```
> restart:
```

Maple calculations regarding example 1
==========
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==========

```
> with(Statistics);
```

[*AbsoluteDeviation*, *AgglomeratedPlot*, *AreaChart*, *AutoCorrelation*, *BarChart*, *Biplot*, *Bootstrap*, *BoxPlot*, *BubblePlot*, *CDF*, *CGF*, *CentralMoment*, *CharacteristicFunction*, *ChiSquareGoodnessOfFitTest*, *ChiSquareIndependenceTest*, *ChiSquareSuitableModelTest*, *ColumnGraph*, *Correlation*, *CorrelationMatrix*, *Count*, *CountMissing*, *Covariance*, *CovarianceMatrix*, *CrossCorrelation*, *Cumulant*, *CumulantGeneratingFunction*, *CumulativeDistributionFunction*, *CumulativeProduct*, *CumulativeSum*, *CumulativeSumChart*, *DataSummary*, *Decile*, *DensityPlot*, *DiscreteValueMap*, *Distribution*, *ErrorPlot*, *EvaluateToFloat*, *Excise*, *ExpectedValue*, *ExponentialFit*, *ExponentialSmoothing*, *FailureRate*, *FisherInformation*, *Fit*, *FivePointSummary*, *FrequencyPlot*, *FrequencyTable*, *GeometricMean*, *GridPlot*, *HarmonicMean*, *HazardRate*, *HeatMap*, *Histogram*, *HodgesLehmann*, *Information*, *InteractiveDataAnalysis*, *InterquartileRange*, *InverseSurvivalFunction*, *Join*, *KernelDensity*, *KernelDensityPlot*, *KernelDensitySample*, *Kurtosis*, *Likelihood*, *LikelihoodRatioStatistic*, *LineChart*, *LinearFilter*, *LinearFit*, *LogLikelihood*, *LogarithmicFit*, *Lowess*, *MGF*, *MLE*, *MakeProcedure*, *MaximumLikelihoodEstimate*, *Mean*, *MeanDeviation*, *Median*, *MedianDeviation*, *MillsRatio*, *Mode*, *Moment*, *MomentGeneratingFunction*, *MovingAverage*, *MovingMedian*, *MovingStatistic*, *NonlinearFit*, *NormalPlot*, *OneSampleChiSquareTest*, *OneSampleTTest*, *OneSampleZTest*, *OneWayANOVA*, *OrderByRank*, *OrderStatistic*, *PCA*, *PDF*, *ParetoChart*, *Percentile*, *PieChart*, *PointPlot*, *PolynomialFit*, *PowerFit*, *PredictiveLeastSquares*, *PrincipalComponentAnalysis*, *Probability*, *ProbabilityDensityFunction*, *ProbabilityFunction*, *ProbabilityPlot*, *ProfileLikelihood*, *ProfileLogLikelihood*, *QuadraticMean*, *Quantile*, *QuantilePlot*, *Quartile*, *RandomVariable*, *Range*, *Rank*, *Remove*, *RemoveInRange*, *RemoveNonNumeric*, *RepeatedMedianEstimator*, *RousseeuwCrouxQn*, *RousseeuwCrouxSn*, *Sample*, *Scale*, *ScatterPlot*, *ScatterPlot3D*, *Score*, *ScreePlot*, *Select*, *SelectInRange*, *SelectNonNumeric*, *ShapiroWilkWTest*, *Shuffle*, *Skewness*, *Sort*, *Specialize*, *SplitByColumn*, *StandardDeviation*, *StandardError*, *StandardizedMoment*, *SunflowerPlot*, *Support*, *SurfacePlot*, *SurvivalFunction*, *SymmetryPlot*, *Tally*, *TallyInto*, *TreeMap*, *Trim*, *TrimmedMean*, *TwoSampleFTest*, *TwoSamplePairedTTest*, *TwoSampleTTest*, *TwoSampleZTest*, *Variance*, *Variation*, *VennDiagram*, *ViolinPlot*, *WeibullPlot*, *WeightedMovingAverage*, *Winsorize*, *WinsorizedMean*] **(1)**

The objective function for example 1 is based on Eq. (18).

```
> Prob_ex1:=exp(-(8/2)*ln(2*Pi)-8*ln(s)-((y1-a)^2+(y2-1)^2+(y3-a)
  ^2+(y4-a)^2+(y5-a)^2+(y6-a)^2+(y7-a)^2+(y8-a)^2)/(2*s^2)):
```

The modelled y values equal a=1.

```
> a:=1:
```

The simulated y values are introduced:

```
> y1:=1.2529423:y2:=0.6062315:y3:=0.8870169:y4:=0.7365572:y5:=
  1.0504223:y6:=0.7883198:y7:=0.912006:y8:=1.2955773:
```

The mean value is calculated.

```
> Mean([y1,y2,y3,y4,y5,y6,y7,y8]);
```

$$0.941134162500000 \tag{2}$$

The standard deviation is calculated.

```
> std:=StandardDeviation([y1,y2,y3,y4,y5,y6,y7,y8]);
```

$$std := 0.243761896003632 \tag{3}$$

Normalizing the objective function to obtain the probability density:

```
> int(Prob_ex1,s=0..infinity);normf:=1/%;
```

$$0.2073738318$$

$$normf := 4.822209202 \tag{4}$$

Probability density Prob_ex1_norm

```
> Prob_ex1_norm:=Prob_ex1*normf;
```

$$Prob\_ex1\_norm := 4.822209202\, e^{-4\ln(2\pi) - 8\ln(s) - \frac{0.2218302641}{s^2}} \tag{5}$$

```
> with(VectorCalculus);
```

$$[\&x, `*`, `+`, `-`, `.`, <,>, <|>, About, AddCoordinates, ArcLength, BasisFormat, Binormal, \tag{6}$$
$$ConvertVector, CrossProduct, Curl, Curvature, \mathrm{D}, Del, DirectionalDiff, Divergence,$$
$$DotProduct, Flux, GetCoordinateParameters, GetCoordinates, GetNames,$$
$$GetPVDescription, GetRootPoint, GetSpace, Gradient, Hessian, IsPositionVector,$$
$$IsRootedVector, IsVectorField, Jacobian, Laplacian, LineInt, MapToBasis, \nabla, Norm,$$
$$Normalize, PathInt, PlotPositionVector, PlotVector, PositionVector, PrincipalNormal,$$
$$RadiusOfCurvature, RootedVector, ScalarPotential, SetCoordinateParameters,$$
$$SetCoordinates, SpaceCurve, SurfaceInt, TNBFrame, TangentLine, TangentPlane,$$
$$TangentVector, Torsion, Vector, VectorField, VectorPotential, VectorSpace, Wronskian,$$
$$diff, eval, evalVF, int, limit, series]$$

Gradient of the probability density with respect to variable s:

```
> g1 := Gradient(Prob_ex1_norm, [s]);
```

$$g1 := \left(4.822209202\left(-\frac{8}{s} + \frac{0.4436605282}{s^3}\right) e^{-4\ln(2\pi) - 8\ln(s) - \frac{0.2218302641}{s^2}}\right)\bar{e}_s \tag{7}$$

Derivative of the probability density with respect to variable s:

```
> delpdels:=(-8/s+.4436605282/(s^3))*exp(-4*ln(2*Pi)-8*ln(s)
  -.2218302641/(s^2));
```

$$delpdels := \left(-\frac{8}{s} + \frac{0.4436605282}{s^3}\right) e^{-4\ln(2\pi) - 8\ln(s) - \frac{0.2218302641}{s^2}} \tag{8}$$

```
> f:=delpdels=0;
```

$$f := \left(-\frac{8}{s} + \frac{0.4436605282}{s^3}\right) e^{-4\ln(2\pi) - 8\ln(s) - \frac{0.2218302641}{s^2}} = 0 \tag{9}$$

Maximum of the probability density

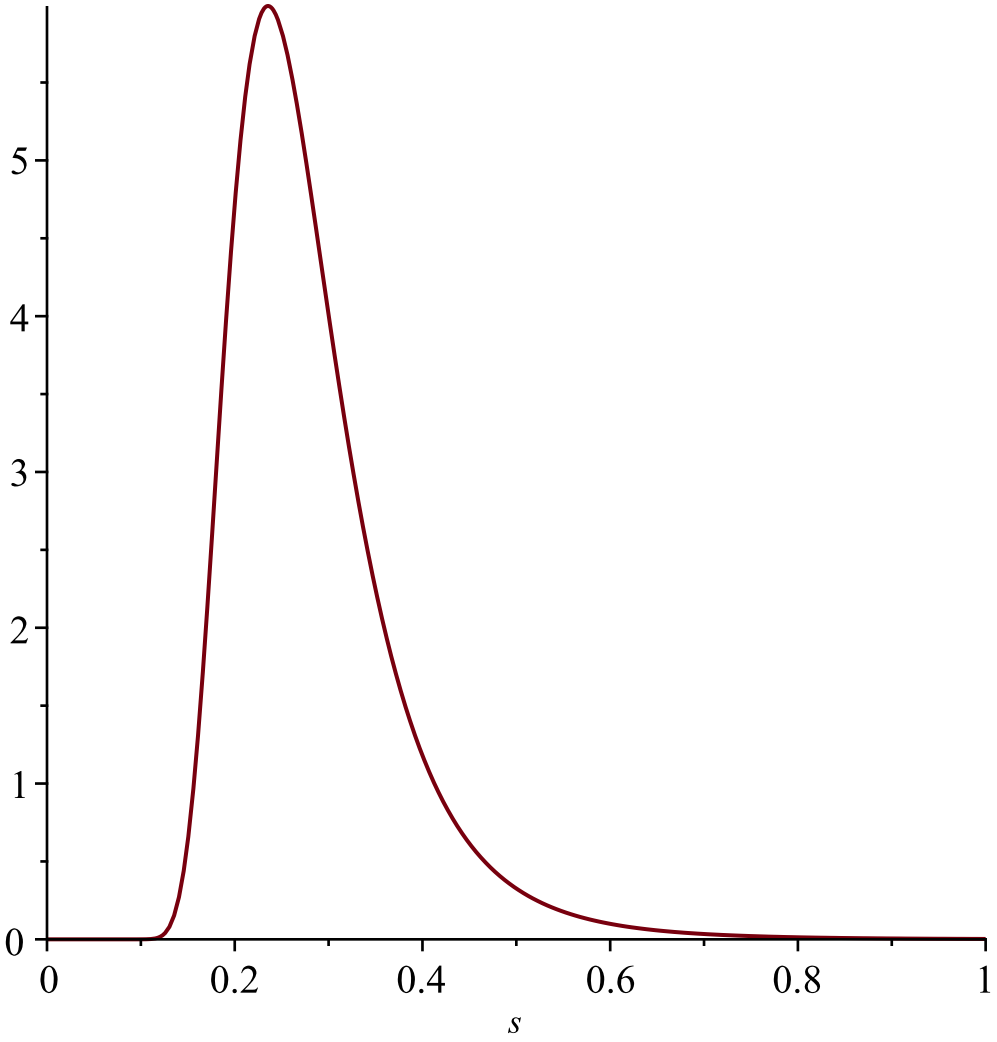
```
> fsolve({f}, {s = 0.01 .. 0.5});
```

(10)

$$\{s = 0.2354943015\} \quad \textbf{(10)}$$

Plot of the probability density

```
> plot(Prob_ex1_norm, s=0..1);
```




```
> int(Prob_ex1_norm, s=0..100);
```

$$0.9999999997 \quad \textbf{(11)}$$

```
> deriv2nd:=diff(delpdels,s);
```

$$deriv2nd := \left(\frac{8}{s^2} - \frac{1.330981585}{s^4} \right) e^{-4\ln(2\pi) - 8\ln(s) - \frac{0.2218302641}{s^2}} + \left(-\frac{8}{s} + \frac{0.4436605282}{s^3} \right)^2 e^{-4\ln(2\pi) - 8\ln(s) - \frac{0.2218302641}{s^2}} \quad \textbf{(12)}$$

```
> deriv2nd:=(8/s^2-1.330981585/(s^4))*exp(-4*ln(2*Pi)-8*ln(s)
  -.2218302641/(s^2))+(-8/s+.4436605282/(s^3))^2*exp(-4*ln(2*Pi)-8*
  ln(s)-.2218302641/(s^2));
```

$$deriv2nd := \left(\frac{8}{s^2} - \frac{1.330981585}{s^4} \right) e^{-4\ln(2\pi) - 8\ln(s) - \frac{0.2218302641}{s^2}} + \left(-\frac{8}{s} \right. \quad \textbf{(13)}$$

$$+\frac{0.4436605282}{s^3}\Bigg)^2 e^{-4\ln(2\pi)-8\ln(s)-\frac{0.2218302641}{s^2}}$$

```
> g:=deriv2nd=0;
```

$$g := \left(\frac{8}{s^2}-\frac{1.330981585}{s^4}\right) e^{-4\ln(2\pi)-8\ln(s)-\frac{0.2218302641}{s^2}} + \left(-\frac{8}{s}+\frac{0.4436605282}{s^3}\right)^2 e^{-4\ln(2\pi)-8\ln(s)-\frac{0.2218302641}{s^2}} = 0 \quad \textbf{(14)}$$

```
> fsolve({g}, {s = 0.01 .. 0.3});
```

$$\{s = 0.1794844627\} \quad \textbf{(15)}$$

```
> sleft:=.1794844627;
```

$$sleft := 0.1794844627 \quad \textbf{(16)}$$

```
> fsolve({g}, {s = 0.2 .. 1});
```

$$\{s = 0.2913115368\} \quad \textbf{(17)}$$

```
> sright:=.2913115368;
```

$$sright := 0.2913115368 \quad \textbf{(18)}$$