UNITELMA SAPIENZA

SAPIENZA
UNIVERSITÀ DI ROMA
DIPARTIMENTO DI INFORMATICA

Probabilità

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23. Variabili geometriche

successo S $P(S) = p$ $0 < p \leq 1$

fallimento F $P(F) = 1 - p$

$F\,F\,F\,F \ldots F\,S$

esiti indipendenti

X = tempo di primo successo

$X \in \mathbb{N}$ $X \geq 1$

$$P(X=K)=(1-p)^{K-1}p$$

$K=1,2,3\ldots$

$$\{X=K\}=\underbrace{FF\ldots F}_{K-1}S$$

Variabile aleatoria geometrica

$$P(X=K)\geq 0 \quad \forall K \quad \checkmark \qquad p,\ 1-p\geq 0$$

$$\sum_K P(X=K)=1$$

$$\sum_{k} P(X=k) = \sum_{k=1}^{\infty} p(1-p)^{k-1} =$$

$$p \sum_{k=0}^{\infty} (1-p)^{k} = p \frac{1}{1-(1-p)} = 1$$

$$P(X=\infty)=0$$

$$P(X>k) = (1-p)^{k} \xrightarrow[k\to\infty]{} 0$$

$$P(X \leq k) = 1-(1-p)^{k}$$

b, n estrazioni <u>con</u> reinserimento

$$P(B) = \frac{b}{b+n}\,; \quad P(N) = \frac{n}{b+n}$$

X = numero di estrazioni necessarie per ottenere una palla nera

$$P(X=K) = \left(\frac{b}{b+n}\right)^{K-1} \frac{n}{b+n}$$

$$P(X \geq K) = \left(\frac{b}{b+n}\right)^{K-1}$$

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \qquad |x| < 1$$

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{d}{dx}\,\frac{1}{1-x} = \frac{1}{(1-x)^2}$$

$$\| \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$$

$$\sum_{n=0}^{\infty} \frac{d}{dx} x^n = \sum_{n=1}^{\infty} \frac{d}{dx} x^n = \sum_{n=1}^{\infty} n\, x^{n-1}$$

$$\sum_{n=2}^{\infty} n(n-1)\, x^{n-2} = \frac{2}{(1-x)^3}$$

$$X \sim \mathcal{G}(p) \qquad P(X=k) = p(1-p)^{k-1}$$

$$E(X) = \sum_{k=1}^{\infty} k\,P(X=k) = \sum_{k=1}^{\infty} k p (1-p)^{k-1}$$

$$= \frac{p}{(1-(1-p))^2} = \frac{p}{p^2} = \frac{1}{p}$$

$$\boxed{E(X) = \frac{1}{p}}$$

$$Var(X) = E(X^2) - E(X)^2 =$$

$$E(X^2) - E(X) + E(X) - E(X)^2 =$$

$$E(X(X-1)) + E(X) - E(X)^2$$

$$E[X(X-1)) = \sum_{k=1}^{\infty} k(k-1)P(X=k)$$

$$= \sum_{k=2}^{\infty} k(k-1)P(X=k)$$

$$= \sum_{k=2}^{\infty} k(k-1)\, p(1-p)^{k-1} = p(1-p) \sum_{k=2}^{\infty} k(k-1)(1-p)^{k-2}$$

$$= p(1-p) \frac{2}{(1-(1-p))^3} = \frac{2p(1-p)}{p^3} = \frac{2(1-p)}{p^2}$$

$$Var(X) = E(X(X-1)) + E(X) - E(X)^2$$

$$= \frac{2(1-p)}{p^2} + \frac{1}{p} - \frac{1}{p^2} = \frac{2(1-p)+p-1}{p^2}$$

$$= \frac{1-p}{p^2}$$

coppia di dadi ; T il tempo necessario per ottenere almeno 11.

$$E(T) = ?$$

$$P(X \geq 11) = P(X = 11) + P(X = 12) =$$

$$\frac{2}{36} + \frac{1}{36} = \frac{3}{36} = \frac{1}{12}$$

$$E(T) = 12$$

A, B vince il primo che ottiene $\underline{1}$

6 4

X = tempo di primo succ. per A

Y = tempo di primo succ. per B

$\{X > Y\}$ = vince B

$\{X < Y\}$ = vince A

$\{X = Y\}$ = pareggio

$$P(\text{vince } A) + P(\text{vince } B) + P(\text{pareggio}) = 1$$

$$X \sim \mathcal{G}\left(\frac{1}{6}\right) \; ; \quad Y \sim \mathcal{G}\left(\frac{1}{4}\right)$$

$$P(X = k) = \frac{1}{6}\left(\frac{5}{6}\right)^{k-1} \; ; \quad P(Y = h) = \frac{1}{4}\left(\frac{3}{4}\right)^{h-1}$$

$$X \perp\!\!\!\perp Y$$

$$P(X = Y) = ? \qquad \{X = Y\} = \bigcup_{k=1}^{\infty} \{X = k \cap Y = k\}$$

$$P(X=Y) = \sum_{k=1}^{\infty} P(X=k \wedge Y=k) =$$

$$= \sum_{k=1}^{\infty} P(X=k)\,P(Y=k) = \sum_{k=1}^{\infty} \frac{1}{6}\left(\frac{5}{6}\right)^{k-1} \cdot \frac{1}{4} \cdot \left(\frac{3}{4}\right)^{k-1} =$$

$$\frac{1}{24} \sum_{k=0}^{\infty} \left(\frac{5}{8}\right)^{k} = \frac{1}{24} \cdot \frac{1}{1-\frac{5}{8}} = \frac{1}{24} \cdot \frac{1}{\frac{3}{8}} =$$

$$\frac{1}{24} \cdot \frac{8}{3} = \frac{1}{9}$$

$$P(X>Y)=\sum_{k=1}^{\infty}P(X>Y\mid Y=k)\,P(Y=k)=$$

$$\sum_{k=1}^{\infty}P(X>k\mid Y=k)\,P(Y=k)=$$

$$=\sum_{k=1}^{\infty}P(X>k)\,P(Y=k)=\sum_{k=1}^{\infty}\left(\frac{5}{6}\right)^{k}\frac{1}{4}\left(\frac{3}{4}\right)^{k-1}=$$

$$\frac{1}{4}\cdot\frac{5}{6}\sum_{k=0}^{\infty}\left(\frac{5}{6}\right)^{k}\left(\frac{3}{4}\right)^{k}=\frac{5}{24}\sum_{k=0}^{\infty}\left(\frac{5}{8}\right)^{k}=$$

$$=\frac{5}{24}\cdot\frac{1}{1-\frac{5}{8}}=\frac{5}{24}\cdot\frac{8}{3}=\frac{5}{9}$$

$$P(X > Y) = \frac{5}{9} \; ; \quad P(X = Y) = \frac{1}{9}$$

$$\Rightarrow \quad P(X < Y) = \frac{1}{3}$$

Mancanza di memoria

$$X \sim \mathcal{G}(p)$$

$$P(X > n+k \mid X > k) = \frac{P(X > n+k \wedge X > k)}{P(X > k)}$$

$$= \frac{P(X > n+k)}{P(X > k)} = \frac{(1-p)^{n+k}}{(1-p)^k} = (1-p)^n = P(X > n)$$

$$P(X > n+k \mid X > k) = P(X > n)$$

Distribuzione geometrica è l'<u>unica</u> distr. discreta che gode della proprietà della mancanza di memoria.

IL CASO NON HA MEMORIA

tempo necessario per ottenere r successi X

$P(X=K) = P($ r-esimo successo si verifica al tempo $K)$

$p = P($successo in una prova$)$

$K-1$ prove $\quad r-1$ successi $\quad K-r$ insuccessi

$$p^{r-1}(1-p)^{K-r}\binom{K-1}{r-1}$$

$$P(X=K) = \binom{K-1}{r-1} p^r (1-p)^{K-r}$$

binomiale negativa.

è la somma di r v.a. geometrice (tutte di parametro p) indipendenti

$$E(X) = \frac{r}{p} \qquad E(X) = \frac{r(1-p)}{p^2}.$$