Assignment #1 - Maneesh Wijewardhana

Q1. $(a_i)_{i\in \mathbb{N}}$ and $(b_i)_{i\in \mathbb{Z}^+}$.

a) 7, 11, 15, 19, 23

$a_i = 4(i)+3,\ i\in \mathbb{N}$

$b_i = 4(i-1)+3,\ i\in \mathbb{Z}^+$

b) 3, −6, 12, −24, 48

$a_i = -3(-1)^i\cdot 2^{i-1},\ i\in \mathbb{N}$ | $b_i = -3(-1)^{i-1}\cdot 2^{i-2},\ i\in \mathbb{Z}^+$

c) 0, 11, 99, 1001, 9999

$a_i = 10^i-(-1)^i,\ i\in \mathbb{N}$ | $b_i = 10^{i-1}-(-1)^{i-1},\ i\in \mathbb{Z}^+$

d) 1, 0, 3, 0, 5, 0, 7, 0

$a_i = i+1 \bmod 2\cdot(i+1),\ i\in \mathbb{N}$

$b_i = i-1+1 \bmod 2\cdot(i-1+1),\ i\in \mathbb{N}$

$b_i = i \bmod 2\cdot(i),\ i\in \mathbb{N}$

Q2. **a)** $\sum_{j=0}^{8}(1+(-1)^j)$

$= \sum_{j=0}^{8}(1) + \sum_{j=0}^{8}(-1)^j$, $a_0 = 1$ for first sum, $a_0 = (-1)^0$ for second sum

$= a_0(n+1) + a_0\left[\frac{r^{n+1}-1}{r-1}\right]$

$= 1(8+1)+1\left[\frac{(-1)^9-1}{-1-1}\right]$

$= 10$

b) $\sum_{j=0}^{8}(3^j-2^j)$

$= \sum_{j=0}^{8}(3^j) - \sum_{j=0}^{8}(2^j)$, $a_0 = 3^0$ for first sum, $a_0 = 2^0$ for second sum

$= a_0\left[\frac{r^{n+1}-1}{r-1}\right] - a_0\left[\frac{r^{n+1}-1}{r-1}\right]$

$= 1\left[\frac{3^9-1}{3-1}\right] - 1\left[\frac{2^9-1}{2-1}\right]$

$= 9330$

c) $\sum_{j=0}^{8} (2 \cdot 3^j + 3 \cdot 2^j)$

$= 2 \sum_{j=0}^{8} (3^j) + 3 \sum_{j=0}^{8} (2^j)$, $a_0 = 2(3^0) = 2$ for first sum, $a_0 = 3(2^0) = 3$ for second sum

$= a_0 \left[\frac{r^{n+1} - 1}{r-1} \right] + a_0 \left[\frac{r^{n+1} - 1}{r-1} \right]$

$= 2 \left[\frac{3^9 - 1}{3-1} \right] + 3 \left[\frac{2^9 - 1}{2-1} \right]$

$= 2(9841) + 3(511)$

$= 21215$

d) $\sum_{j=0}^{8} (2^{j+1} - 2^j)$

$= \left[\frac{r^{n+1} - 1}{r-1} \right]$

$= \left[\frac{2^9 - 1}{2-1} \right]$

$= 511$

Q3.

a) $\sum_{i=1}^{3} \sum_{j=1}^{2} i$

$\sum_{j=1}^{2} i = i \cdot 2$

$\sum_{i=1}^{3} i \cdot 2$

$\sum_{i=1}^{3} i \cdot 2 = 12$

b) $\sum_{i=1}^{3} \sum_{j=1}^{2} j$

$\sum_{j=1}^{2} j = 3$

$\sum_{i=1}^{3} 3$

$= 9$

c) $\sum_{i=1}^{3} \sum_{j=1}^{2} (i+j)$

$\sum_{j=1}^{2} (i+j) = 3i + 6$

$= \sum_{i=1}^{3} 3i + 6$

$= 3 \sum_{i=1}^{3} i + 2$

$= 21$

d) $\sum_{i=1}^{3} \sum_{j=1}^{i} (i+j)$

$\sum_{j=1}^{i} i+j = \frac{3i^2 + i}{2}$

$= \sum_{i=1}^{3} \frac{3i^2 + i}{2}$

$= 24$

e) $\sum_{i=1}^{3} \sum_{j=i}^{i+1} (i+j)$

$= \sum_{i=1}^{3} \left(\sum_{j=1}^{i+1} (i+j) \right)$

$= \sum_{i=1}^{3} \left[\left(\sum_{j=1}^{i} (i+j) \right) + i + i \right]$

$= \sum_{i=1}^{3} \left[\left(\sum_{i} 2i \right) + 2i + 1 \right.$

$= \sum_{i=1}^{3} (2i + 2i + 1)$

$= \sum_{i=1}^{3} (4i+1)$

$= \sum_{i=1}^{3} 4i + \sum_{i=1}^{3} (1)$

$= 4 \sum_{i=1}^{3} i + 3$

$= 4 \left(\frac{3(3+1)}{2} \right) + 3$

$= 27$

Q4. We know that $\prod_{i=1}^{n} i$ can be denoted by $n!$ which is the factorial of a positive integer n.

e.g. $1! = 1$, $2! = 1\times 2 = 2$, $3! = 1\times 2\times 3 = 6$ ...

$\therefore \frac{i}{i+1}$ becomes $\frac{n!}{(n+1)!}$

$= \frac{n!}{n! \cdot n+1}$

$= \frac{1}{n+1}$

Q5.

Let n represent any positive integer.
If n can be written as a sum of distinct powers of 2, that equals to $P(n)$

Base case: When $n=1$, $1 = 2^0$. This statement is true

Inductive step: Assume this statement is true when $n=j$, for $1 \le j \le k$. Then j can be written as a sum of distinct powers of two.

Case 1: $k+1$ is odd positive integer
$\therefore k$ is an even integer

From this, k can be written as a sum of distinct powers of two.
Since k is even, 2^0 is not included because that is odd.
Then $k+1$ can be written as $P(k+1) = P(k) + 2^0$ which is the sum of distinct powers of 2

Case 2: $k+1$ is even
$\therefore (k+1)/2$ is a positive integer, and $1 \le (k+1)/2 \le k$
from this, $(k+1)/2$ can be written as the sum of distinct powers of two
Then $k+1$ can be written as $P(k+1) = 2 \cdot P((k+1)/2)$.
Since all powers in $P((k+1)/2)$ are distinct, then all powers in $2 \cdot P((k+1)/2)$ are distinct as well

By this strong induction, the statement is true.

Q6

such that $i \geq 3$

a) Consider a string of length i that contain three consecutive 0s such that a string either ends with: 1, 10, 100, or with 000

1st case: S_{i-1} possibilities
2nd case: S_{i-2} possibilities
3rd case: S_{i-3} possibilities
4th case: 2^{i-3} possibilities

$\therefore$ the sequence is $S_i = S_{i-1} + S_{i-2} + S_{i-3} + 2^{i-3}$ for $i \geq 3$

b) Let S_i rep. the number of bit strings of length i that contain the string 01.

1st case: string is a sequence beginning with 1, then the bit string of length $i-1$ has S_{i-1} possible strings that contain 01

2nd case: String is a sequence starting with k zeros followed by a 1. There are 2^{i-k-1} bit strings of length $i-k-1$.
$\therefore$ there are 2^{i-k-1} bit strings starting with k zeros followed by a 1 and then followed by a bit string of length $i-k-1$.

3rd case: The bit string is a sequence ending in 01. There are 2^{i-2} bit strings of length $i-2$ $\therefore$ there are 2^{i-2} bit strings of length $i-2$ followed by 01

Adding all cases together...

$$S_i = S_{i-1} + \sum_{k=1}^{i-1} 2^{i-k-1} = S_{i-1} + 2^{i-1} - 1$$

c) $U_i = U_{i-1} + 2U_{i-2}$, where $U_1 = 1$, $U_2 = 3$. Solving the root equation $x^2 - x - 2 = 0$ we get $x_1 = -1$, $x_2 = 2$.
$\therefore$ $U_i = U_1(-1)^i + U_2 2^i$.
Solving the initial conditions for U_1, U_2, we get $U_1 = \frac{1}{3}$, $U_2 = \frac{2}{3}$

$\therefore$ $U_i = \frac{1}{3}(-1)^i + \frac{2}{3}2^i$

Q7. let $P(n)$

a) Prove that: $\forall n \in \mathbb{N}, \sum_{i=0}^{n} i2^i = (n-1)2^{n+1} + 2$

$$\sum_{i=0}^{n} i2^i = (n-1)2^{n+1} + 2$$

Base case: $n=1$

LHS: $\sum_{i=0}^{1} i2^i = 0\cdot 2^0 + 1\cdot 2^1 = 2$

RHS: $(1-1)\,2^2 + 2 = 2$

Since LH and RH are equal, this holds for $n=1$

Suppose $P(n)$ holds for $n=k$

then... $\sum_{i=0}^{k} i2^i = (k-1)2^{k+1} + 2$

We must prove $P(n)$ for $n=k+1$

LHS: $\sum_{i=0}^{k+1} i2^i = \sum_{i=0}^{k} i2^i + (k+1)2^{k+1}$

$= (k-1)2^{k+1} + 2 + 2^{k+1}\cdot k + 2^{k+1}$

$= k\cdot 2^{k+1} - 2^{k+1} + 2 + k\cdot 2^{k+1}$

$= 2k2^{k+1} + 2 = k\cdot 2^{k+2} + 2$

RHS: for $n=k+1 = (k+1-1)2^{k+2} + 2 = k\cdot 2^{k+2} + 2$

$\therefore$ LH and RH are equal and $P(n)$ holds for $k+1$ whenever it holds for k

by induction, the statement $\forall n \in \mathbb{N}, \sum_{i=0}^{n} i2^i = (n-1)2^{n+1} + 2$ is true

b) $S_1 = 1 \quad S_n = 2S_{n-1} + n$

We show that $S_n = 4S_{n-2} + (2n+n) - 2$ and $S_n = 8S_{n-3} + (4n+2n+2) - (4\times 2+2)$

Now $S_{n-1} = 2S_{n-2} + (n-1)$

$\Rightarrow 2S_{n-1} = 4S_{n-2} + 2n - 2$

$\Rightarrow 2S_{n+1} + n = 4S_{n-2} + (2n+n) - 2$

$\therefore S_n = 4S_{n-2} + (2n+n) - 2$

$S_{n-2} = 2S_{n-3} + (n-2)$

$\Rightarrow 4S_{n-2} = 8S_{n-3} + (4n-8)$

$\Rightarrow 4S_{n-2} + (2n+n) - 2 = 8S_{n-3} + (4n-8) + (2n+n) - 2$

$\therefore S_n = 8S_{n-3} + (4n+2n+n) - (4\cdot 2+3)$

A150 : $S_{n-4} = 2S_{n-1-4} + (n-4)$

$S_{n-4} = 2S_{n-5} + (n-4)$

$S_{n-5} = 2S_{n-6}(n-6)$

c) not sure how to approach this question...