CHAPTER 5: Linear Programming

Maximising or minimising a linear function $b_1x_1 + b_2x_2 + b_3x_3 + \cdots + b_nx_n$ with the $b's$ the coefficients and the $x's$ the variables.

Example 1

Method 1: The geometric solution method

Model Type			
	L	**GL**	**Available hours**
Assembling time	25	50	1100
Finishing-off time	35	30	1050
Number of units	x	y	

- Let x be the number of units for model L and y be the number of units for model GL produced per day.

- The profit (in R1000) for model L is 10 and model GL it is 15.

- Select x and y such that the profit is maximised. Profit function: $10x + 15y$.

Restrictions:

$25x + 50y \leq 1100$
$35x + 30y \leq 1050$
$x, y \geq 0$

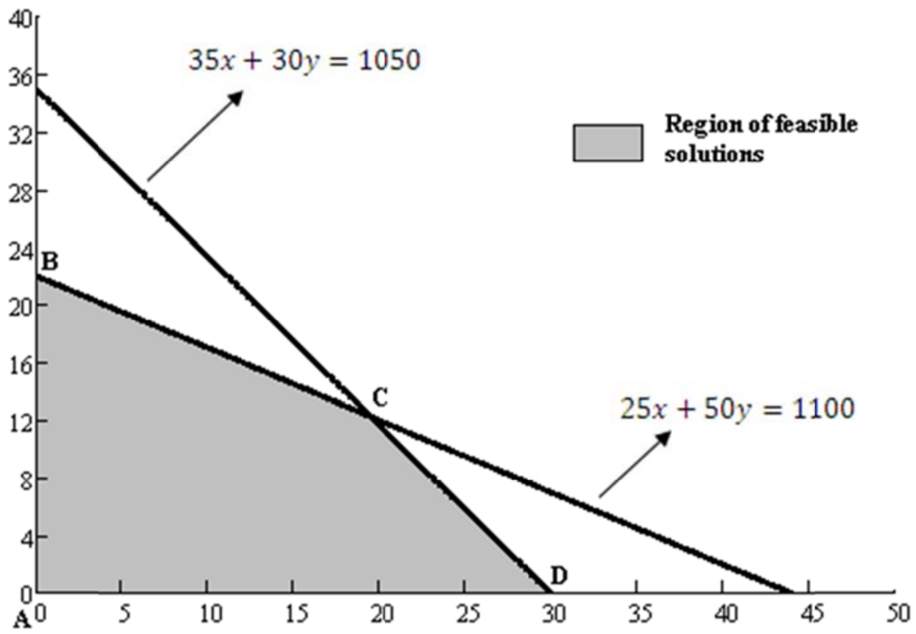
Note: $x, y \geq 0$ means that both x and y are greater than or equal to 0, i.e. $x \geq 0$ and $y \geq 0$.

Feasible solution area:

For $25x + 50y = 1100$ at $x = 0$ we get $y = \frac{1100}{50} = 22$ and at $y = 0$ we get $x = \frac{1100}{25} = 44$. Therefore, we have the coordinates: (0, 22) and (44, 0).

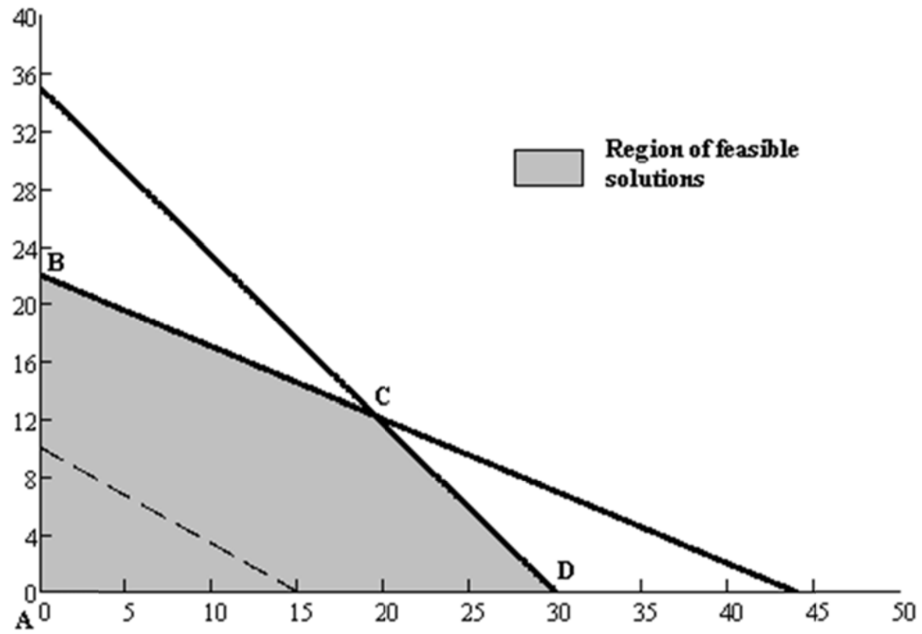
For $35x + 30y = 1050$ at $x = 0$ we get $y = \frac{1050}{30} = 35$ and at $y = 0$ we get $x = \frac{1050}{35} = 30$. Therefore, we have the coordinates: (0, 35) and (30, 0).

Graphical representation:



Objective function: Maximise $10x + 15y$

- Set the profit function equal to any constant ($10x + 15y = c$).
- Then $y = \frac{c}{15} - \frac{10}{15}x$ (see the dotted line on the graph).
- To maximise the profit function, move the function away from the origin.
- Note: If we wanted to minimise the function, we'd move the function nearer to the origin.



Calculate the co-ordinates of point C using Cramer's rule:

$$25x + 50y = 1100$$
$$35x + 30y = 1050$$

Matrix notation: $AX = B$ with $A = \begin{pmatrix} 25 & 50 \\ 35 & 30 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 1100 \\ 1050 \end{pmatrix}$.

$|A| = \begin{vmatrix} 25 & 50 \\ 35 & 30 \end{vmatrix} = (25)(30) - (35)(50) = -1000$

$x = \dfrac{\begin{vmatrix} 1100 & 50 \\ 1050 & 30 \end{vmatrix}}{-1000} = 19.5$ and $y = \dfrac{\begin{vmatrix} 25 & 1100 \\ 35 & 1050 \end{vmatrix}}{-1000} = 12.25.$

Hence the profit is maximised at $x = 19.5$ and $y = 12.25$.

- Profit: $10x + 15y = 10(19.5) + 15(12.25) = 378.75$.

- Assembly time: $25x + 50y = 25(19.5) + 50(12.25) = 1100$.

- Finishing-off time: $35x + 30y = 35(19.5) + 30(12.25) = 1050$.

Note the assembly and finishing-off time is optimally used.

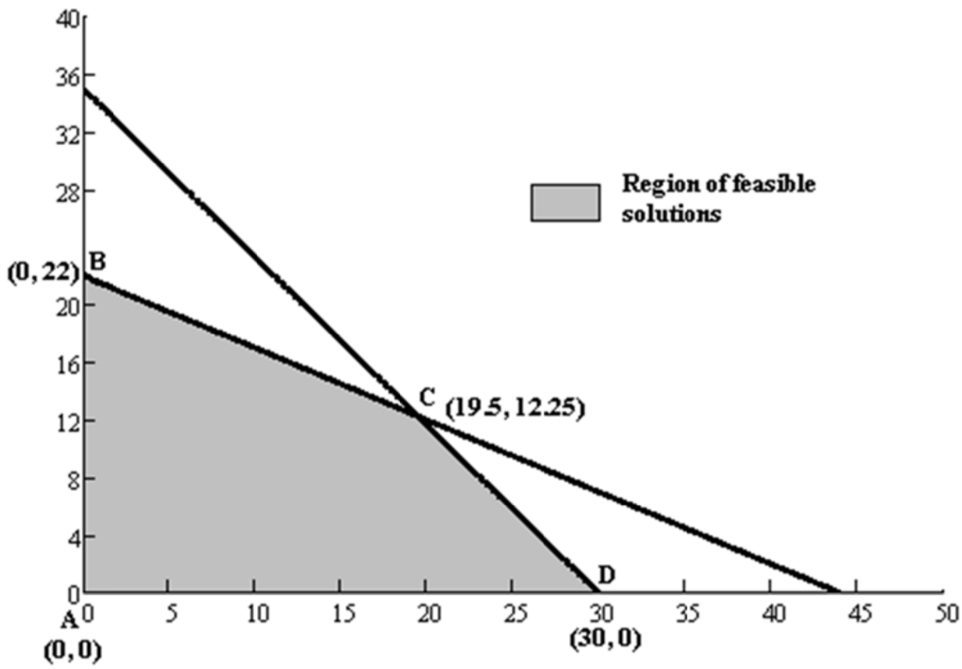
<u>Method 2: The extreme point method</u>

- The corners of the feasible region are known as the **extreme points**.

- The solution of the linear equations can **always** be found in at least one of the extreme points.

- If two points *C* and *D* give both an optimal solution then all the points on $\overline{CD}$ will give the optimal solutions.

- Method
    - Determine the corners of the feasible region.
    - Calculate the values of the objective function.
    - Identify the co-ordinates that give the optimal solution.



Extreme points	Coordinates (x, y)	Profit function $10x + 15y$
A	(0, 0)	10(0) + 15(0) = 0
B	(0, 22)	10(0) + 15(22) = 330
C	(19.5, 12.25)	10(19.5) + 15(12.25) = 378.75
D	(30, 0)	10(30) + 15(0) = 300

The profit function has a maximum of R378.75 (in 1000) at $x = 19.5$ and $y = 12.25$.

Example 2

Method 1: The geometric solution method

Question:
Minimise the cost function: $3x_1 + 5x_2$

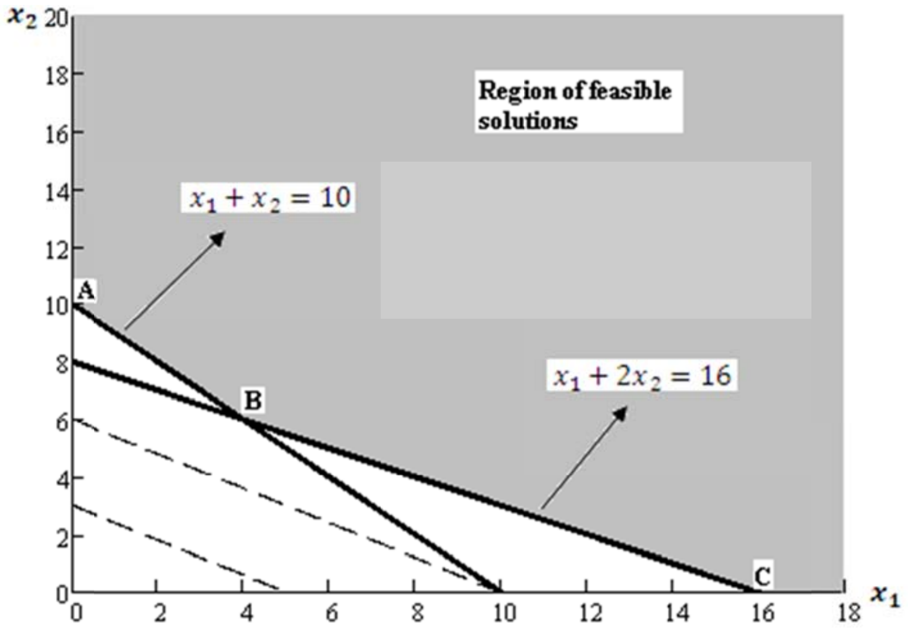
Subject to:
$x_1 + 2x_2 \geq 16$
$x_1 + x_2 \geq 10$
$x_1, x_2 \geq 0$

Answer:
Obtain the region of feasible solutions.

For $x_1 + 2x_2 = 16$ at $x_1 = 0$ we get $x_2 = 8$ and at $x_2 = 0$ we get $x_1 = 16$.
Therefore, we have the coordinates: (0, 8) and (16, 0).

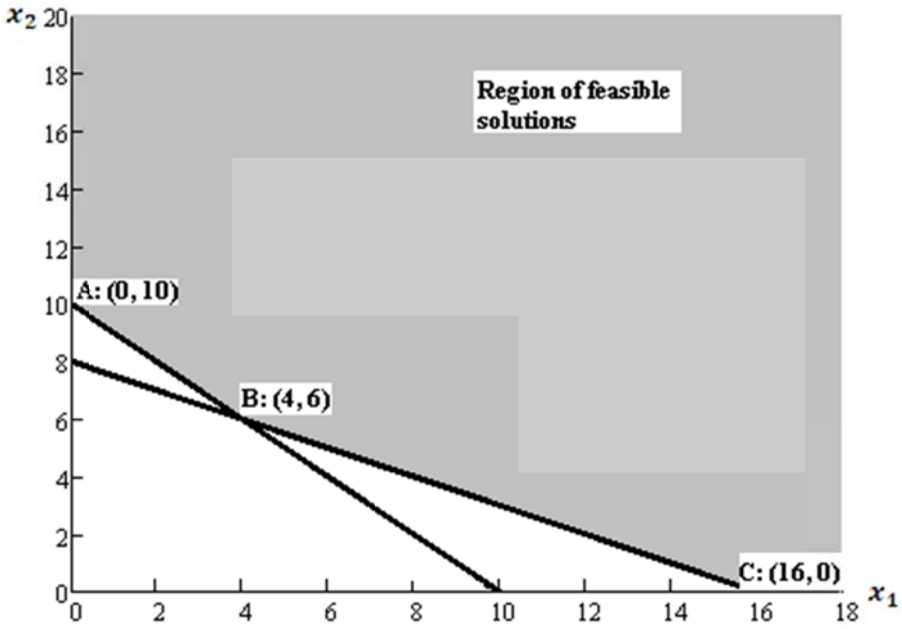
For $x_1 + x_2 = 10$ at $x_1 = 0$ we get $x_2 = 10$ and at $x_2 = 0$ we get $x_1 = 10$.
Therefore, we have the coordinates: (0, 10) and (10, 0).

Set $3x_1 + 5x_2 = c$ and we find $x_2 = \frac{c}{5} - \frac{3}{5}x_1$.



The objective function will be a minimum at point B.
The coordinates of point B can be found using Cramer's Rule as (4, 6).
Therefore, the minimum cost is 3(4) + 5(6) = 42.

<u>Method 2: The extreme point method</u>



Extreme points	Coordinates (x_1, x_2)	Cost function $3x_1 + 5x_2$
A	(0, 10)	3(0) + 5(10) = 50
B	(4, 6)	3(4) + 5(6) = 42
C	(16, 0)	3(16) + 5(0) = 48

The cost function has a maximum of R42 at $x_1 = 4$ and $x_2 = 6$.

Note on Example 2: To obtain the coordinate (4, 6) it is necessary to solve the following system of linear equations:

$$x_1 + 2x_2 = 16$$
$$x_1 + x_2 = 10$$

Use Cramer's rule to solve this system of linear equations.

Matrix notation: $AX = B$ with $A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $B = \begin{pmatrix} 16 \\ 10 \end{pmatrix}$.

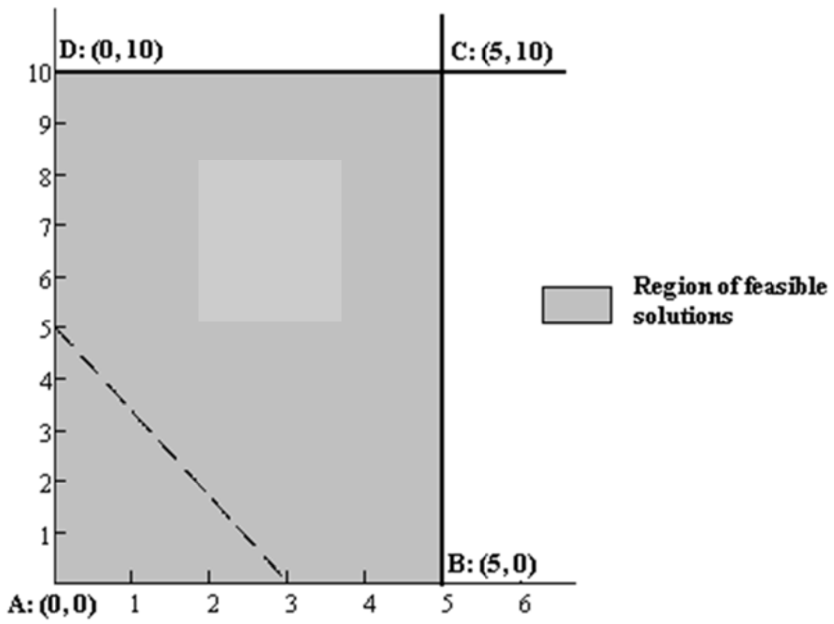
You will find the answers to be $x_1 = 4$ and $x_2 = 6$ (try this on your own using Cramer's rule).

Example 3

Question:
Maximise $5x + 3y$ subject to
$x \leq 5$
$y \leq 10$
$x, y \geq 0$
by using the geometric solution method.

Answer:

Set $5x + 3y = c$ so that $y = \frac{c}{3} - \frac{5}{3}x$.



Answer = Point C. Therefore, $x = 5$ and $y = 10$. The maximum is 5(5) + (3)(10) = 55.

Example 4

A construction company has two teams, namely team A and team B. Each team consists of at least 1 builder. Each builder in team A works 9 hours per day and gets paid R84. Each builder in team B works 12 hours per day and gets paid R60. The budget for daily wages for the company is R840. Suppose that on a given day there are 108 man-hours available. The profit for the construction company on work done by a builder in team A and B is respectively R40 and R70.

Let x denote the number of builders in team A and let y denote the number of builders in team B.

Hint: A question of this format is easier to answer if you construct a table:

Table:

	Team A	**Team B**	**Total**
Hours	9	12	108
Pay	84	60	840

Question
The objective function is given by: **Maximise** $40x + 70y$

Question
The constraints that lead to the region of feasible solutions are:

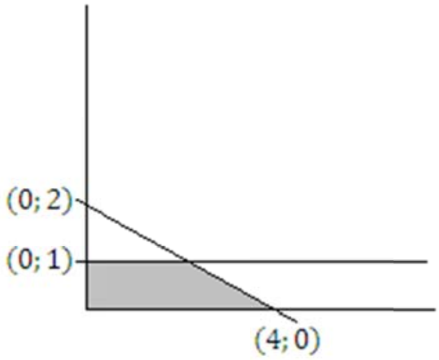
$$84x + 60y \leq 840$$
$$9x + 12y \leq 108$$
$$x, y \geq 1$$

Question (Note: We can only ask for one constraint at a time)
The time (in hours) constraint for the LP problem is: $9x + 12y \leq 108$

Question (Note: We can only ask for one constraint at a time)
The payment (in Rand) constraint for the LP problem is: $84x + 60y \leq 840$

Example 5

Find the constraints to the following LP problem:



Answer

The first constraint can easily be obtained from the horizontal line intersecting the y-axis at $y = 1$. The constraint: $0 \leq y \leq 1$.

For the second constraint we need to find the equation of the straight line intersecting the y-axis at $y = 2$ and the x-axis at $x = 4$. We have two coordinates:

$(x_1, y_1) = (4, 0)$ and $(x_2, y_2) = (0, 2)$. We substitute these coordinates into the equation of a straight line:

$$y = y_2 + \frac{y_2 - y_1}{x_2 - x_1}(x - x_2)$$

$$y = 2 + \frac{2 - 0}{0 - 4}(x - 0)$$

$$y = 2 - \frac{1}{2}x$$

Therefore, the constraints are:

$$0 \leq y \leq 1$$
$$y + \frac{1}{2}x \leq 2$$
$$x \geq 0$$

The Excel for Linear Programming is self-study: See the Assignment Book

Typical exam questions:

Question 1 is based on the following information:
Consider the following LP-problem:

Maximise $x + y$ subject to constraints:

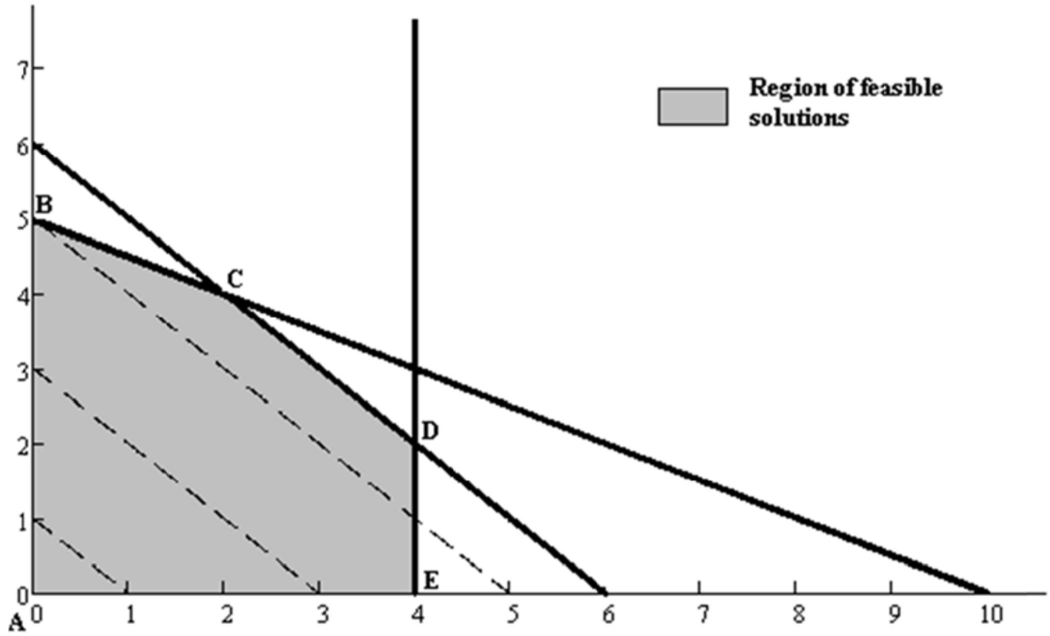
$$x \leq 4$$
$$x + y \leq 6$$
$$\frac{1}{2}x + y \leq 5$$
$$x, y \geq 0$$

Question 1:
The point or line segment that maximises the objection function is:

Answer 1:
For $x + y = 6$: Set $x = 0$ and find $y = 6$. Set $y = 0$ and find $x = 6$.
Coordinates (0, 6) and (6, 0).

For $0.5x + y = 5$: Set $x = 0$ and find $y = 5$. Set $y = 0$ and find $x = 10$.
Coordinates (0, 5) and (10, 0).

Using the objective function (**Maximise $\boldsymbol{x + y}$**) we set $x + y = c$ so that $y = \boldsymbol{c} - x$.

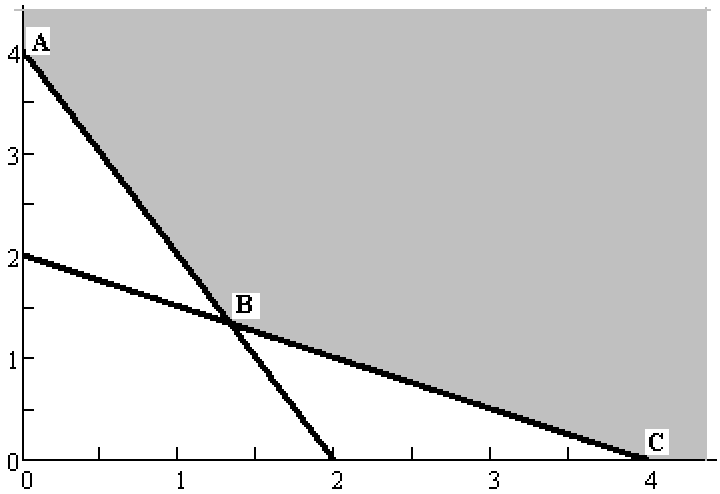


Any point on the line segment $\overline{CD}$ will give a maximum.

Question 2 is based on the following information:
Consider the following LP-problem:

Minimise $x + y$ subject to
$2x + y \geq 4$
$\frac{1}{2}x + y \geq 2$
$x, y \geq 0$

The region of feasible solutions is indicated by the shaded area ABC.



Question 2:
The objective function will reach a minimum solution at the point B. The value of the y co-ordinate at the point B is:

(A) 4	(B) 0
(C) 2	(D) $0.\dot{8}$
(E) $1.\dot{3}$	

Answer 2:
To obtain the coordinate of point B it is necessary to solve the following system of linear equations:

$$2x + y = 4$$
$$\frac{1}{2}x + y = 2$$

Use Cramer's rule to solve this system of linear equations.

Matrix notation: $AX = B$ with $A = \begin{pmatrix} 2 & 1 \\ 0.5 & 1 \end{pmatrix}$, $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and $B = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

You will find the answers to be $x = 1.\dot{3}$ and $y = 1.\dot{3}$ (try this on your own using Cramer's rule).
Answer = E