```
> restart:
> with(Statistics);
```

```
[AbsoluteDeviation, AgglomeratedPlot, AreaChart, AutoCorrelation, BarChart, Biplot,        (1)
    Bootstrap, BoxPlot, BubblePlot, CDF, CGF, CentralMoment, CharacteristicFunction,
    ChiSquareGoodnessOfFitTest, ChiSquareIndependenceTest, ChiSquareSuitableModelTest,
    ColumnGraph, Correlation, CorrelationMatrix, Count, CountMissing, Covariance,
    CovarianceMatrix, CrossCorrelation, Cumulant, CumulantGeneratingFunction,
    CumulativeDistributionFunction, CumulativeProduct, CumulativeSum,
    CumulativeSumChart, DataSummary, Decile, DensityPlot, DiscreteValueMap, Distribution,
    ErrorPlot, EvaluateToFloat, Excise, ExpectedValue, ExponentialFit,
    ExponentialSmoothing, FailureRate, FisherInformation, Fit, FivePointSummary,
    FrequencyPlot, FrequencyTable, GeometricMean, GridPlot, HarmonicMean, HazardRate,
    HeatMap, Histogram, HodgesLehmann, Information, InteractiveDataAnalysis,
    InterquartileRange, InverseSurvivalFunction, Join, KernelDensity, KernelDensityPlot,
    KernelDensitySample, Kurtosis, Likelihood, LikelihoodRatioStatistic, LineChart,
    LinearFilter, LinearFit, LogLikelihood, LogarithmicFit, Lowess, MGF, MLE,
    MakeProcedure, MaximumLikelihoodEstimate, Mean, MeanDeviation, Median,
    MedianDeviation, MillsRatio, Mode, Moment, MomentGeneratingFunction,
    MovingAverage, MovingMedian, MovingStatistic, NonlinearFit, NormalPlot,
    OneSampleChiSquareTest, OneSampleTTest, OneSampleZTest, OneWayANOVA,
    OrderByRank, OrderStatistic, PCA, PDF, ParetoChart, Percentile, PieChart, PointPlot,
    PolynomialFit, PowerFit, PredictiveLeastSquares, PrincipalComponentAnalysis,
    Probability, ProbabilityDensityFunction, ProbabilityFunction, ProbabilityPlot,
    ProfileLikelihood, ProfileLogLikelihood, QuadraticMean, Quantile, QuantilePlot, Quartile,
    RandomVariable, Range, Rank, Remove, RemoveInRange, RemoveNonNumeric,
    RepeatedMedianEstimator, RousseeuwCrouxQn, RousseeuwCrouxSn, Sample, Scale,
    ScatterPlot, ScatterPlot3D, Score, ScreePlot, Select, SelectInRange, SelectNonNumeric,
    ShapiroWilkWTest, Shuffle, Skewness, Sort, Specialize, SplitByColumn, StandardDeviation,
    StandardError, StandardizedMoment, SunflowerPlot, Support, SurfacePlot,
    SurvivalFunction, SymmetryPlot, Tally, TallyInto, TreeMap, Trim, TrimmedMean,
    TwoSampleFTest, TwoSamplePairedTTest, TwoSampleTTest, TwoSampleZTest, Variance,
    Variation, VennDiagram, ViolinPlot, WeibullPlot, WeightedMovingAverage, Winsorize,
    WinsorizedMean]
```

The objective function for example 1 is based on Eq. (18).

```
> Prob_ex1:=exp(-(8/2)*ln(2*Pi)-8*ln(s)-((y1-a)^2+(y2-1)^2+(y3-a)
  ^2+(y4-a)^2+(y5-a)^2+(y6-a)^2+(y7-a)^2+(y8-a)^2)/(2*s^2)):
```

The modelled y values equal a=1.

```
> a:=1:
```

The simulated y values are introduced:

```
> y1:=1.2529423:y2:=0.6062315:y3:=0.8870169:y4:=0.7365572:y5:=
  1.0504223:y6:=0.7883198:y7:=0.912006:y8:=1.2955773:
```

The mean value is calculated.

```
> Mean([y1,y2,y3,y4,y5,y6,y7,y8]);
                              0.941134162500000                                  (2)
```

The standard deviation is calculated.

```
> std:=StandardDeviation([y1,y2,y3,y4,y5,y6,y7,y8]);
```

$$std := 0.243761896003632 \tag{3}$$

Normalizing the objective function to obtain the probability density:

```
> int(Prob_ex1,s=0..infinity);normf:=1/%;
```

$$0.2073738318$$

$$normf := 4.822209202 \tag{4}$$

Probability density Prob_ex1_norm

```
> Prob_ex1_norm:=Prob_ex1*normf;
```

$$Prob\_ex1\_norm := 4.822209202\, \mathrm{e}^{-4\ln(2\pi) - 8\ln(s) - \frac{0.2218302641}{s^2}} \tag{5}$$

```
> with(VectorCalculus);
```

$$[\&x, \text{`*`}, \text{`+`}, \text{`-`}, \text{`.`}, <,>, <|>, About, AddCoordinates, ArcLength, BasisFormat, Binormal, ConvertVector, CrossProduct, Curl, Curvature, \mathrm{D}, Del, DirectionalDiff, Divergence, DotProduct, Flux, GetCoordinateParameters, GetCoordinates, GetNames, GetPVDescription, GetRootPoint, GetSpace, Gradient, Hessian, IsPositionVector, IsRootedVector, IsVectorField, Jacobian, Laplacian, LineInt, MapToBasis, \nabla, Norm, Normalize, PathInt, PlotPositionVector, PlotVector, PositionVector, PrincipalNormal, RadiusOfCurvature, RootedVector, ScalarPotential, SetCoordinateParameters, SetCoordinates, SpaceCurve, SurfaceInt, TNBFrame, TangentLine, TangentPlane, TangentVector, Torsion, Vector, VectorField, VectorPotential, VectorSpace, Wronskian, diff, eval, evalVF, int, limit, series] \tag{6}$$

Gradient of the probability density with respect to variable s:

```
> g1 := Gradient(Prob_ex1_norm, [s]);
```

$$g1 := \left(4.822209202 \left(-\frac{8}{s} + \frac{0.4436605282}{s^3}\right) \mathrm{e}^{-4\ln(2\pi) - 8\ln(s) - \frac{0.2218302641}{s^2}}\right) \bar{e}_s \tag{7}$$

Derivative of the probability density with respect to variable s:

```
> delpdels:=(-8/s+.4436605282/(s^3))*exp(-4*ln(2*Pi)-8*ln(s)
  -.2218302641/(s^2));
```

$$delpdels := \left(-\frac{8}{s} + \frac{0.4436605282}{s^3}\right) \mathrm{e}^{-4\ln(2\pi) - 8\ln(s) - \frac{0.2218302641}{s^2}} \tag{8}$$

```
> f:=delpdels=0;
```

$$f := \left(-\frac{8}{s} + \frac{0.4436605282}{s^3}\right) \mathrm{e}^{-4\ln(2\pi) - 8\ln(s) - \frac{0.2218302641}{s^2}} = 0 \tag{9}$$

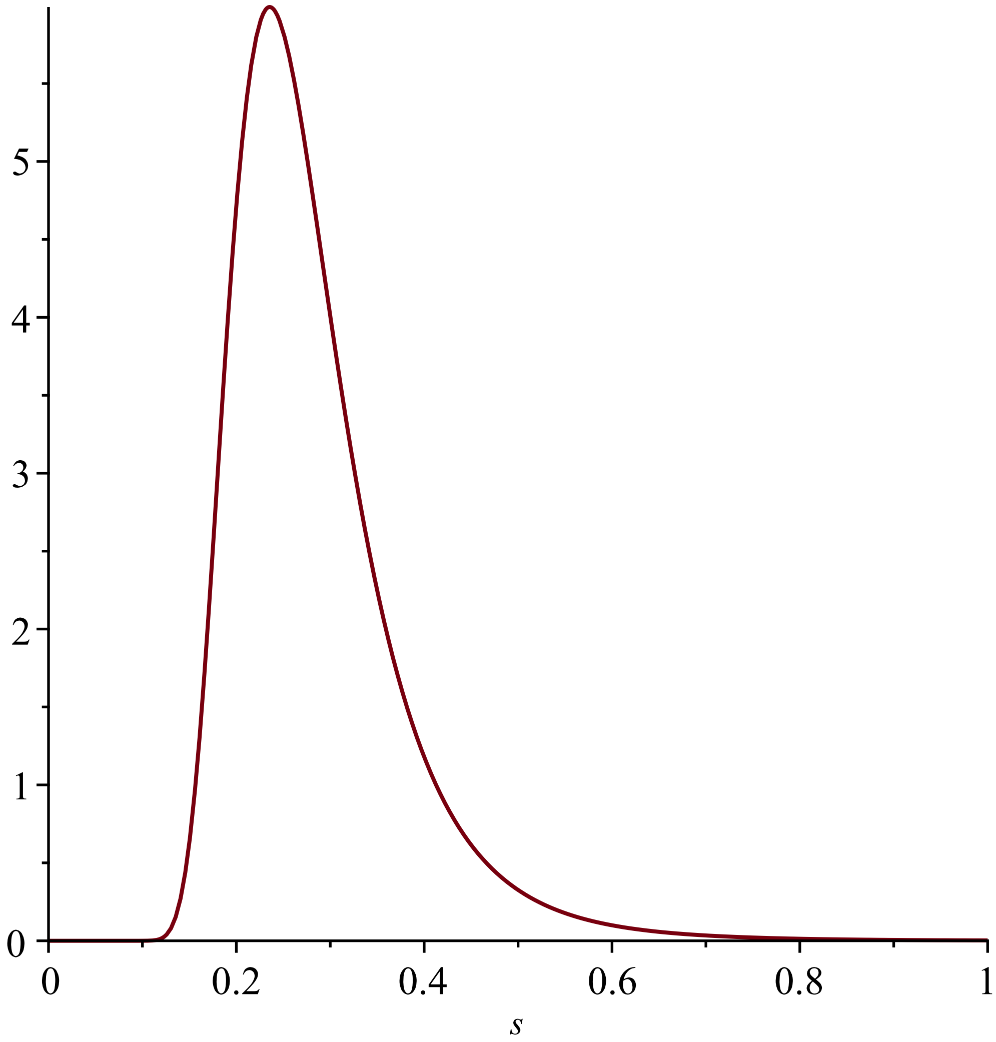
Maximum of the probability density

```
> fsolve({f}, {s = 0.01 .. 0.5});
```

$$\{s = 0.2354943015\} \tag{10}$$

Plot of the probability density

```
> plot(Prob_ex1_norm, s=0..1);
```



```
> int(Prob_ex1_norm, s=0..100);
```

$$0.9999999997 \qquad \textbf{(11)}$$

```
> deriv2nd:=diff(delpdels,s);
```

$$deriv2nd := \left(\frac{8}{s^2} - \frac{1.330981585}{s^4}\right) \mathrm{e}^{-4 \ln(2 \pi) - 8 \ln(s) - \frac{0.2218302641}{s^2}} + \left(-\frac{8}{s} + \frac{0.4436605282}{s^3}\right)^2 \mathrm{e}^{-4 \ln(2 \pi) - 8 \ln(s) - \frac{0.2218302641}{s^2}} \qquad \textbf{(12)}$$

```
> deriv2nd:=(8/s^2-1.330981585/(s^4))*exp(-4*ln(2*Pi)-8*ln(s)
  -.2218302641/(s^2))+(-8/s+.4436605282/(s^3))^2*exp(-4*ln(2*Pi)-8*
  ln(s)-.2218302641/(s^2));
```

$$deriv2nd := \left(\frac{8}{s^2} - \frac{1.330981585}{s^4}\right) \mathrm{e}^{-4 \ln(2 \pi) - 8 \ln(s) - \frac{0.2218302641}{s^2}} + \left(-\frac{8}{s} + \frac{0.4436605282}{s^3}\right)^2 \mathrm{e}^{-4 \ln(2 \pi) - 8 \ln(s) - \frac{0.2218302641}{s^2}} \qquad \textbf{(13)}$$

```
> g:=deriv2nd=0;
```

$$g := \left(\frac{8}{s^2} - \frac{1.330981585}{s^4}\right) \mathrm{e}^{-4\ln(2\pi) - 8\ln(s) - \frac{0.2218302641}{s^2}} + \left(-\frac{8}{s} + \frac{0.4436605282}{s^3}\right)^2 \mathrm{e}^{-4\ln(2\pi) - 8\ln(s) - \frac{0.2218302641}{s^2}} = 0 \qquad \textbf{(14)}$$

```
> fsolve({g}, {s = 0.01 .. 0.3});
```

$$\{s = 0.1794844627\} \qquad \textbf{(15)}$$

```
> sleft:=.1794844627;
```

$$sleft := 0.1794844627 \qquad \textbf{(16)}$$

```
> fsolve({g}, {s = 0.2 .. 1});
```

$$\{s = 0.2913115368\} \qquad \textbf{(17)}$$

```
> sright:=.2913115368;
```

$$sright := 0.2913115368 \qquad \textbf{(18)}$$