# Anisotropic adaptation via a Zienkiewicz-Zhu error estimator for 2D elliptic problems

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**Abstract** We propose a Zienkiewicz-Zhu a posteriori error estimator in 2D, which shares the computational advantages typical of the original estimator. The novelty is the inclusion of the geometrical features of the computational mesh, useful for an anisotropic mesh adaptation. The adapted triangulations are shown numerically to be quasi-optimal with respect to the error-vs-number of elements behaviour.

## 1 Motivations

Among the various a posteriori error estimation techniques available in the literature, one of the most popular in practice is the one proposed by Zienkiewicz and Zhu ([11, 12]). The idea behind this estimator is quite simple: for example, consider the finite element approximation $u_h$ to the solution $u$ of an advection-diffusion-reaction (ADR) equation. Since the gradient $\nabla u_h$ is less accurate than the solution, we recover an improved gradient, say $\nabla^* u_h$, by suitably fitting $\nabla u_h$ over some patches of elements. The discrepancy $\|\nabla^* u_h - \nabla u_h\|_{L^2(\Omega)}$ then identifies an estimator for the $H^1(\Omega)$-seminorm of the discretization error $u - u_h$.

The popularity of this methodology can be attributed to various factors: the method is independent of the problem, of the governing equations and of most details of the finite element formulation (except for the finite element space), it is cheap to compute and easy to implement, and works very well in practice.

On the other hand, ADR problems often exhibit strong directional features (e.g., internal or boundary layers). In these cases the effectiveness of the finite element approximation benefits from a suitable anisotropic computational mesh, fitting size, shape and orientation of its triangles to the directional features of the solution at hand ([9, 1, 5]).

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In this paper we propose some gradient recovery techniques suited to define an anisotropic counterpart of the Zienkiewicz-Zhu estimator. The novelty is the inclusion of the geometrical information of the mesh triangles, maintaining the above good properties of the standard Zienkiewicz-Zhu estimator. Despite the somewhat heuristic nature of the proposed estimator, the overall anisotropic adaptation procedure turns out to be effective in practice. The adapted meshes, built through a metric-based optimisation algorithm, are shown numerically to be quasi-optimal with respect to the error-vs-number of elements behaviour.

## 2 Recovery procedures

According to a Zienkiewicz-Zhu approach, we distinguish between two steps: first we furnish a procedure for obtaining an approximate recovered gradient; second, we employ this recovered gradient for a posteriori error purposes.
To fix ideas, we consider the standard ADR problem completed with homogeneous Dirichlet boundary conditions, i.e., find $u \in V$, such that

$$\int_\Omega \mu \nabla u \cdot \nabla v \, d\mathbf{x} + \int_\Omega \mathbf{b} \cdot \nabla u \, v \, d\mathbf{x} + \int_\Omega \gamma u \, v \, d\mathbf{x} = \int_\Omega f v \, d\mathbf{x} \quad \forall v \in V, \tag{1}$$

with $\Omega$ a polygonal domain in $\mathbb{R}^2$, $\mu > 0$, $\mathbf{b} \in [W^{1,\infty}(\Omega)]^2$, $\gamma \in L^\infty(\Omega)$, and where $V = H_0^1(\Omega)$, standard notation being adopted for the Sobolev spaces and their norms. Proper assumptions are enforced to guarantee the well-posedness of (1).
Let $\mathscr{T}_h = \{K\}$ be a conforming partition of $\Omega$ consisting of triangles and $u_h$ be the Galerkin affine finite element approximation to (1), possibly involving stabilization. We now provide a family of recovery procedures to improve the discrete gradient $\nabla u_h$, using information only related to $u_h$. Several approaches are available in the literature for this purpose (see, e.g., [11, 10, 8]). We propose here a recovered gradient, denoted by $P^r_{\Delta_K}(\nabla u_h)$, which has degree $r$ over the patch $\Delta_K = \{T \in \mathscr{T}_h : T \cap K \neq \emptyset\}$. We seek $P^r_{\Delta_K}(\nabla u_h) \in [\mathbb{P}_r]^2$ such that

$$\int_{\Delta_K} (\nabla u_h - P^r_{\Delta_K}(\nabla u_h)) \cdot \mathbf{w} \, d\mathbf{x} = 0 \quad \forall \mathbf{w} \in [\mathbb{P}_r]^2, \tag{2}$$

with $\mathbb{P}_r = \mathrm{span}\{x_1^i x_2^j \,|\, i + j \leq r\}$. The recovered gradient $P^r_{\Delta_K}(\nabla u_h)$ is strictly associated with $K$, and not to the elements comprising $\Delta_K$ (i.e., for any $T \in \Delta_K$, with $T \neq K$, $P^r_{\Delta_T}(\nabla u_h)$ is, in general, different from $P^r_{\Delta_K}(\nabla u_h)$). In the particular case $r = 0$, we can write out the formula for the recovered gradient, given by

$$P^0_{\Delta_K}(\nabla u_h) = \frac{1}{|\Delta_K|} \sum_{T \in \Delta_K} |T| \, \nabla u_h|_T ,$$

namely, we compute the area-weighted average over the patch $\Delta_K$ of the gradients of the discrete solution.

# 3 The anisotropic estimator

To devise the estimator proposed in this work, we embed the recovery procedures above in a convenient anisotropic setting. This leads to a Zienkiewicz-Zhu-like estimator, automatically including the anisotropic information of the mesh elements. The same potentiality is not so evident in the case of the standard Zienkiewicz-Zhu estimator ([12]).

## 3.1 Anisotropic source

We employ the anisotropic setting in [4]. The size, shape and orientation of each element $K$ of $\mathscr{T}_h$ are characterized by the affine map $T_K : \widehat{K} \to K$, where $\widehat{K}$ is the equilateral reference triangle centred at the origin, with coordinates $(-\sqrt{3}/2, -1/2)$, $(\sqrt{3}/2, -1/2)$, $(0, 1)$ and edge length $\sqrt{3}$. It holds $\mathbf{x} = T_K(\widehat{\mathbf{x}}) = M_K\,\widehat{\mathbf{x}} + \mathbf{t}_K$, with $M_K \in \mathbb{R}^{2\times 2}$ the Jacobian and $\mathbf{t}_K \in \mathbb{R}^2$ the shift vector. Matrix $M_K$ is factorized as $M_K = B_K Z_K$ via the polar decomposition, where $B_K \in \mathbb{R}^{2\times 2}$ is symmetric positive definite, and $Z_K \in \mathbb{R}^{2\times 2}$ is orthogonal. Then $B_K$ is spectrally decomposed as $B_K = R_K^T \Lambda_K R_K$, with $R_K^T = [\mathbf{r}_{1,K}, \mathbf{r}_{2,K}]$ and $\Lambda_K = \mathrm{diag}(\lambda_{1,K}, \lambda_{2,K})$ the eigenvector and eigenvalue matrix, respectively.
Through $T_K$ the unit circle circumscribing $\widehat{K}$ is changed into an ellipse circumscribing $K$: the unit vectors $\{\mathbf{r}_{i,K}\}$ define the corresponding principal directions, whereas the quantities $\{\lambda_{i,K}\}$ measure the length of the ellipse semi-axes. Without loss of generality, we assume $\lambda_{1,K} \geq \lambda_{2,K} > 0$ so that the stretching factor, $s_K = \lambda_{1,K}/\lambda_{2,K}$, satisfies $s_K \geq 1$, for any $K \in \mathscr{T}_h$, equality holding when $K$ is equilateral.

The estimator proposed in Sect. 3.2 is inspired by an anisotropic interpolation error estimate derived in this setting ([4]). In particular, let $I_h^1$ be the Clément interpolant of degree 1 for functions $v \in H^1(\Omega)$ .

**Proposition 1.** *Let $v \in H^1(\Omega)$. Then, if $\#\Delta_K \leq \mathscr{D}$ and $\mathrm{diam}(\widehat{\Delta}_K) \leq \delta$, for any $K \in \mathscr{T}_h$, there exists a constant $C = C(\mathscr{D}, \delta)$, such that*

$$\|v - I_h^1(v)\|_{L^2(K)} \leq C\Big(\sum_{i=1}^{2} \lambda_{i,K}^2 \,(\mathbf{r}_{i,K}^T G_{\Delta_K}(\nabla v)\,\mathbf{r}_{i,K})\Big)^{1/2}, \tag{3}$$

*with $G_K(\cdot)$ the symmetric semidefinite positive matrix with entries*

$$[G_{\Delta_K}(\mathbf{w})]_{i,j} = \sum_{T\in\Delta_K} \int_T w_i\, w_j\, d\mathbf{x}, \quad \textit{with } i, j = 1, 2, \tag{4}$$

*for any vector-valued function $\mathbf{w} = (w_1, w_2)^T \in [L^2(\Omega)]^2$, $\#\Delta_K$ the cardinality of the patch, $\mathrm{diam}(\widehat{\Delta}_K)$ the diameter of $\widehat{\Delta}_K = T_K^{-1}(\Delta_K)$, the pullback of $\Delta_K$ via the map $T_K$.*

*Remark 1.* The hypotheses of Proposition 1 constrain the variation of $\{\mathbf{r}_{i,K}\}$ and $\{\lambda_{i,K}\}$ over $\Delta_K$ but do not limit the anisotropy of $K$.

### 3.2 The estimator

Driven by Proposition 1 we devise the anisotropic a posteriori error estimator. Let $\mathbf{E}^r_{\Delta_K} = P^r_{\Delta_K}(\nabla u_h) - \nabla u_h|_{\Delta_K}$ be the approximation for the error on the gradient, over $\Delta_K$. We define the anisotropic Zienkiewicz-Zhu local estimator for the $H^1$-seminorm of the discretization error as

$$\left[\eta^r_{K,\mathrm{A}}\right]^2 = \frac{1}{\lambda_{1,K}\lambda_{2,K}} \sum_{i=1}^{2} \lambda^2_{i,K} \left(\mathbf{r}^T_{i,K}\, G_{\Delta_K}(\mathbf{E}^r_{\Delta_K})\mathbf{r}_{i,K}\right), \tag{5}$$

where the matrix $G_{\Delta_K}(\cdot)$ is defined as in (4). Then the corresponding global error estimator is given by

$$\left[\eta^r_{\mathrm{A}}\right]^2 = \sum_{K\in\mathscr{T}_h} \left[\eta^r_{K,\mathrm{A}}\right]^2. \tag{6}$$

The estimator (5)-(6) is essentially heuristic. The terms summed on the right-hand side of (5) are suggested by (3) with $v = u - u_h$, after substituting the partial derivatives of $u$ with the corresponding components of $P^r_{\Delta_K}(\nabla u_h)$. However some rationale can be provided. The scaling factor $\lambda_{1,K}\lambda_{2,K}$ guarantees a *consistency* with respect to the isotropic case, i.e., when $\lambda_{1,K} = \lambda_{2,K}$, (5) turns into an isotropic Zienkiewicz-Zhu-like estimator based on the patchwise recovered gradient (2), that is

$$\left[\eta^r_{K,\mathrm{I}}\right]^2 = \int_{\Delta_K} |\mathbf{E}^r_{\Delta_K}|^2\, d\mathbf{x} \quad \text{and} \quad \left[\eta^r_{\mathrm{I}}\right]^2 = \sum_{K\in\mathscr{T}_h} \left[\eta^r_{K,\mathrm{I}}\right]^2.$$

Moreover a sort of *equivalence* between $\eta^r_{K,\mathrm{A}}$ and $|u-u_h|_{H^1(\Delta_K)}$ can be proved. In more detail, given a function $v \in H^1(\Omega)$, let $\widehat{v} = v \circ T_K$ be the associated pullback. Virtually, we would like to choose $v = u - u_h$. Then, we have

$$\int_{\widehat{\Delta}_K} |\widehat{\nabla}\,\widehat{v}|^2\, d\,\widehat{\mathbf{x}} = \frac{1}{\lambda_{1,K}\lambda_{2,K}} \sum_{i=1}^{2} \lambda^2_{i,K} \left(\mathbf{r}^T_{i,K} G_{\Delta_K}(\nabla v)\,\mathbf{r}_{i,K}\right),$$

$$s_K^{-1}\, |v|^2_{H^1(\Delta_K)} \le \frac{1}{\lambda_{1,K}\lambda_{2,K}} \sum_{i=1}^{2} \lambda^2_{i,K} \left(\mathbf{r}^T_{i,K} G_{\Delta_K}(\nabla v)\,\mathbf{r}_{i,K}\right) \le s_K\, |v|^2_{H^1(\Delta_K)},$$

where the middle term mimics estimator (5), on replacing $\nabla v$ with $\mathbf{E}^r_{\Delta_K}$.

*The patch test*

We aim to check the consistency of the recovery procedure by computing the local effectivity index $\mathrm{E.I.}^r_{K,\mathrm{A}} = \eta^r_{K,\mathrm{A}}/|u-u_h|_{H^1(K)}$, for $r = 0,1$. To avoid a bias effect due to the grid, we consider the case when $u$ is isotropic and the regular patch $\Delta_K$ consists of 13 equilateral triangles, each of area $3\sqrt{3}/4$, with pivot element $\widehat{K}$. In particular let $u = ax_1^2 + 2bx_1x_2 + cx_2^2$, with $a,b,c \in \mathbb{R}$, picked such that the Hessian, $H = [a\ b; b\ c]$, has eigenvalues with the same modulus. This happens only when *i)* $a = c$ and $b = 0$ or when *ii)* $a = -c$ and $b$ is arbitrary. As typical in a patch test, let $u_h$ coincide with the Lagrange affine interpolant of $u$. It turns out that

$|u-u_h|_{H^1(K)} = |a|\sqrt{|K|}$, $\|P^0_{\Delta_K}(\nabla u_h) - \nabla u_h\|_{L^2(\Delta_K)} = |a|\sqrt{132\,|K|}$, i.e., $\mathrm{E.I.}^0_{K,\mathrm{A}} = \sqrt{132} \simeq 11.5$ in the case *i)*; the case *ii)* leads to $|u-u_h|_{H^1(K)} = \sqrt{2\,|K|\,(a^2+b^2)}$, $\|P^0_{\Delta_K}(\nabla u_h) - \nabla u_h\|_{L^2(\Delta_K)} = \sqrt{1884\,|K|\,(a^2+b^2)/13}$, i.e., $\mathrm{E.I.}^0_{K,\mathrm{A}} = \sqrt{1884/26} \simeq 8.51$. Analogously, for the case $r=1$, we obtain $\mathrm{E.I.}^1_{K,\mathrm{A}} \simeq 3.44$ in the case *i)* and $\mathrm{E.I.}^1_{K,\mathrm{A}} \simeq 3.52$ in the case *ii)*.
It can be checked that the same values can be obtained after applying either roto-translations or homotheties to $\Delta_K$.
Although this isotropic context may seem favorable, we expect a similar behaviour also in the anisotropic case, provided that the mesh is adapted to the solution.

Estimator (5)-(6) is problem-free, i.e., it can be applied to more general problems, such as elasticity or Navier-Stokes equations. In such a case one could replace, e.g., the gradient with the stress (rate) tensor ([11]). Alternately, the adaptation can be driven by the gradient of a scalar variable representative of the problem, like the pressure or the speed for the Navier-Stokes equations.
The estimator corresponding to $r=0$ is extended to the 3D case in [2]. Here an adaptation driven by a scalar quantity (speed for the Navier-Stokes equations and density for a multimaterial application) is also assessed.

## 4 The adaptive procedure

We employ a metric-based adaptive procedure driven by estimator $\eta^r_{\mathrm{A}}$. In particular, for a fixed accuracy on the numerical solution, we look for the mesh with the least number of elements. The tensor field $\widetilde{M} : \Omega \to \mathrm{I\!R}^2$, is the actual unknown. According to a predictive procedure, at each iteration, $j$, of the adaptive process, we deal with: *i)* the actual mesh $\mathscr{T}_h^{(j)}$, where problem (1) is approximated; *ii)* the new metric $\widetilde{M}^{(j+1)}$ piecewise constant on $\mathscr{T}_h^{(j)}$, predicted elementwise through a suitable local optimization procedure; *iii)* the new mesh $\mathscr{T}_h^{(j+1)}$ guaranteeing that all the edges are unit length with respect to $\widetilde{M}^{(j+1)}$ ([7]).
We focus on step *ii)*, which is at the heart of the whole adaptive procedure. We minimize $\left[\eta^r_{K,\mathrm{A}}\right]^2$ in (5) with respect to stretching and orientation, and then, via an equidistribution criterion, we compute the actual values of $\lambda_{1,K}$ and $\lambda_{2,K}$. For this purpose we first rewrite the estimator as

$$\begin{aligned}\left[\eta^r_{K,\mathrm{A}}\right]^2 &= s_K\,(\mathbf{r}^T_{1,K}\,G_{\Delta_K}(\mathbf{E}^r_{\Delta_K})\,\mathbf{r}_{1,K}) + s_K^{-1}\,(\mathbf{r}^T_{2,K}\,G_{\Delta_K}(\mathbf{E}^r_{\Delta_K})\,\mathbf{r}_{2,K})\\ &= \lambda_{1,K}\,\lambda_{2,K}\,|\widehat{\Delta}_K|\left[s_K\,(\mathbf{r}^T_{1,K}\,\widehat{G}_{\Delta_K}(\mathbf{E}^r_{\Delta_K})\,\mathbf{r}_{1,K}) + s_K^{-1}\,(\mathbf{r}^T_{2,K}\,\widehat{G}_{\Delta_K}(\mathbf{E}^r_{\Delta_K})\,\mathbf{r}_{2,K})\right], \qquad (7)\end{aligned}$$

where $\widehat{G}_{\Delta_K}(\cdot)$ is the scaled matrix $G_{\Delta_K}(\cdot)/|\Delta_K|$, and $|\Delta_K| = \lambda_{1,K}\lambda_{2,K}\,|\widehat{\Delta}_K|$. The idea is that we single out the area-dependent information (the multiplicative term) from the quantity in brackets, depending on orientation and stretching. Then we minimize this last term with respect to $s_K$ and $\{\mathbf{r}_{i,K}\}$, as stated by the following result.

**Proposition 2.** *Let*

$$J(s_K, \{\mathbf{r}_{i,K}\}_{i=1,2}) = s_K\,(\mathbf{r}_{1,K}^T \widehat{G}_{\Delta_K}(\mathbf{E}_{\Delta_K}^r)\mathbf{r}_{1,K}) + s_K^{-1}\,(\mathbf{r}_{2,K}^T \widehat{G}_{\Delta_K}(\mathbf{E}_{\Delta_K}^r)\mathbf{r}_{2,K}), \tag{8}$$

*and let* $\{g_i, \mathbf{g}_i\}_{i=1,2}$ *be the eigen-pairs associated with* $\widehat{G}_{\Delta_K}(\mathbf{E}_{\Delta_K}^r)$*, where it is understood* $g_1 \geq g_2 > 0$ *and* $\{\mathbf{g}_i\}_{i=1,2}$ *are orthonormal. Then* $J(\cdot)$ *is minimized when*

$$s_K = \sqrt{g_1/g_2}, \quad \mathbf{r}_{1,K} = \mathbf{g}_2, \quad \mathbf{r}_{2,K} = \mathbf{g}_1. \tag{9}$$

*Proof.* The result follows from Proposition 14 in [3].

Notice that the optimal values in (9) equalize the two terms in (8), i.e., $s_K\, g_2 = s_K^{-1}\, g_1 = \sqrt{g_1\, g_2}$. This implies that the minimum of $J(\cdot)$ does not depend on $s_K$. To construct $\widetilde{M}^{(j+1)}$, we just have to compute $\{\lambda_{i,K}\}_{i=1,2}$. For this purpose we employ the equidistribution criterion, according to which $\left[\eta_{K,\mathrm{A}}^r\right]^2 = \tau^2/\#\mathscr{T}_h^{(j)}$, where $\tau$ is the fixed accuracy and $\#\mathscr{T}_h^{(j)}$ is the cardinality of the background mesh. Thanks to Proposition 2, we obtain $\lambda_{1,K}\,\lambda_{2,K} = \tau^2/(2\#\mathscr{T}_h^{(j)}|\widehat{\Delta}_K|\sqrt{g_1 g_2})$. Since $s_K = \lambda_{1,K}/\lambda_{2,K}$, we have

$$\lambda_{1,K} = g_2^{-1/2}\left(\frac{\tau^2}{2\#\mathscr{T}_h^{(j)}|\widehat{\Delta}_K|}\right)^{1/2}, \quad \lambda_{2,K} = g_1^{-1/2}\left(\frac{\tau^2}{2\#\mathscr{T}_h^{(j)}|\widehat{\Delta}_K|}\right)^{1/2}. \tag{10}$$

The predicted metric $\widetilde{M}^{(j+1)}$ is formed, elementwise, by $\widetilde{M}_K^{(j+1)} = \widetilde{M}^{(j+1)}|_K = R_K^T \Lambda_K^{-2} R_K$, with $R_K^T = [\mathbf{r}_{1,K}, \mathbf{r}_{2,K}]$ and $\Lambda_K = \mathrm{diag}(\lambda_{1,K}, \lambda_{2,K})$, where $\{\mathbf{r}_{i,K}\}_{i=1,2}$ and $\{\lambda_{i,K}\}_{i=1,2}$ are provided by (9) and (10), respectively.
Now, for task *iii)*, we employ the function `adaptmesh` in [6]. Since it takes as input a nodewise representation of $\widetilde{M}^{(j+1)}$, we have to average the elementwise information. The nodewise metric is thus $\widetilde{M}_N^{(j+1)} = (3\,|\Delta_N|)^{-1} \sum_{K\in\Delta_N} |K|\,\widetilde{M}_K^{(j+1)}$, where $\Delta_N$ is the patch of elements sharing node $N$ and $|\Delta_N|$ is the corresponding area. The scaling factor $1/3$ shrinks the reference triangle to a unit edge one.

*Remark 2.* The hypothesis on the eigenvalues in Proposition 2 can be relaxed by assuming $g_1 \geq g_2 \geq 0$, i.e., that $\widehat{G}_{\Delta_K}(\mathbf{E}_{\Delta_K}^r)$ is actually positive semidefinite. This degenerate case is tackled by choosing $g_i = \max(g_i, g_{\min})$, for $i = 1, 2$, where $g_{\min} = \tau^2/\left(h_\Omega^2\, 2\#\mathscr{T}_h^{(j)}|\widehat{\Delta}_K|\right)$, with $h_\Omega$ the diameter of the domain. Thus, if $g_i$ is degenerate, $\lambda_{i,K} = h_\Omega$.

**Test case 1: pure diffusion**

We solve (1) with $\mu = 1$, $\mathbf{b} = \mathbf{0}$, $\gamma = 0$ on $\Omega = (-1,1)^2$, with $f$ chosen such that $u(x_1, x_2) = \tanh(10x_2^2 - 20x_1^3)(x_2^2 - 1)$ and Dirichlet compatible boundary conditions. We apply the above adaptive procedure with the choices $\tau = 2, 1, 0.5$ and $r = 0, 1$. Figure 1 gathers the final adapted grids for $\tau = 1$, obtained after eight iterations. The meshes match the anisotropic features of $u$, as highlighted by the detail on the right. In Tables 1 and 2 a more quantitative analysis is provided. The effectivity index $\mathrm{E.I.}_\mathrm{A}^r = \eta_\mathrm{A}^r / |u-u_h|_{H^1(\Omega)}$ is essentially independent of $\tau$. In the case $r=1$ the meshes are coarser, $\mathrm{E.I.}_\mathrm{A}^1$ being closer to 1. The error-vs-number of elements behaviour is quasi-optimal in both cases, i.e., of the order of about $-0.5$.

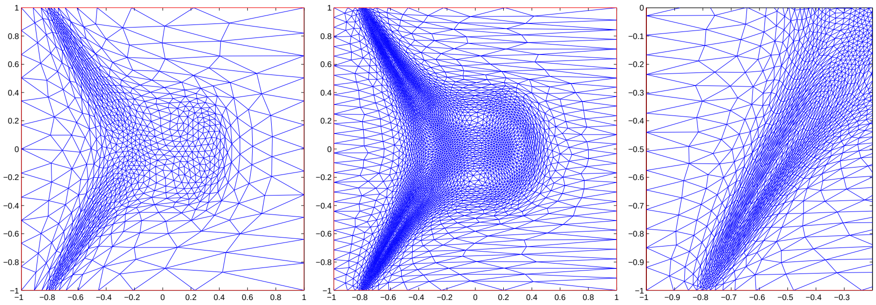

**Fig. 1** Final adapted grids for test case 1: $\tau=1$, $r=1$ (left), $r=0$ (middle and right)

**Table 1** Test case 1: $r=0$

| $\tau$ | $\#\mathscr{T}_h$ | max $s_K$ | $\|u-u_h\|_{H^1(\Omega)}$ | $\eta_\mathrm{A}^0$ | $\mathrm{E.I.}_\mathrm{A}^0$ | $\eta_\mathrm{I}^0$ | $\mathrm{E.I.}_\mathrm{I}^0$ |
|---|---|---|---|---|---|---|---|
| 2 | 1861 | 25.15 | $0.3038\cdot 10^{+0}$ | $0.2108\cdot 10^{+1}$ | 6.939 | $0.3019\cdot 10^{+1}$ | 9.938 |
| 1 | 6220 | 31.13 | $0.1552\cdot 10^{+0}$ | $0.1081\cdot 10^{+1}$ | 6.965 | $0.1572\cdot 10^{+1}$ | 10.12 |
| 0.5 | 22388 | 48.13 | $0.8024\cdot 10^{-1}$ | $0.5522\cdot 10^{+0}$ | 6.882 | $0.8149\cdot 10^{+0}$ | 10.16 |

**Table 2** Test case 1: $r=1$

| $\tau$ | $\#\mathscr{T}_h$ | max $s_K$ | $\|u-u_h\|_{H^1(\Omega)}$ | $\eta_\mathrm{A}^1$ | $\mathrm{E.I.}_\mathrm{A}^1$ | $\eta_\mathrm{I}^1$ | $\mathrm{E.I.}_\mathrm{I}^1$ |
|---|---|---|---|---|---|---|---|
| 2 | 533 | 19.64 | $0.6753\cdot 10^{+0}$ | $0.1747\cdot 10^{+1}$ | 2.586 | $0.2393\cdot 10^{+1}$ | 3.543 |
| 1 | 1541 | 17.34 | $0.3503\cdot 10^{+0}$ | $0.8802\cdot 10^{+0}$ | 2.512 | $0.1209\cdot 10^{+1}$ | 3.450 |
| 0.5 | 4699 | 27.71 | $0.1893\cdot 10^{+0}$ | $0.4408\cdot 10^{+0}$ | 2.328 | $0.6180\cdot 10^{+0}$ | 3.264 |

**Test case 2: advection-diffusion**

We now consider an instance of (1) more complex than test case 1, choosing $\mu = 10^{-3}$, $\mathbf{b} = (x_2, -x_1)^\mathrm{T}$, $\gamma = 0$, $f = 1$ on $\Omega = (0,1)^2$. The exact solution, not explicitly available, exhibits two boundary layers and a circular internal layer. We discretize (1) by the SUPG method. The adaptive procedure is run, picking $\tau = 2, 1$ and $r = 1$. All the layers are sharply detected by the anisotropic estimator (see Fig. 2). The results in Table 3 confirm the reliability of both the estimator and the adaptive procedure. Notice also the large values of the stretching factor, the maximum being reached in correspondence with the two boundary layers.

**Table 3** Test case 2 : $r=1$

| $\tau$ | $\#\mathscr{T}_h$ | max $s_K$ | $\eta_\mathrm{A}^1$ | $\eta_\mathrm{I}^1$ |
|---|---|---|---|---|
| 2 | 482 | 57.33 | $0.1727\cdot 10^{+1}$ | $0.3884\cdot 10^{+1}$ |
| 1 | 1273 | 109.98 | $0.8925\cdot 10^{+0}$ | $0.2062\cdot 10^{+1}$ |

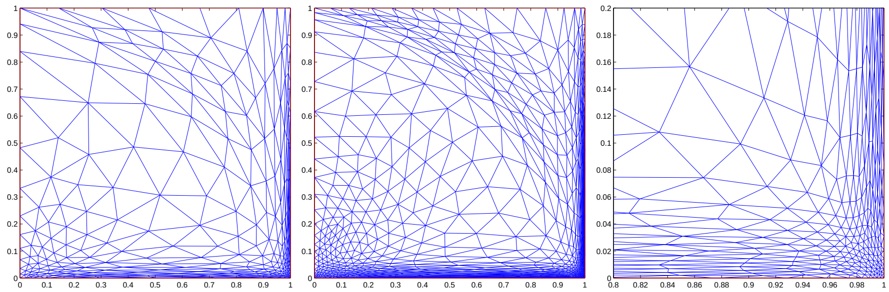

**Fig. 2** Final adapted grids for test case 2: $r = 1$, $\tau = 2$ (left), $\tau = 1$ (middle and right)

## 5 Conclusions

Despite its heuristic nature, the proposed anisotropic Zienkiewicz-Zhu a posteriori estimator provides satisfactory results. Indeed it detects the anisotropic features of the problem at hand, exhibiting a quasi-optimal error-vs-number of elements behaviour as well. This occurs even in the case $r = 0$, which identifies the roughest gradient recovery in the proposed class of estimators.

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