CIS*4720 Image Processing and Vision

Assignment 2 Part 1 & 2

Maneesh Wijewardhana (1125828)

I have read and understood the Academic Misconduct section in the course outline. I assert this work is my own.

1a)

- Let $h = G_1 + G_2$ which by definition, is infinite
- Then for any $(x, y) \in 0..M-1 \to 0..N-1$ we have
  - $T_f(h)(x,y) = \sum_{u=0}^{m-1} \sum_{v=0}^{n-1} f(u,v)h(x+u-\frac{m-1}{2}, y+v-\frac{n-1}{2})$
  - $= \sum_{u=0}^{m-1} \sum_{v=0}^{n-1} f(u,v)(G_1(x+u-\frac{m-1}{2}, y+v-\frac{n-1}{2}) + G_2(x+u-\frac{m-1}{2}, y+v-\frac{n-1}{2}))$
  - Then by distributing the sum over addition we get:
  - $= \sum_{u=0}^{m-1} \sum_{v=0}^{n-1} f(u,v)(G_1(x+u-\frac{m-1}{2}, y+v-\frac{n-1}{2}) + f(u,v)G_2(x+u-\frac{m-1}{2}, y+v-\frac{n-1}{2}))$
  - $= \sum_{u=0}^{m-1} \sum_{v=0}^{n-1} f(u,v)G_1(x+u-\frac{m-1}{2}, y+v-\frac{n-1}{2}) + \sum_{u=0}^{m-1} \sum_{v=0}^{n-1} f(u,v)G_2(x+u-\frac{m-1}{2}, y+v-\frac{n-1}{2})$
  - $= T_f(G_1)(x,y) + T_f(G_2)(x,y)$
  - $\therefore T_f(G_1 + G_2) = T_f(G_1) + T_f(G_2)$

1b)

- Consider a zero padded infinite image $G_1 = \begin{bmatrix} 3 & 9 & 5 \\ 3 & 3 & 6 \\ 4 & 5 & 3 \end{bmatrix} \wedge G_2 = \begin{bmatrix} 8 & 6 & 9 \\ 3 & 10 & 15 \\ 6 & 1 & 20 \end{bmatrix}$
- Consider the neighborhood $f = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
  - $\therefore S_f\{(0,1),(1,0),(1,2),(2,1)\}$
- Then $G_1 + G_2 = \begin{bmatrix} 11 & 15 & 14 \\ 6 & 13 & 21 \\ 10 & 6 & 23 \end{bmatrix}$
- When we apply our neighborhood f, we get the set $MIN_f(G_1 + G_2)\{15, 6, 21, 6\} = \mathbf{6}$
- When we calculate $MIN_f(G_1) + MIN_f(G_2)$, we get $3 + 1 = \mathbf{4}$
- $\therefore MIN_f(G_1 + G_2) \neq MIN_f(G_1) + MIN_f(G_2)$ due to a counterexample

1c)

- Consider the same zero padded infinite images and neighborhood defined in **1b)**
- When we apply our neighborhood f, we get the set $MED_f(G_1 + G_2)\{6, 6, 15, 21\} = \mathbf{13.5}$
- When we calculate $MED_f(G_1) + MED_f(G_2)$, we get $5.5 + 4.5 = \mathbf{10}$
- $\therefore MED_f(G_1 + G_2) \neq MED_f(G_1) + MED_f(G_2)$ due to a counterexample

1d)

- Consider the same zero padded infinite images and neighborhood defined in **1b)**
- When we apply our neighborhood f, we get the set $MAX_f(G_1 + G_2)\{15, 6, 21, 6\} = \mathbf{21}$
- When we calculate $MAX_f(G_1) + MAX_f(G_2)$, we get $9 + 15 = \mathbf{24}$
- $\therefore MAX_f(G_1 + G_2) \neq MAX_f(G_1) + MAX_f(G_2)$ due to a counterexample

2a)

- Let's consider the case where $k = 0$, then $T_f(k * G)(x,y) = \sum_{u=0}^{m-1} \sum_{v=0}^{n-1} f(u,v)(k * G)(x+u-\frac{m-1}{2}, y+v-\frac{n-1}{2}) = 0 \; \forall (x,y) \in 0..M-1 \times 0..N-1$ since $k * G$ is zero everywhere

- Similarly, $k * T_f(G)(x,y) = k\sum_{u=0}^{m-1}\sum_{v=0}^{n-1} f(u,v)G(x+u-\frac{m-1}{2}, y+v-\frac{n-1}{2}) = 0\ \forall(x,y) \in 0..M-1 \times 0..N-1$ since k is zero
- $\therefore T_f(k*G) = k*T_f(G)$ holds when $k=0$
- Now let's consider the case where k is not zero, then we have:
  - ◇ $T_f(k*G)(x,y) = \sum_{u=0}^{m-1}\sum_{v=0}^{n-1} f(u,v)(k*G)(x+u-\frac{m-1}{2}, y+v-\frac{n-1}{2})$ **(factor)**
  - ◇ $= k\sum_{u=0}^{m-1}\sum_{v=0}^{n-1} f(u,v)G(x+u-\frac{m-1}{2}, y+v-\frac{n-1}{2})$
  - ◇ $= k*T_f(G)(x,y)$
- $\therefore$ we have proven that $T_f(k*G) = k*T_f(G)$

2b)

- Consider a zero padded infinite image $G = \begin{bmatrix} 3 & 9 & 5 \\ 3 & 3 & 6 \\ 4 & 5 & 3 \end{bmatrix}$
- Consider the neighborhood $f = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
  - ◇ $\therefore S_f\{(0,1),(1,0),(1,2),(2,1)\}$
- When we apply our neighborhood f, we get the set $MIN_f(k*G)\{-27,-9,-18,-15\} = \mathbf{-27}$
- When we calculate $k*MIN_f(G)$, we get $(-3)*3 = \mathbf{-9}$
- $\therefore MIN_f(k*G) \neq k*MIN_f(G)$ due to a counterexample

2c)

- We have $G(x+u-\frac{m-1}{2}, y+v-\frac{n-1}{2})$
- We need to show that multiplying all the values by k does not change the number of values below or above the median
- Let's consider the set of values below the median of the unscaled values, suppose this set has m values
- Then, the set of scaled values below the median is:
  - ◇ $k*G(x+u-\frac{m-1}{2}, y+v-\frac{n-1}{2})$ for $(u,v) \in S_f$, sorted in increasing order where there are m values in this set
- Multiplying all these values by k does not change their order, so the value that was previously the median of the unscaled values is now the $(m/2)$-th largest value in the scaled set
- $\therefore$ the number of values below the median is still m, and the number of values above the median is still the same as well
- The same argument applies to the set of values above the median, so the number of values below and above the median is unchanged when we scale the values by k
- $\therefore$ we have shown that $MED_f(k*G) = k*MED_f(G)$

2d)

- Consider the same zero padded infinite images and neighborhood defined in **2b)**
- When we apply our neighborhood f, we get the set $MAX_f(k*G)\{-27,-9,-18,-15\} = \mathbf{-9}$
- When we calculate $k*MAX_f(G)$, we get $(-3)*(-9) = \mathbf{27}$
- $\therefore MAX_f(k*G) \neq k*MAX_f(G)$ due to a counterexample

3a)

- Since $F(x,y) = 0\ \forall(x,y) \in \mathbb{Z}^2$, any infinite image G where $G\ R\ F$ must satisfy $G(x+a, y+b) = 0\ \forall(x,y)$ and some $(a,b) \in \mathbb{Z}^2$
- This means that $G(x,y)$ must also equal to $0\ \forall(x,y) \in \mathbb{Z}^2$
- $\therefore$ there is only one infinite image that is related to F, more specifically, the function $G(x,y) = 0\ \forall(x,y) \in \mathbb{Z}^2$

3b)

- Let H be an infinite image where $H\ R\ G$

- Then $\exists$ a pair of integers $(a, b) \mid \forall(x, y) \in \mathbb{Z}^2, H(x+a, y+b) = G(x, y)$. In particular, we have $H(a, b) = G(0, 0) = 1$
- If we consider the values of H in the first row $(y = 0)$ and the first column $(x = 0)$, we have:
  - ◇ $H(x+a, b) = G(x, 0) = 0 \ \forall x \neq 0$
  - ◇ $H(a, y+b) = G(0, y) = 0 \ \forall y \neq 0$
- $\therefore H$ must be constant along each row and each column, with the value of 0 everywhere except for $H(a, b) = 1$
- This means that any infinite image like above is related to G
- $\therefore$ the number of infinite images that are related to G is the same as the number of choices for the cell (a, b) which is infinite since $\mathbb{Z}^2$ is infinite

3c)

- Consider the infinite image H defined as follows:
  - ◇ $H(x, y) = 0$ if $(x, y) \in \{(2n, 2m) | n, m \in \mathbb{Z}\}$ (i.e, H is 0 at all even coordinates)
  - ◇ $H(x, y) = 1$ if $(x, y) \in \{(2n+1, 2m+1) | n, m \in \mathbb{Z}\}$ (i.e, H is 1 at all odd coordinates)
- We claim that H is related to exactly $F \wedge G$
- To see this, note that for any $(x, y) \in \mathbb{Z}^2$, either (x, y) is even or odd
  - ◇ If (x, y) is even:
    - ∗ $(x - 2n, y - 2m)$ is also even for any $(n, m) \in \mathbb{Z}^2$
    - ∗ $\therefore$ for any $(a, b) \in \mathbb{Z}^2$ we have $H(x+a, y+b) = 0 = F(x+a-a, y+b-b)$
  - ◇ If (x, y) is odd:
    - ∗ $(x - 2n - 1, y - 2m - 1)$ is also odd for any $(n, m) \in \mathbb{Z}^2$
    - ∗ $\therefore$ for any $(a, b) \in \mathbb{Z}^2$ we have $H(x+a, y+b) = 1 = G(x+a-a, y+b-b)$
- Thus, H is related to both $F \wedge G$
- To show that H is not realted to any other infinite image, let's take another image K such that $H \ R \ K$
- This means $H(x+a, y+b) = K(x, y)$
- Consider the case where $(a, b) = (0, 0)$, then $\forall(x, y) \in \mathbb{Z}^2$ we have $H(x, y) = K(x, y)$
- In particular, this implies that for all odd $(x, y) \in \mathbb{Z}^2, K(x, y) = 1$ and for all even $(x, y) \in \mathbb{Z}^2, K(x, y) = 0$
- However, this means that K has the same pattern as G which we have already shown is related to H

4a)

- f^C can be formally defined as follows:
  - ◇ $\forall(x, y) \in 0..M-1 \times 0..N-1, f^C(x, y) = f(x, y)$
  - ◇ $\forall(x, y) \notin 0..M-1 \times 0..N-1, \forall(i, j) \in \mathbb{Z}^2, f^C(x+iM, y+jN) = f(x, y)$

4b)

- f^R can be formally defined as follows:
  - ◇ $f^R(x, y) = f(|x'|, |y'|)$ where $x' \wedge y'$ are defined as follows:
    - ∗ $x' = x$ if $0 \leq x < M$
    - ∗ $x' = -x$ if $-M \leq x < 0$
    - ∗ $x' = 2M - x - 1$ if $x \geq M$
    - ∗ $y' = y$ if $0 \leq y < N$
    - ∗ $y' = -y$ if $-N \leq y < 0$
    - ∗ $y' = 2N - x - 1$ if $y \geq N$

5a)

- There is only **1** Z-image that are also C-images which is the 0 image
- [...0 0 0...]

5b)

- There is only **1** Z-image that are also R-images which is the 0 image
- [...0 0 0...]

5c)

- R-images are also C-images due to this concept:
- If we take an R-image like $[1\ 2\ 3]^R$, we can expand it out to $[...1\ 2\ 3\ 3\ 2\ 1\ 1\ 2\ 3...]^R$
- We can see that in this case, the values [3 2 1] are seen when $2m \times 2n$ which is reflected
- However, we can also see that the values [1 2 3] are seen when $4m \times 4n$ which corresponds to the C-image because the original was $[1\ 2\ 3]^R$

5d)

- If we take a C-image defined as $[1\ 2\ 3]^C$, expanding it out gets us $[1\ 2\ 3\ 1\ 2\ 3]^C$ which is not reflected and thus, not an R-image

6a)

- The infinite image that has no discernible patterns such as zero padding, circular indexing, or reflected indexing, but rather consists of random values would allow for no minimum generator
- For example, if we add 1s in random spots:
  - [...1 3 1 1 2 1 1...] does not have a minimum generator
  - The 1 values can be replaced by any integer which would allow for it to be an infinite image

6b)

- An infinite image with 1 value in it would by definition have exactly 1 minimum generator which would be itself
- For example:
  - The infinite image $[...1...]^C$
  - This image has exactly 1 minimum generator defined as [1]

6c)

- An infinite image with 2 values repeating would by definition have exactly 2 minimum generators which would be the 2 values in either direction
- For example:
  - The infinite image $[...1\ 0\ 1\ 0\ 1\ 0...]^C$
  - This image has exactly 2 minimum generators defined as [1 0] and [0 1]