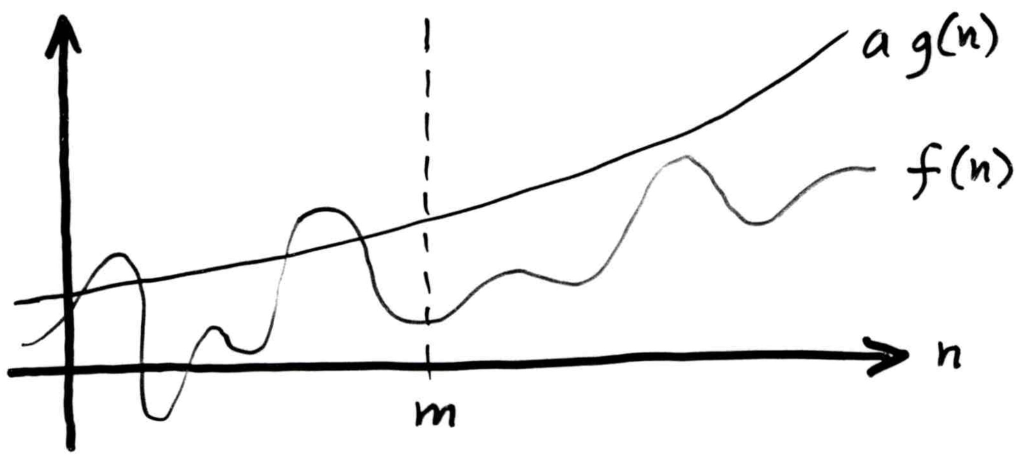
Asymptotic approximation of functions

We use the following notations to describe classes of functions.

Definition (Big oh)

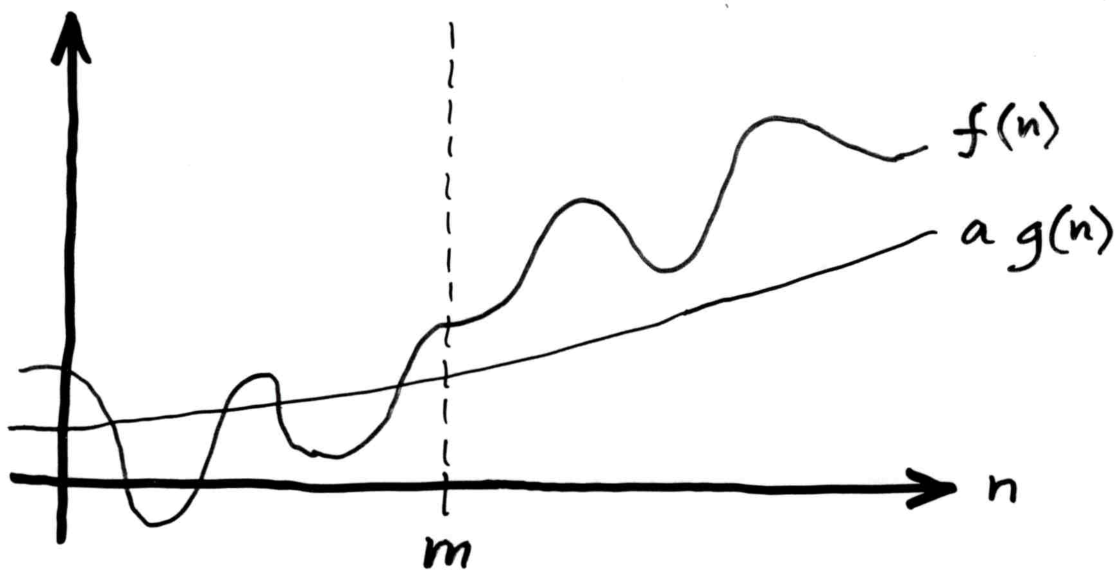
$$\mathcal{O}(g(n)) := \left\{ f(n) : \exists \text{ constants } a>0, m \;\; \forall n \geq m \;\; \left(0 \leq f(n) \leq a\, g(n) \right) \right\}$$



Note $\mathcal{O}(g(n))$ denotes the class of functions that are asymptotically upper-bounded by (some multiple of) $g(n)$.

Definition (Big Omega)

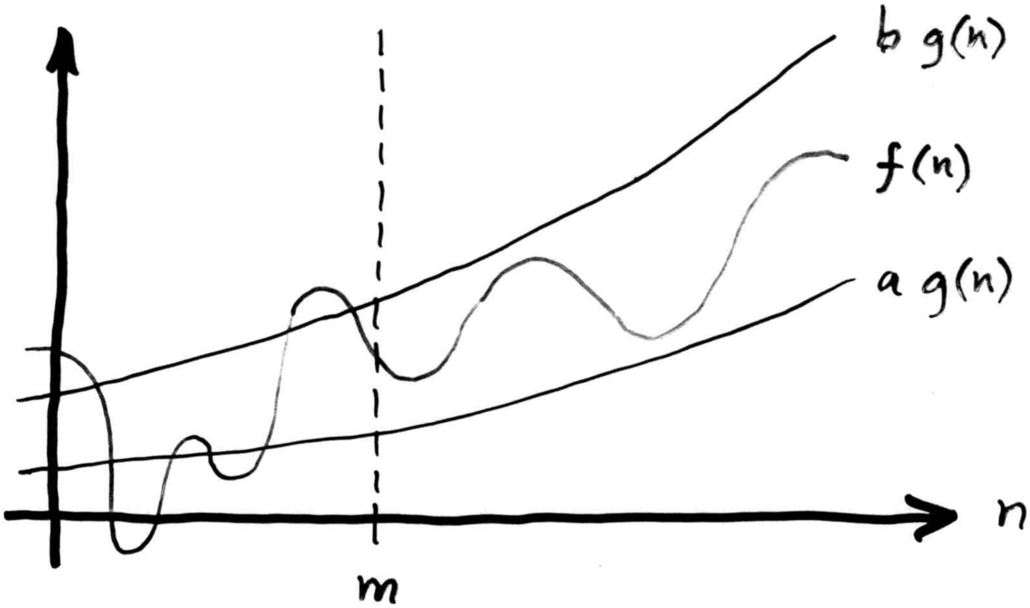
$$\Omega(g(n)) := \left\{ f(n) : \exists \text{ constants } a>0, m \ \forall n \geq m \ \big(f(n) \geq a\,g(n) \text{ and } f(n) \geq 0 \big) \right\}$$



Note $\Omega(g(n))$ denotes the class of functions that are asymptotically lower-bounded by $g(n)$.

Definition (Big Theta)

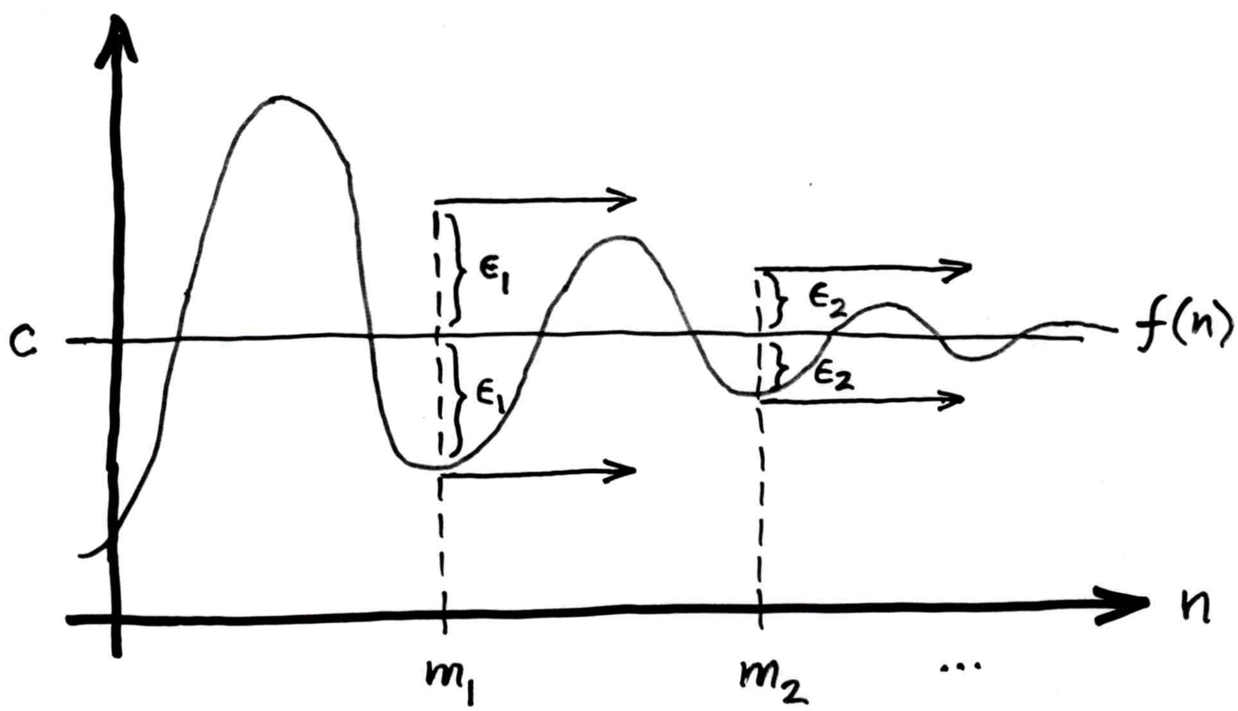
$$\Theta(g(n)) := \left\{ f(n) : \exists \text{ constants } a>0, b>0, m \;\; \forall\, n \geq m \;\; \left(a\,g(n) \leq f(n) \leq b\,g(n) \text{ and } f(n) \geq 0 \right) \right\}$$



Note $\Theta(g(n))$ denotes the class of functions that are asymptotically upper- and lower-bounded by $g(n)$.

Definition (limit)

$$\lim_{n \to \infty} f(n) = c \quad \text{iff} \quad \begin{array}{l} \forall\, \epsilon > 0 \\ \exists\, m \\ \forall\, n \geq m \\ \left(\left| f(n) - c \right| \leq \epsilon \right). \end{array}$$



Theorem 1 (Proving Θ using limits)

Suppose $g(n)$ is positive for all large n, and

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = c > 0,$$

where c is a constant.

Then

$$f(n) \in \Theta(g(n)).$$

Proof Suppose $\lim_{n \to \infty} \frac{f(n)}{g(n)} = c > 0$.

Then $\forall \epsilon > 0 \ \exists m \ \forall n \geq m$

$$c - \epsilon \leq \frac{f(n)}{g(n)} \leq c + \epsilon.$$

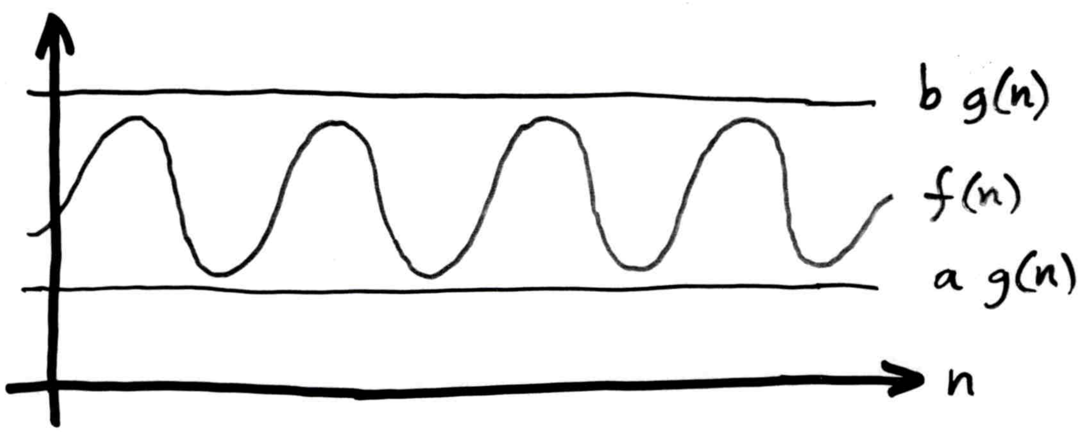
So pick any ϵ such that $0 < \epsilon < c$.

Then the constants $c-\epsilon$, $c+\epsilon$, and m above fulfill the roles of a, b, m in the definition of Θ.

$\square$

Note Theorem 1 is useful in proving $f \in \Theta(g)$, but it's only a sufficient condition!

Example (Counterexample to $\Leftarrow$-direction for Th$^{\underline{m}}$ 1)



$$f(n) \in \Theta(g(n)), \text{ but } \lim_{n\to\infty} \frac{f(n)}{g(n)} \neq c.$$

Definition (little oh)

$$o(g(n)) := \left\{ f(n) : \begin{array}{l} \forall \text{ constants } a > 0 \\ \exists \text{ constant } m \\ \forall\, n \geq m \\ \left(0 \leq f(n) \leq a\, g(n) \right) \end{array} \right\}$$

Note $o(g)$ denotes the class of functions that are asymptotically strictly upper-bounded by g.

(These functions grow slower than any multiple of g.)

Theorem 2 (Proving o using limits)

Suppose $f(n)$ and $g(n)$ are positive for all large n, and that

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 .$$

Then $f(n) \in o(g(n))$.

Proof

$$\lim_{n \to \infty} \frac{f(n)}{g(n)} = 0 \Rightarrow \forall \epsilon > 0 \; \exists m \; \forall n \geq m \left(-\epsilon \leq \frac{f(n)}{g(n)} \leq \epsilon \right)$$

$$\Rightarrow f(n) \in o(g(n))$$

by definition.

$\square$

Definition (little omega)

$$\omega(g(n)) := \left\{ f(n) : \begin{array}{l} \forall \text{ constants } a > 0 \\ \exists \text{ constant } m \\ \forall\, n \geq m \\ \left(f(n) \geq a\, g(n) \right. \\ \quad \left. \text{and } f(n) \geq 0 \right) \end{array} \right\}.$$

Note $\omega(g)$ denotes the class of functions that are asymptotically strictly lower-bounded by g.

(These functions grow faster than any multiple of g.)

Example 1 Show $\lg n \in o(n)$.

Using Theorem 2, evaluate

$$\lim_{n\to\infty} \frac{\lg n}{n} = \lim_{n\to\infty} \frac{1}{\ln 2} \frac{\ln n}{n} \quad \text{by change of base}$$

$$= \frac{1}{\ln 2} \lim_{n\to\infty} \frac{\frac{d}{dn}(\ln n)}{\frac{d}{dn}(n)}$$

by L'Hospital's Rule

$$= \frac{1}{\ln 2} \lim_{n\to\infty} \frac{\frac{1}{n}}{1}$$

$$= \frac{1}{\ln 2} \cdot 0$$

$$= 0. \quad \square$$

Example 2 Show $10n + 1000 \ln n \in \Theta(n)$.

Using Theorem 1, evaluate

$$\lim_{n\to\infty} \frac{10n + 1000 \ln n}{n}$$

$$= \lim_{n\to\infty} \left(10 + 1000 \frac{\ln n}{n} \right)$$

$$= 10 + 1000 \lim_{n\to\infty} \frac{\ln n}{n}$$

$$= 10 + 1000 \cdot 0 \quad \text{by Example 1}$$

$$= 10$$

$$> 0.$$

$$\square$$

Basic properties of asymptotic operators

$$\left\{\begin{array}{l} f := f(n) \\ g := g(n) \\ h := h(n) \end{array}\right.$$

- $f \in \Theta(g) \iff f \in O(g)$ and $f \in \Omega(g)$

$$\left.\begin{array}{l} \bullet\ f \in O(g) \iff g \in \Omega(f) \\ \bullet\ f \in o(g) \iff g \in \omega(f) \end{array}\right\} \text{Symmetry}$$

$$\left.\begin{array}{l} \bullet\ f \in o(g) \Rightarrow f \in O(g) \\ \bullet\ f \in \omega(g) \Rightarrow f \in \Omega(g) \end{array}\right\} \text{Weakening}$$

- $f \in X(g)$ and $g \in X(h) \Rightarrow f \in X(h)$

  where X is Θ, O, Ω, o, or ω.

$\underbrace{\hspace{6em}}_{\text{Transitivity}}$

Interpretation of expressions involving asymptotic operators

Definition

Let $f_1, f_2, \cdots, f_p$ and $g_1, g_2, \cdots, g_q$

be functions of n,

$X_1, X_2, \cdots, X_p$ and $Y_1, Y_2, \cdots, Y_q$

be asymptotic operators, and

$P(X_1(f_1), \cdots, X_p(f_p))$ and

$Q(Y_1(g_1), \cdots, Y_q(g_q))$

be expressions in these functions and operators.

Then "$P(X_1(f_1), \cdots, X_p(f_p)) = Q(Y_1(g_1), \cdots, Y_q(g_q))$"

means $\forall\ x_1 \in X_1(f_1), \cdots, x_p \in X_p(f_p)$

$\exists\ y_1 \in Y_1(g_1), \cdots, y_q \in Y_q(g_q)$

s.t. $P(x_1, \cdots, x_p) = Q(y_1, \cdots, y_q)$. $\square$

Example "$n^{O(1)} = o(2^n)$"

means for any $f \in O(1)$,

there is a $g \in o(2^n)$

such that $n^{f(n)} = g(n)$

(or more simply, if $f \in O(1)$, then

$n^{f(n)} \in o(2^n)$). $\square$

Example "$\log \Theta\left(\left(\frac{n}{e}\right)^n \sqrt{n}\right) = O(n \log n)$"

means for any $f \in \Theta\left(\left(\frac{n}{e}\right)^n \sqrt{n}\right)$,

there is a $g \in O(n \log n)$

such that $\log f(n) = g(n)$

(or more simply, if $f \in \Theta\left(\left(\frac{n}{e}\right)^n \sqrt{n}\right)$,

then $\log f(n) \in O(n \log n)$). $\square$

Arithmetic of asymptotic operators

$$\left\{ \begin{array}{l} f = f(n),\ g = g(n),\ h = h(n), \\ X \text{ is } \Theta, O, \Omega, o, \text{ or } \omega. \end{array} \right.$$

Equality

- $X(X(f)) = X(f)$
- $X(\Theta(f)) = X(f)$
- $\Theta(X(f)) = X(f)$

Addition

- $X(f) + X(g) = X(f+g)$
- $f + O(f) = \Theta(f)$
- $f + \Omega(f) = \Omega(f)$

Subtraction

- $f - o(f) = \Theta(f)$

Multi-plication

- $X(f)\, X(g) = X(fg)$
- $f\, o(g) = o(fg)$
- $f\, \omega(g) = \omega(fg)$

Arithmetic, cont'd

Logarithm • $f = \Theta(g)$ where $f(n) > 1$ and $g(n) > 1$ for all large n

$$\Rightarrow \log f = \Theta(\log g).$$

Exponential • $f = 2^h$ and $g = 2^{\omega(h)}$ where $h = \omega(1)$

$$\Rightarrow f = o(g).$$

Composition • $f = o(g)$ and $h = \omega(1)$

$$\Rightarrow f \circ h = o(g \circ h).$$

Polynomials vs. Exponentials • $\forall\, a > 0,\ b > 1 \left(n^a = o(b^n) \right).$

Stirling's approximation of $n!$

- Asymptotics :

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + \Theta\left(\frac{1}{n}\right)\right)$$

$$= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + o(1)\right)$$

(looser)

$$= \Theta\left(n^{\frac{1}{2}}\left(\frac{n}{e}\right)^n\right).$$

(even looser)

- Hard inequality :

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \le n! \le \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n + \frac{1}{12}n}$$

for all $n \geq 1$.

Example Show $\log n! = \Theta(n \log n)$.

By Stirling's approximation,

$$n! = \Theta\left(n^{\frac{1}{2}} \left(\frac{n}{e}\right)^n\right).$$

Applying the Logarithm Property from our "Arithmetic of ..." table,

$$\log n! = \Theta\left(\log\left(n^{\frac{1}{2}} \left(\frac{n}{e}\right)^n\right)\right)$$

$$= \Theta\left(\frac{1}{2}\log n + n \log n - n \log e\right)$$

$$= \Theta\left(\Theta(n \log n)\right)$$ by the Addition and Subtraction Properties

$$= \Theta(n \log n).$$

$\square$

Example Show $e^n = o(n!)$.

- $e^n = \left(2^{\lg e}\right)^n = 2^{(\lg e)n} = 2^{\Theta(n)}$.
- $n! = 2^{\lg n!} = 2^{\Theta(n \log n)}$ by prev. ex.
- Since $\Theta(n) = o(n \log n)$,

  $e^n = o(n!)$ by our Exponentials Prop.

$\square$

Example Show $(\lg n)^{\lg n} = o\left(\left(\frac{3}{2}\right)^n\right)$.

- $(\lg n)^{\lg n} = \left(2^{\lg \lg n}\right)^{\lg n} = 2^{\Theta(\lg n \lg \lg n)}$.
- $\left(\frac{3}{2}\right)^n = \left(2^{\lg \frac{3}{2}}\right)^n = 2^{\Theta(n)}$.
- Since $\Theta(\lg n \lg\lg n) = o(n)$,

  $(\lg n)^{\lg n} = o\left(\left(\frac{3}{2}\right)^n\right)$ by our Exponentials Prop.

$\square$