```
> restart;
```

Maple calculations regarding example 3
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==========

```
> for i from 1 to 20 do yc[i]:=a0+a1*x[i] end do:
> for i from 1 to 20 do s[i]:=s0+s1*abs(yc[i]) end do:
> lns:=add(ln(s[i]),i=1..20):
> lt:=add((y[i]-yc[i])^2/2/s[i]^2,i=1..20):
```

The probability density function 'Prob_ex3' is defined:

```
> Prob_ex3:=exp(-(20/2)*ln(2*Pi)-lns-lt):
```

The data points x[i] and y[i] are introduced:

```
> y[1]:=-6.65647123;y[2]:=-0.011729798;y[3]:=4.34198;y[4]:=4.56334;
  y[5]:=8.37979;y[6]:=10.2627;y[7]:=10.22623;y[8]:=14.28611;y[9]:=
  13.79561;y[10]:=21.09856;y[11]:=17.50144;y[12]:=20.10634;y[13]:=
  23.32865;y[14]:=18.41583;y[15]:=36.10625;y[16]:=34.69508;y[17]:=
  33.13127;y[18]:=43.87842;y[19]:=34.06836;y[20]:=43.07314;
```

$$y_1 := -6.65647123$$
$$y_2 := -0.011729798$$
$$y_3 := 4.34198$$
$$y_4 := 4.56334$$
$$y_5 := 8.37979$$
$$y_6 := 10.2627$$
$$y_7 := 10.22623$$
$$y_8 := 14.28611$$
$$y_9 := 13.79561$$
$$y_{10} := 21.09856$$
$$y_{11} := 17.50144$$
$$y_{12} := 20.10634$$
$$y_{13} := 23.32865$$
$$y_{14} := 18.41583$$
$$y_{15} := 36.10625$$
$$y_{16} := 34.69508$$
$$y_{17} := 33.13127$$
$$y_{18} := 43.87842$$
$$y_{19} := 34.06836$$
$$y_{20} := 43.07314 \quad \textbf{(1)}$$

```
> x[1]:=-10;x[2]:=-5;x[3]:=0;x[4]:=5;x[5]:=10;x[6]:=15;x[7]:=20;x
```

```
[8]:=25;x[9]:=30;x[10]:=35;x[11]:=40;x[12]:=45;x[13]:=50;x[14]:=
55;x[15]:=60;x[16]:=65;x[17]:=70;x[18]:=75;x[19]:=80;x[20]:=85;
```

$$x_1 := -10$$
$$x_2 := -5$$
$$x_3 := 0$$
$$x_4 := 5$$
$$x_5 := 10$$
$$x_6 := 15$$
$$x_7 := 20$$
$$x_8 := 25$$
$$x_9 := 30$$
$$x_{10} := 35$$
$$x_{11} := 40$$
$$x_{12} := 45$$
$$x_{13} := 50$$
$$x_{14} := 55$$
$$x_{15} := 60$$
$$x_{16} := 65$$
$$x_{17} := 70$$
$$x_{18} := 75$$
$$x_{19} := 80$$
$$x_{20} := 85 \quad \textbf{(2)}$$

```
> with(LinearAlgebra):
> with(VectorCalculus);
```

[*&x*, `*`, `+`, `-`, `.`, *<,>*, *<|>*, *About*, *AddCoordinates*, *ArcLength*, *BasisFormat*, *Binormal*, *ConvertVector*, *CrossProduct*, *Curl*, *Curvature*, D, *Del*, *DirectionalDiff*, *Divergence*, *DotProduct*, *Flux*, *GetCoordinateParameters*, *GetCoordinates*, *GetNames*, *GetPVDescription*, *GetRootPoint*, *GetSpace*, *Gradient*, *Hessian*, *IsPositionVector*, *IsRootedVector*, *IsVectorField*, *Jacobian*, *Laplacian*, *LineInt*, *MapToBasis*, ∇, *Norm*, *Normalize*, *PathInt*, *PlotPositionVector*, *PlotVector*, *PositionVector*, *PrincipalNormal*, *RadiusOfCurvature*, *RootedVector*, *ScalarPotential*, *SetCoordinateParameters*, *SetCoordinates*, *SpaceCurve*, *SurfaceInt*, *TNBFrame*, *TangentLine*, *TangentPlane*, *TangentVector*, *Torsion*, *Vector*, *VectorField*, *VectorPotential*, *VectorSpace*, *Wronskian*, *diff*, *eval*, *evalVF*, *int*, *limit*, *series*] **(3)**

```
> g1 := Gradient(Prob_ex3, [s0, s1,a0,a1]):
```

The gradient 'g1' is calculated, i.e. the partial derivatives 'delpdela0=partial(Prob_ex3)/partial(a0)', 'delpdela1=partial(Prob_ex3)/partial(a1)', 'delpdels0=partial(Prob_ex3)/partial(s0)' and 'delpdels1=partial(Prob_ex3)/partial(s1)' with respect to 'a0', 'a1', 's0' and 's1' are computed:

```
> delpdela1:=((-a1*x[4]-a0+y[4])*x[4]/(s0+s1*abs(a1*x[4]+a0))^2+(-
```

```
a1*x[20]-a0+y[20])*x[20]/(s0+s1*abs(a1*x[20]+a0))^2+(-a1*x[9]-a0+
y[9])*x[9]/(s0+s1*abs(a1*x[9]+a0))^2+(-a1*x[16]-a0+y[16])*x[16]/
(s0+s1*abs(a1*x[16]+a0))^2+(-a1*x[13]-a0+y[13])*x[13]/(s0+s1*abs
(a1*x[13]+a0))^2+(-a1*x[19]-a0+y[19])*x[19]/(s0+s1*abs(a1*x[19]+
a0))^2+(-a1*x[15]-a0+y[15])*x[15]/(s0+s1*abs(a1*x[15]+a0))^2+(-
a1*x[10]-a0+y[10])*x[10]/(s0+s1*abs(a1*x[10]+a0))^2+(-a1*x[18]-
a0+y[18])*x[18]/(s0+s1*abs(a1*x[18]+a0))^2+(-a1*x[17]-a0+y[17])*x
[17]/(s0+s1*abs(a1*x[17]+a0))^2+(-a1*x[7]-a0+y[7])*x[7]/(s0+s1*
abs(a1*x[7]+a0))^2+(-a1*x[3]-a0+y[3])*x[3]/(s0+s1*abs(a1*x[3]+a0)
)^2+(-a1*x[6]-a0+y[6])*x[6]/(s0+s1*abs(a1*x[6]+a0))^2+(-a1*x[11]-
a0+y[11])*x[11]/(s0+s1*abs(a1*x[11]+a0))^2+(-a1*x[14]-a0+y[14])*x
[14]/(s0+s1*abs(a1*x[14]+a0))^2+(-a1*x[5]-a0+y[5])*x[5]/(s0+s1*
abs(a1*x[5]+a0))^2+(-a1*x[2]-a0+y[2])*x[2]/(s0+s1*abs(a1*x[2]+a0)
)^2+(-a1*x[12]-a0+y[12])*x[12]/(s0+s1*abs(a1*x[12]+a0))^2+(-a1*x
[1]-a0+y[1])*x[1]/(s0+s1*abs(a1*x[1]+a0))^2+(-a1*x[8]-a0+y[8])*x
[8]/(s0+s1*abs(a1*x[8]+a0))^2+(-a1*x[10]-a0+y[10])^2*s1*x[10]*abs
(1, a1*x[10]+a0)/(s0+s1*abs(a1*x[10]+a0))^3+(-a1*x[11]-a0+y[11])
^2*s1*x[11]*abs(1, a1*x[11]+a0)/(s0+s1*abs(a1*x[11]+a0))^3+(-a1*x
[12]-a0+y[12])^2*s1*x[12]*abs(1, a1*x[12]+a0)/(s0+s1*abs(a1*x[12]
+a0))^3+(-a1*x[13]-a0+y[13])^2*s1*x[13]*abs(1, a1*x[13]+a0)/(s0+
s1*abs(a1*x[13]+a0))^3+(-a1*x[14]-a0+y[14])^2*s1*x[14]*abs(1, a1*
x[14]+a0)/(s0+s1*abs(a1*x[14]+a0))^3+(-a1*x[15]-a0+y[15])^2*s1*x
[15]*abs(1, a1*x[15]+a0)/(s0+s1*abs(a1*x[15]+a0))^3+(-a1*x[16]-
a0+y[16])^2*s1*x[16]*abs(1, a1*x[16]+a0)/(s0+s1*abs(a1*x[16]+a0))
^3+(-a1*x[17]-a0+y[17])^2*s1*x[17]*abs(1, a1*x[17]+a0)/(s0+s1*abs
(a1*x[17]+a0))^3+(-a1*x[18]-a0+y[18])^2*s1*x[18]*abs(1, a1*x[18]+
a0)/(s0+s1*abs(a1*x[18]+a0))^3+(-a1*x[19]-a0+y[19])^2*s1*x[19]*
abs(1, a1*x[19]+a0)/(s0+s1*abs(a1*x[19]+a0))^3+(-a1*x[20]-a0+y
[20])^2*s1*x[20]*abs(1, a1*x[20]+a0)/(s0+s1*abs(a1*x[20]+a0))^3+
(-a1*x[1]-a0+y[1])^2*s1*x[1]*abs(1, a1*x[1]+a0)/(s0+s1*abs(a1*x
[1]+a0))^3+(-a1*x[2]-a0+y[2])^2*s1*x[2]*abs(1, a1*x[2]+a0)/(s0+
s1*abs(a1*x[2]+a0))^3+(-a1*x[3]-a0+y[3])^2*s1*x[3]*abs(1, a1*x[3]
+a0)/(s0+s1*abs(a1*x[3]+a0))^3+(-a1*x[4]-a0+y[4])^2*s1*x[4]*abs
(1, a1*x[4]+a0)/(s0+s1*abs(a1*x[4]+a0))^3+(-a1*x[5]-a0+y[5])^2*
s1*x[5]*abs(1, a1*x[5]+a0)/(s0+s1*abs(a1*x[5]+a0))^3+(-a1*x[6]-
a0+y[6])^2*s1*x[6]*abs(1, a1*x[6]+a0)/(s0+s1*abs(a1*x[6]+a0))^3+
(-a1*x[7]-a0+y[7])^2*s1*x[7]*abs(1, a1*x[7]+a0)/(s0+s1*abs(a1*x
[7]+a0))^3+(-a1*x[8]-a0+y[8])^2*s1*x[8]*abs(1, a1*x[8]+a0)/(s0+
s1*abs(a1*x[8]+a0))^3+(-a1*x[9]-a0+y[9])^2*s1*x[9]*abs(1, a1*x[9]
+a0)/(s0+s1*abs(a1*x[9]+a0))^3-s1*x[9]*abs(1, a1*x[9]+a0)/(s0+s1*
abs(a1*x[9]+a0))-s1*x[10]*abs(1, a1*x[10]+a0)/(s0+s1*abs(a1*x[10]
+a0))-s1*x[11]*abs(1, a1*x[11]+a0)/(s0+s1*abs(a1*x[11]+a0))-s1*x
[12]*abs(1, a1*x[12]+a0)/(s0+s1*abs(a1*x[12]+a0))-s1*x[13]*abs(1,
a1*x[13]+a0)/(s0+s1*abs(a1*x[13]+a0))-s1*x[14]*abs(1, a1*x[14]+
a0)/(s0+s1*abs(a1*x[14]+a0))-s1*x[15]*abs(1, a1*x[15]+a0)/(s0+s1*
abs(a1*x[15]+a0))-s1*x[16]*abs(1, a1*x[16]+a0)/(s0+s1*abs(a1*x
[16]+a0))-s1*x[17]*abs(1, a1*x[17]+a0)/(s0+s1*abs(a1*x[17]+a0))-
s1*x[18]*abs(1, a1*x[18]+a0)/(s0+s1*abs(a1*x[18]+a0))-s1*x[19]*
abs(1, a1*x[19]+a0)/(s0+s1*abs(a1*x[19]+a0))-s1*x[20]*abs(1, a1*x
[20]+a0)/(s0+s1*abs(a1*x[20]+a0))-s1*x[1]*abs(1, a1*x[1]+a0)/(s0+
s1*abs(a1*x[1]+a0))-s1*x[2]*abs(1, a1*x[2]+a0)/(s0+s1*abs(a1*x[2]
+a0))-s1*x[3]*abs(1, a1*x[3]+a0)/(s0+s1*abs(a1*x[3]+a0))-s1*x[4]*
abs(1, a1*x[4]+a0)/(s0+s1*abs(a1*x[4]+a0))-s1*x[5]*abs(1, a1*x[5]
+a0)/(s0+s1*abs(a1*x[5]+a0))-s1*x[6]*abs(1, a1*x[6]+a0)/(s0+s1*
abs(a1*x[6]+a0))-s1*x[7]*abs(1, a1*x[7]+a0)/(s0+s1*abs(a1*x[7]+
a0))-s1*x[8]*abs(1, a1*x[8]+a0)/(s0+s1*abs(a1*x[8]+a0)))*exp(-10*
ln(2*Pi)-ln(s0+s1*abs(a1*x[12]+a0))-ln(s0+s1*abs(a1*x[13]+a0))-ln
```

```
(s0+s1*abs(a1*x[14]+a0))-ln(s0+s1*abs(a1*x[15]+a0))-ln(s0+s1*abs
(a1*x[16]+a0))-ln(s0+s1*abs(a1*x[17]+a0))-ln(s0+s1*abs(a1*x[18]+
a0))-ln(s0+s1*abs(a1*x[19]+a0))-ln(s0+s1*abs(a1*x[20]+a0))-ln(s0+
s1*abs(a1*x[1]+a0))-ln(s0+s1*abs(a1*x[2]+a0))-ln(s0+s1*abs(a1*x
[3]+a0))-ln(s0+s1*abs(a1*x[4]+a0))-ln(s0+s1*abs(a1*x[5]+a0))-ln
(s0+s1*abs(a1*x[6]+a0))-ln(s0+s1*abs(a1*x[7]+a0))-ln(s0+s1*abs
(a1*x[8]+a0))-ln(s0+s1*abs(a1*x[9]+a0))-ln(s0+s1*abs(a1*x[10]+a0)
)-ln(s0+s1*abs(a1*x[11]+a0))-(-a1*x[20]-a0+y[20])^2/(2*(s0+s1*abs
(a1*x[20]+a0))^2)-(-a1*x[19]-a0+y[19])^2/(2*(s0+s1*abs(a1*x[19]+
a0))^2)-(-a1*x[18]-a0+y[18])^2/(2*(s0+s1*abs(a1*x[18]+a0))^2)-(-
a1*x[17]-a0+y[17])^2/(2*(s0+s1*abs(a1*x[17]+a0))^2)-(-a1*x[16]-
a0+y[16])^2/(2*(s0+s1*abs(a1*x[16]+a0))^2)-(-a1*x[15]-a0+y[15])
^2/(2*(s0+s1*abs(a1*x[15]+a0))^2)-(-a1*x[14]-a0+y[14])^2/(2*(s0+
s1*abs(a1*x[14]+a0))^2)-(-a1*x[13]-a0+y[13])^2/(2*(s0+s1*abs(a1*x
[13]+a0))^2)-(-a1*x[12]-a0+y[12])^2/(2*(s0+s1*abs(a1*x[12]+a0))
^2)-(-a1*x[11]-a0+y[11])^2/(2*(s0+s1*abs(a1*x[11]+a0))^2)-(-a1*x
[10]-a0+y[10])^2/(2*(s0+s1*abs(a1*x[10]+a0))^2)-(-a1*x[9]-a0+y[9]
)^2/(2*(s0+s1*abs(a1*x[9]+a0))^2)-(-a1*x[8]-a0+y[8])^2/(2*(s0+s1*
abs(a1*x[8]+a0))^2)-(-a1*x[7]-a0+y[7])^2/(2*(s0+s1*abs(a1*x[7]+
a0))^2)-(-a1*x[6]-a0+y[6])^2/(2*(s0+s1*abs(a1*x[6]+a0))^2)-(-a1*x
[5]-a0+y[5])^2/(2*(s0+s1*abs(a1*x[5]+a0))^2)-(-a1*x[4]-a0+y[4])
^2/(2*(s0+s1*abs(a1*x[4]+a0))^2)-(-a1*x[3]-a0+y[3])^2/(2*(s0+s1*
abs(a1*x[3]+a0))^2)-(-a1*x[2]-a0+y[2])^2/(2*(s0+s1*abs(a1*x[2]+
a0))^2)-(-a1*x[1]-a0+y[1])^2/(2*(s0+s1*abs(a1*x[1]+a0))^2)):
> delpdela0:=((-a1*x[10]-a0+y[10])^2*s1*abs(1, a1*x[10]+a0)/(s0+s1*
abs(a1*x[10]+a0))^3+(-a1*x[20]-a0+y[20])^2*s1*abs(1, a1*x[20]+a0)
/(s0+s1*abs(a1*x[20]+a0))^3+(-a1*x[18]-a0+y[18])^2*s1*abs(1, a1*x
[18]+a0)/(s0+s1*abs(a1*x[18]+a0))^3+(-a1*x[19]-a0+y[19])^2*s1*abs
(1, a1*x[19]+a0)/(s0+s1*abs(a1*x[19]+a0))^3+(-a1*x[16]-a0+y[16])
^2*s1*abs(1, a1*x[16]+a0)/(s0+s1*abs(a1*x[16]+a0))^3+(-a1*x[17]-
a0+y[17])^2*s1*abs(1, a1*x[17]+a0)/(s0+s1*abs(a1*x[17]+a0))^3+(-
a1*x[9]-a0+y[9])^2*s1*abs(1, a1*x[9]+a0)/(s0+s1*abs(a1*x[9]+a0))
^3+(-a1*x[7]-a0+y[7])^2*s1*abs(1, a1*x[7]+a0)/(s0+s1*abs(a1*x[7]+
a0))^3+(-a1*x[8]-a0+y[8])^2*s1*abs(1, a1*x[8]+a0)/(s0+s1*abs(a1*x
[8]+a0))^3+(-a1*x[5]-a0+y[5])^2*s1*abs(1, a1*x[5]+a0)/(s0+s1*abs
(a1*x[5]+a0))^3+(-a1*x[6]-a0+y[6])^2*s1*abs(1, a1*x[6]+a0)/(s0+
s1*abs(a1*x[6]+a0))^3+(-a1*x[3]-a0+y[3])^2*s1*abs(1, a1*x[3]+a0)/
(s0+s1*abs(a1*x[3]+a0))^3+(-a1*x[4]-a0+y[4])^2*s1*abs(1, a1*x[4]+
a0)/(s0+s1*abs(a1*x[4]+a0))^3+(-a1*x[2]-a0+y[2])^2*s1*abs(1, a1*x
[2]+a0)/(s0+s1*abs(a1*x[2]+a0))^3+(-a1*x[15]-a0+y[15])^2*s1*abs
(1, a1*x[15]+a0)/(s0+s1*abs(a1*x[15]+a0))^3+(-a1*x[1]-a0+y[1])^2*
s1*abs(1, a1*x[1]+a0)/(s0+s1*abs(a1*x[1]+a0))^3+(-a1*x[13]-a0+y
[13])^2*s1*abs(1, a1*x[13]+a0)/(s0+s1*abs(a1*x[13]+a0))^3+(-a1*x
[14]-a0+y[14])^2*s1*abs(1, a1*x[14]+a0)/(s0+s1*abs(a1*x[14]+a0))
^3+(-a1*x[12]-a0+y[12])^2*s1*abs(1, a1*x[12]+a0)/(s0+s1*abs(a1*x
[12]+a0))^3+(-a1*x[11]-a0+y[11])^2*s1*abs(1, a1*x[11]+a0)/(s0+s1*
abs(a1*x[11]+a0))^3+(-a1*x[11]-a0+y[11])/(s0+s1*abs(a1*x[11]+a0))
^2+(-a1*x[10]-a0+y[10])/(s0+s1*abs(a1*x[10]+a0))^2+(-a1*x[9]-a0+y
[9])/(s0+s1*abs(a1*x[9]+a0))^2+(-a1*x[8]-a0+y[8])/(s0+s1*abs(a1*x
[8]+a0))^2+(-a1*x[7]-a0+y[7])/(s0+s1*abs(a1*x[7]+a0))^2+(-a1*x[6]
-a0+y[6])/(s0+s1*abs(a1*x[6]+a0))^2+(-a1*x[5]-a0+y[5])/(s0+s1*abs
(a1*x[5]+a0))^2+(-a1*x[4]-a0+y[4])/(s0+s1*abs(a1*x[4]+a0))^2+(-
a1*x[3]-a0+y[3])/(s0+s1*abs(a1*x[3]+a0))^2+(-a1*x[2]-a0+y[2])/
(s0+s1*abs(a1*x[2]+a0))^2+(-a1*x[1]-a0+y[1])/(s0+s1*abs(a1*x[1]+
a0))^2+(-a1*x[17]-a0+y[17])/(s0+s1*abs(a1*x[17]+a0))^2+(-a1*x[16]
-a0+y[16])/(s0+s1*abs(a1*x[16]+a0))^2+(-a1*x[15]-a0+y[15])/(s0+
```

```
s1*abs(a1*x[15]+a0))^2+(-a1*x[14]-a0+y[14])/(s0+s1*abs(a1*x[14]+
a0))^2+(-a1*x[13]-a0+y[13])/(s0+s1*abs(a1*x[13]+a0))^2+(-a1*x[12]
-a0+y[12])/(s0+s1*abs(a1*x[12]+a0))^2+(-a1*x[18]-a0+y[18])/(s0+
s1*abs(a1*x[18]+a0))^2+(-a1*x[20]-a0+y[20])/(s0+s1*abs(a1*x[20]+
a0))^2+(-a1*x[19]-a0+y[19])/(s0+s1*abs(a1*x[19]+a0))^2-s1*abs(1,
a1*x[8]+a0)/(s0+s1*abs(a1*x[8]+a0))-s1*abs(1, a1*x[3]+a0)/(s0+s1*
abs(a1*x[3]+a0))-s1*abs(1, a1*x[12]+a0)/(s0+s1*abs(a1*x[12]+a0))-
s1*abs(1, a1*x[18]+a0)/(s0+s1*abs(a1*x[18]+a0))-s1*abs(1, a1*x[6]
+a0)/(s0+s1*abs(a1*x[6]+a0))-s1*abs(1, a1*x[20]+a0)/(s0+s1*abs
(a1*x[20]+a0))-s1*abs(1, a1*x[17]+a0)/(s0+s1*abs(a1*x[17]+a0))-
s1*abs(1, a1*x[1]+a0)/(s0+s1*abs(a1*x[1]+a0))-s1*abs(1, a1*x[7]+
a0)/(s0+s1*abs(a1*x[7]+a0))-s1*abs(1, a1*x[5]+a0)/(s0+s1*abs(a1*x
[5]+a0))-s1*abs(1, a1*x[11]+a0)/(s0+s1*abs(a1*x[11]+a0))-s1*abs
(1, a1*x[4]+a0)/(s0+s1*abs(a1*x[4]+a0))-s1*abs(1, a1*x[15]+a0)/
(s0+s1*abs(a1*x[15]+a0))-s1*abs(1, a1*x[2]+a0)/(s0+s1*abs(a1*x[2]
+a0))-s1*abs(1, a1*x[10]+a0)/(s0+s1*abs(a1*x[10]+a0))-s1*abs(1,
a1*x[13]+a0)/(s0+s1*abs(a1*x[13]+a0))-s1*abs(1, a1*x[16]+a0)/(s0+
s1*abs(a1*x[16]+a0))-s1*abs(1, a1*x[14]+a0)/(s0+s1*abs(a1*x[14]+
a0))-s1*abs(1, a1*x[9]+a0)/(s0+s1*abs(a1*x[9]+a0))-s1*abs(1, a1*x
[19]+a0)/(s0+s1*abs(a1*x[19]+a0)))*exp(-10*ln(2*Pi)-ln(s0+s1*abs
(a1*x[12]+a0))-ln(s0+s1*abs(a1*x[13]+a0))-ln(s0+s1*abs(a1*x[14]+
a0))-ln(s0+s1*abs(a1*x[15]+a0))-ln(s0+s1*abs(a1*x[16]+a0))-ln(s0+
s1*abs(a1*x[17]+a0))-ln(s0+s1*abs(a1*x[18]+a0))-ln(s0+s1*abs(a1*x
[19]+a0))-ln(s0+s1*abs(a1*x[20]+a0))-ln(s0+s1*abs(a1*x[1]+a0))-ln
(s0+s1*abs(a1*x[2]+a0))-ln(s0+s1*abs(a1*x[3]+a0))-ln(s0+s1*abs
(a1*x[4]+a0))-ln(s0+s1*abs(a1*x[5]+a0))-ln(s0+s1*abs(a1*x[6]+a0))
-ln(s0+s1*abs(a1*x[7]+a0))-ln(s0+s1*abs(a1*x[8]+a0))-ln(s0+s1*abs
(a1*x[9]+a0))-ln(s0+s1*abs(a1*x[10]+a0))-ln(s0+s1*abs(a1*x[11]+
a0))-(-a1*x[20]-a0+y[20])^2/(2*(s0+s1*abs(a1*x[20]+a0))^2)-(-a1*x
[19]-a0+y[19])^2/(2*(s0+s1*abs(a1*x[19]+a0))^2)-(-a1*x[18]-a0+y
[18])^2/(2*(s0+s1*abs(a1*x[18]+a0))^2)-(-a1*x[17]-a0+y[17])^2/(2*
(s0+s1*abs(a1*x[17]+a0))^2)-(-a1*x[16]-a0+y[16])^2/(2*(s0+s1*abs
(a1*x[16]+a0))^2)-(-a1*x[15]-a0+y[15])^2/(2*(s0+s1*abs(a1*x[15]+
a0))^2)-(-a1*x[14]-a0+y[14])^2/(2*(s0+s1*abs(a1*x[14]+a0))^2)-(-
a1*x[13]-a0+y[13])^2/(2*(s0+s1*abs(a1*x[13]+a0))^2)-(-a1*x[12]-
a0+y[12])^2/(2*(s0+s1*abs(a1*x[12]+a0))^2)-(-a1*x[11]-a0+y[11])
^2/(2*(s0+s1*abs(a1*x[11]+a0))^2)-(-a1*x[10]-a0+y[10])^2/(2*(s0+
s1*abs(a1*x[10]+a0))^2)-(-a1*x[9]-a0+y[9])^2/(2*(s0+s1*abs(a1*x
[9]+a0))^2)-(-a1*x[8]-a0+y[8])^2/(2*(s0+s1*abs(a1*x[8]+a0))^2)-(-
a1*x[7]-a0+y[7])^2/(2*(s0+s1*abs(a1*x[7]+a0))^2)-(-a1*x[6]-a0+y
[6])^2/(2*(s0+s1*abs(a1*x[6]+a0))^2)-(-a1*x[5]-a0+y[5])^2/(2*(s0+
s1*abs(a1*x[5]+a0))^2)-(-a1*x[4]-a0+y[4])^2/(2*(s0+s1*abs(a1*x[4]
+a0))^2)-(-a1*x[3]-a0+y[3])^2/(2*(s0+s1*abs(a1*x[3]+a0))^2)-(-a1*
x[2]-a0+y[2])^2/(2*(s0+s1*abs(a1*x[2]+a0))^2)-(-a1*x[1]-a0+y[1])
^2/(2*(s0+s1*abs(a1*x[1]+a0))^2)):
> delpdels0:=(-1/(s0+s1*abs(a1*x[4]+a0))-1/(s0+s1*abs(a1*x[3]+a0))
-1/(s0+s1*abs(a1*x[2]+a0))-1/(s0+s1*abs(a1*x[1]+a0))-1/(s0+s1*abs
(a1*x[20]+a0))-1/(s0+s1*abs(a1*x[19]+a0))-1/(s0+s1*abs(a1*x[18]+
a0))-1/(s0+s1*abs(a1*x[17]+a0))-1/(s0+s1*abs(a1*x[16]+a0))-1/(s0+
s1*abs(a1*x[15]+a0))-1/(s0+s1*abs(a1*x[14]+a0))-1/(s0+s1*abs(a1*x
[13]+a0))-1/(s0+s1*abs(a1*x[12]+a0))-1/(s0+s1*abs(a1*x[11]+a0))
-1/(s0+s1*abs(a1*x[10]+a0))-1/(s0+s1*abs(a1*x[9]+a0))-1/(s0+s1*
abs(a1*x[8]+a0))-1/(s0+s1*abs(a1*x[7]+a0))-1/(s0+s1*abs(a1*x[6]+
a0))-1/(s0+s1*abs(a1*x[5]+a0))+(-a1*x[19]-a0+y[19])^2/(s0+s1*abs
(a1*x[19]+a0))^3+(-a1*x[20]-a0+y[20])^2/(s0+s1*abs(a1*x[20]+a0))
^3+(-a1*x[7]-a0+y[7])^2/(s0+s1*abs(a1*x[7]+a0))^3+(-a1*x[8]-a0+y
```

```
[8])^2/(s0+s1*abs(a1*x[8]+a0))^3+(-a1*x[9]-a0+y[9])^2/(s0+s1*abs
(a1*x[9]+a0))^3+(-a1*x[10]-a0+y[10])^2/(s0+s1*abs(a1*x[10]+a0))
^3+(-a1*x[11]-a0+y[11])^2/(s0+s1*abs(a1*x[11]+a0))^3+(-a1*x[12]-
a0+y[12])^2/(s0+s1*abs(a1*x[12]+a0))^3+(-a1*x[13]-a0+y[13])^2/
(s0+s1*abs(a1*x[13]+a0))^3+(-a1*x[14]-a0+y[14])^2/(s0+s1*abs(a1*x
[14]+a0))^3+(-a1*x[15]-a0+y[15])^2/(s0+s1*abs(a1*x[15]+a0))^3+(-
a1*x[16]-a0+y[16])^2/(s0+s1*abs(a1*x[16]+a0))^3+(-a1*x[17]-a0+y
[17])^2/(s0+s1*abs(a1*x[17]+a0))^3+(-a1*x[18]-a0+y[18])^2/(s0+s1*
abs(a1*x[18]+a0))^3+(-a1*x[1]-a0+y[1])^2/(s0+s1*abs(a1*x[1]+a0))
^3+(-a1*x[2]-a0+y[2])^2/(s0+s1*abs(a1*x[2]+a0))^3+(-a1*x[3]-a0+y
[3])^2/(s0+s1*abs(a1*x[3]+a0))^3+(-a1*x[4]-a0+y[4])^2/(s0+s1*abs
(a1*x[4]+a0))^3+(-a1*x[5]-a0+y[5])^2/(s0+s1*abs(a1*x[5]+a0))^3+(-
a1*x[6]-a0+y[6])^2/(s0+s1*abs(a1*x[6]+a0))^3)*exp(-10*ln(2*Pi)-ln
(s0+s1*abs(a1*x[12]+a0))-ln(s0+s1*abs(a1*x[13]+a0))-ln(s0+s1*abs
(a1*x[14]+a0))-ln(s0+s1*abs(a1*x[15]+a0))-ln(s0+s1*abs(a1*x[16]+
a0))-ln(s0+s1*abs(a1*x[17]+a0))-ln(s0+s1*abs(a1*x[18]+a0))-ln(s0+
s1*abs(a1*x[19]+a0))-ln(s0+s1*abs(a1*x[20]+a0))-ln(s0+s1*abs(a1*x
[1]+a0))-ln(s0+s1*abs(a1*x[2]+a0))-ln(s0+s1*abs(a1*x[3]+a0))-ln
(s0+s1*abs(a1*x[4]+a0))-ln(s0+s1*abs(a1*x[5]+a0))-ln(s0+s1*abs
(a1*x[6]+a0))-ln(s0+s1*abs(a1*x[7]+a0))-ln(s0+s1*abs(a1*x[8]+a0))
-ln(s0+s1*abs(a1*x[9]+a0))-ln(s0+s1*abs(a1*x[10]+a0))-ln(s0+s1*
abs(a1*x[11]+a0))-(-a1*x[20]-a0+y[20])^2/(2*(s0+s1*abs(a1*x[20]+
a0))^2)-(-a1*x[19]-a0+y[19])^2/(2*(s0+s1*abs(a1*x[19]+a0))^2)-(-
a1*x[18]-a0+y[18])^2/(2*(s0+s1*abs(a1*x[18]+a0))^2)-(-a1*x[17]-
a0+y[17])^2/(2*(s0+s1*abs(a1*x[17]+a0))^2)-(-a1*x[16]-a0+y[16])
^2/(2*(s0+s1*abs(a1*x[16]+a0))^2)-(-a1*x[15]-a0+y[15])^2/(2*(s0+
s1*abs(a1*x[15]+a0))^2)-(-a1*x[14]-a0+y[14])^2/(2*(s0+s1*abs(a1*x
[14]+a0))^2)-(-a1*x[13]-a0+y[13])^2/(2*(s0+s1*abs(a1*x[13]+a0))
^2)-(-a1*x[12]-a0+y[12])^2/(2*(s0+s1*abs(a1*x[12]+a0))^2)-(-a1*x
[11]-a0+y[11])^2/(2*(s0+s1*abs(a1*x[11]+a0))^2)-(-a1*x[10]-a0+y
[10])^2/(2*(s0+s1*abs(a1*x[10]+a0))^2)-(-a1*x[9]-a0+y[9])^2/(2*
(s0+s1*abs(a1*x[9]+a0))^2)-(-a1*x[8]-a0+y[8])^2/(2*(s0+s1*abs(a1*
x[8]+a0))^2)-(-a1*x[7]-a0+y[7])^2/(2*(s0+s1*abs(a1*x[7]+a0))^2)-
(-a1*x[6]-a0+y[6])^2/(2*(s0+s1*abs(a1*x[6]+a0))^2)-(-a1*x[5]-a0+y
[5])^2/(2*(s0+s1*abs(a1*x[5]+a0))^2)-(-a1*x[4]-a0+y[4])^2/(2*(s0+
s1*abs(a1*x[4]+a0))^2)-(-a1*x[3]-a0+y[3])^2/(2*(s0+s1*abs(a1*x[3]
+a0))^2)-(-a1*x[2]-a0+y[2])^2/(2*(s0+s1*abs(a1*x[2]+a0))^2)-(-a1*
x[1]-a0+y[1])^2/(2*(s0+s1*abs(a1*x[1]+a0))^2)):
> delpdels1:=((-a1*x[3]-a0+y[3])^2*abs(a1*x[3]+a0)/(s0+s1*abs(a1*x
[3]+a0))^3+(-a1*x[16]-a0+y[16])^2*abs(a1*x[16]+a0)/(s0+s1*abs(a1*
x[16]+a0))^3+(-a1*x[18]-a0+y[18])^2*abs(a1*x[18]+a0)/(s0+s1*abs
(a1*x[18]+a0))^3+(-a1*x[6]-a0+y[6])^2*abs(a1*x[6]+a0)/(s0+s1*abs
(a1*x[6]+a0))^3+(-a1*x[11]-a0+y[11])^2*abs(a1*x[11]+a0)/(s0+s1*
abs(a1*x[11]+a0))^3+(-a1*x[4]-a0+y[4])^2*abs(a1*x[4]+a0)/(s0+s1*
abs(a1*x[4]+a0))^3+(-a1*x[7]-a0+y[7])^2*abs(a1*x[7]+a0)/(s0+s1*
abs(a1*x[7]+a0))^3+(-a1*x[13]-a0+y[13])^2*abs(a1*x[13]+a0)/(s0+
s1*abs(a1*x[13]+a0))^3+(-a1*x[15]-a0+y[15])^2*abs(a1*x[15]+a0)/
(s0+s1*abs(a1*x[15]+a0))^3+(-a1*x[8]-a0+y[8])^2*abs(a1*x[8]+a0)/
(s0+s1*abs(a1*x[8]+a0))^3+(-a1*x[10]-a0+y[10])^2*abs(a1*x[10]+a0)
/(s0+s1*abs(a1*x[10]+a0))^3+(-a1*x[20]-a0+y[20])^2*abs(a1*x[20]+
a0)/(s0+s1*abs(a1*x[20]+a0))^3+(-a1*x[12]-a0+y[12])^2*abs(a1*x
[12]+a0)/(s0+s1*abs(a1*x[12]+a0))^3+(-a1*x[17]-a0+y[17])^2*abs
(a1*x[17]+a0)/(s0+s1*abs(a1*x[17]+a0))^3+(-a1*x[14]-a0+y[14])^2*
abs(a1*x[14]+a0)/(s0+s1*abs(a1*x[14]+a0))^3+(-a1*x[5]-a0+y[5])^2*
abs(a1*x[5]+a0)/(s0+s1*abs(a1*x[5]+a0))^3+(-a1*x[19]-a0+y[19])^2*
abs(a1*x[19]+a0)/(s0+s1*abs(a1*x[19]+a0))^3+(-a1*x[1]-a0+y[1])^2*
```

```
abs(a1*x[1]+a0)/(s0+s1*abs(a1*x[1]+a0))^3+(-a1*x[2]-a0+y[2])^2*
abs(a1*x[2]+a0)/(s0+s1*abs(a1*x[2]+a0))^3+(-a1*x[9]-a0+y[9])^2*
abs(a1*x[9]+a0)/(s0+s1*abs(a1*x[9]+a0))^3-abs(a1*x[6]+a0)/(s0+s1*
abs(a1*x[6]+a0))-abs(a1*x[7]+a0)/(s0+s1*abs(a1*x[7]+a0))-abs(a1*x
[8]+a0)/(s0+s1*abs(a1*x[8]+a0))-abs(a1*x[9]+a0)/(s0+s1*abs(a1*x
[9]+a0))-abs(a1*x[10]+a0)/(s0+s1*abs(a1*x[10]+a0))-abs(a1*x[11]+
a0)/(s0+s1*abs(a1*x[11]+a0))-abs(a1*x[12]+a0)/(s0+s1*abs(a1*x[12]
+a0))-abs(a1*x[13]+a0)/(s0+s1*abs(a1*x[13]+a0))-abs(a1*x[14]+a0)/
(s0+s1*abs(a1*x[14]+a0))-abs(a1*x[15]+a0)/(s0+s1*abs(a1*x[15]+a0)
)-abs(a1*x[16]+a0)/(s0+s1*abs(a1*x[16]+a0))-abs(a1*x[17]+a0)/(s0+
s1*abs(a1*x[17]+a0))-abs(a1*x[18]+a0)/(s0+s1*abs(a1*x[18]+a0))-
abs(a1*x[19]+a0)/(s0+s1*abs(a1*x[19]+a0))-abs(a1*x[20]+a0)/(s0+
s1*abs(a1*x[20]+a0))-abs(a1*x[1]+a0)/(s0+s1*abs(a1*x[1]+a0))-abs
(a1*x[2]+a0)/(s0+s1*abs(a1*x[2]+a0))-abs(a1*x[3]+a0)/(s0+s1*abs
(a1*x[3]+a0))-abs(a1*x[4]+a0)/(s0+s1*abs(a1*x[4]+a0))-abs(a1*x[5]
+a0)/(s0+s1*abs(a1*x[5]+a0)))*exp(-10*ln(2*Pi)-ln(s0+s1*abs(a1*x
[12]+a0))-ln(s0+s1*abs(a1*x[13]+a0))-ln(s0+s1*abs(a1*x[14]+a0))-
ln(s0+s1*abs(a1*x[15]+a0))-ln(s0+s1*abs(a1*x[16]+a0))-ln(s0+s1*
abs(a1*x[17]+a0))-ln(s0+s1*abs(a1*x[18]+a0))-ln(s0+s1*abs(a1*x
[19]+a0))-ln(s0+s1*abs(a1*x[20]+a0))-ln(s0+s1*abs(a1*x[1]+a0))-ln
(s0+s1*abs(a1*x[2]+a0))-ln(s0+s1*abs(a1*x[3]+a0))-ln(s0+s1*abs
(a1*x[4]+a0))-ln(s0+s1*abs(a1*x[5]+a0))-ln(s0+s1*abs(a1*x[6]+a0))
-ln(s0+s1*abs(a1*x[7]+a0))-ln(s0+s1*abs(a1*x[8]+a0))-ln(s0+s1*abs
(a1*x[9]+a0))-ln(s0+s1*abs(a1*x[10]+a0))-ln(s0+s1*abs(a1*x[11]+
a0))-(-a1*x[20]-a0+y[20])^2/(2*(s0+s1*abs(a1*x[20]+a0))^2)-(-a1*x
[19]-a0+y[19])^2/(2*(s0+s1*abs(a1*x[19]+a0))^2)-(-a1*x[18]-a0+y
[18])^2/(2*(s0+s1*abs(a1*x[18]+a0))^2)-(-a1*x[17]-a0+y[17])^2/(2*
(s0+s1*abs(a1*x[17]+a0))^2)-(-a1*x[16]-a0+y[16])^2/(2*(s0+s1*abs
(a1*x[16]+a0))^2)-(-a1*x[15]-a0+y[15])^2/(2*(s0+s1*abs(a1*x[15]+
a0))^2)-(-a1*x[14]-a0+y[14])^2/(2*(s0+s1*abs(a1*x[14]+a0))^2)-(-
a1*x[13]-a0+y[13])^2/(2*(s0+s1*abs(a1*x[13]+a0))^2)-(-a1*x[12]-
a0+y[12])^2/(2*(s0+s1*abs(a1*x[12]+a0))^2)-(-a1*x[11]-a0+y[11])
^2/(2*(s0+s1*abs(a1*x[11]+a0))^2)-(-a1*x[10]-a0+y[10])^2/(2*(s0+
s1*abs(a1*x[10]+a0))^2)-(-a1*x[9]-a0+y[9])^2/(2*(s0+s1*abs(a1*x
[9]+a0))^2)-(-a1*x[8]-a0+y[8])^2/(2*(s0+s1*abs(a1*x[8]+a0))^2)-(-
a1*x[7]-a0+y[7])^2/(2*(s0+s1*abs(a1*x[7]+a0))^2)-(-a1*x[6]-a0+y
[6])^2/(2*(s0+s1*abs(a1*x[6]+a0))^2)-(-a1*x[5]-a0+y[5])^2/(2*(s0+
s1*abs(a1*x[5]+a0))^2)-(-a1*x[4]-a0+y[4])^2/(2*(s0+s1*abs(a1*x[4]
+a0))^2)-(-a1*x[3]-a0+y[3])^2/(2*(s0+s1*abs(a1*x[3]+a0))^2)-(-a1*
x[2]-a0+y[2])^2/(2*(s0+s1*abs(a1*x[2]+a0))^2)-(-a1*x[1]-a0+y[1])
^2/(2*(s0+s1*abs(a1*x[1]+a0))^2)):
```

The partial derivatives are all set to zero, and the objective function Z_1 (Eq. 18) is minimized providing the optimized parameters a0, a1,s0 and s1 (see also Table 1):

```
> f:=delpdels1=0:g:=delpdela1=0:u:=delpdels0=0:v:=delpdela0=0:
> fsolve({f,g,u,v}, {s0 = 1.5 .. 2, s1 = 0.07 .. 0.1, a0 = 1.5 ..
  2, a1 = 0.4 .. 0.6});
```

$$\{a0 = 1.767158847,\ a1 = 0.4681491315,\ s0 = 1.723975788,\ s1 = 0.08721863022\} \quad \textbf{(4)}$$

```
> Z:=-2*ln(Prob_ex3):
> hess:=Hessian(Z, [a0, a1, s0,s1]):

> hess0:=subs(a0= 1.767158847,hess):
> hess1:=subs(a1= .4681491315,hess0):hess2:=subs(s0= 1.723975788,
  hess1):hess3:=subs(s1= 0.8721863022e-1,hess2):
```

The Hessian matrix is calculated for the minimum of the objective function Z1:

```
> hess_matrix:=evalf(hess3);
```

$$hess\_matrix := \begin{bmatrix} 5.576189343 & 86.25003034 & 0.6919358702 & 4.007823943 \\ 86.25003034 & 4799.968285 & 27.54237605 & 240.2800389 \\ 0.6919358702 & 27.54237605 & 9.997150598 & 103.2304434 \\ 4.007823943 & 240.2800389 & 103.2304434 & 2529.680397 \end{bmatrix} \quad (5)$$

The covariance matrix (inverse of the Hessian matrix) is calculated:

```
> cov:=MatrixInverse(hess_matrix);
```

$$cov := [[0.248839419565509, -0.00443904009413467, -0.00911868397342866, 0.000399510520696810], [-0.00443904009413467, 0.000290928587417226, -0.000486589630174699, -7.44195928457812\ 10^{-7}], [-0.00911868397342866, -0.000486589630174698, 0.175199154424154, -0.00708880954311573], [0.000399510520696810, -7.44195928457812\ 10^{-7}, -0.00708880954311573, 0.000684022614857737]] \quad (6)$$

```
> with(Statistics);
```

$$[AbsoluteDeviation, AgglomeratedPlot, AreaChart, AutoCorrelation, BarChart, Biplot, Bootstrap, BoxPlot, BubblePlot, CDF, CGF, CentralMoment, CharacteristicFunction, ChiSquareGoodnessOfFitTest, ChiSquareIndependenceTest, ChiSquareSuitableModelTest, ColumnGraph, Correlation, CorrelationMatrix, Count, CountMissing, Covariance, CovarianceMatrix, CrossCorrelation, Cumulant, CumulantGeneratingFunction, CumulativeDistributionFunction, CumulativeProduct, CumulativeSum, CumulativeSumChart, DataSummary, Decile, DensityPlot, DiscreteValueMap, Distribution, ErrorPlot, EvaluateToFloat, Excise, ExpectedValue, ExponentialFit, ExponentialSmoothing, FailureRate, FisherInformation, Fit, FivePointSummary, FrequencyPlot, FrequencyTable, GeometricMean, GridPlot, HarmonicMean, HazardRate, HeatMap, Histogram, HodgesLehmann, Information, InteractiveDataAnalysis, InterquartileRange, InverseSurvivalFunction, Join, KernelDensity, KernelDensityPlot, KernelDensitySample, Kurtosis, Likelihood, LikelihoodRatioStatistic, LineChart, LinearFilter, LinearFit, LogLikelihood, LogarithmicFit, Lowess, MGF, MLE, MakeProcedure, MaximumLikelihoodEstimate, Mean, MeanDeviation, Median, MedianDeviation, MillsRatio, Mode, Moment, MomentGeneratingFunction, MovingAverage, MovingMedian, MovingStatistic, NonlinearFit, NormalPlot, OneSampleChiSquareTest, OneSampleTTest, OneSampleZTest, OneWayANOVA, OrderByRank, OrderStatistic, PCA, PDF, ParetoChart, Percentile, PieChart, PointPlot, PolynomialFit, PowerFit, PredictiveLeastSquares, PrincipalComponentAnalysis, Probability, ProbabilityDensityFunction, ProbabilityFunction, ProbabilityPlot, ProfileLikelihood, ProfileLogLikelihood, QuadraticMean, Quantile, QuantilePlot, Quartile, RandomVariable, Range, Rank, Remove, RemoveInRange, RemoveNonNumeric, RepeatedMedianEstimator, RousseeuwCrouxQn, RousseeuwCrouxSn, Sample, Scale, \quad (7)$$

ScatterPlot, *ScatterPlot3D*, *Score*, *ScreePlot*, *Select*, *SelectInRange*, *SelectNonNumeric*, *ShapiroWilkWTest*, *Shuffle*, *Skewness*, *Sort*, *Specialize*, *SplitByColumn*, *StandardDeviation*, *StandardError*, *StandardizedMoment*, *SunflowerPlot*, *Support*, *SurfacePlot*, *SurvivalFunction*, *SymmetryPlot*, *Tally*, *TallyInto*, *TreeMap*, *Trim*, *TrimmedMean*, *TwoSampleFTest*, *TwoSamplePairedTTest*, *TwoSampleTTest*, *TwoSampleZTest*, *Variance*, *Variation*, *VennDiagram*, *ViolinPlot*, *WeibullPlot*, *WeightedMovingAverage*, *Winsorize*, *WinsorizedMean*]

The correlation coefficients (Eq. 24) follow from the covariance matrix (compare Table 2):

```
> rho12:=cov(1,2)/sqrt(cov(1,1))/sqrt(cov(2,2)); rho13:=cov(1,3)
  /sqrt(cov(1,1))/sqrt(cov(3,3)); rho14:=cov(1,4)/sqrt(cov(1,1))
  /sqrt(cov(4,4));rho23:=cov(2,3)/sqrt(cov(2,2))/sqrt(cov(3,3));
  rho24:=cov(2,4)/sqrt(cov(2,2))/sqrt(cov(4,4));rho34:=cov(3,4)
  /sqrt(cov(3,3))/sqrt(cov(4,4));
```

$$\rho 12 := -0.521718550137058$$

$$\rho 13 := -0.0436723510109006$$

$$\rho 14 := 0.0306219893043441$$

$$\rho 23 := -0.0681559298616130$$

$$\rho 24 := -0.00166824126790481 \quad \textbf{(8)}$$

The uncertainties of the parameters follow from the diagonal terms of the covariance matrix (compare Table 1):

```
> dela0:=sqrt(cov(1,1));
```

$$dela0 := 0.498838069482982 \quad \textbf{(9)}$$

```
> dela1:=sqrt(cov(2,2));
```

$$dela1 := 0.0170566288409294 \quad \textbf{(10)}$$

```
> dels0:=sqrt(cov(3,3));
```

$$dels0 := 0.418567980648489 \quad \textbf{(11)}$$

```
> dels1:=sqrt(cov(4,4));
```

$$dels1 := 0.0261538260080191 \quad \textbf{(12)}$$

```
> with(LinearAlgebra);
```

[*&x*, *Add*, *Adjoint*, *BackwardSubstitute*, *BandMatrix*, *Basis*, *BezoutMatrix*, *BidiagonalForm*, *BilinearForm*, *CARE*, *CharacteristicMatrix*, *CharacteristicPolynomial*, *Column*, *ColumnDimension*, *ColumnOperation*, *ColumnSpace*, *CompanionMatrix*, *CompressedSparseForm*, *ConditionNumber*, *ConstantMatrix*, *ConstantVector*, *Copy*, *CreatePermutation*, *CrossProduct*, *DARE*, *DeleteColumn*, *DeleteRow*, *Determinant*, *Diagonal*, *DiagonalMatrix*, *Dimension*, *Dimensions*, *DotProduct*, *EigenConditionNumbers*, *Eigenvalues*, *Eigenvectors*, *Equal*, *ForwardSubstitute*, *FrobeniusForm*, *FromCompressedSparseForm*, *FromSplitForm*, *GaussianElimination*, *GenerateEquations*, *GenerateMatrix*, *Generic*, *GetResultDataType*, *GetResultShape*, *GivensRotationMatrix*, *GramSchmidt*, *HankelMatrix*, *HermiteForm*, *HermitianTranspose*, *HessenbergForm*, *HilbertMatrix*, *HouseholderMatrix*, *IdentityMatrix*, *IntersectionBasis*, *IsDefinite*, *IsOrthogonal*, *IsSimilar*, *IsUnitary*, *JordanBlockMatrix*, *JordanForm*, *KroneckerProduct*, *LA_Main*, *LUDecomposition*, *LeastSquares*, *LinearSolve*, *LyapunovSolve*, *Map*, *Map2*, *MatrixAdd*, *MatrixExponential*, *MatrixFunction*, *MatrixInverse*, *MatrixMatrixMultiply*, *MatrixNorm*, *MatrixPower*, *MatrixScalarMultiply*, *MatrixVectorMultiply*, **(13)**

MinimalPolynomial, *Minor*, *Modular*, *Multiply*, *NoUserValue*, *Norm*, *Normalize*, *NullSpace*, *OuterProductMatrix*, *Permanent*, *Pivot*, *PopovForm*, *ProjectionMatrix*, *QRDecomposition*, *RandomMatrix*, *RandomVector*, *Rank*, *RationalCanonicalForm*, *ReducedRowEchelonForm*, *Row*, *RowDimension*, *RowOperation*, *RowSpace*, *ScalarMatrix*, *ScalarMultiply*, *ScalarVector*, *SchurForm*, *SingularValues*, *SmithForm*, *SplitForm*, *StronglyConnectedBlocks*, *SubMatrix*, *SubVector*, *SumBasis*, *SylvesterMatrix*, *SylvesterSolve*, *ToeplitzMatrix*, *Trace*, *Transpose*, *TridiagonalForm*, *UnitVector*, *VandermondeMatrix*, *VectorAdd*, *VectorAngle*, *VectorMatrixMultiply*, *VectorNorm*, *VectorScalarMultiply*, *ZeroMatrix*, *ZeroVector*, *Zip*]

```
> Determinant(hess_matrix);
```

$$2.777231249\ 10^{8} \quad \textbf{(14)}$$

```
> ConditionNumber(hess_matrix);
```

$$1354.464659 \quad \textbf{(15)}$$