Math Fundamentals I

- Exponents & Logarithms
- Summations
- Recurrence Relations

Exponents & Logarithms

What is an exponent?

An exponent *x* denotes multiplication by the same number (called the *base*) *x* times.

Examples:

- $3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$
- $b^3 = b \cdot b \cdot b$
- $a^1 = a$
- $34^0 = 1$

Rules of Exponents

(1) $b^x \cdot b^y = b^{x+y}$

(2) $\dfrac{b^x}{b^y} = b^{x-y}$

(3) $b^{-x} = \dfrac{1}{b^x}$

(4) $(b \cdot c)^x = b^x c^x$

(5) $b^{1/2} = \sqrt{b}$

$b^{1/3} = \sqrt[3]{b}$

(6) $b^0 = 1$; $b \neq 0$

(7) $(b^x)^y = b^{xy}$

What is an logarithm?

$b^x = y \iff \log_b y = x$

A logarithm inverts an exponent. This allows us to rearrange exponents so that we can solve for the exponent. It also means that exponents and logarithms cancel each other out.

Examples:

- $3^5 = 243$ and $\log_3 243 = 5$
- If $b^3 = 12$, then $\log_b 12 = 3$
- $a^1 = a$ and $\log_a a^1 = 1$
- $34^0 = 1$ and $\log_{34} 1 = 0$

$b^{\log_b x} = x$

$\log_b b^x = x$

Rules of Logarithms

① $\log_b(x \cdot y) = \log_b x + \log_b y$

② Change of Base $\log_b n = \dfrac{\log_a n}{\log_a b}$

③ $\log_b n^x = x \log_b n$

④ $\log_b(x/y) = \log_b x - \log_b y$

⑤ $\log_b \frac{1}{n} = \overset{0}{\cancel{\log_b 1}} - \log_b n = -\log_b n$

⑥ $b^{\log_b x} = x$

⑦ $\log_b b^x = x$

1. Solve for x. $4^x = 16$

$\log_4 4^x = \log_4 16$

$x = \log_4 16$

$= 2$

$4^{\log_4 16} = 16$

Solve for x.

2. $4^x = N$

$\log_4 4^x = \log_4 N$

$x = \log_4 N$

3. Simplify:

$2^{\log_2 N + 1} - 1$

$= 2^{\log N} \cdot 2^1 - 1$

$= N \cdot 2 - 1$

$= 2N - 1$

4. $4^{\log N + 1} - 2$

$(2^2)^{\log N + 1} - 2 \quad 4N^2 - 2$

$= (2^2)^{\log N} \cdot (2^2)^1$

$= 4^{\log N} \cdot 4^1 - 2$

$= (2^2)^{\log N} \cdot 4^1 - 2$

$= (2^{\log N})^2 \cdot 4^1 - 2$

$= N^2 \cdot 4 - 2$

$= 4N^2 - 2$

5. $\log_{16} 32 = \dfrac{\log_2 32}{\log_2 16} = \dfrac{5}{4}$

$\log\left(\dfrac{N^2}{2}\right) = \log N^2 - \log 2$

$= 2\log N - \log 2$

Determine what's wrong with the following "proof."

Conjecture: 8 = 16

Direct Proof.

Let a and k be any positive integers.

a^k

$= a^{k+1}/a$

$= a^{k+2}/a^2$

$= a^k \cdot a^k/a^2$

$= a^{2k}/a^2$

$= \left(a^2\right)^k/a^2$

$= \left(a^2\right)^{k-1}$

Let a = 2 and k = 3.

Then $2^3 = 8 = \left(2^2\right)^2 = 4^2 = 16$.

$\left(\frac{a^{k+1}}{a}\right)^2$

Summations

$$\sum_{k=0}^{N} f(k) = f(0) + f(1) + f(2) + \dots + f(N)$$

“Rules” of Summations

① $$\sum_{k=m}^{N} \underline{\underline{c}} \cdot \boxed{f(k)} = cf(0) + cf(1) + \ldots + cf(N) = c \sum_{k=m}^{N} f(k)$$

② $$\sum_{k=m}^{N} \left[f(k) + g(k)\right] = \sum_{k=m}^{N} f(k) + \sum_{k=m}^{N} g(k)$$

③ $$\sum_{k=m}^{N} f(k) = \sum_{k=0}^{N} f(k) - \sum_{k=0}^{m-1} f(k)$$

Closed Form

$$① \sum_{k=0}^{N} c = c \sum_{k=0}^{N} 1 = c(N+1) \leftarrow$$

$$2 \overset{\times 2}{+} 4 \overset{\times 2}{+} 8 \overset{\times 2}{+} 16 + \dots$$

$$2 \overset{+2}{+} 4 \overset{+2}{+} 6 \overset{+2}{+} 8 + \dots$$

$$\rightarrow ② \sum_{k=0}^{N} k = \frac{N(N+1)}{2}$$

$$③ \sum_{k=0}^{N} r^k = \frac{r^{N+1} - 1}{r-1} \, ; \, r \neq 0, 1$$

$$④ \sum_{k=1}^{N} \log k = \log 1 + \log 2 + \dots + \log N = \log [1 \cdot 2 \cdot 3 \cdot \dots \cdot N] = \log(N!)$$

1. $1 \overset{+1}{+} 2 \overset{+1}{+} 3 + 4 + \dots + N$

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}$$

$$-5 + \sum_{k=0}^{N/5} 5k$$

2. $-5 \overset{+5}{+} 0 \overset{+5}{+} 5 \overset{+5}{+} \dots \overset{+5}{+} N$

$$\sum_{k=0}^{\frac{N+5}{5}} (-5 + 5k) = -5 \sum_{k=0}^{\frac{N+5}{5}} 1 + 5 \sum_{k=0}^{\frac{N+5}{5}} k = -5\left[\frac{N+5}{5} + 1\right] + 5\left[\frac{\left(\frac{N+5}{5}\right)\left(\frac{N+5}{5} + 1\right)}{2}\right]$$

$$-5 + 5k = N$$

$$k = \frac{N+5}{5}$$

3. $1 + 2 + 4 + 8 + \dots + N$ $\qquad 2N-1$

$$\sum_{k=0}^{\log N} 2^k = 2^{\log N + 1} - 1 = 2N - 1$$

$k = \log N/4$

$\log_2 2^k = \log_2 \frac{N}{4}$

4. $\underbrace{\overset{\times 2}{4} + 8 \overset{\cdot 2}{+} 16 \overset{\cdot 2}{+} 32 \overset{2}{+} \dots + N}$ $= 4 + (1 + 2 + 4 + \dots + N/4)$

$4 \cdot 1 + 4 \cdot 2 + 4 \cdot 4 \dots$

$$\sum 4 (2^k) = 4 \sum_{k=0}^{\log_2 N/4} 2^k = 4 \left[2^{\log N/4 + 1} - 1 \right]$$

Recurrence Relations

What is a recurrence relation?

$$T(N) = T(N-1) + 1; \quad T(1) = 1$$

$$T(N) = T(N/2) + 1$$

$$a_n = a_{n/2} + 1$$

Solving Recurrence Relations: Expansion & Summation

$$X = N - 1$$

$$T(X) = T(X-1) + X$$

$$\boxed{T(N-1) = T(N-2) + (N-1)}$$

$$T(N) = \underline{T(N-1)} + N \; ; \quad T(1) = 1$$

$$= \underline{T(N-2)} + (N-1) + N$$

$$= T(N-3) + (N-2) + (N-1) + N$$

$$\vdots$$

$$= \underbrace{1}_{T(1)} + 2 + \ldots + (N-2) + (N-1) + N$$

$$= \sum_{k=1}^{N} k = \frac{N(N+1)}{2}$$

+1 +1 +1 +1

Solving Recurrence Relations: Using a Tree

Solving Recurrence Relations: Using the Master Theorem

Given:

- a and b are integers
- $a \geq 1$ and $b > 1$
- c is a positive real number
- d is a nonnegative real number
- $N = b^k$ where k is a positive integer
- $T(N)$ is an increasing function
- $T(N) = aT(N/b) + cN^d$

Then:

- if $a < b^d$, $T(N)$ is $O(N^d)$
- if $a = b^d$, $T(N)$ is $O(N^d logN)$
- if $a > b^d$, $T(N)$ is $O(N^{log_b a})$

Preview

Prove that if $T(N) = T(floor(N/2)) + 1$ and $T(1) = 1$, then $T(N)$ is $O(logN)$.