2. Two-scale asymptotic expansions method

We will illustrate this method by applying it to an elliptic boundary value problem for the unknown $u^\varepsilon : \Omega \to \mathbb{R}$ given by

$$\begin{cases} -\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) = f & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

The above model can be an efficient model for various phenomena in continuum mechanics. In the below table, we list some of the scientific domains in which the above elliptic problem can be used and the meaning of the coefficient $A(x)$ and the unknown $u^\varepsilon(x)$ in those respective fields.

Domain	**Unknown** $u^\varepsilon(x)$	**Material coefficient** $A(x)$
Electrostatics	Electric potential	Dielectric coefficient
Magnetostatics	Magnetic potential	Magnetic permeability
Steady heat transfer	Temperature	Thermal conductivity

As an helpful example, let us consider a one-dimensional problem which I borrowed from a lecture by Daniel Peterseim[2] (Bonn).

Example 2.1. *For the unknown $u^\varepsilon(x)$, consider the boundary value problem*

$$\begin{cases} -\frac{\mathrm{d}}{\mathrm{d}x}\left(a\left(\frac{x}{\varepsilon}\right)\frac{\mathrm{d}u^\varepsilon}{\mathrm{d}x}\right) = 1 & \textit{in } (0,1), \\ u^\varepsilon(0) = u^\varepsilon(1) = 0, \end{cases} \tag{2}$$

with the conductivity coefficient given as

$$a\left(\frac{x}{\varepsilon}\right) = \frac{1}{2+\cos\left(\frac{2\pi x}{\varepsilon}\right)}$$

which is plotted below

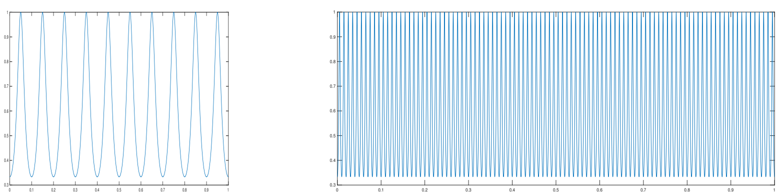

Figure 3. Plot of $\left(2+\cos\left(\frac{2\pi x}{\varepsilon}\right)\right)^{-1}$ with $\varepsilon = 10^{-1}$ (left), 10^{-2} (right).

[2]He gave this example during a scientific workshop at the Hausdorff Research Institute for Mathematics, Bonn.

Observe that the coefficient $a\left(\frac{x}{\varepsilon}\right)$ *given above has more and more oscillations in the* $\varepsilon \ll 1$ *regime. It is indeed of interest to learn the effect of these high frequency oscillations on the solution* $u^\varepsilon(x)$*. A lucky accident in this example is that we can explicitly integrate the differential equation* (2) *yielding*

$$u^\varepsilon(x) = x - x^2 + \varepsilon\left(\frac{1}{4\pi}\sin\left(\frac{2\pi x}{\varepsilon}\right) - \frac{x}{2\pi}\sin\left(\frac{2\pi x}{\varepsilon}\right) - \frac{\varepsilon}{4\pi^2}\cos\left(\frac{2\pi x}{\varepsilon}\right) + \frac{\varepsilon}{4\pi^2}\right)$$

which is plotted below

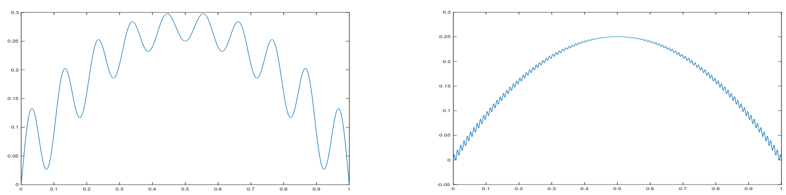

FIGURE 4. Plot of $u^\varepsilon(x)$ on $[0,1]$ with $\varepsilon = 10^{-1}$ (left), 10^{-2} (right).

Remark that the solution $u^\varepsilon(x)$ *has a parabolic profile* $x - x^2$ *with small amplitude oscillations of* $\mathcal{O}(\varepsilon)$ *along the profile. Indeed we have the uniform convergence*

$$u^\varepsilon(x) \to x - x^2 \qquad \textit{as } \varepsilon \to 0.$$

So, the above parabolic profile is the homogenized approximation of the solution profile $u^\varepsilon(x)$*.*

In the above example, instead of characterizing the homogenized operator, we have explicitly computed the homogenized profile. This is possible, thanks to the explicit integration of the differential equation (2). There is no such luck in obtaining explicit analytical expressions for solutions to differential equations in general, even in one dimension.

Let us briefly describe the setting of periodic homogenization. Denote the unit cube $Y := (0,1)^d \subset \mathbb{R}^d$. Consider a Y-periodic function $A : \Omega \to \mathbb{R}^{d\times d}$, i.e.,

$$A(x + k\mathbf{e}_i) = A(x) \qquad \text{for each } x \in \mathbb{R}^d, \quad \forall k \in \mathbb{Z} \quad \forall i \in \{1, \dots, d\}$$

where $\{\mathbf{e}_i\}$ denotes the canonical basis in $\mathbb{R}^d$. Taking $\varepsilon > 0$ as a parameter, consider the dilated sequence defined as

$$A^\varepsilon(x) := A\left(\frac{x}{\varepsilon}\right).$$

Remark that the dilated function $A^\varepsilon(x)$ is εY-periodic because

$$A^\varepsilon(x + k\varepsilon\mathbf{e}_i) = A\left(\frac{x + k\varepsilon\mathbf{e}_i}{\varepsilon}\right) = A\left(\frac{x}{\varepsilon} + k\mathbf{e}_i\right) = A\left(\frac{x}{\varepsilon}\right) = A^\varepsilon(x)$$

for all $k \in \mathbb{Z}$.

In all the theory that is being presented in this course, a fundamental assumption is that of *separated scales*. To be precise, we consider x to be a *slow variable*

and introduce a new auxiliary variable $y := \frac{x}{\varepsilon}$ which we call a *fast variable*. The assumption of separated scales essentially implies that the variables x and y are to be treated as independent variables.
With the above understanding of the slow and fast variables, in the periodic setting, we let the variable $y \in Y$, the unit cube in $\mathbb{R}^d$ introduced earlier. Then, the periodic conductivity coefficient A can be treated as a function of the fast spatial variable y. We further assume that the conductivity coefficient $A(y)$ is uniformly bounded and uniformly positive definite, i.e., there exist constants $\alpha, \beta > 0$ (uniform in the y variable) such that

$$\alpha\,|\xi|^2 \le A(y)\xi\cdot\xi \le \beta\,|\xi|^2 \qquad \forall y \in Y \quad \text{ and } \forall \xi \in \mathbb{R}^d \setminus \{0\}. \tag{3}$$

Employing the Lax-Milgram theorem, we can prove that the boundary value problem (1) is well-posed.

Theorem 2.2. *Suppose the source term $f \in \mathrm{L}^2(\Omega)$. Suppose that the periodic conductivity matrix $A(y)$ is positive definite and that it is uniformly bounded, i.e., it satisfies (3). Then for each fixed $\varepsilon > 0$, there exists a unique solution $u^\varepsilon \in \mathrm{H}^1_0(\Omega)$ to the elliptic boundary value problem (1). Furthermore, we have the uniform estimate*

$$\|u^\varepsilon\|_{\mathrm{H}^1_0(\Omega)} \le C\,\|f\|_{\mathrm{L}^2(\Omega)} \tag{4}$$

with constant $C > 0$ being independent of the parameter ε.

2.1. **A postulate and matched asymptotic.** The model conductivity problem (1) has two spatial scales and it has a small parameter $\varepsilon > 0$. Hence the essential idea of the method of asymptotic expansions is to consider a power series expansion for the unknown u^ε in the parameter ε and with coefficients that depend on both the spatial scales, i.e., on both the slow and fast variables. More precisely, we postulate that the solution to (1) can be written as

$$u^\varepsilon(x) = \sum_{i\ge 0} \varepsilon^i u_i\left(x, \frac{x}{\varepsilon}\right) \tag{5}$$

under the assumption that the coefficient functions $u_i(x,y)$ are Y-periodic in the y variable. The coefficient functions $u_i\left(x, \frac{x}{\varepsilon}\right)$ are referred to as locally periodic functions. At least formally, for $\varepsilon \ll 1$, we have from (5) the approximation

$$u^\varepsilon(x) \approx u_0(x,y)\Big|_{y=\frac{x}{\varepsilon}}$$

Then, in the $\varepsilon \ll 1$ regime, one can argue that the y dependance of u_0 gets "averaged" out over the periodic cell Y. One could then find the homogenized approximation for u^ε as

$$u_{\text{hom}}(x) := \int_Y u_0(x,y)\,\mathrm{d}y.$$

The above heuristics essentially motivates the postulated asymptotic expansion (5) for the solution $u^\varepsilon(x)$. These hand-wavy heuristics are not at all rigorous. More analysis oriented approach to derive homogenized limits will be the objective of the chapters to come. As far as this chapter is concerned, we admit the two-scale asymptotic expansion for the solution and carry on with the computations. Observe that the explicit solution obtained in Example 2.1 has the structure postulated above in (5). That is a mere coincidence. The principle behind the asymptotic expansions method is to plug the expansion (5) in the model problem (1) and match the coefficients of various powers of ε. Before we proceed, we need to keep the following in mind while applying the asymptotic expansion method.

- For any locally periodic function $\psi\left(x, \frac{x}{\varepsilon}\right)$, its total derivative becomes
$$\nabla\left[\psi\left(x, \frac{x}{\varepsilon}\right)\right] = \nabla_x\psi(x,y)\Big|_{y=\frac{x}{\varepsilon}} + \frac{1}{\varepsilon}\nabla_y\psi(x,y)\Big|_{y=\frac{x}{\varepsilon}}$$
Note that the above expression is a consequence of the chain rule for differentiation.
- Suppose $\Phi(x,y)$ is a function of both the slow and fast variables. Then the following are equivalent
$$(i) \qquad \Phi\left(x, \frac{x}{\varepsilon}\right) = 0 \quad \forall x \in \Omega,\, \forall \varepsilon > 0,$$
$$(ii) \qquad \Phi(x,y) = 0 \quad \forall x \in \Omega,\, \forall y \in Y.$$

Let us get on with the task of plugging the asymptotic expansion (5) in the model problem (1) yielding

$$\begin{aligned}
-\mathrm{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon\right) = & -\frac{1}{\varepsilon^2}\mathrm{div}_y\Big(A(y)\,\nabla_y u_0(x,y)\Big)\Big|_{y=\frac{x}{\varepsilon}} \\
& -\frac{1}{\varepsilon}\mathrm{div}_y\Big(A(y)\left(\nabla_x u_0(x,y) + \nabla_y u_1(x,y)\right)\Big)\Big|_{y=\frac{x}{\varepsilon}} \\
& -\frac{1}{\varepsilon}\mathrm{div}_x\Big(A(y)\,\nabla_y u_0(x,y)\Big)\Big|_{y=\frac{x}{\varepsilon}} \\
& -\mathrm{div}_y\Big(A(y)\left(\nabla_x u_1(x,y) + \nabla_y u_2(x,y)\right)\Big)\Big|_{y=\frac{x}{\varepsilon}} \\
& -\mathrm{div}_x\Big(A(y)\left(\nabla_x u_0(x,y) + \nabla_y u_1(x,y)\right)\Big)\Big|_{y=\frac{x}{\varepsilon}} \\
& -\varepsilon\,\mathrm{div}_y\Big(A(y)\left(\nabla_x u_2(x,y) + \nabla_y u_3(x,y)\right)\Big)\Big|_{y=\frac{x}{\varepsilon}} \\
& -\varepsilon\,\mathrm{div}_x\Big(A(y)\left(\nabla_x u_1(x,y) + \nabla_y u_2(x,y)\right)\Big)\Big|_{y=\frac{x}{\varepsilon}} \\
& -\cdots
\end{aligned} \tag{6}$$

where we will not explicitly write down the $\mathcal{O}(\varepsilon^2)$ terms and further on (but the readers can naturally extrapolate). In the computations to follow, only the $\mathcal{O}(1)$ terms and the terms of higher order play significant role. In the boundary value problem (1), $f(x)$ is a given source term of $\mathcal{O}(1)$. Hence equating the coefficients of various powers of ε, we obtain a cascade of equations. As it will become apparent in the computations to follow, the orders in ε that are essential for arriving at the

homogenized equation are that of $\mathcal{O}(\varepsilon^{-2})$, $\mathcal{O}(\varepsilon^{-1})$ and $\mathcal{O}(1)$.
At $\mathcal{O}(\varepsilon^{-2})$, we obtain

$$\begin{cases} \operatorname{div}_y\Big(A\,(y)\,\nabla_y u_0(x,y)\Big) = 0 & \text{for } y \in Y, \\ \qquad\qquad y \mapsto u_0(x,y) & \text{is } Y\text{-periodic.} \end{cases} \tag{7}$$

Note that the periodic boundary conditions are because of our periodicity assumption on the coefficient functions in the postulated asymptotic expansion. The variable x is treated as a parameter in (7) which is treated as a periodic boundary value problem in the y variable.
Before proceeding further, let us remark that u_0 is a function independent of the y variable. To see that, let us multiply the equation (7) by u_0 and integrate over Y yielding

$$\int_Y A(y)\nabla_y u_0(x,y)\cdot\nabla_y u_0(x,y)\,\mathrm{d}y - \int_{\partial Y} A(y)\nabla_y u_0(x,y)\cdot\mathbf{n}(y)\,\mathrm{d}\sigma(y) = 0.$$

By assumption $A(\cdot)$ and $u_0(x,\cdot)$ are periodic. As the normal $\mathbf{n}(y)$ takes opposite signs on the opposite ends of the boundary ∂Y, we have

$$\int_{\partial Y} A(y)\nabla_y u_0(x,y)\cdot\mathbf{n}(y)\,\mathrm{d}\sigma(y) = 0.$$

Furthermore, the coercivity assumption on $A(y)$ implies

$$\alpha\,\|\nabla_y u_0(x,\cdot)\|^2_{\mathrm{L}^2(Y)} \le \int_Y A(y)\nabla_y u_0(x,y)\cdot\nabla_y u_0(x,y)\,\mathrm{d}y$$

with the right hand side of the above inequality vanishing by the above energy calculations. Hence we indeed have that

$$u_0(x,y) \equiv u_0(x).$$

Continuing our method of matched asymptotic, at $\mathcal{O}(\varepsilon^{-1})$, we have

$$\begin{cases} -\operatorname{div}_y\Big(A\,(y)\,\nabla_y u_1(x,y)\Big) = \operatorname{div}_y\Big(A\,(y)\,\nabla_x u_0(x)\Big) & \text{for } y \in Y, \\ \qquad\qquad y \mapsto u_1(x,y) & \text{is } Y\text{-periodic.} \end{cases} \tag{8}$$

Note that we have dropped the term involving the $\nabla_y u_0(x,y)$ from (6) even though it was apparently of order $\mathcal{O}(\varepsilon{-}1)$. This is because of our earlier observation that u_0 is independent of the y variable. Equation (8) should be treated as an equation for the unknown $u_1(x,y)$. Because of linearity, observe that we can separate the slow and fast variables in u_1 as

$$u_1(x,y) = \sum_{i=1}^{d} \omega_i(y)\frac{\partial u_0}{\partial x_i}(x) \tag{9}$$

where the unknown functions $w_i(y)$ for each $i \in \{1, \ldots, d\}$ solve the so-called cell problems

$$
\left\{
\begin{aligned}
-\mathrm{div}_y\Big(A\left(y\right)\nabla_y\omega_i(y)\Big) &= \mathrm{div}_y\Big(A\left(y\right)\mathbf{e}_i\Big) && \text{for } y \in Y,\\
y \mapsto \omega_i(y) & && \text{is } Y\text{-periodic.}
\end{aligned}
\right.
\tag{10}
$$

Here $\{\mathbf{e}_i\}_{i=1}^d$ denotes the canonical basis of $\mathbb{R}^d$. The d equations in (10) are called cell problems as they are posed on the periodicity cell Y. The solutions to (10) are henceforth called cell solutions.

Finally, at $\mathcal{O}(1)$, we obtain

$$
\left\{
\begin{aligned}
-\mathrm{div}_y\Big(A\left(y\right)\nabla_y u_2(x,y)\Big) &= \mathrm{div}_y\Big(A\left(y\right)\nabla_x u_1(x,y)\Big)\\
&\quad + \mathrm{div}_x\Big(A\left(y\right)\Big(\nabla_x u_0(x) + \nabla_y u_1(x,y)\Big)\Big) + f(x),\\
y \mapsto u_1(x,y) & \qquad\qquad \text{is } Y\text{-periodic.}
\end{aligned}
\right.
\tag{11}
$$

So far, we have not spoken about the solvability of the periodic boundary value problems (7)-(8)-(10)-(11) that we derived by plugging the asymptotic expansion in the model problem. The following lemma will address this issue.

Lemma 2.3. *Suppose the periodic conductivity coefficient $A(y)$ is positive definite and that it is uniformly bounded, i.e., it satisfies (3). Let $g \in \mathrm{L}^2(Y)$ be a given function. Then there exists a unique solution $v \in \mathrm{H}^1_{\mathrm{per}}(Y)/\mathbb{R}$ to the following periodic boundary value problem*

$$
\left\{
\begin{aligned}
-\mathrm{div}_y\Big(A\left(y\right)\nabla_y v(y)\Big) &= g(y) && \textit{for } y \in Y,\\
y \mapsto v(y) & && \textit{is } Y\textit{-periodic}
\end{aligned}
\right.
\tag{12}
$$

if and only if the source term $g(y)$ in (12) satisfies the compatibility condition

$$
\langle g \rangle := \int_Y g(y)\,\mathrm{d}y = 0. \tag{13}
$$

Proof. We know that the the quotient space $\mathrm{H}^1_{\mathrm{per}}(Y)$ has the equivalent norm

$$
\|v\|_{\mathrm{H}^1_{\mathrm{per}}(Y)} = \|\nabla_y v\|_{\mathrm{L}^2(Y)}\,.
$$

Solving the periodic boundary value problem (12) is equivalent to solving the variational problem:

Find a unique $v \in \mathrm{H}^1_{\mathrm{per}}(Y)$ such that $\forall w \in \mathrm{H}^1_{\mathrm{per}}(Y)$, we have

$$
\mathfrak{a}(v,w) = l(w)
$$

$$
\text{with} \quad \mathfrak{a}(v,w) := \int_Y A(y)\nabla_y v(y)\cdot\nabla_y w(y)\,\mathrm{d}y \quad \text{and} \quad l(w) := \int_Y g(y)w(y)\,\mathrm{d}y.
$$

To employ the Lax-Milgram theorem, we need to show that the bilinear form $\mathfrak{a}(\cdot,\cdot)$ is coercive on $\mathrm{H}^1_{\mathrm{per}}(Y)$. This is straightforward. Next, we need to show that the linear form $l(\cdot)$ is bounded. Compute the modulus of the linear form

$$
\begin{aligned}
|l(w)| = \left|\int_Y g(y)w(y)\,\mathrm{d}y\right| &= \left|\int_Y \Big(g(y)w(y) - \langle w\rangle g(y) + \langle w\rangle g(y)\Big)\,\mathrm{d}y\right| \\
&= \int_Y \left|\Big(g(y)w(y) - \langle w\rangle g(y)\Big)\right|\,\mathrm{d}y \\
&\le \|w - \langle w\rangle\|_{\mathrm{L}^2(Y)} \|g\|_{\mathrm{L}^2(Y)} \\
&\le C\,\|\nabla_y w\|_{\mathrm{L}^2(Y)} \|g\|_{\mathrm{L}^2(Y)}
\end{aligned}
$$

where we have used the Hölder inequality and the Poincaré inequality. Note that the above computation only works under the assumption that $g(y)$ is of zero mean. The existence and uniqueness of $v \in \mathrm{H}^1_{\mathrm{per}}(Y)/\mathbb{R}$ then follows from Lax-Milgram. $\square$

Remark 2.4. *This remark is about the solvability of the boundary value problems (7)-(8)-(10)-(11). As Lemma 2.3 asserts, only detail that one needs to check is that the source terms in all these periodic boundary value problems are of zero average on the periodicity cell.*

- *The source term in the $\mathcal{O}(\varepsilon^{-2})$ equation (7) is*

$$g(y) \equiv 0.$$

  Hence the compatibility condition (13) is trivially satisfied. The result of Lemma 2.3 then implies that the unique solution to (7) is

$$0 \in \mathrm{H}^1_{\mathrm{per}}(Y)/\mathbb{R} \qquad i.e.,\ u_0(x,y) \equiv u_0(x)$$

  which is what we deduced earlier via energy computations.
- *The source term in the $\mathcal{O}(\varepsilon^{-1})$ equation (8) is*

$$g(y) = \mathrm{div}_y\Big(A\,(y)\,\nabla_x u_0(x)\Big).$$

  As the variable x plays the role of a parameter, we will not make explicit the dependence of the source g on the x variable. Let us now compute the average of $g(y)$ over the periodic cell

$$\int_Y \mathrm{div}_y\Big(A\,(y)\,\nabla_x u_0(x)\Big)\,\mathrm{d}y = \int_{\partial Y} A\,(y)\,\nabla_x u_0(x)\cdot \mathbf{n}(y)\,\mathrm{d}\sigma(y) = 0.$$

  Thus the compatibility condition (13) is satisfied. Hence the existence of a unique solution $u_1(x,\cdot) \in \mathrm{H}^1_{\mathrm{per}}(Y)/\mathbb{R}$ to the $\mathcal{O}(\varepsilon^{-1})$ equation (8).
- *A similar arguments work for the cell problems (10) whose source terms are*

$$g(y) = \mathrm{div}_y\Big(A\,(y)\,\mathbf{e}_i\Big).$$

- *Finally, to solve the periodic boundary value problem for $u_2(x,\cdot) \in \mathrm{H}^1_{\mathrm{per}}(Y)/\mathbb{R}$, let us make sure that the source term in (11) satisfies the compatibility condition. In fact, writing down the compatibility condition for (11) yields the*

homogenized equation for $u_0(x)$.
The source term in the $\mathcal{O}(1)$ equation (11) is

$$g(y) = \mathrm{div}_y\Big(A\,(y)\,\nabla_x u_1(x,y)\Big) + \mathrm{div}_x\Big(A\,(y)\,\left(\nabla_x u_0(x) + \nabla_y u_1(x,y)\right)\Big) + f(x)$$

Integrating the above given $g(y)$ over the periodicity cell yields

$$\int_Y g(y)\,\mathrm{d}y = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 := \int_Y \mathrm{div}_y\Big(A\,(y)\,\nabla_x u_1(x,y)\Big)\,\mathrm{d}y + \int_Y \mathrm{div}_x\Big(A\,(y)\,\left(\nabla_x u_0(x) + \nabla_y u_1(x,y)\right)\Big)\,\mathrm{d}y + \int_Y f(x)\,\mathrm{d}y.$$

Remark that, because $A(\cdot)$ and $u_1(x,\cdot)$ are Y-periodic, we have

$$\mathcal{I}_1 = 0.$$

Furthermore, as f is independent of the y variable,

$$\mathcal{I}_3 = f(x).$$

From Lemma 2.3, we can solve for $u_2(x,\cdot)$ uniquely, provided we satisfy the compatibility condition which translates here as

$$-\mathcal{I}_2 = \mathcal{I}_3$$

which is nothing but

$$-\,\mathrm{div}_x \int_Y \Big(A\,(y)\,\Big(\nabla_x u_0(x) + \nabla_y u_1(x,y)\Big)\Big)\,\mathrm{d}y = f(x) \qquad \text{for } x \in \Omega. \tag{14}$$

Observe that substituting for $u_1(x,y)$ in the above equation in terms of the cell solutions $\omega_i(y)$ and the gradient of $u_0(x)$, i.e., using (9) results in a second order differential equation for $u_0(x)$.

Let us pick up our calculations from (14). Substituting for $u_1(x,y)$ using (9) results in

$$\begin{aligned}
-\,\mathrm{div}_x \int_Y \Big(A\,(y)\Big(\nabla_x u_0(x,y) + \nabla_y u_1(x,y)\Big)\Big)\,\mathrm{d}y \\
&= -\,\mathrm{div}_x \int_Y \left(A\,(y)\left(\nabla_x u_0(x,y) + \sum_{k=1}^{d} \nabla_y \omega_k(y)\frac{\partial u_0}{\partial x_k}(x)\right)\right)\mathrm{d}y \\
&= -\,\mathrm{div}_x\Big(A_{\mathrm{hom}}\nabla_x u_0(x)\Big)
\end{aligned}$$

with the constant matrix A_{hom} – called the *homogenized matrix* – which is given as

$$A_{\mathrm{hom}}\xi = \int_Y A\,(y)\left(\xi + \sum_{k=1}^{d} \xi_k \nabla_y \omega_k(y)\right)\mathrm{d}y$$

for any $\xi \in \mathbb{R}^d$. Note in particular that the elements of the matrix A_{hom} are

$$[A_{\text{hom}}]_{ij} = \int_Y A(y)\Big(\mathbf{e}_j + \nabla_y \omega_j(y)\Big) \cdot \mathbf{e}_i \, \mathrm{d}y \tag{15}$$

for $i, j \in \{1, \dots, d\}$. The calculations so far are summarised below.

Proposition 2.5. *The solution $u^\varepsilon(x)$ to the boundary value problem (1) is approximated as*

$$u^\varepsilon(x) \approx u_0(x) + \varepsilon \sum_{j=1}^{d} \omega_j\left(\frac{x}{\varepsilon}\right) \frac{\partial u_0}{\partial x_j}(x)$$

where $u_0(x)$ is called the homogenized approximation which solves the homogenized boundary value problem

$$\begin{cases} -\mathrm{div}\Big(A_{\text{hom}} \nabla u_0(x)\Big) = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{16}$$

with A_{hom} being the constant homogenized conductivity whose elements are

$$[A_{\text{hom}}]_{ij} = \int_Y A(y)\Big(\mathbf{e}_j + \nabla_y \omega_j(y)\Big) \cdot \mathbf{e}_i \, \mathrm{d}y$$

for $i, j \in \{1, \dots, d\}$. The functions $\omega_j(y)$ are called the cell solutions which solve the cell problems

$$\begin{cases} -\mathrm{div}_y\Big(A(y)\, \nabla_y \omega_i(y)\Big) = \mathrm{div}_y\Big(A(y)\, \mathbf{e}_i\Big) & \text{for } y \in Y, \\ y \mapsto \omega_i(y) & \text{is } Y\text{-periodic} \end{cases}$$

for each $i \in \{1, \dots, d\}$. Furthermore, the homogenized conductivity is uniformly positive definite, i.e., there exists a positive $\alpha > 0$ such that

$$A_{\text{hom}} \xi \cdot \xi \geq \alpha \, |\xi|^2 \tag{17}$$

for all $\xi \in \mathbb{R}^d \setminus \{0\}$.

Proof. A lion share of the proof of Proposition 2.5 is already present in the calculations preceding the statement of the proposition. We are left, however, to show that the homogenized approximation $u_0(x)$ satisfies zero Dirichlet data on the boundary and to show that A_{hom} is uniformly positive definite.
As we are postulating an asymptotic expansion (5), let us simply equate the expansion to zero on the boundary

$$0 = u_0(x) + \varepsilon\, u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2\, u_2\left(x, \frac{x}{\varepsilon}\right) + \dots \qquad \text{for } x \in \partial\Omega.$$

As the above equality should hold in the $\varepsilon \to 0$ limit, at least formally, we obtain

$$u_0(x) = 0 \qquad \text{for } x \in \partial\Omega.$$

In order to prove the positive definiteness of A_{hom}, let us multiply by $\omega_i(y)$ the cell problem for $\omega_j(y)$ and integrate over the periodicity cell Y yielding

$$\int_Y A(y)\Big(\nabla_y \omega_j(y) + \mathbf{e}_j\Big) \cdot \nabla_y \omega_i(y) \, \mathrm{d}y = 0.$$

Add the above zero to the expression (15) yielding an alternate expression for the elements of the homogenized conductivity

$$[A_{\mathrm{hom}}]_{ij} = \int_Y A(y)\Big(\mathbf{e}_j + \nabla_y \omega_j(y)\Big)\cdot\Big(\mathbf{e}_i + \nabla_y \omega_i(y)\Big)\,\mathrm{d}y \tag{18}$$

for $i, j \in \{1, \ldots, d\}$. Using the above alternate expression for the homogenized conductivity, let us compute the dot product

$$A_{\mathrm{hom}}\xi\cdot\xi = \int_Y A(y)\Big(\xi + \nabla_y \omega_\xi(y)\Big)\cdot\Big(\xi + \nabla_y \omega_\xi(y)\Big)\,\mathrm{d}y$$

where the function $\omega_\xi(y)$ is defined as

$$\omega_\xi(y) = \sum_{k=1}^{d} \omega_k(y)\xi_k.$$

The coercivity assumption (3) on the conductivity coefficient $A(y)$ implies

$$\begin{aligned} A_{\mathrm{hom}}\xi\cdot\xi &\geq \alpha \int_Y \left|\xi + \nabla_y \omega_\xi(y)\right|^2 \,\mathrm{d}y \\ &= \alpha \underbrace{\int_Y \left|\nabla_y \omega_\xi(y)\right|^2 \,\mathrm{d}y}_{\geq 0} + \alpha \int_Y |\xi|^2 \,\mathrm{d}y + 2\alpha \underbrace{\int_Y \xi\cdot\nabla_y \omega_\xi(y)\,\mathrm{d}y}_{=0 \text{ as } \omega_\xi \text{ periodic}} \end{aligned}$$

Hence

$$A_{\mathrm{hom}}\xi\cdot\xi \geq \alpha\,|\xi|^2 .$$

Suppose now that

$$A_{\mathrm{hom}}\xi\cdot\xi = 0$$

which implies that

$$\xi + \nabla_y \omega_\xi(y) = 0 \qquad \forall y \in Y$$

i.e.

$$\omega_\xi(y) = C - \xi\cdot y$$

for some constant C. This says that $\omega_\xi(y)$ is an affine function in the y variable. This is a contradiction to the periodicity of $\omega_\xi(y)$ except if $\xi = 0$. $\square$

Remark 2.6. *If the conductivity coefficient $A(y)$ is assumed to be symmetric, then the expression (18) for the homogenized conductivity implies that A_{hom} is symmetric as well.*

Exercise 2.7. *Using the method of two-scale asymptotic expansions, prove a result similar in flavour to Proposition 2.5 for the following periodic parabolic problem:*

$$\begin{cases} \rho\left(\dfrac{x}{\varepsilon}\right)\partial_t u^\varepsilon(t,x) - \operatorname{div}\left(A\left(\dfrac{x}{\varepsilon}\right)\nabla u^\varepsilon(t,x)\right) = f(t,x) & \text{in } (0,T)\times\Omega, \\ u^\varepsilon(t,x) = 0 & \text{on } (0,T)\times\partial\Omega, \\ u^\varepsilon(0,x) = u^{\mathrm{in}}(x) & \text{in } \Omega, \end{cases} \tag{19}$$

where the coefficients $\rho(\cdot)$ and $A(\cdot)$ are Y-periodic. Assume that the coefficient ρ is bounded from above and away from zero, i.e.,

$$0 < \mathfrak{c}_1 \le \rho(\cdot) \le \mathfrak{c}_2 < \infty.$$

Assume further that the coefficient $A(\cdot)$ satisfies the assumption (3). The bulk source $f(t,x)$ and the initial datum $u^{\rm in}(x)$ are nice functions and are as regular as the computations demand.
Hint: Treat the time variable t as a parameter while performing the asymptotic expansions, i.e., postulate that the solution $u^\varepsilon(t,x)$ to the above initial boundary value problem can be written as

$$u^\varepsilon(t,x) = u_0\left(t,x,\frac{x}{\varepsilon}\right) + \varepsilon\, u_1\left(t,x,\frac{x}{\varepsilon}\right) + \varepsilon^2\, u_2\left(t,x,\frac{x}{\varepsilon}\right) + \dots$$

where the coefficient functions $u_i(t,x,y)$ are assumed to be Y-periodic in the y variable.

Exercise 2.8. *Assume that the coefficients $\rho(\cdot)$, $A(\cdot)$ satisfy the same assumptions as in Exercise 2.7. Assume further that the data $f(t,x)$, $u^{\rm in}(x)$ and $u^{\rm in}_1(x)$ are nice functions and that they are as regular as the computations demand. Using the method of two-scale asymptotic expansions, prove a result similar in flavour to Proposition 2.5 for the following periodic hyperbolic problem:*

$$(20)\qquad \begin{cases} \rho\left(\frac{x}{\varepsilon}\right)\partial^2_{tt}u^\varepsilon(t,x) - \operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon(t,x)\right) = f(t,x) & \text{in } (0,T)\times\Omega,\\ u^\varepsilon(t,x) = 0 & \text{on } (0,T)\times\partial\Omega,\\ u^\varepsilon(0,x) = u^{\rm in}(x) & \text{in } \Omega,\\ \partial_t u^\varepsilon(0,x) = u^{\rm in}_1(x) & \text{in } \Omega. \end{cases}$$

2.2. **One dimensional problem.** In this section, we will present some explicit computations in a one dimensional setting. Let the strictly positive and bounded conductivity coefficient be denoted by $a(y)$ which is 1-periodic. Then one dimensional analogue of the boundary value problem (1) is

$$(21)\qquad \begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}x}\left(a\left(\frac{x}{\varepsilon}\right)\dfrac{\mathrm{d}u^\varepsilon}{\mathrm{d}x}\right) = f(x) & \text{in } (\ell_1,\ell_2),\\ u^\varepsilon(\ell_1) = u^\varepsilon(\ell_2) = 0, \end{cases}$$

for an interval $(\ell_1,\ell_2)\subset\mathbb{R}$.
The one-dimensional analogue of Proposition 2.5 is stated here.

Corollary 2.9. *The cell solution in the one-dimensional setting is given by*

$$(22)\qquad \omega(y) = C - y + a_{\rm hom}\int_0^y \frac{\mathrm{d}\tau}{a(\tau)}$$

for a constant C. Furthermore, the constant a_{hom} is the homogenized conductivity given by

$$a_{\text{hom}} = \left(\int_0^1 \frac{\mathrm{d}y}{a(y)} \right)^{-1} \tag{23}$$

Proof. Let us write down the one-dimensional analogue of the cell problem (10).

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}y} \left(a\,(y) \left(\dfrac{\mathrm{d}\omega}{\mathrm{d}y}(y) + 1 \right) \right) = 0 \quad \text{in } (0,1), \\ \omega(0) = \omega(1). \end{cases}$$

Explicitly integrating the above differential equation yields for some constant C_1,

$$a\,(y) \left(\frac{\mathrm{d}\omega}{\mathrm{d}y}(y) + 1 \right) = C_1 \implies \frac{\mathrm{d}\omega}{\mathrm{d}y}(y) = -1 + \frac{C_1}{a(y)}$$

Integrating again would yield

$$\omega(y) = C - y + \int_0^y \frac{C_1}{a(\tau)}\,\mathrm{d}\tau.$$

The constant of integration C is a dummy variable (for now). It can be chosen by normalization as we look for solutions in the quotient space $\mathrm{H}^1_{\text{per}}(0,1)/\mathbb{R}$. The constant C_1, however, is determined by the periodic boundary condition in the cell problem.

$$C_1 = \left(\int_0^1 \frac{\mathrm{d}y}{a(y)} \right)^{-1}.$$

Now, let us write down the one dimensional analogue of the homogenized conductivity (15).

$$a_{\text{hom}} = \int_0^1 a\,(y) \left(\frac{\mathrm{d}\omega}{\mathrm{d}y}(y) + 1 \right) \mathrm{d}y$$

which is nothing but

$$C_1 \int_0^1 \mathrm{d}y.$$

Hence we have deduced that the one dimensional homogenized conductivity is nothing but the harmonic average over the periodicity cell, i.e.,

$$a_{\text{hom}} = \left(\int_0^1 \frac{\mathrm{d}y}{a(y)} \right)^{-1}$$

This proves the corollary. □

Remark 2.10. *Note that the following strict inequality holds*

$$\left(\int_0^1 \frac{\mathrm{d}y}{a(y)} \right)^{-1} < \int_0^1 a(y)\,\mathrm{d}y$$

for any non-trivial function $a(y)$. By, trivial, we mean functions that are identically constants in which case the harmonic average and the simple average coincide. This observation implies that the homogenized conductivity of a periodic composite is *always less than the associated simple average (arithmetic average), at least in one dimension.*

Remark 2.11. *Let us reconsider the one dimensional illustrative Example 2.1. We chose the periodic conductivity to be*

$$a(y) = \frac{1}{2 + \cos(2\pi y)}.$$

Computing the homogenized conductivity in this case yields

$$a_{\text{hom}} = \left(\int_0^1 \Big(2 + \cos(2\pi y) \Big) \, \mathrm{d}y \right)^{-1} = \frac{1}{2}.$$

Hence the homoegnized equation for the Example 2.1 becomes

$$\begin{cases} -\dfrac{\mathrm{d}^2 u_0}{\mathrm{d}x^2} = 2 & \text{in } (0,1), \\ u_0(0) = u_0(1) = 0 & \end{cases}$$

which integrates explicitly as

$$u_0(x) = x - x^2$$

This is the parabolic profile which we found as a macroscopic approximation to the solution $u^\varepsilon(x)$ in Example 2.1.

Exercise 2.12. *Compute the homogenized conductivity a_{hom} for the following periodic conductivity which is piecewise constant:*

$$a(y) = \begin{cases} 1 & \text{for } y \in \left(0, \frac{1}{3}\right) \\ 2 & \text{for } y \in \left[\frac{1}{3}, \frac{1}{2}\right) \\ 3 & \text{for } y \in \left[\frac{1}{2}, \frac{3}{4}\right) \\ 4 & \text{for } y \in \left[\frac{3}{4}, 1\right) \end{cases}$$

Compute also its arithmetic average.

2.3. **Laminated geometries in higher dimensions.** The one dimensional setting of section 2.2 is pretty special in the sense that given a periodic conductivity $a(y)$, we can straightaway compute the homogenized conductivity a_{hom} without having to go though the cell problem. In this section, we consider the examples of periodic laminated structures in higher dimensions, i.e., scenarios where the conductivity coefficient depends on a single direction (i.e. varying only in one direction). We take

$$A(y) = A(y_1).$$

In this section, we further make the assumption that the above conductivity coefficient is isotropic, i.e., there exists a scalar function $a(y_1)$ such that

$$A(y) = a(y_1)\,\mathrm{Id}. \tag{24}$$

We will prove the following interesting result on the above chosen laminated structures.

Proposition 2.13. *The homogenized conductivity associated with the isotropic conductivity coefficient* (24) *is the following diagonal matrix*

$$[A_{\mathrm{hom}}]_{ii} = \begin{cases} \left(\displaystyle\int_0^1 \frac{\mathrm{d}y_1}{a(y_1)}\right)^{-1} & \text{for } i = 1, \\ \displaystyle\int_0^1 a(y_1)\,\mathrm{d}y_1 & \text{for } i \neq 1. \end{cases}$$

Remark 2.14. *It is to be noted that even though we chose an isotropic periodic conductivity coefficient to begin with, the homogenization procedure destroys the isotropy as can be noticed in Proposition 2.13.*

Proof of Proposition 2.13. Let us rewrite the cell problem (10) with the chosen isotropic conductivity (24) as

$$\begin{cases} -\mathrm{div}_y\Big(a\,(y_1)\,\nabla_y\omega_i(y)\Big) = \mathrm{div}_y\Big(a\,(y_1)\,\mathbf{e}_i\Big) & \text{for } y \in Y, \\ y \mapsto \omega_i(y) & \text{is } Y\text{-periodic} \end{cases}$$

for each $i \in \{1, \ldots, d\}$. In particular, for $i = 1$, we have

$$\begin{cases} -\mathrm{div}_y\Big(a\,(y_1)\,\nabla_y\omega_1(y)\Big) = \dfrac{\partial}{\partial y_1}a\,(y_1) & \text{for } y \in Y, \\ y \mapsto \omega_1(y) & \text{is } Y\text{-periodic} \end{cases}$$

Observe that the solution $\omega_1(y)$ depends only on the y_1 direction, i.e.,

$$\omega_1(y) = \chi(y_1)$$

where $\chi(y_1)$ solves the one dimensional cell problem

$$\begin{cases} -\dfrac{\mathrm{d}}{\mathrm{d}y_1}\left(a\,(y_1)\left(\dfrac{\mathrm{d}\chi}{\mathrm{d}y_1}(y_1) + 1\right)\right) = 0 & \text{in } (0,1), \\ \chi(0) = \chi(1). \end{cases}$$

Next, for $i \neq 1$, we have

$$\begin{cases} -\mathrm{div}_y\Big(a\,(y_1)\,\nabla_y\omega_i(y)\Big) = \dfrac{\partial}{\partial y_i}a\,(y_1) = 0 & \text{for } y \in Y, \\ y \mapsto \omega_i(y) & \text{is } Y\text{-periodic} \end{cases}$$

Observe that the solutions to the above $(d-1)$ cell problems are all in the zero equivalence class (see Lemma 2.3). Hence we have

$$\omega_i(y) \equiv 0 \qquad \text{for } i \neq 1.$$

The above information on the cell solutions help us get some insight into the homogenized coefficient which in this case is a diagonal matrix. Let us compute $[A_{\mathrm{hom}}]_{11}$ using the formula (15).

$$[A_{\mathrm{hom}}]_{11} = \int_Y a(y_1)\left(\frac{\mathrm{d}\chi}{\mathrm{d}y_1}(y_1) + 1\right)\mathrm{d}y_1 = \left(\int_0^1 \frac{\mathrm{d}y_1}{a(y_1)}\right)^{-1}$$

where the last equality follows from Corollary 2.9. Next, let us compute $[A_{\mathrm{hom}}]_{ii}$ for $i \neq 1$ using the formula (15).

$$[A_{\mathrm{hom}}]_{ii} = \int_Y a(y_1)\Big(\nabla_y \omega_i(y) + \mathbf{e}_i\Big) \cdot \mathbf{e}_i \,\mathrm{d}y = \int_0^1 a(y_1)\,\mathrm{d}y_1$$

because the cell solutions $\omega_i(y) \equiv 0$ for $i \neq 1$. □

Exercise 2.15. *Using Proposition 2.13, compute the homogenized conductivity A_{hom} for the following laminar periodic conductivity:*

$$A(y) = \begin{cases} d_1 & \textit{for } y_1 \in \big(0, \frac{1}{2}\big) \\ d_2 & \textit{for } y_1 \in \big[\frac{1}{2}, 1\big) \end{cases}$$

with strictly positive constants d_1, d_2.

Exercise 2.16. *Mimicking the proof of Proposition 2.13, obtain analytic expressions for the homogenized conductivity A_{hom} in two dimensions when the periodic conductivity coefficient is laminar, i.e., $A(y) = A(y_1)$, but not necessarily isotropic as was the case in (24).*