Assignment #3 Maneesh Wijewardhana 2021/11/18

1. unordered unique subsets: $\binom{n-1+k}{k}$

SMALL

- 0 toppings: $\binom{18-1+0}{0}\cdot 5\cdot 3$
- 1 topping: $\binom{18-1+1}{1}\cdot 5\cdot 3$
- 2 topping: $\binom{18-1+2}{2}\cdot 5\cdot 3$
- 3 topping: $\binom{18-1+3}{3}\cdot 5\cdot 3$

$15+270+2565+17100=19950$

Medium

- 4 topping: $\binom{18-1+4}{4}\cdot 5\cdot 3=89775$

$89775+19950=104725 \longrightarrow 104725+19950=129675$

have to do like small because 2 crust options

Large

- 0 topping: $\binom{18-1+0}{0}\cdot 5\cdot 2$
- 1 topping: $\binom{18-1+1}{1}\cdot 5\cdot 2$
- 2 topping: $\binom{18-1+2}{2}\cdot 5\cdot 2$
- 3 topping: $\binom{18-1+3}{3}\cdot 5\cdot 2$
- 4 topping: $\binom{18-1+4}{4}\cdot 5\cdot 2$

$10+180+1710+11400+59850=73150$

Extra Large

- 5 topping: $\binom{18-1+5}{5}\cdot 5\cdot 2=263340$

Now... $263340+73150$
$=336490$
then... $336490+73150$
$=409640$

Finally... $129675+409640=539315$

$\therefore$ 539315 possible varieties

2 a) If A has n elements, then $A\times A$ has n^2 pairs. This relation could include none, any, or all of those pairs. The # of subsets of a set with K elements is 2^k
$\therefore$ # of unique relations is $2^{(n^2)}$

b) For a given value of n, we can use bell numbers. Bell numbers count the possible partitions of a given set and also the number of equivalent relations on it. For example, $B_3=5$ because the 3-element set $\{a,b,c\}$ can be partioned in 5 distinct ways:

$\{\{a\},\{b\},\{c\}\}$
$\{\{a\},\{b,c\}\}$
$\{\{b\},\{a,c\}\}$
$\{\{c\},\{a,b\}\}$
$\{\{a,b,c\}\}$

\# of equivalence relations

$\therefore$ to calculate B_n, we can use a Bell triangle as follows

1 B_1
1 2 B_2
2 3 5 B_3
5 7 10 15 B_4 ... B_n

c) Recall, strict total order is not reflexive, transitive, and connective.
For the empty set ($n=0$), there only exists one order since there are no elements to order. There are $n!$ strict total orders for this example as $0!=1$.
Lets assume for n elements, there are $n!$ strict total orders. Lets prove for $n+1$ elements. If we select n elements, every order on the $n+1$ elements will induce an order on those n elements. Also, if the induced order was different, the original order must have been different too. By assumption, there are $n!$ ways to order them.
When we put back element $n+1$, no matter the order, there are always $n+1$ positions to insert the additional element
$\therefore$ the set of $n+1$ elements has exactly $n!(n+1)=(n+1)!$ different strict total orders
$\therefore$ by induction, a set of n elements always has $n!$ strict total orders.

d) Let $f: A \to A$ and suppose $A = \{a_1, a_2, a_3\}$. f can map a_1 to any of the elements which means 3 choices for the value of $f(a_1)$. Similarly, there are 3 choices for the value of $f(a_2)$. $\therefore$ the total number of functions $f: A \to A$ is $3 \times 3 \times 3 = 3^3$
In general, if X has m elements and Y has n elements, then the # of unique functions from $X \to Y$ is n^m. In $A \to A$, it would be n^n

e) In a), our size for the # of different unique relations was $2^{(n^2)}$.
In d), our size for the # of unique functions was n^n
For a general choice of n, there will always be more unique relations than unique functions for $A \to A$.

3. If $f: A \to B$ is surjective, then every element in A maps to an element in B which means $|A| \geq |B|$.
If $g: B \to A$ is surjective, then every element in B maps to an element in A which means $|B| \geq |A|$.

If $|A| \geq |B|$ and $|B| \geq |A|$ then $|A| = |B|$.
We know that if 2 sets have the same cardinality and a function between them is surjective, then it is also injective.

Schroeder - Bernstein Theorem states that if $f: A \to B$ is injective and $g: B \to A$ is also injective, then $\exists$ a bijective function between A and B.

4 a) Let S be an infinite set and let $a_0 \in S$.
Since S is infinite, $\exists a_1 \in S, a_1 \neq a_0 \land \exists a_2 \in S, a_2 \neq a_0, a_2 \neq a_1$ and so on
$\therefore$ we can keep picking elements out of S and assign to $a_0, a_1, a_2 \ldots$ but it will never terminate the fact that S is infinite. $\therefore$ the set $T = \{a_0, a_1, a_2 \ldots\} \subseteq S$

b) Suppose $\exists$ an injection $f: \mathbb{N} \to T$. Let I be the image of f.
From knowing that the injection of an image is a bijection, we say $I^{-1}: I \to \mathbb{N}$ is a bijection
$\therefore$ I is a countably infinite subset of T.
Now suppose $\exists$ a surjection $\phi: \mathbb{N} \to T$.
We know that if T is a non-empty set, then T is countable iff $\exists$ a surjection $f: \mathbb{N} \to T$
This means T is countably infinite and since a set is a subset of itself, T is a countably infinite subset of itself

c) not sure how to answer it

d) Let $S = \mathbb{R}$ and T = even numbers in $\mathbb{R}$.
$\therefore \exists 2 \in S, 2 \neq 0 \land \exists 4 \in S, 4 \neq 2, 4 \neq 0$, and so on
we can keep picking elements out of S but S will always be infinite so $T \subseteq S$

5. A countable set is infinite if it can be placed in a bijection with the natural numbers

i) $S = \{T_1, T_2, T_3, \dots\}$
$S = \{T_n \mid n \in \mathbb{N}\}$
T_n is an uncountably infinite set
$T_1 = \{x \in \mathbb{N} \mid 0 < x \leq 1\}$
$T_2 = \{x \in \mathbb{N} \mid 0 < x \leq 2\}$
$T_3 = \{x \in \mathbb{N} \mid 0 < x \leq 3\}$
$\vdots$

$\bigcap_{i=1}^{\infty} T_i = \{1\}$ which is finite but non-empty

ii) Does not exist because it is not possible.

iii) $S = \{T_1, T_2, T_3 \dots\}$
$S = \{T_n \mid n \in \mathbb{N}\}$
T_n is an uncountably infinite set
Lets say:
$T_1 = \{x \in \mathbb{R} \mid 0 \leq x \leq 0.1\}$
$T_2 = \{x \in \mathbb{R} \mid 0 \leq x \leq 0.01\}$
$T_3 = \{x \in \mathbb{R} \mid 0 \leq x \leq 0.001\}$
$\vdots$

We see that $|T_1| = |T_2| = |T_3|$ are all uncountably infinite
so... $\bigcap_{i=1}^{\infty} T_i =$ another uncountably infinite set