Homework 1 Solutions
Due: Thursday 8 September 2022 by 11:59 PM

Instructions. Type your answers to the following questions and submit as a PDF on Gradescope by the due date and time listed above. (You may write your solutions by hand, but remember that it is at your own risk as illegible solutions will not receive credit.) Assign pages to questions when you submit. **For all questions, you must show your work and/or provide a justification for your answers.**

Note on Academic Dishonesty. Although you are allowed to discuss these problems with other people, your work must be entirely your own. It is considered academic dishonesty to read anyone else's solution to any of these problems or to share your solution with another student or to look for solutions to these questions online. See the syllabus for more information on academic dishonesty.

Grading. Some of these questions may be graded for completion and some for accuracy.

Question 1.
Jamal and Erica are trying to save up money for a fun vacation. Both of them decide to save pennies in a jar (separate jars), but they each use a different strategy. The following shows how many additional pennies each one adds to their jar each week.

Week	What Jamal adds to his jar	What Erica adds to her jar
1	1 penny	5 pennies
2	2 pennies	10 pennies
3	4 pennies	15 pennies
4	8 pennies	20 pennies
5	16 pennies	25 pennies
…	…	…

(a) If they keep this up for a full year (52 weeks), how much money will each of them have? Show your work. It's okay to give a good approximation since these numbers may get very large.
(b) If they keep this up for *W* weeks, how much money will each of them have (in terms of *W*)? Show your work.

(c) Jamal realizes his plan is not very feasible since he would have to be putting in more money than he actually makes in a week. One week, he puts in $163.84 worth of pennies and realizes it's time to quit. If that is the last week he puts money in the jar, how much money does he have in the jar? Show your work.
(d) If Jamal quits putting money in the jar as described in (c) above and Erica sticks with her original plan, for how many weeks will she have to put money in the jar in order to have a total that is at least as much as Jamal's total? Show your work.
(e) Let's say that Jamal goes forward with his plan until one week he puts in *P* pennies, and then he quits. In terms of *P*, how much money does Jamal have in the jar? Show your work.

(a)
After the 52nd week, Jamal will have

$$1 + 2 + 4 + ... + 2^{51} = \sum_{k=0}^{51} 2^k = \frac{2^{52}-1}{1} = 2^{52} - 1 \approx 4.5 \times 10^{15} cents = 4.5 \times 10^{13} dollars$$

$$= \$4,500,000,000,000$$

After the 52nd week, Erica will have

$$5 + 10 + 15 + ... + 260 = \sum_{k=1}^{52} 5k = 5\left(\frac{52(53)}{2}\right) = 6,890\, cents = \$68.90.$$

(b)

After week *W*, Jamal will have $\sum_{k=0}^{W-1} 2^k = 2^W - 1\, cents = \frac{2^w-1}{10}\, dollars.$

After week *W*, Erica will have $\sum_{k=1}^{W} 5k = 5\left(\frac{W(W+1)}{2}\right) cents = \frac{W(W+1)}{4}\, dollars.$

(c)
Jamal's total would be

$$1 + 2 + 4 + ... + 16,384 = \sum_{k=0}^{14} 2^k = \frac{2^{15}-1}{1} = 32,767\, cents = \$327.67.$$

(d)
For this, we need to solve for *w* in:

$$\sum_{k=1}^{w} 5k \geq 32,767$$

$$5\left(\frac{w(w+1)}{2}\right) \geq 32,767$$

$$w(w + 1) \geq 13,106.8$$

$$w^2 + w - 13,106.8 \geq 0$$

When I graph this function, I see that it hits 0 around *w = 114*. To get the exact value, I should plug in *w = 113* and *w = 114.*

$113^2 + 113 - 13,106.8 = -224.8 < 0$

$114^2 + 114 - 13,106.8 = 3.2 > 0$

So clearly, Erica would have to follow her plan for 114 weeks in order to end with about the same amount of money as Jamal.

(e)

$\sum_{k=0}^{logP} 2^k = 2^{logP+1} - 1 = 2P - 1\, cents = \frac{2P-1}{10}\, dollars$

Question 2

Prove by contradiction that $\sqrt[3]{2}$ is irrational. Note: In order to prove this, you will likely have to prove another conjecture first. State all the proof methods/strategies that you use.

First, we must prove that if n^3 is even, then *n* is even.
Proof by Contraposition.
Let $n = 2k + 1$, where *k* is an integer.
Then $n^3 = (2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1 = 2j + 1$, where *j* is an integer. So n^3 is an odd integer.

Proof by Contradiction.
Assume that $\sqrt[3]{2}$ is rational. Then $\sqrt[3]{2} = \frac{p}{q},\ q \neq 0$ where *p* and *q* are integers and are coprime.

$\sqrt[3]{2} = \frac{p}{q}$

$\left(\sqrt[3]{2}\right)^3 = \left(\frac{p}{q}\right)^3$

$2 = \frac{p^3}{q^3}$

$2q^3 = p^3$

So p^3 is even, which means that p is even and $p = 2k,\ k \in Z$.

$\sqrt[3]{2} = \frac{2k}{q}$

$\left(\sqrt[3]{2}\right)^3 = \left(\frac{2k}{q}\right)^3$

$2 = \frac{8k^3}{q^3}$

$2q^3 = 8k^3$

$q^3 = 4k^3 = 2(2k^3)$, which means that q^3 is even and so is q.

If *p* and *q* are both even, then they are not coprime, which contradicts the assumption. Therefore, $\sqrt[3]{2}$ is irrational.

Question 3

Find a formula for the following sum and then prove it using induction:
$\frac{1}{1*2} + \frac{1}{2*3} + ... + \frac{1}{n(n+1)}$. (Hint: To find the formula, write out the first few sums--that is, figure out the sum when $n = 1$, then when $n = 2$, etc. until you see the pattern.)

$n = 1$: $\frac{1}{2}$

$n = 2$: $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$

$n = 3$: $\frac{2}{3} + \frac{1}{12} = \frac{3}{4}$

…

Conjecture: $\sum_{n=1}^{m} \frac{1}{n(n+1)} = \frac{m}{m+1}$ for all $m \geq 1$

Basis Step: $\sum_{n=1}^{1} \frac{1}{n(n+1)} = \frac{1}{2}$

Inductive Step: Assume as the IH that $\sum_{n=1}^{k} \frac{1}{n(n+1)} = \frac{k}{k+1}$ for some $k \geq 1$. We must show that $\sum_{n=1}^{k+1} \frac{1}{n(n+1)} = \frac{k+1}{k+2}$.

$\sum_{n=1}^{k+1} \frac{1}{n(n+1)} = \sum_{n=1}^{k} \frac{1}{n(n+1)} + \frac{1}{(k+1)(k+2)}$

$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$ by the IH

$= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)} = \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$

Question 4

(a) Explain the difference between a proof by contraposition and a proof by contradiction. You should reference formal logic in your answer.

(b) Imagine an algorithm called AlgorithmX that is meant to find a specific element in a collection of elements. These collections can be of any size and the search element can be any element that matches in type. Since the set of possible inputs is huge, it would not be feasible to run the algorithm on all possible inputs to prove that it works correctly. Theoretically, how could a proof by contradiction be used to prove that AlgorithmX works correctly on all valid inputs?

(a) In a proof by contraposition, we are trying to prove that p → q is true, so we prove that $\neg q \rightarrow \neg p$ because it is the contrapositive of the statement we are trying to prove and the contrapositive is logically equivalent to the original implication.
In a proof by contradiction, we assume the opposite of what we are trying to prove and show that it leads to false. In other words, to prove p, we assume $\neg p$ and show that it leads to F. If We know that $\neg p \rightarrow F$, we can conclude that $\neg p$ must be false because F→F is a true statement but T→F is a false statement.

(b) Since we are trying to prove that the algorithm works on all possible inputs, we can assume that there is some input for which it does not work. If we can show that that leads to a contradiction, then we know that such an input does not exist.

Question 5.
Let $T(N) = 3T(floor(N/3)) + N$ and $T(1) = T(2) = 1$.
Prove by induction that $T(N) \leq NlogN + N$ for all $N \geq 1$.

Basis Step.
$T(1) = 1 \leq 1log1 + 1 = 1$
$T(2) = 1 \leq 2log2 + 1 = 3$

Inductive Step.
Assume as the IH that $T(j) \leq jlogj + j$ for $1 \leq j \leq k - 1$ for some $k \geq 3$.
$T(k) = 3T(floor(k/3)) + k$
$\leq 3(floor(k/3)log(floor(k/3)) + floor(k/3)) + k$ by the IH
$\leq 3((k/3)log(k/3) + k/3) + k$
$= klog(k/3) + k + k$
$= klogk - klog3 + k + k$
$= klogk + k - klog3 + k$
$\leq klogk + k$