MATH*2000 Assignment #4, Due Friday December 3rd

The following assignment is to be submitted through courselink before midnight on Friday December 3^{rd}. Please ensure that your work is legible before submitting. Any work which is illegible to the point where it is next to impossible to read will be discarded. The student will be properly informed of this evaluation and asked to resubmit their assignment subject to the normal late penalites of this course.

1. *Infinite Sets of Minimal Cardinality* **(10 Marks)**

   Garfield the cat is sitting in bed. While doing so, he postulates the existence of a set S. He supposes that S is defined in a specific way such that $|S| = \infty$ and $|S| < |\mathbb{N}|$; this suggests that S is an infinite set of a smaller cardinality than a countably infinite set. Can S truly exist? If yes, give an example of S. If no, prove by contradiction that this set cannot exist.

2. *Adjacency of Elements* **(10 Marks)**

   Suppose we had a set $S \subset \mathbb{R}$. Let us define the relation $\mathcal{R}$ on S such that $(x, y) \in \mathcal{R}$ if $x < y$. We recognize that $\mathcal{R}$ is a strict total order relation on S *(be sure to verify this first if it isn't clear to you)*. Now let us examine the relation more closely. We can define two elements $(x, z) \in \mathcal{R}$ to be *adjacent* if there does not exist $y \in S$ such that $(x, y), (y, z) \in \mathcal{R}$.

   (a) If $S = \mathbb{Z}$, is it possible for adjacent elements to exist? If it is possible, give examples of such pairs of elements that can be found in $\mathcal{R}$. If it is not possible, prove why this is the case.

   (b) If $S = \mathbb{Q}$, is it possible for adjacent elements to exist? If it is possible, give examples of such pairs of elements that can be found in $\mathcal{R}$. If it is not possible, prove why this is the case.

3. *Independent Axioms* **(10 Marks)**

   Consider the three following axioms:

   (a) $A \to B$

   (b) $B \to C$

   (c) $A \to C$

   By considering pairs of these axioms, Answer the following questions about axiom indepedence:

   - Is axiom c) independent from axiom a) and b)? Be sure to justify your answer.
   - Is axiom b) independent from axiom a) and c)? Be sure to justify your answer.
   - Is axiom a) independent from axiom b) and c)? Be sure to justify your answer.

4. *Omission of Axioms* **(10 Marks)**

   The following question is highly exploratory in nature. Consequentially, you can discuss your suppositions without necessarily defending them formally. I am more interested in what you think when you are presented with unintuitive premises. Recall that Peano has defined five axioms which construct the set $\mathbb{N}$. He based his axioms around the idea that each value n in the set is "succeeded" by n'.

   (a) $1 \in \mathbb{N}$

   (b) $n \in \mathbb{N} \to n' \in \mathbb{N}$

   (c) There does not exist n such that $n' = 1$.

   (d) If $m, n \in \mathbb{N}$ and if $m' = n'$, then $m = n$

   (e) Let P be any property. If $P(1)$ is true and $\forall i \in \mathbb{N}, P(i) \to P(i')$, then $P(n)$ is true $\forall n \in \mathbb{N}$.

Consider five different number systems. Each one is made by choosing four of Peano's axioms and leaving one of them out. Briefly discuss what oddities each of these number systems would have compared to the standard system of $\mathbb{N}$.

5. *Commutative Multiplication* **(10 Marks)**

   Formally prove the following statement: $\forall n, m \in \mathbb{N}, n \cdot m = m \cdot n$. When proving this statement, you may only use the following axioms and properties in your work.

   - $\forall n \in \mathbb{N},\ \ n + 1 = n'$ (first axiom of addition)
   - $\forall n, m \in \mathbb{N},\ \ n + m' = (n + m)'$ (second axiom of addition)
   - You may assume the commutative law and associative law for the addition operator.
   - $\forall n \in \mathbb{N},\ \ n \cdot 1 = n$ (first axiom of multiplication)
   - $\forall n, m \in \mathbb{N},\ \ n \cdot m' = n \cdot m + n$ (second axiom of multiplication)
   - $\forall n, m, p \in \mathbb{N},\ \ p \cdot (n + m) = p \cdot n + p \cdot m \wedge (n + m) \cdot p = n \cdot p + m \cdot p$ (distributive property)