ISE-5405 – Optimization 1 Project

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Problem Statement

A local school district needs to assign 10th, 11th, and 12th grade students to three high schools. The district provides buses for all high school students who travel strictly more than one mile, so the school district wants a plan for assigning students that will minimize the total transportation cost. The annual cost of transporting a student from one of the six residential areas of the city to each of the schools is $100 per mile. Distances between areas and schools are shown in the following table (along with other basic data for next year).

Area	Number of Students	Percent 10th	Percent 11th	Percent 12th	Distance (mi) to School 1	Distance (mi) to School 2	Distance (mi) to School 3
1	450	31	39	30	2.5	0.5	5
2	500	36	27	37	N/A[1]	2.5	3
3	550	32	33	35	4	1.5	2
4	450	29	41	30	2.5	3	N/A
5	550	38	35	27	1.5	N/A	2.5
6	400	35	29	36	3	2	0.5
				School Capacity:	900	1100	1000

The school district also has imposed the restriction that each grade must constitute between 30 and 36 percent of each school's population. For example, if 890 students are assigned to School 1, there should be between 890×0.30=267 and 890×0.36=320.4 students in each grade (10th, 11th, and 12th). The above table shows the percentage of each area's high school population for next year that falls into each of the three grades. The school attendance zone boundaries can be drawn to split any given area among more than one school but assume that the percentages shown in the table will continue to hold for any partial assignment of an area to a school. For example, if 200 students from Area 2 are assigned to School 2, then you should assume that 200×0.36=72 of those students are in 10th grade, 200×0.27=54 students are in 11th grade, and 200×0.37=74 students are in 12th grade. You have been hired as an operations research consultant to assist the school district in determining how many students in each area should be assigned to each school.

a) Formulate this problem as a linear program.

Let xij be the number of students of Area 'i' allotted to School 'j'.

Let dij be the distance from Area 'i' to School 'j'.

The students in each area should be divided among the 3 different schools. The following constrains will satisfy this condition:

x11 + x12 + x13 == 450 **– (1)**

x21 + x22 + x23 == 500 **– (2)**

x31 + x32 + x33 == 550 **– (3)**

x41 + x42 + x43 == 450 **– (4)**

x51 + x52 + x53 == 550 **– (5)**

x61 + x62 + x63 == 400 **– (6)**

The total number of students assigned to each school should not exceed the maximum limit of each school. The following constrains will satisfy this condition:

x11 + x21 + x31 + x41 + x51 + x61 <= 900 **– (7)**

x12 + x22 + x32 + x42 + x52 + x62 <= 1100 **– (8)**

x13 + x23 + x33 + x43 + x53 + x63 <= 1000 **– (9)**

The school district also has imposed the restriction that each grade must constitute between 30 and 36 percent of each school's population. The following constrains will satisfy this condition:

For School-1:

Grade 10:

0.3*(x11 + x21 + x31 + x41 + x51 + x61) <= 0.31*x11 + 0.36*x21 + 0.32*x31 + 0.29*x41 + 0.38*x51 + 0.35*x61 **– (10)**

0.31*x11 + 0.36*x21 + 0.32*x31 + 0.29*x41 + 0.38*x51 + 0.35*x61 <=0.36*(x11 + x21 + x31 + x41 + x51 + x61) **– (11)**

Grade 11:

0.3*(x12 + x22 + x32 + x42 + x52 + x62) <= 0.31*x12 + 0.36*x22 + 0.32*x32 + 0.29*x42 + 0.38*x52 + 0.35*x62 **– (12)**

0.31*x12 + 0.36*x22 + 0.32*x32 + 0.29*x42 + 0.38*x52 + 0.35*x62 <=0.36*(x12 + x22 + x32 + x42 + x52 + x62) **– (13)**

Grade 12:

0.3*(x13 + x23 + x33 + x43 + x53 + x63) <= 0.31*x13 + 0.36*x23 + 0.32*x33 + 0.29*x43 + 0.38*x53 + 0.35*x63 **– (14)**

0.31*x13 + 0.36*x23 + 0.32*x33 + 0.29*x43 + 0.38*x53 + 0.35*x63 <= 0.36*(x13 + x23 + x33 + x43 + x53 + x63) **– (15)**

For School-2:

Grade 10:

0.3*(x11 + x21 + x31 + x41 + x51 + x61) <= 0.39*x11 + 0.27*x21 + 0.33*x31 + 0.41*x41 + 0.35*x51 + 0.29*x61 **– (16)**

0.39*x11 + 0.27*x21 + 0.33*x31 + 0.41*x41 + 0.35*x51 + 0.29*x61 <=0.36*(x11 + x21 + x31 + x41 + x51 + x61) **– (17)**

Grade 11:

0.3*(x12 + x22 + x32 + x42 + x52 + x62) <= 0.39*x12 + 0.27*x22 + 0.33*x32 + 0.41*x42 + 0.35*x52 + 0.29*x62 **– (18)**

0.39*x12 + 0.27*x22 + 0.33*x32 + 0.41*x42 + 0.35*x52 + 0.29*x62 <= 0.36*(x12 + x22 + x32 + x42 + x52 + x62) **– (19)**

Grade 12:

0.3*(x13 + x23 + x33 + x43 + x53 + x63) <= 0.39*x13 + 0.27*x23 + 0.33*x33 + 0.41*x43 + 0.35*x53 + 0.29*x63 **– (20)**

0.39*x13 + 0.27*x23 + 0.33*x33 + 0.41*x43 + 0.35*x53 + 0.29*x63 <= 0.36*(x13 + x23 + x33 + x43 + x53 + x63) **– (21)**

For School-3:

Grade 10:

$0.3*(x11 + x21 + x31 + x41 + x51 + x61) <= 0.3*x11 + 0.37*x21 + 0.35*x31 + 0.3*x41 + 0.27*x51 + 0.36*x61$ **– (22)**

$0.3*x11 + 0.37*x21 + 0.35*x31 + 0.3*x41 + 0.27*x51 + 0.36*x61 <= 0.36*(x11 + x21 + x31 + x41 + x51 + x61)$ **– (23)**

Grade 11:

$0.3*(x12 + x22 + x32 + x42 + x52 + x62) <= 0.3*x12 + 0.37*x22 + 0.35*x32 + 0.3*x42 + 0.27*x52 + 0.36*x62$ **– (24)**

$0.3*x12 + 0.37*x22 + 0.35*x32 + 0.3*x42 + 0.27*x52 + 0.36*x62 <= 0.36*(x12 + x22 + x32 + x42 + x52 + x62)$ **– (25)**

Grade 12:

$0.3*(x13 + x23 + x33 + x43 + x53 + x63) <= 0.3*x13 + 0.37*x23 + 0.35*x33 + 0.3*x43 + 0.27*x53 + 0.36*x63$ **– (26)**

$0.3*x13 + 0.37*x23 + 0.35*x33 + 0.3*x43 + 0.27*x53 + 0.36*x63 <= 0.36*(x13 + x23 + x33 + x43 + x53 + x63)$ **– (27)**

The district provides buses for all high school students who travel strictly more than one mile. So, let us introduce a new variable ‘yij’.

$yij = 1$ if $dij > 1$; else $yij = 0$ **– (28)**

Also, the non-negativity constraints: All $xij >= 0$ **– (29)**

Objective Function:

The goal is to minimize the total transportation cost.

$$\begin{aligned} \text{Min. } z = 100*(&d11*x11*y11 + d12*x12*y12 + d13*x13*y13 \\ &+ d21*x21*y21 + d22*x22*y22 + d23*x23*y23 \\ &+ d31*x31*y31 + d32*x32*y32 + d33*x33*y33 \\ &+ d41*x41*y41 + d42*x42*y42 + d43*x43*y43 \\ &+ d51*x51*y51 + d52*x52*y52 + d53*x53*y53 \\ &+ d61*x61*y61 + d62*x62*y62 + d63*x63*y63) \end{aligned}$$

b) Solve the model using Excel or Gurobi and clearly state your recommendation to the school district.

Solved the model using Gurobi.

The Objective Function value = $507,500

```
Optimal objective  5.075000000e+05
x11 0
x12 450
x13 0
x21 0
x22 272.222
x23 227.778
x31 250
x32 161.111
x33 138.889
x41 233.333
x42 216.667
x43 0
x51 416.667
x52 0
x53 133.333
x61 0
x62 0
x63 400
Obj: 507500
```

The school district is considering some revised transportation plans to reduce costs. Option 1 is to provide transportation only for students who live strictly more than 1.5 miles away. Option 2 is to only provide transportation for those living strictly more than 2 miles away.

c) Revise the model from part (a) to fit Option 1 and solve. Compare these results with those from part (b), including the reduction in total transportation cost.

Keep the original constraints unchanged and modifying constraint number **(28)** to

$y_{ij} = 1$ if $d_{ij} > 1.5$; else $y_{ij} = 0$

The results obtained are as follows –

```
Optimal objective  3.906250000e+05
x11 0
x12 450
x13 0
x21 0
x22 133.333
x23 366.667
x31 0
x32 404.167
x33 145.833
x41 337.5
x42 112.5
x43 0
x51 375
x52 0
x53 175
x61 187.5
x62 0
x63 212.5
Obj: 390625
```

The transportation cost decreases by $116,875.

The number of students allocated to each school from different areas also reflects a change as shown above.

d) Repeat part (c) for Option 2. The school district now needs to choose among the three alternative transportation plans (the current one or Option 1 or Option 2).

Keep the original constraints unchanged and modifying constraint number **(28)** to

yij = 1 if dij > 2; else yij = 0

The results obtained are as follows –

```
Optimal objective  3.425000000e+05
x11 0
x12 450
x13 0
x21 0
x22 500
x23 0
x31 0
x32 0
x33 550
x41 300
x42 150
x43 0
x51 400
x52 0
x53 150
x61 200
x62 0
x63 200
Obj: 342500
```

The transportation cost decreases by $165,000.

The number of students allocated to each school from different areas also reflects a change as shown above.

The school district now needs to choose among the three alternative transportation plans (the current one or Option 1 or Option 2). One important factor is transportation costs. However, the school district also wants to place equal weight on a second factor: the inconvenience and safety problems caused by forcing students to travel by foot or bicycle a substantial distance (at least a mile, and especially at least 1.5 miles). Therefore, they want to choose a plan that provides the best trade-off between these two factors.

e) Use your results from parts (b), (c), and (d) to summarize the key information related to these two factors and make your recommendation to the school district.

We will choose the current transportation plan amongst the three alternatives in front of us.

Miles	Cost	No. of Students Walking
More than 1	507500	850
More than 1.5	390625	1441.667
More than 2	342500	1600

Objective – to minimize the transportation cost and the number of students traveling by foot.

To analyse how much we are paying for the safety of students in each scenario,

Total number of students to be transported = 2900 (for all scenarios)

<u>Scenario 1 –</u>

Number of students provided with transportation = 2900-850 = 2050

Cost of transportation for each student = 507500/2050 = $247.56 (safety cost for each student)

<u>Scenario 2 –</u>

Number of students provided with transportation = 2900-1441.667 = 1458.333

Cost of transportation for each student = 390625/1458.333 = $267.85 (safety cost for each student)

<u>Scenario 3 –</u>

Number of students provided with transportation = 2900-1600 = 1300

Cost of transportation for each student = 342500/1300 = $263.461 (safety cost for each student)

As we need to find the trade-off between the costs incurred in transporting a single student (minimum cost for safety of students using the public transportation) and the safety of the students walking by foot (minimum number of students), we choose the best scenario to be the **SCENARIO 1.**

Cost (least cost) = $247.56 for transporting one student with the public transport.

Number of students walking (least number) = 850

Hence, both the attributes are minimized and hence we find our ideal transportation plan.

One concern of the school district is the ongoing road construction in area 6. These construction projects have been delaying traffic considerably and are likely to affect the cost of transporting students from area 6, perhaps increasing them as much as 10 percent.

f) Use the sensitivity report from part (b) to determine how much the transportation cost from area 6 to school 1 can increase (assuming no change in the costs for the other schools) before the current optimal solution would no longer be optimal. If the allowable increase is less than 10 percent, re-solve to find the new optimal solution with a 10 percent increase.

Variable	X	RC	Obj	SAObjLow	SAObjUp
x11	0	266.667	250	-16.6667	inf
x12	450	0	0	-inf	266.667
x13	0	450	500	50	inf
x21	0	1e+12	1e+12	666.667	inf
x22	272.222	0	250	250	250
x23	227.778	0	300	300	300
x31	250	0	400	100	400
x32	161.111	0	150	150	150
x33	138.889	0	200	200	342.105
x41	233.333	0	250	250	314.286
x42	216.667	0	300	235.714	300
x43	0	1e+12	1e+12	350	inf
x51	416.667	0	150	-30	150
x52	0	1e+12	1e+12	200	inf
x53	133.333	0	250	250	430
x61	0	0	300	300	inf
x62	0	250	200	-50	inf
x63	400	0	0	-inf	0

From the sensitivity report,

Allowable increase in transportation cost for variable x61 in the objective function is infinity. Hence, any increase in the variable will not change the current optimal solution.

Since the allowable increase is greater than 10 percent, we need not re-solve the model.

g) Repeat part (f) for school 2 (assuming no change in the costs for the other schools).

From the sensitivity report,

Allowable increase in transportation cost for variable x62 in the objective function is infinity. Hence, any increase in the variable will not change the current optimal solution.

Since the allowable increase is greater than 10 percent, we need not re-solve the model.

The school district has the option of adding portable classrooms to increase the capacity of one or more of the high schools for a few years. However, this is costly, and the district leaders would consider it only if it would significantly decrease transportation costs. Each portable classroom holds 20 students and has a leasing cost of $800 per year. To analyze this option, the school district decides to assume that the road construction in area 6 will wind down without significantly increasing the transportation costs from that area.

h) For each school, use the corresponding shadow price from the report obtained in part (b) to determine whether it would be worthwhile to add any portable classrooms.

Constraint	Sense	Slack	Pi	RHS	SARHSLow	SARHSUp
R0	=	0	50	450	172.222	505.556
R1	=	0	300	500	333.333	600
R2	=	0	200	550	272.222	650
R3	=	0	350	450	233.333	495.455
R4	=	0	250	550	416.667	650
R5	=	0	-0	400	0	500
R6	<	0	-16.6667	900	816.667	1188
R7	<	0	-50	1100	1016.67	1377.78
R8	<	100	0	1000	900	inf
R9	<	36	0	0	-36	inf
R10	>	-18	0	0	-inf	18
R11	<	21.8889	0	0	-21.8889	inf
R12	>	-44.1111	0	0	-inf	44.1111
R13	<	47.1111	0	0	-47.1111	inf
R14	>	-6.88889	0	0	-inf	6.88889
R15	<	54	0	0	-54	inf
R16	>	0	1666.67	0	-10.1455	5
R17	<	61	0	0	-61	inf
R18	>	-5	0	0	-inf	5
R19	<	0	9.09495e-13	0	-9.66667	5
R20	>	-54	0	0	-inf	54
R21	<	0	-3333.33	0	-6.34091	9
R22	>	-54	0	0	-inf	54
R23	<	27.1111	0	0	-27.1111	inf
R24	>	-38.8889	0	0	-inf	38.8889
R25	<	42.8889	0	0	-42.8889	inf
R26	>	-11.1111	0	0	-inf	11.1111

One portable classroom costs the school district = $800/year.

__School 1:__

Decrease in transportation cost with an increase of 20 students = 16.6667 x 20 = $333.334

Comparing the two values, we can see that we are incurring a loss of (800-333.334 =) $466.66 to add one portable classroom consisting of 20 students. Hence, we do not recommend adding the classroom.

__School 2:__

Decrease in transportation cost with an increase of 20 students = 50 x 20 = $1000

Comparing the two values, we can see that we are making a profit of (1000-800=) $200 to add one portable classroom consisting of 20 students. Hence, we do recommend adding the classroom.

__School 3:__

Decrease in transportation cost with an increase of 20 students = 0 x 20 = $0

Comparing the two values, we can see that we are incurring a loss of $800 to add one portable classroom consisting of 20 students. Hence, we do not recommend adding the classroom.

i) For each school where it is worthwhile to add any portable classrooms, use the report from part (b) to determine how many could be added before the shadow price (optimal dual solution) would no longer be valid (assuming this is the only school receiving portable classrooms).

As seen above, it is worthwhile to add portable classrooms to school 2.

Since the allowable increase in the number of students going to school 2 is 1377.38 – 1100 = 277.38.

Each classroom can seat 20 students.

Hence, number of classrooms that can be added without changing the optimality conditions = 277.38/20 = 13.869

Therefore, maximum number of classrooms that can be added without violating the optimality constraints = 13