1}{(z - z_0)^{m-1}} + \dots \quad 0 < |z - z_0| < R$$

This implies that z_0 is a pole of order m.

Conversely if f has a pole of order m at z_0, then the Laurent expansion of f about z_0 is

$$\begin{aligned} f(z) &= \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-(m-1)}}{(z - z_0)^{m-1}} + \dots \\ &= \frac{1}{(z - z_0)^m}(a_{-m} + a_{-(m-1)}(z - z_0) + \dots) \end{aligned}$$

■

Zeros of holomorphic functions

Definition (Zero of order m). We say that f has a zero of order m at $z_0 \in \mathbb{C}$ if $f^{(k)}(z_0) = 0, k = 0, 1, \dots m - 1$ and $f^{(m)}(z_0) \neq 0$.

Theorem

A holomorphic function f has a zero of order m at z_0 iff it can be written in the form $f(z) = (z - z_0)^m g(z)$ where g is holomorphic at z_0 and $g(z_0) \neq 0$.

Proof. ($\Longrightarrow$)

$$\begin{aligned} f(z) &= \frac{f^{(m)}(z_0)}{m!}(z - z_0)^m + \frac{f^{m+1}(z_0)}{(m+1)!}(z - z_0)^{m+1} + \dots \\ &= (z - z_0)^m \left(\frac{f^{(m)}(z_0)}{m!} + \frac{f^{(m+1}(z_0)}{(m+1)!}(z - z_0) + \dots \right) \\ &= (z - z_0)^m g(z) \end{aligned}$$

($\Longleftarrow$): Conversely if $f(z) = (z - z_0)^m g(z)$ where $g(z) \neq 0$, then $f^{(k)}(z_0) = 0$ for $k = 0, \dots, m - 1$ and $f^{(m)}(z_0) = m! g(z_0) \neq 0$. ■

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Corollary *The zeros of a non-constant holomorphic function are isolated. That is every zero has a neighbourhood inside of which it is the only zero.*

Proof. If z_0 is a zero of f of order m, then $f(z) = (z - z_0)^m g(z)$ where $g(z)$ is holomorphic at z_0 and $g(z_0) \neq 0 \implies g$ is continuous $\implies \exists$ a neighbourhood $U_{z_0} \subset\in \mathbb{C}$ of z_0 such that $g(z) \neq 0, z \in U_{z_0} \implies f(z) \neq 0$ except for $z = z_0$. ∎

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Lecture-7

Residue Theory

We have
$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n, \quad 0 < |z - z_0| < R$$

Definition (Residue). $\mathrm{Res}[f, z_0] = a_{-1}$.

Theorem:

Let $\gamma \subset \{z : 0 < |z - z_0| < R\}, R > 0$ be a simple, closed piecewise-smooth curve that contains z_0 and f is holomorphic in $\{z : 0 < |z - z_0| < R\}$. Then
$$\mathrm{Res}[f, z_0] = \frac{1}{2\pi i} \oint_\gamma f(z)\, \mathrm{d}z$$

Proof. Let $0 < r < R$. By using the Deformation Theorem, we obtain
$$\begin{aligned}\frac{1}{2\pi i} \oint_\gamma f(z)\, \mathrm{d}z &= \frac{1}{2\pi i} \oint_{|z-z_0|=r} f(z)\, \mathrm{d}z \\ &= \frac{1}{2\pi i} \oint_{|z-z_0|=r} \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n\, \mathrm{d}z\end{aligned}$$

Parameterising $z - z_0 = re^{i\theta}$, $\theta \in [0, 2\pi]$
$$\begin{aligned}&= \frac{1}{2\pi i} \sum_{n=-\infty}^{\infty} \int_0^{2\pi} a_n r^n e^{in\theta} ire^{i\theta}\, \mathrm{d}\theta \\ &= a_{-1} = \mathrm{Res}[f, z_0]\end{aligned}$$

Since if $n \neq -1$, $\int_0^{2\pi} e^{i\theta(n+1)} = 0$ by periodicity of $\sin\theta, \cos\theta$ and thus $e^{i\theta}$. ∎

Residue

Let f be holomorphic inside and on a simple, closed piecewise-smooth curve γ except at the singularities $z_1, z_2, \ldots, z_n$ in its interior. Then
$$\oint_\gamma f(z)\, \mathrm{d}z = 2\pi i \sum_{j=1}^{n} \mathrm{Res}[f, z_j]$$

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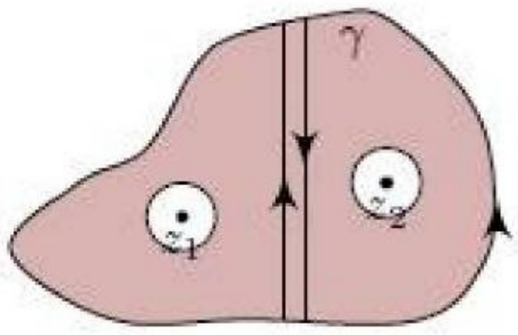
Proof. Let $\gamma_j = \{z : |z - z_j| = r_j\}$, where r_j are small enough.



Using the Deformation theorem we find

$$\oint_\gamma f(z)\ \mathrm{d}z = \sum_{j=1}^{n} \oint_{\gamma_j} f(z)\ \mathrm{d}z = 2\pi i \sum_{j=1}^{n} \mathrm{Res}[f, z_j] \qquad \blacksquare$$

Example $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!z^2} + \dots$. Then

$$\oint_{|z|=1} e^{1/z}\ \mathrm{d}z = 2\pi i$$

since $a_{-1} = 1$.

Calculating Residues

Let

$$f(z) = a_{-m}(z - z_0)^{-m} + \dots + a_{-1}(z - z_0)^{-1} + a_0 + a_1(z - z_0) + \dots$$

Introduce $g(z) = (z - z_0)^m f(z)$.

For $m = 1$,

$$g(z) = a_{-1} + a_0(z - z_0) + \dots$$

Then

$$\boxed{\mathrm{Res}[f, z_0] = \lim_{z \to z_0} g(z) = \lim_{z \to z_0} (z - z_0) f(z)}$$

For $m = 2$, then

$$g(z) = a_{-2} + a_{-1}(z - z_0) + a_0(z - z_0)^2 + \dots$$

$$\mathrm{Res}[f, z_0] = a_{-1} = \frac{\mathrm{d}}{\mathrm{d}z} g(z)\bigg|_{z=z_0} = \lim_{z \to z_0} \frac{\mathrm{d}}{\mathrm{d}z}((z - z_0)^2 f(z))$$

Residues for repeated roots: Inductively for m,

$$\boxed{\mathrm{Res}[f, z_0] = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}}((z - z_0)^m f(z))}$$

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Examples

(i) Evaluate

$$\oint_\gamma \frac{1}{z^5 - z^3}\,\mathrm{d}z, \quad \gamma = \{z : |z| = 1/2\}$$

We have

$$\frac{1}{z^5 - z^3} = \frac{1}{z^3}\frac{1}{(z-1)(z+1)}$$

So three singularities: $z_1 = 1, z_2 = -1, z_3 = 0$, only z_3 is inside γ.

$m = 3$, so

$$\begin{aligned}
\mathrm{Res}[f, 0] &= \lim_{z\to 0} \frac{1}{2!}\frac{\mathrm{d}^2}{\mathrm{d}z^2}(z^3 \cdot f(z)) \\
&= \lim_{z\to 0} \frac{1}{2}\left(\frac{1}{z^2-1}\right)'' \\
&= \frac{1}{2}\lim_{z\to 0}\left(\frac{-2z}{(z^2-1)^2}\right)' \\
&= \lim_{z\to 0} \frac{-(z^2-1)^2 - (z)2(z^2-1)\cdot 2z}{(z^2-1)^4} \\
&= -1
\end{aligned}$$

So

$$\oint_\gamma \frac{1}{z^5 - z^3}\,\mathrm{d}z = 2\pi i \cdot \mathrm{Res}[f, 0] = -2\pi i$$

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(ii) Evaluate

$$\oint_\gamma \frac{1}{(z+5)(z^2-1)}\,\mathrm{d}z, \quad \gamma = \{z : |z| = 2\}$$

We see that

$$f(z) = \frac{1}{(z+5)(z^2-1)} = \frac{1}{(z+5)(z+1)(z-1)}$$

We have poles at $z = -1, z = +1$ inside γ, so we write

$$\oint_\gamma f(z)\,\mathrm{d}z = 2\pi i\{\mathrm{Res}[f,-1] + \mathrm{Res}[f,1]\}$$

We compute

$$\begin{aligned}\mathrm{Res}[f,1] &= \lim_{z\to 1}(z-1)\cdot\frac{1}{(z+5)(z^2-1)} \\ &= \lim_{z\to 1}\frac{1}{(z+5)(z+1)} = \frac{1}{12}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Res}[f,-1] &= \lim_{z\to -1}(z+1)\cdot\frac{1}{(z+5)(z^2-1)} \\ &= \lim_{z\to -1}\frac{1}{(z+5)(z-1)} = -\frac{1}{8}\end{aligned}$$

Thus

$$\oint_\gamma \frac{1}{(z+5)(z^2-1)}\,\mathrm{d}z = 2\pi i\left(\frac{1}{12} - \frac{1}{8}\right) = -\frac{\pi i}{12}$$

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Lecture-8

Integrals of the Form $\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta$

Put $z = e^{i\theta}$, $0 \le \theta \le 2\pi$ We can then write

$dz = ie^{i\theta}d\theta$, $\cos\theta = \dfrac{e^{i\theta}+e^{-i\theta}}{2}$, $\sin\theta = \dfrac{e^{i\theta}-e^{-i\theta}}{2i}$. Since $dz = ie^{i\theta}d\theta = iz\,d\theta$ and $z^{-1} = 1/z = e^{-i\theta}$, these three quantities are equivalent to

$$d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{1}{2}(z+z^{-1}), \quad \sin\theta = \frac{1}{2i}(z-z^{-1}).$$

We have

$$\oint_C F\left(\frac{1}{2}(z+z^{-1}), \frac{1}{2i}(z-z^{-1})\right)\frac{dz}{iz},$$

where C is the unit circle $|z| = 1$.

EXAMPLE

Evaluate $\displaystyle\int_0^{2\pi} \frac{1}{(2+\cos\theta)^2}\, d\theta.$

Solution When we use the substitutions given in (4), the given trigonometric integral becomes the contour integral

$$\oint_C \frac{1}{\left(2+\frac{1}{2}(z+z^{-1})\right)^2}\frac{dz}{iz} = \oint_C \frac{1}{\left(2+\dfrac{z^2+1}{2z}\right)^2}\frac{dz}{iz}.$$

Carrying out the algebraic simplification of the integrand then yields

$$\frac{4}{i}\oint_C \frac{z}{(z^2+4z+1)^2}\, dz.$$

From the quadratic formula we can factor the polynomial z^2+4z+1 as $z^2+4z+1 = (z-z_1)(z-z_2)$, where $z_1 = -2-\sqrt{3}$ and $z_2 = -2+\sqrt{3}$. Thus, the integrand can be written

$$\frac{z}{(z^2+4z+1)^2} = \frac{z}{(z-z_1)^2(z-z_2)^2}.$$

Because only z_2 is inside the unit circle C, we have

$$\oint_C \frac{z}{(z^2+4z+1)^2}\, dz = 2\pi i\,\mathrm{Res}(f(z), z_2)..$$

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$$\begin{aligned}\mathrm{Res}(f(z), z_2) &= \lim_{z\to z_2} \frac{d}{dz}(z-z_2)^2 f(z) = \lim_{z\to z_2} \frac{d}{dz}\frac{z}{(z-z_1)^2} \\ &= \lim_{z\to z_2} \frac{-z-z_1}{(z-z_1)^3} = \frac{1}{6\sqrt{3}}.\end{aligned}$$

Hence, $$\frac{4}{i}\oint_C \frac{z}{(z^2+4z+1)}\,dz = \frac{4}{i}\cdot 2\pi i \mathrm{Res}(f(z), z_1) = \frac{4}{i}\cdot 2\pi i \cdot \frac{1}{6\sqrt{3}}$$

and, finally, $$\int_0^{2\pi} \frac{1}{(2+\cos\theta)^2}\,d\theta = \frac{4\pi}{3\sqrt{3}}.$$

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Lecture-9

Integrals of the Form $\int_{-\infty}^{+\infty} f(x)dx$

Suppose $y = f(x)$ is a real

function that is defined and continuous on the interval $[0, \infty)$. In elementary calculus the improper integral $I_1 = \int_0^\infty f(x)\,dx$ is defined as the limit

$$I_1 = \int_0^\infty f(x)\,dx = \lim_{R\to\infty}\int_0^R f(x)\,dx. \tag{1}$$

If the limit exists, the integral I_1 is said to be **convergent**; otherwise, it is **divergent**. The improper integral $I_2 = \int_{-\infty}^0 f(x)\,dx$ is defined similarly:

$$I_2 = \int_{-\infty}^0 f(x)\,dx = \lim_{R\to\infty}\int_{-R}^0 f(x)\,dx. \tag{2}$$

Finally, if f is continuous on $(-\infty, \infty)$, then $\int_{-\infty}^\infty f(x)\,dx$ is defined to be

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx = I_1 + I_2, \tag{3}$$

provided *both* integrals I_1 and I_2 are convergent. If either one, I_1 or I_2, is divergent, then $\int_{-\infty}^\infty f(x)\,dx$ is divergent. It is important to remember that the right-hand side of (3) is *not* the same as

$$\lim_{R\to\infty}\left[\int_{-R}^0 f(x)\,dx + \int_0^R f(x)\,dx\right] = \lim_{R\to\infty}\int_{-R}^R f(x)\,dx. \tag{4}$$

For the integral $\int_{-\infty}^\infty f(x)\,dx$ to be convergent, the limits (1) and (2) must exist independently of one another. *But*, in the event that we know (a priori) that an improper integral $\int_{-\infty}^\infty f(x)\,dx$ converges, we can then evaluate it by means of the single limiting process given in (4):

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{R\to\infty}\int_{-R}^R f(x)\,dx. \tag{5}$$

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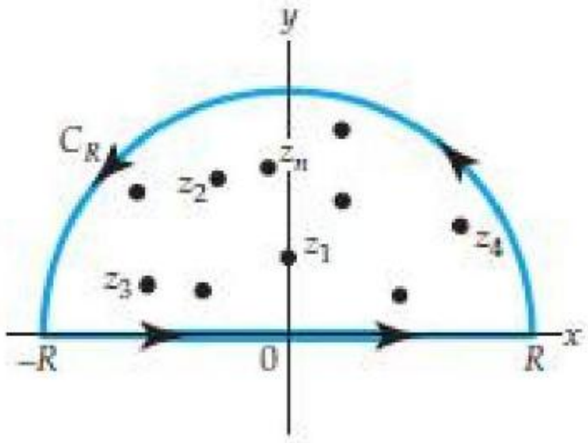
On the other hand, the symmetric limit in (5) may exist even though the improper integral $\int_{-\infty}^{\infty} f(x)\,dx$ is divergent. For example, the integral $\int_{-\infty}^{\infty} x\,dx$ is divergent since $\lim_{R\to\infty}\int_0^R x\,dx = \lim_{R\to\infty}\frac{1}{2}R^2 = \infty$. However, (5) gives

$$\lim_{R\to\infty}\int_{-R}^{R} x\,dx = \lim_{R\to\infty}\frac{1}{2}[R^2-(-R)^2] = 0. \qquad (6)$$

The limit in (5), if it exists, is called the **Cauchy principal value** (P.V.) of the integral and is written

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx. \qquad (7)$$

To evaluate an integral $\int_{-\infty}^{\infty} f(x)\,dx$, where the rational function $f(x) = p(x)/q(x)$ is continuous on $(-\infty,\ \infty)$, by residue theory we replace x by the complex variable z and integrate the complex function f over a closed contour C that consists of the interval $[-R,\ R]$ on the real axis and a semicircle C_R of radius large enough to enclose all the poles of $f(z) = p(z)/q(z)$ in the upper



half-plane $\text{Im}(z) > 0$.

we have

$$\oint_C f(z)\,dz = \int_{C_R} f(z)\,dz + \int_{-R}^{R} f(x)\,dx = 2\pi i\sum_{k=1}^{n}\text{Res}(f(z), z_k),$$

where z_k, $k = 1, 2, \ldots, n$ denotes poles in the upper half-plane. If we c show that the integral $\int_{C_R} f(z)\,dz \to 0$ as $R\to\infty$, then we have

$$\text{P.V.}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = 2\pi i\sum_{k=1}^{n}\text{Res}(f(z), z_k).$$

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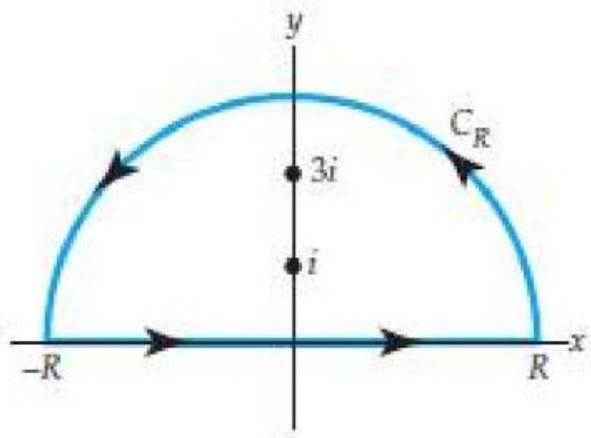
EXAMPLE

Evaluate the Cauchy principal value of $\displaystyle\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)}\,dx$.

Solution Let $f(z) = 1/(z^2+1)(z^2+9)$. Since

$$(z^2+1)(z^2+9) = (z-i)(z+i)(z-3i)(z+3i),$$

we take C be the closed contour consisting of the interval $[-R,\ R]$ on the



x-axis and the semicircle C_R of radius $R > 3$.

$$\oint_C \frac{1}{(z^2+1)(z^2+9)}\,dz = \int_{-R}^{R} \frac{1}{(x^2+1)(x^2+9)}\,dx + \int_{C_R} \frac{1}{(z^2+1)(z^2+9)}\,dz$$

$$= I_1 + I_2$$

and $$I_1 + I_2 = 2\pi i\,[\mathrm{Res}(f(z), i) + \mathrm{Res}(f(z), 3i)].$$

At the simple poles $z = i$ and $z = 3i$ we find, respectively,

$$\mathrm{Res}(f(z),\ i) = \frac{1}{16i} \quad \text{and} \quad \mathrm{Res}(f(z),\ 3i) = -\frac{1}{48i},$$

so that $$I_1 + I_2 = 2\pi i\left[\frac{1}{16i} + \left(-\frac{1}{48i}\right)\right] = \frac{\pi}{12}.$$

This last result shows that $|I_2| \to 0$ as $R \to \infty$, and so we conclude that $\lim_{R\to\infty} I_2 = 0$.

$$\lim_{R\to\infty}\int_{-R}^{R} \frac{1}{(x^2+1)(x^2+9)}\,dx = \frac{\pi}{12} \quad \text{or} \quad \text{P.V.}\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+9)}\,dx = \frac{\pi}{12}.$$

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MODEL QUESTION PAPER

B. TECH.

SECOND SEMESTER EXAMINATION

MATHEMATICS-II (PAPER CODE – KAS-203)

Time: 3 Hours Total Marks: 100

SECTION – A

Note: Attempt **all** questions. Each question carries equal marks: **(10×2=20)**

1.(a) Find the complimentary function of $(D-3)^2(D^2+1)y=0$.

(b) Find the P.I. of $(D^2+1)y=e^{-x}+Sin3x+x^2$.

(c) Find the value of $\int_0^{\frac{\pi}{2}} \sin^3 x \cos^{\frac{5}{2}} x dx$.

(d) Test the convergence of the Improper Integral $\int_0^{\infty} \frac{dx}{(1+x)\sqrt{x}}$.

(e) Check, whether the series $[1-\frac{1}{n}]^{n^2}$ is convergent or divergent.

(f) If $f(x)=x\,Sinx \quad 0 \le x \le 2\pi$ .Find a_0 *and* a_n.

(g) Find the fixed point of the mapping $w=\frac{(5z+4)}{(z+5)}$.

(h) Show that $u=-\sin x \sinh y$ is harmonic. Find the conjugate harmonic of v.

(i) State Cauchy integral theorem and Cauchy integral formula.

(j) If $f(z)=\frac{z^2}{(z-1)^2(z+2)}$ find $\operatorname{Re}s\, f(-2)$.

SECTION – B

Note: Attempt **any three** parts from this section. Each part carry equal marks: **(3×10=30)**

2. (a) Solve in Series $2x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}-(x+1)y=0$

(b) Apply Dirichlet's integral to find the mass of an octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, the density at any point being $\rho=kxyz$.

(c) Find the Fourier series for the function $f(t)=\begin{cases} t, & 0<t<1 \\ 1-t, & 1<t<2 \end{cases}$. Hence deduce that $\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=\frac{\pi^2}{8}$.

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(d) Show that the function $f(z)$ defined by $f(z)=\begin{cases} \dfrac{x^3y^5(x+iy)}{x^6+y^{10}}, & z\neq 0 \\ 0 & z=0 \end{cases}$ is not analytic at the origin even though it satisfies Cauchy-Riemann equation at the origin.

(e) Evaluate by contour integration $\int_0^{2\pi} \dfrac{d\theta}{a+b\sin\theta},\ a>|b|$

SECTION - C

Note: Attempt **any two** parts from all questions of this section. Each part carry equal marks: $(5\times2\times5=50)$

1.(a) Using variation of parameter method $\dfrac{d^2y}{dx^2}-y=\dfrac{2}{1+e^x}$.

(b) Solve $(1+x^2)\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos\log(1+x)$ by changing the independent variable.

(c) Solve $\dfrac{dx}{dt}+4x+3y=t, \dfrac{dy}{dt}+2x+5y=e^t$.

2. (a) Prove that the relation between beta and gamma function.

(b) Prove that $\int_0^1 \dfrac{x^2\,dx}{(1-x^4)^{\frac{1}{2}}} * \int_0^1 \dfrac{dx}{(1+x^4)^{\frac{1}{2}}} = \dfrac{\pi}{4\sqrt{2}}$

(c) The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axex in A, B, and C. Apply Dirichlet's integral to find the volume of the tetrahedron OABC . Also find its mass if the density at any point is $kxyz$.

3. (a) Test the convergence of the series: $1+\frac{2}{3}.\frac{1}{4}+\frac{2}{3}.\frac{4}{5}.\frac{1}{6}+\frac{2}{3}.\frac{4}{5}.\frac{6}{7}.\frac{1}{8}+.....$

(b) Obtain a half-range cosine series for $f(x)=\begin{cases} kx \ \ for \ \ 0\le x\le \dfrac{l}{2} \\ k(l-x) \ \ for \dfrac{l}{2}\le x\le l \end{cases}$.

(c) Find the Fourier series for the function $f(x) = x + x^2$, $-\pi < x < \pi$. Hence show that

(i) $\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots\ldots$ (ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + -----$.

4. (a) Determine the analytic function $f(z)=u+iv$, in terms of z, whose real part is $e^{-x}(x\sin y-y\sin y)$.

(b) State and Prove the necessary condition for a function to be analytic.

(c) Find the image of the circle $|z-1|=1$ in the complex plane under the map $w=\frac{1}{z}$

5. (a) Evaluate the following integral $\int_C \dfrac{12z-7}{(z-1)^2(2z+3)}dz$, where C is the circle $|z|=2$.

(b) Expand $f(z)=\dfrac{z}{(z^2-1)(z^2+4)}$ in Laurent series about $1<|z|<2$.

(c) Using Complex Integration, evaluate $\int_0^\infty \dfrac{\cos mx}{1+x^2}dx$.

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B. TECH.
(SEM II) THEORY EXAMINATION 2018-19
MATHEMATICS-II

Time: 3 Hours ***Total Marks: 100***

Note: Attempt all Sections. If require any missing data; then choose suitably.

SECTION A

1. Attempt *all* questions in brief. 2 x 10 = 20

QNo.	Question	Marks	CO
a.	Find the P.I of $\frac{d^2y}{dx^2} + 4y = sin2x$	2	1
b.	Solve simultaneous equations $\frac{dx}{dt} = 3y$, $\frac{dy}{dt} = 3x$	2	1
c.	Find the volume of solid generated by revolving the circle $x^2 + y^2 = 25$ about y-axis.	2	2
d.	Evaluate $\Gamma\left(-\frac{5}{2}\right)$. where Γ is gamma function	2	2
e.	Find the Fourier constant a_1 of $f(x) = x^2$, $-\pi \le x \le \pi$	2	3
f.	Discuss the convergence of sequence $a_n = \frac{2n}{n^2+1}$.	2	3
g.	Show that complex function $f(z) = z^3$ is analytic.	2	4
h.	Define Conformal mapping.	2	4
i.	Evaluate $\int_0^{1+i}(x^2 - iy)dz$ along the path $y = x$.	2	5
j.	Find residue of $f(z) = \frac{cosz}{z(z+5)}$ at $z = 0$	2	5

SECTION B

2. Attempt any *three* of the following:

QNo.	Question	Marks	CO
a.	Use Frobenius method to solve $9x(1-x)\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0$	10	1
b.	Apply Dirichlet integral to find the volume of an octant of the sphere $x^2 + y^2 + z^2 = 25$.	10	2
c.	Find half range sine series of $f(x) = \begin{cases} x & 0 < x < 2 \\ 4 - x & 2 < x < 4 \end{cases}$	10	3
d.	Show that $u = x^4 - 6x^2y^2 + y^4$ is harmonic function. Find complex function $f(z)$ whose u is a real part.	10	4
e.	Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in regions (i) $1 < \|z\| < 2$ (ii) $2 < \|z\|$	10	5

SECTION C

3. Attempt any *one* part of the following:

QNo.	Question	Marks	CO
a.	Solve $\frac{d^2y}{dx^2} + y = tanx$ by method of variation of parameter.	10	1

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b.	Solve $x^2\frac{d^2y}{dx^2} - 2(x^2+x)\frac{dy}{dx} + (x^2+2x+2)y = 0$ by Normal Form.	10	1

4. Attempt any *one* part of the following:

QNo.	Question	Marks	CO
a.	Prove that $\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$ where Γ is gamma function	10	2
b.	Use Beta and Gamma function to solve $\int_0^\infty \frac{1}{1+x^4}dx \int_0^{\frac{\pi}{2}} \sqrt{\cot\theta}\, d\theta$	10	2

5. Attempt any *one* part of the following:

QNo.	Question	Marks	CO
a.	Find the Fourier series of $f(x) = x\sin x \quad , -\pi \le x \le \pi$	10	3
b.	State D' Alembert's test. Test the series $1 + \frac{x}{2} + \frac{x^2}{5} + \frac{x^3}{10} \dots\dots + \frac{x^n}{n^2+1} + \dots\dots\dots\dots$	10	3

6. Attempt any *one* part of the following:

QNo.	Question	Marks	CO
a.	Let $f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}}$ $when\ z \neq 0$, $f(z) = 0\ when\ z = 0$. Prove that Cauchy Riemann satisfies at $z = 0$ but function is not differentiable at $z = 0$.	10	4
b.	Find Mobius transformation that maps points $z = 0, -i, 2i$ into the points $w = 5i, \infty, -\frac{i}{3}$ respectively.	10	4

7. Attempt any *one* part of the following:

QNo.	Question	Marks	CO
a.	Using Cauchy Integral formula evaluate $\int_c \frac{\sin z}{(z^2+25)^2}dz$ where c is circle $\lvert z\rvert = 8$	10	5
b.	Apply residue theorem to evaluate $\int_{-\infty}^{\infty} \frac{x^2dx}{(x^2+1)(x^2+4)}$	10	5

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