Sums

Definition

For a sequence $a_0, a_1, a_2, \cdots$ of real numbers,

(Finite sum) • $\sum_{0 \leq i \leq n} a_i := a_0 + a_1 + \cdots + a_n$.

(Infinite sum) • $\sum_{i \geq 0} a_i := \lim_{n \to \infty} \sum_{0 \leq i \leq n} a_i$.

(Empty sum) • $\sum_{i \in \emptyset} a_i := 0$.

Property (Linearity)

$$\sum_{0 \leq i \leq n} \left(c\, a_i + b_i \right) = c \sum_{0 \leq i \leq n} a_i + \sum_{0 \leq i \leq n} b_i$$

Basic sums

- Arithmetic

$$\sum_{0 \le i \le n} i = \frac{n(n+1)}{2}$$

- Finite geometric

$$\sum_{0 \le i \le n} a^i = \begin{cases} \dfrac{a^{n+1} - 1}{a-1}, & a \ne 1; \\ n+1, & a = 1. \end{cases}$$

- Infinite geometric

$$\sum_{i \ge 0} a^i = \frac{1}{1-a}, \quad |a| < 1.$$

Basic sums, cont'd

- Finite arithmetic-geometric

$$\sum_{0 \le i \le n} i a^i = \begin{cases} \dfrac{a\left(n a^{n+1} - (n+1) a^n + 1\right)}{(a-1)^2}, & a \ne 1; \\[2ex] \dfrac{n(n+1)}{2}, & a = 1. \end{cases}$$

- Infinite arithmetic-geometric

$$\sum_{i \ge 0} i a^i = \frac{a}{(1-a)^2}, \qquad |a| < 1.$$

- Harmonic

$$\sum_{1 \le i \le n} \frac{1}{i} = \ln n + \Theta(1).$$

Basic sums, cont'd

- Sums of powers

  For k an integer,

$$\sum_{1 \le i \le n} i^k = \begin{cases} \Theta(1), & k \le -2; \\ \ln n + \Theta(1), & k = -1; \\ \dfrac{n^{k+1}}{k+1} + \Theta(n^k), & k \ge 0. \end{cases}$$

Theorem (Summing an asymptotic approximation)

Suppose f is a function satisfying

$$\sum_{1 \le i \le n} f(i) = \omega(1).$$

Then

$$\sum_{1 \le i \le n} \Theta\big(f(i)\big) = \Theta\Big(\sum_{1 \le i \le n} f(i)\Big).$$

Comment

With our interpretation of asymptotic expressions, this means the following.

Let $h(n) := \sum_{1 \le i \le n} f(i)$.

Then for any $g \in \Theta(f)$,

$$\sum_{1 \le i \le n} g(i) \in \Theta\big(h(n)\big). \quad \square$$

Bounding by the extreme term

- $\displaystyle\sum_{1 \le i \le n} a_i \le \sum_{1 \le i \le n} \max_{1 \le i \le n} \{a_i\} = n \max_{1 \le i \le n} \{a_i\}.$

- $\displaystyle\sum_{1 \le i \le n} a_i \ge \sum_{1 \le i \le n} \min_{1 \le i \le n} \{a_i\} = n \min_{1 \le i \le n} \{a_i\}.$

Example Show $\displaystyle\sum_{1 \le i \le n} i = O(n^2)$.

$$\sum_{1 \le i \le n} i \le \sum_{1 \le i \le n} n = n^2 = O(n^2).$$

$\square$

Bounding by splitting the domain

- Partition the domain of the index variable.
- Bound the sum over each part of the domain.

Example   Show $\sum_{1 \le i \le n} i = \Omega(n^2)$.

First notice that bounding by the extreme term directly is insufficient:

$$\sum_{1 \le i \le n} i \geq \sum_{1 \le i \le n} 1 = n \neq \Omega(n^2).$$

But let us partition the domain into $\left\{ \left[1, \frac{n}{2}\right), \left[\frac{n}{2}, n\right] \right\}$:

$$\sum_{1 \le i \le n} i = \sum_{1 \le i < \frac{n}{2}} i + \sum_{\frac{n}{2} \le i \le n} i$$

$$\geq \sum_{1 \le i < \frac{n}{2}} 1 + \sum_{\frac{n}{2} \le i \le n} \frac{n}{2}$$

$$\geq 0 + \left(\frac{n}{2}\right)\left(\frac{n}{2}\right) = \Omega(n^2).$$

$\square$