Assignment #1 | Due date: 2022-01-31 | CIS* 3490 | Maneesh Wijewardhana (1125828)

1. a) $(n^2+1)^{10}$

$(n^2+1)^{10} \approx (n^2)^{10} = n^{20} \in \Theta(n^{20})$

Proof:

$$\lim_{n\to\infty} \frac{(n^2+1)^{10}}{n^{20}}$$
$$= \lim_{n\to\infty} \frac{(n^2+1)^{10}}{(n^2)^{10}}$$
$$= \lim_{n\to\infty} \left(\frac{(n^2+1)}{n^2}\right)^{10}$$
$$= \lim_{n\to\infty} \left(n^2/n^2 + 1/n^2\right)^{10}$$
$$= \lim_{n\to\infty} \left(1 + 1/n^2\right)^{10} = 1$$

∵ result is a constant, $(n^2+1)^{10}$ and n^{20} have the same asymptotic growth and thus, n^{20} can be the effeciency class of $(n^2+1)^{10}$

b) $\sqrt{10n^2+7n+3}$

$\sqrt{10n^2+7n+3} \approx \sqrt{10n^2} = \sqrt{10}n \in \Theta(n)$

Proof:

$$\lim_{n\to\infty} \frac{\sqrt{10n^2+7n+3}}{n}$$
$$= \lim_{n\to\infty} \frac{\sqrt{10n^2+7n+3}}{n^2}$$
$$= \lim_{n\to\infty} \sqrt{10 + \frac{7}{n} + \frac{3}{n^2}} = \sqrt{10}$$

∴ $\sqrt{10n^2+7n+3} \in \Theta(n)$

c) $2n\lg(n+2)^2 + (n+2)^2 \lg\frac{n}{2}$

$= 2n2\lg(n+2) + (n+2)^2(\lg n - 1) \in \Theta(n\lg n) + \Theta(n^2\lg n) = \Theta(n^2\lg n)$

d) $2^{n+1} + 3^{n-1}$

$= 2^n \cdot 2 + 3^n \cdot \frac{1}{3} \in \Theta(2^n) + \Theta(3^n) = \Theta(3^n)$

e) $\lfloor \log_2 n \rfloor \approx \log_2 n \in \Theta(\log n)$

Proof:

$x-1 < \lfloor x \rfloor \le x$ so $\lfloor \log_2 n \rfloor \le \log_2 n$ upper bound and lower bound

$\lfloor \log_2 n \rfloor > \log_2 n - 1 \ge \log_2 n - \frac{1}{2}\log_2 n$ (for every $n \ge 4$) $= \frac{1}{2}\log_2 n$

∴ $\lfloor \log_2 n \rfloor \in \Theta(\log_2 n) = \Theta(\log n)$

2. $(n-2)! \in \Theta(n!)$, $5\lg(n+100)^{10} \in \Theta(\log n)$, $2^{2n} \in \Theta(4^n)$, $0.001n^4+3n^3+1 \in \Theta(n^4)$, $\ln^2 n \in \Theta(\log^2 n)$, $\sqrt[3]{n} \in \Theta(n^{1/3})$, $3^n \in \Theta(3^n)$

Ordered:

$5\lg(n+100)^{10}$, $\ln^2 n$, $\sqrt[3]{n}$, $0.001n^4+3n^3+1$, 3^n, 2^{2n}, $(n-2)!$

3a) $= \sum_{i=1}^{500} (2i-1) = \sum_{i=1}^{500} 2i - \sum_{i=1}^{500} 1 = 2\left(\frac{500 \cdot 501}{2}\right) - 500 = 250000$

b) $= \sum_{i=1}^{10} 2^i = \sum_{i=0}^{10} 2^i - 1 = (2^{11}-1) - 1 = 2046$

c) $\sum_{i=3}^{n+1} 1 = (n+1) - 3 + 1 = n-1$

d) $\sum_{i=3}^{n+1} i = \sum_{i=0}^{n+1} i - \sum_{i=0}^{2} i = \frac{(n+1)(n+2)}{2} - 3 = \frac{n^2+3n-4}{2}$

e) $\sum_{i=0}^{n-1} i(i+1) = \sum_{i=0}^{n-1} i^2 + i = \sum_{i=0}^{n-1} i^2 + \sum_{i=0}^{n-1} i = \frac{(n-1)n(2n-1)}{6} + \frac{(n-1)n}{2} = \frac{(n^2-1)n}{3}$

f) $\sum_{j=1}^{n} 3^{j+1} = 3\sum_{j=1}^{n} 3^j = 3\left(\sum_{j=0}^{n} 3^j - 1\right) = 3\left(\frac{3^{n+1}-1}{3-1} - 1\right) = \frac{3^{n+2}-9}{2}$

g) $\sum_{i=1}^{n}\sum_{j=1}^{n} ij = \sum_{i=1}^{n} i \sum_{j=1}^{n} j = \sum_{i=1}^{n} i \frac{n(n+1)}{2} = \frac{n(n+1)}{2} \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \frac{n(n+1)}{2} = \frac{n^2(n+1)^2}{4}$

h) $\sum_{i=1}^{n} 1/i(i+1) = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1}\right)$

$= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) + \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1 - \frac{1}{n+1} = \frac{n}{n+1}$

4.a) $\sum_{i=0}^{n-1} (i^2+1)^2 = \sum_{i=0}^{n-1} (i^4 + 2i^2 + 1) = \sum_{i=0}^{n-1} i^4 + 2\sum_{i=0}^{n-1} i^2 + \sum_{i=0}^{n-1} 1 \in \Theta(n^5) + \Theta(n^3) + \Theta(n) = \Theta(n^5)$

b) $\sum_{i=2}^{n-1} lg\, i^2 = \sum_{i=2}^{n-1} 2\log_2 i = 2\sum_{i=2}^{n-1} \log_2 i = 2\sum_{i=1}^{n} \log_2 i - 2\log_2 n \in 2\Theta(n \log n) - \Theta(\log n) = \Theta(n \log n)$

c) $\sum_{i=1}^{n} (i+1)2^{i-1} = \sum_{i=1}^{n} i2^{i-1} + \sum_{i=1}^{n} 2^{i-1} = \frac{1}{2}\sum_{i=1}^{n} i2^i + \sum_{j=0}^{n-1} 2^j \in \Theta(n2^n) + \Theta(2^n) = \Theta(n2^n)$

d) $\sum_{i=0}^{n-1}\sum_{j=0}^{i-1} (i+j) = \sum_{i=0}^{n-1}\left(\sum_{j=0}^{i-1} i + \sum_{j=0}^{i-1} j\right) = \sum_{i=0}^{n-1}\left(i^2 + \frac{(i-1)i}{2}\right) = \sum_{i=0}^{n-1}\left(\frac{3}{2}i^2 - \frac{1}{2}i\right) = \frac{3}{2}\sum_{i=0}^{n-1} i^2 - \frac{1}{2}\sum_{i=0}^{n-1} i \in \Theta(n^3) + \Theta(n^2) = \Theta(n^3)$

5. First Formula:
$D(n) = 2$, $M(n) = n$, $A(n) + S(n) = [(n-1) + (n-1)] + (n+1) = 3n-1$

Second Formula:
$D(n) = 2$, $M(n) = n+1$, $A(n) + S(n) = [(n-1) + (n-1)] + 2 = 2n$

6. i) $A(n) = 4A(n-1)$ for $n > 1$, $A(1) = 5$

$A(n) = 4A(n-1)$
$= 4[4A(n-2)] = 4^2 A(n-2)$
$= 4^2[4A(n-3)] = 4^3 A(n-3)$
$= \ldots$
$= 4^i A(n-i)$
$= \ldots$
$= 4^{n-1} A(1) = 5(4^{n-1}) \in \Theta(4^n)$ let $n - i = 1$

ii) $A(n) = A(n-1) + 3$ for $n > 1$, $A(1) = 0$

$A(n) = A(n-1) + 3$

$= [A(n-2)+3] + 3 = A(n-2) + 2 \cdot 3$

$= [A(n-3)+3] + 2 \cdot 3 = A(n-3) + 3 \cdot 3$

$= \ldots$

$= A(n-i) + i \cdot 3$

$= \ldots$

$= A(1) + (n-1) \cdot 3$ let $n-i=1$

$= 3(n-1) \in \Theta(n)$

iii) $A(n) = A(n-1) + 2n$ for $n > 0$, $A(0) = 0$

$A(n) = A(n-1) + 2n$

$= [A(n-2) + (n-1)] + 2n = A(n-2) + (n-1) + 2n$

$= [A(n-3) + (n-2)] + (n-1) + 2n = A(n-3) + (n-2) + (n-1) + 2n$

$= \ldots$

$= A(n-i) + (n-i+1) + (n-i+2) + \ldots + 2n$

$= \ldots$

$= A(0) + 1 + 2 + \ldots + 2n$ let $n-i=0$

$= 2n(n+1)/2 \in \Theta(n^2)$

iv) $A(n) = A(n/5) + 1$ for $n > 1$, $A(1) = 1$ (solve for $n = 5^k$)

$A(5^k) = A(5^{k-1}) + 1$

$= [A(5^{k-2}) + 1] + 1 = A(5^{k-2}) + 2$

$= [A(5^{k-3}) + 1] + 2 = A(5^{k-3}) + 3$

$= \ldots$

$= A(5^{k-i}) + i$

$= \ldots$

$= A(5^{k-k}) + k$ let $k-i=0$

$= 1 + \log_5 n \in \Theta(\log n)$

v) $A(n) = 2A(n/2) + \log_2 n$ for $n > 1$, $A(1) = 1$ (solve for $n = 2^k$)

$A(2^k) = 2A(2^{k-1}) + \log_2(2^k)$

$= 2(2A(2^{k-2}) + \log_2(2^k)) + \log_2(2^k)$

$= 2^2(A(2^{k-2}) + \log_2(2^k)) + \log_2(2^k)$

$= 2^2((2A(2^{k-3}) + \log_2(2^k)) + \log_2(2^k)) + \log_2(2^k)$

$= \ldots$

$= 2^i A(2^{k-i}) + (i)\log_2(2^k)$

$= 2^i A(2^{k-i}) + (i)k \log_2(2)$

$= 2^i A(2^{k-i}) + (i)k(1)$

$= 2^k A(2^{k-k}) + k^2$

$= 2\log_2 n \in \Theta(\log n)$

7. i) Sorts the numbers in descending order using selection sort

ii) n input size

iii) Comparison

iv) $C(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} [(n-1)-(i+1)+1] = \sum_{i=0}^{n-2} (n-1-i)$

$= \sum_{i=0}^{n-2} (n-1) - \sum_{i=0}^{n-2} i = (n-1) \sum_{i=0}^{n-2} 1 - \sum_{i=0}^{n-2} i$

$= (n-1)(n-1) + \frac{(n-2)(n-1)}{2} = (n-1)\left[(n-1) + \frac{n-2}{2}\right] = (n-1)\left(\frac{n}{2}\right) = \frac{n(n-1)}{2}$

v) $\Theta(n^2)$

8. i) sorts the number in ascending order using insertion sort

ii) n input size

iii) comparison

iv) $C(n) = \sum_{i=1}^{n-1} \sum_{j=0}^{i-1} 1 = \sum_{i=1}^{n-1} [(i-1)-0+1] = \sum_{i=1}^{n-1} [i-1+1] = \sum_{i=1}^{n-1} i$

$= \frac{n(n-1)}{2}$

v) $\Theta(n^2)$

9. i) Computes if given array is sorted from lowest to highest using recursion

ii) n input size

iii) comparison

iv) $A(1) = 1$, $A([1..n]) \rightarrow A([1..n-1]) \rightarrow A([1..n-2]) \ldots A([1,2]) \rightarrow A([1])$
every call but $A([1])$ executes 2 times but $A(1)$ only once so
$2(n-1)+1$ total

v) $\Theta(n)$

10. i) recursively searches for integer K using binary search

ii) $r+1$ input size

iii) Comparison

iv) Since $A(1)=1$, when $n=2^k$, $A(n) = A(1) + k = 1 + \log_2(n)$

$\log_2(n) \leq 1 + \log_2(n) \leq 2\log_2(n)$ for all $n \geq 2$

$\therefore A(n) = \Theta(\log_2(n))$

v) $\Theta(\log_2 n)$ because of the smoothness rule (Appendix B in TB)