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> restart;
Maple calculations regarding example 5
=====

> for i from 1 to 20 do yc[i]:=a1*x[i] end do:
> for i from 1 to 20 do s[i]:=s0 end do:
> lns:=add(ln(s[i]),i=1..20);
lns := 20 ln(s0) (1)

> lt:=add((y[i]-yc[i])^2/2/s[i]^2,i=1..20):
The probability density function 'Prob_ex5' is defined:
> Prob_ex5:=exp(-(20/2)*ln(2*Pi)-lns-lt):

The data points x[i] and y[i] are introduced:
> y[1]:=-6.65647123;y[2]:=-0.011729798;y[3]:=4.34198;y[4]:=4.56334;
y[5]:=8.37979;y[6]:=10.2627;y[7]:=10.22623;y[8]:=14.28611;y[9]:=13.79561;y[10]:=21.09856;y[11]:=17.50144;y[12]:=20.10634;y[13]:=23.32865;y[14]:=18.41583;y[15]:=36.10625;y[16]:=34.69508;y[17]:=33.13127;y[18]:=43.87842;y[19]:=34.06836;y[20]:=43.07314;

y1 := -6.65647123
y2 := -0.011729798
y3 := 4.34198
y4 := 4.56334
y5 := 8.37979
y6 := 10.2627
y7 := 10.22623
y8 := 14.28611
y9 := 13.79561
y10 := 21.09856
y11 := 17.50144
y12 := 20.10634
y13 := 23.32865
y14 := 18.41583
y15 := 36.10625
y16 := 34.69508
y17 := 33.13127
y18 := 43.87842
y19 := 34.06836
y20 := 43.07314 (2)

> x[1]:=-10;x[2]:=-5;x[3]:=0;x[4]:=5;x[5]:=10;x[6]:=15;x[7]:=20;x[8]:=25;x[9]:=30;x[10]:=35;x[11]:=40;x[12]:=45;x[13]:=50;x[14]:=55;x[15]:=60;x[16]:=65;x[17]:=70;x[18]:=75;x[19]:=80;x[20]:=85;

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$$\begin{aligned}
x_1 &:= -10 \\
x_2 &:= -5 \\
x_3 &:= 0 \\
x_4 &:= 5 \\
x_5 &:= 10 \\
x_6 &:= 15 \\
x_7 &:= 20 \\
x_8 &:= 25 \\
x_9 &:= 30 \\
x_{10} &:= 35 \\
x_{11} &:= 40 \\
x_{12} &:= 45 \\
x_{13} &:= 50 \\
x_{14} &:= 55 \\
x_{15} &:= 60 \\
x_{16} &:= 65 \\
x_{17} &:= 70 \\
x_{18} &:= 75 \\
x_{19} &:= 80 \\
x_{20} &:= 85
\end{aligned} \tag{3}$$

[> **with(LinearAlgebra) :**

[> #int(Prob\_ex3,s0=0..5, s1=0..5,a0=-10..10,a1=-10..10);normf:=1/%;  
 > with(VectorCalculus);  
 [&x, `\*`, `+`, `-`, ` `, <, >, <|>, About, AddCoordinates, ArcLength, BasisFormat, Binormal, ConvertVector, CrossProduct, Curl, Curvature, D, Del, DirectionalDiff, Divergence, DotProduct, Flux, GetCoordinateParameters, GetCoordinates, GetNames, GetPVDescription, GetRootPoint, GetSpace, Gradient, Hessian, IsPositionVector, IsRootedVector, IsVectorField, Jacobian, Laplacian, LineInt, MapToBasis,  $\nabla$ , Norm, Normalize, PathInt, PlotPositionVector, PlotVector, PositionVector, PrincipalNormal, RadiusOfCurvature, RootedVector, ScalarPotential, SetCoordinateParameters, SetCoordinates, SpaceCurve, SurfaceInt, TNBFrame, TangentLine, TangentPlane, TangentVector, Torsion, Vector, VectorField, VectorPotential, VectorSpace, Wronskian, diff, eval, evalVF, int, limit, series]

[> g1 := Gradient(Prob\_ex5, [s0, a1]);

$$\begin{aligned}
g1 := & \left( \left( -\frac{20}{s0} + \frac{(10 a1 - 6.65647123)^2}{s0^3} + \frac{(5 a1 - 0.011729798)^2}{s0^3} + \frac{18.85279032}{s0^3} \right. \right. \\
& + \frac{(-5 a1 + 4.56334)^2}{s0^3} + \frac{(-10 a1 + 8.37979)^2}{s0^3} + \frac{(-15 a1 + 10.2627)^2}{s0^3} \\
& + \frac{(-20 a1 + 10.22623)^2}{s0^3} + \frac{(-25 a1 + 14.28611)^2}{s0^3} + \frac{(-30 a1 + 13.79561)^2}{s0^3} \\
& + \frac{(-35 a1 + 21.09856)^2}{s0^3} + \frac{(-40 a1 + 17.50144)^2}{s0^3} + \frac{(-45 a1 + 20.10634)^2}{s0^3} \\
& + \frac{(-50 a1 + 23.32865)^2}{s0^3} + \frac{(-55 a1 + 18.41583)^2}{s0^3} + \frac{(-60 a1 + 36.10625)^2}{s0^3} \\
& + \frac{(-65 a1 + 34.69508)^2}{s0^3} + \frac{(-70 a1 + 33.13127)^2}{s0^3} + \frac{(-75 a1 + 43.87842)^2}{s0^3} \\
& + \frac{(-80 a1 + 34.06836)^2}{s0^3} + \frac{(-85 a1 + 43.07314)^2}{s0^3} \Big) \\
& - 10 \ln(2\pi) - 20 \ln(s0) - \frac{(10 a1 - 6.65647123)^2}{2 s0^2} - \frac{(5 a1 - 0.011729798)^2}{2 s0^2} - \frac{9.426395160}{s0^2} \\
e & - \frac{(-5 a1 + 4.56334)^2}{2 s0^2} - \frac{(-10 a1 + 8.37979)^2}{2 s0^2} - \frac{(-15 a1 + 10.2627)^2}{2 s0^2} - \frac{(-20 a1 + 10.22623)^2}{2 s0^2} \\
& - \frac{(-25 a1 + 14.28611)^2}{2 s0^2} - \frac{(-30 a1 + 13.79561)^2}{2 s0^2} - \frac{(-35 a1 + 21.09856)^2}{2 s0^2} - \frac{(-40 a1 + 17.50144)^2}{2 s0^2} \\
& - \frac{(-45 a1 + 20.10634)^2}{2 s0^2} - \frac{(-50 a1 + 23.32865)^2}{2 s0^2} - \frac{(-55 a1 + 18.41583)^2}{2 s0^2} - \frac{(-60 a1 + 36.10625)^2}{2 s0^2} \\
& - \frac{(-65 a1 + 34.69508)^2}{2 s0^2} - \frac{(-70 a1 + 33.13127)^2}{2 s0^2} - \frac{(-75 a1 + 43.87842)^2}{2 s0^2} - \frac{(-80 a1 + 34.06836)^2}{2 s0^2} \\
& - \frac{(-85 a1 + 43.07314)^2}{2 s0^2} \Big) \bar{e}_{s0} + \left( \left( -\frac{10 (10 a1 - 6.65647123)}{s0^2} - \frac{5 (5 a1 - 0.011729798)}{s0^2} \right. \right. \\
& + \frac{5 (-5 a1 + 4.56334)}{s0^2} + \frac{10 (-10 a1 + 8.37979)}{s0^2} + \frac{15 (-15 a1 + 10.2627)}{s0^2}
\end{aligned} \tag{5}$$

$$\begin{aligned}
& + \frac{20 (-20 a1 + 10.22623)}{s0^2} + \frac{25 (-25 a1 + 14.28611)}{s0^2} + \frac{30 (-30 a1 + 13.79561)}{s0^2} \\
& + \frac{35 (-35 a1 + 21.09856)}{s0^2} + \frac{40 (-40 a1 + 17.50144)}{s0^2} + \frac{45 (-45 a1 + 20.10634)}{s0^2} \\
& + \frac{50 (-50 a1 + 23.32865)}{s0^2} + \frac{55 (-55 a1 + 18.41583)}{s0^2} + \frac{60 (-60 a1 + 36.10625)}{s0^2} \\
& + \frac{65 (-65 a1 + 34.69508)}{s0^2} + \frac{70 (-70 a1 + 33.13127)}{s0^2} + \frac{75 (-75 a1 + 43.87842)}{s0^2} \\
& + \frac{80 (-80 a1 + 34.06836)}{s0^2} + \frac{85 (-85 a1 + 43.07314)}{s0^2} \Big) \\
& - 10 \ln(2 \pi) - 20 \ln(s0) - \frac{(10 a1 - 6.65647123)^2}{2 s0^2} - \frac{(5 a1 - 0.011729798)^2}{2 s0^2} - \frac{9.426395160}{s0^2} \\
& e \\
& - \frac{(-5 a1 + 4.56334)^2}{2 s0^2} - \frac{(-10 a1 + 8.37979)^2}{2 s0^2} - \frac{(-15 a1 + 10.2627)^2}{2 s0^2} - \frac{(-20 a1 + 10.22623)^2}{2 s0^2} \\
& - \frac{(-25 a1 + 14.28611)^2}{2 s0^2} - \frac{(-30 a1 + 13.79561)^2}{2 s0^2} - \frac{(-35 a1 + 21.09856)^2}{2 s0^2} - \frac{(-40 a1 + 17.50144)^2}{2 s0^2} \\
& - \frac{(-45 a1 + 20.10634)^2}{2 s0^2} - \frac{(-50 a1 + 23.32865)^2}{2 s0^2} - \frac{(-55 a1 + 18.41583)^2}{2 s0^2} - \frac{(-60 a1 + 36.10625)^2}{2 s0^2} \\
& - \frac{(-65 a1 + 34.69508)^2}{2 s0^2} - \frac{(-70 a1 + 33.13127)^2}{2 s0^2} - \frac{(-75 a1 + 43.87842)^2}{2 s0^2} - \frac{(-80 a1 + 34.06836)^2}{2 s0^2} \\
& - \frac{(-85 a1 + 43.07314)^2}{2 s0^2} \Big) \bar{e}_{a1}
\end{aligned}$$

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> delpdels0:=(-20/s0+(10*a1-6.65647123)^2/s0^3+(5*a1-0.11729798e-1)
^2/s0^3+18.85279032/(s0^3)+(-5*a1+4.56334)^2/s0^3+(-10*
a1+8.37979)^2/s0^3+(-15*a1+10.2627)^2/s0^3+(-20*a1+10.22623)
^2/s0^3+(-25*a1+14.28611)^2/s0^3+(-30*a1+13.79561)^2/s0^3+(-35*
a1+21.09856)^2/s0^3+(-40*a1+17.50144)^2/s0^3+(-45*a1+20.10634)
^2/s0^3+(-50*a1+23.32865)^2/s0^3+(-55*a1+18.41583)^2/s0^3+(-60*
a1+36.10625)^2/s0^3+(-65*a1+34.69508)^2/s0^3+(-70*a1+33.13127)
^2/s0^3+(-75*a1+43.87842)^2/s0^3+(-80*a1+34.06836)^2/s0^3+(-85*
a1+43.07314)^2/s0^3)*exp(-10*ln(2*Pi)-20*ln(s0)-(10*a1
-6.65647123)^2/(2*s0^2)-(5*a1-0.11729798e-1)^2/(2*s0^2)
-9.426395160/(s0^2)-(-5*a1+4.56334)^2/(2*s0^2)-(-10*a1+8.37979)
^2/(2*s0^2)-(-15*a1+10.2627)^2/(2*s0^2)-(-20*a1+10.22623)^2/(2*
s0^2)-(-25*a1+14.28611)^2/(2*s0^2)-(-30*a1+13.79561)^2/(2*s0^2)-

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$$(-35*a1+21.09856)^2/(2*s0^2) - (-40*a1+17.50144)^2/(2*s0^2) - (-45*a1+20.10634)^2/(2*s0^2) - (-50*a1+23.32865)^2/(2*s0^2) - (-55*a1+18.41583)^2/(2*s0^2) - (-60*a1+36.10625)^2/(2*s0^2) - (-65*a1+34.69508)^2/(2*s0^2) - (-70*a1+33.13127)^2/(2*s0^2) - (-75*a1+43.87842)^2/(2*s0^2) - (-80*a1+34.06836)^2/(2*s0^2) - (-85*a1+43.07314)^2/(2*s0^2);$$

$$\begin{aligned} \text{delpdels0} := & \left( -\frac{20}{s0} + \frac{(10 a1 - 6.65647123)^2}{s0^3} + \frac{(5 a1 - 0.011729798)^2}{s0^3} + \frac{18.85279032}{s0^3} \right. \\ & + \frac{(-5 a1 + 4.56334)^2}{s0^3} + \frac{(-10 a1 + 8.37979)^2}{s0^3} + \frac{(-15 a1 + 10.2627)^2}{s0^3} \\ & + \frac{(-20 a1 + 10.22623)^2}{s0^3} + \frac{(-25 a1 + 14.28611)^2}{s0^3} + \frac{(-30 a1 + 13.79561)^2}{s0^3} \\ & + \frac{(-35 a1 + 21.09856)^2}{s0^3} + \frac{(-40 a1 + 17.50144)^2}{s0^3} + \frac{(-45 a1 + 20.10634)^2}{s0^3} \\ & + \frac{(-50 a1 + 23.32865)^2}{s0^3} + \frac{(-55 a1 + 18.41583)^2}{s0^3} + \frac{(-60 a1 + 36.10625)^2}{s0^3} \\ & + \frac{(-65 a1 + 34.69508)^2}{s0^3} + \frac{(-70 a1 + 33.13127)^2}{s0^3} + \frac{(-75 a1 + 43.87842)^2}{s0^3} \\ & \left. + \frac{(-80 a1 + 34.06836)^2}{s0^3} + \frac{(-85 a1 + 43.07314)^2}{s0^3} \right) \\ & - 10 \ln(2\pi) - 20 \ln(s0) - \frac{(10 a1 - 6.65647123)^2}{2 s0^2} - \frac{(5 a1 - 0.011729798)^2}{2 s0^2} - \frac{9.426395160}{s0^2} \\ & - \frac{(-5 a1 + 4.56334)^2}{2 s0^2} - \frac{(-10 a1 + 8.37979)^2}{2 s0^2} - \frac{(-15 a1 + 10.2627)^2}{2 s0^2} - \frac{(-20 a1 + 10.22623)^2}{2 s0^2} \\ & - \frac{(-25 a1 + 14.28611)^2}{2 s0^2} - \frac{(-30 a1 + 13.79561)^2}{2 s0^2} - \frac{(-35 a1 + 21.09856)^2}{2 s0^2} - \frac{(-40 a1 + 17.50144)^2}{2 s0^2} \\ & - \frac{(-45 a1 + 20.10634)^2}{2 s0^2} - \frac{(-50 a1 + 23.32865)^2}{2 s0^2} - \frac{(-55 a1 + 18.41583)^2}{2 s0^2} - \frac{(-60 a1 + 36.10625)^2}{2 s0^2} \\ & - \frac{(-65 a1 + 34.69508)^2}{2 s0^2} - \frac{(-70 a1 + 33.13127)^2}{2 s0^2} - \frac{(-75 a1 + 43.87842)^2}{2 s0^2} - \frac{(-80 a1 + 34.06836)^2}{2 s0^2} \\ & \left. - \frac{(-85 a1 + 43.07314)^2}{2 s0^2} \right) \end{aligned}$$

```
> delpdela1:=(-(10*(10*a1-6.65647123))/s0^2-(5*(5*a1-0.11729798e-1))/s0^2+(5*(-5*a1+4.56334))/s0^2+(10*(-10*a1+8.37979))/s0^2+(15*(-15*a1+10.2627))/s0^2+(20*(-20*a1+10.22623))/s0^2+(25*(-25*a1+14.28611))/s0^2+(30*(-30*a1+13.79561))/s0^2+(35*(-35*a1+21.09856))/s0^2+(40*(-40*a1+17.50144))/s0^2+(45*(-45*a1+43.07314))/s0^2;
```

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a1+20.10634))/s0^2+(50*(-50*a1+23.32865))/s0^2+(55*(-55*
a1+18.41583))/s0^2+(60*(-60*a1+36.10625))/s0^2+(65*(-65*
a1+34.69508))/s0^2+(70*(-70*a1+33.13127))/s0^2+(75*(-75*
a1+43.87842))/s0^2+(80*(-80*a1+34.06836))/s0^2+(85*(-85*
a1+43.07314))/s0^2)*exp(-10*ln(2*Pi)-20*ln(s0)-(10*a1-6.65647123)
^2/(2*s0^2)-(5*a1-0.11729798e-1)^2/(2*s0^2)-9.426395160/(s0^2)-
(-5*a1+4.56334)^2/(2*s0^2)-(-10*a1+8.37979)^2/(2*s0^2)-(-15*
a1+10.2627)^2/(2*s0^2)-(-20*a1+10.22623)^2/(2*s0^2)-(-25*
a1+14.28611)^2/(2*s0^2)-(-30*a1+13.79561)^2/(2*s0^2)-(-35*
a1+21.09856)^2/(2*s0^2)-(-40*a1+17.50144)^2/(2*s0^2)-(-45*
a1+20.10634)^2/(2*s0^2)-(-50*a1+23.32865)^2/(2*s0^2)-(-55*
a1+18.41583)^2/(2*s0^2)-(-60*a1+36.10625)^2/(2*s0^2)-(-65*
a1+34.69508)^2/(2*s0^2)-(-70*a1+33.13127)^2/(2*s0^2)-(-75*
a1+43.87842)^2/(2*s0^2)-(-80*a1+34.06836)^2/(2*s0^2)-(-85*
a1+43.07314)^2/(2*s0^2));

```

$$\begin{aligned}
delpdela1 := & \left( -\frac{100 a1 - 66.56471230}{s0^2} - \frac{25 a1 - 0.058648990}{s0^2} + \frac{-25 a1 + 22.81670}{s0^2} \right. \\
& + \frac{-100 a1 + 83.79790}{s0^2} + \frac{-225 a1 + 153.9405}{s0^2} + \frac{-400 a1 + 204.52460}{s0^2} \\
& + \frac{-625 a1 + 357.15275}{s0^2} + \frac{-900 a1 + 413.86830}{s0^2} + \frac{-1225 a1 + 738.44960}{s0^2} \\
& + \frac{-1600 a1 + 700.05760}{s0^2} + \frac{-2025 a1 + 904.78530}{s0^2} + \frac{-2500 a1 + 1166.43250}{s0^2} \\
& + \frac{-3025 a1 + 1012.87065}{s0^2} + \frac{-3600 a1 + 2166.37500}{s0^2} + \frac{-4225 a1 + 2255.18020}{s0^2} \\
& + \frac{-4900 a1 + 2319.18890}{s0^2} + \frac{-5625 a1 + 3290.88150}{s0^2} + \frac{-6400 a1 + 2725.46880}{s0^2} \\
& \left. + \frac{-7225 a1 + 3661.21690}{s0^2} \right) \\
& - 10 \ln(2\pi) - 20 \ln(s0) - \frac{(10 a1 - 6.65647123)^2}{2 s0^2} - \frac{(5 a1 - 0.011729798)^2}{2 s0^2} - \frac{9.426395160}{s0^2} \\
& e^{-\frac{(-5 a1 + 4.56334)^2}{2 s0^2} - \frac{(-10 a1 + 8.37979)^2}{2 s0^2} - \frac{(-15 a1 + 10.2627)^2}{2 s0^2} - \frac{(-20 a1 + 10.22623)^2}{2 s0^2}} \\
& - \frac{(-25 a1 + 14.28611)^2}{2 s0^2} - \frac{(-30 a1 + 13.79561)^2}{2 s0^2} - \frac{(-35 a1 + 21.09856)^2}{2 s0^2} - \frac{(-40 a1 + 17.50144)^2}{2 s0^2} \\
& - \frac{(-45 a1 + 20.10634)^2}{2 s0^2} - \frac{(-50 a1 + 23.32865)^2}{2 s0^2} - \frac{(-55 a1 + 18.41583)^2}{2 s0^2} - \frac{(-60 a1 + 36.10625)^2}{2 s0^2} \\
& - \frac{(-65 a1 + 34.69508)^2}{2 s0^2} - \frac{(-70 a1 + 33.13127)^2}{2 s0^2} - \frac{(-75 a1 + 43.87842)^2}{2 s0^2} - \frac{(-80 a1 + 34.06836)^2}{2 s0^2}
\end{aligned} \tag{7}$$

$$-\frac{(-85 a1 + 43.07314)^2}{2 s0^2}$$

$$\begin{aligned}
& \text{g :=} \text{delpdel} a1=0 ; \text{u :=} \text{delpdels} 0=0 ; \\
g := & \left( -\frac{100 a1 - 66.56471230}{s0^2} - \frac{25 a1 - 0.058648990}{s0^2} + \frac{-25 a1 + 22.81670}{s0^2} \right. \\
& + \frac{-100 a1 + 83.79790}{s0^2} + \frac{-225 a1 + 153.9405}{s0^2} + \frac{-400 a1 + 204.52460}{s0^2} \\
& + \frac{-625 a1 + 357.15275}{s0^2} + \frac{-900 a1 + 413.86830}{s0^2} + \frac{-1225 a1 + 738.44960}{s0^2} \\
& + \frac{-1600 a1 + 700.05760}{s0^2} + \frac{-2025 a1 + 904.78530}{s0^2} + \frac{-2500 a1 + 1166.43250}{s0^2} \\
& + \frac{-3025 a1 + 1012.87065}{s0^2} + \frac{-3600 a1 + 2166.37500}{s0^2} + \frac{-4225 a1 + 2255.18020}{s0^2} \\
& + \frac{-4900 a1 + 2319.18890}{s0^2} + \frac{-5625 a1 + 3290.88150}{s0^2} + \frac{-6400 a1 + 2725.46880}{s0^2} \\
& \left. + \frac{-7225 a1 + 3661.21690}{s0^2} \right) \\
& - 10 \ln(2 \pi) - 20 \ln(s0) - \frac{(10 a1 - 6.65647123)^2}{2 s0^2} - \frac{(5 a1 - 0.011729798)^2}{2 s0^2} - \frac{9.426395160}{s0^2} \\
e & \\
& - \frac{(-5 a1 + 4.56334)^2}{2 s0^2} - \frac{(-10 a1 + 8.37979)^2}{2 s0^2} - \frac{(-15 a1 + 10.2627)^2}{2 s0^2} - \frac{(-20 a1 + 10.22623)^2}{2 s0^2} \\
& - \frac{(-25 a1 + 14.28611)^2}{2 s0^2} - \frac{(-30 a1 + 13.79561)^2}{2 s0^2} - \frac{(-35 a1 + 21.09856)^2}{2 s0^2} - \frac{(-40 a1 + 17.50144)^2}{2 s0^2} \\
& - \frac{(-45 a1 + 20.10634)^2}{2 s0^2} - \frac{(-50 a1 + 23.32865)^2}{2 s0^2} - \frac{(-55 a1 + 18.41583)^2}{2 s0^2} - \frac{(-60 a1 + 36.10625)^2}{2 s0^2} \\
& - \frac{(-65 a1 + 34.69508)^2}{2 s0^2} - \frac{(-70 a1 + 33.13127)^2}{2 s0^2} - \frac{(-75 a1 + 43.87842)^2}{2 s0^2} - \frac{(-80 a1 + 34.06836)^2}{2 s0^2} \\
& - \frac{(-85 a1 + 43.07314)^2}{2 s0^2} = 0
\end{aligned}$$

$$\begin{aligned}
u := & \left( -\frac{20}{s0} + \frac{(10 a1 - 6.65647123)^2}{s0^3} + \frac{(5 a1 - 0.011729798)^2}{s0^3} + \frac{18.85279032}{s0^3} \right. \\
& \left. + \frac{(-5 a1 + 4.56334)^2}{s0^3} + \frac{(-10 a1 + 8.37979)^2}{s0^3} + \frac{(-15 a1 + 10.2627)^2}{s0^3} \right)
\end{aligned} \tag{8}$$

$$\begin{aligned}
& + \frac{(-20 a1 + 10.22623)^2}{s0^3} + \frac{(-25 a1 + 14.28611)^2}{s0^3} + \frac{(-30 a1 + 13.79561)^2}{s0^3} \\
& + \frac{(-35 a1 + 21.09856)^2}{s0^3} + \frac{(-40 a1 + 17.50144)^2}{s0^3} + \frac{(-45 a1 + 20.10634)^2}{s0^3} \\
& + \frac{(-50 a1 + 23.32865)^2}{s0^3} + \frac{(-55 a1 + 18.41583)^2}{s0^3} + \frac{(-60 a1 + 36.10625)^2}{s0^3} \\
& + \frac{(-65 a1 + 34.69508)^2}{s0^3} + \frac{(-70 a1 + 33.13127)^2}{s0^3} + \frac{(-75 a1 + 43.87842)^2}{s0^3} \\
& + \frac{(-80 a1 + 34.06836)^2}{s0^3} + \frac{(-85 a1 + 43.07314)^2}{s0^3} \Big) \\
& - 10 \ln(2\pi) - 20 \ln(s0) - \frac{(10 a1 - 6.65647123)^2}{2 s0^2} - \frac{(5 a1 - 0.011729798)^2}{2 s0^2} - \frac{9.426395160}{s0^2} \\
& e \\
& - \frac{(-5 a1 + 4.56334)^2}{2 s0^2} - \frac{(-10 a1 + 8.37979)^2}{2 s0^2} - \frac{(-15 a1 + 10.2627)^2}{2 s0^2} - \frac{(-20 a1 + 10.22623)^2}{2 s0^2} \\
& - \frac{(-25 a1 + 14.28611)^2}{2 s0^2} - \frac{(-30 a1 + 13.79561)^2}{2 s0^2} - \frac{(-35 a1 + 21.09856)^2}{2 s0^2} - \frac{(-40 a1 + 17.50144)^2}{2 s0^2} \\
& - \frac{(-45 a1 + 20.10634)^2}{2 s0^2} - \frac{(-50 a1 + 23.32865)^2}{2 s0^2} - \frac{(-55 a1 + 18.41583)^2}{2 s0^2} - \frac{(-60 a1 + 36.10625)^2}{2 s0^2} \\
& - \frac{(-65 a1 + 34.69508)^2}{2 s0^2} - \frac{(-70 a1 + 33.13127)^2}{2 s0^2} - \frac{(-75 a1 + 43.87842)^2}{2 s0^2} - \frac{(-80 a1 + 34.06836)^2}{2 s0^2} \\
& - \frac{(-85 a1 + 43.07314)^2}{2 s0^2} = 0
\end{aligned}$$

The partial derivatives are all set to zero, and the objective function Z\_1 (Eq. 18) is minimized providing the optimized parameters a1 and s0 (see also Table 6):

$$> \text{fsolve}(\{g, u\}, \{s0 = 3 .. 4, a1 = 0.45 .. 0.5\}); \quad (9)$$

$$> Z := -2 * \ln(\text{Prob\_ex5}); \quad (10)$$

$$\begin{aligned}
& -2 \\
& \ln \left( e^{-10 \ln(2\pi) - 20 \ln(s0) - \frac{(10 a1 - 6.65647123)^2}{2 s0^2} - \frac{(5 a1 - 0.011729798)^2}{2 s0^2} - \frac{9.426395160}{s0^2}} \right. \\
& \left. - \frac{(-5 a1 + 4.56334)^2}{2 s0^2} - \frac{(-10 a1 + 8.37979)^2}{2 s0^2} - \frac{(-15 a1 + 10.2627)^2}{2 s0^2} - \frac{(-20 a1 + 10.22623)^2}{2 s0^2} \right. \\
& \left. - \frac{(-25 a1 + 14.28611)^2}{2 s0^2} - \frac{(-30 a1 + 13.79561)^2}{2 s0^2} - \frac{(-35 a1 + 21.09856)^2}{2 s0^2} - \frac{(-40 a1 + 17.50144)^2}{2 s0^2} \right. \\
& \left. - \frac{(-45 a1 + 20.10634)^2}{2 s0^2} - \frac{(-50 a1 + 23.32865)^2}{2 s0^2} - \frac{(-55 a1 + 18.41583)^2}{2 s0^2} - \frac{(-60 a1 + 36.10625)^2}{2 s0^2} \right. \\
& \left. - \frac{(-65 a1 + 34.69508)^2}{2 s0^2} - \frac{(-70 a1 + 33.13127)^2}{2 s0^2} - \frac{(-75 a1 + 43.87842)^2}{2 s0^2} - \frac{(-80 a1 + 34.06836)^2}{2 s0^2} \right. \\
& \left. - \frac{(-85 a1 + 43.07314)^2}{2 s0^2} \right) = 0
\end{aligned}$$

$$\begin{aligned}
& - \frac{(-25 a1 + 14.28611)^2}{2 s0^2} - \frac{(-30 a1 + 13.79561)^2}{2 s0^2} - \frac{(-35 a1 + 21.09856)^2}{2 s0^2} - \frac{(-40 a1 + 17.50144)^2}{2 s0^2} \\
& - \frac{(-45 a1 + 20.10634)^2}{2 s0^2} - \frac{(-50 a1 + 23.32865)^2}{2 s0^2} - \frac{(-55 a1 + 18.41583)^2}{2 s0^2} - \frac{(-60 a1 + 36.10625)^2}{2 s0^2} \\
& - \frac{(-65 a1 + 34.69508)^2}{2 s0^2} - \frac{(-70 a1 + 33.13127)^2}{2 s0^2} - \frac{(-75 a1 + 43.87842)^2}{2 s0^2} - \frac{(-80 a1 + 34.06836)^2}{2 s0^2} \\
& - \frac{(-85 a1 + 43.07314)^2}{2 s0^2}
\end{aligned}$$

The Hessian matrix is calculated for the minimum of the objective function Z1:

```
> hess:=Hessian(Z, [a1,s0]):  
> hess0:=subs(a1= .4970643813,hess);hess1:=subs(s0= 3.791034263,  
hess0);hess_matrix:=evalf(hess1);
```

$$\begin{aligned}
hess0 &:= \begin{bmatrix} \frac{89500}{s0^2} & -\frac{4.68 \cdot 10^{-6}}{s0^3} \\ -\frac{4.68 \cdot 10^{-6}}{s0^3} & -\frac{40}{s0^2} + \frac{1724.632894}{s0^4} \end{bmatrix} \\
hess1 &:= \begin{bmatrix} 6227.412244 & -8.589595735 \cdot 10^{-8} \\ -8.589595735 \cdot 10^{-8} & 5.566402006 \end{bmatrix} \\
hess\_matrix &:= \begin{bmatrix} 6227.412244 & -8.589595735 \cdot 10^{-8} \\ -8.589595735 \cdot 10^{-8} & 5.566402006 \end{bmatrix} \tag{11}
\end{aligned}$$

The covariance matrix (inverse of the Hessian matrix) is calculated:

```
> cov1:=MatrixInverse(hess_matrix);  
cov1 := \begin{bmatrix} 0.000160580343940371 & 2.47793859650863 \cdot 10^{-12} \\ 2.47793859650863 \cdot 10^{-12} & 0.179649259777160 \end{bmatrix} \tag{12}
```

```
> with(Statistics);  
[AbsoluteDeviation, AgglomeratedPlot, AreaChart, AutoCorrelation, BarChart, Biplot,  
Bootstrap, BoxPlot, BubblePlot, CDF, CGF, CentralMoment, CharacteristicFunction,  
ChiSquareGoodnessOfFitTest, ChiSquareIndependenceTest, ChiSquareSuitableModelTest,  
ColumnGraph, Correlation, CorrelationMatrix, Count, CountMissing, Covariance,  
CovarianceMatrix, CrossCorrelation, Cumulant, CumulantGeneratingFunction,  
CumulativeDistributionFunction, CumulativeProduct, CumulativeSum,  
CumulativeSumChart, DataSummary, Decile, DensityPlot, DiscreteValueMap, Distribution,  
ErrorPlot, EvaluateToFloat, Excise, ExpectedValue, ExponentialFit,  
ExponentialSmoothing, FailureRate, FisherInformation, Fit, FivePointSummary,  
FrequencyPlot, FrequencyTable, GeometricMean, GridPlot, HarmonicMean, HazardRate,  
HeatMap, Histogram, HodgesLehmann, Information, InteractiveDataAnalysis,] \tag{13}
```

*InterquartileRange, InverseSurvivalFunction, Join, KernelDensity, KernelDensityPlot,*  
*KernelDensitySample, Kurtosis, Likelihood, LikelihoodRatioStatistic, LineChart,*  
*LinearFilter, LinearFit, LogLikelihood, LogarithmicFit, Lowess, MGF, MLE,*  
*MakeProcedure, MaximumLikelihoodEstimate, Mean, MeanDeviation, Median,*  
*MedianDeviation, MillsRatio, Mode, Moment, MomentGeneratingFunction,*  
*MovingAverage, MovingMedian, MovingStatistic, NonlinearFit, NormalPlot,*  
*OneSampleChiSquareTest, OneSampleTTest, OneSampleZTest, OneWayANOVA,*  
*OrderByRank, OrderStatistic, PCA, PDF, ParetoChart, Percentile, PieChart, PointPlot,*  
*PolynomialFit, PowerFit, PredictiveLeastSquares, PrincipalComponentAnalysis,*  
*Probability, ProbabilityDensityFunction, ProbabilityFunction, ProbabilityPlot,*  
*ProfileLikelihood, ProfileLogLikelihood, QuadraticMean, Quantile, QuantilePlot, Quartile,*  
*RandomVariable, Range, Rank, Remove, RemoveInRange, RemoveNonNumeric,*  
*RepeatedMedianEstimator, RousseeuwCrouxQn, RousseeuwCrouxSn, Sample, Scale,*  
*ScatterPlot, ScatterPlot3D, Score, ScreePlot, Select, SelectInRange, SelectNonNumeric,*  
*ShapiroWilkWTest, Shuffle, Skewness, Sort, Specialize, SplitByColumn, StandardDeviation,*  
*StandardError, StandardizedMoment, SunflowerPlot, Support, SurfacePlot,*  
*SurvivalFunction, SymmetryPlot, Tally, TallyInto, TreeMap, Trim, TrimmedMean,*  
*TwoSampleFTest, TwoSamplePairedTTest, TwoSampleTTest, TwoSampleZTest, Variance,*  
*Variation, VennDiagram, ViolinPlot, WeibullPlot, WeightedMovingAverage, Winsorize,*  
*WinsorizedMean ]*

The correlation coefficient is now close to zero, compare Table 6:

$$> \text{rho12} := \text{cov1}(1, 2) / \sqrt{\text{cov1}(1, 1)} / \sqrt{\text{cov1}(2, 2)} ; \\ \rho_{12} := 4.61351176438973 \cdot 10^{-10} \quad (14)$$

The uncertainties of the parameters follow from the correlation matrix, and are provided in Table 5:

$$> \text{dela1} := \sqrt{\text{cov1}(1, 1)} ; \\ \text{dela1} := 0.0126720299849855 \quad (15)$$

$$> \text{dels0} := \sqrt{\text{cov1}(2, 2)} ; \\ \text{dels0} := 0.423850515839205 \quad (16)$$

$$> \text{with(LinearAlgebra)} ; \\ [&x, \text{Add}, \text{Adjoint}, \text{BackwardSubstitute}, \text{BandMatrix}, \text{Basis}, \text{BezoutMatrix}, \text{BidiagonalForm}, \quad (17)$$

*BilinearForm, CARE, CharacteristicMatrix, CharacteristicPolynomial, Column,*  
*ColumnDimension, ColumnOperation, ColumnSpace, CompanionMatrix,*  
*CompressedSparseForm, ConditionNumber, ConstantMatrix, ConstantVector, Copy,*  
*CreatePermutation, CrossProduct, DARE, DeleteColumn, DeleteRow, Determinant,*  
*Diagonal, DiagonalMatrix, Dimension, Dimensions, DotProduct, EigenConditionNumbers,*  
*Eigenvalues, Eigenvectors, Equal, ForwardSubstitute, FrobeniusForm,*  
*FromCompressedSparseForm, FromSplitForm, GaussianElimination, GenerateEquations,*  
*GenerateMatrix, Generic, GetResultDataType, GetResultShape, GivensRotationMatrix,*  
*GramSchmidt, HankelMatrix, HermiteForm, HermitianTranspose, HessenbergForm,*  
*HilbertMatrix, HouseholderMatrix, IdentityMatrix, IntersectionBasis, IsDefinite,*  
*IsOrthogonal, IsSimilar, IsUnitary, JordanBlockMatrix, JordanForm, KroneckerProduct,*  
*LA\_Main, LUDecomposition, LeastSquares, LinearSolve, LyapunovSolve, Map, Map2,*  
*MatrixAdd, MatrixExponential, MatrixFunction, MatrixInverse, MatrixMatrixMultiply,*

```
MatrixNorm, MatrixPower, MatrixScalarMultiply, MatrixVectorMultiply,
MinimalPolynomial, Minor, Modular, Multiply, NoUserValue, Norm, Normalize,
NullSpace, OuterProductMatrix, Permanent, Pivot, PopovForm, ProjectionMatrix,
QRDecomposition, RandomMatrix, RandomVector, Rank, RationalCanonicalForm,
ReducedRowEchelonForm, Row, RowDimension, RowOperation, RowSpace, ScalarMatrix,
ScalarMultiply, ScalarVector, SchurForm, SingularValues, SmithForm, SplitForm,
StronglyConnectedBlocks, SubMatrix, SubVector, SumBasis, SylvesterMatrix,
SylvesterSolve, ToeplitzMatrix, Trace, Transpose, TridiagonalForm, UnitVector,
VandermondeMatrix, VectorAdd, VectorAngle, VectorMatrixMultiply, VectorNorm,
VectorScalarMultiply, ZeroMatrix, ZeroVector, Zip ]
```

```
> Determinant(hess_matrix); 34664.28001 (18)
```

```
> ConditionNumber(hess_matrix); 1118.750000 (19)
```