Assignment #4 Maneesh Wijewardhana 2021-11-30

1. S cannot truly exist.

Proof, If $|S| < |\mathbb{N}|$, $\exists$ an injection from S to $\mathbb{N}$. We can then take its range which is equal to S. Now we can show that a subset of $\mathbb{N}$ is either finite or equal to $\mathbb{N}$ itself.

If this subset is finite, then its power set is finite. If it is countably infinite, then its power set is uncountable.

$\therefore$ $\mathbb{N}$ is not equal with any power set nor has an infinite cardinal strictly smaller than itself.

2a) Given, • $S \subset \mathbb{R}$

• Relation R on $S \mid (x,y) \in R$ if $x < y$

• R is strict total order relation on S

• 2 elements $(x,z) \in R$ can be adjacent if $\nexists\, y \in S \mid (x,y),(y,z) \in R$

If $S = \mathbb{Z}$, yes it is possible

Proof, For $(x,z) \in R$, $x = 3 \wedge z = 4$

$(3,y),(y,4) \in R$

if $y \in S$ where $S = \mathbb{Z}$ then yes it is possible for (x,z) to be adjacent since theres no $y \in S$ where $S = 4$ s.t. both $(4,y)$ and $(y,4)$ can be related to R.

if $y = 4$, $(3,4) \in R$ but $(4,4) \notin R$

b) If $S = \mathbb{Q}$, no it is not possible

Proof, a number is rational if it can be represented as $\frac{x}{y}$ where $y \neq 0$ and $x, y \in \mathbb{Z}$.

Since x and y are rational, it means there are infinite possible choices for x and y because $|\mathbb{Z}| = \infty$.

$\forall a, b$ where $a = \frac{x_1}{y_1} \wedge b = \frac{x_2}{y_2}$, there would be infinitely many choices of b that would be bigger than a and vice versa. Whatever value chosen for x_1, y_1, x_2, y_2, there will always be a choice of x and y that would be inbetween.

Ex, $1/2 < 2/2$, it looks like there are no numbers between those two but we can add or subtract from the numbers to form something like $3/4$ which is bigger than $1/2$ but smaller than $2/2$.

3.

A	B	C	a) A→B	b) B→C	c) A→C
F	F	F	T	T	T
F	F	T	T	T	T
F	T	F	T	F	T
F	T	T	T	T	T
T	F	F	F	T	F
T	F	T	F	T	T
T	T	F	T	F	F
T	T	T	T	T	T

i) lets look at the cases where a) and b) are true. NO

ii) " " a) and c) " " . YES

iii) " " b) and c) " " . YES

4a) If we exclude this axiom, then we can no longer guarantee that the set is not empty which defines the standard system of $\mathbb{N}$

b) If we exclude this axiom, then we can no longer guarantee that there is more than one element in the set. This breaks axiom e) which generalizes induction as now there may not be $1'$

c) If we exclude this axiom, then it will suggest that $(-\infty, 0]$ are included in our set $\mathbb{N}$. This of course is a contradiction as $\mathbb{N}$ goes from $[1, \infty)$ which is also why it states $n' \neq 1$ because 1 is not greater than anything in $\mathbb{N}$.

d) If we exclude this axiom, then it suggests that elements may appear more than once which is considered a multiset. This is because if m' and n' are equal but m is not equal to n, then it means the same element comes after both. This ideaology contradicts the standard number system $\mathbb{N}$

e) If we exclude this axiom, it would be quite impossible to prove any theorem as we wouldn't be able to get the next natural number. We would lose the ability to apply the successor function which allows us to prove theorems such as $a+b=b+a$ and etc. This ultimately leads us unable to prove that all of the natural numbers follow the other axioms

5. Assume no commutative property for multiplication.
to prove $n \cdot m = m \cdot n$ we first have to do induction over n with a base case of $n=1$.
then, inductive step for $n=k$ so assume $m \cdot k = k \cdot m$ then prove $m \cdot k' = k' \cdot m$

Proof, $\forall n, m \in \mathbb{N}$, $n \cdot m = m \cdot n$

Base case: $n=1$

$\rightarrow 1 \cdot m = m \cdot 1$

$\rightarrow 1 \cdot m = m$ (first axiom of multiplication)

$\rightarrow$ divide both sides by $1 \rightarrow m = m$

Inductive step: assume true for $n=k$ and $k \cdot m = m \cdot k$

Prove true for $(k+1) \cdot m = m \cdot (k+1)$

$k+1 = k'$ (first axiom of addition)

$(k+1) \cdot m = m \cdot k'$

$(k \cdot m) + (1 \cdot m) = m \cdot k'$ (distributive property)

$m \cdot (k+1) = m \cdot k'$ (distributive property)

$m \cdot k' = m \cdot k'$ (first axiom of addition)

$\therefore$ we proved $(k+1) \cdot m = m \cdot (k+1)$ is true using induction so consequently, $n \cdot m = m \cdot n \ \forall n, m$