Sept 13, 2021 Math 2000 Lecture #2

Elementary Connectives

- We use elementary connectives to tie together atomic statements and make a new statements

1. AND Connective $\_ \wedge \_$

Let A and B be atomic statements

$A \Rightarrow x > 0$, $B \Rightarrow x^2 > 0$

$A \wedge B$

" $x > 0$ and $x^2 > 0$

ex: $x = 0$

A	B	$A \wedge B$
F	F	F

$x = -1$

A	B	$A \wedge B$
F	T	F

$x = 1$

A	B	$A \wedge B$
T	T	T

2. OR Connective $\_ \vee \_$

- $A \vee B$ is true if A is true or B is true or both are true

$A \Rightarrow$ cats are animals T
$B \Rightarrow$ cats are plants F
$\} A \vee B \Rightarrow T$

Truth Tables

A	B	$A \wedge B$	$(A \wedge B) \vee B$	
F	F	F	F	
F	T	F	T	
T	F	F	F	
T	T	T	T	

3. NOT Connective $\neg$

- not negates the contents that it is applied to

A	$\neg A$
F	T
T	F

ex: A = I am happily married

$\neg A$ = I am not happily married

ex: $x > 4 = B$

$\neg B = x \leq 4$

Defn: Tautology

- a logical statement that is always true, no matter the value of its component atomic statements

Ex: Carl is a dog OR Carl is not a dog.

$A \vee \neg A$

A	$\neg A$	$A \vee \neg A$
T	F	T
F	T	T

Defn: Contradiction

- a logical statement that is always false

Ex: Balloons are dangerous and balloons are not dangerous

$A \wedge \neg A$

$A \wedge \neg A$	A	$\neg A$
F	F	T
F	T	F

1. Idempotent Law

$A \equiv A \wedge A$ $\quad$ $A \equiv A \vee A$

3. Associative Law

$A \vee B \vee C \vee D$ | $(A \wedge B) \wedge C \equiv A \wedge (B \wedge C)$

"Can solve between similar connectives in any order"

2. Identity Law

$A \wedge (B \vee \neg B) \equiv A \wedge T \equiv A$

↓ Tautology

$A \vee (B \wedge \neg B) \equiv A \vee F \equiv A$

4. Distributive Law

☆ if no brackets, do OR first

ex: $6(3+5) \Rightarrow (18+30)$

$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$

5. De Morgan's Law

$\neg(A \wedge B) \equiv \neg A \vee \neg B$

$\neg(A \vee B) \equiv \neg A \wedge \neg B$

Defn: Whenever two logical statements have the same T/F values for their column on the truth table, then they are LOGICALLY EQUIVALENT

ex:

A	B	$A \wedge B$	$\neg(A \wedge B)$	$\neg A$	$\neg B$	$\neg A \vee \neg B$
F	F	F	T	T	T	T
T	F	F	T	F	T	T
F	T	F	T	T	F	T
T	T	T	F	F	F	F

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IF - THEN IMPLICATION

$A \land B$ $A \lor B$

commutative

$B \land A$ $B \lor A$

$A \to B$

- "If A, then B"
- "A suffices for B"
- "B is necessary for A"

A	B	$A \to B$
F	F	T
F	T	T
T	F	F
T	T	T

ex: if pigs can fly, then my skin will burn off.
$F \to F$: T

ex: if it rains today, then I will get wet

Logical IF-THEN does not rely on cause effect relationships

$A \to B \equiv \neg A \lor B$

why does this matter?

A is true

B, is it true

we show

$A \to B$

modus ponens

Def'n, the converse of $A \to B$ is a logically distinct statement written as $B \to A$

Ex: if x is a cat, then x has a tail

converse: if x has a tail, then x is a cat

Def'n, the iff ($\Leftrightarrow$)

- is a short hand connective

$A \Leftrightarrow B \equiv (A \to B) \land (B \to A)$

A	B	$A \Leftrightarrow B$
F	F	T
F	T	F
T	F	F
T	T	T

$A \Leftrightarrow B$ is stronger than $A \to B$ as it is a reversible condition

Def'n, The contrapositive

- is a logically equivalent statement to $A \to B$, and it is written as $\neg B \to \neg A$

A	B	$\neg A$	$\neg B$	$A \to B$	$\neg B \to \neg A$
F	F	T	T	T	T
F	T	T	F	T	T
T	F	F	T	F	F
T	T	F	F	T	T

ex: If I eat cheese before bed, then I will have good dreams

$A \to B \equiv \neg B \to \neg A$

if I won't have good dreams, then I did not eat cheese before bed

ex: if $x^2 > 0$, then $x > 0$

if $x \leq 0$, then $x^2 \leq 0$

Even false statements remain false in contrapositive

Quantifiers

when we say x is a number, animal, etc, we must prescribe a universe for x

Defn, ($\forall$) for all
- a symbol which says we refer to all possible x is our universe

Defn, ($\exists$) there exists
- this symbol suggests we refer to at least one x in our universe

Ex: $U=$ all possible birds on earth
$\forall x \in U$, x has wings
$\exists x \in U$, x projectile vomits for defence

$\exists x \in U$, x can fly
$\exists! x \in U$, x is a duck billed geese

Ex, $U = \{0, 1, 6\}$
$\forall x \in U, (x-1)(x-6)x = 0$
"for all can always be relaxed to there exists" "if for all is false, we can make no conclusion about $\exists$"

$\forall \to \exists$, $\nexists \to \cancel{\forall}$
Defn: the symbol $\nexists$
- There exists no values in our universe which will satisfy the equation
Ex: $\nexists x \in \mathbb{Q}, x^2 = 2$
$\equiv$
$\forall x \in \mathbb{Q}, x^2 \neq 2$

Defn: we use the shorthand $\exists$, s.t. in order to mathematical equations easier to read

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Quantifiers (con't)

$\exists$: there exists
$\forall$: for all

$\forall x \in U\ \forall y \in U \equiv \forall x, y \in U$
$\exists x \in U, \exists y \in U \equiv \exists x, y \in U$

Ex: $\forall x\ \exists y \ni x > y$
"For all x, there exists y s.t. x is greater than y"
= T

Ex: $\exists y \in U, \forall x \in U,\ x > y$
"there exists y, s.t. for all x, x is greater than y"
= F

Quantifiers AND Implications

$\forall x \in U : A \to B$
$(\forall x \in U,\ A \to B)$

"whether or not a quantifier is inside an "if-then" can change the statement"

Ex, $\forall x \in \mathbb{R}$, if $x > 0$ then $e^x > 1$ } this example is true
$x > 0\quad T \to T$
$x \leq 0\quad F \to F$

Ex, if $\forall x \in \mathbb{R}\ x > 0$, then $\forall x \in \mathbb{R}\ e^x > 1$
$\downarrow$ F $\quad\downarrow$ F

and quantifiers

Contrapositive:
- Remark
  - When we take the contrapositive and the quantifiers are outside the implication
    ↳ we do not alter those quantifiers
  - But when they are inside of our implication, we must flip our quantifiers

FLIPPING:

$\forall x \in U$, $B(x)$ is true
$\exists x \in U$, $B(x)$ is false } $\neg$

$\exists x \in U$, $B(x)$ is true
$\forall x \in U$, $B(x)$ is false } $\neg$

$\forall x \in U, \exists y \in U$ $B(x)$ is true
$\exists x \in U, \forall y \in U$ $B(x)$ is false } $\neg$

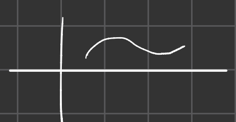
$\forall x \in U$ is actually a large set of $\wedge$ connectives
$\forall x \in \mathbb{N}, x > -1$
$\downarrow$
$(6 > -1) \wedge (7 > -1) \wedge (13 > -1) \wedge \ldots$

$\exists x \in U$ is actually a large set of $\vee$ connectives
$\exists x \in \mathbb{N}, x^2 = 16$
$\downarrow$
$(2^2 \neq 16) \vee (3^2 \neq 16) \vee (4^2 = 16)$ ✓

If we negate quantifiers, we are actually just using de morgans law

Ex, $\forall x \in \mathbb{Z}$ if $x > 0$ then $5x^3 > 0$
contrapositive of implication
$\forall x \in \mathbb{Z}$ if $5x^3 \leq 0$ then $x \leq 0$

Ex: if $\forall x \in (a,b)$, $f(x)$ is differentiable and $|f'(x)| \leq 1$, then $\forall c, d \in (a,b)$ $|f(c) - f(d)| < b - a$



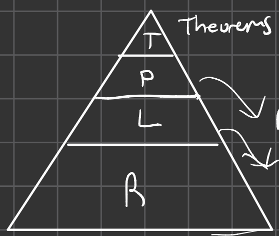
Contra: if $\exists c, d \in (a,b)$, $|f(c) - f(d)| \geq b - a$, then $\exists x \in (a,b)$, $f(x)$ is not differentiable
or
$|f'(x)| > 1$

Sept 20^{th} Math*2000 Lecture #5

Proof techniques

- direct proof
- proof by induction
- proof by contrapositive
- proof by contradiction

- we restrict ourselves to our axioms and our definitions in order to prove things



Theorems: big ideas in math that are general and require a lot of proof

properties: a lesser idea often used to support a theorem

lemma: a lesser version of a property

↳ a simple idea that we dont even bother proving

Definitions: a truth or axiom that we formalize in our math

- The distinction between these terms can sometime be arbitrary

Direct Proof Method

- $A \to B$

↳ assume that A is true

↳ use that knowledge to show that $A \to B$ is true

↳ conclude that B must be true

Property: show that the square of an even number is even

Defn, An even number $n \in \mathbb{Z}$ has the property that $n/2$ has a remainder of 0

↳ "if $\underbrace{n \in \mathbb{Z} \text{ is even}}_{A}$, then n^2 is even"

- assume we have n that is even
- if $n = 2k$ where $k \in \mathbb{Z}$

$A \to B$, to show this we should plug in and see what n^2 equals

$n^2 = (2k)^2 = 4k^2 = 2(2k^2)$

↳ some integer, so this number is even

- So we have shown how $A \to B$ is true

$\therefore$ B is true (n^2 is even)

Defn, a number $n \in \mathbb{Z}$ is odd if $n/2$ has a remainder of 1

Remark, all $n \in \mathbb{Z}$ are either odd or even.

Ex, show that if n is odd, then n^2 is odd

$A \to n = 2k + 1$, where $k \in \mathbb{Z}$

$A \to B \to (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$

Since $2k^2 + 2k$ is some integer, we conclude that $2(2k^2+2k)+1$ is odd

$\therefore n^2$ will be odd when n is odd

Ex, the value $p(n) = n^2 + 3n + 2$ is even for all $n \in \mathbb{Z}$

A: we have $n \in \mathbb{Z}$

$A \to B$: $p(n) = n^2 + 3n + 2$

$(n+1)(n+2)$

- what ends up happening is dependent on our cases

Alternative:
one must be odd
and one must be even
adjacent in $\mathbb{Z}$

CASES in our proof

- sometimes we have to break up a for-all ($\forall$) statement into several cases that when combined together make up our universe of values
- we refer to this as a "proof by cases"

Ex, Case 1: $n=0$
Case 2: $n>0$
Case 3: $n<0$
} FINE

Ex, Case 1: n is even
Case 2: n is odd

Case 1: $P(2k) = \underbrace{(2k+1)}_{\text{odd}}\underbrace{(2k+2)}_{\text{even}}$
$= (2k+1)(k+1)(2)$
= some even value

Case 2: $P(2k+1) = (2k+2)(2k+3)$
$= 2\underbrace{(k+1)(2k+3)}_{\text{some int}}$
= some even value

$\therefore$ $P(n)$ is even in both cases so the statement is true for all ($\forall$) n

Ex, The triangle inequality holds true
$\forall x, y \in \mathbb{R},\ |x+y| \leq |x| + |y|$

So we have another problem that warrants a proof by cases.

Case 1: $x \geq 0$, $y \geq 0$
Case 2: $x \geq 0$, $y < 0$
Case 3: $x < 0$, $y \geq 0$
Case 4: $x < 0$, $y < 0$

C1: $|x+y| = x+y = |x| + |y|$
$\leq |x| + |y|$

C2.1 we have more cases ... where $x+y \geq 0$
$|x+y| = x+y < x+(-y) = |x| + |y|$

C2.2 where $x+y<0$
$|x+y| = -x-y < x-y = |x|+|y|$

C3.1: without loss of generality, we get the same result as 2.1 and 2.2 if we simply swap x and y
C3.2:

C4: $|x+y| = -x-y = (-x)+(-y) = |x| + |y|$

we have proven this inequality for all cases of x and y
$\therefore$ it is true $\forall x, y \in \mathbb{R}$