Using a finite element discretization, the static equilibrium of the unit cell under finite deformation is governed by the following equation:

$$\boldsymbol{r} = \boldsymbol{f}^{ext} - \boldsymbol{f}^{int}(\boldsymbol{u}) = 0 \tag{4}$$

where $\boldsymbol{r}$ is the residual nodal force vector, $\boldsymbol{f}^{ext}$ is the external nodal force vector, $\boldsymbol{f}^{int}(\boldsymbol{u})$ is the internal nodal force vector that is dependent on the nodal displacement vector, $\boldsymbol{u}$. The internal nodal force vector is defined by

$$\boldsymbol{f}^{int}(\boldsymbol{u}) = \frac{\partial(\int_v W(\boldsymbol{u})\, dv)}{\partial \boldsymbol{u}} \tag{5}$$

where $W(\boldsymbol{u})$ is the stored elastic energy. The detailed calculation of $\boldsymbol{f}^{int}(\boldsymbol{u})$ can be found e.g. in Zienkiewicz and Taylor (2005).

The static equilibrium, Eq. (4), is solved iteratively using the Newton–Raphson method, with the incremental equation given as

$$\boldsymbol{K}_t \Delta \boldsymbol{u} = \boldsymbol{r} \tag{6}$$

where the nodal displacement vector is updated by $\boldsymbol{u} = \boldsymbol{u} + \alpha \Delta \boldsymbol{u}$ with $\alpha = 0.5$ to stabilize the Newton–Raphson iterations and $\boldsymbol{K}_t = -\partial \boldsymbol{r} / \partial \boldsymbol{u}$ is the tangent stiffness matrix. The formulation of the tangent stiffness matrix is described in standard books on non-linear finite element theory (Zienkiewicz and Taylor, 2005) and is not presented here.

Problema generico : $f(u) = 0$

Si sceglie u^0

Newton : $J(u^k)\, \delta u = -f(u^k)$, $u^{k+1} = u^k + \delta u$

↳ Jacobiano di f

Nostro problema: $r = \text{residuo} = f^{ext} - f^{int} = 0$

- $f^{ext} = 0$
- $f^{int} = \frac{\partial}{\partial u} \int_Y \Psi = \int_Y \frac{\partial \Psi}{\partial u}$

Il problema sarebbe $\frac{dr}{du}(u^k)\, \delta u = -r(u^k) \iff \boxed{-\int_Y \frac{\partial^2 \Psi}{\partial u^2} \delta u = \int_Y \frac{\partial \Psi}{\partial u}}$ (6 di Sigmund)

Il problema primale, risolto con Newton, diventa in weak form:

$$-\frac{\partial}{\partial u}\left[\int_Y \frac{\partial \Psi(u^k)}{\partial u} \delta u \, dY\right] \varphi = \int_Y \frac{\partial \Psi(u^k)}{\partial u} \varphi \, dY \quad \forall \varphi$$

la derivata seconda l'abbiamo già calcolata perché ci serviva per il problema duale

$$\begin{aligned}
\frac{\partial}{\partial \vec{u}}\left(\int_Y \frac{\partial \vec{\Psi}\left(\hat{\vec{u}}\right)}{\partial \vec{u}} \hat{\vec{\lambda}}\, dY\right)\vec{\psi} = &\int_Y \frac{\partial}{\partial u}\left[\lambda_L (J-1) J \left(C_{11}^{-1}\frac{\partial E_{11}}{\partial \vec{u}}\hat{\vec{\lambda}} + C_{12}^{-1}\frac{\partial E_{12}}{\partial \vec{u}}\hat{\vec{\lambda}} + C_{21}^{-1}\frac{\partial E_{21}}{\partial \vec{u}}\hat{\vec{\lambda}} + C_{22}^{-1}\frac{\partial E_{22}}{\partial \vec{u}}\hat{\vec{\lambda}}\right)\right. \\
&\left. + 2\mu_L \left(E_{11}\frac{\partial E_{11}}{\partial \vec{u}}\hat{\vec{\lambda}} + E_{12}\frac{\partial E_{12}}{\partial \vec{u}}\hat{\vec{\lambda}} + E_{21}\frac{\partial E_{21}}{\partial \vec{u}}\hat{\vec{\lambda}} + E_{22}\frac{\partial E_{22}}{\partial \vec{u}}\hat{\vec{\lambda}}\right)\right]\vec{\psi} dY \\
= &\int_Y \left\{\left[\lambda_L \frac{\partial (J-1) J}{\partial \vec{u}}\right]\left[C_{11}^{-1}\frac{\partial E_{11}}{\partial \vec{u}}\hat{\vec{\lambda}} + C_{12}^{-1}\frac{\partial E_{12}}{\partial \vec{u}}\hat{\vec{\lambda}} + C_{21}^{-1}\frac{\partial E_{21}}{\partial \vec{u}}\hat{\vec{\lambda}} + C_{22}^{-1}\frac{\partial E_{22}}{\partial \vec{u}}\hat{\vec{\lambda}}\right]\right. \\
&+ \lambda_L (J-1) J \frac{\partial}{\partial \vec{u}}\left[C_{11}^{-1}\frac{\partial E_{11}}{\partial \vec{u}}\hat{\vec{\lambda}} + C_{12}^{-1}\frac{\partial E_{12}}{\partial \vec{u}}\hat{\vec{\lambda}} + C_{21}^{-1}\frac{\partial E_{21}}{\partial \vec{u}}\hat{\vec{\lambda}} + C_{22}^{-1}\frac{\partial E_{22}}{\partial \vec{u}}\hat{\vec{\lambda}}\right] \\
&\left. + 2\mu_L \frac{\partial}{\partial \vec{u}}\left[E_{11}\frac{\partial E_{11}}{\partial u}\lambda + E_{12}\frac{\partial E_{12}}{\partial \vec{u}}\hat{\vec{\lambda}} + E_{21}\frac{\partial E_{21}}{\partial \vec{u}} + E_{22}\frac{\partial E_{22}}{\partial \vec{u}}\hat{\vec{\lambda}}\right]\right\}\vec{\psi} dY \\
= &\int_Y \lambda_L \frac{\partial\left[J^2 - J\right]}{\partial J}\frac{\partial J}{\partial \vec{u}}\left[C_{11}^{-1}\frac{\partial E_{11}}{\partial \vec{u}}\hat{\vec{\lambda}} + C_{12}^{-1}\frac{\partial E_{12}}{\partial \vec{u}}\hat{\vec{\lambda}} + C_{21}^{-1}\frac{\partial E_{21}}{\partial \vec{u}}\hat{\vec{\lambda}} + C_{22}^{-1}\frac{\partial E_{22}}{\partial \vec{u}}\hat{\vec{\lambda}}\right]\vec{\psi} \\
&+ \lambda_L (J-1) J \left[\frac{\partial C_{11}^{-1}}{\partial \vec{u}}\frac{\partial E_{11}}{\partial \vec{u}}\hat{\vec{\lambda}}\vec{\psi} + C_{11}^{-1}\frac{\partial^2 E_{11}}{\partial \vec{u}^2}\hat{\vec{\lambda}}\vec{\psi} + \frac{\partial C_{12}^{-1}}{\partial \vec{u}}\frac{\partial E_{12}}{\partial \vec{u}}\hat{\vec{\lambda}}\vec{\psi} + C_{12}^{-1}\frac{\partial^2 E_{12}}{\partial \vec{u}^2}\hat{\vec{\lambda}}\vec{\psi}\right. \\
&\left. + \frac{\partial C_{21}^{-1}}{\partial \vec{u}}\frac{\partial E_{21}}{\partial \vec{u}}\hat{\vec{\lambda}}\vec{\psi} + C_{21}^{-1}\frac{\partial^2 E_{21}}{\partial \vec{u}^2}\hat{\vec{\lambda}}\vec{\psi} + \frac{\partial C_{22}^{-1}}{\partial \vec{u}}\frac{\partial E_{22}}{\partial \vec{u}}\hat{\vec{\lambda}}\vec{\psi} + C_{22}^{-1}\frac{\partial^2 E_{22}}{\partial \vec{u}^2}\hat{\vec{\lambda}}\vec{\psi}\right] \\
&+ 2\mu_L \left[\frac{\partial E_{11}}{\partial \vec{u}}\vec{\psi}\frac{\partial E_{11}}{\partial \vec{u}}\hat{\vec{\lambda}} + E_{11}\frac{\partial^2 E_{11}}{\partial \vec{u}^2}\hat{\vec{\lambda}}\vec{\psi} + \frac{\partial E_{11}}{\partial \vec{u}}\vec{\psi}\frac{\partial E_{12}}{\partial \vec{u}}\hat{\vec{\lambda}} + E_{12}\frac{\partial^2 E_{12}}{\partial \vec{u}^2}\hat{\vec{\lambda}}\vec{\psi} +\right. \\
&\left. \frac{\partial E_{21}}{\partial \vec{u}}\vec{\psi}\frac{\partial E_{21}}{\partial \vec{u}}\hat{\vec{\lambda}} + E_{21}\frac{\partial^2 E_{21}}{\partial \vec{u}^2}\hat{\vec{\lambda}}\vec{\psi} + \frac{\partial E_{2}}{\partial \vec{u}}\vec{\psi}\frac{\partial E_{22}}{\partial \vec{u}}\hat{\vec{\lambda}} + E_{22}\frac{\partial^2 E_{22}}{\partial \vec{u}^2}\hat{\vec{\lambda}}\vec{\psi}\right] dY \\
= &\int_Y \lambda_L (2J-1)\frac{\partial}{\partial \vec{u}}\left[F_{11}F_{22} - F_{12}F_{21}\right]\vec{\psi}\left[C_{11}^{-1}\frac{\partial E_{11}}{\partial \vec{u}}\hat{\vec{\lambda}} + C_{12}^{-1}\frac{\partial E_{12}}{\partial \vec{u}}\hat{\vec{\lambda}} + C_{21}^{-1}\frac{\partial E_{21}}{\partial \vec{u}}\hat{\vec{\lambda}}\right. \\
&\left. + C_{22}^{-1}\frac{\partial E_{22}}{\partial \vec{u}}\hat{\vec{\lambda}}\right] + \ldots dY = (rhsformJ\ldots)\ \forall\ \psi \in V
\end{aligned}$$

where in detail:

$$\begin{aligned}
\frac{\partial}{\partial \vec{u}}\left[F_{11}F_{22} - F_{12}F_{21}\right]\vec{\psi} = &\left[\frac{\partial F_{11}}{\partial \vec{u}}F_{22} + \frac{\partial F_{22}}{\partial \vec{u}}F_{11} - \frac{\partial F_{12}}{\partial \vec{u}}F_{12} - \frac{\partial F_{21}}{\partial \vec{u}}F_{12}\right]\vec{\psi} = \\
&\frac{\partial \psi_1}{\partial x_1}F_{22} + \frac{\partial \psi_2}{\partial x_2}F_{11} - \frac{\partial \psi_1}{\partial x_2}F_{21} - \frac{\partial \psi_2}{\partial x_1}F_{12}
\end{aligned}$$

$\frac{\partial C_{ij}^{-1}}{\partial \vec{u}}\vec{\psi}\frac{\partial E_{ij}}{\partial \vec{u}}\hat{\vec{\lambda}} = \sum_{k,l}\frac{\partial C_{ij}^{-1}}{\partial E_{kl}}\frac{\partial E_{kl}}{\partial \vec{u}}\vec{\psi}\frac{\partial E_{ij}}{\partial \vec{u}}\hat{\vec{\lambda}} = \sum_{k,l} -D_{ijkl}\frac{\partial E_{kl}}{\partial \vec{u}}\vec{\psi}\frac{\partial E_{ij}}{\partial \vec{u}}\hat{\vec{\lambda}}$

$D_{ijkl} = -\frac{\partial C_{ij}^{-1}}{\partial E_{kl}} = C_{ik}^{-1}C_{jl}^{-1} + C_{il}^{-1}C_{jk}^{-1}$ from formula (14) in A.Klarbring, N.Strömberg paper (references ...)