```
> restart;
```

Maple calculations regarding example 1
==========
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==========

```
> with(Statistics);
```

[*AbsoluteDeviation*, *AgglomeratedPlot*, *AreaChart*, *AutoCorrelation*, *BarChart*, *Biplot*, *Bootstrap*, *BoxPlot*, *BubblePlot*, *CDF*, *CGF*, *CentralMoment*, *CharacteristicFunction*, *ChiSquareGoodnessOfFitTest*, *ChiSquareIndependenceTest*, *ChiSquareSuitableModelTest*, *ColumnGraph*, *Correlation*, *CorrelationMatrix*, *Count*, *CountMissing*, *Covariance*, *CovarianceMatrix*, *CrossCorrelation*, *Cumulant*, *CumulantGeneratingFunction*, *CumulativeDistributionFunction*, *CumulativeProduct*, *CumulativeSum*, *CumulativeSumChart*, *DataSummary*, *Decile*, *DensityPlot*, *DiscreteValueMap*, *Distribution*, *ErrorPlot*, *EvaluateToFloat*, *Excise*, *ExpectedValue*, *ExponentialFit*, *ExponentialSmoothing*, *FailureRate*, *FisherInformation*, *Fit*, *FivePointSummary*, *FrequencyPlot*, *FrequencyTable*, *GeometricMean*, *GridPlot*, *HarmonicMean*, *HazardRate*, *HeatMap*, *Histogram*, *HodgesLehmann*, *Information*, *InteractiveDataAnalysis*, *InterquartileRange*, *InverseSurvivalFunction*, *Join*, *KernelDensity*, *KernelDensityPlot*, *KernelDensitySample*, *Kurtosis*, *Likelihood*, *LikelihoodRatioStatistic*, *LineChart*, *LinearFilter*, *LinearFit*, *LogLikelihood*, *LogarithmicFit*, *Lowess*, *MGF*, *MLE*, *MakeProcedure*, *MaximumLikelihoodEstimate*, *Mean*, *MeanDeviation*, *Median*, *MedianDeviation*, *MillsRatio*, *Mode*, *Moment*, *MomentGeneratingFunction*, *MovingAverage*, *MovingMedian*, *MovingStatistic*, *NonlinearFit*, *NormalPlot*, *OneSampleChiSquareTest*, *OneSampleTTest*, *OneSampleZTest*, *OneWayANOVA*, *OrderByRank*, *OrderStatistic*, *PCA*, *PDF*, *ParetoChart*, *Percentile*, *PieChart*, *PointPlot*, *PolynomialFit*, *PowerFit*, *PredictiveLeastSquares*, *PrincipalComponentAnalysis*, *Probability*, *ProbabilityDensityFunction*, *ProbabilityFunction*, *ProbabilityPlot*, *ProfileLikelihood*, *ProfileLogLikelihood*, *QuadraticMean*, *Quantile*, *QuantilePlot*, *Quartile*, *RandomVariable*, *Range*, *Rank*, *Remove*, *RemoveInRange*, *RemoveNonNumeric*, *RepeatedMedianEstimator*, *RousseeuwCrouxQn*, *RousseeuwCrouxSn*, *Sample*, *Scale*, *ScatterPlot*, *ScatterPlot3D*, *Score*, *ScreePlot*, *Select*, *SelectInRange*, *SelectNonNumeric*, *ShapiroWilkWTest*, *Shuffle*, *Skewness*, *Sort*, *Specialize*, *SplitByColumn*, *StandardDeviation*, *StandardError*, *StandardizedMoment*, *SunflowerPlot*, *Support*, *SurfacePlot*, *SurvivalFunction*, *SymmetryPlot*, *Tally*, *TallyInto*, *TreeMap*, *Trim*, *TrimmedMean*, *TwoSampleFTest*, *TwoSamplePairedTTest*, *TwoSampleTTest*, *TwoSampleZTest*, *Variance*, *Variation*, *VennDiagram*, *ViolinPlot*, *WeibullPlot*, *WeightedMovingAverage*, *Winsorize*, *WinsorizedMean*] **(1)**

```
> with(LinearAlgebra):
```

```
> Prob_ex2:=exp(-(8/2)*ln(2*Pi)-8*ln(s)-((y1-a)^2+(y2-a)^2+(y3-a)
  ^2+(y4-a)^2+(y5-a)^2+(y6-a)^2+(y7-a)^2+(y8-a)^2)/(2*s^2));
```

Prob_ex2 := **(2)**

$$e^{-4\ln(2\pi)-8\ln(s)}\, e^{-\frac{(y1-a)^2+(y2-a)^2+(y3-a)^2+(y4-a)^2+(y5-a)^2+(y6-a)^2+(y7-a)^2+(y8-a)^2}{2s^2}}$$

```
> y1:=1.2529423;y2:=0.6062315;y3:=0.8870169;y4:=0.7365572;y5:=
  1.0504223;y6:=0.7883198;y7:=0.912006;y8:=1.2955773;
```

$$y1 := 1.2529423$$
$$y2 := 0.6062315$$
$$y3 := 0.8870169$$
$$y4 := 0.7365572$$
$$y5 := 1.0504223$$
$$y6 := 0.7883198$$
$$y7 := 0.912006$$
$$y8 := 1.2955773 \quad \textbf{(3)}$$

```
> Mean([y1,y2,y3,y4,y5,y6,y7,y8]);
```

$$0.941134162500000 \quad \textbf{(4)}$$

```
> std:=StandardDeviation([y1,y2,y3,y4,y5,y6,y7,y8]);
```

$$std := 0.243761896003632 \quad \textbf{(5)}$$

```
> std_large:=evalf((std^2*7/8)^(1/2));
```

$$std\_large := 0.228018374712779 \quad \textbf{(6)}$$

```
> int(Prob_ex2,s=0..infinity, a=0.01..100);normf:=1/%;
```

$$0.06321408573$$
$$normf := 15.81925909 \quad \textbf{(7)}$$

```
> with(VectorCalculus);
```

[&x, `*`, `+`, `-`, `.`, <,>, <|>, *About*, *AddCoordinates*, *ArcLength*, *BasisFormat*, *Binormal*, *ConvertVector*, *CrossProduct*, *Curl*, *Curvature*, D, *Del*, *DirectionalDiff*, *Divergence*, *DotProduct*, *Flux*, *GetCoordinateParameters*, *GetCoordinates*, *GetNames*, *GetPVDescription*, *GetRootPoint*, *GetSpace*, *Gradient*, *Hessian*, *IsPositionVector*, *IsRootedVector*, *IsVectorField*, *Jacobian*, *Laplacian*, *LineInt*, *MapToBasis*, ∇, *Norm*, *Normalize*, *PathInt*, *PlotPositionVector*, *PlotVector*, *PositionVector*, *PrincipalNormal*, *RadiusOfCurvature*, *RootedVector*, *ScalarPotential*, *SetCoordinateParameters*, *SetCoordinates*, *SpaceCurve*, *SurfaceInt*, *TNBFrame*, *TangentLine*, *TangentPlane*, *TangentVector*, *Torsion*, *Vector*, *VectorField*, *VectorPotential*, *VectorSpace*, *Wronskian*, *diff*, *eval*, *evalVF*, *int*, *limit*, *series*] **(8)**

```
> g1 := Gradient(Prob_ex2, [s, a]);
```

$$g1 := \left(\left(-\frac{8}{s} + \frac{1}{s^3}\left((1.2529423-a)^2 + (0.6062315-a)^2 + (0.8870169-a)^2 \right.\right.\right. \quad \textbf{(9)}$$

$$\left.\left.\left. + (0.7365572-a)^2 + (1.0504223-a)^2 + (0.7883198-a)^2 + (0.912006-a)^2 \right.\right.\right.$$

$$+ (1.2955773 - a)^2))$$

$$e^{-4\ln(2\pi) - 8\ln(s) - \frac{1}{2s^2}((1.2529423 - a)^2 + (0.6062315 - a)^2 + (0.8870169 - a)^2}$$

$$+ (0.7365572 - a)^2 + (1.0504223 - a)^2 + (0.7883198 - a)^2 + (0.912006 - a)^2 + (1.2955773 - a)^2) \Bigg) \bar{e}_s$$

$$+ \left(-\frac{1}{2s^2} \left((-15.0581466 \right.\right.$$

$$+ 16a)$$

$$e^{-4\ln(2\pi) - 8\ln(s) - \frac{1}{2s^2}((1.2529423 - a)^2 + (0.6062315 - a)^2 + (0.8870169 - a)^2}$$

$$+ (0.7365572 - a)^2 + (1.0504223 - a)^2 + (0.7883198 - a)^2 + (0.912006 - a)^2 + (1.2955773 - a)^2) \Bigg) \Bigg) \bar{e}_a$$

```
> delpdels:=(-8/s+((1.2529423-a)^2+(.6062315-a)^2+0.1276518089e-1+
  (.7365572-a)^2+(1.0504223-a)^2+(.7883198-a)^2+(.912006-a)^2+
  (1.2955773-a)^2)/s^3)*exp(-4*ln(2*Pi)-8*ln(s)-((1.2529423-a)^2+
  (.6062315-a)^2+0.1276518089e-1+(.7365572-a)^2+(1.0504223-a)^2+
  (.7883198-a)^2+(.912006-a)^2+(1.2955773-a)^2)/(2*s^2));
```

$$delpdels := \left(-\frac{8}{s} + \frac{1}{s^3}((1.2529423 - a)^2 + (0.6062315 - a)^2 + 0.01276518089 \right. \tag{10}$$

$$+ (0.7365572 - a)^2 + (1.0504223 - a)^2 + (0.7883198 - a)^2 + (0.912006 - a)^2$$

$$+ (1.2955773 - a)^2))$$

$$e^{-4\ln(2\pi) - 8\ln(s) - \frac{1}{2s^2}((1.2529423 - a)^2 + (0.6062315 - a)^2 + 0.01276518089}$$

$$+ (0.7365572 - a)^2 + (1.0504223 - a)^2 + (0.7883198 - a)^2 + (0.912006 - a)^2 + (1.2955773 - a)^2)$$

```
> delpdela:=-(-13.2841128+14*a)*exp(-4*ln(2*Pi)-8*ln(s)-((1.2529423
  -a)^2+(.6062315-a)^2+0.1276518089e-1+(.7365572-a)^2+(1.0504223-a)
  ^2+(.7883198-a)^2+(.912006-a)^2+(1.2955773-a)^2)/(2*s^2))/(2*s^2)
  ;
```

$$delpdela := -\frac{1}{2s^2} \left((-13.2841128 \right. \tag{11}$$

$$+ 14a)$$

$$e^{-4\ln(2\pi)-8\ln(s)-\frac{1}{2s^2}\left((1.2529423-a)^2+(0.6062315-a)^2+0.01276518089+(0.7365572-a)^2+(1.0504223-a)^2+(0.7883198-a)^2+(0.912006-a)^2+(1.2955773-a)^2\right)}$$

```
> f:=delpdels=0;g:=delpdela=0;
```

$$f := \left(-\frac{8}{s}+\frac{1}{s^3}\left((1.2529423-a)^2+(0.6062315-a)^2+0.01276518089+(0.7365572-a)^2+(1.0504223-a)^2+(0.7883198-a)^2+(0.912006-a)^2+(1.2955773-a)^2\right)\right)$$

$$e^{-4\ln(2\pi)-8\ln(s)-\frac{1}{2s^2}\left((1.2529423-a)^2+(0.6062315-a)^2+0.01276518089+(0.7365572-a)^2+(1.0504223-a)^2+(0.7883198-a)^2+(0.912006-a)^2+(1.2955773-a)^2\right)}=0$$

$$g := -\frac{1}{2s^2}\left((-13.2841128+14a)e^{-4\ln(2\pi)-8\ln(s)-\frac{1}{2s^2}\left((1.2529423-a)^2+(0.6062315-a)^2+0.01276518089+(0.7365572-a)^2+(1.0504223-a)^2+(0.7883198-a)^2+(0.912006-a)^2+(1.2955773-a)^2\right)}\right)=0 \quad \textbf{(12)}$$

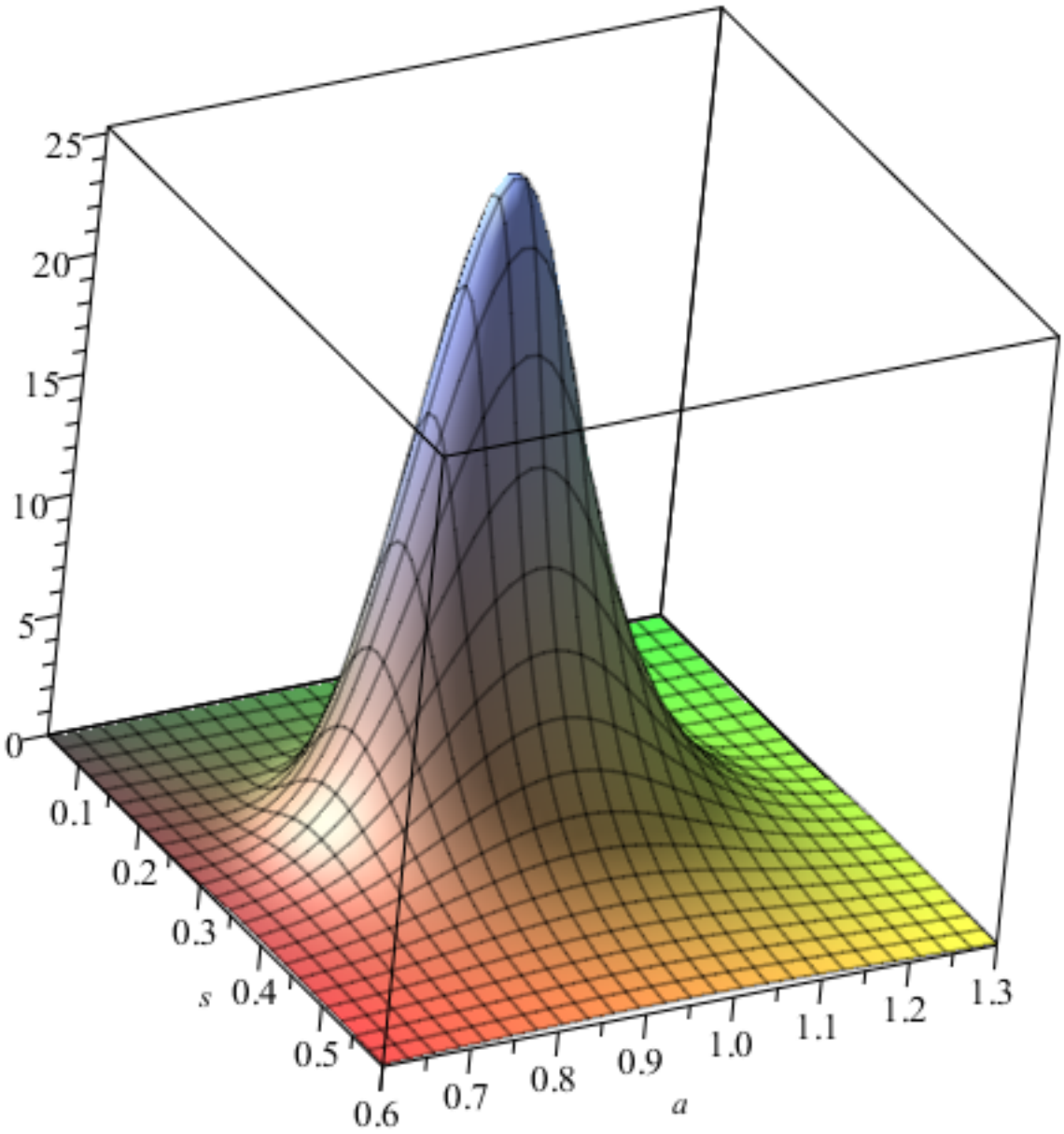
```
> fsolve({f, g}, {s = 0.05 .. 0.5, a = 0.8 .. 1.2});
```

$$\{a = 0.9488652000,\ s = 0.2305854380\} \quad \textbf{(13)}$$

```
> with(plottools);
```

[*annulus*, *arc*, *arrow*, *circle*, *cone*, *cuboid*, *curve*, *cutin*, *cutout*, *cylinder*, *disk*, *dodecahedron*, *ellipse*, *ellipticArc*, *exportplot*, *extrude*, *getdata*, *hemisphere*, *hexahedron*, *homothety*, *hyperbola*, *icosahedron*, *importplot*, *line*, *octahedron*, *parallelepiped*, *pieslice*, *point*, *polygon*, *prism*, *project*, *rectangle*, *reflect*, *rotate*, *scale*, *sector*, *semitorus*, *sphere*, *stellate*, *tetrahedron*, *torus*, *transform*, *translate*] **(14)**

```
> P:=plot3d(Prob_ex2*normf,s=.05..0.6,a=0.6..1.3, orientation =
  [-20, 60,10]);
```




```
> Z:=-2*ln(Prob_ex2*normf)-8*ln(2*Pi);
```

$$Z := \tag{15}$$

$$-2\ln\left(15.81925909\, e^{-4\ln(2\pi)-8\ln(s)-\frac{1}{2s^2}\left((1.2529423-a)^2+(0.6062315-a)^2+(0.8870169-a)^2+(0.7365572-a)^2+(1.0504223-a)^2+(0.7883198-a)^2+(0.912006-a)^2+(1.2955773-a)^2\right)}\right)-8\ln(2\pi)$$

```
> hess:=Hessian(Z, [a, s]);
```

$$hess := \left[\left[\frac{16.00000000}{s^2}, -\frac{2.000000000\,(-15.0581466+16\,a)}{s^3}\right],\right.$$
$$\left[-\frac{2.000000000\,(-15.0581466+16\,a)}{s^3}, -\frac{16.00000000}{s^2}\right.$$
$$+\frac{1}{s^4}\left(6.000000000\left((1.2529423-a)^2+(0.6062315-a)^2+(0.8870169-a)^2\right.\right.$$
$$+(0.7365572-a)^2+(1.0504223-a)^2+(0.7883198-a)^2+(0.912006-a)^2$$
$$\left.\left.+(1.2955773-a)^2\right)\right)\left.\right]\left.\right] \tag{16}$$

```
> subs(a=.9488652,subs(s=0.2305854380,hess));
```

$$\begin{bmatrix} 300.9235859 & -20.178651 \\ -20.178651 & 582.8731355 \end{bmatrix} \tag{17}$$

```
>  cov:=MatrixInverse(hess);
```

$$cov := \left[\left[-\frac{s^2\,(-16.\,s^2+45.01084277-90.34887960\,a+48.\,a^2)}{256.\,s^2+186.8176317-481.8606910\,a+256.\,a^2},\right.\right.$$
$$\left.-\frac{2.000000000\,s^3\,(-15.05814660+16.\,a)}{256.\,s^2+186.8176317-481.8606910\,a+256.\,a^2}\right],$$
$$\left[-\frac{2.000000000\,s^3\,(-15.05814660+16.\,a)}{256.\,s^2+186.8176317-481.8606910\,a+256.\,a^2},\right.$$
$$\left.\left.-\frac{16.00000000\,s^4}{256.0000000\,s^2+186.8176317-481.860691\,a+256.0000000\,a^2}\right]\right] \tag{18}$$

```
> cov1:=subs(a=.9488652,subs(s=0.2305854380,cov));
```

$$cov1 := \begin{bmatrix} 0.003330835049 & 0.0001153111264 \\ 0.0001153111264 & 0.001719631200 \end{bmatrix} \tag{19}$$

```
> s1:=sqrt(cov1(1,1));
```

$$s1 := 0.05771338709 \tag{20}$$

```
> s2:=sqrt(cov1(2,2));
```

$$s2 := 0.04146843619 \tag{21}$$

```
> rho:=cov1(1,2)/s1/s2;
```

$$\rho := 0.04818113135 \tag{22}$$