

```

> restart;
Maple calculations regarding example 5
=====
> for i from 1 to 20 do yc[i]:=a1*x[i] end do:
> for i from 1 to 20 do s[i]:=s0 end do:
> lns:=add(ln(s[i]),i=1..20);
                                     lns := 20 ln(s0)
> lt:=add((y[i]-yc[i])^2/2/s[i]^2,i=1..20):
The probability density function 'Prob_ex5' is defined:
> Prob_ex5:=exp(-(20/2)*ln(2*Pi)-lns-lt):
The data points x[i] and y[i] are introduced:
> y[1]:=-6.65647123;y[2]:=-0.011729798;y[3]:=4.34198;y[4]:=4.56334;
y[5]:=8.37979;y[6]:=10.2627;y[7]:=10.22623;y[8]:=14.28611;y[9]:=
13.79561;y[10]:=21.09856;y[11]:=17.50144;y[12]:=20.10634;y[13]:=
23.32865;y[14]:=18.41583;y[15]:=36.10625;y[16]:=34.69508;y[17]:=
33.13127;y[18]:=43.87842;y[19]:=34.06836;y[20]:=43.07314;

                                      $y_1 := -6.65647123$ 
                                      $y_2 := -0.011729798$ 
                                      $y_3 := 4.34198$ 
                                      $y_4 := 4.56334$ 
                                      $y_5 := 8.37979$ 
                                      $y_6 := 10.2627$ 
                                      $y_7 := 10.22623$ 
                                      $y_8 := 14.28611$ 
                                      $y_9 := 13.79561$ 
                                      $y_{10} := 21.09856$ 
                                      $y_{11} := 17.50144$ 
                                      $y_{12} := 20.10634$ 
                                      $y_{13} := 23.32865$ 
                                      $y_{14} := 18.41583$ 
                                      $y_{15} := 36.10625$ 
                                      $y_{16} := 34.69508$ 
                                      $y_{17} := 33.13127$ 
                                      $y_{18} := 43.87842$ 
                                      $y_{19} := 34.06836$ 
                                      $y_{20} := 43.07314$ 
> x[1]:=-10;x[2]:=-5;x[3]:=0;x[4]:=5;x[5]:=10;x[6]:=15;x[7]:=20;x
[8]:=25;x[9]:=30;x[10]:=35;x[11]:=40;x[12]:=45;x[13]:=50;x[14]:=
55;x[15]:=60;x[16]:=65;x[17]:=70;x[18]:=75;x[19]:=80;x[20]:=85;

```

```

 $x_1 := -10$ 
 $x_2 := -5$ 
 $x_3 := 0$ 
 $x_4 := 5$ 
 $x_5 := 10$ 
 $x_6 := 15$ 
 $x_7 := 20$ 
 $x_8 := 25$ 
 $x_9 := 30$ 
 $x_{10} := 35$ 
 $x_{11} := 40$ 
 $x_{12} := 45$ 
 $x_{13} := 50$ 
 $x_{14} := 55$ 
 $x_{15} := 60$ 
 $x_{16} := 65$ 
 $x_{17} := 70$ 
 $x_{18} := 75$ 
 $x_{19} := 80$ 
 $x_{20} := 85$ 

```

(3)

```
[> with (LinearAlgebra) :
```

```
[> #int(Prob_ex3,s0=0..5, s1=0..5,a0=-10..10,a1=-10..10);normf:=1/%;
> with (VectorCalculus) ;
[&x, `*`, `+`, `^`, `.` , < , > , <|> , About, AddCoordinates, ArcLength, BasisFormat, Binormal,
  ConvertVector, CrossProduct, Curl, Curvature, D, Del, DirectionalDiff, Divergence,
  DotProduct, Flux, GetCoordinateParameters, GetCoordinates, GetNames,
  GetPVDescription, GetRootPoint, GetSpace, Gradient, Hessian, IsPositionVector,
  IsRootedVector, IsVectorField, Jacobian, Laplacian, LineInt, MapToBasis,  $\nabla$ , Norm,
  Normalize, PathInt, PlotPositionVector, PlotVector, PositionVector, PrincipalNormal,
  RadiusOfCurvature, RootedVector, ScalarPotential, SetCoordinateParameters,
  SetCoordinates, SpaceCurve, SurfaceInt, TNBFrame, TangentLine, TangentPlane,
  TangentVector, Torsion, Vector, VectorField, VectorPotential, VectorSpace, Wronskian,
  diff, eval, evalVF, int, limit, series]
```

(4)

```
[> g1 := Gradient(Prob_ex5, [s0, a1]);
```

$$\begin{aligned}
g1 := & \left(\left(-\frac{20}{s0} + \frac{(10\, a1 - 6.65647123)^2}{s0^3} + \frac{(5\, a1 - 0.011729798)^2}{s0^3} + \frac{18.85279032}{s0^3} \right. \right. \\
& + \frac{(-5\, a1 + 4.56334)^2}{s0^3} + \frac{(-10\, a1 + 8.37979)^2}{s0^3} + \frac{(-15\, a1 + 10.2627)^2}{s0^3} \\
& + \frac{(-20\, a1 + 10.22623)^2}{s0^3} + \frac{(-25\, a1 + 14.28611)^2}{s0^3} + \frac{(-30\, a1 + 13.79561)^2}{s0^3} \\
& + \frac{(-35\, a1 + 21.09856)^2}{s0^3} + \frac{(-40\, a1 + 17.50144)^2}{s0^3} + \frac{(-45\, a1 + 20.10634)^2}{s0^3} \\
& + \frac{(-50\, a1 + 23.32865)^2}{s0^3} + \frac{(-55\, a1 + 18.41583)^2}{s0^3} + \frac{(-60\, a1 + 36.10625)^2}{s0^3} \\
& + \frac{(-65\, a1 + 34.69508)^2}{s0^3} + \frac{(-70\, a1 + 33.13127)^2}{s0^3} + \frac{(-75\, a1 + 43.87842)^2}{s0^3} \\
& \left. + \frac{(-80\, a1 + 34.06836)^2}{s0^3} + \frac{(-85\, a1 + 43.07314)^2}{s0^3} \right) \\
& -10 \ln(2\pi) - 20 \ln(s0) - \frac{(10\, a1 - 6.65647123)^2}{2\, s0^2} - \frac{(5\, a1 - 0.011729798)^2}{2\, s0^2} - \frac{9.426395160}{s0^2} \\
& e \\
& - \frac{(-5\, a1 + 4.56334)^2}{2\, s0^2} - \frac{(-10\, a1 + 8.37979)^2}{2\, s0^2} - \frac{(-15\, a1 + 10.2627)^2}{2\, s0^2} - \frac{(-20\, a1 + 10.22623)^2}{2\, s0^2} \\
& - \frac{(-25\, a1 + 14.28611)^2}{2\, s0^2} - \frac{(-30\, a1 + 13.79561)^2}{2\, s0^2} - \frac{(-35\, a1 + 21.09856)^2}{2\, s0^2} - \frac{(-40\, a1 + 17.50144)^2}{2\, s0^2} \\
& - \frac{(-45\, a1 + 20.10634)^2}{2\, s0^2} - \frac{(-50\, a1 + 23.32865)^2}{2\, s0^2} - \frac{(-55\, a1 + 18.41583)^2}{2\, s0^2} - \frac{(-60\, a1 + 36.10625)^2}{2\, s0^2} \\
& - \frac{(-65\, a1 + 34.69508)^2}{2\, s0^2} - \frac{(-70\, a1 + 33.13127)^2}{2\, s0^2} - \frac{(-75\, a1 + 43.87842)^2}{2\, s0^2} - \frac{(-80\, a1 + 34.06836)^2}{2\, s0^2} \\
& - \frac{(-85\, a1 + 43.07314)^2}{2\, s0^2} \Big) \bar{e}_{s0} + \left(\left(-\frac{10\, (10\, a1 - 6.65647123)}{s0^2} - \frac{5\, (5\, a1 - 0.011729798)}{s0^2} \right. \right. \\
& + \frac{5\, (-5\, a1 + 4.56334)}{s0^2} + \frac{10\, (-10\, a1 + 8.37979)}{s0^2} + \frac{15\, (-15\, a1 + 10.2627)}{s0^2}
\end{aligned} \tag{5}$$

$$\begin{aligned}
& + \frac{20 (-20 a l + 10.22623)}{s0^2} + \frac{25 (-25 a l + 14.28611)}{s0^2} + \frac{30 (-30 a l + 13.79561)}{s0^2} \\
& + \frac{35 (-35 a l + 21.09856)}{s0^2} + \frac{40 (-40 a l + 17.50144)}{s0^2} + \frac{45 (-45 a l + 20.10634)}{s0^2} \\
& + \frac{50 (-50 a l + 23.32865)}{s0^2} + \frac{55 (-55 a l + 18.41583)}{s0^2} + \frac{60 (-60 a l + 36.10625)}{s0^2} \\
& + \frac{65 (-65 a l + 34.69508)}{s0^2} + \frac{70 (-70 a l + 33.13127)}{s0^2} + \frac{75 (-75 a l + 43.87842)}{s0^2} \\
& + \frac{80 (-80 a l + 34.06836)}{s0^2} + \frac{85 (-85 a l + 43.07314)}{s0^2} \Big) \\
& e^{-10 \ln(2\pi) - 20 \ln(s0) - \frac{(10 a l - 6.65647123)^2}{2 s0^2} - \frac{(5 a l - 0.011729798)^2}{2 s0^2} - \frac{9.426395160}{s0^2}} \\
& - \frac{(-5 a l + 4.56334)^2}{2 s0^2} - \frac{(-10 a l + 8.37979)^2}{2 s0^2} - \frac{(-15 a l + 10.2627)^2}{2 s0^2} - \frac{(-20 a l + 10.22623)^2}{2 s0^2} \\
& - \frac{(-25 a l + 14.28611)^2}{2 s0^2} - \frac{(-30 a l + 13.79561)^2}{2 s0^2} - \frac{(-35 a l + 21.09856)^2}{2 s0^2} - \frac{(-40 a l + 17.50144)^2}{2 s0^2} \\
& - \frac{(-45 a l + 20.10634)^2}{2 s0^2} - \frac{(-50 a l + 23.32865)^2}{2 s0^2} - \frac{(-55 a l + 18.41583)^2}{2 s0^2} - \frac{(-60 a l + 36.10625)^2}{2 s0^2} \\
& - \frac{(-65 a l + 34.69508)^2}{2 s0^2} - \frac{(-70 a l + 33.13127)^2}{2 s0^2} - \frac{(-75 a l + 43.87842)^2}{2 s0^2} - \frac{(-80 a l + 34.06836)^2}{2 s0^2} \\
& - \frac{(-85 a l + 43.07314)^2}{2 s0^2} \Big) \bar{e}_{al}
\end{aligned}$$

```

> delpdel s0:=(-20/s0+(10*a1-6.65647123)^2/s0^3+(5*a1-0.11729798e-1)
^2/s0^3+18.85279032/(s0^3)+(-5*a1+4.56334)^2/s0^3+(-10*
a1+8.37979)^2/s0^3+(-15*a1+10.2627)^2/s0^3+(-20*a1+10.22623)
^2/s0^3+(-25*a1+14.28611)^2/s0^3+(-30*a1+13.79561)^2/s0^3+(-35*
a1+21.09856)^2/s0^3+(-40*a1+17.50144)^2/s0^3+(-45*a1+20.10634)
^2/s0^3+(-50*a1+23.32865)^2/s0^3+(-55*a1+18.41583)^2/s0^3+(-60*
a1+36.10625)^2/s0^3+(-65*a1+34.69508)^2/s0^3+(-70*a1+33.13127)
^2/s0^3+(-75*a1+43.87842)^2/s0^3+(-80*a1+34.06836)^2/s0^3+(-85*
a1+43.07314)^2/s0^3)*exp(-10*ln(2*Pi)-20*ln(s0)-(10*a1
-6.65647123)^2/(2*s0^2)-(5*a1-0.11729798e-1)^2/(2*s0^2)
-9.426395160/(s0^2)-(-5*a1+4.56334)^2/(2*s0^2)-(-10*a1+8.37979)
^2/(2*s0^2)-(-15*a1+10.2627)^2/(2*s0^2)-(-20*a1+10.22623)^2/(2*
s0^2)-(-25*a1+14.28611)^2/(2*s0^2)-(-30*a1+13.79561)^2/(2*s0^2)-

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$(-35*a1+21.09856)^2/(2*s0^2) - (-40*a1+17.50144)^2/(2*s0^2) - (-45*a1+20.10634)^2/(2*s0^2) - (-50*a1+23.32865)^2/(2*s0^2) - (-55*a1+18.41583)^2/(2*s0^2) - (-60*a1+36.10625)^2/(2*s0^2) - (-65*a1+34.69508)^2/(2*s0^2) - (-70*a1+33.13127)^2/(2*s0^2) - (-75*a1+43.87842)^2/(2*s0^2) - (-80*a1+34.06836)^2/(2*s0^2) - (-85*a1+43.07314)^2/(2*s0^2) ;$

$$\begin{aligned} delpdels0 := & \left(-\frac{20}{s0} + \frac{(10\,al - 6.65647123)^2}{s0^3} + \frac{(5\,al - 0.011729798)^2}{s0^3} + \frac{18.85279032}{s0^3} \right. \\ & + \frac{(-5\,al + 4.56334)^2}{s0^3} + \frac{(-10\,al + 8.37979)^2}{s0^3} + \frac{(-15\,al + 10.2627)^2}{s0^3} \\ & + \frac{(-20\,al + 10.22623)^2}{s0^3} + \frac{(-25\,al + 14.28611)^2}{s0^3} + \frac{(-30\,al + 13.79561)^2}{s0^3} \\ & + \frac{(-35\,al + 21.09856)^2}{s0^3} + \frac{(-40\,al + 17.50144)^2}{s0^3} + \frac{(-45\,al + 20.10634)^2}{s0^3} \\ & + \frac{(-50\,al + 23.32865)^2}{s0^3} + \frac{(-55\,al + 18.41583)^2}{s0^3} + \frac{(-60\,al + 36.10625)^2}{s0^3} \\ & + \frac{(-65\,al + 34.69508)^2}{s0^3} + \frac{(-70\,al + 33.13127)^2}{s0^3} + \frac{(-75\,al + 43.87842)^2}{s0^3} \\ & \left. + \frac{(-80\,al + 34.06836)^2}{s0^3} + \frac{(-85\,al + 43.07314)^2}{s0^3} \right) \\ & - 10 \ln(2\pi) - 20 \ln(s0) - \frac{(10\,al - 6.65647123)^2}{2\,s0^2} - \frac{(5\,al - 0.011729798)^2}{2\,s0^2} - \frac{9.426395160}{s0^2} \\ & - \frac{(-5\,al + 4.56334)^2}{2\,s0^2} - \frac{(-10\,al + 8.37979)^2}{2\,s0^2} - \frac{(-15\,al + 10.2627)^2}{2\,s0^2} - \frac{(-20\,al + 10.22623)^2}{2\,s0^2} \\ & - \frac{(-25\,al + 14.28611)^2}{2\,s0^2} - \frac{(-30\,al + 13.79561)^2}{2\,s0^2} - \frac{(-35\,al + 21.09856)^2}{2\,s0^2} - \frac{(-40\,al + 17.50144)^2}{2\,s0^2} \\ & - \frac{(-45\,al + 20.10634)^2}{2\,s0^2} - \frac{(-50\,al + 23.32865)^2}{2\,s0^2} - \frac{(-55\,al + 18.41583)^2}{2\,s0^2} - \frac{(-60\,al + 36.10625)^2}{2\,s0^2} \\ & - \frac{(-65\,al + 34.69508)^2}{2\,s0^2} - \frac{(-70\,al + 33.13127)^2}{2\,s0^2} - \frac{(-75\,al + 43.87842)^2}{2\,s0^2} - \frac{(-80\,al + 34.06836)^2}{2\,s0^2} \\ & - \frac{(-85\,al + 43.07314)^2}{2\,s0^2} \end{aligned} \quad (6)$$

> delpdela1:=(- (10*(10*a1-6.65647123))/s0^2-(5*(5*a1-0.11729798e-1))/s0^2+(5*(-5*a1+4.56334))/s0^2+(10*(-10*a1+8.37979))/s0^2+(15*(-15*a1+10.2627))/s0^2+(20*(-20*a1+10.22623))/s0^2+(25*(-25*a1+14.28611))/s0^2+(30*(-30*a1+13.79561))/s0^2+(35*(-35*a1+21.09856))/s0^2+(40*(-40*a1+17.50144))/s0^2+(45*(-45*

$a1+20.10634)) / s0^2 + (50 * (-50*a1+23.32865)) / s0^2 + (55 * (-55*$
 $a1+18.41583)) / s0^2 + (60 * (-60*a1+36.10625)) / s0^2 + (65 * (-65*$
 $a1+34.69508)) / s0^2 + (70 * (-70*a1+33.13127)) / s0^2 + (75 * (-75*$
 $a1+43.87842)) / s0^2 + (80 * (-80*a1+34.06836)) / s0^2 + (85 * (-85*$
 $a1+43.07314)) / s0^2) * \exp(-10*\ln(2*\Pi) - 20*\ln(s0) - (10*a1-6.65647123)$
 $^2/(2*s0^2) - (5*a1-0.11729798e-1)^2/(2*s0^2) - 9.426395160/(s0^2) -$
 $(-5*a1+4.56334)^2/(2*s0^2) - (-10*a1+8.37979)^2/(2*s0^2) - (-15*$
 $a1+10.2627)^2/(2*s0^2) - (-20*a1+10.22623)^2/(2*s0^2) - (-25*$
 $a1+14.28611)^2/(2*s0^2) - (-30*a1+13.79561)^2/(2*s0^2) - (-35*$
 $a1+21.09856)^2/(2*s0^2) - (-40*a1+17.50144)^2/(2*s0^2) - (-45*$
 $a1+20.10634)^2/(2*s0^2) - (-50*a1+23.32865)^2/(2*s0^2) - (-55*$
 $a1+18.41583)^2/(2*s0^2) - (-60*a1+36.10625)^2/(2*s0^2) - (-65*$
 $a1+34.69508)^2/(2*s0^2) - (-70*a1+33.13127)^2/(2*s0^2) - (-75*$
 $a1+43.87842)^2/(2*s0^2) - (-80*a1+34.06836)^2/(2*s0^2) - (-85*$
 $a1+43.07314)^2/(2*s0^2)) ;$

$$\begin{aligned}
 delpdelal := & \left(-\frac{100\,al - 66.56471230}{s0^2} - \frac{25\,al - 0.058648990}{s0^2} + \frac{-25\,al + 22.81670}{s0^2} \right. \\
 & + \frac{-100\,al + 83.79790}{s0^2} + \frac{-225\,al + 153.9405}{s0^2} + \frac{-400\,al + 204.52460}{s0^2} \\
 & + \frac{-625\,al + 357.15275}{s0^2} + \frac{-900\,al + 413.86830}{s0^2} + \frac{-1225\,al + 738.44960}{s0^2} \\
 & + \frac{-1600\,al + 700.05760}{s0^2} + \frac{-2025\,al + 904.78530}{s0^2} + \frac{-2500\,al + 1166.43250}{s0^2} \\
 & + \frac{-3025\,al + 1012.87065}{s0^2} + \frac{-3600\,al + 2166.37500}{s0^2} + \frac{-4225\,al + 2255.18020}{s0^2} \\
 & + \frac{-4900\,al + 2319.18890}{s0^2} + \frac{-5625\,al + 3290.88150}{s0^2} + \frac{-6400\,al + 2725.46880}{s0^2} \\
 & \left. + \frac{-7225\,al + 3661.21690}{s0^2} \right) \\
 & e^{-10\ln(2\pi) - 20\ln(s0) - \frac{(10\,al - 6.65647123)^2}{2\,s0^2} - \frac{(5\,al - 0.011729798)^2}{2\,s0^2} - \frac{9.426395160}{s0^2}} \\
 & - \frac{(-5\,al + 4.56334)^2}{2\,s0^2} - \frac{(-10\,al + 8.37979)^2}{2\,s0^2} - \frac{(-15\,al + 10.2627)^2}{2\,s0^2} - \frac{(-20\,al + 10.22623)^2}{2\,s0^2} \\
 & - \frac{(-25\,al + 14.28611)^2}{2\,s0^2} - \frac{(-30\,al + 13.79561)^2}{2\,s0^2} - \frac{(-35\,al + 21.09856)^2}{2\,s0^2} - \frac{(-40\,al + 17.50144)^2}{2\,s0^2} \\
 & - \frac{(-45\,al + 20.10634)^2}{2\,s0^2} - \frac{(-50\,al + 23.32865)^2}{2\,s0^2} - \frac{(-55\,al + 18.41583)^2}{2\,s0^2} - \frac{(-60\,al + 36.10625)^2}{2\,s0^2} \\
 & - \frac{(-65\,al + 34.69508)^2}{2\,s0^2} - \frac{(-70\,al + 33.13127)^2}{2\,s0^2} - \frac{(-75\,al + 43.87842)^2}{2\,s0^2} - \frac{(-80\,al + 34.06836)^2}{2\,s0^2}
 \end{aligned}$$

(7)

$$- \frac{(-85 a l + 43.07314)^2}{2 s 0^2}$$

> **g:=delpdelal=0;u:=delpdels0=0;**

$$g := \left(- \frac{100 a l - 66.56471230}{s 0^2} - \frac{25 a l - 0.058648990}{s 0^2} + \frac{-25 a l + 22.81670}{s 0^2} \right. \\ + \frac{-100 a l + 83.79790}{s 0^2} + \frac{-225 a l + 153.9405}{s 0^2} + \frac{-400 a l + 204.52460}{s 0^2} \\ + \frac{-625 a l + 357.15275}{s 0^2} + \frac{-900 a l + 413.86830}{s 0^2} + \frac{-1225 a l + 738.44960}{s 0^2} \\ + \frac{-1600 a l + 700.05760}{s 0^2} + \frac{-2025 a l + 904.78530}{s 0^2} + \frac{-2500 a l + 1166.43250}{s 0^2} \\ + \frac{-3025 a l + 1012.87065}{s 0^2} + \frac{-3600 a l + 2166.37500}{s 0^2} + \frac{-4225 a l + 2255.18020}{s 0^2} \\ + \frac{-4900 a l + 2319.18890}{s 0^2} + \frac{-5625 a l + 3290.88150}{s 0^2} + \frac{-6400 a l + 2725.46880}{s 0^2} \\ \left. + \frac{-7225 a l + 3661.21690}{s 0^2} \right)$$

$$e^{-10 \ln(2 \pi) - 20 \ln(s 0) - \frac{(10 a l - 6.65647123)^2}{2 s 0^2} - \frac{(5 a l - 0.011729798)^2}{2 s 0^2} - \frac{9.426395160}{s 0^2}}$$

$$- \frac{(-5 a l + 4.56334)^2}{2 s 0^2} - \frac{(-10 a l + 8.37979)^2}{2 s 0^2} - \frac{(-15 a l + 10.2627)^2}{2 s 0^2} - \frac{(-20 a l + 10.22623)^2}{2 s 0^2}$$

$$- \frac{(-25 a l + 14.28611)^2}{2 s 0^2} - \frac{(-30 a l + 13.79561)^2}{2 s 0^2} - \frac{(-35 a l + 21.09856)^2}{2 s 0^2} - \frac{(-40 a l + 17.50144)^2}{2 s 0^2}$$

$$- \frac{(-45 a l + 20.10634)^2}{2 s 0^2} - \frac{(-50 a l + 23.32865)^2}{2 s 0^2} - \frac{(-55 a l + 18.41583)^2}{2 s 0^2} - \frac{(-60 a l + 36.10625)^2}{2 s 0^2}$$

$$- \frac{(-65 a l + 34.69508)^2}{2 s 0^2} - \frac{(-70 a l + 33.13127)^2}{2 s 0^2} - \frac{(-75 a l + 43.87842)^2}{2 s 0^2} - \frac{(-80 a l + 34.06836)^2}{2 s 0^2}$$

$$- \frac{(-85 a l + 43.07314)^2}{2 s 0^2} = 0$$

$$u := \left(- \frac{20}{s 0} + \frac{(10 a l - 6.65647123)^2}{s 0^3} + \frac{(5 a l - 0.011729798)^2}{s 0^3} + \frac{18.85279032}{s 0^3} \right. \\ \left. + \frac{(-5 a l + 4.56334)^2}{s 0^3} + \frac{(-10 a l + 8.37979)^2}{s 0^3} + \frac{(-15 a l + 10.2627)^2}{s 0^3} \right)$$

(8)

$$\begin{aligned}
& + \frac{(-20 a l + 10.22623)^2}{s0^3} + \frac{(-25 a l + 14.28611)^2}{s0^3} + \frac{(-30 a l + 13.79561)^2}{s0^3} \\
& + \frac{(-35 a l + 21.09856)^2}{s0^3} + \frac{(-40 a l + 17.50144)^2}{s0^3} + \frac{(-45 a l + 20.10634)^2}{s0^3} \\
& + \frac{(-50 a l + 23.32865)^2}{s0^3} + \frac{(-55 a l + 18.41583)^2}{s0^3} + \frac{(-60 a l + 36.10625)^2}{s0^3} \\
& + \frac{(-65 a l + 34.69508)^2}{s0^3} + \frac{(-70 a l + 33.13127)^2}{s0^3} + \frac{(-75 a l + 43.87842)^2}{s0^3} \\
& + \frac{(-80 a l + 34.06836)^2}{s0^3} + \frac{(-85 a l + 43.07314)^2}{s0^3} \Big) \\
& e^{-10 \ln(2\pi) - 20 \ln(s0) - \frac{(10 a l - 6.65647123)^2}{2 s0^2} - \frac{(5 a l - 0.011729798)^2}{2 s0^2} - \frac{9.426395160}{s0^2}} \\
& - \frac{(-5 a l + 4.56334)^2}{2 s0^2} - \frac{(-10 a l + 8.37979)^2}{2 s0^2} - \frac{(-15 a l + 10.2627)^2}{2 s0^2} - \frac{(-20 a l + 10.22623)^2}{2 s0^2} \\
& - \frac{(-25 a l + 14.28611)^2}{2 s0^2} - \frac{(-30 a l + 13.79561)^2}{2 s0^2} - \frac{(-35 a l + 21.09856)^2}{2 s0^2} - \frac{(-40 a l + 17.50144)^2}{2 s0^2} \\
& - \frac{(-45 a l + 20.10634)^2}{2 s0^2} - \frac{(-50 a l + 23.32865)^2}{2 s0^2} - \frac{(-55 a l + 18.41583)^2}{2 s0^2} - \frac{(-60 a l + 36.10625)^2}{2 s0^2} \\
& - \frac{(-65 a l + 34.69508)^2}{2 s0^2} - \frac{(-70 a l + 33.13127)^2}{2 s0^2} - \frac{(-75 a l + 43.87842)^2}{2 s0^2} - \frac{(-80 a l + 34.06836)^2}{2 s0^2} \\
& - \frac{(-85 a l + 43.07314)^2}{2 s0^2} = 0
\end{aligned}$$

The partial derivatives are all set to zero, and the objective function Z_1 (Eq. 18) is minimized providing the optimized parameters a1 and s0 (see also Table 6):

```
> fsolve({g,u}, {s0 = 3 .. 4, a1 = 0.45 .. 0.5});
      {a1 = 0.4970643813, s0 = 3.791034263}
```

(9)

```
> Z:=-2*ln(Prob_ex5);
```

```
Z :=
```

(10)

$$\begin{aligned}
& -2 \\
& \ln \left(e^{-10 \ln(2\pi) - 20 \ln(s0) - \frac{(10 a l - 6.65647123)^2}{2 s0^2} - \frac{(5 a l - 0.011729798)^2}{2 s0^2} - \frac{9.426395160}{s0^2}} \right. \\
& \quad \left. - \frac{(-5 a l + 4.56334)^2}{2 s0^2} - \frac{(-10 a l + 8.37979)^2}{2 s0^2} - \frac{(-15 a l + 10.2627)^2}{2 s0^2} - \frac{(-20 a l + 10.22623)^2}{2 s0^2} \right.
\end{aligned}$$

$$\begin{aligned}
& - \frac{(-25 a l + 14.28611)^2}{2 s 0^2} - \frac{(-30 a l + 13.79561)^2}{2 s 0^2} - \frac{(-35 a l + 21.09856)^2}{2 s 0^2} - \frac{(-40 a l + 17.50144)^2}{2 s 0^2} \\
& - \frac{(-45 a l + 20.10634)^2}{2 s 0^2} - \frac{(-50 a l + 23.32865)^2}{2 s 0^2} - \frac{(-55 a l + 18.41583)^2}{2 s 0^2} - \frac{(-60 a l + 36.10625)^2}{2 s 0^2} \\
& - \frac{(-65 a l + 34.69508)^2}{2 s 0^2} - \frac{(-70 a l + 33.13127)^2}{2 s 0^2} - \frac{(-75 a l + 43.87842)^2}{2 s 0^2} - \frac{(-80 a l + 34.06836)^2}{2 s 0^2} \\
& - \frac{(-85 a l + 43.07314)^2}{2 s 0^2} \Big)
\end{aligned}$$

The Hessian matrix is calculated for the minimum of the objective function Z1:

```

> hess:=Hessian(Z, [a1,s0]):
> hess0:=subs(a1=.4970643813,hess);hess1:=subs(s0= 3.791034263,
hess0);hess_matrix:=evalf(hess1);

```

$$\begin{aligned}
hess0 &:= \begin{bmatrix} \frac{89500}{s0^2} & -\frac{4.68 \cdot 10^{-6}}{s0^3} \\ -\frac{4.68 \cdot 10^{-6}}{s0^3} & -\frac{40}{s0^2} + \frac{1724.632894}{s0^4} \end{bmatrix} \\
hess1 &:= \begin{bmatrix} 6227.412244 & -8.589595735 \cdot 10^{-8} \\ -8.589595735 \cdot 10^{-8} & 5.566402006 \end{bmatrix} \\
hess_matrix &:= \begin{bmatrix} 6227.412244 & -8.589595735 \cdot 10^{-8} \\ -8.589595735 \cdot 10^{-8} & 5.566402006 \end{bmatrix} \tag{11}
\end{aligned}$$

The covariance matrix (inverse of the Hessian matrix) is calculated:

```

> cov1:=MatrixInverse(hess_matrix);

```

$$cov1 := \begin{bmatrix} 0.000160580343940371 & 2.47793859650863 \cdot 10^{-12} \\ 2.47793859650863 \cdot 10^{-12} & 0.179649259777160 \end{bmatrix} \tag{12}$$

```

> with(Statistics);
[AbsoluteDeviation, AgglomeratedPlot, AreaChart, AutoCorrelation, BarChart, Biplot,
Bootstrap, BoxPlot, BubblePlot, CDF, CGF, CentralMoment, CharacteristicFunction,
ChiSquareGoodnessOfFitTest, ChiSquareIndependenceTest, ChiSquareSuitableModelTest,
ColumnGraph, Correlation, CorrelationMatrix, Count, CountMissing, Covariance,
CovarianceMatrix, CrossCorrelation, Cumulant, CumulantGeneratingFunction,
CumulativeDistributionFunction, CumulativeProduct, CumulativeSum,
CumulativeSumChart, DataSummary, Decile, DensityPlot, DiscreteValueMap, Distribution,
ErrorPlot, EvaluateToFloat, Excise, ExpectedValue, ExponentialFit,
ExponentialSmoothing, FailureRate, FisherInformation, Fit, FivePointSummary,
FrequencyPlot, FrequencyTable, GeometricMean, GridPlot, HarmonicMean, HazardRate,
HeatMap, Histogram, HodgesLehmann, Information, InteractiveDataAnalysis,

```

(13)

InterquartileRange, InverseSurvivalFunction, Join, KernelDensity, KernelDensityPlot, KernelDensitySample, Kurtosis, Likelihood, LikelihoodRatioStatistic, LineChart, LinearFilter, LinearFit, LogLikelihood, LogarithmicFit, Lowess, MGF, MLE, MakeProcedure, MaximumLikelihoodEstimate, Mean, MeanDeviation, Median, MedianDeviation, MillsRatio, Mode, Moment, MomentGeneratingFunction, MovingAverage, MovingMedian, MovingStatistic, NonlinearFit, NormalPlot, OneSampleChiSquareTest, OneSampleTTest, OneSampleZTest, OneWayANOVA, OrderByRank, OrderStatistic, PCA, PDF, ParetoChart, Percentile, PieChart, PointPlot, PolynomialFit, PowerFit, PredictiveLeastSquares, PrincipalComponentAnalysis, Probability, ProbabilityDensityFunction, ProbabilityFunction, ProbabilityPlot, ProfileLikelihood, ProfileLogLikelihood, QuadraticMean, Quantile, QuantilePlot, Quartile, RandomVariable, Range, Rank, Remove, RemoveInRange, RemoveNonNumeric, RepeatedMedianEstimator, RousseeuwCrouxQn, RousseeuwCrouxSn, Sample, Scale, ScatterPlot, ScatterPlot3D, Score, ScreePlot, Select, SelectInRange, SelectNonNumeric, ShapiroWilkWTest, Shuffle, Skewness, Sort, Specialize, SplitByColumn, StandardDeviation, StandardError, StandardizedMoment, SunflowerPlot, Support, SurfacePlot, SurvivalFunction, SymmetryPlot, Tally, TallyInto, TreeMap, Trim, TrimmedMean, TwoSampleFTest, TwoSamplePairedTTest, TwoSampleTTest, TwoSampleZTest, Variance, Variation, VennDiagram, ViolinPlot, WeibullPlot, WeightedMovingAverage, Winsorize, WinsorizedMean]

The correlation coefficient is now close to zero, compare Table 6:

$$\begin{aligned} > \text{rho12} := \text{cov1}(1, 2) / \sqrt{\text{cov1}(1, 1)} / \sqrt{\text{cov1}(2, 2)} ; \\ & \quad \rho_{12} := 4.61351176438973 \cdot 10^{-10} \end{aligned} \quad (14)$$

The uncertainties of the parameters follow from the correlation matrix, and are provided in Table 5:

$$\begin{aligned} > \text{del1} := \sqrt{\text{cov1}(1, 1)} ; \\ & \quad \text{del1} := 0.0126720299849855 \end{aligned} \quad (15)$$

$$\begin{aligned} > \text{dels0} := \sqrt{\text{cov1}(2, 2)} ; \\ & \quad \text{dels0} := 0.423850515839205 \end{aligned} \quad (16)$$

$$> \text{with}(\text{LinearAlgebra}) ; \quad (17)$$

[&x, Add, Adjoint, BackwardSubstitute, BandMatrix, Basis, BezoutMatrix, BidiagonalForm, BilinearForm, CARE, CharacteristicMatrix, CharacteristicPolynomial, Column, ColumnDimension, ColumnOperation, ColumnSpace, CompanionMatrix, CompressedSparseForm, ConditionNumber, ConstantMatrix, ConstantVector, Copy, CreatePermutation, CrossProduct, DARE, DeleteColumn, DeleteRow, Determinant, Diagonal, DiagonalMatrix, Dimension, Dimensions, DotProduct, EigenConditionNumbers, Eigenvalues, Eigenvectors, Equal, ForwardSubstitute, FrobeniusForm, FromCompressedSparseForm, FromSplitForm, GaussianElimination, GenerateEquations, GenerateMatrix, Generic, GetResultDataType, GetResultShape, GivensRotationMatrix, GramSchmidt, HankelMatrix, HermiteForm, HermitianTranspose, HessenbergForm, HilbertMatrix, HouseholderMatrix, IdentityMatrix, IntersectionBasis, IsDefinite, IsOrthogonal, IsSimilar, IsUnitary, JordanBlockMatrix, JordanForm, KroneckerProduct, LA_Main, LUDecomposition, LeastSquares, LinearSolve, LyapunovSolve, Map, Map2, MatrixAdd, MatrixExponential, MatrixFunction, MatrixInverse, MatrixMatrixMultiply,

MatrixNorm, MatrixPower, MatrixScalarMultiply, MatrixVectorMultiply, MinimalPolynomial, Minor, Modular, Multiply, NoUserValue, Norm, Normalize, NullSpace, OuterProductMatrix, Permanent, Pivot, PopovForm, ProjectionMatrix, QRDecomposition, RandomMatrix, RandomVector, Rank, RationalCanonicalForm, ReducedRowEchelonForm, Row, RowDimension, RowOperation, RowSpace, ScalarMatrix, ScalarMultiply, ScalarVector, SchurForm, SingularValues, SmithForm, SplitForm, StronglyConnectedBlocks, SubMatrix, SubVector, SumBasis, SylvesterMatrix, SylvesterSolve, ToeplitzMatrix, Trace, Transpose, TridiagonalForm, UnitVector, VandermondeMatrix, VectorAdd, VectorAngle, VectorMatrixMultiply, VectorNorm, VectorScalarMultiply, ZeroMatrix, ZeroVector, Zip]

```
> Determinant(hess_matrix) ;
                                     34664.28001 (18)
```

```
> ConditionNumber(hess_matrix) ;
                                     1118.750000 (19)
```