Probabilità

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24. Variabili di Poisson

X Poisson $P(X=k)=e^{-\lambda}\frac{\lambda^k}{k!}$

$\lambda>0 \qquad k=0,1,2,3\ldots\ldots$

$$\sum_{k=0}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!} \overset{?}{=} 1 \qquad \sum_{n=0}^{\infty}\frac{x^k}{k!}=e^x$$

$$\parallel$$

$$e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}=e^{-\lambda}e^{\lambda}=1$$

Poisson 1837

1898 Bortkevic

morti da calcio di cavallo

1) prob. molto bassa del singolo evento

2) numero molto alto di eventi potenziali

3) indipendenza

incidenti = X

$$E(X) = ? \; ; \quad X = \sum_{i=1}^{N} Y_i$$

$$P(Y_i = 1) = \text{molto piccola} = \frac{\lambda}{N}$$

$$E(X) = \sum_{i=1}^{N} E(Y_i) = N \cdot \frac{\lambda}{N} = \lambda$$

$$X \sim B\left(N; \frac{\lambda}{N}\right)$$

$$P(X = k) = \binom{N}{k}\left(\frac{\lambda}{N}\right)^{k}\left(1 - \frac{\lambda}{N}\right)^{N-k}$$

$$\lim_{N \to \infty} P_N(X = k)$$

$$\binom{N}{k} = \frac{N!}{k!(N-k)!} = \frac{N(N-1)\ldots(N-k+1)}{k!}$$

$$\frac{N(N-1)\cdots(N-K+1)}{K!}\,\frac{\lambda^K}{N^K}\left(1-\frac{\lambda}{N}\right)^N \underbrace{\left(1-\frac{\lambda}{N}\right)^{-K}}_{\to 1} =$$

$$\frac{\lambda^K}{K!}\,\underbrace{\frac{N(N-1)\cdots(N-K+1)}{N^K}}_{\to 1}\,\underbrace{\left(1-\frac{\lambda}{N}\right)^N}_{\to e^{-\lambda}}$$

$$\boxed{\left(1+\frac{x}{n}\right)^n \to e^x}$$

$$P_N(X=K) \xrightarrow[N\to\infty]{} e^{-\lambda}\frac{\lambda^K}{K!}$$

Poisson ~ Binomiale

Applicazioni: assicurazioni

incidenti automobilistici

$$e^{-\lambda} \frac{\lambda^k}{k!} \longrightarrow 0 \quad \text{molto velocemente}$$

$$k \longrightarrow \infty$$

$$X \sim \mathcal{P}(\lambda) \qquad \lambda \in \mathbb{R}^+$$

$$E(X) = \sum_{k=0}^{\infty} k e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=1}^{\infty} k \frac{\lambda^k}{k!} =$$

$$= e^{-\lambda} \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda \underbrace{e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}_{=1}$$

$$= \lambda$$

$$Var(X) = E(X^2) - E(X)^2$$

$$E(X^2) = \sum_{k=0}^{\infty} k^2 e^{-\lambda} \frac{\lambda^k}{k!} = \lambda^2 + \lambda$$

$$Var(X) = \lambda$$

Se X è di Poisson $\Longrightarrow E(X) = Var(X)$

$X \sim \mathcal{P}(\lambda)$, $Y \sim \mathcal{P}(\mu)$

X, Y indipendenti

$\Longrightarrow X + Y \sim \mathcal{P}(\lambda+\mu)$

$Z = X + Y$

$$\{Z = k\} = \bigcup_{j=0}^{k} \{X = j \cap Y = k-j\}$$

$$P(Z=k) = \sum_{j=0}^{k} P(X=j)P(Y=k-j) =$$

$$= \sum_{j=0}^{k} e^{-\lambda} \frac{\lambda^j}{j!} e^{-\mu} \frac{\mu^{k-j}}{(k-j)!} =$$

$$\frac{e^{-(\lambda+\mu)}}{k!} \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \lambda^j \mu^{k-j} =$$

$$\frac{e^{-(\lambda+\mu)}}{k!} (\lambda+\mu)^k \Rightarrow Z \sim \mathcal{P}(\lambda+\mu)$$

$X \sim \mathcal{P}(\lambda)$

$X = Y_1 + Y_2 + Y_3$

Y_1, Y_2, Y_3 indip. $\quad Y_i \sim \mathcal{P}\left(\frac{\lambda}{3}\right)$

eventi che si verificano in un intervallo di tempo

$X = \#$ eventi in un giorno e $X \sim \mathcal{P}(\lambda)$

$\Rightarrow$ # eventi in una sett. $\sim \mathcal{P}(7\lambda)$

eventi in un'ora $\sim \mathcal{P}\left(\frac{\lambda}{24}\right)$

eventi in 5 ore $\sim \mathcal{P}\left(\frac{5}{24}\lambda\right)$