#physics

The wave functions for both the electrodes: $\psi_1 \propto e^{i\varphi_1}$ and $\psi_2 \propto e^{i\varphi_2}$ with phase difference $\Delta\varphi = \varphi_2 - \varphi_1$ (or simply φ)

$$E_{kin} = 2eV$$

and we know $E = \hbar\dot{\varphi}$ hence

$$V_{Josephson} = \frac{\hbar}{2e}\dot{\varphi}$$

and current

$$I_{Josphson} = I_C sin\varphi$$

Model the Josephson Junction as Resistor, capacitor and a shunt in parallel. We know from classical electrodynamics that $I_{capacitor} = C\frac{dV}{dt}$ therefore

$$I_{capacitor} = \frac{C\hbar}{2e}\ddot{\varphi} \tag{1}$$

$$I_{resistor} = \frac{\hbar}{2eR}\dot{\varphi} \tag{2}$$

$$I_{shunt} = I_C sin\varphi \tag{3}$$

by Kirchoff's law,

$$I_{exit} = I_{capacitor} + I_{resistor} + I_{shunt}$$

Hence

$$I_{exit} = \frac{C\hbar}{2e}\ddot{\varphi} + \frac{\hbar}{2eR}\dot{\varphi} + I_C sin\varphi \tag{4}$$

(4) is a second order differential equation and is analogous to the Pendulum

$$\Gamma = mL^2\ddot{\theta} + b\dot{\theta} + mgL\sin\theta$$

Josephson	Pendulum
Bias current: I_C	Applied torque: Γ
phase difference: φ	angle from the vertical: θ
capacitance: C	mass: m
Voltage: $\frac{\hbar}{2e}\dot{\varphi}$	Angular velocity: $\dot{\theta}$
Conductance: $\frac{1}{R}$	damping constant: b

Josephson	Pendulum
Critical Current: I_C	Max gravitational torque mgL
Change in voltage: $\dot{V} = \frac{\hbar}{2e}\ddot{\varphi}$	Angular acceleration: $L^2\ddot{\theta}$

The above table makes the two equations completely analogous. Hence the overdamped pendulum gives an a physical interpretation of the RCSJ model.

Dimensionless Formulation:

we want to find conditions when it is safe to neglect the second order term. For this we use dimensionless formulation.
Just like in the overdamped oscillator case, we replace time t with a dimensionless constant:

$$\tau = \frac{2eI_CR}{\hbar}t \tag{7}$$

this gives us

$$\beta\varphi'' + \varphi' + \sin\varphi = \frac{I}{I_C}$$

where $\beta = \frac{2eI_CR^2C}{\hbar}$ which is called as the ***Mc-Cumber Parameter***.
We can safely neglect the second order term when $\beta \ll 1$, the overdamped limit.
in that situation, we obtain the first order system,

$$\varphi' = f(\varphi) = \frac{I}{I_C} - \sin\varphi \tag{8}$$

when $I > I_C$ we have no fixed points.
when $I = I_C$ we have one fixed point which is half stable,
when $I < I_C$ we have two fixed points, one stable and one unstable.
Hence a pitchfork bifurcation has occurred at $(I, \varphi^*) = \left(I_C, \frac{\pi}{2}\right)$

Current Voltage Curve

we want to find $\langle V\rangle$ as a function of the applies constant current I.
$\langle V\rangle = I_CR\langle\varphi'\rangle$ \
Hence it is sufficient to find $\langle\varphi'\rangle$. There are two cases two consider: \
When $I \leq I_C$ all solutions approach a fixed point $\varphi^* = \sin^{-1}\left(\frac{I}{I_C}\right)$ Thus $\varphi' = 0$ in steady state and hence $\langle V\rangle = 0$ for all $I \leq I_C$ \

When $I \geq I_C$, the solutions are periodic with a period T, we find T by finding the time required for φ to change by 2π,

$$T = \int dt = \int_0^{2\pi} \frac{d\varphi}{dt}\, d\varphi = \int_0^{2\pi} \frac{d\varphi}{\frac{I}{I_C} - \sin\varphi} = \frac{2\pi}{\sqrt{\left(\frac{I}{IC}\right)^2 - 1}}$$

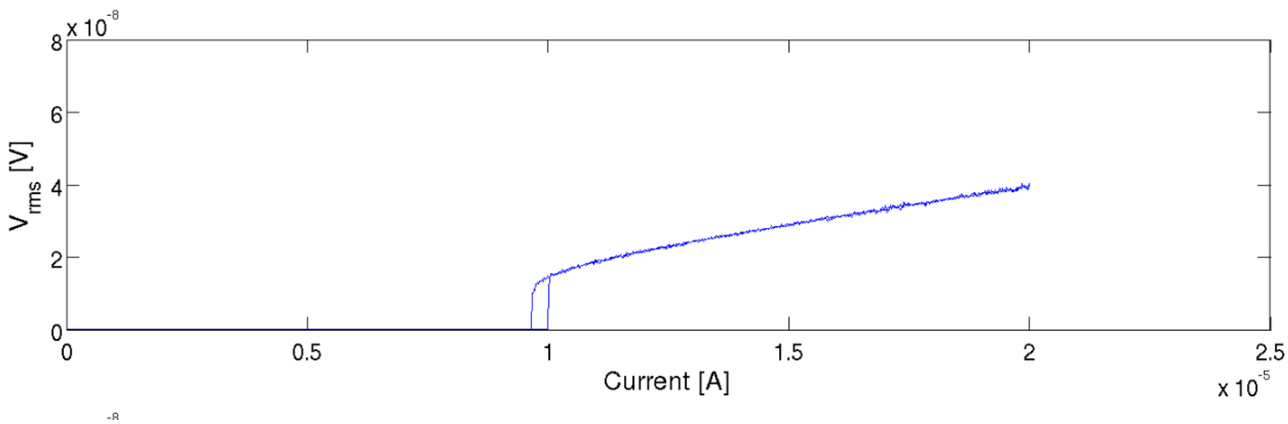
we compute $\langle \varphi' \rangle$ by taking average over one cycle:

$$\langle \varphi' \rangle = \frac{1}{T}\int_0^T \frac{d\varphi}{d\tau} d\tau = \frac{1}{T}\int_{\varphi(0)}^{\varphi(T)} d\varphi = \frac{1}{t}\int_0^{2\pi} d\varphi = \frac{2\pi}{T}$$

(by change of variables formula)
Substituting T and $\langle \varphi' \rangle$ to the $\langle V \rangle$ formula we obtain

$$\langle V \rangle = I_C R \sqrt{\left(\frac{I}{I_C}\right) - 1}$$

hence the I-V curve looks like:



General RCSJ

Governing equation.

$$I_{exit} = \frac{C\hbar}{2e}\ddot{\varphi} + \frac{\hbar}{2eR}\dot{\varphi} + I_C sin\varphi \tag{4}$$

If we introduce dimensionless $\tilde{t} = \sqrt{\frac{2eI_C}{\hbar C}}t$ and $I = \frac{I_{exit}}{I_C}$, and $\alpha = \sqrt{\frac{\hbar}{2eI_c R^2 C}}$.
Then we obtain

$$\phi'' + \alpha\phi' + \sin\phi = I$$

We can rewrite this as a system:

$$\phi' = y$$
$$y' = I - \sin\phi - \alpha y$$

Fixed Points:

To understand the fixed points we need

Linear Dynamical Systems

As shown in 2D linear system, $\dot{X} = AX$ is a vector valued linear differential equation. We can convert higher order differential equations into vector valued differential equations as shown in System of ODE.

Example1:

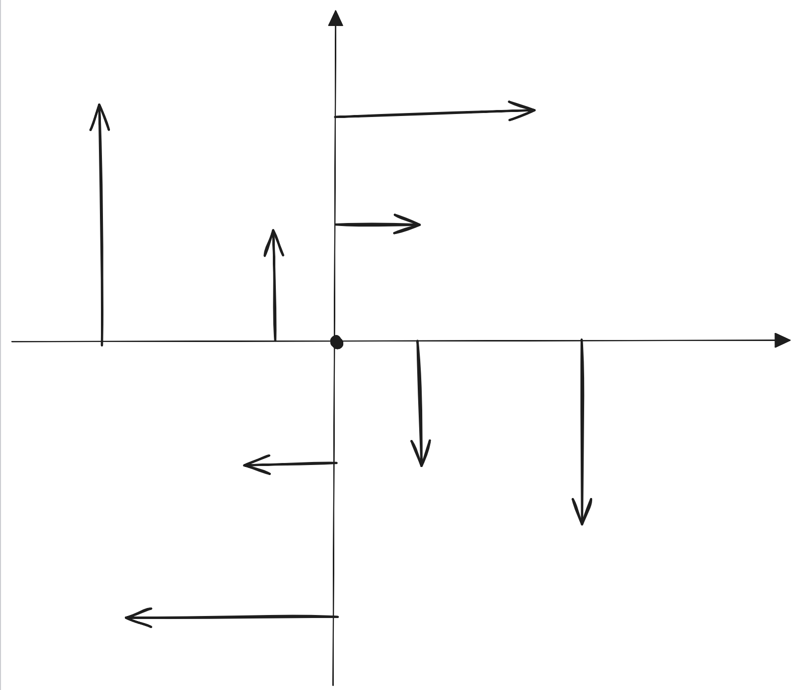
consider the simple harmonic oscillator: $m\ddot{x} + kx = 0$, where $\ddot{x}$ is the acceleration of the particle at position x, only knowing the position is insufficient as an initial condition. We must know the velocity as well. Velocity is acting like a parameter independent of x and hence we can introduce velocity as a variable to our differential equation.

$$v = \dot{x}$$
$$\dot{v} = -\frac{k}{m}x$$

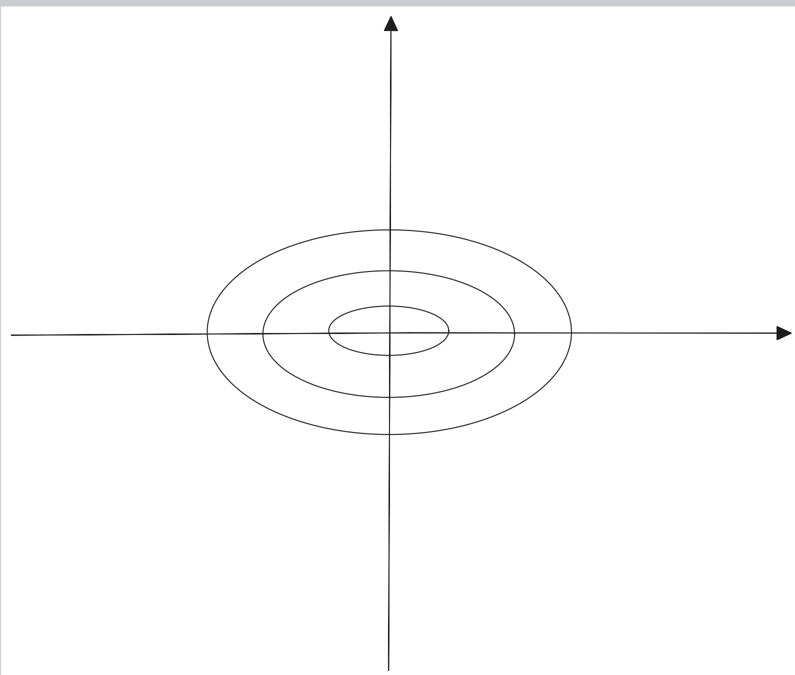
we let $\omega^2 = \frac{k}{m}$.
Now we have obtained a vector valued linear differential equation:

$$\dot{x} = v$$
$$\dot{v} = -\omega^2 x$$

we study this as a vector field on a plane, where each vector on the plane corresponds to a phase point $(\dot{x}, \dot{v})$
when $\dot{v} = 0$ then we get vertical flow of $-\omega^2 x$ when $\dot{x} = 0$ we get horizontal flow of $\pm v$ this is how it looks pictorially:



the origin is a fixed point but any other point outside the origin never reaches a fixed point, they are rather stuck in closed orbits.



Example2:

consider the system $\dot{X} = AX$ where $A = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix}$

we get the system

$$\dot{x} = ax$$
$$\dot{y} = -y$$

both the parameters are independent of each other hence we can solve them separately, which gives us $x = x_0 e^{at}$ and $y = y_0 e^{-t}$. So the behaviour of the vector field all depends on a.
When a is negative, it approaches to be straight line and then deviates away from origin. the all points tend to collapse at $\dot{y} = 0$, in fact when $a < 0$ all the values will collapse to the origin which is a ***Stable node***.

Star:

when the decay rates in both the directions are equal, as in the above example when $a = -1$. In this case all vectors approach the stable node in a straight line.

Saddle Point:

in the above example origin was the saddle point since it was a stable fixed point and later turned into an unstable fixed point.

Some useful definitions:

Stable Manifold

The set of conditions x_0 such that $x \to x^*$ as $t \to \infty$.

Unstable Manifold

The set of conditions x_0 such that $x \to x*$ as $t \to -\infty$

in our example, y-axis is the ***stable manifold***
whereas the x-axis is an ***unstable manifold***.

Attracting point:

x^* is said to be a gathering point if $x(t) \to x^*$ as $t \to \infty$, for all $x(t) \in B(x^*, \epsilon)$.
x^* is ***Globally attracting***.
in example 2, origin is globally attracting if $a < 0$.

Liapunov Stable Point:

A phase point x^* is said to be Liapunov stable if all the trajectories that start sufficiently close to x^* remain close to it for all time t.

Example: Liapunov stable does not imply attracting:

in example 1, origin is a liapunov stable point but not a gathering point.

Example: Attracting point does not imply Liapunov stable:

consider the system: $\dot{\theta} = 1 - \cos\theta$. $\theta = 0$ is the only fixed point. It is a half stable fixed point a.k.a ***Neutral Point***. It is a gathering point since trajectory has to pass through that point at some time. It is in a fixed orbit hence always approaches $\theta^* = 0$. But at any give point the point is away from $\theta^* = 0$ hence it cannot be a Liapunov Stable point.

Neutrally stable point.

If a point is liapunov stable but not attracting, it is called a neutrally stable point.
For example: origin is a Neutrally stable point in Example1.

In practice, we say that a point is ***Stable*** if it is both attracting and Liapunov stable And a point is ***Unstable*** if it is neither.

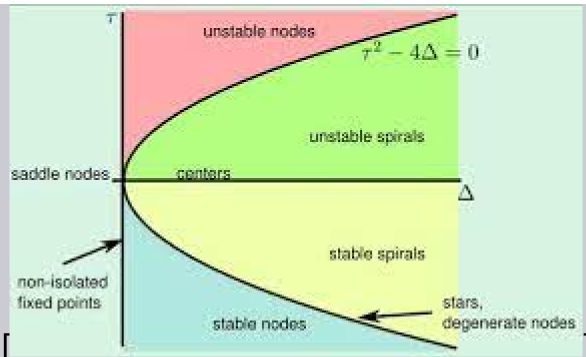
Classification of Linear Dynamical Systems

in the above examples we encountered some instances of straight lined flow, and we would like to characterise that.
A typical straight lined flow looks like $\mathbf{x(t)} = e^{\lambda t}\mathbf{v}$. where $\mathbf{v}$ is a vector in our phase plane. Hence the flow is along $\mathbf{v}$ where the rate depends on the time.
We want to characterise $\mathbf{x(t)} = e^{\lambda t}\mathbf{v}$ in the general form of $\dot{X} = AX$. Here $\dot{\mathbf{x}} = \lambda e^{\lambda t}\mathbf{v}$. Since $\dot{\mathbf{x}} = A\mathbf{x}$, we have $\dot{\mathbf{x}} = \lambda e^{\lambda t}\mathbf{v} = Ae^{\lambda t}\mathbf{v}$, we can cancel the scaler $e^{\lambda t}$ from both the sides to obtain $\lambda\mathbf{v} = A\mathbf{v}$. This means that the desired straight line solution only exists only if $\mathbf{v}$ is an eigenvector of A with its corresponding eigenvalue λ.
The most general situation that we will encounter is that after solving the characteristic equation we obtain two distinct eigenvalues, say λ_1, λ_2. The the corresponding eigenvectors are linearly independent, say $\mathbf{v_1, v_2}$. Then the general solution we obtain for $\dot{\mathbf{x}} = A\mathbf{x}$ is $\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{v_1} + c_2 e^{\lambda_2 t}\mathbf{v_2}$. Since $A\mathbf{x}$ if a linear functional, it is smooth hence by existence and uniqueness theorem, the solution is unique.

Generalisation:

consider a. two dimensional flow $\dot{X} = Ax$ lets say it has eigenvectors v_1 and v_2 with their respective eigenvalues λ_1 and λ_2. If v_1 and v_2 exist and are unique, then v_1 and v_2 are linearly independent, hence they span the phase plane. Let $B = [v_1 \quad v_2]$, then the vector field on the plane is governed by $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = B \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda}_2 t \end{bmatrix}$. Where c_1 and c_2 depend on the initial condition i.e at $t = 0$.
Now suppose $\lambda_1 \lambda_2 = -1$ then one of our solutions is expanding and the other is decaying. Without loss of generality, assume $\lambda_1 > 0$ and $\lambda_2 < 0$. Let $L_2 = kv_2$, the line

from the origin along the vector v_2. For that the initial condition $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
then the vector field along L_2 is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} ae^{\lambda_2 t} \\ be^{\lambda_2 t} \end{bmatrix}$. As $t \to \infty$, $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \to 0$. Hence L_2 is a stable manifold. Similarly, let $L_1 = k' v_1$ the line from the origin along the vector v_1. For this, the initial condition $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The vector field along L_1 is $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} a' e^{\lambda\backslash t} \\ b' e^{\lambda_1 t} \end{bmatrix}$. As $t \to \infty$, $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \to \infty$. Hence L_1 is an unstable manifold.
Now suppose $\lambda_2 < \lambda_1 < 0$, then both the eigensolutions are decaying. Since $|\lambda_2| > |\lambda_1|$, line along v_2 decays faster, hence its called the *Fast eigendirection* where as v_1 is called the *Slow eigendirection*. Since all the points end up in the origin, origin is a *Stable Node*.
The node has two independent eigendirections (the fast and the slow eigendirection) as $t \to -\infty$ points move along the fast eigendirection, and as $t \to \infty$ points move along the slow eigendirection.
If λ_1 and λ_2 are complex valued. $\lambda_{1,2} = \frac{1}{2}(\tau \pm \sqrt{\tau^2 - 4\Delta})$. That means $4\Delta > \tau^2$. This we rewrite as $\alpha \pm i\omega$. Where $\alpha = \frac{\tau}{2}$ and $\omega = \sqrt{4\Delta - \tau^2}$. Hence the solutions can be written in form of $c_1 v_1 e^{\alpha + i\omega t} + c_2 v_2 e^{\alpha - i\omega t}$. i.e
$[X(t)] = e^{\alpha}[\cos\omega t(c_1 v_1 + c_2 v_2) + i \sin\omega t(c_1 v_1 - c_2 v_2)]$. where $\alpha \in \mathbb{R}$. So α determines the rate of flow. If $\alpha > 0$ its an *unstable spiral*. $\alpha < 0$ its a *stable spiral.*
Suppose now that $\lambda_1 = \lambda_2 = \lambda$, then we have two cases:
Case 1: We have distinct eigenvectors. Since $\dot{X} = AX$, and since λ is an eigenvector, we have $Av = \lambda V$ for some v. For initial condition i.e $X_0 = c_1 v_1 + c_2 v_2$, where v_1 and v_2 are the two distinct eigenvectors. $AX_0 = A(c_1 v_1 + c_2 v_2)$=
$Ac_1 v_1 + Ac_2 v_2 = c_1 \lambda v_1 + c_2 \lambda v_2 = \lambda(c_1 v_1 + c_2 v_2) = \lambda X_0$. (since $Av_k = \lambda v_k$, for $k = 1, 2$).
Hence we have $AX_0 = \lambda X_0$, so every X_0 is an eigenvector to A. Assume $\lambda \neq 0$, all trajectories are straight lines through origin ($X(t) = e^{\lambda t} X_0$). And the fixed point is called ***Star Node***. If $\lambda = 0$ then every point is a fixed point (as $\dot{X} = 0$).
Case2: We have a unique eigenvector. i.e the eigenspace corresponding to λ is one dimensional. For example any matrix of the form $A = \begin{bmatrix} \lambda & b \\ 0 & \lambda \end{bmatrix}$ where $b \neq 0$ has only one-dimensional eigenspace. When theres only one dimensional eigenspace, the fixed point is called a ***Degenerate Node***. In the degenerate node, all the trajectories become parallel to the eigendirection. i.e the trajectories collapse. A degenerate node is a spiral between a spiral and a node.

[

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since $\Delta = \lambda_1\lambda_2$ and $\tau = \lambda_1 + \lambda_2$, let $P = \tau^2 - 4\Delta$
When $\Delta < 0$ means $\lambda_1\lambda_2 < 0$ so we get saddle points.
When $\Delta > 0$, the node are real with same sign or complex valued, this can be classified in the following cases:
Case 1: $\tau > 0$ (everything is unstable)
in this we have two sub cases:
1a) $P > 0$, roots are real, and same signed, hence positive roots hence we get unstable nodes
1b) $P < 0$, roots are complex. We get unstable spirals.
Case 2: $\tau < 0$ (everything is stable)
2a) $P > 0$ roots are real, same signed, hence both negative. We get stable nodes.
2b) $P < 0$ roots are complex, we get stable spirals
Boderline cases:
$\tau = 0$: Eigen values are purely imaginary hence we get Centers, (aka, neutral points) (closed orbits)
$P = 0$: We get degenerate nodes.
$\Delta = 0$: Non isolated fixed points.

For fixed points to occur $y^* = 0$ so $\sin\phi^* = I$. Hence we have two fixed points when $I < 1$ and no fixed points when $I > 1$.
Assume $I < 1$ so the fixed points exist.
Since our system is non linear, we will have to linearise it using

Linear Stability Analysis

Linearisation about x^* (in 1-D flow)

We would like to understand how (fast) a phase point is approaching a given fixed point x^* Hence we define a perturbation $\eta(t) = x(t) - x^*$ i.e the how far is $x(t)$ away from x^*
.
differentiating gives us $\dot{\eta}(t) = \dot{x}(t) = f(x) = f(\eta + x^*)$ Taylor series around $\eta + x^*$ gives us:

$$f(\eta + x^*) = f(x^*) + \eta f'(x^*) + O(\eta^2)$$

we know that $f(x^*) = 0$ hence we have

$$\dot{\eta}(t) = \eta f'(x^*) + O(\eta^2)$$

now if $f'(x^*) \neq 0$ then the $O(\eta^2)$ terms are negligible. Hence we may write the approximation as

$$\dot{\eta}(t) \approx \eta f'(x^*)$$

*if $f'(x^*)>0$ then the perturbation $\eta(t)$ grows exponentially.
if $f'(x^*) < 0$ then the perturbation $\eta(t)$ decays.* This is supports the observation we had made in [Flows on the Line aka One dimensional Flow > General observation]

Characteristic Time Scale:

the reciprocal $\frac{1}{|f'(x^*)|}$ is called the characteristic time scale.
it determines the time required for $x(t)$ to vary significantly in the neighbourhood of x^*.

Linearisation in 2 dimensional flows:

Consider the system $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$ where $X \in \mathbb{R}^2$. Let (x^*, y^*) be the fixed points of the flow, i.e $f(x^*, y^*) = 0$. Let $u = x - x^*$ and $v = y - y^*$.
Now
$\dot{u} = \dot{x} = f(x, y) = f(x^* + u, y^* + v) = f(x^*, y^*) + u\frac{\partial f}{\partial x}|_{(x^*,y^*)} + v\frac{\partial f}{\partial y}|_{(x^*,y^*)} + O(u^2, v^2$
.
Since $f(x^*, y^*) = 0$, we obtain $\dot{u} = u\frac{\partial f}{\partial x}|_{(x^*,y^*)} + v\frac{\partial f}{\partial y}|_{(x^*,y^*)} + O(u^2, v^2, uv)$.
Similarly, we obtain $\dot{v} = u\frac{\partial g}{\partial x}|_{(x^*,y^*)} + v\frac{\partial g}{\partial y}|_{(x^*,y^*)} + O(u^2, v^2, uv)$. Hence the disturbance (u, v) evolves according to

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + O(u^2, v^2, uv)$$

The matrix

$$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$$

is called the ***Jacobian Matrix*** at the fixed point (x^*, y^*).
Hence we obtain the form $\dot{X} = AX$ that we can analyse using [Linear Dynamical Systems].

Example: Linearised system may incorrectly predict the fixed point it it is not a borderline case

Consider the system

$$\begin{aligned}\dot{x} &= -y + ax(x^2+y^2) \\ \dot{y} &= \ \ x + ay(x^2+y^2)\end{aligned}$$

Since the fixed point is the origin, we can simply linearise by simply omitting nonlinear terms in x and y. Hence we obtain the Jacobian $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ so we have $\tau = 0$ and $\Delta = 1$. Hence the origin has to be the centre always.
To analyse the non linear system we change to polar coordinates. So $x = r\cos\theta$ and $y = r\sin\theta$.
We know $x^2 + y^2 = r^2$, differentiating this we get $x\dot{x} + y\dot{y} = r\dot{r}$. Substituting $\dot{x} = -y + axr^2$ and $\dot{y} = x + ayr^2$ we get $\dot{r} = ar^3$.
Now we have to solve for $\dot{\theta}$. We know $\theta = \arctan\left(\frac{y}{x}\right)$. hence $\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2} = 1$.
Hence we obtain a system in polar coordinates:

$$\begin{aligned}\dot{r} &= ar^3 \\ \dot{\theta} &= 1\end{aligned}$$

So the flow has a constant angular velocity $\dot{\theta} = 1$. If $a < 0$, origin is a stable spiral. If $a = 0$, origin is the centre (as predicted by linearisation). And if $a > 0$ we have unstable spirals.

Hyperbolic fixed points:

To know whether our linearisation fails or not, we introduce another classification called hyperbolic fixed point.
A point is said to be a hyperbolic fixed point if $\mathrm{Re}(\lambda_i) \neq 0$.
The example in the above case is not a hyperbolic fixed point, and hence or linearisation failed.
A phase portrait is said to be ***Structurally stable*** if its topology cannot be changed by arbitrarily small perturbation to the vector field. For example, the phase portrait of a saddle point is structurally stable but for a centre its not.

Then the jacobian $A = \begin{bmatrix} 0 & 1 \\ -\cos\phi^* & -\alpha \end{bmatrix}$.
We obtain $\tau = -\alpha < 0$ and $\Delta = \cos\phi^* = \pm\sqrt{1 - I^2}$. So when $\Delta < 0$ the fixed point is a saddle point.

When $\Delta > 0$: two cases
We have a stable node if $\tau^2 - 4\Delta = \alpha^2 - 4\sqrt{1 - I^2} > 0$ i.e if the damping is strong enough or I is close to 1.
Otherwise the sink is a stable spiral.
At $I = 1$ the stable node and the saddle point coalesce, so a

Saddle-Node Bifurcation

aka Fold Bifurcation or Blue Sky bifurcation

1-D Case

A saddle node bifurcation is a local bifurcation i which two fixed points of a dynamical system collide and annihilate each other*

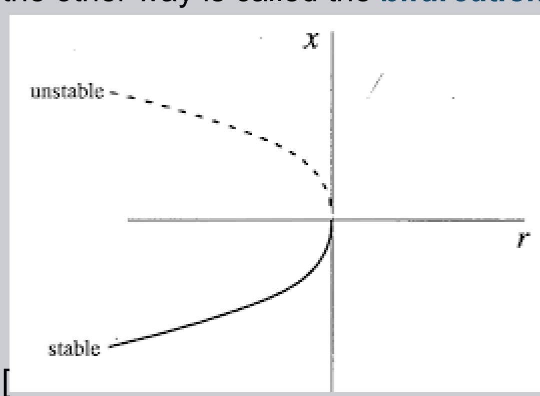
Example:

$$\dot{x} = r + x^2$$

a) if $r < 0$ we have two fixed points, one stable and one unstable
b) if $r = 0$ we have one fixed point which is **half stable**
c) if $r > 0$ we have no fixed points.
in this case we say that a ***bifurcation occurred at*** $r = 0$

Representation:

one way to represent this is a continuous stack of vector fields from the phase $r < 0$ to $r > 0$
the other way is called the ***bifurcation diagram:***

[

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Examples:

1. consider the system $\dot{x} = r - x^2$. The fixed points are $x^* = \pm\sqrt{r}$. Now $f'(x^*) = -2x^*$ so $x^* = \sqrt{r}$ is a stable fixed point and $x^* = -\sqrt{r}$ is an unstable

fixed point. At $r=0$ $f(x^*)=0$ so the linearisation vanishes when the fixed points coalesce.

2. consider the system $\dot{x}=r-x-e^{-x}$. We will get fixed points when $r-x=e^{-x}$ i.e their graphs intersect. To find a point where the fixed point coalesce we want the tangents of both the curves e^{-x} and $r-x$ to be equal. So $\frac{d}{dx}e^{-x}=\frac{d}{dx}(r-x)$ which happens only when $x=0$ which gives us $r=1$. Hence we say that the bifurcation occurred when $r_C=1$.
Example 1 is also called as

> Normal Forms
> We attempt to generalise every bifurcation model with its respective generalised form which is called the
>
> ## Example: Normal form for saddle node bifurcation.
>
> We understand that in the context of Saddle-Node Bifurcation $\dot{x}$ is not just dependent on x but on r too. Hence we have $\dot{x}=f(x,r)$. When $r=r_c$ we achieve a bifurcation point. x^* is a root of the function.
>
> $$f(x,y)=f(x^*,r_C)+(x-x^*)\frac{\partial f}{\partial x}\mid_{(x^*,r_c)}+(r-r_c)\frac{\partial f}{\partial r}\mid_{(x^*,r_c)}+\frac{1}{2}(x-x^*$$
>
> Since $f(x^*,r)=0\,\forall r$ so $f(x^*,r_c=0)$, thats the first term of our Taylor expansion.
> Secondly, for a bifurcation to occur, we need the function to attain a local maxima/minima at (x^*,r). This is also called the tangency condition of a saddle node bifurcation. Hence $\frac{\partial f}{\partial x}\mid_{(x^*,r_C)}=0$ which is the second term of out Taylor expansion.
> This gives us the form:
>
> $$\dot{x}=a(r-r_C)+b(x-x^*)+\dots$$
>
> This looks like a generalisation of Saddle-Node Bifurcation > Example the first example.
> Hence this is called as the **Normal Form for saddle node bifurcation.**

2D-Case

In many respects bifurcation is a one-dimensional event. The bifurcation occurs on a line just like the 1-D case.
A general case for saddle node bifurcation in the 2D-phase space is:
Consider a two dimensional system $\dot{x}=f(x,y)$ and $\dot{y}=g(x,y)$ that depends on the parameter μ. The nullclines would be $f(x,y)=0$ and $g(x,y)=0$. The points of

intersections of these two nullclines give us the fixed points. Suppose that as μ increases the nullclines pull away from each other, becoming tangent at $\mu = \mu_c$. The fixed point collide at $\mu = \mu_c$ and annihilate. We say that a saddle node bifurcation has occurred at $\mu = \mu_c$.
If we extend the normal form of the 1D case to the 2D case, here is a prototype:

$$\dot{x} = \mu x - x^2$$
$$\dot{y} = -y$$

All the trajectories move towards the x-axis as $t \to \infty$. The fixed points we obtain are $(x^*, y^*) = (\pm\sqrt{\mu}, 0)$. When $\mu > 0$ we have two fixed points one stable node and the other is a saddle point. These two collide when $\mu = 0$ and disappear when $\mu < 0$.
The bifurcation still affects the flow even when the fixed points are annihilated. A bottleneck does occur in the region of bifurcation.

Example: (Genetic Control System)

The concentrations of protein and messenger RNA is modelled by the following system:

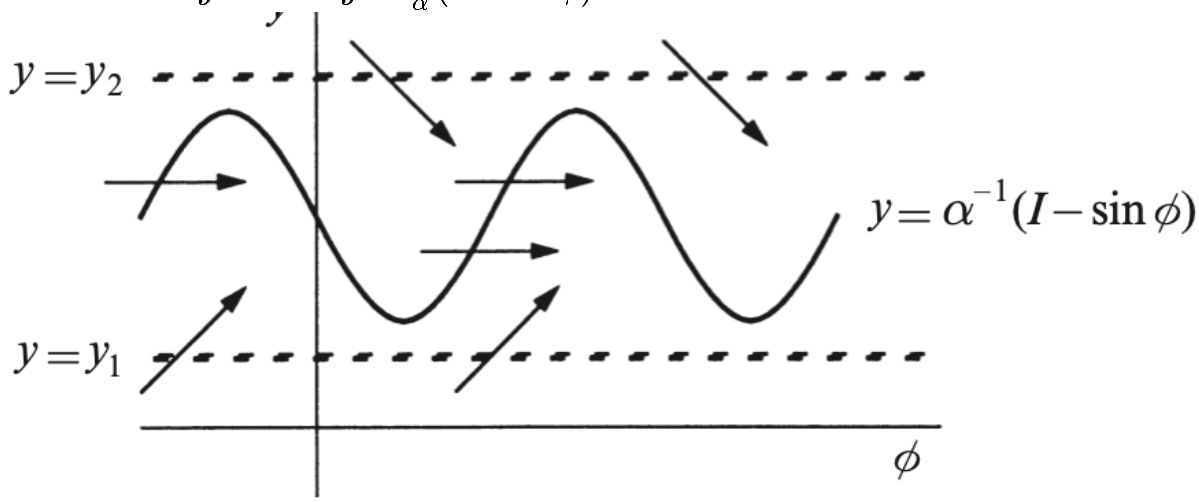
$$\dot{x} = -ax + y$$
$$\dot{y} = \frac{x^2}{1+x^2} - by$$

If we plot the nullclines $y = ax$ (the line) and $y = \frac{x^2}{b(1+x^2)}$ (the sigmoid). These two curves intersect at three points for small values of a. As a increases the line start moving away from the sigmoid it becomes tangent to the sigmoid at $a = a_c$. To find a_c we can compute directly where the fixed points coalesce. The nullcline intersect when $ax = \frac{x^2}{b(1+x^2)}$. Which give us the quadratic equation: $ab(1 + x^2) = x$ which has two solutions $x^* = \frac{1 \pm \sqrt{1-4a^2b^2}}{2ab}$. The solutions coalesce when $2ab = 1$ i.e $a_c = \frac{1}{2b}$. The fixed point $x^* = 1$ is where the bifurcation takes place.
We find the jacobian $A = \begin{bmatrix} -a & 1 \\ \frac{2x}{(1+x^2)^2} & -b \end{bmatrix}$. Here $\tau = -(a + b) < 0$.
At origin $\Delta = ab > 0$ hence it is always a stable node since $\tau^2 - 4\Delta = (a - b)^2 > 0$.
$\Delta = ab - \frac{2x}{(1+x^2)^2} = ab\left(1 - \frac{2x}{ab(1+x^2)^2}\right) = ab\left(1 - \frac{2}{1+x^2}\right)$ since $ab(1 + x^2) = x$ for $x = x^*$.
Which means $\Delta = ab\left(\frac{x^2-1}{x^2+1}\right)$. So $\Delta < 0$ when $0 < x^* < 1$ which is the middle fixed point, that means that the middle fixed point is a saddle point.
The fixed point $x^* > 1$ is a stable node since $\Delta < ab$ so $\tau^2 - 4\Delta > (a - b)^2 > 0$.

has occurred.

Existence of Orbits:

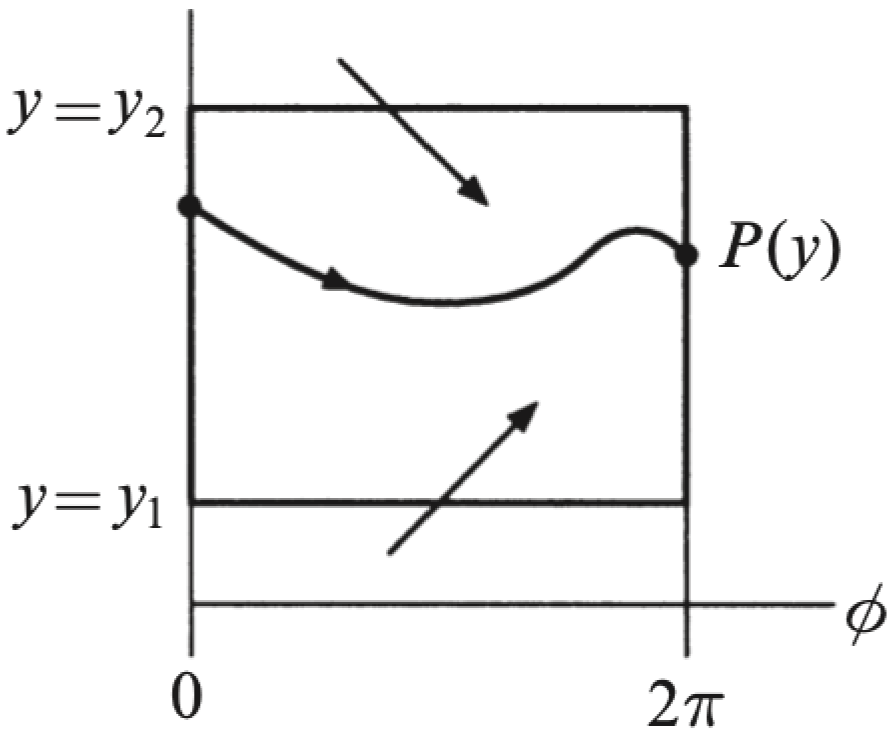
In the case when $I > 1$:
No fixed points are available.
We want to show that periodic solutions exist.
Observe the nullcline $y' = 0$ i.e $y = \frac{1}{\alpha}(I - \sin\phi)$.



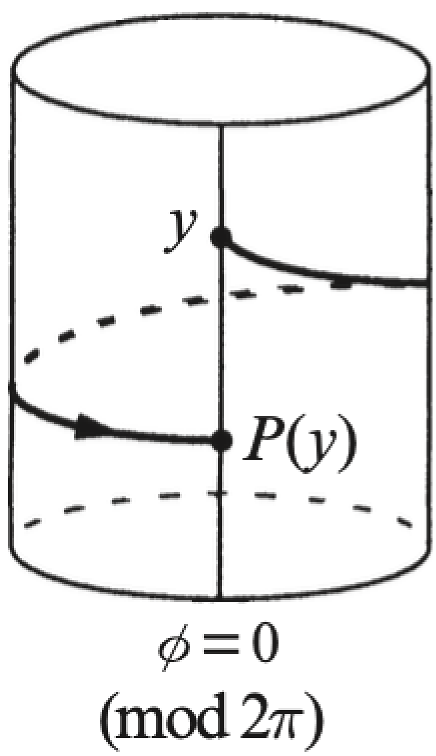
The flow is downward above the nullcline and upward below it.

Constructing a Poincare map:

All the trajectories confine themselves in $y_1 < y < y_2$ as $t \to \infty$.
Since ϕ is a phase difference, we are only interested in the interval $(0, 2\pi)$, as the rest of the patch is just its repetition.
Consider a trajectory starting at some height y of the box until it intersects the right side of the box at new height $P(y)$. This is called the Poincare map.



We cannot compute $P(y)$ explicitly, but observe that $y_1 < P(y_1) < y_2$.
$P(y)$ is a continuous function. Since solutions of differential equations depend continuously on initial conditions. Then by fixed Brouwer Fixed point theorem, there exists y^* such that $P(y^*) = y^*$.
So a periodic solution exists and is a closed orbit in the cylindrical phase space:

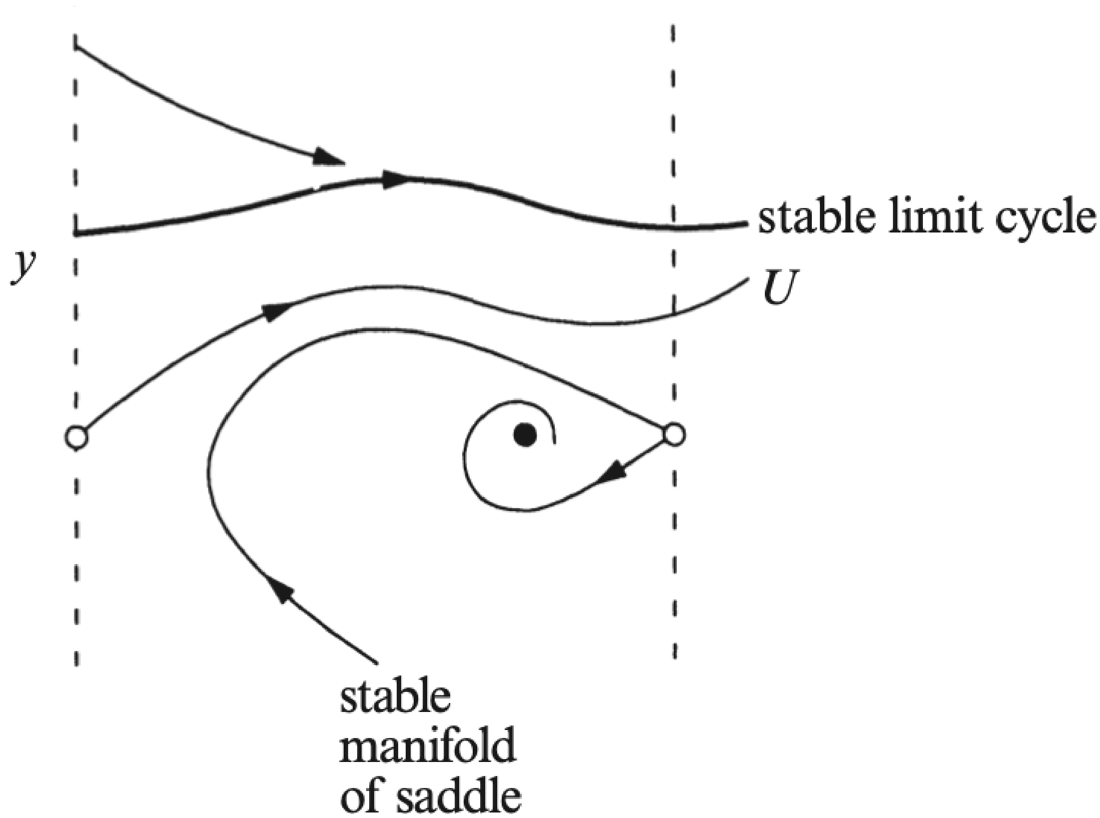


$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$f^n(x) = \frac{1}{h^n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+kh)$$

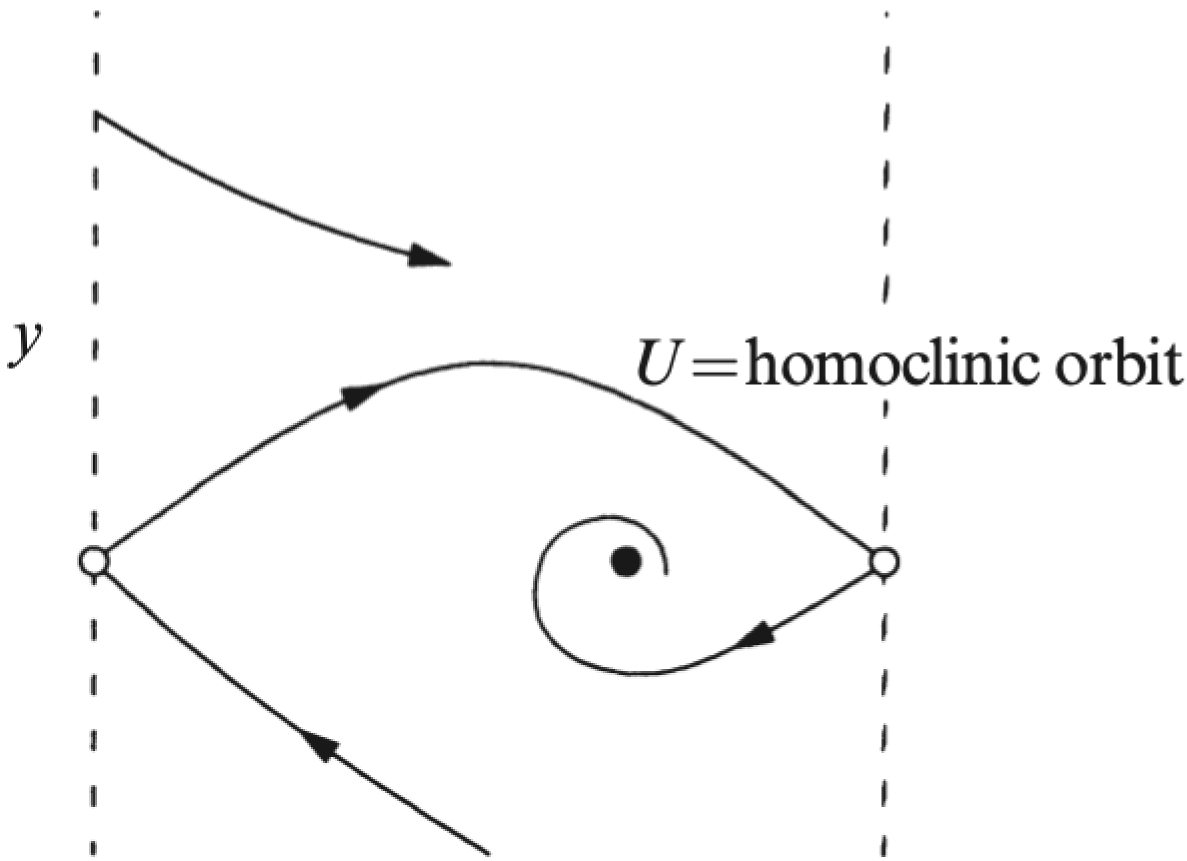
Homoclinic Bifurcation

A part of the limit cycle moves closer to a saddle point. At the bifurcation the cycle touches the saddle point and becomes a homoclinic orbit. Thus is another kind of infinite period bifurcation.
We slowly decrease I from some value $I > 1$.
We have shown that initially it has a stable limit cycle.
The system is bistable when $I_c < I < 1$, i.e a sink coexists with the stable limit cycle.

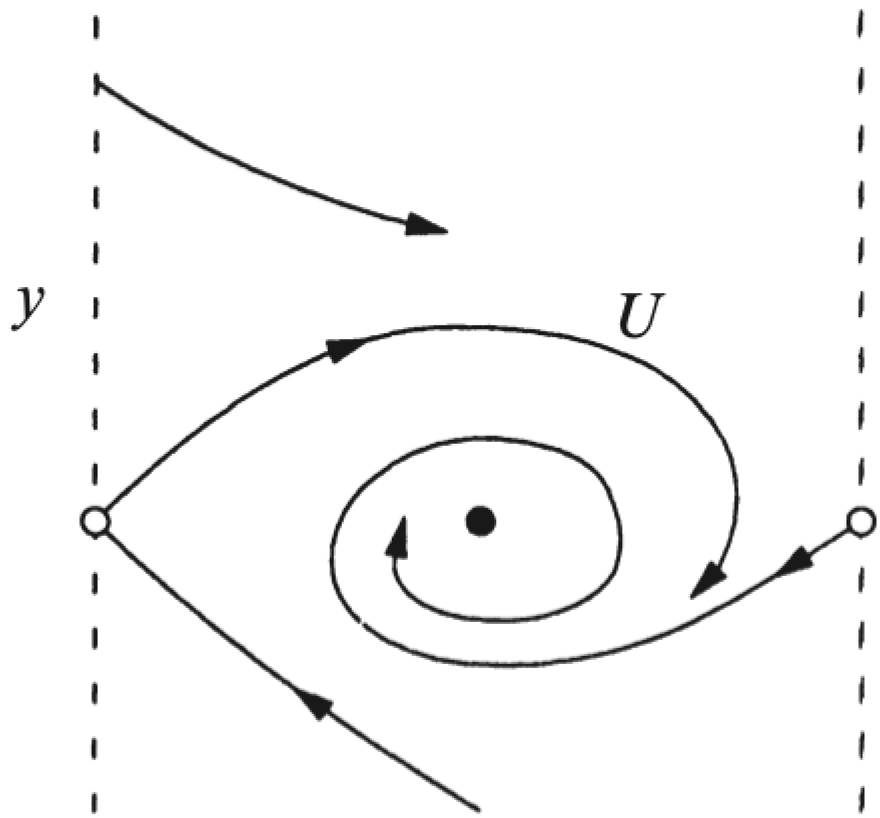


U is a branch of the unstable manifold of the saddle.
As I decreases, the stable limit cycle moves down and squeezes U closer to the stable

manifold of the saddle.

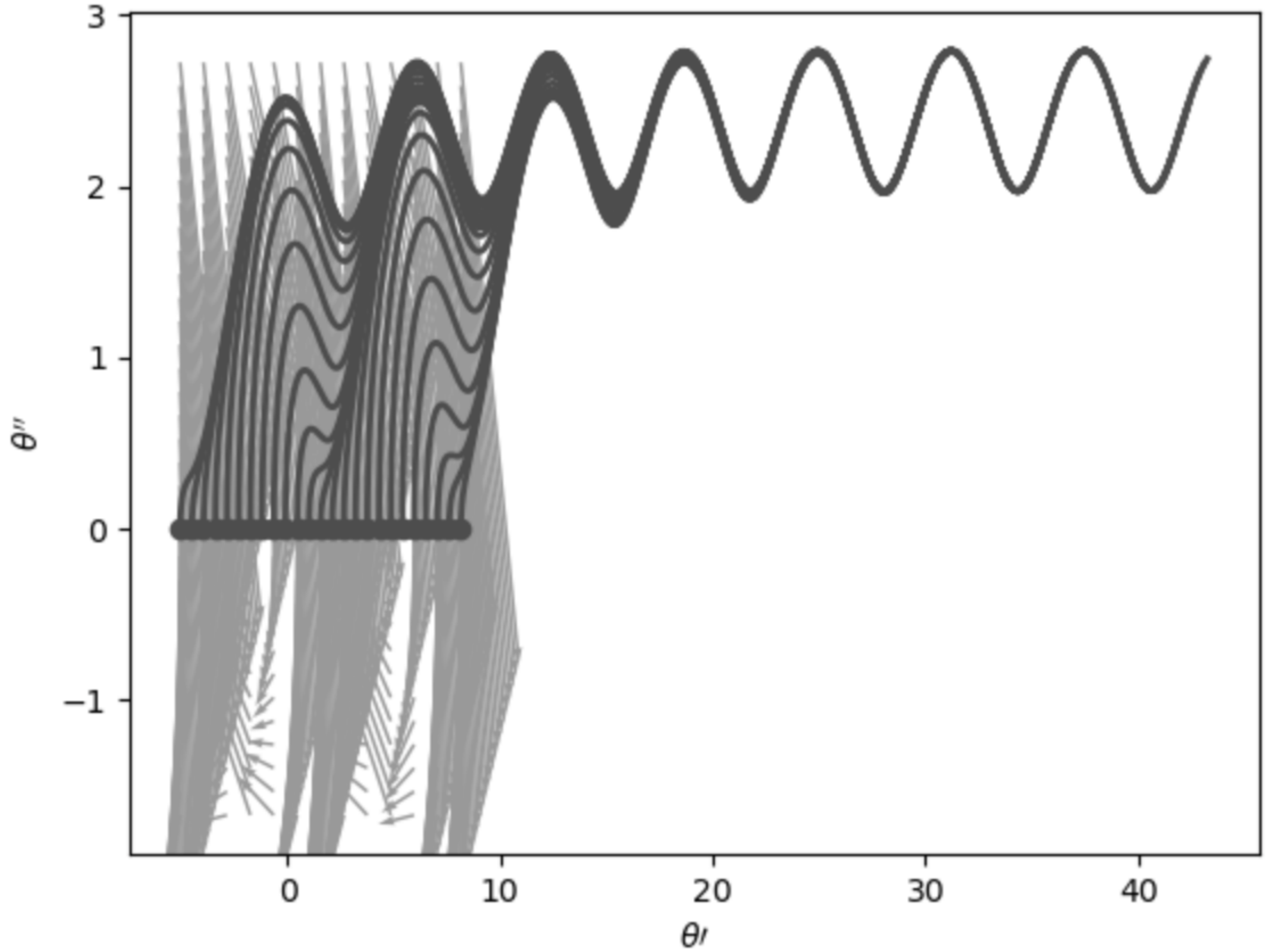


At $I = I_c$ the limit cycle merges with U in a homoclinic bifurcation. Now U is a homoclinic orbit. Finally when $I < I_c$ the connection breaks and U spirals into the sink.

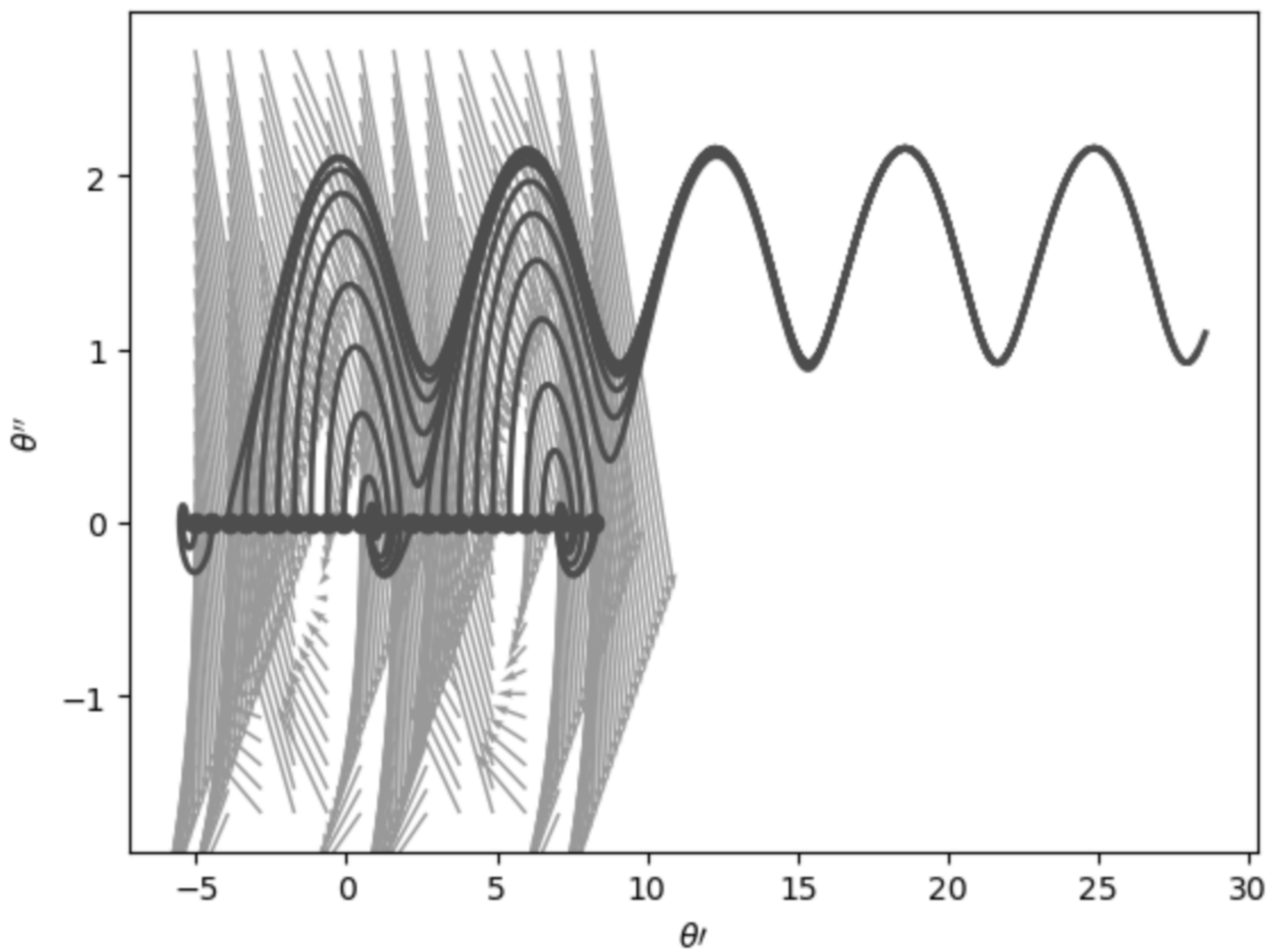


Python Plots for some α and I values:

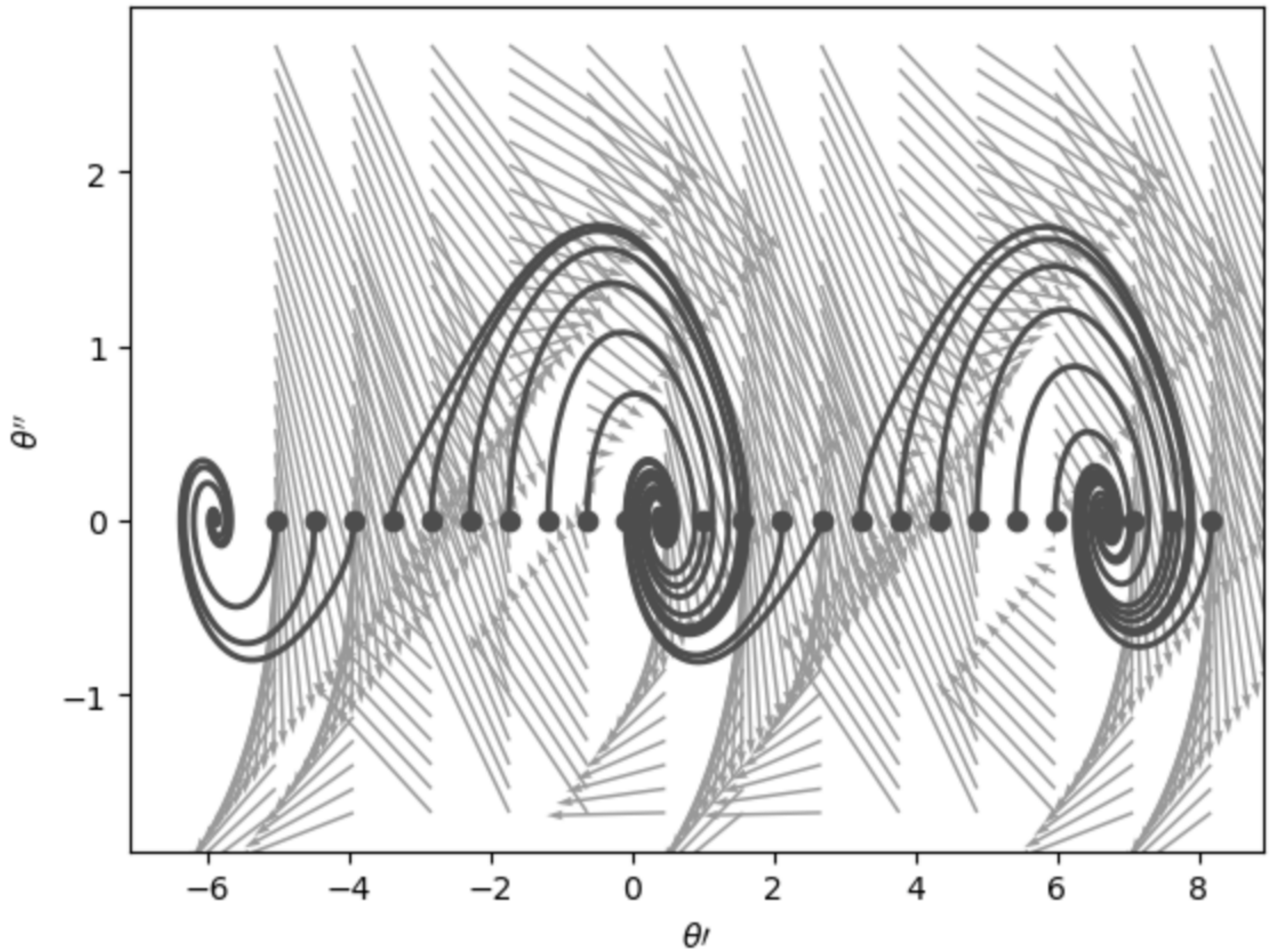
$\alpha = 0.5$ and $I = 1.2$



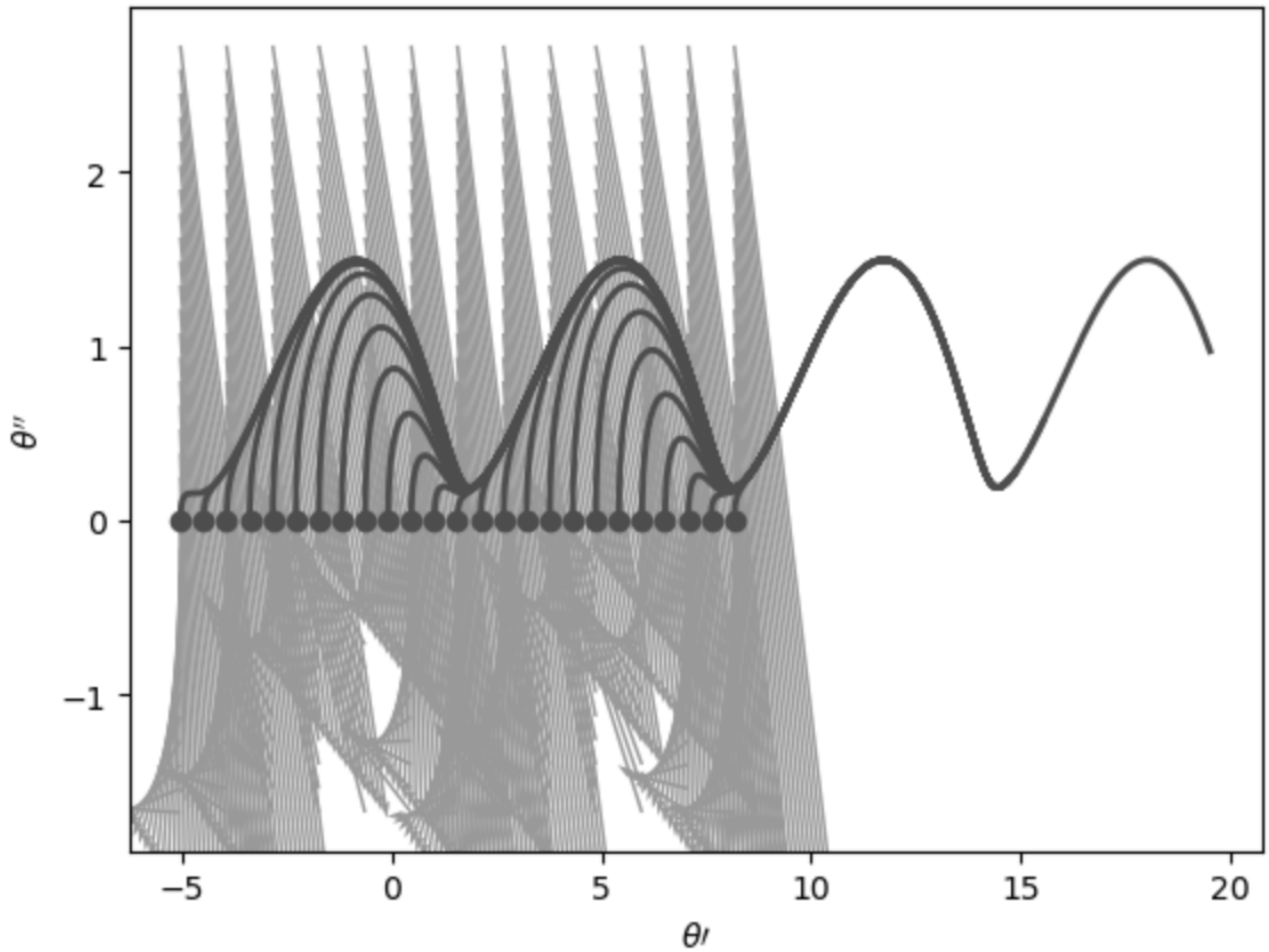
$\alpha = 0.5$ and $I = 0.8$ (bistable)



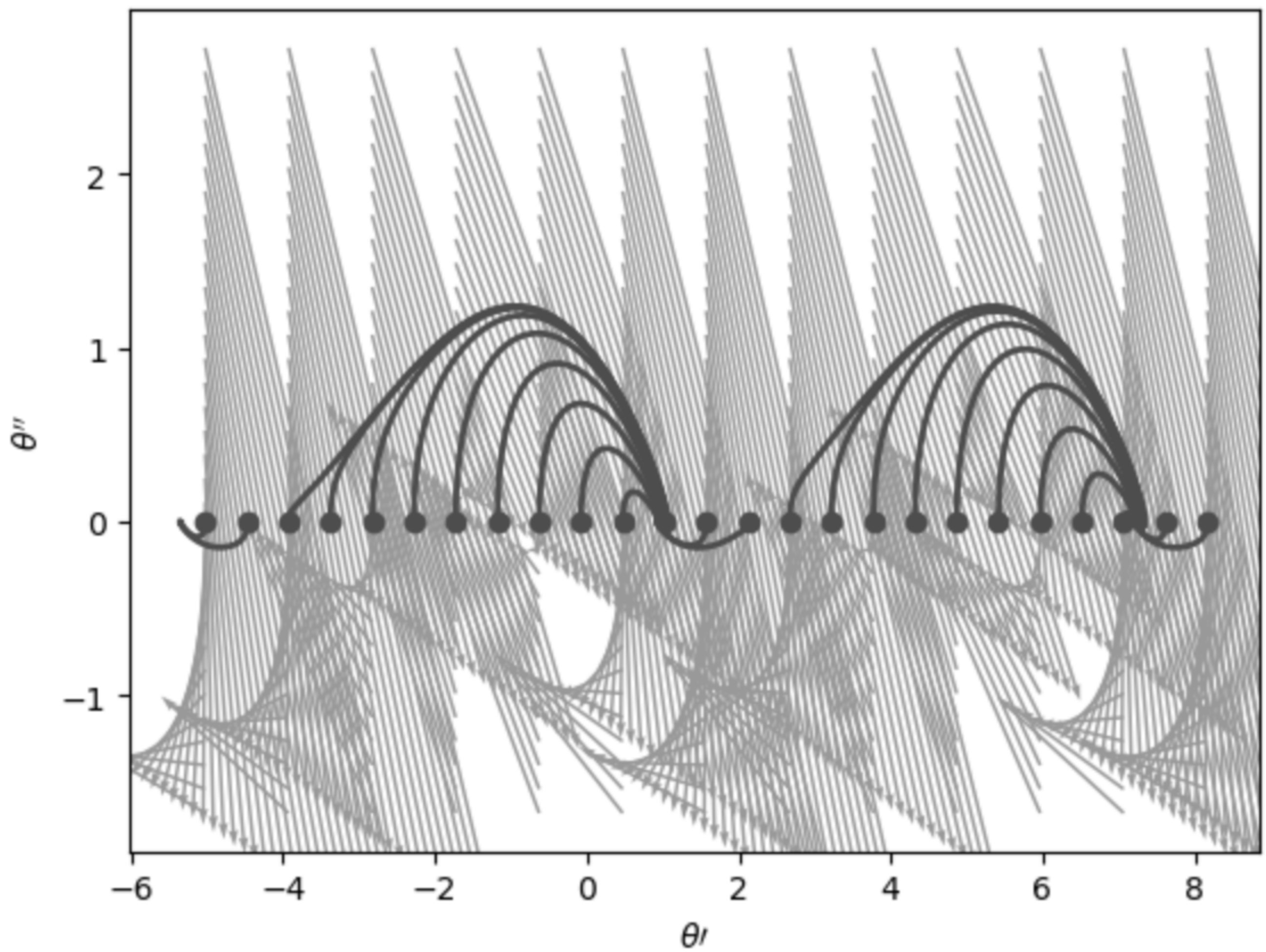
$\alpha = 0.5$ and $I = 0.4$



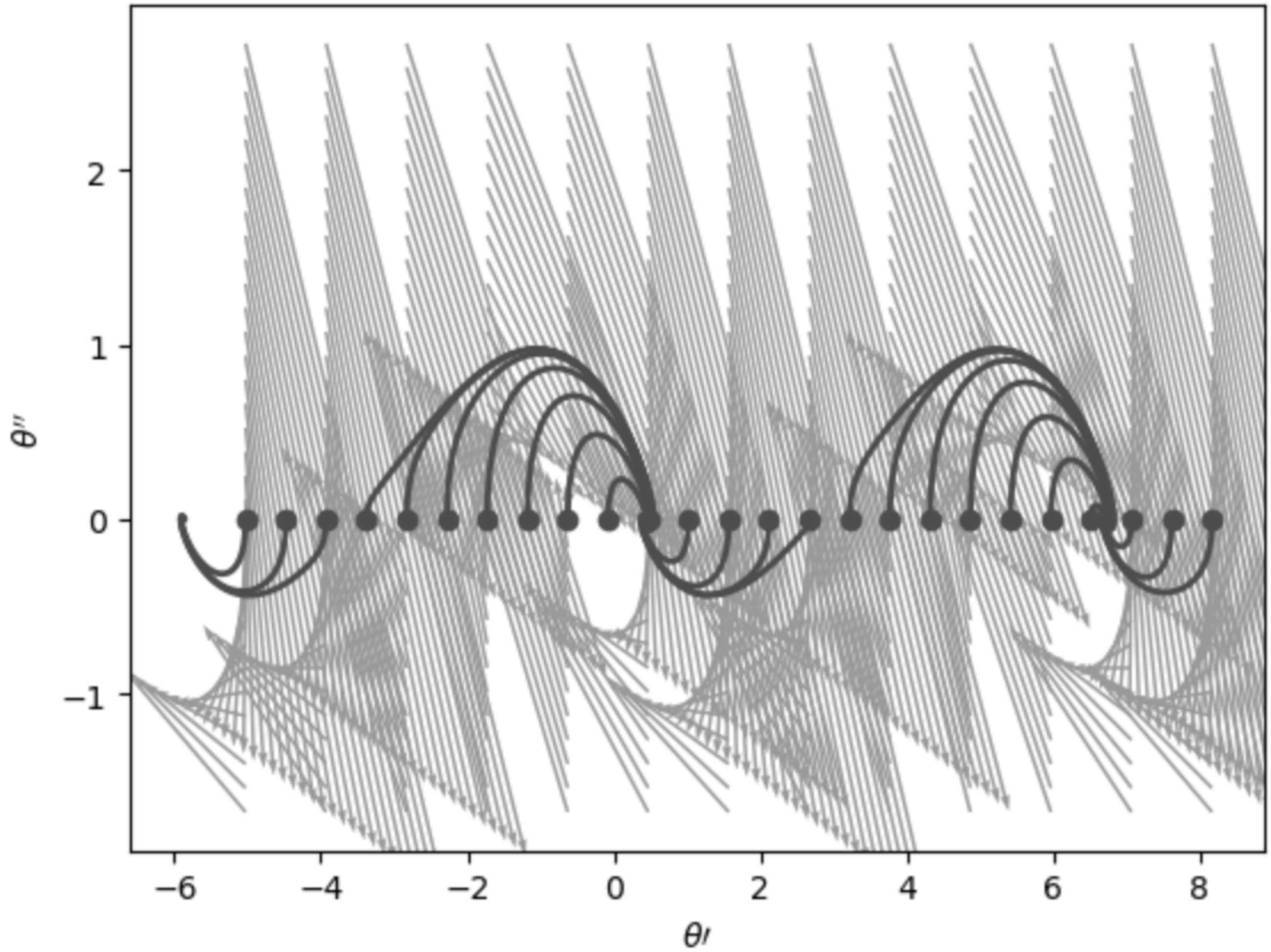
$\alpha = 1.3$ and $I = 1.2$



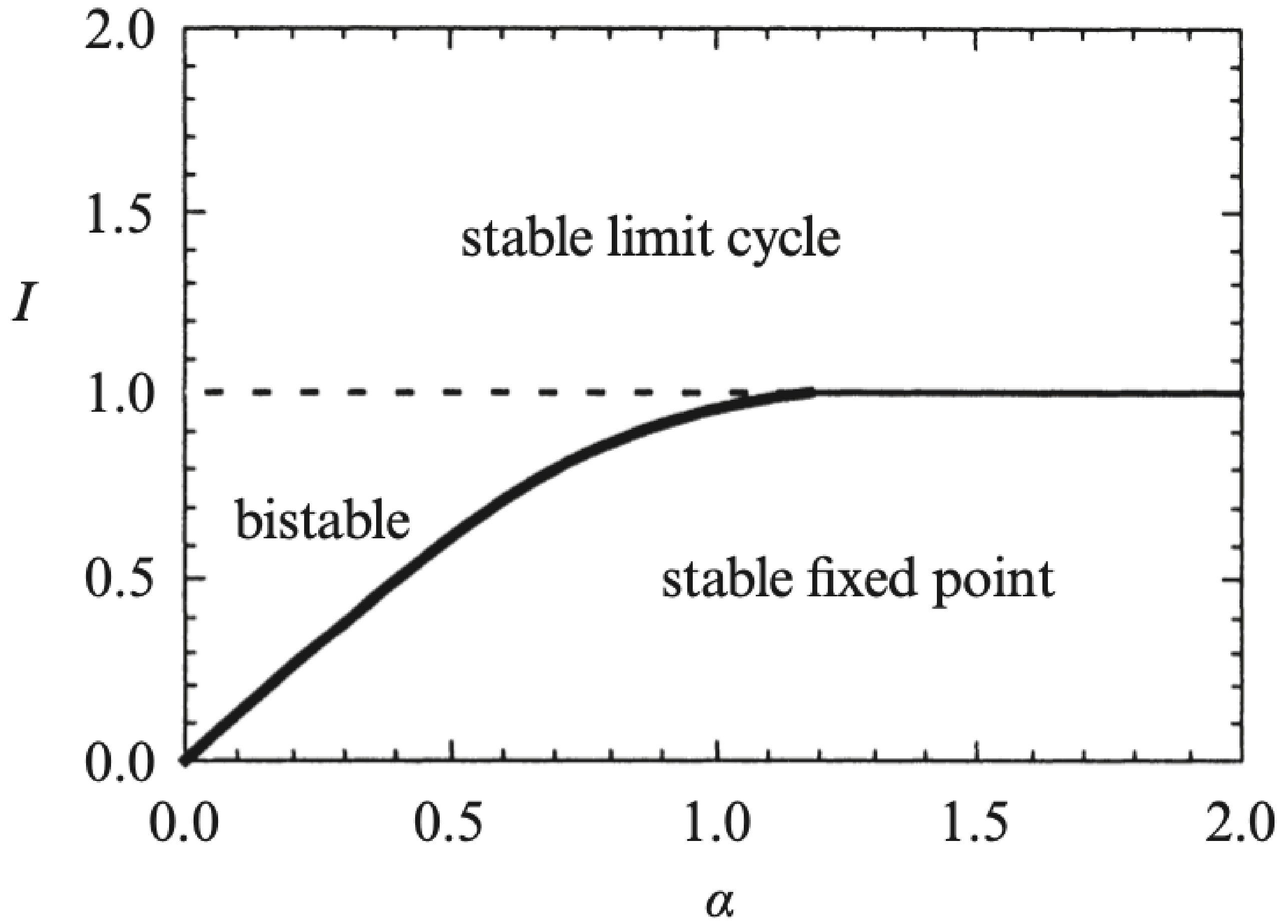
$\alpha = 1.3$ and $I = 0.8$



$\alpha = 1.3$ and $I = 0.4$



Notice that we do not get Homoclinic Bifurcation for $\alpha = 1.3$. We obtain Bistable regions for particular values of α and I.
If we plot α vs I we obtain:



The curve that separates the bistable and the stable fixed points can be approximated analytically using an advanced Technique known as Melinkov's method (See Guckenheimer and Holmes 1983, p.202).

Hysteretic Current Voltage Curve:

If we take a small value of α, and start from $I = 0$, we know that there exists a fixed point hence for the time average $\langle V \rangle = 0$. It will remain zero until $I = 1$.Because now the limit cycle exists, Now if we decrease back from $I = 1$, the limit cycle will still persist for $I < 1$, but the frequency will tend to zero as we are approaching an infinite period bifurcation. We know that

$\langle V \rangle$ is proportional to frequency, $\langle V \rangle$ also drops to zero continuously.