```
> restart;
```

Maple calculations regarding example 4
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```
> for i from 1 to 20 do yc[i]:=a1*x[i] end do:
> for i from 1 to 20 do s[i]:=s0+s1*abs(yc[i]) end do:
> lns:=add(ln(s[i]),i=1..20):
> lt:=add((y[i]-yc[i])^2/2/s[i]^2,i=1..20);
```

$$lt := \frac{(-a1\,x_1+y_1)^2}{2\,(s0+s1\,|a1\,x_1|)^2}+\frac{(-a1\,x_2+y_2)^2}{2\,(s0+s1\,|a1\,x_2|)^2}+\frac{(-a1\,x_3+y_3)^2}{2\,(s0+s1\,|a1\,x_3|)^2}+\frac{(-a1\,x_4+y_4)^2}{2\,(s0+s1\,|a1\,x_4|)^2}+\frac{(-a1\,x_5+y_5)^2}{2\,(s0+s1\,|a1\,x_5|)^2}+\frac{(-a1\,x_6+y_6)^2}{2\,(s0+s1\,|a1\,x_6|)^2}+\frac{(-a1\,x_7+y_7)^2}{2\,(s0+s1\,|a1\,x_7|)^2}+\frac{(-a1\,x_8+y_8)^2}{2\,(s0+s1\,|a1\,x_8|)^2}+\frac{(-a1\,x_9+y_9)^2}{2\,(s0+s1\,|a1\,x_9|)^2}+\frac{(-a1\,x_{10}+y_{10})^2}{2\,(s0+s1\,|a1\,x_{10}|)^2}+\frac{(-a1\,x_{11}+y_{11})^2}{2\,(s0+s1\,|a1\,x_{11}|)^2}+\frac{(-a1\,x_{12}+y_{12})^2}{2\,(s0+s1\,|a1\,x_{12}|)^2}+\frac{(-a1\,x_{13}+y_{13})^2}{2\,(s0+s1\,|a1\,x_{13}|)^2}+\frac{(-a1\,x_{14}+y_{14})^2}{2\,(s0+s1\,|a1\,x_{14}|)^2}+\frac{(-a1\,x_{15}+y_{15})^2}{2\,(s0+s1\,|a1\,x_{15}|)^2}+\frac{(-a1\,x_{16}+y_{16})^2}{2\,(s0+s1\,|a1\,x_{16}|)^2}+\frac{(-a1\,x_{17}+y_{17})^2}{2\,(s0+s1\,|a1\,x_{17}|)^2}+\frac{(-a1\,x_{18}+y_{18})^2}{2\,(s0+s1\,|a1\,x_{18}|)^2}+\frac{(-a1\,x_{19}+y_{19})^2}{2\,(s0+s1\,|a1\,x_{19}|)^2}+\frac{(-a1\,x_{20}+y_{20})^2}{2\,(s0+s1\,|a1\,x_{20}|)^2} \tag{1}$$

The probability density function 'Prob_ex4' is defined:

```
> Prob_ex4:=exp(-(20/2)*ln(2*Pi)-lns-lt):
```

The data points x[i] and y[i] are introduced:

```
> y[1]:=-6.65647123;y[2]:=-0.011729798;y[3]:=4.34198;y[4]:=4.56334;
  y[5]:=8.37979;y[6]:=10.2627;y[7]:=10.22623;y[8]:=14.28611;y[9]:=
  13.79561;y[10]:=21.09856;y[11]:=17.50144;y[12]:=20.10634;y[13]:=
  23.32865;y[14]:=18.41583;y[15]:=36.10625;y[16]:=34.69508;y[17]:=
  33.13127;y[18]:=43.87842;y[19]:=34.06836;y[20]:=43.07314;
```

$$y_1 := -6.65647123$$
$$y_2 := -0.011729798$$
$$y_3 := 4.34198$$
$$y_4 := 4.56334$$

$$y_5 := 8.37979$$
$$y_6 := 10.2627$$
$$y_7 := 10.22623$$
$$y_8 := 14.28611$$
$$y_9 := 13.79561$$
$$y_{10} := 21.09856$$
$$y_{11} := 17.50144$$
$$y_{12} := 20.10634$$
$$y_{13} := 23.32865$$
$$y_{14} := 18.41583$$
$$y_{15} := 36.10625$$
$$y_{16} := 34.69508$$
$$y_{17} := 33.13127$$
$$y_{18} := 43.87842$$
$$y_{19} := 34.06836$$
$$y_{20} := 43.07314 \quad \textbf{(2)}$$

```
> x[1]:=-10;x[2]:=-5;x[3]:=0;x[4]:=5;x[5]:=10;x[6]:=15;x[7]:=20;x
  [8]:=25;x[9]:=30;x[10]:=35;x[11]:=40;x[12]:=45;x[13]:=50;x[14]:=
  55;x[15]:=60;x[16]:=65;x[17]:=70;x[18]:=75;x[19]:=80;x[20]:=85;
```

$$x_1 := -10$$
$$x_2 := -5$$
$$x_3 := 0$$
$$x_4 := 5$$
$$x_5 := 10$$
$$x_6 := 15$$
$$x_7 := 20$$
$$x_8 := 25$$
$$x_9 := 30$$
$$x_{10} := 35$$
$$x_{11} := 40$$
$$x_{12} := 45$$
$$x_{13} := 50$$
$$x_{14} := 55$$
$$x_{15} := 60$$
$$x_{16} := 65$$

$$x_{17} := 70$$
$$x_{18} := 75$$
$$x_{19} := 80$$
$$x_{20} := 85 \quad (3)$$

```
> with(LinearAlgebra):
> with(VectorCalculus);
```

[&x, `*`, `+`, `-`, `.`, <,>, <|>, About, AddCoordinates, ArcLength, BasisFormat, Binormal, ConvertVector, CrossProduct, Curl, Curvature, D, Del, DirectionalDiff, Divergence, DotProduct, Flux, GetCoordinateParameters, GetCoordinates, GetNames, GetPVDescription, GetRootPoint, GetSpace, Gradient, Hessian, IsPositionVector, IsRootedVector, IsVectorField, Jacobian, Laplacian, LineInt, MapToBasis, ∇, Norm, Normalize, PathInt, PlotPositionVector, PlotVector, PositionVector, PrincipalNormal, RadiusOfCurvature, RootedVector, ScalarPotential, SetCoordinateParameters, SetCoordinates, SpaceCurve, SurfaceInt, TNBFrame, TangentLine, TangentPlane, TangentVector, Torsion, Vector, VectorField, VectorPotential, VectorSpace, Wronskian, diff, eval, evalVF, int, limit, series] **(4)**

The gradient 'g1' is calculated, i.e. the partial derivatives 'delpdela1=partial(Prob_ex4)/partial(a1)', 'delpdels0=partial(Prob_ex3)/partial(s0)' and 'delpdels1=partial(Prob_ex3)/partial(s1)' with respect to 'a1', 's0' and 's1' are computed:

```
> g1 := Gradient(Prob_ex4, [s0, s1,a1]):
> delpdels0:=(-1/s0-2/(s0+10*s1*abs(a1))-1/(s0+45*s1*abs(a1))-1/
  (s0+40*s1*abs(a1))-1/(s0+35*s1*abs(a1))-1/(s0+30*s1*abs(a1))-1/
  (s0+25*s1*abs(a1))-1/(s0+20*s1*abs(a1))-1/(s0+15*s1*abs(a1))-2/
  (s0+5*s1*abs(a1))-1/(s0+85*s1*abs(a1))-1/(s0+80*s1*abs(a1))-1/
  (s0+75*s1*abs(a1))-1/(s0+70*s1*abs(a1))-1/(s0+65*s1*abs(a1))-1/
  (s0+60*s1*abs(a1))-1/(s0+55*s1*abs(a1))-1/(s0+50*s1*abs(a1))+
  (-30*a1+13.79561)^2/(s0+30*s1*abs(a1))^3+(-35*a1+21.09856)^2/
  (s0+35*s1*abs(a1))^3+(-40*a1+17.50144)^2/(s0+40*s1*abs(a1))^3+
  (-45*a1+20.10634)^2/(s0+45*s1*abs(a1))^3+(-50*a1+23.32865)^2/
  (s0+50*s1*abs(a1))^3+(-55*a1+18.41583)^2/(s0+55*s1*abs(a1))^3+
  (-60*a1+36.10625)^2/(s0+60*s1*abs(a1))^3+(-65*a1+34.69508)^2/
  (s0+65*s1*abs(a1))^3+(-70*a1+33.13127)^2/(s0+70*s1*abs(a1))^3+
  (-75*a1+43.87842)^2/(s0+75*s1*abs(a1))^3+(-80*a1+34.06836)^2/
  (s0+80*s1*abs(a1))^3+(-85*a1+43.07314)^2/(s0+85*s1*abs(a1))^3+
  (10*a1-6.65647123)^2/(s0+10*s1*abs(a1))^3+(5*a1-0.11729798e-1)^2/
  (s0+5*s1*abs(a1))^3+(-5*a1+4.56334)^2/(s0+5*s1*abs(a1))^3+(-10*
  a1+8.37979)^2/(s0+10*s1*abs(a1))^3+(-15*a1+10.2627)^2/(s0+15*s1*
  abs(a1))^3+(-20*a1+10.22623)^2/(s0+20*s1*abs(a1))^3+(-25*
  a1+14.28611)^2/(s0+25*s1*abs(a1))^3+18.85279032/(s0^3))*exp(-(10*
  a1-6.65647123)^2/(2*(s0+10*s1*abs(a1))^2)-(-85*a1+43.07314)^2/(2*
  (s0+85*s1*abs(a1))^2)-(-80*a1+34.06836)^2/(2*(s0+80*s1*abs(a1))
  ^2)-(-75*a1+43.87842)^2/(2*(s0+75*s1*abs(a1))^2)-(-70*
  a1+33.13127)^2/(2*(s0+70*s1*abs(a1))^2)-(-65*a1+34.69508)^2/(2*
  (s0+65*s1*abs(a1))^2)-(-60*a1+36.10625)^2/(2*(s0+60*s1*abs(a1))
  ^2)-(-55*a1+18.41583)^2/(2*(s0+55*s1*abs(a1))^2)-(-50*
  a1+23.32865)^2/(2*(s0+50*s1*abs(a1))^2)-(-45*a1+20.10634)^2/(2*
  (s0+45*s1*abs(a1))^2)-(-40*a1+17.50144)^2/(2*(s0+40*s1*abs(a1))
  ^2)-(-35*a1+21.09856)^2/(2*(s0+35*s1*abs(a1))^2)-(-30*
```

```
a1+13.79561)^2/(2*(s0+30*s1*abs(a1))^2)-ln(s0+85*s1*abs(a1))-ln
(s0+80*s1*abs(a1))-ln(s0+75*s1*abs(a1))-ln(s0+70*s1*abs(a1))-ln
(s0+65*s1*abs(a1))-ln(s0+60*s1*abs(a1))-ln(s0+55*s1*abs(a1))-ln
(s0+50*s1*abs(a1))-ln(s0+45*s1*abs(a1))-ln(s0+40*s1*abs(a1))-ln
(s0+35*s1*abs(a1))-ln(s0+30*s1*abs(a1))-ln(s0+25*s1*abs(a1))-ln
(s0+20*s1*abs(a1))-ln(s0+15*s1*abs(a1))-2*ln(s0+5*s1*abs(a1))-ln
(s0)-2*ln(s0+10*s1*abs(a1))-9.426395160/(s0^2)-10*ln(2*Pi)-(-25*
a1+14.28611)^2/(2*(s0+25*s1*abs(a1))^2)-(-20*a1+10.22623)^2/(2*
(s0+20*s1*abs(a1))^2)-(-15*a1+10.2627)^2/(2*(s0+15*s1*abs(a1))^2)
-(-10*a1+8.37979)^2/(2*(s0+10*s1*abs(a1))^2)-(-5*a1+4.56334)^2/
(2*(s0+5*s1*abs(a1))^2)-(5*a1-0.11729798e-1)^2/(2*(s0+5*s1*abs
(a1))^2)):
> delpdels1:=(80*(-80*a1+34.06836)^2*abs(a1)/(s0+80*s1*abs(a1))
^3+85*(-85*a1+43.07314)^2*abs(a1)/(s0+85*s1*abs(a1))^3+10*(10*a1
-6.65647123)^2*abs(a1)/(s0+10*s1*abs(a1))^3+5*(5*a1
-0.11729798e-1)^2*abs(a1)/(s0+5*s1*abs(a1))^3+5*(-5*a1+4.56334)
^2*abs(a1)/(s0+5*s1*abs(a1))^3+10*(-10*a1+8.37979)^2*abs(a1)/
(s0+10*s1*abs(a1))^3+15*(-15*a1+10.2627)^2*abs(a1)/(s0+15*s1*abs
(a1))^3+20*(-20*a1+10.22623)^2*abs(a1)/(s0+20*s1*abs(a1))^3+25*
(-25*a1+14.28611)^2*abs(a1)/(s0+25*s1*abs(a1))^3+30*(-30*
a1+13.79561)^2*abs(a1)/(s0+30*s1*abs(a1))^3+35*(-35*a1+21.09856)
^2*abs(a1)/(s0+35*s1*abs(a1))^3+40*(-40*a1+17.50144)^2*abs(a1)/
(s0+40*s1*abs(a1))^3+45*(-45*a1+20.10634)^2*abs(a1)/(s0+45*s1*abs
(a1))^3+50*(-50*a1+23.32865)^2*abs(a1)/(s0+50*s1*abs(a1))^3+55*
(-55*a1+18.41583)^2*abs(a1)/(s0+55*s1*abs(a1))^3+60*(-60*
a1+36.10625)^2*abs(a1)/(s0+60*s1*abs(a1))^3+65*(-65*a1+34.69508)
^2*abs(a1)/(s0+65*s1*abs(a1))^3+70*(-70*a1+33.13127)^2*abs(a1)/
(s0+70*s1*abs(a1))^3+75*(-75*a1+43.87842)^2*abs(a1)/(s0+75*s1*abs
(a1))^3-20*abs(a1)/(s0+10*s1*abs(a1))-85*abs(a1)/(s0+85*s1*abs
(a1))-80*abs(a1)/(s0+80*s1*abs(a1))-75*abs(a1)/(s0+75*s1*abs(a1))
-70*abs(a1)/(s0+70*s1*abs(a1))-65*abs(a1)/(s0+65*s1*abs(a1))-60*
abs(a1)/(s0+60*s1*abs(a1))-55*abs(a1)/(s0+55*s1*abs(a1))-50*abs
(a1)/(s0+50*s1*abs(a1))-45*abs(a1)/(s0+45*s1*abs(a1))-40*abs(a1)/
(s0+40*s1*abs(a1))-35*abs(a1)/(s0+35*s1*abs(a1))-30*abs(a1)/
(s0+30*s1*abs(a1))-25*abs(a1)/(s0+25*s1*abs(a1))-20*abs(a1)/
(s0+20*s1*abs(a1))-15*abs(a1)/(s0+15*s1*abs(a1))-10*abs(a1)/
(s0+5*s1*abs(a1)))*exp(-9.426395160/(s0^2)-(-25*a1+14.28611)^2/
(2*(s0+25*s1*abs(a1))^2)-(-20*a1+10.22623)^2/(2*(s0+20*s1*abs(a1)
)^2)-(-15*a1+10.2627)^2/(2*(s0+15*s1*abs(a1))^2)-(-10*a1+8.37979)
^2/(2*(s0+10*s1*abs(a1))^2)-(-5*a1+4.56334)^2/(2*(s0+5*s1*abs(a1)
)^2)-(5*a1-0.11729798e-1)^2/(2*(s0+5*s1*abs(a1))^2)-(10*a1
-6.65647123)^2/(2*(s0+10*s1*abs(a1))^2)-(-85*a1+43.07314)^2/(2*
(s0+85*s1*abs(a1))^2)-(-80*a1+34.06836)^2/(2*(s0+80*s1*abs(a1))
^2)-(-75*a1+43.87842)^2/(2*(s0+75*s1*abs(a1))^2)-(-70*
a1+33.13127)^2/(2*(s0+70*s1*abs(a1))^2)-(-65*a1+34.69508)^2/(2*
(s0+65*s1*abs(a1))^2)-(-60*a1+36.10625)^2/(2*(s0+60*s1*abs(a1))
^2)-(-55*a1+18.41583)^2/(2*(s0+55*s1*abs(a1))^2)-ln(s0+85*s1*abs
(a1))-ln(s0+80*s1*abs(a1))-ln(s0+75*s1*abs(a1))-ln(s0+70*s1*abs
(a1))-ln(s0+65*s1*abs(a1))-ln(s0+60*s1*abs(a1))-ln(s0+55*s1*abs
(a1))-ln(s0+50*s1*abs(a1))-ln(s0+45*s1*abs(a1))-ln(s0+40*s1*abs
(a1))-ln(s0+35*s1*abs(a1))-ln(s0+30*s1*abs(a1))-ln(s0+25*s1*abs
(a1))-ln(s0+20*s1*abs(a1))-ln(s0+15*s1*abs(a1))-2*ln(s0+5*s1*abs
(a1))-ln(s0)-2*ln(s0+10*s1*abs(a1))-(-50*a1+23.32865)^2/(2*
(s0+50*s1*abs(a1))^2)-(-45*a1+20.10634)^2/(2*(s0+45*s1*abs(a1))
^2)-(-40*a1+17.50144)^2/(2*(s0+40*s1*abs(a1))^2)-(-35*
a1+21.09856)^2/(2*(s0+35*s1*abs(a1))^2)-(-30*a1+13.79561)^2/(2*
```

```
  (s0+30*s1*abs(a1))^2)-10*ln(2*Pi)):
> delpdela1:=((15*(-15*a1+10.2627))/(s0+15*s1*abs(a1))^2+(10*(-10*
  a1+8.37979))/(s0+10*s1*abs(a1))^2+(5*(-5*a1+4.56334))/(s0+5*s1*
  abs(a1))^2-(5*(5*a1-0.11729798e-1))/(s0+5*s1*abs(a1))^2-(10*(10*
  a1-6.65647123))/(s0+10*s1*abs(a1))^2+(85*(-85*a1+43.07314))/
  (s0+85*s1*abs(a1))^2+(80*(-80*a1+34.06836))/(s0+80*s1*abs(a1))
  ^2+5*(5*a1-0.11729798e-1)^2*s1*abs(1, a1)/(s0+5*s1*abs(a1))^3+5*
  (-5*a1+4.56334)^2*s1*abs(1, a1)/(s0+5*s1*abs(a1))^3+10*(-10*
  a1+8.37979)^2*s1*abs(1, a1)/(s0+10*s1*abs(a1))^3+15*(-15*
  a1+10.2627)^2*s1*abs(1, a1)/(s0+15*s1*abs(a1))^3+20*(-20*
  a1+10.22623)^2*s1*abs(1, a1)/(s0+20*s1*abs(a1))^3+25*(-25*
  a1+14.28611)^2*s1*abs(1, a1)/(s0+25*s1*abs(a1))^3+30*(-30*
  a1+13.79561)^2*s1*abs(1, a1)/(s0+30*s1*abs(a1))^3+35*(-35*
  a1+21.09856)^2*s1*abs(1, a1)/(s0+35*s1*abs(a1))^3+40*(-40*
  a1+17.50144)^2*s1*abs(1, a1)/(s0+40*s1*abs(a1))^3+45*(-45*
  a1+20.10634)^2*s1*abs(1, a1)/(s0+45*s1*abs(a1))^3+50*(-50*
  a1+23.32865)^2*s1*abs(1, a1)/(s0+50*s1*abs(a1))^3+55*(-55*
  a1+18.41583)^2*s1*abs(1, a1)/(s0+55*s1*abs(a1))^3+60*(-60*
  a1+36.10625)^2*s1*abs(1, a1)/(s0+60*s1*abs(a1))^3+65*(-65*
  a1+34.69508)^2*s1*abs(1, a1)/(s0+65*s1*abs(a1))^3+70*(-70*
  a1+33.13127)^2*s1*abs(1, a1)/(s0+70*s1*abs(a1))^3+75*(-75*
  a1+43.87842)^2*s1*abs(1, a1)/(s0+75*s1*abs(a1))^3+80*(-80*
  a1+34.06836)^2*s1*abs(1, a1)/(s0+80*s1*abs(a1))^3+85*(-85*
  a1+43.07314)^2*s1*abs(1, a1)/(s0+85*s1*abs(a1))^3+10*(10*a1
  -6.65647123)^2*s1*abs(1, a1)/(s0+10*s1*abs(a1))^3+(75*(-75*
  a1+43.87842))/(s0+75*s1*abs(a1))^2+(70*(-70*a1+33.13127))/(s0+70*
  s1*abs(a1))^2+(65*(-65*a1+34.69508))/(s0+65*s1*abs(a1))^2+(60*
  (-60*a1+36.10625))/(s0+60*s1*abs(a1))^2+(55*(-55*a1+18.41583))/
  (s0+55*s1*abs(a1))^2+(50*(-50*a1+23.32865))/(s0+50*s1*abs(a1))^2+
  (45*(-45*a1+20.10634))/(s0+45*s1*abs(a1))^2+(40*(-40*a1+17.50144)
  )/(s0+40*s1*abs(a1))^2+(35*(-35*a1+21.09856))/(s0+35*s1*abs(a1))
  ^2-35*s1*abs(1, a1)/(s0+35*s1*abs(a1))-30*s1*abs(1, a1)/(s0+30*
  s1*abs(a1))-25*s1*abs(1, a1)/(s0+25*s1*abs(a1))-20*s1*abs(1, a1)/
  (s0+20*s1*abs(a1))-15*s1*abs(1, a1)/(s0+15*s1*abs(a1))-10*s1*abs
  (1, a1)/(s0+5*s1*abs(a1))-20*s1*abs(1, a1)/(s0+10*s1*abs(a1))-85*
  s1*abs(1, a1)/(s0+85*s1*abs(a1))-80*s1*abs(1, a1)/(s0+80*s1*abs
  (a1))-75*s1*abs(1, a1)/(s0+75*s1*abs(a1))-70*s1*abs(1, a1)/
  (s0+70*s1*abs(a1))-65*s1*abs(1, a1)/(s0+65*s1*abs(a1))-60*s1*abs
  (1, a1)/(s0+60*s1*abs(a1))-55*s1*abs(1, a1)/(s0+55*s1*abs(a1))
  -50*s1*abs(1, a1)/(s0+50*s1*abs(a1))-45*s1*abs(1, a1)/(s0+45*s1*
  abs(a1))-40*s1*abs(1, a1)/(s0+40*s1*abs(a1))+(30*(-30*
  a1+13.79561))/(s0+30*s1*abs(a1))^2+(25*(-25*a1+14.28611))/(s0+25*
  s1*abs(a1))^2+(20*(-20*a1+10.22623))/(s0+20*s1*abs(a1))^2)*exp(-
  (-25*a1+14.28611)^2/(2*(s0+25*s1*abs(a1))^2)-(-20*a1+10.22623)^2/
  (2*(s0+20*s1*abs(a1))^2)-(-15*a1+10.2627)^2/(2*(s0+15*s1*abs(a1))
  ^2)-(-10*a1+8.37979)^2/(2*(s0+10*s1*abs(a1))^2)-ln(s0+85*s1*abs
  (a1))-ln(s0+80*s1*abs(a1))-ln(s0+75*s1*abs(a1))-ln(s0+70*s1*abs
  (a1))-ln(s0+65*s1*abs(a1))-ln(s0+60*s1*abs(a1))-ln(s0+55*s1*abs
  (a1))-ln(s0+50*s1*abs(a1))-ln(s0+45*s1*abs(a1))-ln(s0+40*s1*abs
  (a1))-ln(s0+35*s1*abs(a1))-ln(s0+30*s1*abs(a1))-ln(s0+25*s1*abs
  (a1))-ln(s0+20*s1*abs(a1))-ln(s0+15*s1*abs(a1))-2*ln(s0+5*s1*abs
  (a1))-ln(s0)-2*ln(s0+10*s1*abs(a1))-(-5*a1+4.56334)^2/(2*(s0+5*
  s1*abs(a1))^2)-(5*a1-0.11729798e-1)^2/(2*(s0+5*s1*abs(a1))^2)-
  (10*a1-6.65647123)^2/(2*(s0+10*s1*abs(a1))^2)-(-85*a1+43.07314)
  ^2/(2*(s0+85*s1*abs(a1))^2)-(-80*a1+34.06836)^2/(2*(s0+80*s1*abs
  (a1))^2)-(-75*a1+43.87842)^2/(2*(s0+75*s1*abs(a1))^2)-(-70*
```

```
a1+33.13127)^2/(2*(s0+70*s1*abs(a1))^2)-(-65*a1+34.69508)^2/(2*
(s0+65*s1*abs(a1))^2)-(-60*a1+36.10625)^2/(2*(s0+60*s1*abs(a1))
^2)-(-55*a1+18.41583)^2/(2*(s0+55*s1*abs(a1))^2)-(-50*
a1+23.32865)^2/(2*(s0+50*s1*abs(a1))^2)-(-45*a1+20.10634)^2/(2*
(s0+45*s1*abs(a1))^2)-(-40*a1+17.50144)^2/(2*(s0+40*s1*abs(a1))
^2)-(-35*a1+21.09856)^2/(2*(s0+35*s1*abs(a1))^2)-(-30*
a1+13.79561)^2/(2*(s0+30*s1*abs(a1))^2)-10*ln(2*Pi)-9.426395160/
(s0^2)):
> f:=delpdels1=0:g:=delpdela1=0:u:=delpdels0=0:
```

The partial derivatives are all set to zero, and the objective function Z_1 (Eq. 18) is minimized providing the optimized parameters a1,s0 and s1 (see also Table 3):

```
> fsolve({f,g,u}, {s0 = 1.5 .. 3, s1 = 0.04 .. 0.1, a1 = 0.4 ..
0.6});
```

$$\{a1 = 0.4989582900,\ s0 = 2.545834529,\ s1 = 0.05819734350\} \tag{5}$$

```
> Z:=-2*ln(Prob_ex4);
```

$$Z := \tag{6}$$

$$-2 \ln\left(e^{-\ln(s0+30\,s1\,|a1|)-\ln(s0+35\,s1\,|a1|)-\ln(s0+40\,s1\,|a1|)-\ln(s0+45\,s1\,|a1|)-\ln(s0+50\,s1\,|a1|)} \right.$$

$$-\ln(s0+55\,s1\,|a1|)-\ln(s0+60\,s1\,|a1|)-\ln(s0+65\,s1\,|a1|)-\ln(s0+70\,s1\,|a1|)-\ln(s0+75\,s1\,|a1|)-\ln(s0$$

$$+80\,s1\,|a1|)-\ln(s0+85\,s1\,|a1|)-\frac{9.426395160}{s0^2}-2\ln(s0+10\,s1\,|a1|)-2\ln(s0+5\,s1\,|a1|)-\ln(s0)-\ln(s0$$

$$+15\,s1\,|a1|)-\ln(s0+20\,s1\,|a1|)-\ln(s0+25\,s1\,|a1|)-\frac{(-60\,a1+36.10625)^2}{2\,(s0+60\,s1\,|a1|)^2}-\frac{(-65\,a1+34.69508)^2}{2\,(s0+65\,s1\,|a1|)^2}$$

$$-\frac{(-70\,a1+33.13127)^2}{2\,(s0+70\,s1\,|a1|)^2}-\frac{(-75\,a1+43.87842)^2}{2\,(s0+75\,s1\,|a1|)^2}-\frac{(-80\,a1+34.06836)^2}{2\,(s0+80\,s1\,|a1|)^2}-\frac{(-85\,a1+43.07314)^2}{2\,(s0+85\,s1\,|a1|)^2}$$

$$-\frac{(-25\,a1+14.28611)^2}{2\,(s0+25\,s1\,|a1|)^2}-\frac{(-30\,a1+13.79561)^2}{2\,(s0+30\,s1\,|a1|)^2}-\frac{(-35\,a1+21.09856)^2}{2\,(s0+35\,s1\,|a1|)^2}-\frac{(-40\,a1+17.50144)^2}{2\,(s0+40\,s1\,|a1|)^2}$$

$$-\frac{(-45\,a1+20.10634)^2}{2\,(s0+45\,s1\,|a1|)^2}-\frac{(-50\,a1+23.32865)^2}{2\,(s0+50\,s1\,|a1|)^2}-\frac{(-55\,a1+18.41583)^2}{2\,(s0+55\,s1\,|a1|)^2}-\frac{(10\,a1-6.65647123)^2}{2\,(s0+10\,s1\,|a1|)^2}$$

$$-\frac{(5\,a1-0.011729798)^2}{2\,(s0+5\,s1\,|a1|)^2}-\frac{(-5\,a1+4.56334)^2}{2\,(s0+5\,s1\,|a1|)^2}-\frac{(-10\,a1+8.37979)^2}{2\,(s0+10\,s1\,|a1|)^2}-\frac{(-15\,a1+10.2627)^2}{2\,(s0+15\,s1\,|a1|)^2}$$

$$\left.-\frac{(-20\,a1+10.22623)^2}{2\,(s0+20\,s1\,|a1|)^2}-10\ln(2\pi)\right)$$

The Hessian matrix is calculated for the minimum of the objective function Z1:

```
> hess:=Hessian(Z, [a1,s0,s1]):
```

```
> hess0:=subs(a1= .4989582900,hess):hess1:=subs(s0= 2.545834529,
  hess0):hess2:=subs(s1= 0.5819734350e-1,hess1):hess_matrix:=evalf
  (hess2):
```

The covariance matrix (inverse of the Hessian matrix) is calculated:

```
> cov:=MatrixInverse(hess_matrix);
```

$$cov := \begin{bmatrix} 0.000213218991015571 & -0.000650605339946647 & 0.0000142006428288282 \\ -0.000650605339946647 & 0.273325442046615 & -0.00962063551685165 \\ 0.0000142006428288282 & -0.00962063551685165 & 0.000729633182700327 \end{bmatrix} \quad (7)$$

```
> with(Statistics);
```

[*AbsoluteDeviation*, *AgglomeratedPlot*, *AreaChart*, *AutoCorrelation*, *BarChart*, *Biplot*, *Bootstrap*, *BoxPlot*, *BubblePlot*, *CDF*, *CGF*, *CentralMoment*, *CharacteristicFunction*, *ChiSquareGoodnessOfFitTest*, *ChiSquareIndependenceTest*, *ChiSquareSuitableModelTest*, *ColumnGraph*, *Correlation*, *CorrelationMatrix*, *Count*, *CountMissing*, *Covariance*, *CovarianceMatrix*, *CrossCorrelation*, *Cumulant*, *CumulantGeneratingFunction*, *CumulativeDistributionFunction*, *CumulativeProduct*, *CumulativeSum*, *CumulativeSumChart*, *DataSummary*, *Decile*, *DensityPlot*, *DiscreteValueMap*, *Distribution*, *ErrorPlot*, *EvaluateToFloat*, *Excise*, *ExpectedValue*, *ExponentialFit*, *ExponentialSmoothing*, *FailureRate*, *FisherInformation*, *Fit*, *FivePointSummary*, *FrequencyPlot*, *FrequencyTable*, *GeometricMean*, *GridPlot*, *HarmonicMean*, *HazardRate*, *HeatMap*, *Histogram*, *HodgesLehmann*, *Information*, *InteractiveDataAnalysis*, *InterquartileRange*, *InverseSurvivalFunction*, *Join*, *KernelDensity*, *KernelDensityPlot*, *KernelDensitySample*, *Kurtosis*, *Likelihood*, *LikelihoodRatioStatistic*, *LineChart*, *LinearFilter*, *LinearFit*, *LogLikelihood*, *LogarithmicFit*, *Lowess*, *MGF*, *MLE*, *MakeProcedure*, *MaximumLikelihoodEstimate*, *Mean*, *MeanDeviation*, *Median*, *MedianDeviation*, *MillsRatio*, *Mode*, *Moment*, *MomentGeneratingFunction*, *MovingAverage*, *MovingMedian*, *MovingStatistic*, *NonlinearFit*, *NormalPlot*, *OneSampleChiSquareTest*, *OneSampleTTest*, *OneSampleZTest*, *OneWayANOVA*, *OrderByRank*, *OrderStatistic*, *PCA*, *PDF*, *ParetoChart*, *Percentile*, *PieChart*, *PointPlot*, *PolynomialFit*, *PowerFit*, *PredictiveLeastSquares*, *PrincipalComponentAnalysis*, *Probability*, *ProbabilityDensityFunction*, *ProbabilityFunction*, *ProbabilityPlot*, *ProfileLikelihood*, *ProfileLogLikelihood*, *QuadraticMean*, *Quantile*, *QuantilePlot*, *Quartile*, *RandomVariable*, *Range*, *Rank*, *Remove*, *RemoveInRange*, *RemoveNonNumeric*, *RepeatedMedianEstimator*, *RousseeuwCrouxQn*, *RousseeuwCrouxSn*, *Sample*, *Scale*, *ScatterPlot*, *ScatterPlot3D*, *Score*, *ScreePlot*, *Select*, *SelectInRange*, *SelectNonNumeric*, *ShapiroWilkWTest*, *Shuffle*, *Skewness*, *Sort*, *Specialize*, *SplitByColumn*, *StandardDeviation*, *StandardError*, *StandardizedMoment*, *SunflowerPlot*, *Support*, *SurfacePlot*, *SurvivalFunction*, *SymmetryPlot*, *Tally*, *TallyInto*, *TreeMap*, *Trim*, *TrimmedMean*, *TwoSampleFTest*, *TwoSamplePairedTTest*, *TwoSampleTTest*, *TwoSampleZTest*, *Variance*, *Variation*, *VennDiagram*, *ViolinPlot*, *WeibullPlot*, *WeightedMovingAverage*, *Winsorize*, *WinsorizedMean*] **(8)**

The correlation coefficients (Eq. 24) follow from the covariance matrix (compare Table 4):

```
> rho12:=cov(1,2)/sqrt(cov(1,1))/sqrt(cov(2,2)); rho13:=cov(1,3)
  /sqrt(cov(1,1))/sqrt(cov(3,3));rho23:=cov(2,3)/sqrt(cov(2,2))
```

```
/sqrt(cov(3,3));
```

$$\rho 12 := -0.0852245372892805$$

$$\rho 13 := 0.0360033385194894$$

$$\rho 23 := -0.681257730547091 \quad \textbf{(9)}$$

The uncertainties of the parameters follow from the diagonal terms of the covariance matrix (compare Table 3):

```
> dela0:=sqrt(cov(1,1));
```

$$dela0 := 0.0146020201005056 \quad \textbf{(10)}$$

```
> dela1:=sqrt(cov(2,2));
```

$$dela1 := 0.522805357706494 \quad \textbf{(11)}$$

```
> dels0:=sqrt(cov(3,3));
```

$$dels0 := 0.0270117230605589 \quad \textbf{(12)}$$

```
> with(LinearAlgebra);
```

[*&x*, *Add*, *Adjoint*, *BackwardSubstitute*, *BandMatrix*, *Basis*, *BezoutMatrix*, *BidiagonalForm*, *BilinearForm*, *CARE*, *CharacteristicMatrix*, *CharacteristicPolynomial*, *Column*, *ColumnDimension*, *ColumnOperation*, *ColumnSpace*, *CompanionMatrix*, *CompressedSparseForm*, *ConditionNumber*, *ConstantMatrix*, *ConstantVector*, *Copy*, *CreatePermutation*, *CrossProduct*, *DARE*, *DeleteColumn*, *DeleteRow*, *Determinant*, *Diagonal*, *DiagonalMatrix*, *Dimension*, *Dimensions*, *DotProduct*, *EigenConditionNumbers*, *Eigenvalues*, *Eigenvectors*, *Equal*, *ForwardSubstitute*, *FrobeniusForm*, *FromCompressedSparseForm*, *FromSplitForm*, *GaussianElimination*, *GenerateEquations*, *GenerateMatrix*, *Generic*, *GetResultDataType*, *GetResultShape*, *GivensRotationMatrix*, *GramSchmidt*, *HankelMatrix*, *HermiteForm*, *HermitianTranspose*, *HessenbergForm*, *HilbertMatrix*, *HouseholderMatrix*, *IdentityMatrix*, *IntersectionBasis*, *IsDefinite*, *IsOrthogonal*, *IsSimilar*, *IsUnitary*, *JordanBlockMatrix*, *JordanForm*, *KroneckerProduct*, *LA_Main*, *LUDecomposition*, *LeastSquares*, *LinearSolve*, *LyapunovSolve*, *Map*, *Map2*, *MatrixAdd*, *MatrixExponential*, *MatrixFunction*, *MatrixInverse*, *MatrixMatrixMultiply*, *MatrixNorm*, *MatrixPower*, *MatrixScalarMultiply*, *MatrixVectorMultiply*, *MinimalPolynomial*, *Minor*, *Modular*, *Multiply*, *NoUserValue*, *Norm*, *Normalize*, *NullSpace*, *OuterProductMatrix*, *Permanent*, *Pivot*, *PopovForm*, *ProjectionMatrix*, *QRDecomposition*, *RandomMatrix*, *RandomVector*, *Rank*, *RationalCanonicalForm*, *ReducedRowEchelonForm*, *Row*, *RowDimension*, *RowOperation*, *RowSpace*, *ScalarMatrix*, *ScalarMultiply*, *ScalarVector*, *SchurForm*, *SingularValues*, *SmithForm*, *SplitForm*, *StronglyConnectedBlocks*, *SubMatrix*, *SubVector*, *SumBasis*, *SylvesterMatrix*, *SylvesterSolve*, *ToeplitzMatrix*, *Trace*, *Transpose*, *TridiagonalForm*, *UnitVector*, *VandermondeMatrix*, *VectorAdd*, *VectorAngle*, *VectorMatrixMultiply*, *VectorNorm*, *VectorScalarMultiply*, *ZeroMatrix*, *ZeroVector*, *Zip*] **(13)**

```
> Determinant(hess_matrix);
```

$$4.424648160\ 10^{7} \quad \textbf{(14)}$$

```
> ConditionNumber(hess_matrix);
```

$$1375.109792 \quad \textbf{(15)}$$