University of Guelph
CIS 2910 Fall 2016 – FINAL (Dec. 6)
Instructor: Joe Sawada

First Name: ______________________________

Last Name: ______________________________

Student Number: ______________________________

Problem 1: (6 marks)		Problem 6: (4 marks)	
Problem 2: (6 marks)		Problem 7: (5 marks)	
Problem 3: (6 marks)		Problem 8: (6 marks)	
Problem 4: (4 marks)		Problem 9: (5 marks)	
Problem 5: (5 marks)			
		Total (47 marks)	

This test is closed book and lasts 120 minutes.
You may not use any electronic/mechanical computation devices.
There are 11 pages including the cover page.

Problem 1: [6 marks]

(a) TRUE or FALSE:
If a simple graph G can be colored with 4 colors, then G is planar.

Solution: FALSE. Consider $K_{3,3}$

(b) TRUE or FALSE:
A local ultimate frisbee league has 12 teams. If there is a total of 97 players in the league then one team must have exactly 9 players.

Solution: FALSE. 11 teams could have 7 players, and one could have 20 players.

(c) TRUE or FALSE:
If events E and F are dependent then $p(E \cap F) \neq p(E)p(F)$.

Solution: TRUE. It is the opposite of the requirements for independence.

(d) TRUE or FALSE:
If a simple graph G has unique edge weights then it has a unique minimum spanning tree.

Solution: This question had a typo: missed the word "connected". Without it, the answer is FALSE. Thus BOTH ANSWERS are accepted.

(e) TRUE or FALSE:
The number of ways to put 8 identical balls into 3 labeled boxes is $\binom{10}{8}$.

Solution: TRUE. Also equal to $\binom{10}{2}$.

(f) TRUE or FALSE:
If a simple graph G has exactly 2 vertices with odd degree, then the graph has an Euler path.

Solution: FALSE. The graph must also be connected for this to be true.

Problem 2: [6 marks] Multiple Choice.

(a) Which of the following most accurately defines a *directed graph*?

  (a) A graph $G = (V, E)$ consisting of a non-empty set of vertices V and an edge set E of unordered pairs of distinct elements from V.
  (b) A graph $G = (V, E)$ consisting of a non-empty set of vertices V and an edge set E of ordered pairs of elements from V.
  (c) A graph $G = (V, E)$ consisting of a non-empty set of vertices V and an edge set E of ordered pairs of distinct elements from V.
  (d) A graph $G = (V, E)$ consisting of vertices V and an edge set E of directed edges.
  (e) none of the above

  Solution: b

(b) Which of the following corresponds to the *Generalized Pigeonhole Principle*?

  (a) If $k + 1$ objects are placed into k boxes, then there a box containing two objects.
  (b) If more than k objects are placed into k boxes, then then at least one box contains two objects.
  (c) If N objects are placed into k boxes, then there is exactly one box containing at least $\lceil N/k \rceil$ objects.
  (d) If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.
  (e) none of the above

  Solution: d

(c) Which of the following most accurately defines the *chromatic number* of a simple graph G?

  (a) The number of colors that can be used to color the vertices of G.
  (b) The size of the largest clique (complete subgraph) in G.
  (c) The maximum number of colors required to color the vertices of G.
  (d) The minimum number of vertices that require the same color.
  (e) none of the above

  Solution: e

(d) How many binary strings of length 10 have either exactly 4 ones, or begin and end with the same bit?

(a) $\binom{10}{4} + 2^8$

(b) $\binom{10}{4} + 2^9$

(c) $\binom{10}{4} + 2^9 - \binom{8}{2}$

(d) $\binom{10}{4} + 2^9 - \binom{8}{4} - \binom{8}{2}$

(e) $\binom{10}{2} + \binom{8}{4}$

(f) none of the above

Solution: d (inclusion/exclusion)

(e) Consider the Monty Hall problem on 4 doors: there are 4 doors and behind one of the doors is a new car. You are asked to choose a door, after which Monty will open one of the other 3 doors that does not contain the car. What is the probability of winning the car if you now switch doors?

(a) 1/2

(b) 1/3

(c) 1/4

(d) 2/3

(e) 3/8

(f) none of the above

Solution: e (With 3/4 probability you chose wrong initially. In this case by switching there is a 1/2 chance you pick the right door from the remaining 2. So $3/4 * 1/2 = 3/8$).

(f) Consider a 6 by 3 grid graph. How many paths are there starting from the bottom left corner and ending at the top right if only moves up and to the right are allowed?

(a) 18

(b) 20

(c) 36

(d) 56

(e) 84

(f) none of the above

Solution: e ($\binom{9}{3} = \binom{9}{6} = 84$)

Problem 3: [6 marks]

(a) [**2 marks**] Given positive integers n, k with $n \geq k$, state Pascal's Identity.

Solution: $\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$

(b) [**2 marks**] Fill out the first 7 rows of Pascal's triangle with the proper integer values. The first row starts with a single 1.

Solution:

$$\begin{array}{c} 1 \\ 1\ 1 \\ 1\ 2\ 1 \\ 1\ 3\ 3\ 1 \\ 1\ 4\ 6\ 4\ 1 \\ 1\ 5\ 10\ 10\ 5\ 1 \\ 1\ 6\ 15\ 20\ 15\ 6\ 1 \end{array}$$

(c) [**2 marks**] Consider a set S with 25 distinct elements. How many ways can you select a subset of S that has at least 3 elements?

Solution: Take all subsets and subtract those with only 0,1, or 2 elements:

$$2^{25} - \binom{25}{0} - \binom{25}{1} - \binom{25}{2} \quad = \quad 2^{25} - 1 - 25 - 300 \quad = \quad 2^{25} - 326$$

Problem 4: [4 marks]

(a) [**1 mark**] Let $T(n) = 2T(n-1) + T(n-2)$ when $n > 2$ and let $T(1) = T(2) = 2$. What is $T(3)$ and $T(4)$?

Solution: $T(3) = 2T(2) + T(1) = 6$. $T(4) = 2T(3) + T(2) = 12 + 2 = 14$.

(b) [**1 mark**] Suppose event E happens with probability $1/2$. Suppose event F happens with probability $1/3$. Suppose both E and F happen with probability $1/5$. What is the probability that event E happens given that F happens?

Solution: $p(E \mid F) = p(E \cap F)/p(F) = (1/5)/(1/3) = 3/5$.

(c) [**2 marks**] A *ternary* string is a word over the alphabet $\{0, 1, 2\}$. Describe a simple **recursive** algorithm to generate all ternary words of length n in lexicographic order. For $n = 2$ it should list: 00, 01, 02, 10, 11,12, 20, 21, 22.

Solution: With the initial call of $\textsc{Gen}(1)$, and global array $a_1 a_2 \cdots a_n$ to store the current string:

function $\textsc{Gen}(t : \textit{integer})$
    if $t > n$ **then** $\textsc{Print}(a_1 a_2 \cdots a_n)$
    else
        $a_t \leftarrow 0 \quad \textsc{Gen}(t+1)$
        $a_t \leftarrow 1 \quad \textsc{Gen}(t+1)$
        $a_t \leftarrow 2 \quad \textsc{Gen}(t+1)$

Problem 5: [5 marks]

Canada currently has 335 members of parliament (MPs), 247 are male and 88 are female. By party affiliation, there are:

182 Liberal, 97 Conservative, 44 NDP, 10 Bloc, 1 Green, 1 Independent.

(a) **[1 mark]** How many ways can the Prime Minister choose 30 MPs to attend a ribbon cutting ceremony?

Solution: $\binom{335}{30}$

(b) **[2 marks]** Suppose 30 members are attending a special conference on the environment. Of this group exactly 20 are Liberal and 8 are Conservative. The other two members are from the other parties. How many ways can the MPs be seated in a row of 30 seats so that no two Conservatives sit next to each other?

Solution: Line up the 12 non-Conservatives in all 12! ways. That leaves 11 gaps plus 2 spaces on each end to place the 8 Conservatives, which can be ordered in 8! ways. Thus:

$$\binom{23}{8} \cdot 22! \cdot 8! \quad = \quad \frac{23! \cdot 22!}{15!}$$

(c) **[2 marks]** How many ways can the Prime Minister assign seats numbered 1 to 335 to the 335 MPs so that no female is sitting in a seat with an odd number?

Solution: There are 167 even numbered seats and 168 odd numbered seats for a total of 335 seats. There are $\binom{167}{88}$ ways to select the even numbered seats for the females, then 88! ways to assign those seats to the females. There is 247! ways to distribute the males to the remaining seats. Thus:

$$\binom{167}{88} \cdot 88! \cdot 247! \quad = \frac{167! \cdot 247!}{79!}$$

Problem 6: [4 marks]

(a) **[2 marks]** Prove or disprove: Every simple graph with integer edge weights has a unique minimum spanning tree.

Solution: **Counter example**: Take a simple triangle with edge weights all equal to 1. Each of the three possible spanning trees has weight 2.

Alternatively: as the questions did not specify the graph had to be connected, then the statement will be false since a disconnected graph does not have a spanning tree.

(b) **[2 marks]** Prove or disprove: In every simple graph with more than one vertex there exists two vertices with the same degree.

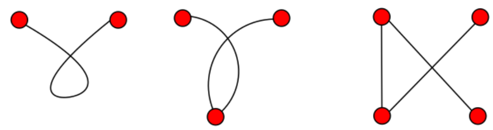
Solution: Consider the number of isolated vertices. If there is more than one, then there are two vertices with same degree of 0. If there is exactly one isolated vertex, then the remaining $n-1$ vertices can have degrees in the range $\{1, 2, \ldots, n-2\}$. Thus, by the PHP, two vertices must have the same degree. Finally, if there is no isolated vertex, then the n vertices must have degrees in the range $\{1, 2, \ldots, n-1\}$. Thus, again by the PHP, two vertices must have the same degree.

Problem 7: [5 marks]

Answer the following questions using the **smallest** number of vertices possible.

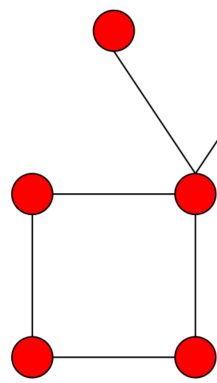
(a) [**1 mark**] Draw a simple graph that is not a planar representation.

Solution: Any of the following 3 were accepted for full marks. A K_5, or something closer to K_4 received half mark.



(b) [**2 marks**] Draw simple graph that has an Euler path with 2 pendant vertices and also contains C_4 as an induced subgraph.

Solution: Start with the C_4, and add two pendant vertices to one of the vertices of the C_4.



(c) [**2 marks**] Draw a simple graph with chromatic number 3 that does not contain either a C_3 or a C_5 as an induced subgraph

Solution: Draw a C_7.

Problem 8: [6 marks]

(a) **[2 marks]** State Euler's Formula.

Solution: $r = m - n + 2$, where r is the number of regions in a planar representation of a simple graph with n vertices and m edges.

(b) **[2 marks]** Use Euler's formula to prove the following statement: If a connected planar simple graph has $n \geq 3$ vertices and m edges and no cycles of length three, then $m \leq 2n - 4$.

Solution: Every edge in a planar representation of a graph contributes twice when summing the degrees of a region. Since every region has degree at least 4, we apply this bound to r, then apply Euler's formula:

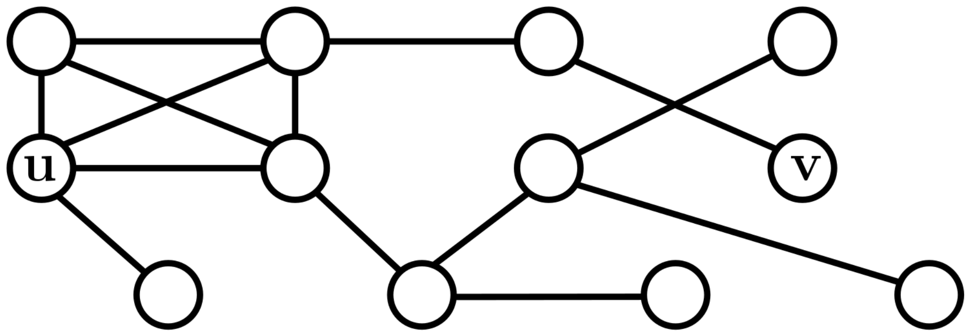
$$\begin{aligned} 2m &= \sum_{\text{all regions } R} deg(R) \\ &\geq 4r \\ &\geq 4(m - n + 2) \\ &\geq 4m - 4n + 8 \end{aligned}$$

Simplifying we get $m \leq 2n - 4$.

(c) **[2 marks]** Use the result from (b) to prove that $K_{3,3}$ is not planar.

Solution: Since $K_{3,3}$ is bipartite, it has no odd cycles and hence no cycles of length 3. Additionally $m = 9$ and $n = 6$ for this graph. Now suppose $K_{3,3}$ has a planar representation. Then by the result from (b), we have $m \leq 2n - 4$ which means that $9 \leq 8$. Contradiction. Thus, $K_{3,3}$ is not planar.

Problem 9: [5 marks] Answer the following questions **in the corresponding boxes** about the following simple graph. Each question is worth 0.5 marks.



(a) Is it a planar ? **Solution:** YES.

(b) Is this graph bipartite? **Solution:** NO.

(c) Does it have an Euler path? **Solution:** NO.

(d) Does it have a Hamilton path? **Solution:** NO.

(e) How many pendant vertices are there? **Solution:** 5.

(f) How many isolated vertices are there? **Solution:** 0.

(g) What is the degree of u? **Solution:** 4.

(h) Does it contain a cut vertex? **Solution:** YES.

(i) Is the vertex u adjacent to v? **Solution:** NO.

(j) What is the chromatic number of this graph? **Solution:** 4.

– END OF FINAL –