CSc 345 Midterm 1

Name	
UA ID Number	
UA email	

General Instructions.

- Put your name and email in the space provided.
- For all questions, read the instructions carefully and write legibly. Illegible answers will not receive credit.
- Do not tear any pages from the exam.
- You may write on any portion of the exam, but only answers in the indicated spaces or in the designated "Extra Space" areas will be graded.
- Assume that all logarithms are base-2 unless specifically indicated otherwise.
- Please read all multiple choice answers carefully.
- Note that when asked for the best-case runtime, you should give the tightest lower bound, and when asked for the worst-case runtime, you should give the tightest upper bound.
- If you need extra space, you can use the last page of the exam, but make sure you make a note in the question that you used the extra space. Also, make sure the work is labeled with the appropriate Section and Question number.
- You are not allowed to use a calculator for this exam.

Some formulas that might be useful:

Summations:

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}$$

$$\sum_{k=0}^{N} r^k = \frac{r^{N+1}-1}{r-1},\ r \neq 1$$

The Master Theorem: Given a function of the form $f(N) = af(N/b) + cN^d$, $f(N)$ is

- $O(N^d)$ if $a < b^d$
- $O(N^d logN)$ if $a = b^d$
- $O(N^{log_b a})$ if $a > b^d$

Other: $log(N!)\ is\ O(NlogN)$

Part 1. Short Answer. (13 points each) Answer each question in the space provided.

1. Write the following series in summation form. Then solve it by putting it into closed-form using the formulas provided. You do not have to simplify the function completely, but your answer should be a function in terms of *N*. Finally, give the big-Theta classification of the resulting function. (Hint: Use the function $3k + 1$ and start the sum at $k = 0$.)

$1 + 4 + 7 + 10 + \ldots + N$

$$\sum_{k=0}^{\frac{N-1}{3}} (3k + 1) = 3\sum_{k=0}^{\frac{N-1}{3}} k + \sum_{k=0}^{\frac{N-1}{3}} 1 = 3\left(\frac{\frac{N-1}{3}\left(\frac{N-1}{3}+1\right)}{2}\right) + \frac{N-1}{3} + 1$$

The final function is $\theta(N^2)$.

2. Use the formal definition of big-Theta, big-Oh, and big-Omega to show that $\frac{1}{3}n^3 + 2n^2 + 5n + 1$ is $\theta(n^3)$.

To show big-Oh:

$\frac{1}{3}n^3 + 2n^2 + 5n + 1 \le 5n^3 + 5n^3 + 5n^3 + 5n^3 = 20n^3$ for all $n \ge 1$

To show big-Omega:

$\frac{1}{3}n^3 + 2n^2 + 5n + 1 \ge \frac{1}{3}n^3$ for all $n \ge 1$

3. Alice is trying to prove that a proposition $P(n)$ is true for all integers $n \geq 1$. First, she shows that $P(1)$ is true. Then she shows that $P(floor(k/3)) \rightarrow P(k)$. Has Alice successfully proven that $P(n)$ is true for all integers $n \geq 1$? If the answer is "no", explain what Alice could do differently to actually prove the proposition for all $n \geq 1$.

No. Alice needs to prove P(2) as a second base case.

4. Let $T(N) = 3T(floor(N/3)) + N$ and $T(1) = T(2) = 1$. Prove by induction that $T(N) \leq NlogN + N$ for all $N \geq 1$. Remember that a majority of the points for this problem will be for appropriately setting up the proof with all the required parts.

Basis Step.
$T(1) = 1 \leq 1log1 + 1 = 1$
$T(2) = 1 \leq 2log2 + 2 = 4$

Inductive Step.
Assume as the IH that $T(j) \leq jlogj + j$ for $2 \leq j \leq k - 1$.
$T(k) = 3T(floor(k/3)) + k$
$\leq 3[floor(k/3)log(floor(k/3)) + floor(k/3)] + k$ by the IH
$\leq 3[(k/3)log(k/3) + k/3] + k$
$= klog(k/3) + k + k$
$= klogk - klog3 + k + k$
$\leq klogk - k + k + k$
$= klogk + k$

5. Answer the questions about the pseudocode below. You can assume that $n > 0$.

```
proc foo(m, n):
   if n = 1 then m
   else m + foo(m, n-1)
end foo
```

(a) Write a recurrence relation for the runtime of the function in terms of n, and explain each part of it.
(b) Give the big-Oh runtime of the function based on the recurrence relation and justify your answer mathematically *without using the Master Theorem.*

(a) $T(n) = T(n - 1) + 1; T(1) = 1$
Explanation: For each call, there is 1 recursive call where n is reduced by 1. Additionally, there is a constant amount of work being done. The base case requires a constant amount of work.
(b) $T(n) = T(n - 1) + 1 = T(n - 2) + 1 + 1 = T(n - 3) + 1 + 1 + 1$
$= \ldots = T(1) + 1 + 1 + \ldots + 1 \approx N$
The big-Oh runtime is $O(N)$.

Part 2. DT/DF/PT (2 points each) For each statement, determine if it is *definitely true (DT)*, *definitely false (DF)*, or *possibly true/possibly false (PT)*.

Given: **1.** $f(n)\ is\ O(g(n))$ **2.** $f(n)\ is\ O(h(n))$ **3.** $f(n)\ is\ \Omega(i(n))$
1. $i(n)\ is\ O(h(n))$ DT. Statement 3 implies that $i(n)\ is\ O(f(n))$. Combining that with Statement 2 means that $i(n)\ is\ O(h(n))$.
2. $g(n)\ is\ \theta(f(n))$ PT. We already know that $g(n)\ is\ \Omega(f(n))$ from Statement 1. It is possible that this is a tight bound, which would make the statement true. But it is also possible that this is not a tight bound, which would make the statement false.
3. $g(n)\ is\ O(h(n))$ PT. We know that g(n) and h(n) are both upper bounds for f(n) but we don't really know the relationship between g and h. h(n) could fall in between g(n) and f(n), making the statement false; or g(n) could fall in between h(n) and f(n), making the statement true.
4. $f(n)\ +\ g(n)\ +\ h(n)\ +\ i(n)\ is\ \Omega(f(n))$ DT. A function cannot grow more slowly than any of its individual terms.
5. $f(n)\ +\ g(n)\ +\ h(n)\ +\ i(n)\ is\ O(f(n))$ PT. If f, g, and h all grow at the same rate, the statement would be true. But if not, we know that h and g don't grow more slowly than f, so f cannot be the fastest growing term, and the overall function could grow faster than f.

[Extra Space]

Part 3. Multiple Choice (5 points each) For each question, circle the letter of the best response. Each question has one best answer. Do not circle multiple answers. You are not required to show your work. However, if your answer is wrong and you do show your work, your work *may* earn you some partial credit.

Questions 1 & 2 refer to the pseudocode below. You can assume that *N* is a power of 4 and that the division is integer division so ¼ = 0.

```
while N > 0:
     for i from 1 to N:
          count++
     end for
     N = N/4
end while
```

1. What is the exact value of *count* after the pseudocode executes?

A. $(4N - 1)/3$ B. $(N - 1)/3$ C. $(4^{N+1} - 1)/3$ D. $(4^{N/4+1} - 1)/3$
E. The answer is not listed.

2. What is the runtime of the pseudocode in big-Theta notation?
A. $\theta(1)$ B. $\theta(logN)$ C. $\theta(N)$ D. $\theta(NlogN)$ E. The answer is not listed.

Questions 3 & 4 refer to the pseudocode below.

```
count = 0
for(i = 0; i < N; i++)
   for(j = i; j < N; j++)
      count++
   end for
end for
```

3. What is the exact final value of *count*?

A. $N(N + 1)/2$ B. $N(N - 1)/2$ C. N^2 D. $2N - 1$ E. The answer is not listed.

4. What is the runtime of the algorithm in big-Theta notation?

A. $O(logN)$ B. $O(N)$ C. $O(NlogN)$ D. $O(N^2)$ E. The answer is not listed.

Questions 5 & 6 refer to the pseudocode below.

```
A = an array of N integers
```

```
count = 1
c = 1
i = 0
while i < N:
      j = i
      while j < N-1 && A[j] <= A[j+1]:
            c++
            j++
      end while
      if c > count
            count = c
      end if
      c = 1
      i = j+1
end while
```

5. What is the worst-case runtime of the algorithm?

A. $O(logN)$ B. $O(N)$ C. $O(NlogN)$ D. $O(N^2)$ E. Answer not listed.

6. What is the best-case runtime of the algorithm?

A. $\Omega(1)$ B. $\Omega(logN)$ C. $\Omega(N)$ D. $\Omega(NlogN)$ E. Answer not listed.

7. Consider the proof below.

Conjecture: $\sum_{j=1}^{N} \frac{1}{j(j+1)} = \frac{N}{N+1}$ for all integers $N \geq 1$.

Proof by Induction.

1 Basis Step.

2 $\sum_{j=1}^{1} \frac{1}{j(j+1)} = \frac{1}{1(2)} = \frac{1}{2}$

3 Inductive Step.

4 Assume as the IH that $\sum_{j=1}^{k-1} \frac{1}{j(j+1)} = \frac{k-1}{k}$.

5 $\sum_{j=1}^{k} \frac{1}{j(j+1)} = \sum_{j=1}^{k-1} \frac{1}{j(j+1)} + \frac{1}{k(k+1)}$

6 $= \frac{k-1}{k} + \frac{1}{k(k+1)}$ by the IH

7 $= \frac{(k-1)(k+1)+1}{k(k+1)}$

8 $= \frac{k^2-1+1}{k(k+1)}$

9 $= \frac{k^2}{k(k+1)}$

10 $= \frac{k}{k+1}$

Which of the following statements is true about the proof?

A. There is an error in Line 2.

B. There is an error in Line 4.

C. There is an error in Line 6.

D. There is an error in Lines 7-10.

E. There is no error in the proof. It is correct.

This question was thrown out.

8. Consider the proof below.

Given: T(N) = T(N-2) + 1; T(1) = T(2) = 1

```
Conjecture: T(N) is O(N)

Basis Step.
  T(1) = 1 ≤ 1

Inductive Step.
  Assume as the IH that T(j) ≤ j for all integers 1 ≤ j ≤ k − 1.
  T(k) = T(k − 2) + 1
  ≤ (k − 2) + 1 by the IH
  = k − 1
  ≤ k
```

Which of the following changes would make this proof correct?

A. Add a second base case showing that $T(2) \leq 2$.

B. Proving that $T(N) \leq N + 1$ instead of N.

C. Using weak induction instead of strong induction.

D. Showing that $P(floor(k/2)) \rightarrow P(k)$ in the Inductive Step.

E. None of the above. The proof is already correct.

9. Consider the proof below.

Conjecture: $3 \mid (3n + 2)$ for all positive integers n.
Proof by induction.
1 Basis Step.
2 Let n = 1. $3 \mid (3n + 2)$ is true.
3 Inductive Step.
4 If $3 \mid (3k + 2)$, then $3 \mid (3(k + 1) + 2)$.
5 In order for $3 \mid (3k + 2)$ to be true, $3k + 2$ must be a multiple of 3.
6 Written another way, $3k + 2 = 3p$, where p is an integer.
7
8 $3(k + 1) + 2$
9 $= 3k + 3 + 2$
10 $= (3k + 2) + 3$
11 $= 3p + 3$
12 $= 3(p + 1)$
13
14 $3(p + 1)$ is clearly a multiple of 3, and $3(k + 1) + 2 = 3(p + 1)$.
15 Thus, $3 \mid (3(k + 1) + 2)$.
16
17 Therefore, $3 \mid (3n + 2)$ for all positive integers n.

Which line in the proof contains an error?

A. Line 2
B. Line 4
C. Line 11
D. Line 14
E. None of the above. The proof is correct.

Questions 10 - 13 refer to the proof below.

This is a proof by induction. It does not explicitly state all the parts such as the basis step and the inductive hypothesis, but everything is there and the proof is valid. Answer the questions below about the proof.

```
Conjecture: The sum of the first N even numbers and the first N multiples
of 3 is equal to the sum of the first N multiples of 5.
Proof by Induction.
1  2 + 3 = 5
```

2 $\sum_{i=1}^{k+1} 2i + \sum_{i=1}^{k+1} 3i = \sum_{i=1}^{k} 2i + 2(k+1) + \sum_{i=1}^{k} 3i + 3(k+1)$

3 $= \left(\sum_{i=1}^{k} 2i + \sum_{i=1}^{k} 3i\right) + (5k + 5)$

4 $= \sum_{i=1}^{k} 5i + 5(k+1)$

5 $= \sum_{i=1}^{k+1} 5i$

10. In which line is the base case proven?
A. Line 1 B. Line 2 C. Line 3 D. Line 4 E. Line 5

11. Which of the following is the inductive hypothesis?

A. $\sum_{i=1}^{N} 2i + \sum_{i=1}^{N} 3i = \sum_{i=1}^{N} 5i$

B. $\sum_{i=1}^{k-1} 2i + \sum_{i=1}^{k-1} 3i = \sum_{i=1}^{k-1} 5i$

C. $\sum_{i=1}^{k} 2i + \sum_{i=1}^{k} 3i = \sum_{i=1}^{k} 5i$

D. $\sum_{i=1}^{k+1} 2i + \sum_{i=1}^{k+1} 3i = \sum_{i=1}^{k+1} 5i$

E. None of the above.

12. In which line is the inductive hypothesis applied?
A. Line 1 B. Line 2 C. Line 3 D. Line 4 E. Line 5

13. This proof uses
A. strong induction. B. weak induction.

14. $8n\log(2^n) + 14n^2 + 15n$ is

A. $O(2^n)$ B. $O(n^2 log n)$ C. $O(n^2)$ D. $\Omega(n)$ E. All of the answers are true.
Partial credit was given to C.

15. Alice is comparing two polynomial functions $f(n)$ and $g(n)$. She draws a log-log plot of the two functions and finds that each of them produces a line. The slope of line for $log(f(n))$ is 2 and the slope of the line for $log(g(n))$ is 4. Which of the following can Alice conclude about $f(n)$ and $g(n)$? A. $f(n)\ is\ O(g(n))$. B. $f(n)\ is\ \Omega(g(n))$. C. $f(n)\ is\ \theta(g(n))$. D. All of the above. E. None of the above.

Part 4. Extra Credit. (5 points) Partial credit is not generally given for extra credit questions. The answer and the work must be correct to get credit.

(a) What does the function from Written Response Question #5 calculate? (b) What does the pseudocode given for Multiple Choice questions 5 & 6 calculate?
(a) It calculates $n \times m$. (b) It calculates the length of the longest sorted subarray.

[Extra Space]

[Extra Space]