```
> restart;
```
Maple calculations regarding example 5
================================
Author: Ernst Gamsjäger
Date: December 2025
Maple 2018
================================
```
> for i from 1 to 20 do yc[i]:=a1*x[i] end do:
> for i from 1 to 20 do s[i]:=s0 end do:
> lns:=add(ln(s[i]),i=1..20);
```
$$lns := 20\ln(s0)$$ **(1)**

```
> lt:=add((y[i]-yc[i])^2/2/s[i]^2,i=1..20):
```
The probability density function 'Prob_ex5' is defined:
```
> Prob_ex5:=exp(-(20/2)*ln(2*Pi)-lns-lt):
```
The data points x[i] and y[i] are introduced:
```
> y[1]:=-6.65647123;y[2]:=-0.011729798;y[3]:=4.34198;y[4]:=4.56334;
  y[5]:=8.37979;y[6]:=10.2627;y[7]:=10.22623;y[8]:=14.28611;y[9]:=
  13.79561;y[10]:=21.09856;y[11]:=17.50144;y[12]:=20.10634;y[13]:=
  23.32865;y[14]:=18.41583;y[15]:=36.10625;y[16]:=34.69508;y[17]:=
  33.13127;y[18]:=43.87842;y[19]:=34.06836;y[20]:=43.07314;
```

$$y_1 := -6.65647123$$

$$y_2 := -0.011729798$$

$$y_3 := 4.34198$$

$$y_4 := 4.56334$$

$$y_5 := 8.37979$$

$$y_6 := 10.2627$$

$$y_7 := 10.22623$$

$$y_8 := 14.28611$$

$$y_9 := 13.79561$$

$$y_{10} := 21.09856$$

$$y_{11} := 17.50144$$

$$y_{12} := 20.10634$$

$$y_{13} := 23.32865$$

$$y_{14} := 18.41583$$

$$y_{15} := 36.10625$$

$$y_{16} := 34.69508$$

$$y_{17} := 33.13127$$

$$y_{18} := 43.87842$$

$$y_{19} := 34.06836$$

$$y_{20} := 43.07314$$ **(2)**

```
> x[1]:=-10;x[2]:=-5;x[3]:=0;x[4]:=5;x[5]:=10;x[6]:=15;x[7]:=20;x
  [8]:=25;x[9]:=30;x[10]:=35;x[11]:=40;x[12]:=45;x[13]:=50;x[14]:=
  55;x[15]:=60;x[16]:=65;x[17]:=70;x[18]:=75;x[19]:=80;x[20]:=85;
```

$$x_1 := -10$$
$$x_2 := -5$$
$$x_3 := 0$$
$$x_4 := 5$$
$$x_5 := 10$$
$$x_6 := 15$$
$$x_7 := 20$$
$$x_8 := 25$$
$$x_9 := 30$$
$$x_{10} := 35$$
$$x_{11} := 40$$
$$x_{12} := 45$$
$$x_{13} := 50$$
$$x_{14} := 55$$
$$x_{15} := 60$$
$$x_{16} := 65$$
$$x_{17} := 70$$
$$x_{18} := 75$$
$$x_{19} := 80$$
$$x_{20} := 85$$

(3)

```
> with(LinearAlgebra):
```

```
> #int(Prob_ex3,s0=0..5, s1=0..5,a0=-10..10,a1=-10..10);normf:=1/%;
> with(VectorCalculus);
```

$[\&x, `*`, `+`, `-`, `.`, <,>, <|>, About, AddCoordinates, ArcLength, BasisFormat, Binormal,$ **(4)**
$ConvertVector, CrossProduct, Curl, Curvature, D, Del, DirectionalDiff, Divergence,$
$DotProduct, Flux, GetCoordinateParameters, GetCoordinates, GetNames,$
$GetPVDescription, GetRootPoint, GetSpace, Gradient, Hessian, IsPositionVector,$
$IsRootedVector, IsVectorField, Jacobian, Laplacian, LineInt, MapToBasis, \nabla, Norm,$
$Normalize, PathInt, PlotPositionVector, PlotVector, PositionVector, PrincipalNormal,$
$RadiusOfCurvature, RootedVector, ScalarPotential, SetCoordinateParameters,$
$SetCoordinates, SpaceCurve, SurfaceInt, TNBFrame, TangentLine, TangentPlane,$
$TangentVector, Torsion, Vector, VectorField, VectorPotential, VectorSpace, Wronskian,$
$diff, eval, evalVF, int, limit, series]$

```
> g1 := Gradient(Prob_ex5, [s0, a1]);
```

$$g1 := \left(\left(-\frac{20}{s0} + \frac{(10\,a1 - 6.65647123)^2}{s0^3} + \frac{(5\,a1 - 0.011729798)^2}{s0^3} + \frac{18.85279032}{s0^3}\right.\right. \tag{5}$$

$$+ \frac{(-5\,a1 + 4.56334)^2}{s0^3} + \frac{(-10\,a1 + 8.37979)^2}{s0^3} + \frac{(-15\,a1 + 10.2627)^2}{s0^3}$$

$$+ \frac{(-20\,a1 + 10.22623)^2}{s0^3} + \frac{(-25\,a1 + 14.28611)^2}{s0^3} + \frac{(-30\,a1 + 13.79561)^2}{s0^3}$$

$$+ \frac{(-35\,a1 + 21.09856)^2}{s0^3} + \frac{(-40\,a1 + 17.50144)^2}{s0^3} + \frac{(-45\,a1 + 20.10634)^2}{s0^3}$$

$$+ \frac{(-50\,a1 + 23.32865)^2}{s0^3} + \frac{(-55\,a1 + 18.41583)^2}{s0^3} + \frac{(-60\,a1 + 36.10625)^2}{s0^3}$$

$$+ \frac{(-65\,a1 + 34.69508)^2}{s0^3} + \frac{(-70\,a1 + 33.13127)^2}{s0^3} + \frac{(-75\,a1 + 43.87842)^2}{s0^3}$$

$$\left.+ \frac{(-80\,a1 + 34.06836)^2}{s0^3} + \frac{(-85\,a1 + 43.07314)^2}{s0^3}\right)$$

$$e^{-10\ln(2\pi) - 20\ln(s0) - \frac{(10\,a1 - 6.65647123)^2}{2\,s0^2} - \frac{(5\,a1 - 0.011729798)^2}{2\,s0^2} - \frac{9.426395160}{s0^2}}$$

$$-\frac{(-5\,a1 + 4.56334)^2}{2\,s0^2} - \frac{(-10\,a1 + 8.37979)^2}{2\,s0^2} - \frac{(-15\,a1 + 10.2627)^2}{2\,s0^2} - \frac{(-20\,a1 + 10.22623)^2}{2\,s0^2}$$

$$-\frac{(-25\,a1 + 14.28611)^2}{2\,s0^2} - \frac{(-30\,a1 + 13.79561)^2}{2\,s0^2} - \frac{(-35\,a1 + 21.09856)^2}{2\,s0^2} - \frac{(-40\,a1 + 17.50144)^2}{2\,s0^2}$$

$$-\frac{(-45\,a1 + 20.10634)^2}{2\,s0^2} - \frac{(-50\,a1 + 23.32865)^2}{2\,s0^2} - \frac{(-55\,a1 + 18.41583)^2}{2\,s0^2} - \frac{(-60\,a1 + 36.10625)^2}{2\,s0^2}$$

$$-\frac{(-65\,a1 + 34.69508)^2}{2\,s0^2} - \frac{(-70\,a1 + 33.13127)^2}{2\,s0^2} - \frac{(-75\,a1 + 43.87842)^2}{2\,s0^2} - \frac{(-80\,a1 + 34.06836)^2}{2\,s0^2}$$

$$\left.-\frac{(-85\,a1 + 43.07314)^2}{2\,s0^2}\right)\overline{e}_{s0} + \left(\left(-\frac{10\,(10\,a1 - 6.65647123)}{s0^2} - \frac{5\,(5\,a1 - 0.011729798)}{s0^2}\right.\right.$$

$$+ \frac{5\,(-5\,a1 + 4.56334)}{s0^2} + \frac{10\,(-10\,a1 + 8.37979)}{s0^2} + \frac{15\,(-15\,a1 + 10.2627)}{s0^2}$$

$$+ \frac{20\,(-20\,a1 + 10.22623)}{s0^2} + \frac{25\,(-25\,a1 + 14.28611)}{s0^2} + \frac{30\,(-30\,a1 + 13.79561)}{s0^2}$$

$$+ \frac{35\,(-35\,a1 + 21.09856)}{s0^2} + \frac{40\,(-40\,a1 + 17.50144)}{s0^2} + \frac{45\,(-45\,a1 + 20.10634)}{s0^2}$$

$$+ \frac{50\,(-50\,a1 + 23.32865)}{s0^2} + \frac{55\,(-55\,a1 + 18.41583)}{s0^2} + \frac{60\,(-60\,a1 + 36.10625)}{s0^2}$$

$$+ \frac{65\,(-65\,a1 + 34.69508)}{s0^2} + \frac{70\,(-70\,a1 + 33.13127)}{s0^2} + \frac{75\,(-75\,a1 + 43.87842)}{s0^2}$$

$$+ \frac{80\,(-80\,a1 + 34.06836)}{s0^2} + \frac{85\,(-85\,a1 + 43.07314)}{s0^2}\Bigg)$$

$$e^{-10\ln(2\pi) - 20\ln(s0) - \frac{(10\,a1 - 6.65647123)^2}{2\,s0^2} - \frac{(5\,a1 - 0.011729798)^2}{2\,s0^2} - \frac{9.426395160}{s0^2}}$$

$$-\frac{(-5\,a1 + 4.56334)^2}{2\,s0^2} - \frac{(-10\,a1 + 8.37979)^2}{2\,s0^2} - \frac{(-15\,a1 + 10.2627)^2}{2\,s0^2} - \frac{(-20\,a1 + 10.22623)^2}{2\,s0^2}$$

$$-\frac{(-25\,a1 + 14.28611)^2}{2\,s0^2} - \frac{(-30\,a1 + 13.79561)^2}{2\,s0^2} - \frac{(-35\,a1 + 21.09856)^2}{2\,s0^2} - \frac{(-40\,a1 + 17.50144)^2}{2\,s0^2}$$

$$-\frac{(-45\,a1 + 20.10634)^2}{2\,s0^2} - \frac{(-50\,a1 + 23.32865)^2}{2\,s0^2} - \frac{(-55\,a1 + 18.41583)^2}{2\,s0^2} - \frac{(-60\,a1 + 36.10625)^2}{2\,s0^2}$$

$$-\frac{(-65\,a1 + 34.69508)^2}{2\,s0^2} - \frac{(-70\,a1 + 33.13127)^2}{2\,s0^2} - \frac{(-75\,a1 + 43.87842)^2}{2\,s0^2} - \frac{(-80\,a1 + 34.06836)^2}{2\,s0^2}$$

$$-\frac{(-85\,a1 + 43.07314)^2}{2\,s0^2}\Bigg)\overline{e}_{a1}$$

```
> delpdels0:=(-20/s0+(10*a1-6.65647123)^2/s0^3+(5*a1-0.11729798e-1)
  ^2/s0^3+18.85279032/(s0^3)+(-5*a1+4.56334)^2/s0^3+(-10*
  a1+8.37979)^2/s0^3+(-15*a1+10.2627)^2/s0^3+(-20*a1+10.22623)
  ^2/s0^3+(-25*a1+14.28611)^2/s0^3+(-30*a1+13.79561)^2/s0^3+(-35*
  a1+21.09856)^2/s0^3+(-40*a1+17.50144)^2/s0^3+(-45*a1+20.10634)
  ^2/s0^3+(-50*a1+23.32865)^2/s0^3+(-55*a1+18.41583)^2/s0^3+(-60*
  a1+36.10625)^2/s0^3+(-65*a1+34.69508)^2/s0^3+(-70*a1+33.13127)
  ^2/s0^3+(-75*a1+43.87842)^2/s0^3+(-80*a1+34.06836)^2/s0^3+(-85*
  a1+43.07314)^2/s0^3)*exp(-10*ln(2*Pi)-20*ln(s0)-(10*a1
  -6.65647123)^2/(2*s0^2)-(5*a1-0.11729798e-1)^2/(2*s0^2)
  -9.426395160/(s0^2)-(-5*a1+4.56334)^2/(2*s0^2)-(-10*a1+8.37979)
  ^2/(2*s0^2)-(-15*a1+10.2627)^2/(2*s0^2)-(-20*a1+10.22623)^2/(2*
  s0^2)-(-25*a1+14.28611)^2/(2*s0^2)-(-30*a1+13.79561)^2/(2*s0^2)-
```

```
(-35*a1+21.09856)^2/(2*s0^2)-(-40*a1+17.50144)^2/(2*s0^2)-(-45*
a1+20.10634)^2/(2*s0^2)-(-50*a1+23.32865)^2/(2*s0^2)-(-55*
a1+18.41583)^2/(2*s0^2)-(-60*a1+36.10625)^2/(2*s0^2)-(-65*
a1+34.69508)^2/(2*s0^2)-(-70*a1+33.13127)^2/(2*s0^2)-(-75*
a1+43.87842)^2/(2*s0^2)-(-80*a1+34.06836)^2/(2*s0^2)-(-85*
a1+43.07314)^2/(2*s0^2));
```

$$delpdels0 := \left(-\frac{20}{s0} + \frac{(10\,a1 - 6.65647123)^2}{s0^3} + \frac{(5\,a1 - 0.011729798)^2}{s0^3} + \frac{18.85279032}{s0^3} \right. \tag{6}$$

$$+ \frac{(-5\,a1 + 4.56334)^2}{s0^3} + \frac{(-10\,a1 + 8.37979)^2}{s0^3} + \frac{(-15\,a1 + 10.2627)^2}{s0^3}$$

$$+ \frac{(-20\,a1 + 10.22623)^2}{s0^3} + \frac{(-25\,a1 + 14.28611)^2}{s0^3} + \frac{(-30\,a1 + 13.79561)^2}{s0^3}$$

$$+ \frac{(-35\,a1 + 21.09856)^2}{s0^3} + \frac{(-40\,a1 + 17.50144)^2}{s0^3} + \frac{(-45\,a1 + 20.10634)^2}{s0^3}$$

$$+ \frac{(-50\,a1 + 23.32865)^2}{s0^3} + \frac{(-55\,a1 + 18.41583)^2}{s0^3} + \frac{(-60\,a1 + 36.10625)^2}{s0^3}$$

$$+ \frac{(-65\,a1 + 34.69508)^2}{s0^3} + \frac{(-70\,a1 + 33.13127)^2}{s0^3} + \frac{(-75\,a1 + 43.87842)^2}{s0^3}$$

$$+ \frac{(-80\,a1 + 34.06836)^2}{s0^3} + \frac{(-85\,a1 + 43.07314)^2}{s0^3} \right)$$

$$e^{-10\ln(2\pi) - 20\ln(s0) - \frac{(10\,a1 - 6.65647123)^2}{2\,s0^2} - \frac{(5\,a1 - 0.011729798)^2}{2\,s0^2} - \frac{9.426395160}{s0^2}}$$

$$- \frac{(-5\,a1 + 4.56334)^2}{2\,s0^2} - \frac{(-10\,a1 + 8.37979)^2}{2\,s0^2} - \frac{(-15\,a1 + 10.2627)^2}{2\,s0^2} - \frac{(-20\,a1 + 10.22623)^2}{2\,s0^2}$$

$$- \frac{(-25\,a1 + 14.28611)^2}{2\,s0^2} - \frac{(-30\,a1 + 13.79561)^2}{2\,s0^2} - \frac{(-35\,a1 + 21.09856)^2}{2\,s0^2} - \frac{(-40\,a1 + 17.50144)^2}{2\,s0^2}$$

$$- \frac{(-45\,a1 + 20.10634)^2}{2\,s0^2} - \frac{(-50\,a1 + 23.32865)^2}{2\,s0^2} - \frac{(-55\,a1 + 18.41583)^2}{2\,s0^2} - \frac{(-60\,a1 + 36.10625)^2}{2\,s0^2}$$

$$- \frac{(-65\,a1 + 34.69508)^2}{2\,s0^2} - \frac{(-70\,a1 + 33.13127)^2}{2\,s0^2} - \frac{(-75\,a1 + 43.87842)^2}{2\,s0^2} - \frac{(-80\,a1 + 34.06836)^2}{2\,s0^2}$$

$$- \frac{(-85\,a1 + 43.07314)^2}{2\,s0^2}$$

```
> delpdela1:=(-(10*(10*a1-6.65647123))/s0^2-(5*(5*a1-0.11729798e-1)
  )/s0^2+(5*(-5*a1+4.56334))/s0^2+(10*(-10*a1+8.37979))/s0^2+(15*
  (-15*a1+10.2627))/s0^2+(20*(-20*a1+10.22623))/s0^2+(25*(-25*
  a1+14.28611))/s0^2+(30*(-30*a1+13.79561))/s0^2+(35*(-35*
  a1+21.09856))/s0^2+(40*(-40*a1+17.50144))/s0^2+(45*(-45*
```

```
a1+20.10634))/s0^2+(50*(-50*a1+23.32865))/s0^2+(55*(-55*
a1+18.41583))/s0^2+(60*(-60*a1+36.10625))/s0^2+(65*(-65*
a1+34.69508))/s0^2+(70*(-70*a1+33.13127))/s0^2+(75*(-75*
a1+43.87842))/s0^2+(80*(-80*a1+34.06836))/s0^2+(85*(-85*
a1+43.07314))/s0^2)*exp(-10*ln(2*Pi)-20*ln(s0)-(10*a1-6.65647123)
^2/(2*s0^2)-(5*a1-0.11729798e-1)^2/(2*s0^2)-9.426395160/(s0^2)-
(-5*a1+4.56334)^2/(2*s0^2)-(-10*a1+8.37979)^2/(2*s0^2)-(-15*
a1+10.2627)^2/(2*s0^2)-(-20*a1+10.22623)^2/(2*s0^2)-(-25*
a1+14.28611)^2/(2*s0^2)-(-30*a1+13.79561)^2/(2*s0^2)-(-35*
a1+21.09856)^2/(2*s0^2)-(-40*a1+17.50144)^2/(2*s0^2)-(-45*
a1+20.10634)^2/(2*s0^2)-(-50*a1+23.32865)^2/(2*s0^2)-(-55*
a1+18.41583)^2/(2*s0^2)-(-60*a1+36.10625)^2/(2*s0^2)-(-65*
a1+34.69508)^2/(2*s0^2)-(-70*a1+33.13127)^2/(2*s0^2)-(-75*
a1+43.87842)^2/(2*s0^2)-(-80*a1+34.06836)^2/(2*s0^2)-(-85*
a1+43.07314)^2/(2*s0^2));
```

$$delpdela1 := \left(-\frac{100\,a1 - 66.56471230}{s0^2} - \frac{25\,a1 - 0.058648990}{s0^2} + \frac{-25\,a1 + 22.81670}{s0^2} \right. \tag{7}$$

$$+ \frac{-100\,a1 + 83.79790}{s0^2} + \frac{-225\,a1 + 153.9405}{s0^2} + \frac{-400\,a1 + 204.52460}{s0^2}$$

$$+ \frac{-625\,a1 + 357.15275}{s0^2} + \frac{-900\,a1 + 413.86830}{s0^2} + \frac{-1225\,a1 + 738.44960}{s0^2}$$

$$+ \frac{-1600\,a1 + 700.05760}{s0^2} + \frac{-2025\,a1 + 904.78530}{s0^2} + \frac{-2500\,a1 + 1166.43250}{s0^2}$$

$$+ \frac{-3025\,a1 + 1012.87065}{s0^2} + \frac{-3600\,a1 + 2166.37500}{s0^2} + \frac{-4225\,a1 + 2255.18020}{s0^2}$$

$$+ \frac{-4900\,a1 + 2319.18890}{s0^2} + \frac{-5625\,a1 + 3290.88150}{s0^2} + \frac{-6400\,a1 + 2725.46880}{s0^2}$$

$$\left. + \frac{-7225\,a1 + 3661.21690}{s0^2} \right)$$

$$e^{-10\ln(2\pi) - 20\ln(s0) - \frac{(10\,a1 - 6.65647123)^2}{2\,s0^2} - \frac{(5\,a1 - 0.011729798)^2}{2\,s0^2} - \frac{9.426395160}{s0^2}}$$

$$- \frac{(-5\,a1 + 4.56334)^2}{2\,s0^2} - \frac{(-10\,a1 + 8.37979)^2}{2\,s0^2} - \frac{(-15\,a1 + 10.2627)^2}{2\,s0^2} - \frac{(-20\,a1 + 10.22623)^2}{2\,s0^2}$$

$$- \frac{(-25\,a1 + 14.28611)^2}{2\,s0^2} - \frac{(-30\,a1 + 13.79561)^2}{2\,s0^2} - \frac{(-35\,a1 + 21.09856)^2}{2\,s0^2} - \frac{(-40\,a1 + 17.50144)^2}{2\,s0^2}$$

$$- \frac{(-45\,a1 + 20.10634)^2}{2\,s0^2} - \frac{(-50\,a1 + 23.32865)^2}{2\,s0^2} - \frac{(-55\,a1 + 18.41583)^2}{2\,s0^2} - \frac{(-60\,a1 + 36.10625)^2}{2\,s0^2}$$

$$- \frac{(-65\,a1 + 34.69508)^2}{2\,s0^2} - \frac{(-70\,a1 + 33.13127)^2}{2\,s0^2} - \frac{(-75\,a1 + 43.87842)^2}{2\,s0^2} - \frac{(-80\,a1 + 34.06836)^2}{2\,s0^2}$$

$$-\frac{(-85\,a1 + 43.07314)^2}{2\,s0^2}$$

```
> g:=delpdela1=0;u:=delpdels0=0;
```

$$g := \left(-\frac{100\,a1 - 66.56471230}{s0^2} - \frac{25\,a1 - 0.058648990}{s0^2} + \frac{-25\,a1 + 22.81670}{s0^2} \right.$$

$$+ \frac{-100\,a1 + 83.79790}{s0^2} + \frac{-225\,a1 + 153.9405}{s0^2} + \frac{-400\,a1 + 204.52460}{s0^2}$$

$$+ \frac{-625\,a1 + 357.15275}{s0^2} + \frac{-900\,a1 + 413.86830}{s0^2} + \frac{-1225\,a1 + 738.44960}{s0^2}$$

$$+ \frac{-1600\,a1 + 700.05760}{s0^2} + \frac{-2025\,a1 + 904.78530}{s0^2} + \frac{-2500\,a1 + 1166.43250}{s0^2}$$

$$+ \frac{-3025\,a1 + 1012.87065}{s0^2} + \frac{-3600\,a1 + 2166.37500}{s0^2} + \frac{-4225\,a1 + 2255.18020}{s0^2}$$

$$+ \frac{-4900\,a1 + 2319.18890}{s0^2} + \frac{-5625\,a1 + 3290.88150}{s0^2} + \frac{-6400\,a1 + 2725.46880}{s0^2}$$

$$\left. + \frac{-7225\,a1 + 3661.21690}{s0^2} \right)$$

$$e^{-10\ln(2\pi) - 20\ln(s0) - \frac{(10\,a1 - 6.65647123)^2}{2\,s0^2} - \frac{(5\,a1 - 0.011729798)^2}{2\,s0^2} - \frac{9.426395160}{s0^2}}$$

$$-\frac{(-5\,a1 + 4.56334)^2}{2\,s0^2} - \frac{(-10\,a1 + 8.37979)^2}{2\,s0^2} - \frac{(-15\,a1 + 10.2627)^2}{2\,s0^2} - \frac{(-20\,a1 + 10.22623)^2}{2\,s0^2}$$

$$-\frac{(-25\,a1 + 14.28611)^2}{2\,s0^2} - \frac{(-30\,a1 + 13.79561)^2}{2\,s0^2} - \frac{(-35\,a1 + 21.09856)^2}{2\,s0^2} - \frac{(-40\,a1 + 17.50144)^2}{2\,s0^2}$$

$$-\frac{(-45\,a1 + 20.10634)^2}{2\,s0^2} - \frac{(-50\,a1 + 23.32865)^2}{2\,s0^2} - \frac{(-55\,a1 + 18.41583)^2}{2\,s0^2} - \frac{(-60\,a1 + 36.10625)^2}{2\,s0^2}$$

$$-\frac{(-65\,a1 + 34.69508)^2}{2\,s0^2} - \frac{(-70\,a1 + 33.13127)^2}{2\,s0^2} - \frac{(-75\,a1 + 43.87842)^2}{2\,s0^2} - \frac{(-80\,a1 + 34.06836)^2}{2\,s0^2}$$

$$-\frac{(-85\,a1 + 43.07314)^2}{2\,s0^2} = 0$$

$$u := \left(-\frac{20}{s0} + \frac{(10\,a1 - 6.65647123)^2}{s0^3} + \frac{(5\,a1 - 0.011729798)^2}{s0^3} + \frac{18.85279032}{s0^3} \right.$$

$$+ \frac{(-5\,a1 + 4.56334)^2}{s0^3} + \frac{(-10\,a1 + 8.37979)^2}{s0^3} + \frac{(-15\,a1 + 10.2627)^2}{s0^3}$$

(8)

$$+ \frac{(-20\,a1 + 10.22623)^2}{s0^3} + \frac{(-25\,a1 + 14.28611)^2}{s0^3} + \frac{(-30\,a1 + 13.79561)^2}{s0^3}$$

$$+ \frac{(-35\,a1 + 21.09856)^2}{s0^3} + \frac{(-40\,a1 + 17.50144)^2}{s0^3} + \frac{(-45\,a1 + 20.10634)^2}{s0^3}$$

$$+ \frac{(-50\,a1 + 23.32865)^2}{s0^3} + \frac{(-55\,a1 + 18.41583)^2}{s0^3} + \frac{(-60\,a1 + 36.10625)^2}{s0^3}$$

$$+ \frac{(-65\,a1 + 34.69508)^2}{s0^3} + \frac{(-70\,a1 + 33.13127)^2}{s0^3} + \frac{(-75\,a1 + 43.87842)^2}{s0^3}$$

$$+ \left. \frac{(-80\,a1 + 34.06836)^2}{s0^3} + \frac{(-85\,a1 + 43.07314)^2}{s0^3} \right)$$

$$e^{\displaystyle -10\ln(2\pi) - 20\ln(s0) - \frac{(10\,a1 - 6.65647123)^2}{2\,s0^2} - \frac{(5\,a1 - 0.011729798)^2}{2\,s0^2} - \frac{9.426395160}{s0^2}}$$

$$-\frac{(-5\,a1 + 4.56334)^2}{2\,s0^2} - \frac{(-10\,a1 + 8.37979)^2}{2\,s0^2} - \frac{(-15\,a1 + 10.2627)^2}{2\,s0^2} - \frac{(-20\,a1 + 10.22623)^2}{2\,s0^2}$$

$$-\frac{(-25\,a1 + 14.28611)^2}{2\,s0^2} - \frac{(-30\,a1 + 13.79561)^2}{2\,s0^2} - \frac{(-35\,a1 + 21.09856)^2}{2\,s0^2} - \frac{(-40\,a1 + 17.50144)^2}{2\,s0^2}$$

$$-\frac{(-45\,a1 + 20.10634)^2}{2\,s0^2} - \frac{(-50\,a1 + 23.32865)^2}{2\,s0^2} - \frac{(-55\,a1 + 18.41583)^2}{2\,s0^2} - \frac{(-60\,a1 + 36.10625)^2}{2\,s0^2}$$

$$-\frac{(-65\,a1 + 34.69508)^2}{2\,s0^2} - \frac{(-70\,a1 + 33.13127)^2}{2\,s0^2} - \frac{(-75\,a1 + 43.87842)^2}{2\,s0^2} - \frac{(-80\,a1 + 34.06836)^2}{2\,s0^2}$$

$$-\frac{(-85\,a1 + 43.07314)^2}{2\,s0^2} = 0$$

The partial derivatives are all set to zero, and the objective function Z_1 (Eq. 18) is minimized providing the optimized parameters a1 and s0 (see also Table 6):

```
> fsolve({g,u}, {s0 = 3 .. 4, a1 = 0.45 .. 0.5});
```
$$\{a1 = 0.4970643813, s0 = 3.791034263\} \tag{9}$$

```
> Z:=-2*ln(Prob_ex5);
```
$$Z := \tag{10}$$

$$-2$$

$$\ln\left(e^{\displaystyle -10\ln(2\pi) - 20\ln(s0) - \frac{(10\,a1 - 6.65647123)^2}{2\,s0^2} - \frac{(5\,a1 - 0.011729798)^2}{2\,s0^2} - \frac{9.426395160}{s0^2}} \right.$$

$$-\frac{(-5\,a1 + 4.56334)^2}{2\,s0^2} - \frac{(-10\,a1 + 8.37979)^2}{2\,s0^2} - \frac{(-15\,a1 + 10.2627)^2}{2\,s0^2} - \frac{(-20\,a1 + 10.22623)^2}{2\,s0^2}$$

$$-\frac{(-25\,a1+14.28611)^2}{2\,s0^2}-\frac{(-30\,a1+13.79561)^2}{2\,s0^2}-\frac{(-35\,a1+21.09856)^2}{2\,s0^2}-\frac{(-40\,a1+17.50144)^2}{2\,s0^2}$$

$$-\frac{(-45\,a1+20.10634)^2}{2\,s0^2}-\frac{(-50\,a1+23.32865)^2}{2\,s0^2}-\frac{(-55\,a1+18.41583)^2}{2\,s0^2}-\frac{(-60\,a1+36.10625)^2}{2\,s0^2}$$

$$-\frac{(-65\,a1+34.69508)^2}{2\,s0^2}-\frac{(-70\,a1+33.13127)^2}{2\,s0^2}-\frac{(-75\,a1+43.87842)^2}{2\,s0^2}-\frac{(-80\,a1+34.06836)^2}{2\,s0^2}$$

$$-\frac{(-85\,a1+43.07314)^2}{2\,s0^2}\Bigg)$$

The Hessian matrix is calculated for the minimum of the objective function Z1:

```
> hess:=Hessian(Z, [a1,s0]):
```
```
> hess0:=subs(a1= .4970643813,hess);hess1:=subs(s0= 3.791034263,
  hess0);hess_matrix:=evalf(hess1);
```

$$hess0 := \begin{bmatrix} \dfrac{89500}{s0^2} & -\dfrac{4.68\ 10^{-6}}{s0^3} \\[2ex] -\dfrac{4.68\ 10^{-6}}{s0^3} & -\dfrac{40}{s0^2}+\dfrac{1724.632894}{s0^4} \end{bmatrix}$$

$$hess1 := \begin{bmatrix} 6227.412244 & -8.589595735\ 10^{-8} \\ -8.589595735\ 10^{-8} & 5.566402006 \end{bmatrix}$$

$$hess\_matrix := \begin{bmatrix} 6227.412244 & -8.589595735\ 10^{-8} \\ -8.589595735\ 10^{-8} & 5.566402006 \end{bmatrix} \qquad \textbf{(11)}$$

The covariance matrix (inverse of the Hessian matrix) is calculated:

```
> cov1:=MatrixInverse(hess_matrix);
```

$$cov1 := \begin{bmatrix} 0.000160580343940371 & 2.47793859650863\ 10^{-12} \\ 2.47793859650863\ 10^{-12} & 0.179649259777160 \end{bmatrix} \qquad \textbf{(12)}$$

```
> with(Statistics);
```

$[AbsoluteDeviation, AgglomeratedPlot, AreaChart, AutoCorrelation, BarChart, Biplot,$ **(13)**
    $Bootstrap, BoxPlot, BubblePlot, CDF, CGF, CentralMoment, CharacteristicFunction,$
    $ChiSquareGoodnessOfFitTest, ChiSquareIndependenceTest, ChiSquareSuitableModelTest,$
    $ColumnGraph, Correlation, CorrelationMatrix, Count, CountMissing, Covariance,$
    $CovarianceMatrix, CrossCorrelation, Cumulant, CumulantGeneratingFunction,$
    $CumulativeDistributionFunction, CumulativeProduct, CumulativeSum,$
    $CumulativeSumChart, DataSummary, Decile, DensityPlot, DiscreteValueMap, Distribution,$
    $ErrorPlot, EvaluateToFloat, Excise, ExpectedValue, ExponentialFit,$
    $ExponentialSmoothing, FailureRate, FisherInformation, Fit, FivePointSummary,$
    $FrequencyPlot, FrequencyTable, GeometricMean, GridPlot, HarmonicMean, HazardRate,$
    $HeatMap, Histogram, HodgesLehmann, Information, InteractiveDataAnalysis,$

InterquartileRange, InverseSurvivalFunction, Join, KernelDensity, KernelDensityPlot, KernelDensitySample, Kurtosis, Likelihood, LikelihoodRatioStatistic, LineChart, LinearFilter, LinearFit, LogLikelihood, LogarithmicFit, Lowess, MGF, MLE, MakeProcedure, MaximumLikelihoodEstimate, Mean, MeanDeviation, Median, MedianDeviation, MillsRatio, Mode, Moment, MomentGeneratingFunction, MovingAverage, MovingMedian, MovingStatistic, NonlinearFit, NormalPlot, OneSampleChiSquareTest, OneSampleTTest, OneSampleZTest, OneWayANOVA, OrderByRank, OrderStatistic, PCA, PDF, ParetoChart, Percentile, PieChart, PointPlot, PolynomialFit, PowerFit, PredictiveLeastSquares, PrincipalComponentAnalysis, Probability, ProbabilityDensityFunction, ProbabilityFunction, ProbabilityPlot, ProfileLikelihood, ProfileLogLikelihood, QuadraticMean, Quantile, QuantilePlot, Quartile, RandomVariable, Range, Rank, Remove, RemoveInRange, RemoveNonNumeric, RepeatedMedianEstimator, RousseeuwCrouxQn, RousseeuwCrouxSn, Sample, Scale, ScatterPlot, ScatterPlot3D, Score, ScreePlot, Select, SelectInRange, SelectNonNumeric, ShapiroWilkWTest, Shuffle, Skewness, Sort, Specialize, SplitByColumn, StandardDeviation, StandardError, StandardizedMoment, SunflowerPlot, Support, SurfacePlot, SurvivalFunction, SymmetryPlot, Tally, TallyInto, TreeMap, Trim, TrimmedMean, TwoSampleFTest, TwoSamplePairedTTest, TwoSampleTTest, TwoSampleZTest, Variance, Variation, VennDiagram, ViolinPlot, WeibullPlot, WeightedMovingAverage, Winsorize, WinsorizedMean]

The correlation coefficient is now close to zero, compare Table 6:

```
> rho12:=cov1(1,2)/sqrt(cov1(1,1))/sqrt(cov1(2,2));
```
$$\rho 12 := 4.61351176438973 \; 10^{-10} \tag{14}$$

The uncertainties of the parameters follow from the correlation matrix, and are provided in Table 5:

```
> dela1:=sqrt(cov1(1,1));
```
$$dela1 := 0.0126720299849855 \tag{15}$$

```
> dels0:=sqrt(cov1(2,2));
```
$$dels0 := 0.423850515839205 \tag{16}$$

```
> with(LinearAlgebra);
```
[&x, Add, Adjoint, BackwardSubstitute, BandMatrix, Basis, BezoutMatrix, BidiagonalForm, BilinearForm, CARE, CharacteristicMatrix, CharacteristicPolynomial, Column, ColumnDimension, ColumnOperation, ColumnSpace, CompanionMatrix, CompressedSparseForm, ConditionNumber, ConstantMatrix, ConstantVector, Copy, CreatePermutation, CrossProduct, DARE, DeleteColumn, DeleteRow, Determinant, Diagonal, DiagonalMatrix, Dimension, Dimensions, DotProduct, EigenConditionNumbers, Eigenvalues, Eigenvectors, Equal, ForwardSubstitute, FrobeniusForm, FromCompressedSparseForm, FromSplitForm, GaussianElimination, GenerateEquations, GenerateMatrix, Generic, GetResultDataType, GetResultShape, GivensRotationMatrix, GramSchmidt, HankelMatrix, HermiteForm, HermitianTranspose, HessenbergForm, HilbertMatrix, HouseholderMatrix, IdentityMatrix, IntersectionBasis, IsDefinite, IsOrthogonal, IsSimilar, IsUnitary, JordanBlockMatrix, JordanForm, KroneckerProduct, LA_Main, LUDecomposition, LeastSquares, LinearSolve, LyapunovSolve, Map, Map2, MatrixAdd, MatrixExponential, MatrixFunction, MatrixInverse, MatrixMatrixMultiply, **(17)**

MatrixNorm, MatrixPower, MatrixScalarMultiply, MatrixVectorMultiply, MinimalPolynomial, Minor, Modular, Multiply, NoUserValue, Norm, Normalize, NullSpace, OuterProductMatrix, Permanent, Pivot, PopovForm, ProjectionMatrix, QRDecomposition, RandomMatrix, RandomVector, Rank, RationalCanonicalForm, ReducedRowEchelonForm, Row, RowDimension, RowOperation, RowSpace, ScalarMatrix, ScalarMultiply, ScalarVector, SchurForm, SingularValues, SmithForm, SplitForm, StronglyConnectedBlocks, SubMatrix, SubVector, SumBasis, SylvesterMatrix, SylvesterSolve, ToeplitzMatrix, Trace, Transpose, TridiagonalForm, UnitVector, VandermondeMatrix, VectorAdd, VectorAngle, VectorMatrixMultiply, VectorNorm, VectorScalarMultiply, ZeroMatrix, ZeroVector, Zip]

> Determinant(hess_matrix);

$$34664.28001 \tag{18}$$

> ConditionNumber(hess_matrix);

$$1118.750000 \tag{19}$$