

Differential Equations

* Differential Equations *

$$\text{eqn} - 2x + y = 2$$

$$\text{diff. eqn} - \frac{dy}{dx} - x + y = 2$$

$$\Rightarrow P \Rightarrow x=1 \quad \& \quad y=1 \quad \text{so} \Rightarrow y=f(x)$$

differential eqn

↓
ordinary diff. eqn

single variable

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + 3y = 0$$

↓
partial diff. eqn

$$\text{more than one var} \\ \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial y^2}\right)^3 = 0$$

order = highest derivative

order = 2

degree = highest power of highest degree
derivative should be free from radical

order = 2 ; degree = 1 f fractions.

order - The highest derivative involved in diff. eqn.

$$\text{Radical} = \left(\frac{dy}{dx}\right)^{1/2, 1/3, 1/4}$$

fraction - derivative should not be in denominator

$$\textcircled{1} \quad \left(\frac{d^2y}{dx^2}\right)^2 = \left(x + \frac{dy}{dx}\right)^{3/2}$$

$$\textcircled{2} \quad \frac{dy}{dx} = x + e^t$$

$$\left(\frac{d^2y}{dx^2}\right)^4 = \left(x + \frac{dy}{dx}\right)^3$$

$$\left(\frac{dy}{dx}\right)^2 = x \frac{dt}{dx} + e^t$$

$$\therefore O = 2 \quad D = 4$$

$$O = 1, D = 2$$

* Differential Equations *

$$\text{Eqn} \rightarrow \text{diff eqn}$$

$$\text{diff eqn} \quad \frac{dy}{dx} + x = 2$$

\Rightarrow $y = f(x)$

Differential eqn

Ordinary diff. eqn

Partial diff. eqn

single variable

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + 3y = 0$$

more than one var

$$\frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial y^2}\right)^3 = 0$$

order = highest derivative

order = 2

degree = highest power of highest degree

degree = 3

order = 2 ; degree = 1
derivative should free from radical & fractions.

order - The highest derivative involved in diff. eqn.

Radical = $\left(\frac{dy}{dx}\right)^{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots}$

fraction - derivative should not in denominator

$$D \left(\frac{d^2y}{dx^2}\right)^2 = \left(x + \frac{dy}{dx}\right)^{3/2}$$

$$\textcircled{1} \quad \frac{dy}{dx} = x + e^t$$

$$\left(\frac{d^2y}{dx^2}\right)^4 = \left(x + \frac{dy}{dx}\right)^3 \quad \left(\frac{dy}{dx}\right)^2 = x + e^t$$

$$\therefore D = 2 \quad D = 4$$

$$D = 1, D = 2$$



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$$2 \left(\frac{d^3y}{dx^3} \right)^2 + \left(\frac{d^2y}{dx^2} \right)^3 + 3y = 0 \quad (4) \quad dy = (t + \sin t) dt$$

$$O=3, D=2$$

$$\frac{dy}{dt} = t + \sin t$$

$$2 \left(\frac{\partial^2 u}{\partial x^2} \right)^3 = c^2 \left(\frac{\partial^3 y}{\partial x^3} \right)^2 \quad O=1, D=1$$

$$O=3, D=2$$

Formation of differential eqns -

$$F(x, y, a, b) = 0 \quad (1)$$

Diff. (1) wrt x

$$F_1(x, y, \frac{dy}{dx}, a, b) = 0 \quad (2)$$

Diff. (2) wrt x

$$F_2(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, a, b) = 0 \quad (3)$$

Eliminate
a, b \Rightarrow

$$\phi(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}) = 0$$

Note:

The no. of arbitrary constants eliminated should be equal to the order of resulting ordinary diff. eqns.

① The diff eqn of family of straight lines passing through origin is

(a) $y = mx$

(b) $y = c_1 x + c_2$

(c) $x dy + y dx = 0$

(d) $x^2 + y^2 = c$

$$\frac{dy}{dx} = \text{slope}$$

$$\therefore y = \frac{dy}{dx} x$$

$$dy/x = dx/y$$

$$x dy - y dx = 0$$

② Find the diff eqn of family of circles centred at (h, k) of radius a where h, k are arbitrary constants.

$$(x-h)^2 + (y-k)^2 = a^2 \quad (1)$$

diff w.r.t. x

$$2(x-h) + 2(y-k)y' = 0$$

$$2(x-h) + 2(y-k)y' = 0 \quad (2)$$

diff w.r.t. x

$$2 + 2y' + 2(y-k)y'' = 0$$

$$2 + 2y' + 2(y-k)y'' = 0$$

$$2 + 2y' + 2(y-k)y'' = 0$$

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$$x-h = \frac{(1+y'^2) - y^2}{y''} \quad \textcircled{1}$$

Put $\textcircled{4}$ in $\textcircled{1}$

$$\frac{(1+y'^2)^2}{y''^2} y'^2 + \frac{(1+y'^2)^2}{y''^2} = a^2$$

$$(1+y'^2)^2 (y'^2 + 1) = a^2 y''^2$$

$$(1+y'^2)^3 = a^2 y''^2$$

$$\textcircled{3} \quad y = c(x-c)^2 \quad \textcircled{1}$$

Dif. w.r.t. x

$$y' = 2c(x-c) \quad \textcircled{2}$$

$$\frac{\textcircled{1}}{\textcircled{2}} \Rightarrow \frac{y}{y'} = \frac{c(x-c)^2}{2c(x-c)} = \frac{x-c}{2} \therefore x-c = \frac{2y}{y'}$$

$$c = x - \frac{2y}{y'} = \frac{xy' - 2y}{y'}$$

$$y = \frac{xy' - 2y}{y'} (x - \frac{xy' - 2y}{y'})^2 = \left(\frac{xy' - 2y}{y'} \right) \frac{y'^2}{y''}$$

$$y'^3 = (xy' - 2y)y''$$

Note:- If the given eqn is of the form $y = c_1 e^{at} + c_2 e^{bt} + c_3 e^{ct} + \dots$

Then D.E. is

$$(D-a)(D-b)(D-c)y = 0$$

$$D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2}$$

e.g.

$$\textcircled{1} \quad y = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{2x}$$

$$\rightarrow (D+1)(D-3)(D-2)y = 0$$

$$(D^2 - 2D - 3)(D-2)y = 0$$

$$(D^3 - 4D^2 + D + 6)y = 0$$

$$\frac{d^3y}{dx^3} - 4 \frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$$

$$\textcircled{2} \quad i = c_1 e^{2t} + c_2 e^{-4t}$$

$$\rightarrow (D-2)(D+4)i = 0$$

$$(D^2 + 2D - 8)i = 0$$

$$\textcircled{3} \quad \frac{d^2i}{dt^2} + \frac{di}{dt} - 8i = 0$$

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Note:-

If the given eqn is of the form

$y = Af(x) + Bg(x)$ Then the D.E. is obtained by simplifying the following determinant.

$$\begin{vmatrix} y & f(x) & g(x) \\ y' & f' & g' \\ y'' & f'' & g'' \end{vmatrix} = 0$$

e.g. ① $y = Ae^{2x} + Bx$

$$\begin{vmatrix} y & e^{2x} & x \\ y' & 2e^{2x} & 1 \\ y'' & 4e^{2x} & 0 \end{vmatrix} = 0 \quad \therefore y(-4e^{2x}) - e^{2x}(-y'') + x(2y'e^{2x}) - 4y''e^{2x} = 0$$

$$\therefore y''(1-2x) + 4xy' - 4y = 0$$

② $y = e^x(A \cos x + B \sin x) \quad \text{--- } ①$

$$y' = e^x(A \cos x + B \sin x) + e^x(-A \sin x + B \cos x) \quad \text{--- } ②$$

$$y' = y + e^x(-A \sin x + B \cos x)$$

$$y'' = y' + e^x(-A \sin x + B \cos x) + e^x(-A \cos x - B \sin x)$$

$$y'' = y' + (y' - y) - y$$

$$\therefore y'' - 2y' + 2y = 0$$

$$\frac{d^2y}{dt^2} = g$$

$$y(s) = \dots$$

$$\frac{dy}{dt} = gt + c_1 \quad \left\{ \begin{array}{l} y(0) = 0 \\ \frac{dy}{dt}(0) = 0 \end{array} \right. \Rightarrow 0 = 0 + c_1 \Rightarrow c_1 = 0$$

$$\textcircled{a} \quad y = \frac{1}{2}gt^2 + c_1 t + c_2 \quad \left. \begin{array}{l} y(0) = 0 \\ \frac{dy}{dt}(0) = 0 \end{array} \right. \Rightarrow 0 = 0 + c_2 \Rightarrow c_2 = 0$$

General sol
of no of arbitrary
const

$$\textcircled{b} \quad y = \frac{1}{2}gt^2 + c_1 t + c_2$$

$$\frac{d^2y}{dt^2} = g$$

$$\textcircled{a} \quad y = \frac{1}{2}gt^2 + c_1 e^t + c_2$$

$$\textcircled{b} \quad y = \frac{1}{2}gt^2 + gt^3 + c_2$$

$$\textcircled{c} \quad y = \frac{1}{2}gt^2 + ct + c$$

\textcircled{d} None

First order first degree diff. eqns.

Notation - $\frac{dy}{dx} = F(x,y)$ or $Mdx + Ndy = 0$ $M(x,y) + N(x,y)$

i) Variable separable method

$$\frac{dy}{dx} = f(x)g(y)$$

$$f(x)dx = g(y)dy$$

$$\int f(x)dx = \int g(y)dy + C$$

$$\textcircled{1} \quad \log\left(\frac{dy}{dx}\right) = 2x - y$$

$$\textcircled{3} \quad \frac{dx}{dy} = e^{2y} + e^y \cdot y$$

$$\log dy - \log dx = 2x - y$$

$$\frac{dx}{dy} = e^x (e^{-2y} + y)$$

$$\frac{dy}{dx} = e^{2x-y} = e^{2x} \cdot e^{-y}$$

$$\int \frac{dx}{e^x} \int (e^{-2x} + y) dy$$

$$\int e^y dy = \int e^{2x} dx$$

$$-e^{-x} = \frac{e^{-2y}}{-2} + \frac{y^2}{2} + C$$

$$\boxed{e^y = \frac{e^{2x}}{2} + C}$$

$$\textcircled{4} \quad \frac{dy}{dx} = 1 + y^2$$

$$\textcircled{2} \quad \frac{dy}{dx} = \frac{y^2}{1-xy}$$

$$\int \frac{dy}{1+y^2} = \int dx$$

$$(1-xy) dy = y^2 dx$$

$$\tan^{-1} y = x + C$$

$$dy = y^2 dx + xy dy$$

$$\boxed{y = \tan(x+C)}$$

$$= y (\underbrace{y dx + x dy}_{d(xy)})$$

$$\textcircled{3} \quad \frac{dy}{dx} = -\frac{x}{y} \quad \text{at } x=1, y=\sqrt{3}$$

$$dy = y d(xy)$$

$$\int y dy = \int -x dx$$

$$\int \frac{dy}{y} = \int d(xy)$$

$$y^2 = -x^2 + 2C$$

$$3 = -1 = 2C$$

$$\boxed{\log y = 2x + C}$$

$$2C = 4$$

$$\boxed{C = 2}$$

$$\boxed{x^2 + y^2 = 4}$$

③ Find the curve passing through the point (0, 1) and satisfying $\sin\left(\frac{dy}{dx}\right) = b$

$$\sin\left(\frac{dy}{dx}\right) = b$$

$$\frac{dy}{dx} = \sin^b b$$

$$\int dy = \int \sin^b b dx$$

$$y = x \sin^b b + c$$

$$\boxed{c=1}$$

$$\boxed{y = x \sin^b b + 1}$$

$$\textcircled{a} \quad \frac{dy}{dx} = e^x \text{ with } y(0)=1$$

then find 'y' when $x=1$

$$\int e^y dy = \int e^x dx$$

$$-e^y = e^x + c$$

$$-e^{-1} = e^1 + c$$

$$-\frac{1}{e} = e + c$$

$$\therefore c = -1 - e^2$$

$$\textcircled{q} \quad \frac{dx}{dt} + 3x = 0$$

$$\textcircled{a} \quad x = 3e^{3t}$$

$$\textcircled{c} \quad x = 2e^{-2t}$$

$$\textcircled{b} \quad x = 3t^3$$

$$\checkmark \textcircled{d} \quad x = 3e^{-3t}$$

$$\frac{dx}{dt} = -3x$$

$$\frac{dx}{dt} = -3dt$$

$$\log x = -3t + \log C$$

$$x = Ce^{-3t}$$

$$\boxed{x = 3e^{-3t}}$$

$$x = -1$$

$$-e^t = e^t - e^{-t}$$

$$e^{-t} = e^t - e^{-t}$$

$$e^{-t} = 2e^{-t}$$

$$\boxed{t = 0}$$

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$$\textcircled{9} \quad \frac{dy}{dx} = 3x^2 - 2x \quad \text{at } x=1, y=1$$

then find $y(3) = ?$

$$\rightarrow \int dy = \int (3x^2 - 2x) dx$$

$$y = x^3 - x^2 + C$$

$$\boxed{C=1}$$

$$y = x^3 - x^2 + 1$$

$$y(3) = 27 - 9 + 1$$

$$\boxed{y(3) = 19}$$

$$\textcircled{10} \quad \frac{dy}{dx} = y^2 \sin x \text{ with } y(0)=1$$

$$\textcircled{a} \quad y \sin x = 1 \quad \textcircled{b} \quad y \cos x = 1$$

$$\textcircled{c} \quad y = \sin x \quad \textcircled{d} \quad x = \sin y$$

$\textcircled{11}$ If the equation contains the terms like $\cos(xy)$, $\sin(xy)$, $(ax+by+c)^2$, etc. can be reduced to variable separable form with a substitution $xy=v$, $x+y=r$, $ax+by+c=u$ resp.

$$\textcircled{1} \quad \frac{dy}{dx} = (4x+y+1)^2 \quad \therefore \frac{dv}{dx} - 4 = v^2$$

$$4x+y+1 = v$$

$$4 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{dv}{dx} - 4$$

$$\frac{dv}{(4+v^2)} = \frac{dx}{dx}$$

$$\frac{1}{2} \tan^{-1} \frac{v}{2} = x + C$$

$$\tan^{-1} \frac{v}{2} = 2x + C \quad \therefore \tan(\frac{v}{2}) = \tan(2x+C)$$

$$\boxed{\frac{v}{2} = \frac{4x+4+1}{2}}$$

Homogeneous diff. method -

$$\frac{dy}{dx} = F(x, y)$$

Homog. fn. of degree 1

sum. degree = den. degree

$$F(kx, ky) = k^n F(x, y)$$

$\frac{dy}{dx} = F(x, y)$ is said to be homogeneous diff eqn if $F(x, y)$ should be a homogeneous function of degree zero.

Note - Eqn $Mdx + Ndy = 0$ is said to be homogeneous diff. eqn if all the terms of M & N should be of same degree.

$$x^2 + y^2 \Rightarrow (kx)^2 + (ky)^2 \therefore \text{degree} = 2.$$

Substitution $y = vx$ or $x = vy$ reduces homogeneous eqn to variable separation form.

$$\frac{dy}{dx} = \frac{x^2y - y^3}{xy^2} = \frac{x^2(\frac{y}{x} - (\frac{y}{x})^3)}{xy^2}$$

Every homogeneous fn. of degree zero can be written as fn. of $\frac{y}{x}$ or $\frac{x}{y}$ & substitution reduces it to variable separation form.

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$$x \frac{dy}{dx} = y [\log y - \log x + 1]$$

$$\frac{dy}{dx} = \frac{y}{x} [\log(y) + 1]$$

$$y = vx$$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$v + x \frac{dv}{dx} = v [\log v + 1]$$

$$x \frac{dv}{dx} = v \log v$$

$$\int \frac{1}{v \log v} dv = \int \frac{dx}{x}$$

$$\log v = t$$

$\therefore v = e^{cx}$

$$\therefore \frac{y}{x} = e^{cx}$$

$$\therefore y = x e^{cx}$$

$$\int dt = \int \frac{dx}{x}$$

$$\int \frac{1}{t} dt = \int \frac{dx}{x}$$

$$\log t = \log x + \log c$$

$$t = cx$$

$$\log t = \log cx$$

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Non-homogeneous diff. eqn method -

$$\boxed{\frac{dy}{dx} = \frac{a_1x + b_1y + g}{a_2x + b_2y + g}}$$

case I :-

$$\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2}}$$

There exists a subst. which reduces given eqn to variable separable form.

case II :-

$$\boxed{\frac{a_1}{a_2} \neq \frac{b_1}{b_2}}$$

$$x = X + h, \quad y = Y + K$$

$$dx = dX, \quad dy = dY$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{a_1(x+h) + b_1(Y+K) + g}{a_2(x+h) + b_2(Y+K) + g} \\ &= \frac{a_1x + b_1Y + (a_1h + b_1K + g)}{a_2x + b_2Y + (a_2h + b_2K + g)} \end{aligned}$$

choose h, K so that

$$\left. \begin{aligned} a_1h + b_1K + g &= 0 \\ a_2h + b_2K + g &= 0 \end{aligned} \right\}$$

$$\frac{dy}{dx} = \frac{a_1x + b_1Y}{a_2x + b_2Y} \quad \text{then } Y = VX$$

$$\textcircled{1} \quad (2x+2y-1)dx = (x+y+1)dy$$

$$\frac{dy}{dx} = \frac{2x+2y-1}{x+y+1} \quad \left(\frac{a_1 - b_1}{a_2 - b_2} \right)$$
$$= \frac{2(x+y-1)}{(x+y)+1}$$

$$x+y = v$$

$$1 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{dv}{dx} - 1$$

$$\frac{dv}{dx} - 1 = \frac{2v-1}{v+1}$$

$$\frac{dv}{dx} = \frac{2v-1+v+1}{v+1} = \frac{3v}{v+1}$$

$$\int \frac{(v+1)}{v} dv = \int 3dx$$

$$v + \log v = 3x + C$$

$$x+y + \log(x+y) = 3x + C$$

$$y - 2x + \log(x+y) = C$$

② Which of following subst. reduces diff. eqn
 $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$ to homogeneous form.



$$x = X + h, \quad y = Y + k$$

$$k + b - 2 = 0$$

$$k - b - 4 = 0$$

$$2k - 6 = 0$$

$$\underline{k=3}, \underline{h=-1}$$

$$\boxed{x = X - 1}, \boxed{y = Y + 3}$$

$$\frac{dy}{dx} = \frac{Y + X}{Y - X} \quad \text{Homo. form}$$

③ (a) $x = X + h, \quad y = Y + k$ reduces
 to homo. form then find h, k .

$$\text{Ans. } h = -1, \quad k = 3$$

(b) $\frac{d^2y}{dx^2} + (h+3)\frac{dy}{dx} + (k-1)y = 0$

Linked que.

Exact differential eqn

$$Mdx + Ndy = 0$$

A differential eqn $Mdx + Ndy = 0$ is said to be exact if it satisfies the condition $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$dI(f(x,y))$ or $f(x,y)$

$$\text{e.g. } y^2 dx + 2xy dy = 0$$

$$d[xy^2] = y^2 dx + 2xy dy$$

The D.E. $Mdx + Ndy = 0$ is exact $\Leftrightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$Mdx + Ndy = 0$$

$$\begin{array}{|c|} \hline \text{exact} \\ \hline \text{i.e. } \left(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \right) \\ \hline \end{array}$$

$\int Mdx + \int \text{[terms of } N \text{ without } dx]$

Integrate w.r.t. x

at constant

D. $\int Mdx + \int \text{[terms of } N \text{ without } dx]$ \rightarrow $\int Mdx$ and $\int Ndy$

$$\frac{\partial M}{\partial y}$$

$$\frac{\partial N}{\partial x}$$

$\int Mdx$ \rightarrow $\int P dx$ \rightarrow $P dx$

$Pdx - (1 + \sin^2 y + \cos^2 x)dy = 0$ is exact then find P .

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\therefore \frac{\partial N}{\partial x} = \sin 2x$$

$$\therefore \frac{\partial P}{\partial y} = \int \sin 2x$$

$$\therefore \boxed{P = 4 \sin 2x} \quad \checkmark$$

$$P = 4 \sin 2x + C \cos x \quad \text{(ans)}$$

③ $(3a^2x^2 + by \cos x)dx + (2\sin x - 4xy^3)dy = 0$ is exact then

Ⓐ Exactness depends on both a and b

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Ⓑ Exactness depends only on a

Ⓒ Exactness depends only on b

Ⓓ Exactness not depends on both a & b .

$$④ (ax+bx+g)dx + (bx+by+f)dy = 0 \quad \rightarrow \text{non-homogeneous}$$

$$\frac{\partial M}{\partial y} = b \quad , \quad \frac{\partial N}{\partial x} = b$$

$$\int (ax+bx+g)dx + \int (bx+by+f)dy = c$$

$$\frac{ax^2}{2} + bxy + gx + \frac{by^2}{2} + fy + c$$

$$⑤ [y(1+\frac{1}{x}) + \cos y]dx + [x + \log x - x \sin y]dy = 0$$

$$\frac{\partial M}{\partial y} = -\sin y + \left(1 + \frac{1}{x}\right) \quad \frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

$$\int \left(y + \frac{1}{x} + \cos y\right)dx + \int 0 dx = c$$

$$\therefore yx + y \log x + x \cos y = c$$

$$⑥ ydx - xdy = 0 \quad \text{---} ①$$

$$\frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -1$$

multiply ① with $\frac{1}{y^2}$

$$\frac{y}{y^2}dx - \frac{x}{y^2}dy = 0$$

$$\frac{1}{y^2}dx - \frac{x}{y^2}dy = 0$$

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$$\frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = -1$$

Integrating factor -

A non exact eqn is converted to the exact by multiplying it with a function $f(x,y)$. Then $f(x,y)$ is called integrating factor.

$$ydx - xdy = 0$$

I.F.

$$\frac{1}{y^2}, \frac{1}{x^2}, \frac{1}{xy}, \frac{1}{x^2+y^2}$$

A constant multiple of an integrating factor is also an integrating factor.

$$Mdx + Ndy = 0$$

Non exact eqn (i.e. $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$)

ii) All terms of M and N should be of same degree.

$$IF = \frac{1}{Mx+Ny}$$

$(Mx+Ny \neq 0)$

ii) The eqn is of the form

$$y f(x,y)dx + x g(x,y)dy = 0$$

$$IF = \frac{1}{Mx-Ny}$$

$(Mx-Ny \neq 0)$

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$$

$\int \frac{\partial M}{\partial x} dx + \int \frac{\partial N}{\partial y} dy$ is not exact

∴

$$I = \int f(x) dx$$

$$I = e^{\int f(x) dx}$$

$$\frac{\partial M}{\partial x} = \frac{\partial N}{\partial y}$$

$\int \frac{\partial M}{\partial x} dx + \int \frac{\partial N}{\partial y} dy$ is not exact

∴

$$(f(x,y))$$

$$I.F = e^{\int f(x,y) dx}$$

v) Inspection method

Note:

A non-exact homogeneous differential eqn independently is converted to exact by multiplying with (V/MN)

q. Find integrated factor of

$$(4x^2y^3 + 2y^4)dx + (2x^3y^3 + 2y^4 - 4x)dy = 0$$

$$\frac{\partial M}{\partial y} = 4y^3 + 2 \quad \frac{\partial N}{\partial x} = 4x^2 - 4$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 4x^2 - 4 - 4y^3 - 2$$

$$4x^2 - 4 - 4y^3 - 2$$

$$2(4x^2 - 2) = -4y^3 - 2$$

$$I.F = e^{\int \frac{2(4x^2 - 2)}{-4y^3 - 2} dx}$$

$$I.F = e^{\int \frac{2(4x^2 - 2)}{-4y^3 - 2} dx}$$

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$$\int f(x) dx$$

$$\frac{1}{2}x^2$$

$$\frac{2}{3}x^3$$

$$\frac{3}{4}x^4$$

$$-\frac{1}{2}x^{-1} = 1/x$$

$$-2/x \quad x^{-2} = 1/x^2$$

$$\textcircled{1} \quad y(1-xy)dx - x(1+xy)dy = 0 \quad -\textcircled{1}$$

$$Mx - Ny = xy - x^2y^2 + 2xy + x^2y^2 = 2xy$$

$$IF = \frac{1}{2xy} \quad \text{or} \quad IF = \frac{1}{xy}$$

Multiply $\textcircled{1}$ with $1/xy$

$$\frac{y(1-xy)}{xy} dx - \frac{x(1+xy)}{xy} dy = 0$$

$$\int \left(\frac{1}{x} - y\right) dx - \int \frac{1}{y} dy = c$$

$$\log x - xy - \log y = c$$

$$\boxed{\log x/y - xy = c}$$

$$I.F = e^{\int \frac{dy}{dx} dx} = e^{\int \frac{y^3}{x^2} dx} = e^{x^2/2 + C}$$

$$\frac{\partial M}{\partial y} = 2y$$

$$\frac{\partial L}{\partial x} = x^2 - 1$$

$$\frac{\partial M}{\partial x} = \frac{\partial L}{\partial y}$$

$$x^2 - 1$$

$$2y^3$$

I.F = $e^{-x^2/2}$

$$x^2 dy - (x^3 + y^3) dx = 0$$

$$\rightarrow Mx + Ny = x^3y - (x^3 + y^3)y = -y^4$$

I.F = $\frac{1}{y^4}$

$$\frac{x^2 y}{-y^4} dy + \frac{x^3 + y^3}{y^4} dy = 0$$

$$-\int \frac{x^2 y}{y^3} dy + \int \frac{1}{y} dy = 0$$

$$-\frac{x^2}{y^2} + \log y = C$$

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$$* \quad ydx - xdy + (1+x)dx + x^2 \cos y dy = 0$$

Multiply with $\frac{1}{x^2}$

$$\frac{ydx - xdy}{x^2} + \frac{(1+x)dx}{x^2} + \frac{x^2 \cos y dy}{x^2} = 0$$

$$\int d\left[-\frac{y}{x}\right] + \int \left(\frac{1}{x^2} + \frac{1}{x}\right) dx + \int \cos y dy = 0$$

$$-\frac{y}{x} - \frac{1}{x} + \log x + \sin y = C$$

$$* \quad ydx - xdy + (y^2 + y)x dx + xy e^x dx = 0$$

Multiply with $\frac{1}{xy}$

$$\frac{ydx - xdy}{xy} + \frac{(y^2 + y)x dx}{xy} + \frac{xy e^x dx}{xy} = 0$$

$$\int d[\log(x/y)] + \int (y+1)dy + \int e^x dx = 0$$

$$\log(x/y) + \frac{y^2 + y}{2} + e^x = C$$

$$* \quad ydx + xdy + x e^x dx = 0$$

$$\int d(xy) + \int x e^{-x} dx = 0$$

$$xy + -e^{-x} - x e^{-x} = C$$

* $y \frac{dy}{dx} + x^2 y^2 \frac{d^2y}{dx^2} = 0$

Multiply with $\frac{1}{y^2}$.

$$\frac{y \frac{dy}{dx} + x^2 y}{x^2} + \frac{x^2 y^2 \frac{d^2y}{dx^2}}{x^2} = 0$$

$$y \frac{dy}{dx} + x^2 = 0$$

$$\log(y) + e^{x^2} = C$$

* Linear differential eqns -

The differential eqn is said to be linear if it satisfies the following two conditions:

i) The dependent variable and all its derivatives should be of 1st degree only i.e.

$$(y')^1, \left(\frac{dy}{dx}\right)^1, \left(\frac{d^2y}{dx^2}\right)^1$$

ii) There is no product of the dependent variable and any of its higher derivative.

$$\left(\frac{y \frac{dy}{dx}}{dx}\right)^1, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots$$

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$$* \frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = x^3$$

\Rightarrow Linear

The eqn containing fn of dependent variable is not linear in that variable.

e.g.

$$\frac{d^2y}{dx^2} + \frac{3dy}{dx} + xe^y = x^3$$

\Rightarrow Non-linear

* Linear in y -

$$\boxed{\frac{dy}{dx} + py = q}$$

soln \downarrow p and q are f' of x or constant.

$$y \cdot e^{\int p dx} = \int q e^{\int p dx} dx + c$$

$$IF = e^{\int p dx}$$

* Linear in x -

$$\boxed{\frac{dx}{dy} + px = q}$$

soln. p and q are f' of y or constant

$$\boxed{e^{-\int p dy} \cdot \left(\int q e^{\int p dy} dy \right) = c}$$

$$y' + \frac{1}{x}y = x^3$$

$$\frac{dy}{dx} + \frac{y}{x} = x^3$$

$$P = \frac{1}{x}, \quad Q = x^3$$

$$e^{\int \frac{1}{x} dx} = x$$

$$\therefore y \cdot x = \int x^3 \cdot x dx + C = \int x^4 dx + C$$

$$yx = \frac{x^5}{5} + C$$

$$\text{at } y(1) = 6/5$$

$$\frac{6}{5} = \frac{1}{5} + C$$

$$C = 1$$

$$y = \frac{x^5}{5} + \frac{1}{5}$$

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$$* \frac{dy}{dx} + y(x\sin x + \cos x) = 1$$

$$\frac{dy}{dx} + \frac{(x\sin x + \cos x)y}{x\cos x} = 1$$

$$\frac{dy}{dx} + \frac{(\tan x + 1)y}{x\cos x} = 1$$

$$e^{\int p dx} = e^{\int (\tan x + 1) dx} = e^{\log \sec x + \log 1} \\ e^{\int p dx} = e^{\log(\sec x \cdot 1)} = e^{\log \sec x}$$

$$\therefore e^{\int p dx} = x \sec x.$$

$$y \cdot x \sec x = \int \frac{1}{x \cos x} \cdot x \sec x dx + C$$

$$\therefore xy \sec x = \tan x + C$$

$$* x^2 \frac{dy}{dx} + 2xy = \frac{2 \log x}{x} \quad \text{with } y(1) = 0 \text{ then find } y \text{ when } x = e$$

→

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{2 \log x}{x^3}$$

$$e^{\int p dx} = e^{\int 2/x dx} = x^2$$

$$y \cdot x^2 = \int \frac{2 \log x}{x^3} \cdot x^2 dx + C$$

$$x = \log x = t \Rightarrow y dx = dt$$

$$x^2 y = \int 2t dt + c$$

$$x^2 y = t^2 + c$$

$$x^2 y = (\log x)^2 + c$$

$$\Rightarrow c = 0$$

$$x^2 y = (\log x)^2$$

when $x=e$ then

$$e^2 y = 1$$

$$\therefore \boxed{y = e^{-2}}$$

$$*(x+2y^3) \frac{dy}{dx} = y \quad \text{with } x(1)=0$$

$$\frac{x dy}{dx} + 2y^3 \frac{dy}{dx} = y \quad \frac{x+2y^3}{y} = \frac{dy}{dx}$$

$$\frac{dx}{dy} - \frac{x}{y} = 2y^2$$

$$e^{\int p dy} = e^{\int -\frac{1}{y} dy} = \frac{1}{y}$$

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$$\frac{x \cdot 1}{y} = \int 2y^2 \cdot \frac{1}{y} dy + c$$

$$\frac{x}{y} = y^2 + c$$

$$x(1) = 0$$

⇒

$$c = -1$$

$$\frac{x}{y} = y^2 - 1$$

$$x = y^3 - y$$

* $\frac{dy}{dx} + y = e^x$ with $y(0) = 1$ find $y(1) = ?$

→

$$e^{\int 1 dx} = e^x$$

$$y \cdot e^x = \int e^x \cdot e^x dx + c$$

$$y e^x = \frac{e^{2x}}{2} + c$$

$$y(0) = 1$$

$$\Rightarrow c = \frac{1}{2}$$

$$y e^x = \frac{e^{2x}}{2} + \frac{1}{2}$$

Art 4EVZ

$$\therefore y = e^x + \tilde{e}^x$$

$$y = e^x + \tilde{e}^x$$

* Bernoulli's differential eqn :-

$$\frac{dy}{dx} + Py = Qy^n \rightarrow \text{Linear if } n=0.$$

P and Q are fn of 'x' (or) constants

\Rightarrow

$$y^{1-n} e^{\int (1-n) P dx} = \int (1-n) Q e^{\int (1-n) P dx} dx + C$$

\circlearrowleft

$$y^{1-n}$$

substitution

The substitution $y^{1-n} = v$ reduces Bernoulli's eqn to linear eqn

GATE

Q.
eqn

$y^{1-n} = v$ reduces the non-linear to which of the following form.

$$\frac{dy}{dx}$$

$$\frac{dv}{dx}$$

$$x^{1-n} v^n$$

$$x^{1-n} v^{n-1}$$

$$e^{\int \frac{1}{x} dx} = e^{\log x}$$

$$\frac{1}{1-n} \frac{dv}{dx} + Pv = q$$

$$\boxed{\frac{dv}{dx} + (1-n)Pv = (1-n)q}$$

Q. Which of the following substitution reduces the non-linear eqn $x \frac{dx}{dy} + x^2y^2 = x^3y^3$ to linear form?

- (a) $y^{-1}=v$ (b) $y^{-2}=v$ (c) $x^1=v$ (d) $x^{-2}=v$

$$\frac{dx}{dy} + xy^2 = x^2y^3$$

$$\therefore y^{1-2} = v$$

$$\therefore x^{-1} = v$$

Q. $\frac{dy}{dx} - 4\tan x = -y^2 \sec x$ with $y(0)=1$

$$e^{\int (1-2)(-\tan x) dx} = e^{\log \sec x} = \sec x$$

$$y^{1-2} \sec x = \int (1-2)(-\sec x)(\sec x) dx + C$$

$$y^{-1} \sec x = \tan x + C$$

$$y(0)=1 \Rightarrow C=1$$

$$\boxed{y = \frac{\sec x}{1 + \tan x}}$$

* Equations of the form

$$\boxed{f'(y) \frac{dy}{dx} + P f(y) = \varphi} \rightarrow \text{Non-linear}$$

$$f(y) = v \Rightarrow \frac{d(v)}{dx} + P(v) \frac{dy}{dx} = \frac{dv}{dx}$$

$$\boxed{\frac{dv}{dx} + Pv = \varphi} \rightarrow \text{Linear}$$

* What substitution reduces the non-linear eqns

$$\frac{dz}{dx} + \frac{z \log z}{x} = \frac{z (\log z)^2}{x^2} \quad \text{Also find I.F. of linear forms.}$$

$$\frac{1}{z (\log z)^2} \frac{dz}{dx} + \frac{1}{z} (\log z)^{-1} = \frac{1}{x^2}$$

$$(\log z)^{-1} = v$$

$$-\frac{1}{z} (\log z)^{-2} \frac{dz}{dx} = \frac{dv}{dx}$$

$$-\frac{dv}{dx} + \frac{1}{x} v = \frac{1}{x^2}$$

$$\frac{dv}{dx} - \frac{1}{x} v = -\frac{1}{x^2}$$

$$e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

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$$* \frac{x dy}{dx} + y \log y = xy e^x$$

$$\rightarrow \frac{x}{y} \frac{dy}{dx} + \log y = xe^x$$

$$\log y = v$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{dv}{dx}$$

$$x \frac{du}{dx} + v = xe^x$$

$$\frac{dv}{dx} + \frac{1}{x} v = e^x$$

$$e^{\int \frac{1}{x} dx} = x$$

$$v \cdot x = \int e^x \cdot x dx + c$$

$$vx = e^x(x-1) + c$$

$$\log y = e^x(x-1) + c$$

* Clairaut's eq?

$$y = x \frac{dy}{dx} + f\left(\frac{dy}{dx}\right)$$

or

$$y = px + f(p)$$

where $p = \frac{dy}{dx}$

so

Directly replace $\frac{dy}{dx}$ by 'c'

$$y = cx + f(c)$$

$$* \left(y - x \frac{dy}{dx} \right) \left(\frac{dy}{dx} - 1 \right) = \frac{dy}{dx}$$

→

$$y - xy' = \frac{y'}{y'-1}$$

$$y = \frac{y'}{y'-1} + xy'$$

$$y = cx + \frac{c}{c-1}$$

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* $P = \sin(y - xp)$ where $P = \frac{dy}{dx}$

$$y - xp = \sin^{-1} P$$

$$y = xp + \sin^{-1} P$$

$$y = cx + \sin^{-1} c$$

* Higher order linear differential eqns with constant coefficients :-

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_{n-1} \frac{dy}{dx} + k_n y = x$$

k_1, k_2, \dots, k_n are constants

x - fn of x or constant.

$$D = \frac{d}{dx}, \quad D^2 = \frac{d^2}{dx^2}, \quad D^3 = \frac{d^3}{dx^3} \Rightarrow \text{e.g. } D e^{2x} = 2 e^{2x}$$

$$\frac{1}{D} = \int, \quad \frac{1}{D^2} = \iint \Rightarrow \text{e.g. } \frac{1}{D} \cos 2x = \frac{\sin 2x}{2}$$

$$(D^n + k_1 D^{n-1} + \dots + k_{n-1} D + k_n) y = x$$

$$F(D) y = x$$

$$F(D) = D^n + k_1 D^{n-1} + \dots + k_{n-1} D + k_n$$

fn of differential operator

- ① The complete soln of $F(D)y = x$ is
- $$\text{C.F} + \text{P.I}$$
- ~~y~~ $y = \text{Complementary f.n} + \text{Particular integral}$
 $= \text{C.F} + \text{P.I}$
- ② If $x = 0$ then $F(D)y = 0$ is called homogeneous linear differential eqn.
- ③ If $x \neq 0$ then $F(D)y = x$ is called non-homogeneous linear differential eqn.
- ④ The solution of homogeneous linear differential eqn $F(D)y = 0$ is called complementary f.n.
- The no. of arbitrary constants in the complementary f.n should be equal to the order of given differential eqn.
- ⑤ The particular integral of $F(D)y = x$ is
- $$\text{P.I} = \left[\frac{1}{F(D)} \right] x . \quad \text{P.I will not contain any arbitrary constants.}$$

- ⑥ If $x \neq 0$ then complete soln of the given eqn is only complementary function.

- ⑦ By assuming ' D ' as an algebraic quantity $F(D) = 0$ becomes an algebraic eqn.
 And is called Auxiliary eqn.

x Procedure to find CF -

By solving an A.E. we get the roots based on nature of this root we write the CF as follows:

Nature of roots	CF
Real and distinct $\lambda = m_1, m_2, m_3$	$CF = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$
2 Real & repeated $\lambda = m_1, m_1$	$CF = (c_1 + c_2 x) e^{m_1 x}$
3 Complex & distinct $\lambda = a \pm ib$	$CF = e^{ax} [c_1 \cos bx + c_2 \sin bx]$
4 Complex & repeated $\lambda = a \pm ib, a \pm ib$	$CF = e^{ax} [(c_1 + c_2 x) \cos bx + (c_3 + c_4 x) \sin bx]$
5 Surds (Real no.) $a \pm \sqrt{b}$	$CF = c_1 e^{(a+\sqrt{b})x} + c_2 e^{(a-\sqrt{b})x}$ OR $CF = e^{ax} [c_1 \cosh \sqrt{b} x + c_2 \sinh \sqrt{b} x]$

Roots

$$D = 1, -\frac{1}{2}, 2$$

$$CF = C_1 e^x + C_2 e^{-\frac{1}{2}x} + C_3 e^{2x}$$

$$D = 2, -2$$

$$CF = C_1 e^{2x} + (C_2 + C_3 x) e^{-2x}$$

$$D = 2 \pm 3i, \frac{1}{3}$$

$$CF = e^{\frac{x}{3}} [C_1 \cos 3x + C_2 \sin 3x] + C_3 e^{2x}$$

$$D = \pm 2i, \pm 4$$

$$CF = C_1 e^{4x} + C_2 e^{-4x} + [C_3 \cos 2x + C_4 \sin 2x]$$

$$* \frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y = 0$$

$$* \frac{d^4y}{dx^4} - 16y = 0$$



$$D^2 - 5D + 6 = 0$$

$$(D^4 - 16)y = 0$$

$$\therefore D = +3, D = -2$$

$$D^4 - 16 = 0$$

$$CF = C_1 e^{+3x} + C_2 e^{-2x}$$

$$\cancel{D^4} = \cancel{16}$$

$$\cancel{(D^2-4)}(D^2+4) = 0$$

$$D = 2, -2, \pm 2i$$

$$CF = C_1 e^{2x} + C_2 e^{-2x} + [C_3 \cos 2x + C_4 \sin 2x]$$

$$x \frac{d^2y}{dx^2} + 2p \frac{dy}{dx} + (p^2 + q^2)y = 0$$

→

$$D^2 + 2pD + (p^2 + q^2) = 0$$

$$D_1, D_2 = \frac{-2p \pm \sqrt{4p^2 - 4p^2 - 4q^2}}{2} = \frac{-2p \pm \sqrt{-4q^2}}{2}$$

$$= \frac{-2p \pm 2qi}{2}$$

$$D_1, D_2 = -p \pm qi$$

$$CF = e^{-px} [q \cos qx + c_2 \sin qx]$$

Note!

If $y = C_1 y_1 + C_2 y_2 + C_3 y_3 + \dots$ is the complete solution of homogeneous differential eqn $F(D)y = 0$, then each one of y_1, y_2, y_3, \dots is linearly independent solution of the homogeneous of linear differential eqn $F(D)y = 0$.

Ques: y_1, y_2 are linearly independent solns of the corresponding linearly indepd homogeneous eqn of $F(D)y = x$ then y_1, y_2 is a soln of which of the following eqns

(a) $F(D)=0$ (b) $x=0$

(c) $F(D)y = x$ (d) $\underline{F(D)y = 0}$

$\sin \theta$ and $\cos \theta$ are linearly independent.

$$\begin{pmatrix} \sin \theta & \cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix}$$

$\begin{pmatrix} \sin \theta & \cos \theta \\ \cos \theta & -\sin \theta \end{pmatrix}^{-1} = \frac{1}{\sin^2 \theta + \cos^2 \theta} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$

$$\begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}^{-1} = \frac{1}{\sin^2 \theta + \cos^2 \theta} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

Since $\sin \theta$, $\cos \theta$ are linearly independent.

CF = Actual Evidence

$$D = 131$$

$$D^2 T_1^2 = 0$$

$$(3^2 + 9^2) T_1^2 = 0$$

$$45^2 T_1^2 = 0$$

If f_1 and f_2 are linearly independent then

$$T_1 = 0$$

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* e^{3x} , e^{-3x} are linearly independent soln of -

$$(a) \frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

$$(b) \frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0$$

$$\checkmark (c) \frac{d^4y}{dx^4} - 81y = 0$$

(d) None

Order \geq No. of indep. solns

$$* y''' + y'' + y' + y = 0$$



$$D^3 + D^2 + D + 1 = 0$$

$$D^2(D+1) + (D+1) = 0 \quad \therefore (D^2+1)(D+1) = 0$$

$$D^2 = 0, D \cdot D^2 = -1, D = -1$$

$$\therefore D = \pm i$$

$$y = C_1 e^{-x} + C_2 \cos x + C_3 \sin x$$

$$* \frac{d^4y}{dx^4} + a^4y = 0$$

$$D^4 + a^4 = 0$$

$$(D^2 + a^2)^2 - 2D^2 a^2 = 0$$

$$(D^2 + a^2)^2 - (\sqrt{2} Da)^2 = 0$$

$$(D^2 + a^2 + \sqrt{2}ad) (D^2 + a^2 - \sqrt{2}ad) = 0$$

$$D^2 + \sqrt{2}ad + a^2 = 0 \quad \text{and} \quad D^2 - \sqrt{2}ad + a^2 = 0$$

$$\frac{-\sqrt{2}a \pm \sqrt{2a^2 - 4a^2}}{2}$$

$$\frac{-\sqrt{2}a \pm \sqrt{-2a^2}}{2}$$

$$\frac{-\sqrt{2}a \pm i\sqrt{2}a}{2}$$

$$\frac{-a \pm ai}{\sqrt{2}}$$

$$\frac{a \pm ai}{\sqrt{2}}$$

$$y = e^{\frac{ax}{\sqrt{2}}} \left[C_1 \cos \frac{a}{\sqrt{2}}x + C_2 \sin \frac{a}{\sqrt{2}}x \right] + e^{\frac{a_1 x}{\sqrt{2}}} \left[C_3 \cos \frac{a_1}{\sqrt{2}}x + C_4 \sin \frac{a_1}{\sqrt{2}}x \right]$$

$$* \frac{d^2f}{dn^2} + 4 \frac{df}{dn} + 4f = 0$$

① $f_1 = e^{2n}$, $f_2 = e^{-2n}$ ✓ ② $f_1 = ne^{-2n}$, $f_2 = e^{-2n}$

③ $f_1 = e^{-2n}$, $f_2 = ne^{2n}$ ④ $f_1 = ne^{-2n}$, $f_2 = e^{-2n}$

* The D.E. $\frac{d^2y}{dx^2} + 2 \cos ax y - 3y = x^2$ is

(a) Homogeneous linear D.E. (b) Non-homogeneous linear D.E. with constant coefficients

(c) Non-linear D.E. order 2 (d) Linear D.E.

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The sol'n of

$$\frac{d^2i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{i}{LC} = 0 \text{ where } R^2C = 4L$$

- (a) $e^{\frac{Rt}{2L}}$ (b) $te^{\frac{Rt}{L}}$
 (c) $e^{-\frac{Rt}{L}}$ (d) $e^{-\frac{Rt}{2L}}$

$$D^2 + \frac{R}{L} D + \frac{1}{LC} = 0$$

$$D_1, D_2 = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2}$$

$$= \frac{-\frac{R}{L} + \sqrt{\frac{R^2 C - 4L}{L^2 C}}}{2}$$

$$= \frac{-\frac{R}{L} \pm 0}{2}$$

$$D_1, D_2 = -\frac{R}{2L}, -\frac{R}{2L} + \frac{-R}{2L}$$

$$\therefore y = C_1 e^{(-\frac{R}{2L})t} + C_2 t e^{(-\frac{R}{2L})t}$$

$$\therefore y = C_1 e^{-\frac{Rt}{2L}} + C_2 t e^{-\frac{Rt}{2L}}$$

$$* \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$$

$$\text{with } y(0)=0 \quad \text{and} \quad y'(0)=1$$

$$D^2 + 4D + 13 = 0$$

$$\lambda_1, \lambda_2 = -\frac{4+6i}{2}$$

$$= -2+3i$$

$$y = e^{-2x} [c_1 \cos 3x + c_2 \sin 3x]$$

$$\text{at } x=0, y=0 \Rightarrow c_1=0$$

$$y = c_2 e^{-2x} \sin 3x$$

$$y' = c_2 [-2e^{-2x} \sin 3x + 3e^{-2x} \cos 3x]$$

$$y'(0) = 1$$

$$\therefore c_2 = \frac{1}{3}$$

$y = \frac{e^{-2x}}{3} \sin 3x$.
---------------------------------	---

$$* \frac{9d^2y}{dx^2} - 6\frac{dy}{dx} + y = 0 \quad \text{with } y(0)=3 \text{ & } y'(0)=1$$

→

$$9D^2 - 6D + 1 = 0$$

$$\therefore D = \lambda_3 - \lambda_3$$

$$y = (c_1 + c_2x) e^{\lambda_3 t}$$

$$y(0)=3 \Rightarrow 3 = c_1$$

$$y = 3 + c_2 x e^{\lambda_3 t}$$

$$y' = c_2 \left[\frac{x}{3} e^{\lambda_3 t} + e^{\lambda_3 t} \right]$$

$$y'(0)=1$$

$$0 = c_2$$

$$y = (3 + 0x) e^{\lambda_3 t}$$

$$y = 3 e^{\lambda_3 t}$$

$$* \frac{d^2y}{dx^2} = n^2 y$$

(A) $y = e^{nx}$

TRUE

(B) $y = e^{-nx}$

TRUE

(C) $y = \sum_{n=1}^{\infty} \frac{x^n}{n!}$

TRUE

(D) $y = \sum_{n=1}^{\infty} \frac{x^n}{n}$

TRUE

* Direct put cond² in options for Y(0)=0

$$\frac{d^2n}{dx^2} = n \Rightarrow \text{where } L \text{ is a constant of } n(0)=k \\ + n'(0)=0$$

$$(A) n = k e^{xt} \quad (B) n = k^2 e^{xt} \quad \text{Here check both cond} \\ \text{then c & d satisfies} \\ (C) n = k e^{\frac{xt}{L}} \quad (D) n = k e^{-\frac{xt}{L}} \quad \text{so from roots of A-E} \\ (D) \text{ is ans.}$$

$$D = \pm \frac{1}{L}$$

$$y = C_1 e^{\frac{xt}{L}} + C_2 e^{-\frac{xt}{L}}$$

* For what value of λ the D.E. $\frac{d^2y}{dx^2} + \lambda^2 y = 0$ with $Y(0)=0$

and $Y(\pi)=0$ will have non-trivial solⁿ

(A) $\lambda = -1$

(B) $\lambda = \pm 1$

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16072023 (D) $\lambda = 0$

(C) $\lambda = 1, 3, 5, \dots$

(D) $\lambda = 0, 1, 4, 9, \dots$

solution (C)

$$D = \pm \lambda i$$

$$y = C_1 \cos \lambda x + C_2 \sin \lambda x$$

$$\lambda = n = 1, 2, 3, \dots$$

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Linked

Q* The complete soln of the differential eqn

 $\frac{d^2y}{dx^2} + \frac{pd\gamma}{dx} + qy = 0$ is $y = C_1 e^{-x} + C_2 e^{-3x}$ then
 values of p & q are -

(a) $p=3, q=3$ (b) $p=4, q=3$

(c) $p=4, q=4$ (d) $p=3, q=4$

$\rightarrow (D+1)(D+3)y = 0$

$(D^2 + 4D + 3)y = 0$

$D^2y + 4Dy + 3y = 0$

$p=4, q=3$

Q* The soln of $\frac{d^2y}{dx^2} + \frac{pd\gamma}{dx} + (q+1)y = 0$ is

\rightarrow (a) $x e^{2x}$ (b) e^{2x} (c) $x e^{-2x}$ (d) $x^2 e^{-2x}$

$\rightarrow y'' + 4y' + 4y = 0$

$D^2 + 4D + 4 = 0$

$(D+2)^2 = 0$

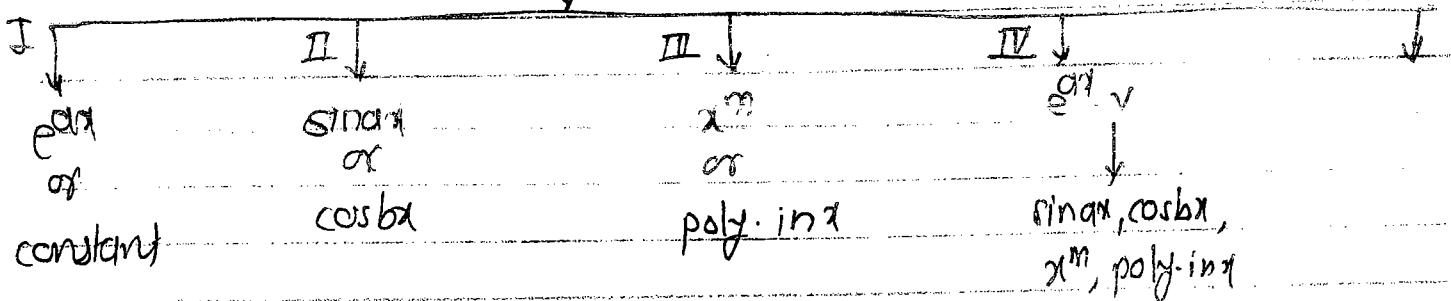
$D = -2, -2$

$y = (C_1 + C_2 x)e^{-2x} = xe^{-2x}$ if $C_1 = 0, C_2 = 1$

Particular Integrals :

$$F(D)y = x$$

$$PI = \left[\frac{1}{F(D)} \right] x$$



Case I - $x = e^{ax}$ or constant

$$PI = \left[\frac{1}{F(D)} \right] e^{ax} = \frac{1}{F(a)} e^{ax} \quad (\text{where } F(a) \neq 0)$$

Replace ' D ' by ' a ' in $f^n f(D)$

if $f(a) = 0$ then

$$PI = \cancel{x f(D)} \times \left[\frac{1}{F'(D)} \right] e^{ax}$$

Replace ' D ' by ' a ' in $F'(D)$

$$PI = x \left[\frac{1}{F'(a)} \right] e^{ax} \quad (F'(a) \neq 0)$$

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$$\text{if } F'(a) = 0 \Rightarrow P.I. = x^2 \left(\frac{1}{F''(a)} \right) e^{dx} \quad (F''(a) \neq 0)$$

* P.I. of $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 7y = e^{-2x}$

 \rightarrow

$$P.I. = \frac{1}{D^2 + 2D + 7} e^{-2x} = \frac{1}{(-2)^2 + 2(-2) + 7} e^{-2x}$$

$P.I. = \frac{e^{-2x}}{7}$

* P.I. of $\frac{d^2y}{dx^2} - 9y = e^{3x} + 3$

$$P.I. = \frac{1}{D^2 - 9} (e^{3x} + 3)$$

$$= x \left(\frac{1}{2D} \right) e^{3x} + \frac{3}{0-9} e^{0x}$$

$$= \frac{x e^{3x}}{6} + \frac{3}{-9}$$

$$\therefore P.I. = \frac{x e^{3x}}{6} - \frac{1}{3}$$

* PI of $\frac{dy}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-2x}$

$$P_1 = \left[\frac{1}{D^2 + 4D + 4} \right] e^{-2x} = \alpha \left(\frac{1}{(2D+4)} \right) e^{-2x} = x^{\alpha} \left(\frac{1}{2} \right)$$

~~(Case I P1)~~

$$\therefore P_2 = x^2 e^{-2x}$$

* 'd' is the root of the A.E. of $F(D)y = e^{dx}$ then PI is
(k constant)

- (a) ke^{-dx}
- (b) ke^{ax}
- (c) kxe^{ax}
- (d) kx^2e^{ax}

* Case II -

$$x = \sin(ax) \cos(bx) \quad (\sin(ax+bx) \text{ or } \cos(ax+bx))$$

$$P_1 = \left[\frac{1}{F(D)} \right] \sin ax = \left[\frac{1}{F(-a^2)} \right] \sin ax \quad (F(-a^2) =$$

Replace ' D^2 ' by ' $-a^2$ ' in $F(D)$

IF $F(-a^2) = 0$ -

$$P_1 = \alpha \left[\frac{1}{F(D)} \right] \sin ax = \alpha \frac{\sin ax}{F'(a^2)}$$

replace ' D^2 ' by ' $-a^2$ ' in $F'(D)$

2A

* PI of $\frac{dy}{dx^4} + \frac{dy}{dx^2} + 3y = \cos 3x$

$$\text{PI} = \frac{1}{D^4 + D^2 + 3} \cos 3x = \frac{1}{(-q)^2 + (-q) + 3} \cos 3x$$

$$\text{PI} = \frac{\cos 3x}{75}$$

* PI of $\frac{dy}{dx^2} + 4y = \cos(2x+4) + e^{-2x}$

$$\text{PI} = \frac{1}{D^2 + 4} (\cos(2x+4) + e^{-2x})$$

$$= x \left(\frac{1}{2D} \right) \cos(2x+4) + \frac{1}{(-2)^2 + 4} e^{-2x}$$

$$= \frac{1}{2} \frac{\sin(2x+4)}{2} + \frac{e^{-2x}}{8}$$

$$\therefore \text{PI} = \frac{x \sin(2x+4)}{4} + \frac{e^{-2x}}{8}$$

* PI of $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + 2y = \sin x$

$$\text{PI} = \frac{1}{D^3 + D^2 + 2D + 2} \sin x$$

$$D^2 + (-1) + 2D + 2$$

$$D^2$$

$$\cos x - \sin x$$

$$P_2 = \frac{\sin x - \cos x}{x}$$

* Case II $x = e^{\lambda t}$ or poly. in x (m is even integer)

$$\begin{aligned} P_1 &= \left[1 \right] [x^m] \\ &\quad [F(D)] * [1 + \phi(D)] \\ &= [1 + \phi(D)]^* x^m \end{aligned}$$

* taking
* is least power common to $F(D)$

$$x = e^{(D+2D+2)x} = e^{(D+2D+2)x}$$

$$\begin{aligned} P_1 &= \left[1 \right] \left[\frac{(D+2D+2)^m}{2} \right] \\ &= \left[1 \right] \left[\frac{(D+2D+2)^m}{2} \right] \end{aligned}$$

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$$= \frac{1}{2} \left[1 - \left(\frac{D^2 + 2D}{2} \right) + \left(\frac{D^2 + 2D}{2} \right)^2 \right] (x^2 + 2)$$

$$= \frac{1}{2} [D^2 + 2 - 1 - 2D + 2]$$

$$\therefore P_2 = \frac{x^2 - 2x + 3}{2}$$

* PI of $\frac{d^3y}{dx^3} + \frac{3d^2y}{dx^2} = x^3 + 3x$

 \rightarrow

$$PI = \frac{1}{D^3 + 3D^2} (x^3 + 3x) = \frac{1}{3D^2 \left[1 + \frac{D^3}{3D^2} \right]} (x^3 + 3x)$$

$$= \frac{1}{3D^2} \left[1 + \frac{D}{3} \right]^{-1} (x^3 + 3x)$$

$$= \frac{1}{3D^2} \left[1 - \frac{D}{3} + \frac{D^2}{9} - \frac{D^3}{27} \right] (x^3 + 3x)$$

$$= \frac{1}{3D^2} \left[x^3 + 3x - \frac{3x^2}{3} - 1 + \frac{5x}{9} - \frac{6}{27} \right]$$

$$= \frac{1}{3D^2} \left[x^3 - x^2 + \frac{33x}{9} - \frac{33}{27} \right] = \frac{1}{3D^2} \left[x^3 - x^2 + \frac{11x}{3} - \frac{11}{9} \right]$$

$$= \frac{1}{3} \left[\frac{x^5}{20} - \frac{x^4}{12} + \frac{11x^3}{18} - \frac{11x^2}{18} \right]$$

* Case IV : $x = e^{\alpha t} \cdot v$

$$PI = \left[\frac{1}{F(D)} \right] \left(\frac{dv}{dt} \right) = e^{at} \left[\frac{1}{F(D+a)} \right] v$$

Replace 'D' by 'D+a' in F(D)

$\frac{1}{F(D+a)} v$	→ II if $v = \sin bt$ or $\cos bt$
$\frac{1}{F(D+a)}$	→ III if $v = x^m$ or poly. ind

DE

$$\textcircled{i} \quad \frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 4y = 0 \quad - \text{ordinary DE}$$

since only one independent variable

$$\textcircled{ii} \quad \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial^2 u}{\partial y^2}\right)^3 = 0 \quad - \text{partial DE}$$

since 2 independent variables.

i) order :- The highest derivative involved in the DE is called order of DE.

ii) degree :- The highest power of highest derivative is called degree of DE (provided the derivatives of the dependent variable should free from radicals & fractions).

$$\textcircled{1} \quad \left(\frac{dy}{dx}\right)^2 = \left[x + 4\left(\frac{dy}{dx}\right)^2\right]^{3/2}$$

$$\left(\frac{dy}{dx}\right)^4 = \left[x + 4\left(\frac{dy}{dx}\right)^2\right]^3$$

order = 2 degree = 4

$$\textcircled{2} \quad \frac{dy}{dx} = x + \frac{2}{\frac{dy}{dx}}$$

$$\left(\frac{dy}{dx}\right)^2 = x \left(\frac{dy}{dx}\right) + 2 \quad \begin{matrix} \text{order} = 1 \\ \text{degree} = 2 \end{matrix}$$

$$\textcircled{3} \quad dr = (\theta + \cos\theta) d\theta$$

$$\frac{dr}{d\theta} = \theta + \cos\theta \quad \begin{matrix} \text{order} = 1 \\ \text{degree} = 1 \end{matrix}$$

$$\textcircled{4} \quad \frac{d^3y}{dx^3} + 2\left(\frac{d^2y}{dx^2}\right)^2 + 3y = 0 \quad \begin{matrix} \text{order} = 3 \\ \text{degree} = 1 \end{matrix}$$

$$\textcircled{5} \quad \frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial y^2}\right)^4$$

order = 2

\Rightarrow Formation of ordinary DE :-

$$F(x, y, a, b) = 0 \quad \text{--- (1)}$$

diff (1) w.r.t x.

$$F_1(x, y, \frac{dy}{dx}, a, b) = 0 \quad \text{--- (2)}$$

diff (2) w.r.t x

$$F_2(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, a, b) = 0 \quad \text{--- (3)}$$

eliminating a & b from above 3 eq,

$$\phi(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}) = 0$$

The no. of arbitrary constants eliminated should be equal to the order of resulting ordinary DE.

* The DE of family of straight line passing through the origin is

(a) $y dy + x dx = 0$

b) none

~~(c)~~ $y dx - x dy = 0$

d) $y dy - x dx = 0$

$$y = mx$$

$$\frac{dy}{dx} = m \Rightarrow y = \frac{dy}{dx} \cdot x$$

$$y dx - x dy = 0$$

(*) $y = c(x-c)^2$ c is arbitrary const.

$$y_1 = \frac{dy}{dx} = 2c(x-c)$$

$$\frac{y}{y_1} = \frac{x-c}{2} \Rightarrow x-c = \frac{2y}{y_1}$$

$$c = x - \frac{2y}{y_1}$$

④ DE of family of circles

Note : If the given eqⁿ of the form $y = Af(x) + Bg(x)$
then the resulting DE is

$$\begin{vmatrix} y & f & g \\ y' & f' & g' \\ y'' & f'' & g'' \end{vmatrix} = 0.$$

⑤ $y = Ae^{2x} + Bx$

$$\begin{vmatrix} y & e^{2x} & x \\ y' & 2e^{2x} & 1 \\ y'' & 4e^{2x} & 0 \end{vmatrix} = 0$$

through $y(-4) - 1(0-y'') + x(4y' - 2y'') = 0$

$$Ay^4(1-2x) + 4xy' - 4y = 0$$

⑥ $y = e^x(A\cos x + B\sin x) \quad \text{--- } ①$

$$y' = e^x(-A\sin x + B\cos x) + e^x(A\cos x + B\sin x)$$

$$y'' = e^x(-A\sin x + B\cos x) + y'$$

$$y''' = y' + e^x(-A\cos x - B\sin x) +$$

$$e^x(-A\sin x + B\cos x)$$

$$y'' = y' - \cancel{y} + y' - \cancel{y}$$

$$y'' - 2y' + 2y = 0$$

Note :- If the given eq of the form

$y = c_1 e^{ax} + c_2 e^{bx} + c_3 e^{cx} + \dots$ then

$$(D-a)(D-b)(D-c)y = 0$$

$$D = \frac{d}{dx} \quad D^2 = \frac{d^2}{dx^2}$$

$$y = Ae^{2x} + Be^{-3x}$$

$$(D-2)(D+3)y = 0$$

$$(D^2 + D - 6)y = 0$$

$$y'' + y' - 6y = 0$$

$$\textcircled{1} \quad y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$$

$$(D+1)(D+2)(D+3)y = 0$$

$$(D^3 + 3D^2 + 2D + 1)y = 0$$

$$(D^3 + 6D^2 + 11D + 6)y = 0$$

↓ ↓ ↓
1+2+3 1+2+3+3+1 1+2+3

$$y''' + 6y'' + 11y' + 6y = 0$$

► Solution of DE :-

$$\frac{d^2y}{dt^2} = g \quad y(0) = 0 \quad y'(0) = 0$$

$$\frac{dy}{dt} = gt + c_1$$

$$y = \frac{1}{2}gt^2 + c_1 t + c_2$$

$$y(0) = 0 \Rightarrow c_2 = 0$$

$$y'(0) = 0 \Rightarrow c_1 = 0$$

$$y = \frac{1}{2}gt^2$$

\Rightarrow 1st order 1st degree DE :-

$$\frac{dy}{dx} = F(x, y)$$

or

$$M(x, y) dx + N(x, y) dy = 0.$$

Variable - separable method :-

$$\textcircled{A} \quad \frac{dy}{dx} = e^{x-y} + x^3 e^{-y}$$
$$= e^{-y} (e^x + x^3).$$

$$\int e^y dy = \int (e^x + x^3) dx.$$

$$e^y = e^x + \frac{x^4}{4} + C.$$

$$\textcircled{B} \quad \log\left(\frac{dy}{dx}\right) = 2x + 3y.$$

$$\frac{dy}{dx} = e^{2x} \cdot e^{3y}.$$

$$\int e^{-3y} dy = \int e^{2x} dx.$$

$$\frac{-1}{3} e^{-3y} = \frac{e^{2x}}{2} + C.$$

$$\textcircled{C} \quad \frac{dy}{dx} \leq \frac{y^2}{1-xy}.$$

$$(1-xy) dy = y^2 dx$$

$$dy = y^2 dx + xy dy.$$

$$dy = y(y dx + x dy)$$

$$\int \frac{1}{y} dy = \int d(xy)$$

$$\log y = xy + C.$$

$$\textcircled{D} \quad \frac{dy}{dx} = \frac{-x}{y} \quad \text{at } x=1, y=\sqrt{3}$$

$$\int y dy = \int -x dx.$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C \Rightarrow C = \frac{3}{2} + \frac{1}{2} = 2.$$

$$y^2 = -x^2 + 4.$$

$$\textcircled{E} \quad \frac{dy}{dx} = y^2 \sin x, \quad y(2\pi) = 1.$$

$$\text{or } y \sin x = 1 \quad \text{or } y \cos x = 1$$

$$\int \frac{1}{y^2} dy = \int dx \sin x$$

$$-\frac{1}{y} = -\cos x + c \Rightarrow c = 0$$

$$y \cos x = 1.$$

$$\frac{dy}{dx} = 3x^2 - 2x \quad \text{passes through } (1,1) \text{ then}$$

find mag of y when $x=3$

$$\int dy = \int 3x^2 - 2x dx$$

$$y = x^3 - x^2 + c$$

$$(1,1) \Rightarrow c = 1$$

$$y = x^3 - x^2 + 1.$$

$$\text{when } x=3 \Rightarrow y = 19.$$

Bio transformation of an organic compound having conc. 'x' can be modelled using DE $\frac{dx}{dt} + kx^2 = 0$.

At $t=0$, the conc is 'a' then the soln. is

$$\frac{dx}{dt} = -kx^2$$

$$\int \frac{1}{x^2} dx = -k/dt$$

$$-\frac{1}{x} = -kt + c \Rightarrow c = -\frac{1}{xa}$$

$$\frac{dx(t)}{dt} + bx(t) = 0$$

$$\int \frac{1}{x} dx = \int b dt$$

$$\log x = -bt + \log c$$

$$x = e^{-bt} c$$

$$d) \frac{dy}{dx} = 1+y^2$$

$$\int \frac{1}{1+y^2} dy = \int dx$$

$$\tan^{-1} y = x + c$$

$$y = \tan(x+c).$$

④ Find the curve passing through the point $(0, 1)$ satisfying $\sin\left(\frac{dy}{dx}\right) = b$.

$$\frac{dy}{dx} = \sin^{-1} b$$

$$\int dy = \int \sin^{-1} b \, dx$$

$$y = (\sin^{-1} b)x + c$$

$$(0, 1) \quad c = 1$$

$$y = (\sin^{-1} b)x + 1.$$

⑤ $\frac{dy}{dx} = e^{x+y}$ given that for $x=1, y=1$;

find y when $x = -1$.

$$\int e^{-y} \, dy = \int e^x \, dx$$

$$-e^{-y} = e^x + c \Rightarrow c = -e^{-1} - e.$$

$$\Rightarrow -e^{-y} = e^x - (e^{-1} + e)$$

$$\text{at } x=1 \Rightarrow -e^{-y} = e - e^{-1} - e$$

$$y = -1.$$

⑥ $\frac{dy}{dx} = (4x+y+1)^2$

$$\frac{dv}{dx} = v^2 + 4$$

$$\int \frac{1}{v^2+4} \, dv = \int dx$$

$$\frac{1}{2} \tan^{-1}\left(\frac{v}{2}\right) = x + c.$$

$$\frac{4x+y+1}{2} = \tan(2x+c)$$

$$4x+y+1 = v$$

$$4 + \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dy}{dx} = \frac{dv}{dx} - 4$$

$$4x+y+1 = 2\tan(2x+c)$$

homogenous differential eqn :-

$$\frac{dy}{dx} = F(x, y)$$

is said to be homogenous DE if $F(x, y)$ should be a homogenous fn of degree '0'.

e.g.: $Mdx + Ndy = 0$ is said to be homogenous if all the terms of M & N should be same degree.

for variable - separable sub $y = vx \text{ or } x = vy$.

$$x \frac{dy}{dx} = y \{ \log y - \log x + 1 \}$$

$$\frac{dy}{dx} = \frac{y}{x} \left\{ \log \left(\frac{y}{x} \right) + 1 \right\}$$

$$y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$v + x \frac{dv}{dx} = v \{ \log v + 1 \}$$

$$x \frac{dv}{dx} = v \log v$$

$$\int \frac{1}{v \log v} dv = \int \frac{1}{x} dx$$

$$\log(\log v) = \log x + \log c.$$

$$\log v = xc \Rightarrow v = e^{cx} \Rightarrow y = xe^{cx}$$

$$(x^3 + 3xy^2) dx + (y^3 + 3x^2y) dy = 0$$

Non-homogeneous DE

An eqn of the form

$$\frac{dy}{dx} = \frac{a_1 x + b_1 y + c_1}{a_2 x + b_2 y + c_2}$$

$$\text{case (i) :- } \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

In this case there exists a substitution which reduces the given eqn to variable - separable form.

$$\text{case (ii) :- } \frac{a_1}{a_2} \neq \frac{b_1}{b_2}$$

Genous

be

$$\text{sub, } x = x + h, \quad y = y + k$$

$$dx = dx, \quad dy = dy$$

'y.

$$\frac{dy}{dx} = \frac{a_1(x+h) + b_1(y+k) + c_1}{a_2(x+h) + b_2(y+k) + c_2}$$

$$\frac{dy}{dx} = \frac{a_1x + b_1y + (a_1h + b_1k + c_1)}{a_2x + b_2y + (a_2h + b_2k + c_2)}$$

choose h, k so that $a_1h + b_1k + c_1 = 0, a_2h + b_2k + c_2 = 0$

$$\frac{dy}{dx} = \frac{a_1x + b_1y}{a_2x + b_2y}$$

$$\textcircled{*} \quad (2x + 2y - 1) dx = (x + y + 1) dy$$

$$\frac{dy}{dx} = \frac{2x + 2y - 1}{x + y + 1} \quad \left(\frac{a_1}{a_2} = \frac{b_1}{b_2} \right)$$

$$\frac{dy}{dx} - 1 = \frac{2v - 1}{v + 1}$$

$$\frac{dv}{dx} = \frac{3v}{v + 1}$$

$$\int \frac{v+1}{v} dv = 3 \int dx \Rightarrow v + \log v = 3x + C$$

$$x + y + \log(x+y) = 3x + C \Rightarrow y - 2x + \log(x+y) =$$

\textcircled{*} which of the following sub reduces the

non-homogeneous eqn $\frac{dy}{dx} = \frac{y+x-2}{y-x-4}$ to hom.

form.

$$\text{sol } a_1 = 1, b_1 = 1$$

$$a_1h + b_1k + c_1 = 0$$

$$k+h-2=0 \quad k=3 \quad h=-1$$

$$k-h-4=0$$

$$x = (x+1) \quad y = (y+3)$$

$$\text{hom form} \Rightarrow \frac{dy}{dx} = \frac{y+x}{y-x}$$

Exact differential eqⁿ :-

An eqⁿ $Mdx + Ndy = 0$ is said to be an exact DE if

$\exists f(x, y)$ such that $d(f(x, y)) = Mdx + Ndy$.

condition to be exact

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

if this satisfies then

find soln.

$$Mdx + Ndy = 0$$

$$\int M dx + \int \cancel{Ndx dy} \int (\text{terms of } N \text{ dy without } x) = C$$

Keep 'y' const ~~keep 'x' const~~

④ $\left\{ y(1 + \frac{1}{x}) + \cos y \right\} dx + \left\{ x + \log x - x \sin y \right\} dy = 0$

$$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y$$

$$\frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

exact!!

solt.

$$\int (y(1 + 1/x) + \cos y) dx + \int 0 dy = C$$

$y \rightarrow \text{const}$

$$= y(x + \log x) + x \cos y + C = C$$

④ $y \sin 2x \, dx - (1 + y^2 + \cos^2 x) \, dy = 0$

M

N

$$\frac{\partial M}{\partial y} = \sin 2x \quad \frac{\partial N}{\partial x} = 2 \sin x \cos x \\ = \sin 2x$$

eqⁿ is exact.

SC

$$\int y \left(\frac{-\cos 2x}{2} \right) + \int (-1 - y^2) \, dy = c.$$

$$-y \frac{\cos 2x}{2} - y - \frac{y^3}{3} = c.$$

④ The eq $P \, dx + (1 + \sin^2 y + \cos^2 x) \, dy = 0$ is exact then

a) $P = \cos 2x$

b) $P = \sin 2x$

c) $P = -\cos 2x$

~~d)~~ p = -sin 2x

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$P = -\sin 2x$$

④ $P \, dx + (1 + \cos^2 x + \sin^2 y) \, dy = 0$ is exact.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial (P)}{\partial y} = 2 \cos x (-\sin x) \\ = -\sin 2x$$

$$P = \int -\sin 2x \, dy$$

$$P = -y \sin 2x$$

④ The DE

$$(3ax^2 z^2 + by \cos x) \, dx + (z \sin x - 4ay^3) \, dy = 0$$

is exact then

a) exactness depends on both a & b

b) " not

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$+ b \cos x = 2 \cos x$$

$b = +2$ not depends on 'a'
only depends on b .

Q) $y dx - x dy = 0$

$$\frac{\partial M}{\partial y} = 1 \quad \frac{\partial N}{\partial x} = -1$$

not exact.

to make it exact; multiply with $1/y^2$

$$\frac{1}{y^2} dx - \frac{x}{y^2} dy = 0$$

$$\frac{\partial M}{\partial y} = -\frac{1}{y^2} = \frac{\partial N}{\partial x} \text{ (exact)}$$

$\therefore 1/y^2$ is called integrating factor.

⇒ Integrating factor:- A non-exact eqn is converted to exact by multiplying a fn $f(x, y)$ then $f(x, y)$ is called integrating factor.

The integrating factors of $y dx - x dy = 0$ are

$$\frac{1}{y^2}, \frac{1}{x^2}, \frac{1}{xy}, \frac{1}{x^2+y^2}$$

$$M dx + N dy = 0$$

non-exact eq

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

I

II

IV

All the terms

eqn is of the

of M & N should

form

be of same

$$\textcircled{a} y f(xy) dx +$$

degree, then

$$x g(xy) dy = 0$$

$$\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = f(x) \quad \text{only}$$

$$(f(x)) dx$$

IV

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = g(y) \text{ only } \Leftrightarrow \text{constant}$$

m

$$\int g(y) dy.$$

$$IF = e^{\int g(y) dy}$$

V Inspection method:

Q. $y(xy - 2) dx + x(x^2y^2 + 2xy + 1) dy = 0 \quad \dots \text{---(1)}$

$$(xy^2 - 2y) dx + (x^3y^2 + 2x^2y + x) dy = 0$$

$$\frac{\partial M}{\partial y} = 2xy - 2$$

$$\frac{\partial N}{\partial x} = 3x^2y^2 + 4xy + 1$$

from (1) it is in form $yf(xy) dx + xg(xy) dy = 0$

it is so $IF = \frac{1}{Mx - Ny}$

form (1)

$$f(xy) = x^2y^2, xy \dots$$

$$g(x, y) = x^2y^2 + 2x \dots$$

parted

n

$$IF = \frac{1}{x^2y^2 - 2xy - (x^3y^3 + 2x^2y^2 + xy)}$$

$$= \frac{-1}{x^3y^3 + 3x^2y^2 + 3xy}$$

Q. The non-exact homogeneous DE $Mdx + Ndy = 0$
is converted to exact by multiplying with
which of following fns.

Ans: $\frac{1}{Mx + Ny}$.

given homogeneous so
degree of M & N will
be same so use II.

Q. $x^2y dx - (x^3 + y^3) dy = 0$

$f(x)$
only

$$\frac{\partial M}{\partial y} = x^2 \quad \frac{\partial N}{\partial x} = -3x^2$$

all the M & N have same degree

$$\frac{x^3y}{-y^4} dx - \frac{(x^3+y^3)}{-y^4} dy = 0 \rightarrow \text{exact.}$$

$\int M dx + \text{terms no } x$

$$\frac{-x^3}{3y^3} + \log y = c \text{ is the soln.}$$

$$(*) x(x-2y) dy + (x^2+y^2+1) dx = 0$$

$$\frac{\partial M}{\partial y} = 2y \quad \frac{\partial N}{\partial x} = 2x - 2y$$

$$\frac{\partial M}{\partial y} + \frac{\partial N}{\partial x} = \frac{2y - 2x + 2y}{x(x-2y)}$$

$$= \frac{-2(x-2y)}{x(x-2y)} = -2$$

$$= -\frac{2}{x} f(x)$$

$$\text{IF } e^{-\int \frac{2}{x} dx} = e^{-2 \log x} = \frac{1}{x^2}$$

$$\frac{x(x-2y)}{x^2} dy + \frac{(x^2+y^2+1)}{x^2} dx = 0$$

$$\text{soln: } x - \frac{y^2 - 1}{x} + \int f(x) dy$$

$$x - \left(\frac{1+y^2}{x} \right) + y = c$$

$$\int f(x) dx$$

$$(*) f(x)$$

$$c$$

$$\frac{1}{x}$$

$$x$$

$$\frac{2}{x}$$

$$x^2$$

$$\frac{3}{x}$$

$$x^3$$

$$\frac{-1}{x}$$

$$x^{-1} = \frac{1}{x}$$

$$\frac{-2}{x}$$

$$x^2$$

$$\textcircled{2} \quad y(1-xy)dx - x(1+xy)dy = 0$$

$$\frac{\partial M}{\partial y} = 1-2xy \quad \frac{\partial N}{\partial x} = -1-2xy.$$

case (ii)

$$Mx-ny = xy - x^2y^2 + xy + x^2y^2 \\ = 2xy.$$

$$IF = \frac{1}{2xy}.$$

$$\frac{y(1-xy)}{2xy} - \frac{x(1+xy)}{2xy} = 0.$$

(don't cancel
variables — cancel
constants ∞)

$$\frac{1-xy}{x} - \frac{1+xy}{y} = 0$$

$$\text{soln is } \log x - xy - \int \frac{1}{y} dy$$

$$\log x - xy - \log y = c.$$

- * By multiplying ^{with} which of the following functions the eq $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$ is converted to exact

$$\frac{\partial M}{\partial y} = Ay^3 + 2 \quad \frac{\partial N}{\partial x} = y^3 - 4$$

case (4)

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{y^3 - 4 - Ay^3 - 2}{y^4 + 2y} \\ = -3(Ay^3 + 2)$$

$$IF = \frac{1}{y^3} \quad \text{or} \quad e^{\int -3/y} = e^{-3 \log y} = \frac{1}{y^3}$$

* The IF of

$$y(x^2y^2 + 2y + 1)dx + 2(x^2y^2 - 2y + 1)dy = 0$$

case ② since $x f(xy) + y g(xy) = 0$.

$$\begin{aligned} Mx - Ny &= x^3y^3 + x^2y^2 + xy - x^3y^3 + x^2y^2 - xy \\ &= 2x^2y^2 \end{aligned}$$

$$IF = \frac{1}{2x^2y^2} \quad \text{or} \quad \frac{1}{x^2y^2} \quad (\text{const does not make difference})$$

$$(*) ydx - xdy + (1+x^2)dx + x^2\sin y dy = 0.$$

$$(xd^x + y d^y) dx + (dx + x^2 \sin y) dy = 0.$$

when there is $ydx - xdy$ is a part of DE

then simply multiply with $\frac{1}{x^2} \cdot \frac{1}{y^2} \cdot \frac{1}{xy} \cdot \frac{1}{x^2+y^2}$
& then integrate

By multiplying with $\frac{1}{x^2}$ we can easily integrate
it is not possible with other terms

Depending on the DE multiply with IF

Note:

$$\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$$

$$\int y e^y = e^y (y-1)$$

$$\frac{ydx - xdy}{x^2} = -d\left(\frac{x}{y}\right)$$

$$\int y e^{-y} = -e^{-y} (y+1)$$

$$\frac{ydx - xdy}{xy} = d\left[\log\left(\frac{x}{y}\right)\right].$$

$$\frac{ydx - xdy}{x^2+y^2} = d\left[\tan^{-1}\left(\frac{x}{y}\right)\right].$$

sol multiply with $\frac{1}{x^2}$

$$\frac{ydx - xdy}{x^2} + \left(\frac{1+x^2}{x^2}\right)dx + \frac{x^2 \sin y dy}{x^2} = 0 \quad (51)$$

$$= -\frac{x}{y} - \frac{1}{x} + x + (-\cos y) = c$$

~~-xy~~

$$\textcircled{(2)} \quad y \frac{dx}{dx} - x \frac{dy}{dx} + y(1+x^2) dx + xy^2 e^y dy = 0$$

multiply with $\frac{1}{xy}$.

~~snat
fence~~

$$y \frac{dx - x dy}{xy} + \left(\frac{1+x^2}{x}\right) dx + \frac{ye^y}{x} dy = 0$$

$$\int d\left[\log\left(\frac{x}{y}\right)\right] + \int \left(\frac{1}{x} + x\right) dx + \int ye^y dy = 0$$

$$\log\left(\frac{x}{y}\right) + \log x + \frac{x^2}{2} + e^y(y-1) = c.$$

~~2~~

$$\textcircled{(3)} \quad y dx + x dy = d(xy)$$

$$\frac{y dx + x dy}{xy} = d \left[\log(xy) \right]$$

~~To~~

$$\textcircled{(4)} \quad y \frac{dx + x dy}{xy} + \int ye^{-y} dy = 0$$

multiply with $\frac{1}{xy}$.

$$\int \frac{y dx + x dy}{xy} + \int ye^{-y} dy = 0$$

$$\log(xy) - e^{-y}(y+1) = c$$

\Rightarrow Linear Differential Equations

A DE is said to be linear if the dependent variable & derivatives should be of 1st degree only & there should be no product of them.

(should) $(y), \left(\frac{dy}{dx}\right)^1, \left(\frac{d^2y}{dx^2}\right)^1 \dots \dots$ degree = 1.

(should not) $y \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^4y}{dx^4}$ (there should not be multiplication)

$$\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right)^2 + e^y = 0$$

is not a linear eqn bcoz ($e^y = 1 + y + \frac{y^2}{2} + \frac{y^3}{3!}$)

multiplication
came

$$\frac{d^2y}{dx^2} + 4\left(\frac{dy}{dx}\right)^2 + 4y = 0 \quad \text{is } \cancel{\text{non-linear eq.}}$$

$\frac{dy}{dx}, \frac{dy}{dx}$ multiplication
came.

soln

\Rightarrow Linear in y :-

$$\frac{dy}{dx} + Py = Q$$

P & Q are fns of x

@ constants

soln.

$$y(e^{\int P dx}) = \int Q e^{\int P dx} dx + C$$

IF

\Rightarrow Linear in x :-

$$\frac{dx}{dy} + Px = Q$$

P & Q are fns
of y @ constants

soln is.

$$x(e^{\int P dy}) = \int Q e^{\int P dy} dy + C$$

IF

soln

) $x \cos x \frac{dy}{dx} + y(x \sin x + \cos x) = 1$

$$\frac{dy}{dx} + y \tan x + \frac{y}{x} = \frac{1}{x \cos x}$$

2

$$\frac{dy}{dx} + y\left(\tan x + \frac{1}{x}\right) = \frac{1}{x \cos x}$$

$\frac{dy}{dx} + y \tan x = \frac{1}{x \cos x}$

P

$\frac{dy}{dx} + y \frac{1}{x} = \frac{1}{x \cos x}$

Q

$\int (\tan x + \frac{1}{x}) dx$

$\log \sec x + \log x$

IF = $e^{\int P dx} = e^{\int \tan x dx} = \sec x$

soln

solution
came

$$(y^2 + y^3)$$

soln : $y(x \sec x) = \int \frac{1}{x \cos x} x \sec x dx + C$
 $= \int \sec^2 x dx$

or eq.

④ $(x + 2y^3) \frac{dy}{dx} = y$ with $x(1) = 0$.

Linear in x .

of x

$$x + 2y^3 = y \frac{dx}{dy}$$

$$\frac{dx}{dy} = \frac{x + 2y^3}{y}$$

$$\frac{dx}{dy} - \frac{x}{y} = 2y^2 \quad P = \frac{-1}{y}$$

IF $e^{-\int \frac{1}{y} dy} = y$

rhs
stands

soln : $x \cdot \frac{1}{y} = \int 2y^2 \cdot \frac{1}{y} dy + C$

$$\frac{x}{y} = y^2 + C \quad \text{at } y=1, x=0 \\ \frac{0}{1} = 1 + C \quad C = -1$$

$$x/y = y^2 - 1$$

⑤ $x^2 \frac{dy}{dx} + 2xy = \frac{2 \log x}{x} ; y(1)=0 \text{ find } y$

when $x = e$.

Linear in y .

$$\frac{dy}{dx} + \frac{2y}{x} = \frac{2 \log x}{x^3}$$

$$\frac{f^2}{x} \quad \cancel{\frac{2}{x}} x^2$$

IF $= e^{\int \frac{2}{x} dx} = e^{2 \log x}$

soln

$$4x^2 = \int \frac{2 \log x}{x^3} x^4$$

$$\int [f(x)]^n \cdot f'(x) dx = f(x)^{n+1}$$

$$\text{at } x=1 ; y=0 \Rightarrow c=0$$

$$x^2y + (\log x)^2 + 0$$

$$\text{at } x=e \quad e^2y = (\log e)^2$$

$$e^2y = 1 \Rightarrow y = e^{-2}$$

④ $x \frac{dy}{dx} + y = x^4 ; y(1) = \frac{6}{5}$

$$\frac{dy}{dx} + \frac{y}{x} = x^3$$

$$IF = e^{\int \frac{1}{x} dx} = \frac{1}{x} \cdot x$$

Soln. $\frac{-y}{x} = \int x^3 \cdot \frac{-1}{x} dx + C$

$$\frac{-y}{x} = -\frac{x^5}{5} + C$$

$$\text{at } x=1 \quad y=6/5 \Rightarrow C=1$$

$$xy = \frac{x^5}{5} + 1$$

$$y = \frac{x^4}{5} + \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} + Py = Qy^n \rightarrow \text{is not a linear eq}$$

↓ Sub $y^{1-n} = v$ reduces Bernoulli eq to linear eq.

$$y^{1-n} \int (1-n)P dx = \int (1-n)Q \cdot v^{\frac{n-1}{n}} dx + C$$

(if y^n is there is q^n multiply $(1-n)$ to every term in soln.)

⑤ The sub $y^{1-n} = v$ reduces the non-linear eq

$$\frac{dy}{dx} + Py = Qy^n \text{ to which of following linear}$$

$$\frac{1}{y^n} \frac{dy}{dx} + P \frac{y}{y^n} = Q.$$

$$y^{1-n} = v \Rightarrow (-n)y^{1-n-1} \frac{dy}{dx} = \frac{dv}{dx}$$

$$\Rightarrow \frac{1}{y^n} \frac{dy}{dx} = \frac{1}{1-n} \frac{dv}{dx}.$$

$$\frac{1}{1-n} \frac{dv}{dx} + Pv = Q$$

$$\boxed{\frac{dv}{dx} + Pv(1-n) = Q(1-n)}.$$

- ② which of the following sub reduces the non-linear eqⁿ to linear form

$$y \frac{dy}{dx} + x^2 y^3 = x^3 y^3.$$

$$\frac{dy}{dx} + x^2 y^2 = x^3 y^2$$

$$y^{1-n} = v \Rightarrow y^{1-2} = v \Rightarrow y^{-1} = v$$

$$③ \frac{dy}{dx} - y \tan x = -y^2 \sec x$$

$$e^{\int (1-z)(-\tan x) dx} = e^{\log \sec x} = \sec x$$

$$y^{1-2} \sec x = \int (1-z)(-\sec x) \sec x dx$$

$$\frac{\sec x}{y} = \tan x + C$$

* eq

\Rightarrow Equation of the form

$$\boxed{f'(y) \frac{dy}{dx} + P f(y) = Q}$$

$$f(y) = v \Rightarrow f'(y) \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dv}{dx} + Pv = q$$

④ The sub $(\log x)^{-1} = v$ reduces the non-linear

$$\text{eq } \frac{dz}{dx} + \frac{z \log z}{x^2} = \frac{z (\log z)^2}{x^2}$$

$$\frac{1}{z (\log z)^2} \frac{dz}{dx} + \frac{(\log z)^{-1}}{x} = \frac{1}{x^2}$$

$$(\log z)^{-1} = v \Rightarrow -(\log z)^{-2} \frac{dz}{dx} = \frac{dv}{dx}$$

$$\frac{1}{z (\log z)^2} \frac{dz}{dx} = -\frac{dv}{dx}$$

$$-\frac{dv}{dx} + \frac{v}{x} = \frac{1}{x^2}$$

$$\frac{dv}{dx} - \frac{v}{x^2} = \frac{1}{x^2}$$

$$x \frac{dy}{dx} + y \log y = xy e^x$$

$$\frac{1}{y} \frac{dy}{dx} + \frac{\log y}{x} = e^x$$

$$\log y = v \Rightarrow \frac{1}{y} \frac{dy}{dx} = \frac{dv}{dx}$$

$$\frac{dv}{dx} + \frac{v}{x} = e^x$$

$$\text{If } v = e^{\int \frac{1}{x} dx} = x$$

$$\log x = \int e^x \cdot x \, dx$$

$$x \log y = e^x(x-1) + c.$$

\Rightarrow Clairaut's eqn :-

An eq of the form

-linear

$$y = x \frac{dy}{dx} + f\left(\frac{dy}{dx}\right).$$

$$\text{or } y = px + f(p)$$

$$\text{replace } \frac{dy}{dx} \rightarrow c.$$

$y = cx + f(c)$ is the fn of soln.

$$\textcircled{a} \quad p = \sin(y - xp)$$

$$y - xp = \sin^{-1} p$$

$$y = xp + \sin^{-1} p$$

$$y = cx + \sin^{-1} c q \quad (c \text{ is a const})$$

$$\textcircled{b} \quad (x-a)y'^2 + (x-y)y' - y = 0$$

$$xy'^2 - ay'^2 + xy' - yy' - y = 0$$

$$xy'(y'+1) - ay'^2 = y(y'+1)$$

$$xy' - \frac{ay'^2}{y'+1} = y$$

$$y = cx - \frac{ac^2}{1+c^2}$$

$$\textcircled{c} \quad \left(y - x \frac{dy}{dx}\right) \left(\frac{dy}{dx} - 1\right) = \frac{dy}{dx}$$

$$(y - xy') (y' - 1) = y'$$

⇒ Higher order linear eqn with constant coefficients

$$\frac{d^n y}{dx^n} + k_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_{n-1} \frac{dy}{dx} + k_n y = x$$

k_1, k_2, \dots, k_{n-1} are constants

x is fn of x .

$$D = \frac{d}{dx} \quad \text{--- differential operator}$$

$$\frac{1}{D} = \int \quad \text{--- inverse differential operator.}$$

$$(D^n + k_1 D^{n-1} + \dots + k_{n-1} D + k_n) y = x$$

$$F(D) y = x$$

$$F(D) = D^n + k_1 D^{n-1} + \dots + k_{n-1} D + k_n = 0$$

is called auxillary eqn.

④ The complete solution of $F(D)y = x$ is

$y = \text{complementary fn} + \text{particular integral}$
(CF) (PI)

⑤ If $x = 0$ then $F(D)y = 0$ is called the homogenous linear DE

⑥ If $x \neq 0$ then $F(D)y = x$ is called non-homogeneous DE

⑦ The soln of homogenous linear DE

i.e., $(F(D)y = 0)$ is called CF.

The no. of arbitrary constants in the CF should

be equal to the higher order of given DE.

fficients

P.I. will not contain any arbitrary constants.

- ④ If $x=0$ the complete soln. of $F(D)y = x$ is
is only CF.

⇒ Procedure to find the CF :-

By assuming 'D' as an algebraic quantity.

$F(D) = 0$ becomes an algebraic eqn & by solving it we get roots for these. Based on the nature of these roots we write CF as follows.

Nature of roots	CF
1. Real & Distinct $D = m_1, m_2, m_3$	$CF = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$
2. Real & Repeated $D = m_1, m_1, m_3$	$CF = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x}$
3. complex & distinct $D = a+ib, m_3$	$e^{ax} [c_1 \cos bx + c_2 \sin bx] + c_3 e^{m_3 x}$
4. complex & repeated $D = a+ib, a+ib, m_5$	$e^{ax} [(c_1 + c_2 x) \cos bx + (c_3 + c_4 x) \sin bx] + c_5 e^{m_5 x}$

Eg:
roots.

$$D = 1, -1/2, 1/2$$

$$CF = c_1 e^x + c_2 e^{-1/2x} + c_3 e^{1/2x}$$

$$D = -2, -2, 1$$

$$CF = (c_1 + c_2 x) e^{-2x} + c_3 e^x$$

should

$$D = 3 \pm 4i, -1 \quad CF = e^{3x} [c_1 \cos 4x + c_2 \sin 4x] + c_3 e^{-x}$$

DE

$$D = \pm 2i, \pm 3$$

$$CF = [c_1 + c_2 \cos 2x + c_3 \sin 2x]$$

G

$$\textcircled{1} \quad \frac{d^2y}{dx^2} - 5 \frac{dy}{dx} + 6y = 0$$

$$D^2 - 5D + 6 = 0 \quad D = 3, 2$$

$$CF = C_1 e^{3x} + C_2 e^{2x}$$

\textcircled{2}

$$\frac{d^4y}{dx^4} - 81y = 0$$

$$D^4 - 81 = 0$$

$$(D^2 - 9)(D^2 + 9) = 0$$

$$D = \pm 3, \pm i3$$

in option it ~~will~~ maybe

$$e^{-3x} + \sin 3x$$

choosing different values

& $C_1, C_2, C_3, C_4, \dots$

$$y = C_1 e^{3x} + C_2 e^{-3x} + (C_3 \cos 3x + C_4 \sin 3x)$$

\textcircled{3}

$$y'' + 2py' + (P^2 + q^2)y = 0$$

$$\therefore D^2 + 2PD + (P^2 + q^2) = 0$$

$$D = \frac{-2P \pm \sqrt{4P^2 - 4(P^2 + q^2)}}{2}$$

$$= -P \pm \sqrt{P^2 - P^2 - q^2}$$

$$= -P \pm iq$$

$$y = e^{-xp}(C_1 \cos qx + C_2 \sin qx)$$

\textcircled{4}

$$y''' - 4y'' + 5y' - 2y = 0$$

$$D^3 - 4D^2 + 5D - 2 = 0$$

$D = 1$ is root

	1	-4	5	-2
$D = 1$	is root			
	1	-3	2	
	0	1	-3	2

$$D^2 - 3D + 2 = 0$$

$$\begin{array}{c} +2 \\ \diagup \\ -2 \quad -1 \end{array}$$

$$D = +1, +2$$

$$D^2 - 2D - 0 \rightarrow 2 \cdot 2 \cdot 2$$

roots are 1, +1, +2.

$$\begin{aligned} D(D-2) &= 1(D-2) = 0 \\ (D-1)(D-2) &= 0 \end{aligned}$$

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{+2x}$$

$$y = (c_1 + c_2 x) e^x + c_3 e^{2x}$$

Note - If $y = c_1 y_1 + c_2 y_2 + c_3 y_3 + \dots$ is a soln.

maybe
values

of $F(D)y = 0$ then each one y_1, y_2, y_3, \dots
is linearly independent soln of $F(D)y = 0$.

④ $\frac{d^2 f}{dt^2} - 4 \frac{df}{dt} + 4f = 0$

3x)

a) $f_1 = e^{2t}, f_2 = e^{-2t}, f_3 = e^{2t}, f_4 = t e^{2t}$

c) $f_1 = e^{-2t}, f_2 = t e^{-2t}, d) f_1 = e^{-2t}, f_2 = t e^{-2t}$

$$D^2 - 4D + 4 = 0$$

$$(D-2)^2 = 0$$

$$D = 2, 2$$

$$y = (c_1 + c_2 t) e^{2t}$$

$$c_1 e^{2t} + c_2 t e^{2t}$$

④ The DE $(D^2 + 4\cos z D + 3)y = x^2$ is b.

a) homogenous DE

b) non-homogeneous DE
with const. coeff.

~~c) linear DE~~

d) Non-linear DE

④ y_1, y_2 are linearly independent solns of

$F(D)y = 0$ then which of the following is
the soln of the same eqn.

④ y_1, y_2 are linearly independent soln of the corresponding homogenous eqⁿ of $F(D)y = x$ then $y_1 + y_2$ is the soln of which of following eqⁿ

- a) $F(D)y = x$
- b) $F(D)y = 0$
- c) $x = 0$
- ~~d) $F(D)y = 0$~~

⑤ e^{2x}, e^{-3x} are the so linearly independent solns

$$D = 2, -3$$

$$(D-2)(D+3)y = 0$$

$$(D^2 + D - 6)y = 0$$

$$y'' + y' - 6y = 0$$

⑥ $e^{2x} + x \cdot e^{2x}$

$$= (C_1 + C_2 x)e^{2x}$$

$$D = 2, 2$$

$$(D-2)^2 y = 0$$

$$(D^2 - 4D + 4)y = 0 \Rightarrow y'' - 4y' + 4y = 0$$

⑦ e^{2x}, e^{-2x} are soln of which

$$D = 2, -2$$

(see options sub $D=2, -2$ which satisfies the eq). It may be

$$(D^2 - 4)y = 0 \Rightarrow y'' - 4y = 0 \quad \frac{d^4y}{dx^4} - 16y = 0$$

⑧ $\sin 3x, \cos 3x$ are LI soln of which of the

following DE

$$y = C_1 \cos 3x + C_2 \sin 3x$$

the

$$\textcircled{4} \quad y'' + 4y' + 13y = 0 ; \quad y(0) = 0 \Rightarrow y'(0) = 1.$$

then

$$(D^2 + 4D + 13) = 0.$$

eqn

$$D = \frac{-4 \pm \sqrt{16 - 52}}{2}$$

$$= \frac{-4 \pm i\sqrt{36}}{2} = -2 \pm 3i.$$

sols

$$y = e^{-2x} (C_1 \cos 3x + C_2 \sin 3x).$$

$$\text{at } x=0 ; y=0.$$

$$0 = C_1$$

$$y = e^{-2x} C_2 \sin 3x.$$

$$\therefore y' = C_2 \left[e^{-2x} (3 \cos 3x) + (-2)e^{-2x} \sin 3x \right]$$

$$\text{at } x=0 \quad y' = 1$$

$$1 = C_2 [3] \Rightarrow C_2 = \frac{1}{3}.$$

$$y = \frac{e^{-2x}}{3} \sin 3x.$$

\textcircled{5}: which of the following is not a soln of

$$\frac{d^2i}{dt^2} + \frac{R}{L} \frac{di}{dt} + \frac{i}{LC} = 0.$$

where: $R^2C = 4L$. R, L, C be the constants.

which

a) $i = e^{-Rt/2L}$

b) $i = t e^{-Rt/2L}$

sy. = 0

~~c) $i = t e^{Rt/2L}$~~

d) $i = e^{-Rt/2L} + t e^{-Rt/2L}$

$$D^2 + \frac{R}{L} D + \frac{1}{LC} = 0$$

the

$$D = \frac{-R/L}{2} \pm \frac{\sqrt{R^2/4 - 1/LC}}{2}$$

$$D = \frac{-R}{2L}, \frac{-R}{2L}$$

$$y = (c_1 + c_2 t) e^{-R/2L}$$

$$= e^{-R/2L} + t e^{-R/2L}$$

④ $\frac{d^2y}{dx^2} + \omega^2 y = 0; \quad y(0) = 0, \quad y(L) = 0$

a) $y = \sum_n c_n \cos \frac{n\pi x}{L}$

b) $y = \sum_n c_n e^{\frac{n\pi x}{L}}$

~~c)~~ $y = \sum_n c_n \sin \frac{n\pi x}{L}$

d) $y = \sum_n c_n x^{\frac{n\pi}{L}}$

Sol sub $y = 0$ at $x=0$ & $x=L \Rightarrow y=0$

④ $ay'' - 6y' + y = 0, \quad y(0) = 3, \quad y'(0) = 1$

$$(9D^2 - 6D + 1) y = 0$$

$$D = \frac{1}{3} \pm \frac{\sqrt{81 - 36}}{18}$$

$$D = \frac{1}{3}, \frac{4}{3}$$

$$y = (c_1 + c_2 x) e^{x/3}$$

$$y(0) = 3 \Rightarrow 3 = c_1$$

$$y'(0) = 1 \Rightarrow 1 = c_2 e^{x/3} + (c_1 + c_2 x) e^{x/3} \cdot \frac{1}{3}$$

$$\Rightarrow 1 = c_2 + 3c_1$$

$$\Rightarrow c_2 = -8$$

$$y = (3 - 8x) e^{x/3}$$

④ For what value of λ the DE $(D^2 + \lambda)y = 0$ will have non-trivial soln.
 $y(0) = 0 \Rightarrow y(x) = 0$ will have non-trivial soln.

$$y = (c_1 \cos \sqrt{\lambda}x + c_2 \sin \sqrt{\lambda}x)$$

$$y(0) = 0 \Rightarrow 0 = c_1$$

$$y(\pi) = 0 \Rightarrow 0 = c_2 \sin \pi \sqrt{\lambda}$$

$$\sin \pi \sqrt{\lambda} = 0$$

$$\pi \sqrt{\lambda} = n\pi$$

~~$$\lambda = \frac{n^2}{4}, \frac{9}{4}, \frac{25}{4}, \dots$$~~

$$\lambda = 1, 4, 9, 16$$

(*) The complete soln of DE $\frac{d^2y}{dx^2} + p \frac{dy}{dx} + qy = 0$

(a)

$$\text{is } y = c_1 e^{-x} + c_2 e^{-3x}$$

P, q are?

$$D = -1, -3$$

$$(D+1)(D+3)y = 0$$

$$(D^2 + 4D + 3)y = 0$$

$$\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 3y = 0 \quad P=4, Q=3.$$

(b) The soln of $y'' + Py' + (Q+1)y = 0$ is

$$y'' + 4y' + 4y = 0$$

$$(D+2)^2 y = 0$$

$$D = -2, -2 \Rightarrow y = (c_1 + c_2 x)e^{-2x}$$

(*) $\frac{d^2n}{dx^2} + \frac{-n}{L^2} = 0; n(0)=k, n(\infty)=0$

$$(D^2 - \frac{1}{L^2})n = 0$$

$$D = \pm \frac{1}{L}$$

$$y = \text{some}$$

0

$$y = c_1 e^{x/L} + c_2 e^{-x/L}$$

sin

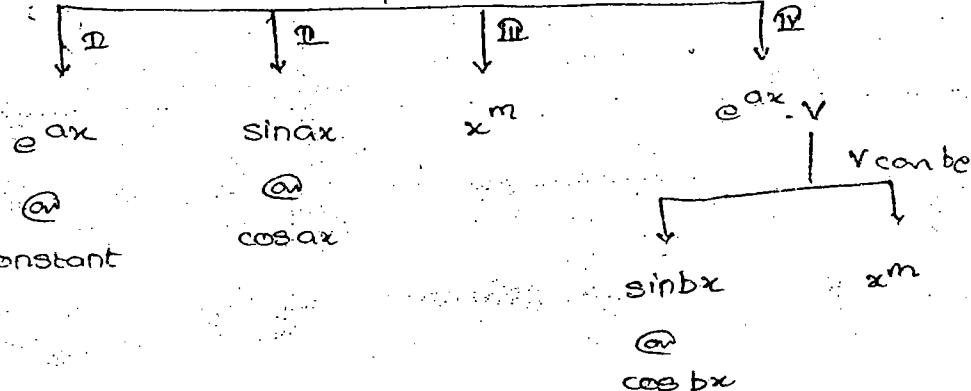
sub. conditions.

* Particular integral :-

$$F(D)y = x$$

$$P.I. = \left[\frac{1}{F(D)} \right] x$$

x may be



case (I) : $x = e^{ax}$ @.. constant

$$P.I. = \left[\frac{1}{F(D)} \right] e^{ax}$$

replace 'D' by 'a' in $F(D)$

$$\boxed{P.I. = \left[\frac{1}{F(a)} \right] e^{ax}} \quad (F(a) \neq 0)$$

if $F(a) = 0$:-

$$P.I. = x \left[\frac{1}{F'(D)} \right] e^{ax}$$

replace $D \rightarrow a$ in $F'(D)$

$$\boxed{P.I. = x \left[\frac{1}{F'(a)} \right] e^{ax}}$$

if $F'(a) = 0$

$$P.I. = x^2 \left[\frac{1}{F''(a)} \right] e^{ax}$$

④ Find PI of $y'' + 3y' - 2y = e^{2x} + 3$

$$PI = \left[\frac{1}{D^2 + 3D - 2} \right] (e^{2x} + 3)$$

$$3 = 3e^{0x}$$

$$= \frac{1}{D^2 + 3D - 2} e^{2x} + \frac{3}{D^2 + 3D - 2} e^{0x}$$

$$= \frac{e^{2x}}{8} - \frac{3}{2}$$

be

⑤ PI of $y'' + 5y' + 6y = e^{-3x}$

$$PI = \left[\frac{1}{D^2 + 5D + 6} \right] e^{-3x} \quad (\text{Dr is } 0)$$

so differentiate

$$= x \cdot \left(\frac{1}{2D+5} \right) e^{-3x}$$

$$= -xe^{-3x}$$

⑥ PI of $y'' + 4y' + 4y = e^{2x}$

$$PI = \left[\frac{1}{D^2 + 4D + 4} \right] e^{-2x} \quad (\text{Dr is } 0)$$

$$= \left(\frac{x}{2D+4} \right) e^{-2x} \quad (\text{Dr is } 0)$$

$$= \frac{x^2}{2} e^{-2x}$$

case(2) :- $x = \sin ax @ \cos ax$

$@ \sin(ax+b)$ $@ \cos(ax+b)$

$$PI = \left[\frac{1}{F(D)} \right] \sin ax$$

replace D^2 by $-a^2$ in $F(D)$.

$$\frac{1}{\sin ax}$$

if $F(-a^2) = 0$

$$\text{then } P_1 = x \left[\frac{1}{F'(D)} \right] \sin ax$$

$$P_1 = x \frac{1}{F'(-a^2)} \sin ax$$

④ $\frac{d^2y}{dx^2} + 5y = \cos(3x+2)$

$$P_1 = \left[\frac{1}{D^2 + 5} \right] \cos(3x+2)$$

$$= \frac{1}{-9+5} \cos(3x+2)$$

$$\Rightarrow -\frac{\cos(3x+2)}{4}$$

⑤ $y'' + 4y = \sin 2x$

$$P_1 = \left[\frac{1}{D^2 + 4} \right] \sin 2x \quad (\text{or } = 0)$$

$$= \frac{x}{2D} \sin 2x = \frac{x}{2} \cdot \left(\frac{1}{D} \sin 2x \right)$$

$$= \frac{x}{2} \left(-\frac{\cos 2x}{2} \right)$$

don't do integ. on x
only do on $\sin 2x$.

$$= -\frac{x \cos 2x}{4}$$

⑥ $(D^3 + D^2 + 2D + 2)y = \cos x$

$$P_1 = \left[\frac{1}{D^3 + D^2 + 2D + 2} \right] \cos x$$

replace $D^2 = -D^2 = -1$

$$= \frac{1}{-D - 1 + 2D + 2} \cos x \quad (D = D \cdot D^2)$$

$$\frac{D-1}{(D^2-1)} \cos x = \frac{(D-1)}{-2} \cos x$$

$$= \frac{1}{2} (\cos x + D(\cos x))$$

$$= \frac{1}{2} [\cos x - \sin x]$$

case : (3) :- $x = x^m$ @ poly. in x (m is +ve).

$$\begin{aligned} P_1 &= \left[\frac{1}{F(D)} \right] x^m \\ &= \frac{1}{*} x^m \\ &\quad * (1 + \phi(D)) \\ &= \frac{1}{*} [1 + \phi(D)]^{-1} x^m. \end{aligned}$$

(* is the least power in $F(D)$)

Q) $\frac{d^2y}{dx^2} + 2y = x^3$

$$P_1 = \left[\frac{1}{D^2 + 2} \right] x^3.$$

$$= \frac{1}{2} \left[\frac{1}{1 + 0.5D^2} \right] x^3$$

$$= \frac{1}{2} \left[1 + \frac{D^2}{2} \right]^{-1} x^3.$$

$$= \frac{1}{2} \left[1 - \frac{D^2}{2} + \frac{D^4}{4} \dots \right] x^3.$$

$$= \frac{1}{2} \left[x^3 - 3x + 0 \dots \right] = x^3 - \frac{3x}{2}.$$

Q) $D^2 + D = x^2 + 2$

$$P_1 = \left[\frac{1}{D^2 + D} \right] (x^2 + 2)$$

$$\begin{aligned}
 &= \frac{1}{D} \left(1 + \frac{D}{D+2} \right) \\
 &= \frac{1}{D} (1+D)^{-1} (x^2 + 2) \\
 &= \frac{1}{D} (1-D+D^2)(x^2 + 2) \\
 &= \frac{1}{D} \{ x^2 + 2 - 2x + 2 \} \\
 &= \frac{x^3}{3} + 4x - x^2
 \end{aligned}$$

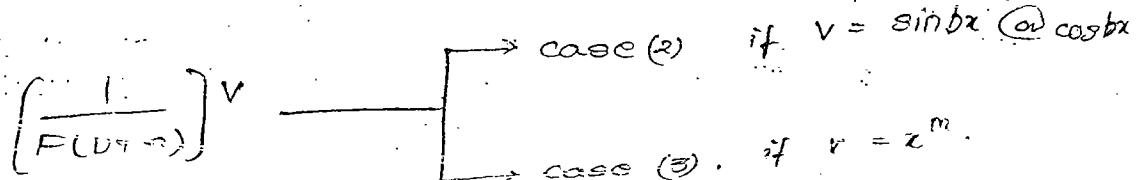
case (4) :- $x = e^{ax} \cdot v$

($v = \cos bx$ @ $\sin bx$) @ ($v = x^m$ @ poly in x)

$$P_1 = \left[\frac{1}{F(D)} \right] e^{ax} v.$$

replace $D \rightarrow 'D+a'$ in $F(D)$.

$$= e^{ax} \left[\frac{1}{F(D+a)} \right] v.$$



* $\frac{d^2y}{dx^2} + 3y = e^x \sin 2x.$

$$P_1 = \left[\frac{1}{D^2 + 3} \right] e^x \sin 2x$$

$$= e^x \left[\frac{1}{(D+1)^2 + 3} \right] \sin 2x$$

$$= e^x \left[\frac{1}{D^2 + 2D + 4} \right] \sin 2x$$

$$= \frac{e^x}{2} \left(-\frac{\cos 2x}{2} \right)$$

$$= -\frac{e^x \cos 2x}{4}$$

④ $\frac{d^2y}{dx^2} - 4y = \cos^2 x$

$$F(D) = D^2 - 4 = 0 \quad D = \pm 2$$

$$CF = C_1 e^{2x} + C_2 e^{-2x}$$

$$PI = \frac{1}{D^2 - 4} \cos^2 x = \frac{1}{2} \left[\frac{1}{D^2 - 4} \right] (1 + \cos 2x)$$

$$= \frac{1}{2} \left[\frac{1}{-4} \right] + \frac{1}{2} \left[\frac{1}{-4} \right] \cos 2x$$

$$PI = \frac{-1}{8} - \frac{1}{16} \cos 2x$$

$$y = C_1 e^{2x} + C_2 e^{-2x} - \frac{1}{8} - \frac{1}{16} \cos 2x.$$

2. costbx
⑤ The PI of $\frac{d^2y}{dx^2} + a^2y = \sin ax$ is

$$PI = \frac{1}{D^2 + a^2} \sin ax = \frac{x}{2a} \sin ax$$

$$= \frac{-x}{2a} \cos ax.$$

⑥ The PI of $\frac{d^2y}{dx^2} + 4y = \sin 2x + \cos 3x$ is.

$$(Ax \cos 2x + B \sin 3x) \quad A, B = \frac{\text{cof}}$$

$$(PI)_1 = \frac{x}{20} \sin 2x.$$

$$(PI)_2 = \frac{1}{-5} \cos 3x$$

$$= \frac{-x}{4} \cos 2x$$

$$\textcircled{A} \quad (D-2)^3 y = x e^{2x}$$

$$D = 2, 2, 2$$

$$CF = (c_1 + c_2 x + c_3 x^2) e^{2x}$$

$$PI = \frac{1}{(D-2)^3} x e^{2x}$$

$$= e^{2x} \frac{1}{((D+2)-2)^3} x.$$

$$= e^{2x} \frac{1}{D^3} x.$$

$$= e^{2x} \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{x^4}{4}$$

$$y = (c_1 + c_2 x + c_3 x^2) e^{2x} + \frac{x^4 e^{2x}}{24}.$$

$$\textcircled{B} \quad y'' - 2y' + y = x - 1$$

$$(D^2 - 2D + 1) = 0$$

$$(D-1)^2 = 0 \quad D = 1, 1$$

$$CF = (c_1 + c_2 x) e^x$$

$$PI = \frac{1}{(D^2 - 2D + 1)} (x-1)$$

$$= (1 + D^2 - 2D)^{-1} (x-1)$$

$$= (1 + (D^2 - 2D)) (x-1)$$

$$PI = (x-1) + 2 = x+1$$

$$\textcircled{C} \quad \frac{dy}{dx^2} = e^x \cos x$$

$$P_1 = \frac{1}{D^2} e^x \cos x.$$

$$= e^x \frac{1}{(D+1)^2} \cos x.$$

$$= e^x \left[\frac{1}{D^2 + 2D + 1} \right] \cos x = \frac{e^x}{x} \sin x.$$

④ $\frac{d^2y}{dx^2} + y = x$; at $x=0, y=1$
 $x=\pi/2, y=\pi/2$

$$D^2 + 1 = 0$$

$$D \pm i \pm i$$

$$CF = (c_1 \cos x + c_2 \sin x)$$

$$P_1 = \frac{1}{D^2 + 1} x$$

$$= (1 + D^2)^{-1} x$$

$$= (1 - D^2)^{-1} x \Rightarrow x = 0 = x.$$

$$y = c_1 \cos x + c_2 \sin x + x$$

$$x=0, y=1 \Rightarrow 1 = c_1$$

$$x=\pi/2, y=\pi/2 \Rightarrow \frac{\pi}{2} = c_2 + \pi/2 \Rightarrow c_2 = 0$$

$$y = \cos x + x/4$$

⑤ $y'' - 3y' + 2y = \cosh x$

$$= \frac{e^x + e^{-x}}{2}$$

$$D^2 - 3D + 2 = 0 \Rightarrow D^2 - 2D - D + 2 = 0$$

$$D = 1, 2$$

$$CF = c_1 e^x + c_2 e^{2x}$$

$$P_1 = \int \frac{1}{e^x} e^x dx + \int \frac{1}{e^{2x}} e^{-x} dx$$

$$\textcircled{*} \quad y'' - 3y' + 2y = e^x ; \quad y = 3 \quad \text{at } x=0 \quad \frac{dy}{dx} = 3 \quad \text{at } x=0$$

$$D^2 - 3D + 2 = 0$$

$$D = 1, 2$$

$$CF = C_1 e^x + C_2 e^{2x}$$

$$P.I. = \left[\frac{1}{D^2 - 3D + 2} \right] e^x$$

$$= -x e^x$$

$$y = C_1 e^x + C_2 e^{2x} - x e^x$$

$$y' = C_1 e^x + 2C_2 e^{2x} - e^x(x+1)$$

$$x=0, y=3 \Rightarrow 3 = C_1 + C_2$$

$$x=0, y'=3 \Rightarrow 3 = C_1 + 2C_2 - 1$$

$$C_2 - 1 = 0$$

$$C_2 = 1, \quad C_1 = 2$$

\Rightarrow Cauchy's homogenous D.Eqⁿ :-

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_{n-1} x \frac{dy}{dx} + k_n y = 0$$

k_1, k_2, \dots are constants.

$x = e^t$ x is function of t .

it reduces to linear DE with constant coef.

$$\frac{dy}{dx} = D_1 y = \frac{dy}{dt}$$

$$x^2 \frac{d^2 y}{dx^2} = D_1(D_1 - 1)y$$

$$x^3 \frac{d^3 y}{dx^3} = D_1(D_1 - 1)(D_1 - 2)y$$

$$(D_1^2 + D_1 + 1)y = 4 \cos t$$

$$\frac{dy}{dx}$$

$$P_1 = \left[\frac{1}{D_1^2 + D_1 + 1} \right] 4 \cos t$$

$$= 4 \left[\frac{1}{D_1} \cos t \right]$$

$$= 4 \sin t,$$

$$(D_1(D_1 - 1) + D_1 + 1)y = 4 \cos t.$$

$$(D_1^2 + 1)y = 4 \cos t.$$

$$P_1 = 4 \left(\frac{1}{D_1^2 + 1} \right) 4 \cos t.$$

$$= 2t \sin t$$

$$= 2 \log(1+x) \sin[\log(1+x)].$$

⇒ Method of variation of parameters :-

This is a method to find the P1

of

$$\boxed{\frac{d^2y}{dx^2} + p \frac{dy}{dx} + qy = R}$$

$$1. CF = C_1 y_1 + C_2 y_2$$

$$2. P_1 = A y_1 + B y_2$$

$$A = - \int \frac{R y_2}{\omega} dx, \quad B = \int \frac{R y_1}{\omega} dx$$

$$\omega = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \Rightarrow y_1 y_2' - y_2 y_1'$$

$$3. y = CF + P_1$$

Note :- If P & Q are functions of x then we should know two independent soln y_1, y_2 of the corresponding homogenous eqn

$y'' + Py' + Qy = 0$ so that we can write CF
(CF = $C_1 y_1 + C_2 y_2$)

$$\textcircled{1} \quad y'' - 6y' + 9y = \frac{e^{3x}}{x^2}$$

$$D^2 - 6D + 9 = 0$$

$$D = 3, 3$$

$$CF = C_1 e^{3x} + C_2 x e^{3x}$$

$$y_1 = e^{3x} \quad y_2 = x e^{3x}$$

$$W = \begin{vmatrix} e^{3x} & x e^{3x} \\ 3e^{3x} & e^{3x} + 3x e^{3x} \end{vmatrix}$$

$$= e^{6x} + 3x e^{6x} = 3x e^{6x}$$

$$W = e^{6x}$$

$$A = - \int \frac{e^{3x}}{x^2} \cdot x e^{3x} dx = - \int \frac{1}{x} = -\log x$$

$$B = + \int \frac{e^{3x}}{x^2} \cdot \frac{e^{3x}}{e^{6x}} dx = \frac{1}{x}$$

$$P_I = -\log x e^{3x} - \frac{1}{x} x e^{3x}$$

$$= -e^{3x} (\log x + 1)$$

The PI of $\frac{d^2y}{dx^2} + 4y = \sec^2 2x$ is $Ay_1 + By_2$

where $y_1 = \cos 2x$, $y_2 = \sin 2x$ find A & B

Elm

b :-

$$\textcircled{8} \quad x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} + 4y = 0$$

$$\text{Sub. } x = e^t \Rightarrow \log x = t$$

$$(D_1(D_1-1) - 3D_1 + 4)y = 0$$

$$(D_1^2 - 4D_1 + 4)y = 0$$

$$D_1 = 2, 2$$

$$y = (c_1 + c_2 t) e^{2t}$$

$$y = (c_1 + c_2 \log x) x^2$$

$$\textcircled{9} \quad \text{Find } P_1 \text{ of } x^2 y'' - xy' + 3y = x^2 \log x$$

$$\text{Sub. } x = e^t \Rightarrow \log x = t$$

$$(D_1(D_1-1) - D_1 + 3)y = e^{2t} \cdot t$$

$$(D_1^2 - 2D_1 + 3)y = t e^{2t}$$

$$D_1 = -1, 3$$

$$\begin{aligned} & -3 \\ & D^2 - 5D + 3 \\ & D(D-1) \end{aligned}$$

$$P_1 = \left[\frac{1}{D_1^2 - 2D_1 + 3} \right] t e^{2t}$$

$$= e^{2t} \left[\frac{1}{(D_1+2)^2 - 2(D_1+2) + 3} \right] t$$

$$= e^{2t} \left[\frac{1}{D_1^2 + 2D_1 + 3} \right] t$$

cof.

$$= \frac{e^{2t}}{3} \left[1 - \left(\frac{D_1^2}{3} + \frac{2D_1}{3} \right) \right] t$$

$$= \frac{e^{2t}}{3} \left[t - \frac{2}{3} \right].$$

$$\textcircled{10} \quad P_1 = \frac{x^2}{3} \left\{ \log x - \frac{2}{3} \right\}$$

Legendre's eqⁿ :-

$$\Rightarrow (a+bx)^n \frac{d^n y}{dx^n} + k_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + k_{n-1} (a+bx) \frac{dy}{dx} + k_n y = x.$$

$(a+bx) = e^t \downarrow$

Linear DE with const coef.

$$(a+bx) \frac{dy}{dx} = bD_1 y$$

$$t = \log(a+bx)$$

$$(a+bx)^2 \frac{d^2 y}{dx^2} = b^2 D_1 (D_1 - 1)y$$

$$(a+bx)^3 \frac{d^3 y}{dx^3} = b^3 D_1 (D_1 - 1)(D_1 - 2)y$$

$$D_1 = \frac{d}{dt}, \quad D_t^2 = \frac{d}{dt^2}$$

$$(1+2x)^2 y'' - 6(1+2x)y' + 16y = 0$$

$$4D_t(D_1 - 1) - 6(2)D_1 + 16 = 0$$

$$4D_1^2 - 8D_1 + 16 = 0 \quad (1+2x) = e^{2t}$$

$$D_1^2 - 2D_1 + 4 = 0$$

$$D_1 = 2, 2$$

$$CF = (C_1 + C_2 t) e^{2t}$$

$$\cancel{D_1} = \cancel{(1/2, 1/2, 7)}$$

$$CF = [C_1 + C_2 \log(1+2x)] (1+2x)^2$$

$$P.I. \text{ of } (1+x^2)^2 y'' + (1+x)y' + y = 4 \cos(\log(1+x))$$

$$(1+x) = e^t$$

D Res. of $f(z) = \frac{1}{(z^2+1)^2}$ @ $z=i$ is

$$f(z) = \frac{1}{(z+i)^2(z-i)^2} \rightarrow \underset{z \rightarrow i}{\text{Lt}} \frac{d}{dz} \left\{ (z-i)^2 \times \frac{1}{(z+i)^2(z-i)^2} \right\}$$

$$\underset{z \rightarrow i}{\text{Lt}} \frac{-2}{(z+i)^3} \Rightarrow \frac{-2}{(2i)^3} = \frac{-2}{8i^3} = \frac{1}{4i}$$

* f(z) = $\frac{z^2}{(z-1)^2(z+2)}$ calculate res. at each of the poles.

$$z=1, z=-2$$

$$\text{Res at } z=1 \rightarrow \underset{z \rightarrow 1}{\text{Lt}} \frac{d}{dz} \left\{ (z-1)^2 \frac{z^2}{(z-1)^2(z+2)} \right\}$$

$$\Rightarrow \underset{z \rightarrow 1}{\text{Lt}} \frac{d}{dz} \left[\frac{z^2}{z+2} \right] \Rightarrow \underset{z \rightarrow 1}{\text{Lt}} \left[\frac{2z}{z+2} - \frac{z^2}{(z+2)^2} \right]$$

$$= \frac{2}{3} - \frac{1}{9} = \frac{6-1}{9} = \frac{5}{9}$$

$$\text{Res at } z=-2 \rightarrow \underset{z \rightarrow -2}{\text{Lt}} \frac{(z+2)}{(z+2)^2(z+2)} \Rightarrow \frac{4}{9}$$

D Res. of $f(z) = \frac{1-2z}{z(z-1)(z-2)}$ at its poles

- a) $1/2, -1/2, 1$ b) $1/2, 1/2, -1$ ~~c) $1/2, 1, -3/2$~~ d) $1/2, -1, 3/2$

$$\text{At } z \rightarrow 0 \quad \underset{z \rightarrow 0}{\text{Lt}} \frac{1-2z}{(z-1)(z-2)} = \frac{1}{2}$$

$$\text{At } z \rightarrow 1 \quad \underset{z \rightarrow 1}{\text{Lt}} \frac{1-2z}{z(z-2)} = \frac{-1}{-1} = 1$$

$$\text{At } z \rightarrow 2 \quad \underset{z \rightarrow 2}{\text{Lt}} \frac{1-2z}{z(z-1)} = \frac{-3}{2}$$

$$\textcircled{R} \quad f(z) = \frac{1}{(z+2)^2(z-2)^2} \quad \text{at } z=2 \text{ (8)}$$

$$\lim_{z \rightarrow 2} \frac{d}{dz} \left\{ (z-2)^2 \times \frac{1}{(z+2)^2(z-2)^2} \right\} \rightarrow \lim_{z \rightarrow 2} \frac{-2}{(z+2)^3} = \frac{-2}{64} = \underline{\underline{-\frac{1}{32}}}$$

Cauchy's Residue Theorem.

$f(z)$ is analytic in a closed curve C except at a finite no. of points, then $\int_C f(z) dz = 2\pi i \{ \text{sum of residues } f(z) \text{ at each of the poles} \}$

$$\begin{aligned} \textcircled{R} \quad & \oint_C \frac{1}{(z+2)^2(z-2)^2} dz \quad |z|=5 \\ &= 2\pi i [\text{Res}(2) + \text{Res}(-2)] \\ &= 2\pi i \left[\lim_{z \rightarrow 2} \frac{d}{dz} \frac{1}{(z+2)^2} + \lim_{z \rightarrow -2} \frac{d}{dz} \frac{1}{(z-2)^2} \right] \\ &= 2\pi i \left[\lim_{z \rightarrow 2} \frac{-2}{(z+2)^3} + \lim_{z \rightarrow -2} \frac{-2}{(z-2)^3} \right] = 2\pi i \left[\frac{-2}{64} + \frac{2}{64} \right] = \underline{\underline{0}} \end{aligned}$$

$$\textcircled{R} \quad \oint_C \frac{e^z}{z^2+1} dz \quad \text{where } C \text{ is the region } |z|=2$$

$$\begin{aligned} \oint_C \frac{e^z}{(z-i)(z+i)} dz &= 2\pi i \{ \text{Res}(i) + \text{Res}(-i) \} \\ &= 2\pi i \left\{ \lim_{z \rightarrow i} \frac{e^z}{z+i} + \lim_{z \rightarrow -i} \frac{e^z}{z-i} \right\} \\ &= 2\pi i \left\{ \frac{e^i}{2i} + \frac{\bar{e}^{-i}}{-2i} \right\} = \pi [e^i - e^{-i}] \end{aligned}$$

$$\textcircled{R} \quad \oint_C \frac{z^2+1}{z(az+1)} dz \quad |z|=1$$

$$\begin{aligned} \frac{1}{a} \oint_C \frac{z^2+1}{z(z+\frac{1}{a})} dz &= \frac{2\pi i}{2} \{ \text{Res}(0) + \text{Res}(-\frac{1}{a}) \} \\ &= \pi i \left\{ \lim_{z \rightarrow 0} \frac{z^2+1}{z+\frac{1}{a}} + \lim_{z \rightarrow -\frac{1}{a}} \frac{z^2+1}{z} \right\} \\ &= \pi i \left\{ 2 + \frac{-5}{2} \right\} = \underline{\underline{-\frac{7\pi i}{2}}} \end{aligned}$$