Minimum Spanning Tree

- Terminology and Properties
- Prim’s Algorithm
- Kruskal’s Algorithm

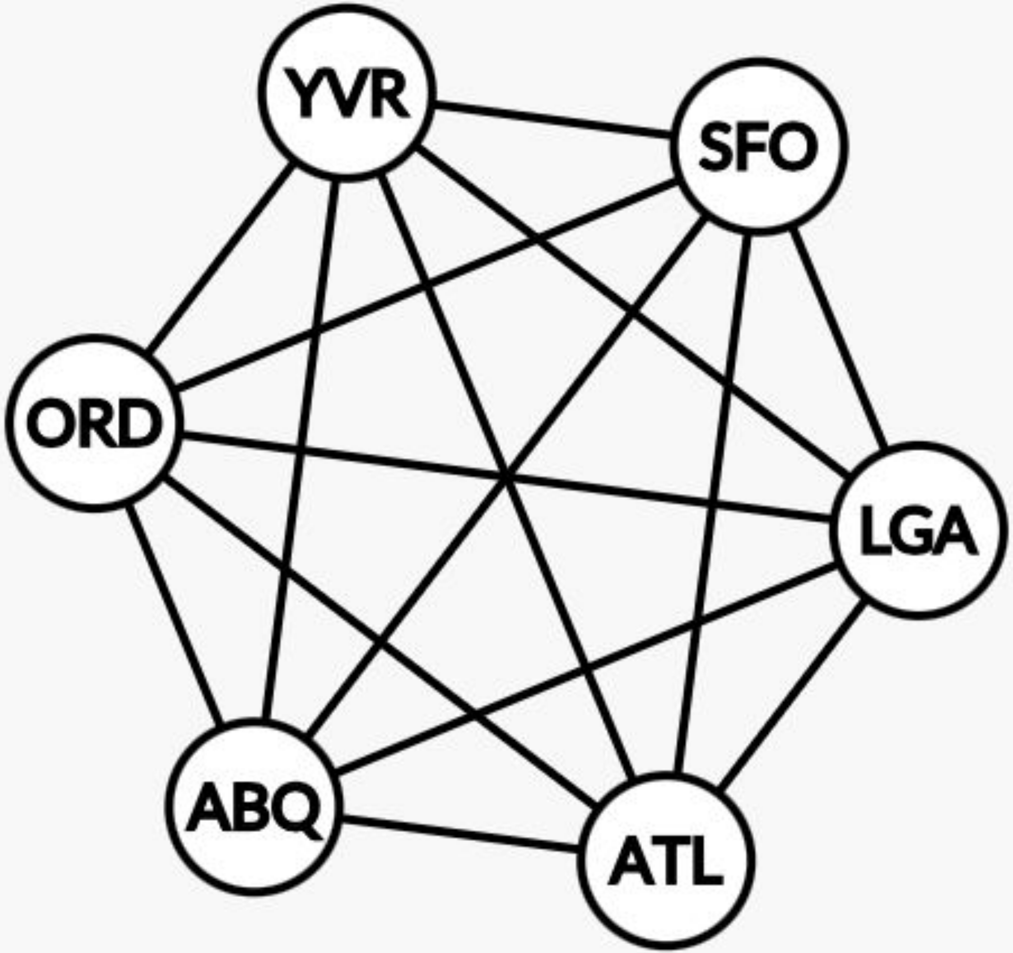
Imagine...

You are employed at a brand new airline, and your first task is to determine which flights the airline should provide according to the following guidelines:

- There needs to be a way to get between each pair of cities/airports in the following list: San Francisco, CA (SFO); Vancouver, BC (YVR); Albuquerque, NM (ABQ); Chicago, IL (ORD); Atlanta, GA (ATL); New York, NY (LGA)
- The route between two cities does not have to be a direct flight.
- The total number of flights should be minimized.
- The total distance covered by all flights should be minimized.
- Assume that if a flight exists, it goes both ways (e.g. SFO→ YVR and YVR→ SFO)

How would you go about solving this problem?

	SFO	YVR	ABQ	ORD	ATL	LGA
SFO	0	1280	443	2960	3440	4139
YVR	1280	0	2072	2820	3600	3920
ABQ	443	2072	0	1799	2048	2940
ORD	2960	2820	1799	0	951	1163
ATL	3440	3600	2048	951	0	1219
LGA	4139	3920	2940	1163	1219	0


YVR
SFO
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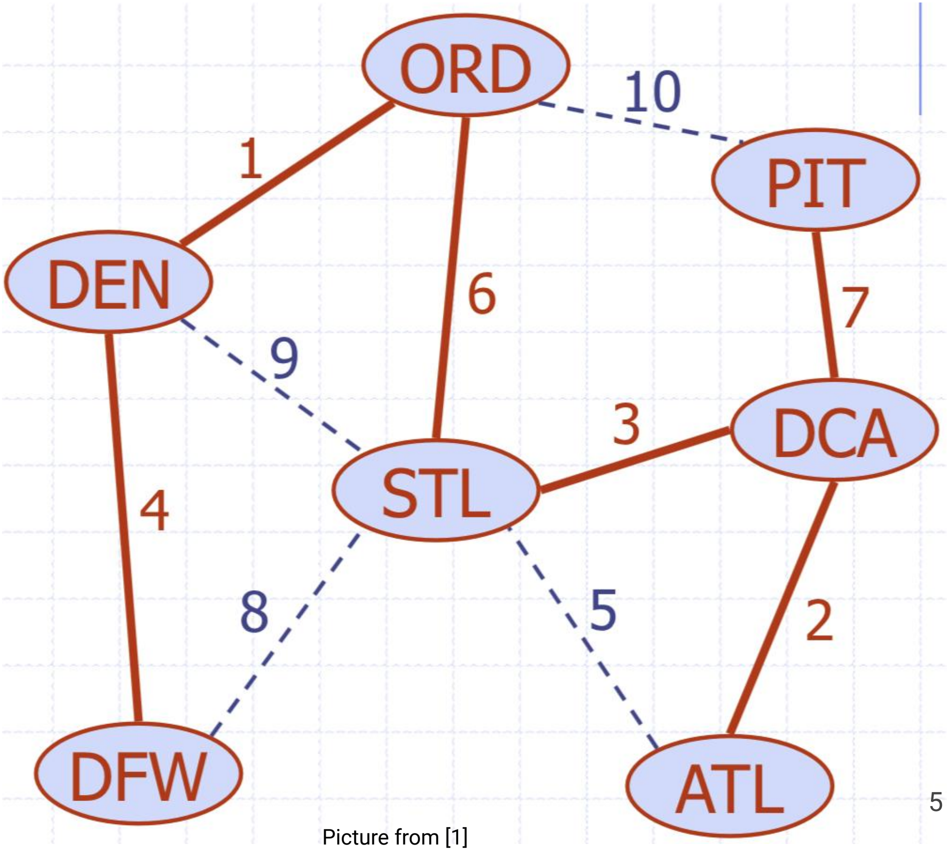
LGA
ABQ
ATL

What we really want is a minimum spanning tree (MST)...

A minimum spanning tree of a graph G...

- ...is a spanning subgraph (i.e. it contains all the vertices of G)
- ...is a tree (i.e. it has no cycles)
- ...is minimal weight-wise (i.e. the total weight of all the edges it uses is minimized so that no other spanning tree has a smaller total weight)
- ...does NOT guarantee the shortest path between two vertices!



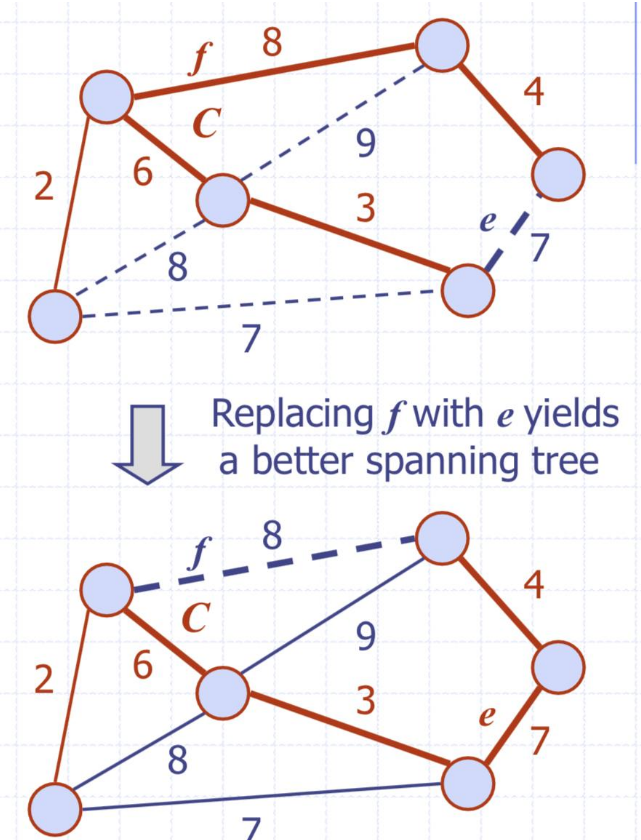
Picture from [1]

Property 1: The Cycle Property

- Let **T** be a minimum spanning tree of a weighted graph **G**
- Let **e** be an edge of **G** that is not in **T**
- Let **C** be the cycle formed by adding **e** to **T**
- Claim: For every edge **f** of **C**:

  weight(f) <= weight(e)

Proof by Contradiction: ???



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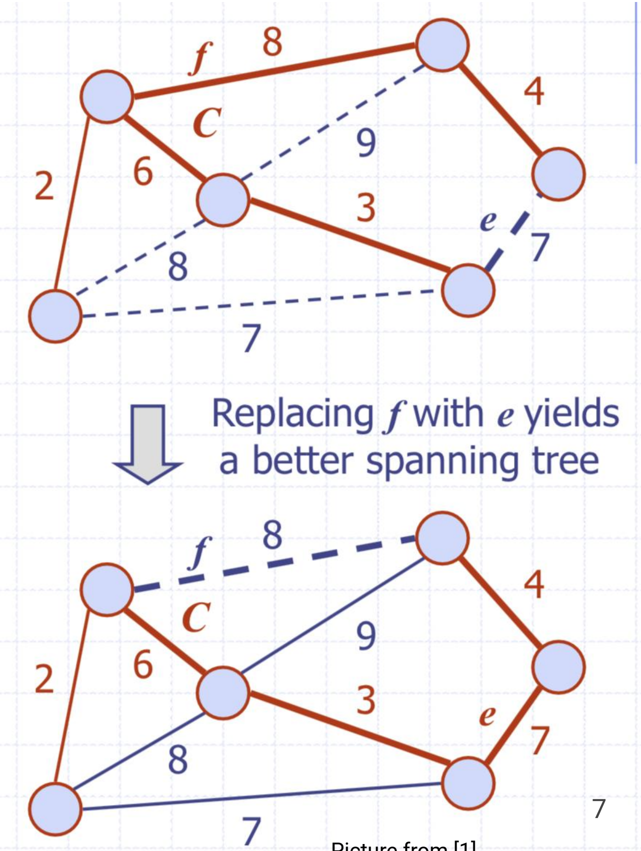
Proof by Contradiction: Given all the properties above, assume that for some edge **f** in **C**, ***weight(f) > weight(e)***. Then replacing **f** with **e** will produce a spanning tree **T'** such that the total weight of **T'** is smaller than the total weight of **T**. But that contradicts the definition of **T** as an MST of **G.**

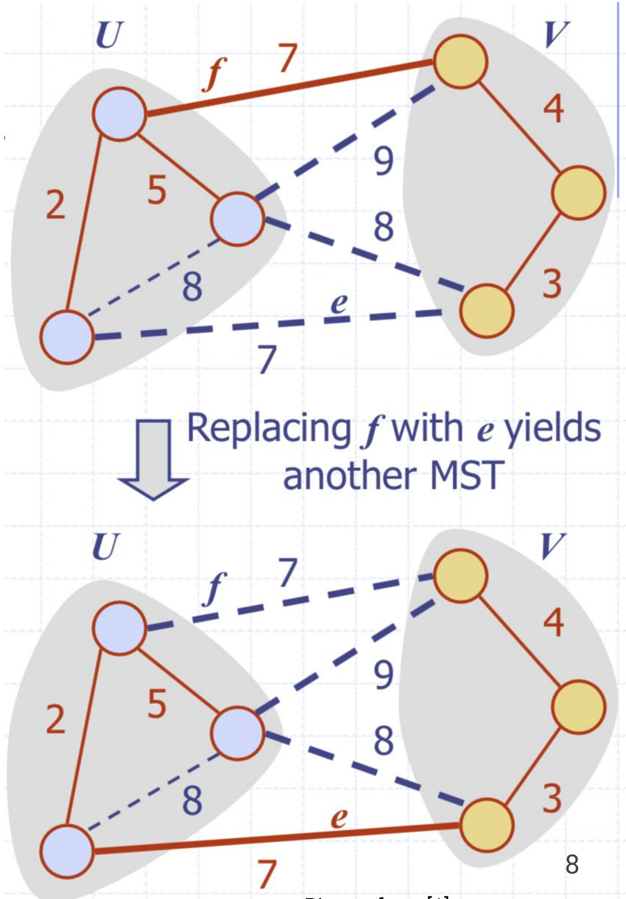


Picture from [1]

Property 2: The Partition Property

- Consider a partition of a graph **G** into subsets **U** and **V**
- Let **e** be an edge of minimum weight across the partition (i.e. **e** has one endpoint in **U** and one endpoint in **V**)
- Claim: There is a minimum spanning tree of **G** that includes edge **e**

Proof: ???



Picture from [1]

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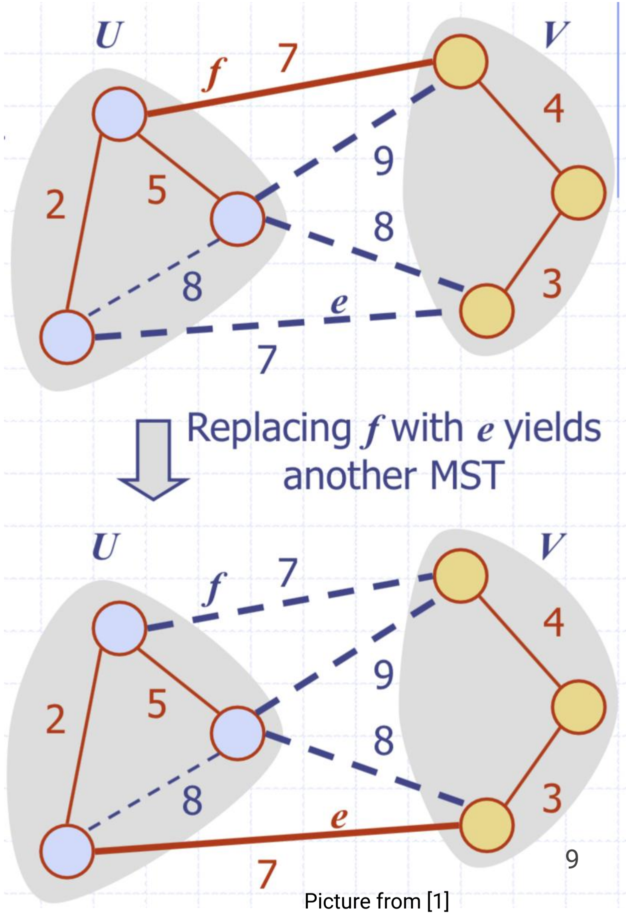
Proof: Let **T** be an MST of **G** and let the definitions above be true. If **T** does not contain **e**, consider the cycle **C** formed by **e** with **T** and let **f** be an edge of **C** across the partition. By the cycle property, ***weight(f)<=weight(e)***. Thus, ***weight(f) = weight(e)***, and we obtain another MST by replacing **f** with **e**.



Picture from [1]

Questions to think about...

1. Given an undirected, weighted graph **G**, let **T** be an MST of **G**. Is **T** unique?

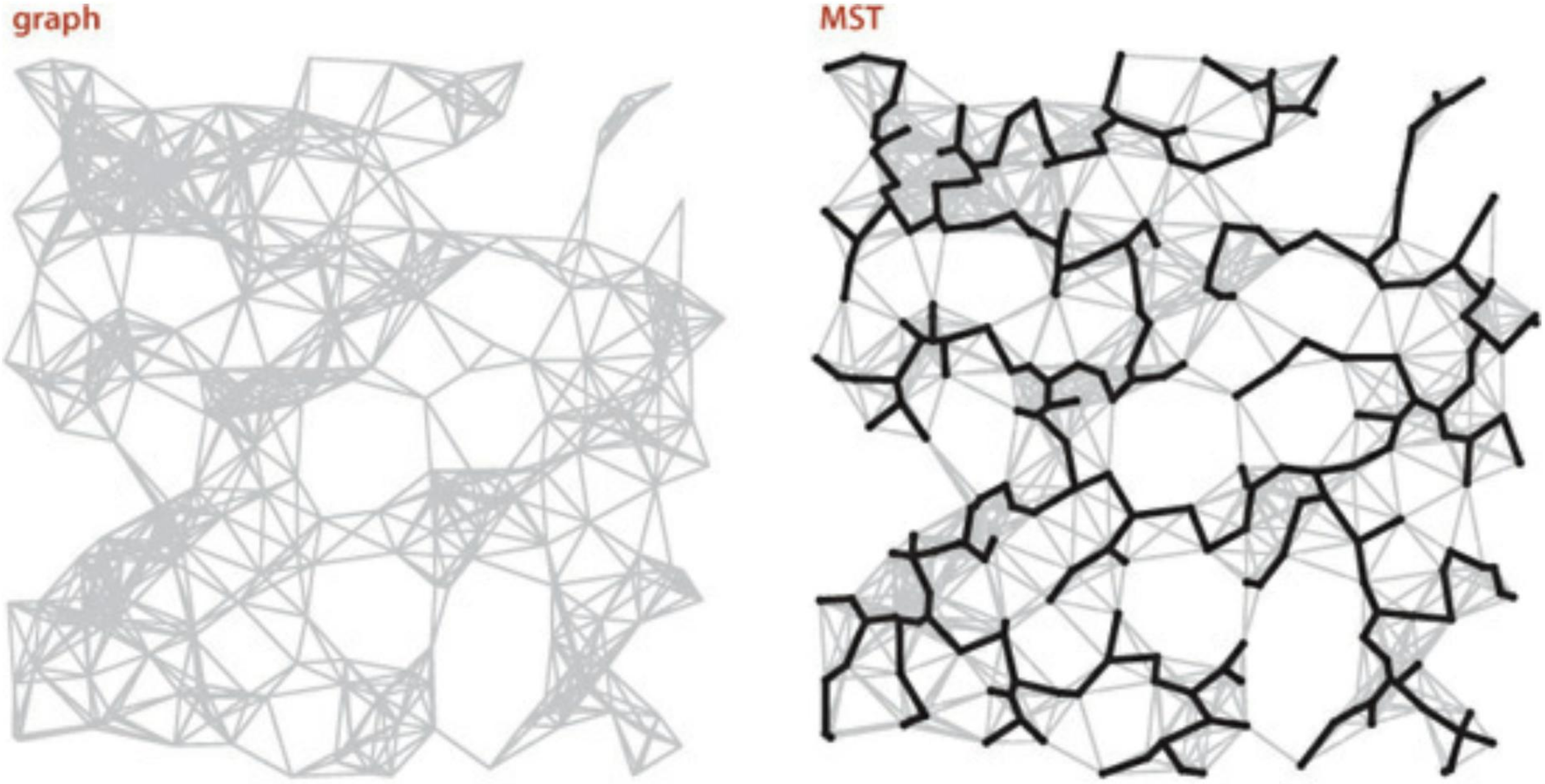
2. Does an MST also give you the shortest path between a pair of vertices?

3. Does every weighted undirected graph have an MST?

4. Can you find an MST of a weighted undirected graph if there are negative weight edges?

5. Can you find an MST in a directed graph?

6. How would you find an MST of a weighted undirected graph?

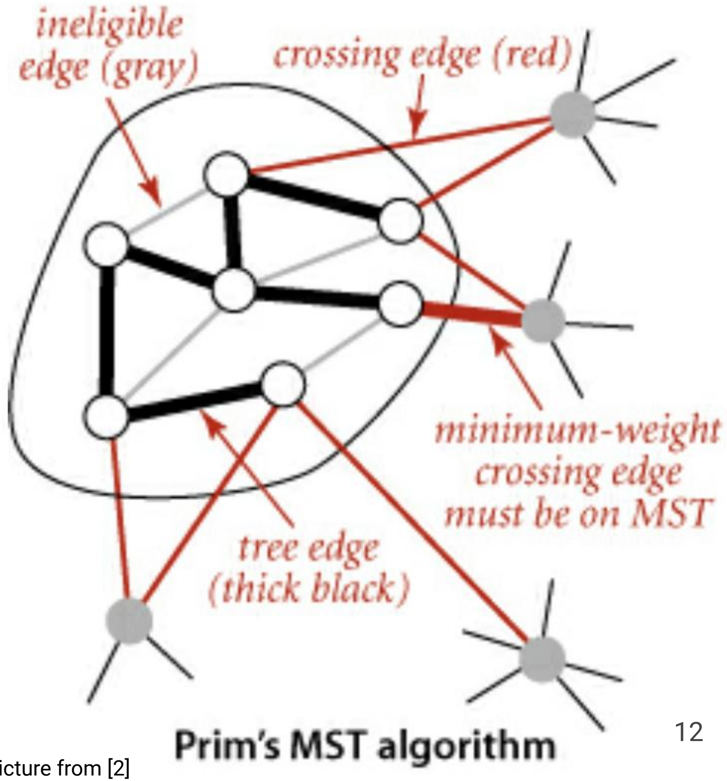


A 250-vertex Euclidean graph (with 1,273 edges) and its MST

Picture from [2]

Prim's Algorithm (aka Jarnik's Algorithm)

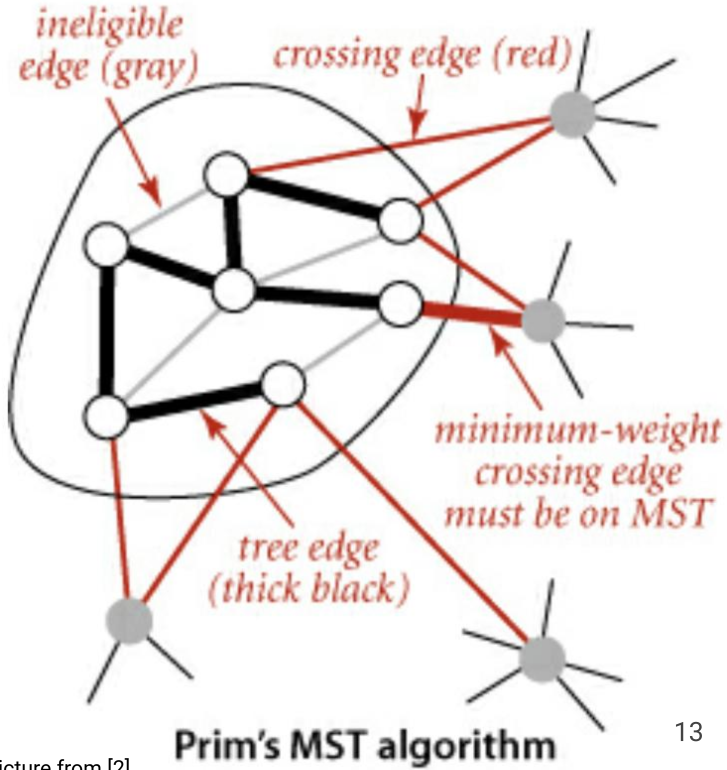
- Grow the tree one edge at a time
- Start with a single vertex (counts as a tree)
- Add |V| - 1 edges to it by adding the minimum-weight edge that connects a new vertex to the tree (crossing the partition that is defined by the current tree vertices--called a *crossing edge*)
- NOTE: When you add an edge to the tree, you are also adding a vertex to the tree.
- Like Dijkstra's Shortest Path, this is a greedy algorithm.



Picture from [2]

Prim's Algorithm (aka Jarnik's Algorithm)

Proof of Correctness: Follows directly from the Partition Property because we are choosing the minimum weight edge across the tree-defined partition.



Picture from [2]

Implementation of Prim's Algorithm

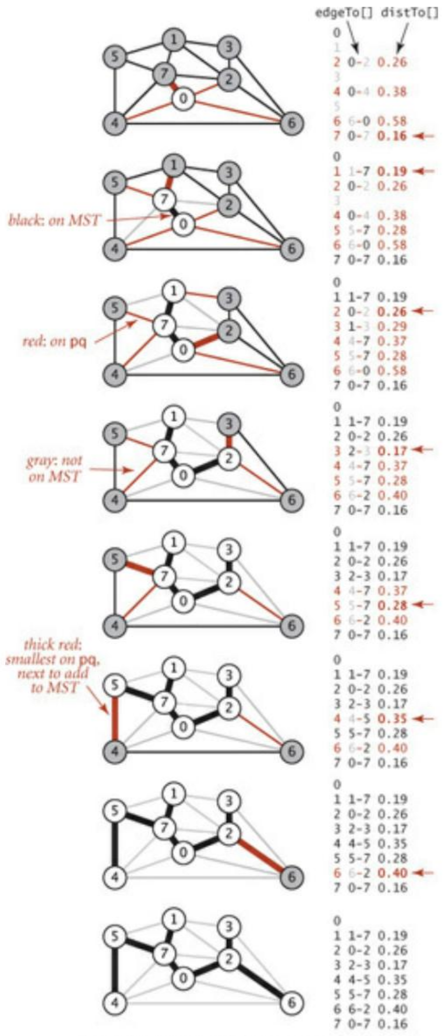
- **marked[]**: an array of booleans to keep track of vertices on the tree
- **edgeTo[]**: an array to keep track of the lightest edge connecting a new vertex to the tree
- **distTo[]**: an array to keep track of the weight of the lightest edge connecting a new vertex to the tree
- **pq**: a minimum priority queue to keep track of eligible crossing edges (key is the weight of the edges)

Algorithm $PrimsMST(G)$
Input: $G = (V, E)$, a weighted, undirected graph
Output: A minimum-weight spanning tree of G

$edgeTo[] :=$ a $|V|-$sized array to store the edge connecting a vertex to the tree
$distTo[] :=$ a $|V|-$sized array to keep track of the distance of the edge connecting a vertex to a tree
$marked[] :=$ a $|V|-$sized array to keep track of which vertices have been visited
$pq :=$ a heap-based min priority queue with weights as keys and vertices as values

//initialize structures
for all $v \in V$:
  $distTo[v] = \infty$
$distTo[0] = 0$
$pq.insert(0, 0)$

//main loop
while $!pq.isEmpty$:
  $v = pq.delMin()$
  $marked[v] = T$
  for each edge $e = (v, u)$ adjacent to v:
    if $marked[u]$
      continue
    if $e.weight() < distTo[u]$:
      $edgeTo[u] = e$
      $distTo[u] = e.weight()$
      if $pq.contains(u)$:
        $pq.changeKey(u, distTo[u])$
      else:
        $pq.insert(u, distTo[u])$



Trace of Prim's algorithm (eager version)

Picture from [2]

```
//main loop
while !pq.isEmpty:
    v = pq.delMin()
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    for each edge e = (v, u) adjacent to v:
        if marked[u]
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        if e.weight() < distTo[u]:
            edgeTo[u] = e
            distTo[u] = e.weight()
            if pq.contains(u):
                pq.changeKey(u, distTo[u])
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```

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O(|V|)

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                pq.changeKey(u, distTo[u])
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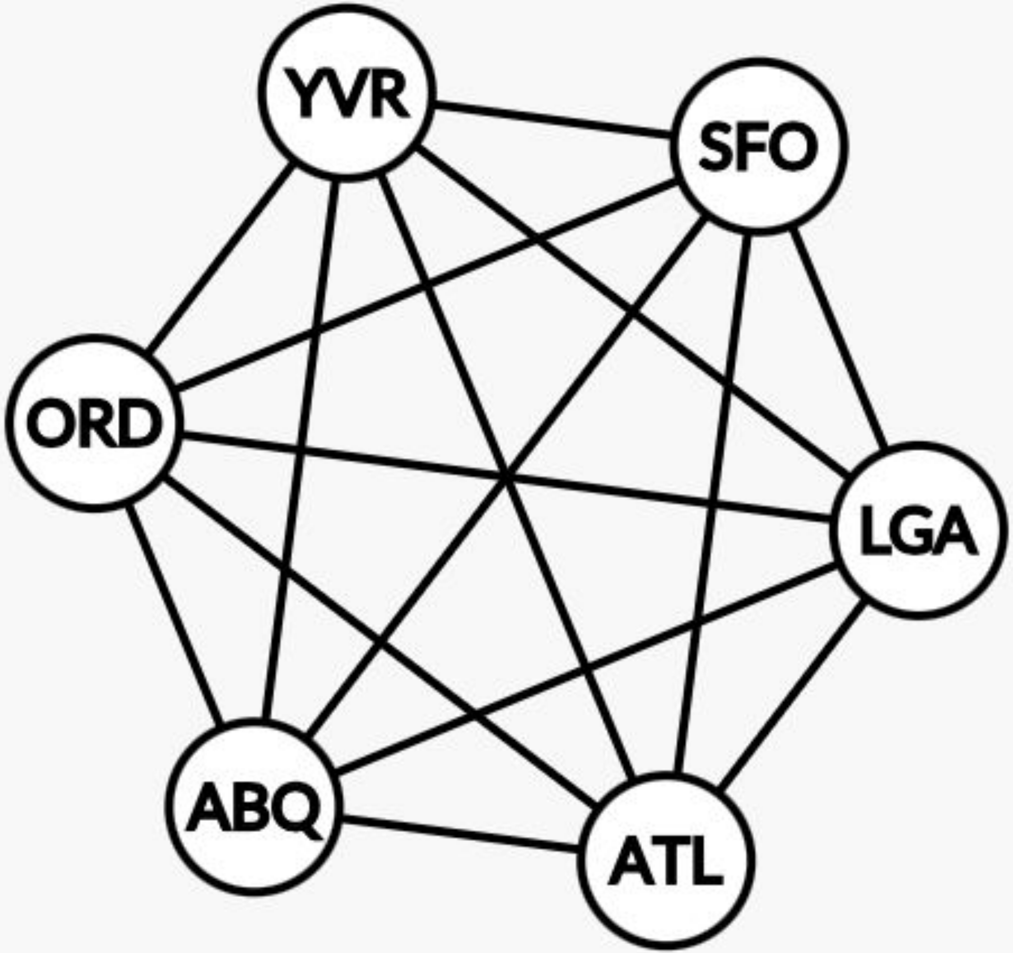
```

Assuming adjacency list representation, total time is **O((V + E)logV) = O(ElogV)** for a connected graph

- delMin in a heap-based PQ: O(logV)
- checking if PQ contains a vertex and changing a key is O(logV) as long as there is an auxiliary data structure keeping track of positions in the queue
- key of any vertex *v* is updated at most *deg(v)* times and sum of all degrees is O(E)


YVR
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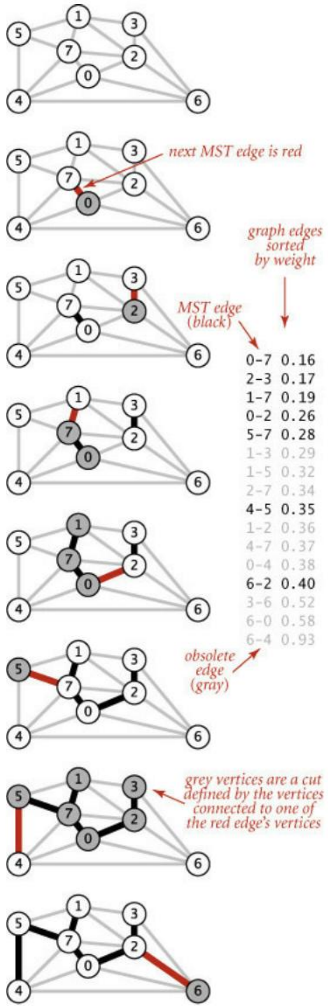
Kruskal’s Algorithm

- Process edges, adding edges to the tree smallest-weight first **as long as the new edge does not form a cycle.**
- The result is that the algorithm creates a forest that eventually merges into a single tree.
- What would be some of the challenges in implementing this?
- What data structures would be useful?
- How would you represent the graph?

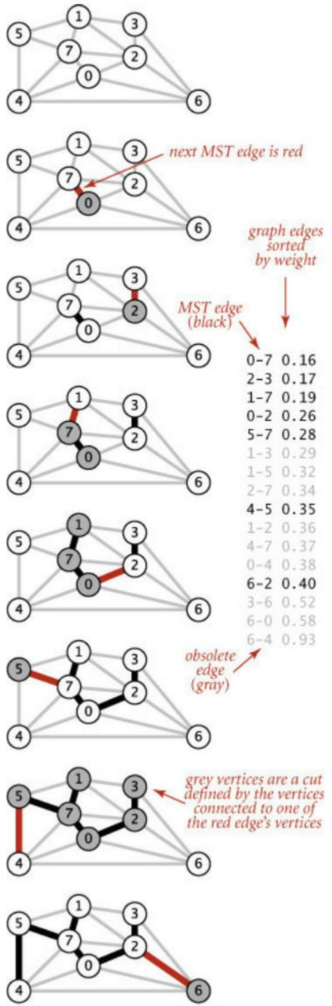


Trace of Kruskal's algorithm

Picture from [2]

Kruskal's Algorithm

- Can be implemented using disjoint sets for each separate component
- Start with |V| disjoint sets, each containing a single vertex
- Combine sets (union) by adding edges, reducing the number of separate components until there is only one



Trace of Kruskal's algorithm

Picture from [2]

Algorithm $KruskalsMST(G)$
Input: $G = (V, E)$, a connected, weighted, undirected graph
Output: A minimum-weight spanning tree of G

$pq :=$ a minimum priority queue of edges where keys are weights
for each edge $e = (u, v) \in E$:
    $pq.insert(e, e.weight())$

$A = \emptyset$
for each $v \in V$:
    $makeSet(v)$
while $|A| < |V| - 1$
    $e = (u, v) = pq.delMin()$
    if $findSet(u) \neq findSet(v)$:
        $A = A \cup (u, v)$
        $union(u, v)$
return A

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- **O(ElogE)** for doing **E** inserts into the **E**-sized PQ
- Alternatively, we could just sort the edges in **O(ElogE)** time

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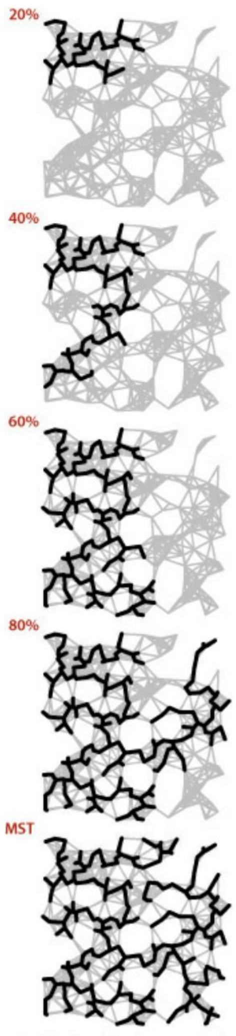
- *makeSet* is an O(1) operation, so the total time is **O(V)**
- The total time required for **E** *union* and *findSet* operations is **O(ElogV)**

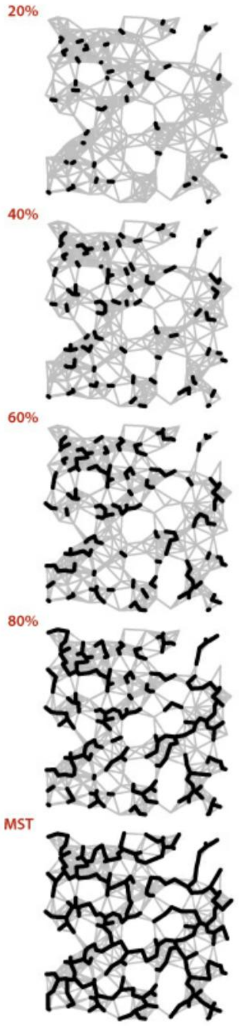
- **Total time: O(ElogE + ElogV)=O(ElogE)**



Prim's algorithm (250 vertices)

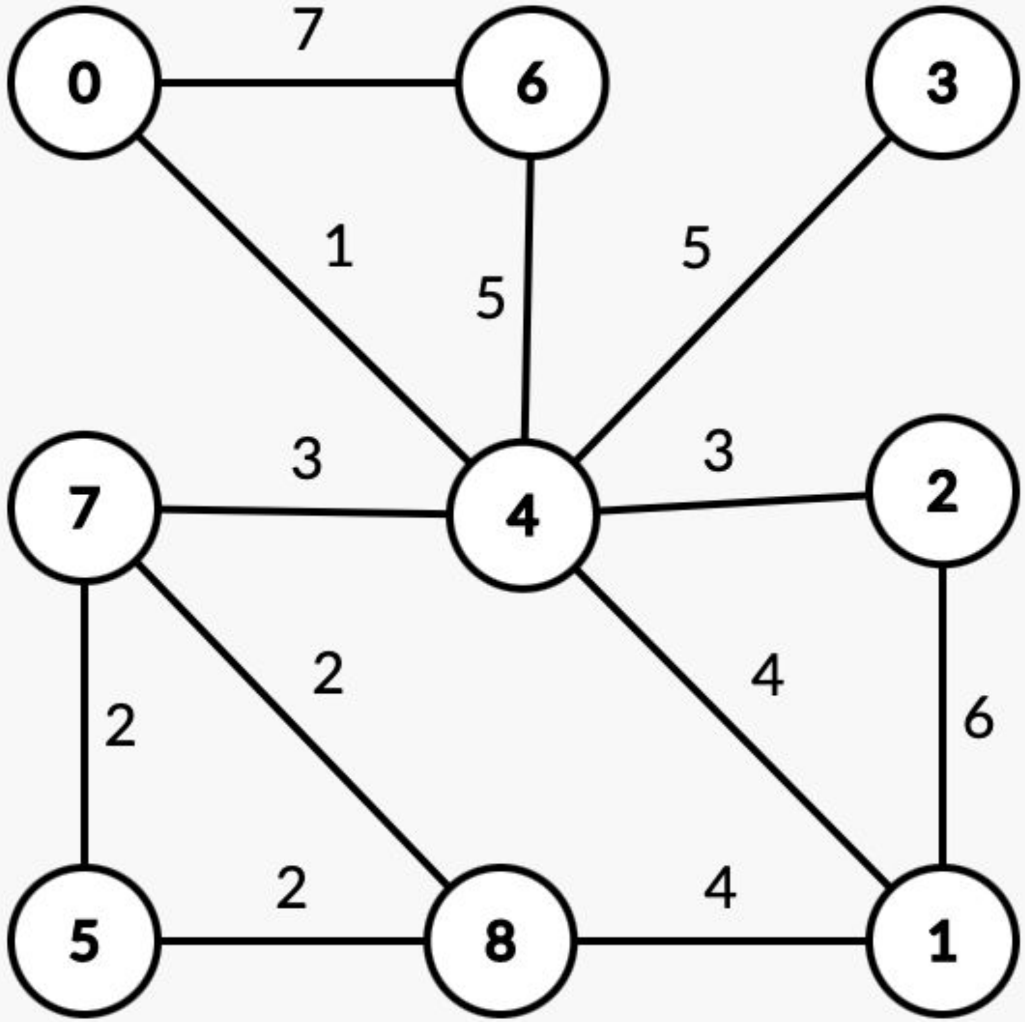
Prim's vs. Kruskal's

- Space: |**V**| vs. |**E**|
- Time: |**E**|**log**|**V**| vs. |**E**|**log**|**E**|



Kruskal's algorithm (250 vertices)

Pictures from [2]



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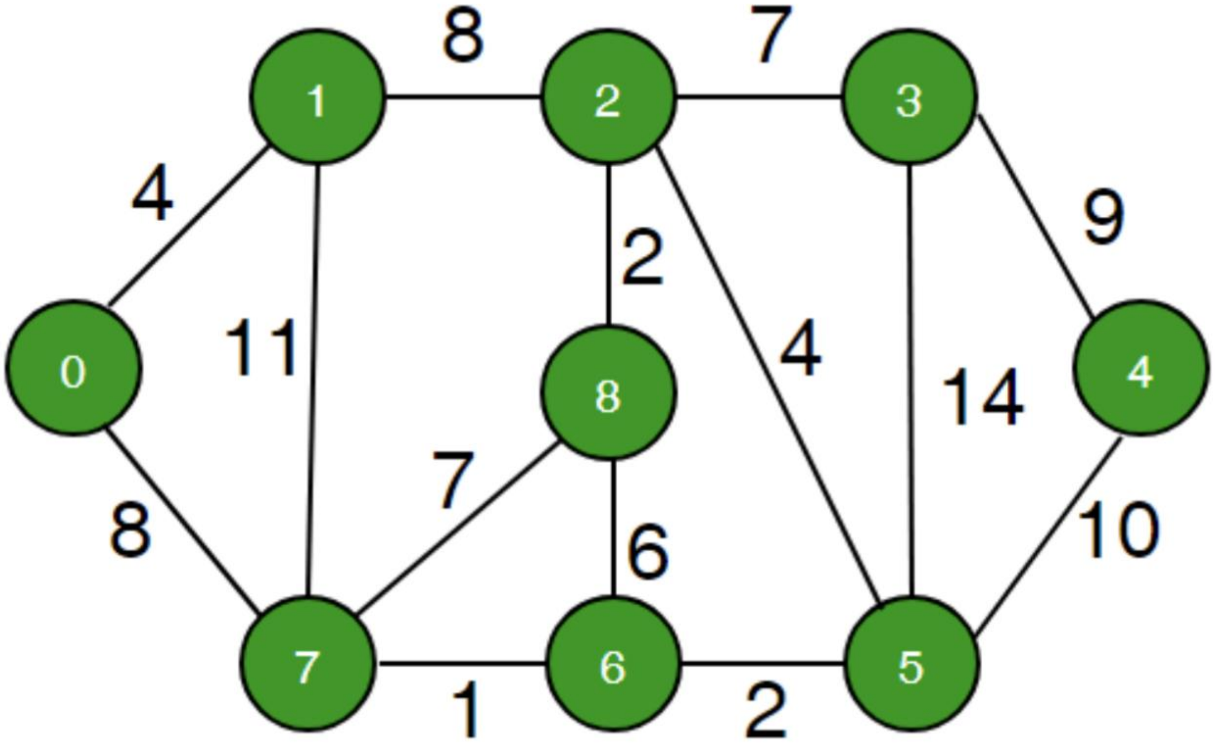
More Examples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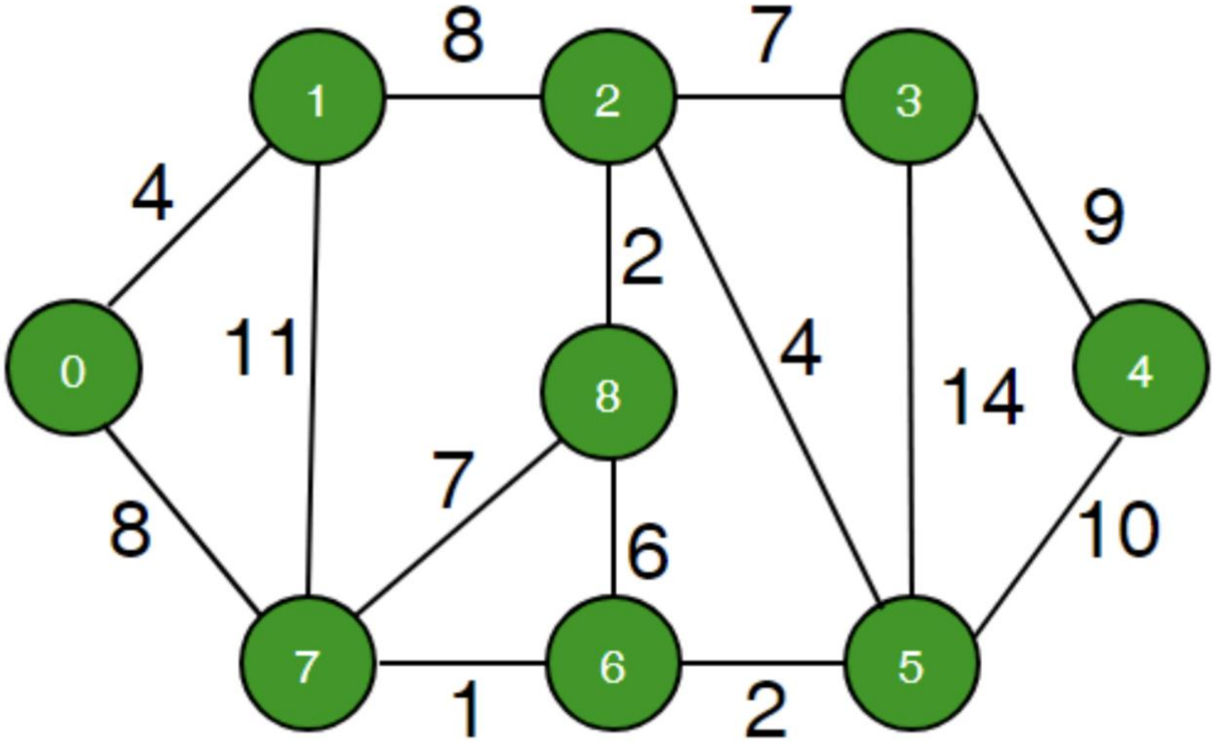

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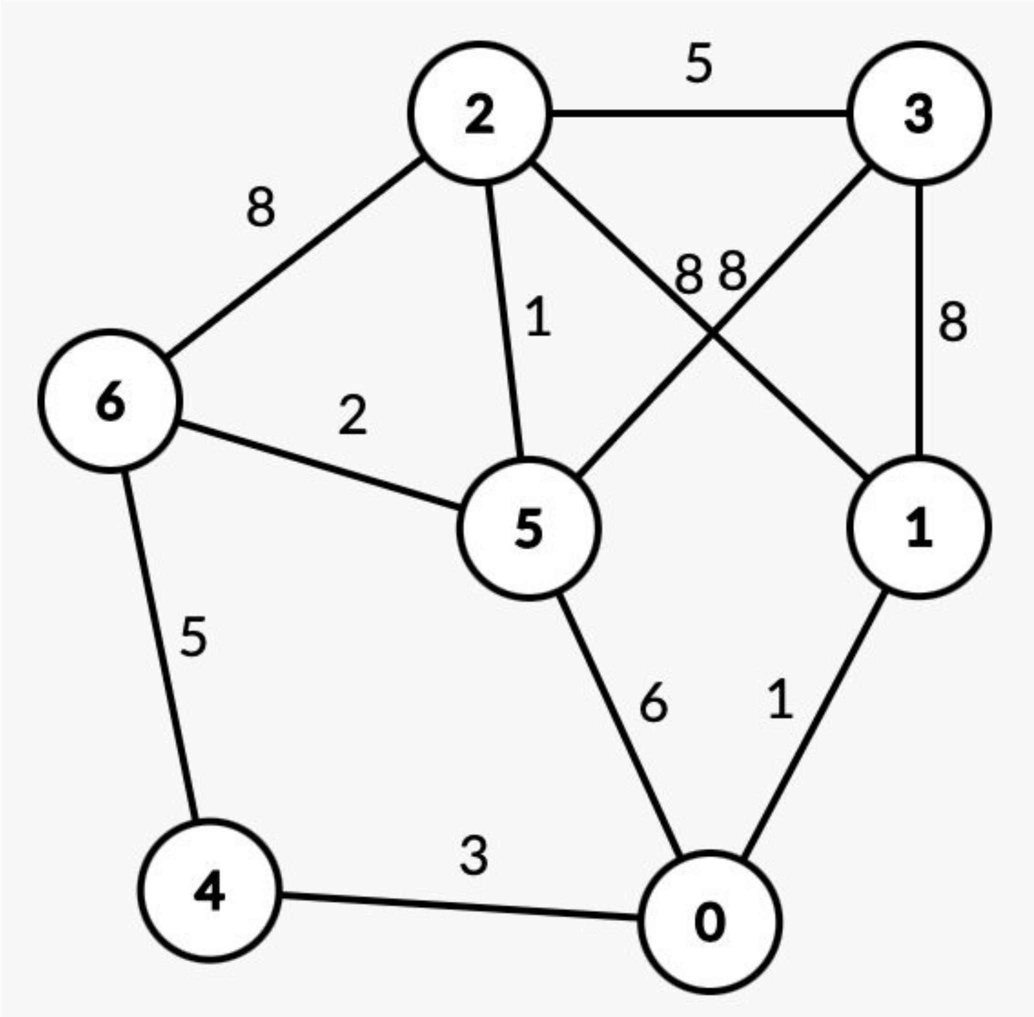
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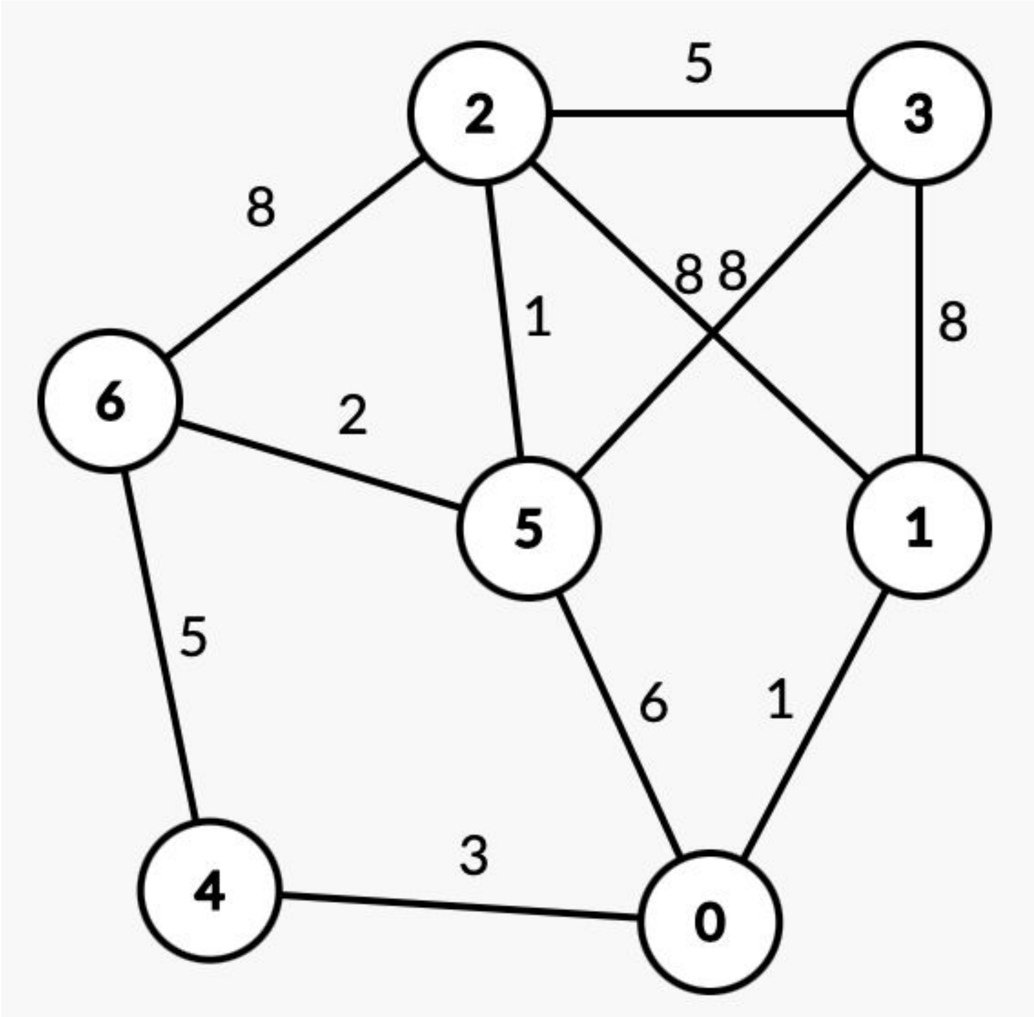
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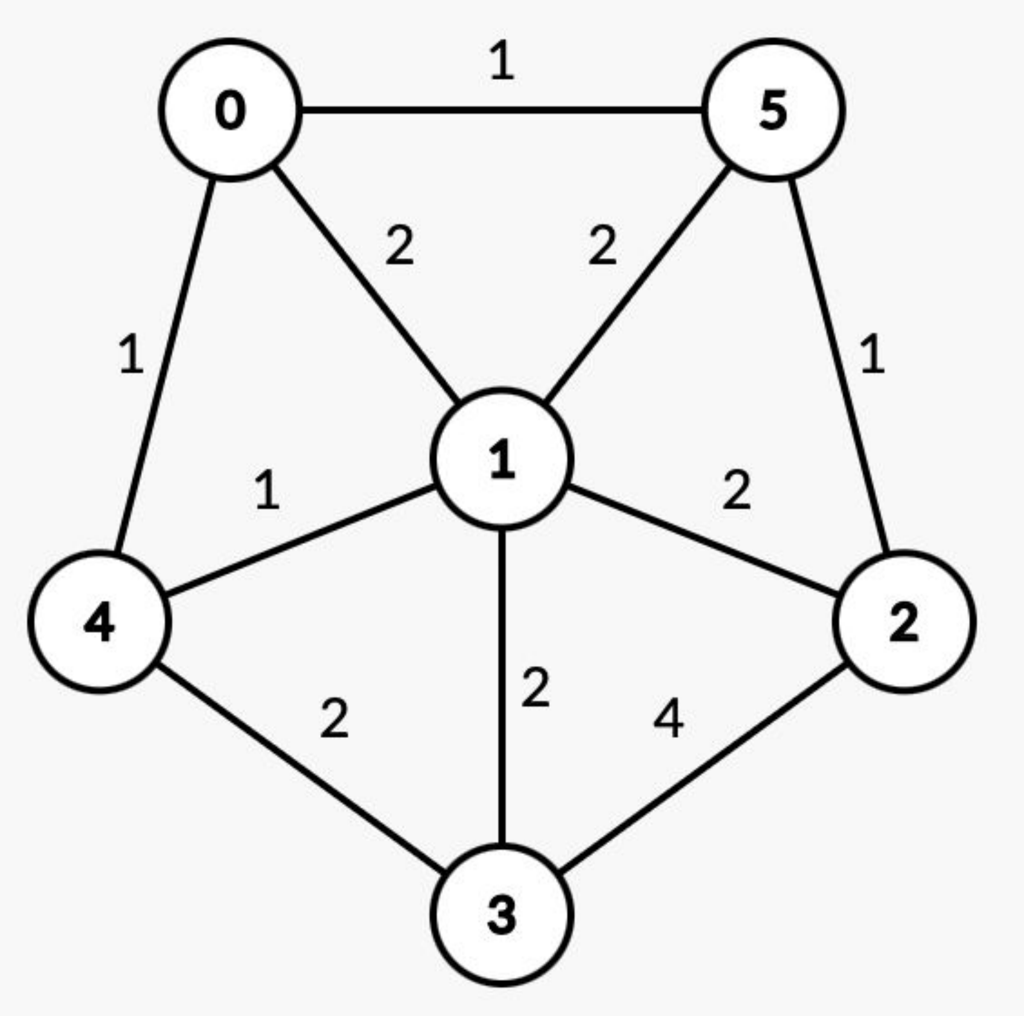
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References

[1] Goodrich and Tamassia
[2] Sedgewick and Wayne
[3] en.wikipedia.org