CSC 545 **Homework 4 (version 1)** **Spring 2022**

This homework is due Thursday, April 28, at 3:30pm MST. Please upload a single PDF file containing your submission (ensuring scans of handwritten work are legible) on Gradescope by that time.

The questions are drawn from the material in the lectures on *graph matchings*.

The homework is worth a total of 100 points. When point breakdowns are not given for the parts of a problem, each part has equal weight.

Please remember to start each problem on a *new page*, and mark the *corresponding pages* in your submission for each homework problem on Gradescope. Conciseness counts!

(1) **(Augmenting a matching)** (50 points) For an undirected graph $G = (V, E)$, let M be a matching, P be an augmenting path for M, and consider the augmentation $\widetilde{M} = M \oplus P$.

(Note: Recall that operation '$\oplus$' denotes the symmetric difference between two sets: namely, $A \oplus B := (A - B) \cup (B - A)$.)

(a) (30 points) Prove that the augmentation $\widetilde{M}$ is also a matching.

(b) (20 points) Prove that the augmented matching $\widetilde{M}$ has size $|\widetilde{M}| = |M| + 1$.

(2) **(Path and cycle cover)** (50 points) Given a directed graph $G = (V, E)$, a *path and cycle cover* of G is an edge subset $C \subseteq E$ such that the subgraph (V, C) consists of vertex-disjoint directed paths and directed cycles. In other words, in C, every vertex has both in-degree and out-degree at most 1.

Given a directed graph G with edge weights ω, design an algorithm that finds a path and cycle cover C of maximum total weight $\sum_{e \in C} \omega(e)$ in $O(mn + n^2 \log n)$ time, where n and m are the number of vertices and edges of G.

(Hint: Reduce the problem of computing a maximum weight path and cycle cover to the problem of computing a maximum weight matching in a bipartite graph.)

(Note: If graph G is acyclic, then an algorithm for path and cycle cover will find an optimal covering of G by disjoint paths.)