University of Guelph
CIS 2910 F16 – Midterm (Oct. 13)
Instructor: Joe Sawada

First Name: ______________________________

Last Name: ______________________________

Student Number: ______________________________

Problem 1: (5 marks)		Problem 6: (4 marks)	
Problem 2: (6 marks)		Problem 7: (4 marks)	
Problem 3: (4 marks)			
Problem 4: (6 marks)			
Problem 5: (4 marks)			
		Total (33 marks)	

This test is closed book and lasts 75 minutes.
You may not use any electronic/mechanical computation devices.
There are 8 pages including the cover page.

Problem 1: [5 marks]

(a) TRUE or FALSE: In a class of 30 students, at least 3 must be born in December.

Solution: **False**, they all could be born in January.

(b) TRUE or FALSE: In a class of 30 students, there exists at least one day of the week on which exactly 5 were born.

Solution: **False**, all 30 could be born on Monday.

(c) TRUE or FALSE: $\binom{10}{0} \cdot 0! = 1$

Solution: **True**

(d) TRUE or FALSE: Let A be a set with 30 elements and let B be a set with 60 elements. Then the number of different subsets of A is greater than the number of subsets of B with exactly 30 elements.

Solution: **False**, there are 2^{30} subsets of a 30 element set and $\binom{60}{30}$ 30-element subsets of a set of size 60. Suppose the statement is true, then:

$$\begin{aligned}
2^{30} &> \frac{60!}{30! \cdot 30!} = \frac{60 \cdot 59 \cdot 58 \cdots 31}{30!} \\
2^{30} \cdot 30! &> 60 \cdot 59 \cdot 58 \cdots 31 \\
2(30) \cdot 2(29) \cdot 2(28) \cdots 2(1) &> 60 \cdot 59 \cdot 58 \cdots 31 \\
60 \cdot 58 \cdot 56 \cdots 2 &> 60 \cdot 59 \cdot 58 \cdots 31
\end{aligned}$$

This inequality is clearly false.

(e) TRUE or FALSE: The number of ways to rearrange the letters in the word RAPTORS is the same as the number of ways to rearrange the letters ABCDEFG such that A comes before B.

Solution: **True**, each string has 7 characters. In the first, the R is repeated twice so we have $7!/2!$. In the second, the characters are distinct, and half the strings will have A before B to get $7!/2$.

Problem 2: [6 marks]

A local bakery wants to display 15 pies in a row in its front window. Because it is near Thanksgiving, they can choose from a selection of Pumpkin, Apple, and Cherry pies. They have an unlimited number of each. How many ways can they set up their display of 15 pies if:

(a) [**1 mark**] There are no cherry pies?

Solution: 2^{15}

(b) [**1 mark**] The first and last pie is pumpkin?

Solution: 3^{13}

(c) [**1 mark**] There are exactly 3 pumpkin pies?

Solution: Choose the 3 positions for the pumpkin pie first: $\binom{15}{3} \cdot 2^{12}$

(d) [**1 mark**] There are at least 2 pumpkin pies?

Solution: Total ways minus those with 1 pumpkin pie minus those with 0 pumpkin pies:

$$3^{15} - 15 \cdot 2^{14} - 2^{15}$$

(e) [**1 mark**] Each pie appears at least 4 times?

Solution: Possible combinations for the number of each (447, 474, 744, 456, 465, 546, 564, 645, 654, 555) =

$$3 \cdot \binom{15}{4} \cdot \binom{11}{4} + 6 \cdot \binom{15}{4} \cdot \binom{11}{5} + \binom{15}{5} \cdot \binom{10}{5}$$

(f) [**1 mark**] Either the first pie is apple, or the middle pie is apple.

Solution: Inclusion/exclusion: $3^{14} + 3^{14} - 3^{13}$.

Problem 3: [4 marks]

A local market is selling the following 8 different items: **A**pples, **B**ananas, **C**herries, **D**onuts, **E**ggs, **F**ries, **G**um, and **H**otdogs. It's a great day to go shopping. For this question, when you pack your (infinitely large) knapsack with items, the order of items does not matter.

(a) [**1 mark**] How many different ways can you fill your knapsack so that you have at most one of every item?

Solution: All possible subsets (including the empty set) of the 8 items = $2^8 = 256$.

(b) [**1 mark**] How many different ways can you fill your knapsack so that it has at least 3 items, but no item is repeated?

Solution: Take the answer to (a) and subtract packing with 0 items, 1 item and 2 items = $2^8 - \binom{8}{0} - \binom{8}{1} - \binom{8}{2} = 256 - 1 - 8 - 28 = 219$.

(c) [**1 mark**] How many different ways can you fill your knapsack with a total of 20 items, and exactly 2 different items? For example, it may contain 18 eggs and 2 apples.

Solution: Select the two items: $\binom{8}{2}$. Then there are 19 choices for how much of the first item you take from 1 to 19. Thus $19 \cdot \binom{8}{2}$.

(d) [**1 mark**] How many different ways can you fill your knapsack with 100 items so it contains exactly 1 apple?

Solution: This material had not yet been covered. But there is a simple formula. The idea is to consider 7 bins, one for each remaining item, and then fill the bins with 99 balls in all possible ways. You need 6 partitions between the bins (the 1s). The 99 balls are 0s. Each selection corresponds to a binary string of length 105 with 6 ones. So there are $\binom{105}{6}$ ways to do this.

Problem 4: [6 marks]

(a) [**1 mark**] Express $\binom{20}{17}$ as a decimal number.

Solution: $20/(17! \cdot 3!) = 20 \cdot 19 \cdot 18/6 = 20 \cdot 19 \cdot 3 = 1140$

(b)[**1 mark**] How many ways are there to order the 7 letters TORONTO?

Solution: $7!/(3! \cdot 2!)$

(c)[**1 mark**] What is the probability of rolling a 4 with two regular six-sided dice?

Solution: $3/36 = 1/12 = 8.333\%$

Alternatively, if you assumed that a 4 needed be on one of the dice (instead of sum as intended), then you apply inclusion exclusion. 6 ways for first dice to be 4, plus 6 ways for second dice to be 4, minus case when both are $4 = 11/36$.

(d)[**1 mark**] It is a beautiful Sunday at the track. In the final race, there are 8 horses competing, and the race ends in a three-way tie for first. There are no other ties. Given these constraints, how many different ways can the horses finish?

Solution: Choose the three horses that tie for first $\binom{8}{3}$, then consider the 5 ways to order the remaining horses $5! = \binom{8}{3} \cdot 5!$

(e)[**1 mark**] How many binary strings of length $n > 100$ are there such that the first 4 bits are 0's and the last 90 bits are 1's?

Solution: 2^{n-94}

(f)[**1 mark**] How many permutations of the 10 letters ABCDEFGHIJ have the A next to J?

Solution: $2 \cdot 9!$ (consider AJ as one character, then consider JA as one character)

Problem 5: [4 marks]

(a) **[2 marks]** State the Pigeon Hole Principle.

Solution: If N objects are placed into k boxes, then there is **at least** one box containing **at least** $\lceil N/k \rceil$ objects.

(b) **[2 marks]** State Pascal's identity.

Solution:

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

Problem 6: [4 marks]

(a) [**1 mark**] Given an example of a sequence that is an geometric progression.

Solution: $2, 4, 8, 16, 32, 64, \ldots$

(b) [**1 mark**] Given an example of a sequence that is an arithmetic progression.

Solution: $2, 4, 6, 8, 10, \ldots$

(c) [**2 marks**] Simplify the following sum, where $n > 3$:

$$\sum_{i=3}^{n-1} 2^i$$

Solution:

$$\begin{aligned}
\sum_{i=3}^{n-1} 2^i &= \sum_{i=0}^{n} 2^i - 2^0 - 2^1 - 2^2 - 2^n \\
&= (2^{n+1} - 1) - 1 - 2 - 4 - 2^n \\
&= 2^{n+1} - 2^n - 8 \\
&= 2^n - 8
\end{aligned}$$

Problem 7: [4 marks]

Prove the following identity by induction for all $n \geq 2$:

$$\sum_{i=0}^{n} 2^{n-i} = 2^{n+1} - 1$$

Solution:

First observe that $\sum_{i=0}^{n} 2^{n-i} = 2^n + 2^{n-1} + \cdots 2^0 = \sum_{i=0}^{n} 2^i$. Now the proof follows directly as in the notes, but starting at the base case of $n = 2$.

Base case $n = 2$.

$$\sum_{i=0}^{2} 2^{n-i} = 2^2 + 2^1 + 2^0 = 7 = 2^3 - 1.$$

Inductive Hypothesis: assume that $\sum_{i=0}^{k} 2^{k-i} = 2^{k+1} - 1$ for $k \geq 2$.

Consider $k+1$:

$$\begin{aligned}
\sum_{i=0}^{k+1} 2^{k+1-i} &= \sum_{i=0}^{k} 2^{k-i} + 2^{k+1} \\
&= 2^{k+1} - 1 + 2^{k+1} \quad \text{by I. H.} \\
&= 2^{k+2} - 1
\end{aligned}$$

Thus, by the principle of mathematical induction, the statement is true for $n \geq 2$.

– END OF MIDTERM –