UNITELMA SAPIENZA
SAPIENZA
UNIVERSITÀ DI ROMA
DIPARTIMENTO DI INFORMATICA

Metodi matematici per l'Informatica

Modulo 16.2 – Logica Predicativa (parte II: metodo dei tableau)

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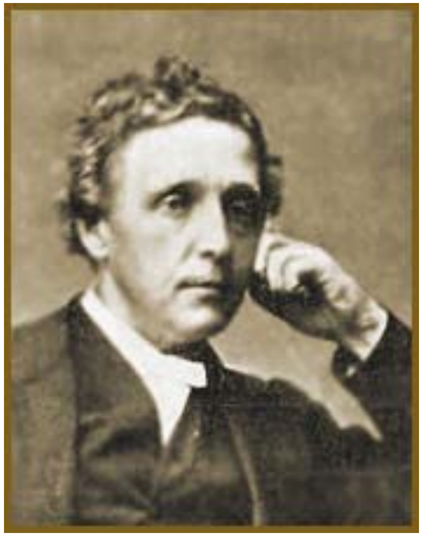
$$\frac{U(S) \qquad \dfrac{\forall x \,.\, U(x) \rightarrow M(x)}{U(S) \rightarrow M(S)}}{M(S)}$$

Aristotele (384 – 322 a.C.)

Tutti gli uomini sono mortali.
Socrate è un uomo.
Socrate è mortale.

Οργανον



Lewis Carroll (1832 - 1898)

Aristotele (384 – 322 a.C.)

Tutti i leoni sono creature feroci.
Qualche leone non beve caffè.
Qualche creatura feroce non beve caffè.

Symbolic Logic

Tutti gli uomini sono mortali.
Socrate è un uomo.
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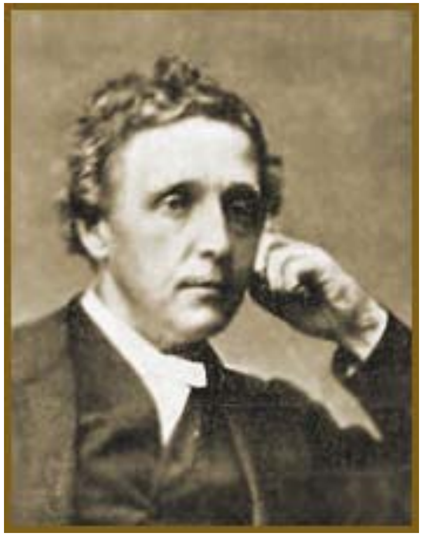
Οργανον



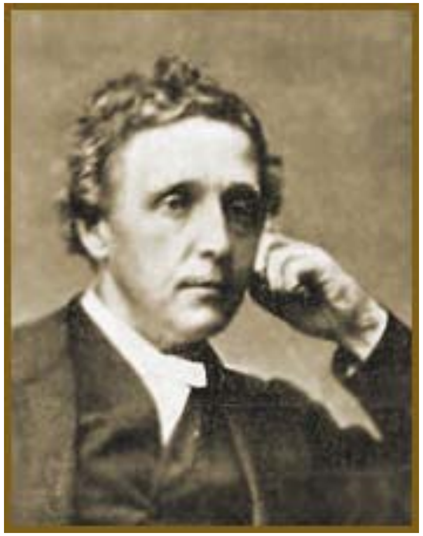
Lewis Carroll (1832 - 1898)

$$\frac{\exists x\,.\,L(x) \land \neg C(x)}{\dfrac{L(a) \land \neg C(a)}{\neg C(a) \qquad L(a)}} \qquad \frac{\forall x\,.\,L(x) \rightarrow F(x)}{L(a) \rightarrow F(a)}$$

$$\frac{L(a) \qquad L(a) \rightarrow F(a)}{F(a)}$$

Tutti i leoni sono creature feroci.
Qualche leone non beve caffè.
Qualche creatura feroce non beve caffè.

Symbolic Logic



Lewis Carroll (1832 - 1898)

$$\frac{\exists x \,.\, L(x) \wedge \neg C(x)}{L(a) \wedge \neg\, C(a)} \qquad \frac{\forall x \,.\, L(x) \rightarrow F(x)}{L(a) \rightarrow F(a)}$$

che differenza c'è ?

Tutti i leoni sono creature feroci.
Qualche leone non beve caffè.
Qualche creatura feroce non beve caffè.

Symbolic Logic

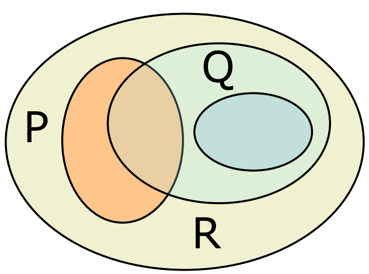
$$\dfrac{\dfrac{\exists x \,.\, P(x)}{P(a)} \qquad \dfrac{\exists x \,.\, Q(x)}{Q(a)}}{P(a) \wedge Q(a)}$$

☹

$$\dfrac{\exists x \,.\, L(x) \wedge \neg C(x)}{L(a) \wedge \neg C(a)} \qquad \dfrac{\forall x \,.\, L(x) \to F(x)}{L(a) \to F(a)}$$

che differenza c'è ?

n :



$Q_n(x)$ = *x è un uomo*
$R_n(x)$ = *x è mortale*
$P_n(x)$ = *x ha almeno 100 arti*

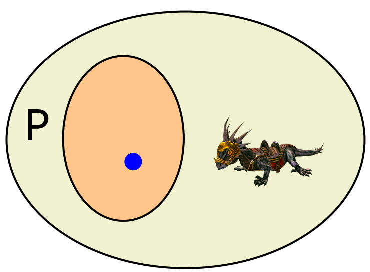
$\not\models_n \exists x.\, P(x) \wedge Q(x)$

esiste un uomo con almeno 100 arti

$$\frac{\forall x\,.\,P(x)}{P(a)} \qquad \frac{\exists x\,.\,Q(x)}{Q(b)} \qquad\qquad \frac{\exists x\,.\,P(x)}{P(f\cancel{(b)})}$$ ☹

a qualunque b *nuovo!*

n :



$f(x) =$ 

$P_n(x)$ = x *ha almeno 100 arti*

esiste un essere con almeno 100 arti

per semplicità eliminiamo le funzioni dal linguaggio

$$\frac{\forall x \,.\, A}{[a/x]\, A} \qquad \frac{\exists x \,.\, A}{[b/x]\, A} \qquad \frac{\exists x \,.\, (P(x) \wedge R(x, y))}{P(b) \wedge R(b, y)}$$

a qualunque b *nuovo!*

data una qualunque formula A, denotiamo con [b/x]A la formula ottenuta sostituendo b ad ogni occorrenza *libera* di x in A

termini $t_1, t_2, \ldots ::= x \mid y \mid z \mid \ldots \mid a \mid b \mid c \mid \ldots$

formule $A, B, \ldots ::= P\,(t_1, t_2, \ldots, t_n) \mid \ldots \mid \mathit{falso} \mid A \vee B$
$\mid A \wedge B \mid A \rightarrow B \mid \neg A \mid \forall x.\, A \mid \exists x.\, A$

$$\frac{\forall x \,.\, A}{[a/x]\, A} \qquad \frac{\exists x \,.\, A}{[b/x]\, A} \qquad \frac{\exists x \,.\, (P(x) \wedge R(x, y))}{P(b) \wedge R(b, y)}$$

a qualunque b *nuovo!*

data una qualunque formula A, denotiamo con [b/x]A la formula ottenuta sostituendo b ad ogni occorrenza *libera* di x in A

$$[b/x]\, (P(x) \wedge R(x, y)) = P(b) \wedge R(b, y)$$

$$[b/x]\, (P(z) \wedge R(z, y)) = P(z) \wedge R(z, y)$$

$$[b/x]\, (P(x) \wedge \forall y \,.\, R(x, y)) = P(b) \wedge \forall y \,.\, R(b, y)$$

$$[b/x]\, (P(x) \wedge \forall x \,.\, R(x, y)) = P(b) \wedge \forall x \,.\, R(x, y)$$

Tableau predicativi

(take one)

$\forall x\ .\ A$
|
$[a/x]\ A$

a qualunque

$\exists x\ .\ A$
|
$[b/x]\ A$

b *nuovo*

$\neg\exists x\ .\ A$

?

$\forall$ & $\exists$

(semantica)

$[\![\, \forall x.\ A \,]\!]_{\rho} = T$ se, *per ogni* $v \in U$, $[\![\, A \,]\!]_{\rho[x \mapsto v]} = T$

$= F$ *altrimenti*

$[\![\, \exists x.\ A \,]\!]_{\rho} = T$ se, *esiste* $v \in U$, $[\![\, A \,]\!]_{\rho[x \mapsto v]} = T$

$= F$ *altrimenti*

∀ & ∃

(semantica)

$[\![\forall x.\ A]\!]_\rho = T$ se, *per ogni* $v \in U$, $[\![A]\!]_{\rho[x \mapsto v]} = T$
$= F$ *altrimenti*

$[\![\forall x.\ A]\!]_\rho = T$ se, *per ogni* $v \in U$, $[\![A]\!]_{\rho[x \mapsto v]} = T$
$= F$ se *esiste* $v \in U$ t.c. $[\![A]\!]_{\rho[x \mapsto v]} = F$

$[\![\forall x.\ \neg A]\!]_\rho = T$ se, *per ogni* $v \in U$, $[\![\neg A]\!]_{\rho[x \mapsto v]} = T$
$= F$ se *esiste* $v \in U$ t.c. $[\![\neg A]\!]_{\rho[x \mapsto v]} = F$

$[\![\forall x.\ \neg A]\!]_\rho = T$ se, *per ogni* $v \in U$, $[\![A]\!]_{\rho[x \mapsto v]} = F$
$= F$ se *esiste* $v \in U$ t.c. $[\![A]\!]_{\rho[x \mapsto v]} = T$

$\forall$ & $\exists$

(semantica)

$$
\begin{aligned}
[\![\neg\forall x.\ \neg A]\!]_\rho &= \neg\, [\![\forall x.\ \neg A]\!]_\rho \\
&= F \ \text{ se, } \textit{per ogni } v \in U,\ [\![A]\!]_{\rho[x\mapsto v]} = F \\
&= T \ \text{ se } \textit{esiste } v \in U \text{ t.c. } [\![A]\!]_{\rho[x\mapsto v]} = T \\
&= [\![\exists x.\ A]\!]_\rho
\end{aligned}
$$

$$
\begin{aligned}
[\![\forall x.\ \neg A]\!]_\rho &= T \ \text{ se, } \textit{per ogni } v \in U,\ [\![A]\!]_{\rho[x\mapsto v]} = F \\
&= F \ \text{ se } \textit{esiste } v \in U \text{ t.c. } [\![A]\!]_{\rho[x\mapsto v]} = T
\end{aligned}
$$

$\forall$ & $\exists$

(semantica)

$$
\begin{aligned}
[\![\neg\forall x.\ \neg A]\!]_\rho &= \neg\,[\![\forall x.\ \neg A]\!]_\rho \\
&= F \ \text{ se, } \textit{per ogni } v \in U,\ [\![A]\!]_{\rho[x\mapsto v]} = F \\
&= T \ \text{ se } \textit{esiste } v \in U \text{ t.c. } [\![A]\!]_{\rho[x\mapsto v]} = T \\
&= [\![\exists x.\ A]\!]_\rho
\end{aligned}
$$

per ogni modello m: $\vDash_m \neg\forall x.\ \neg A$ se e solo se $\vDash_m \exists x.\ A$

$\exists x.\ A \equiv \neg\forall x.\ \neg A$ (*equivalenza semantica*)

$\forall x.\ A \equiv \neg\neg\forall x.\ \neg\neg A \equiv \neg\exists x.\ \neg A$

Tableau predicativi

(take one)

$$\begin{array}{c} \forall x \,.\, A \\ | \\ [a/x]\, A \end{array} \qquad \begin{array}{c} \neg\exists x \,.\, A \\ | \\ [a/x]\, \neg A \end{array}$$

a qualunque

$$\begin{array}{c} \exists x \,.\, A \\ | \\ [b/x]\, A \end{array} \qquad \begin{array}{c} \neg\forall x \,.\, A \\ | \\ [b/x]\, \neg A \end{array}$$

b *nuovo*

$$\begin{array}{c} A \wedge B \\ | \\ A,\ B \end{array} \qquad \begin{array}{c} A \vee B \\ / \quad \backslash \\ A \qquad B \end{array} \qquad \ldots$$

Tableau predicativi

(take one)

$\forall x \,.\, A$
|
$[a/x]\, A$

a qualunque

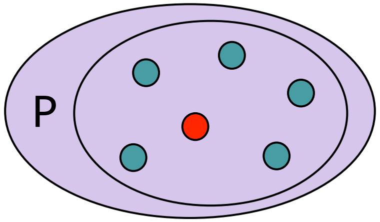
$\neg\exists x \,.\, A$
|
$[a/x]\, \neg A$

$\neg((\forall x \,.\, P(x)) \rightarrow (\exists x \,.\, P(x)))$ ☺

$\exists x \,.\, A$
|
$[b/x]\, A$

b *nuovo*

$\neg\forall x \,.\, A$
|
$[b/x]\, \neg A$



$A \wedge B$
|
A, B

$A \vee B$
/ \
A B

...

Tableau predicativi

(take one)

$\forall x \,.\, A$
$|$
$[a/x]\, A$

$\neg\exists x \,.\, A$
$|$
$[a/x]\, \neg A$

a qualunque

$\exists x \,.\, A$
$|$
$[b/x]\, A$

$\neg\forall x \,.\, A$
$|$
$[b/x]\, \neg A$

b *nuovo*

$A \wedge B$
$|$
A, B

$A \vee B$
$/ \quad \backslash$
$A \quad\quad B$

...

$\neg((\forall x \,.\, P(x)) \rightarrow (\exists x \,.\, P(x)))$
$|$
$\forall x.\, P(x)\,,\, \neg\exists x.\, P(x)$
$|$
$P(a)\,,\, \neg\exists x.\, P(x)$
$|$
$P(a)\,,\, \neg P(a)$

$\diamond$

Tableau predicativi

(take one)

$$\dfrac{\forall x\,.\,A}{[a/x]\,A} \qquad \dfrac{\neg\exists x\,.\,A}{[a/x]\,\neg A}$$

a qualunque

$$\dfrac{\exists x\,.\,A}{[b/x]\,A} \qquad \dfrac{\neg\forall x\,.\,A}{[b/x]\,\neg A}$$

b *nuovo*

$$\dfrac{A \wedge B}{A,\ B} \qquad \dfrac{A \vee B}{A \quad\mid\quad B} \qquad \ldots$$

$$\neg((\exists x.\ \neg P(x)) \vee (\forall x.\ P(x)))$$
$$\neg\exists x.\ \neg P(x)\ ,\ \neg\forall x.\ P(x)$$
$$\neg\neg P(a)\ ,\ \neg\forall x.\ P(x)$$
$$\neg\neg P(a)\ ,\ \neg P(b)$$

☹

Tableau predicativi

(take one)

$$\forall x\,.\,A \;\Big|\; [a/x]\,A \qquad \neg\exists x\,.\,A \;\Big|\; [a/x]\,\neg A$$

a qualunque

$$\exists x\,.\,A \;\Big|\; [b/x]\,A \qquad \neg\forall x\,.\,A \;\Big|\; [b/x]\,\neg A$$

b *nuovo*

$$A \wedge B \;\Big|\; A, B \qquad A \vee B \;\Big/\; A \;\Big\backslash\; B \qquad \ldots$$

$$\neg((\exists x.\, \neg P(x)) \vee (\forall x.\, P(x)))$$
$$|$$
$$\neg\exists x.\, \neg P(x)\ ,\ \neg\forall x.\, P(x)$$
$$|$$
$$\neg\exists x.\, \neg P(x)\ ,\ \neg P(a)$$
$$|$$
$$\neg\neg P(a)\ ,\ \neg P(a)$$
$$\diamond$$

Tableau predicativi

(take one)

$$\begin{array}{c} \forall x\,.\,A \\ | \\ [a/x]\,A \end{array} \qquad \begin{array}{c} \neg\exists x\,.\,A \\ | \\ [a/x]\,\neg A \end{array}$$

a qualunque

$$\begin{array}{c} \exists x\,.\,A \\ | \\ [b/x]\,A \end{array} \qquad \begin{array}{c} \neg\forall x\,.\,A \\ | \\ [b/x]\,\neg A \end{array}$$

b *nuovo*

$$\begin{array}{c} A \wedge B \\ | \\ A,\ B \end{array} \qquad \begin{array}{ccc} & A \vee B & \\ \swarrow & & \searrow \\ A & & B \end{array} \qquad \ldots$$

$$\begin{array}{c} \neg(\neg\forall x.\ [P(x) \wedge (\exists y.\ \neg P(y)]) \\ | \\ \forall x.\ [P(x) \wedge \exists y.\ \neg P(y)] \\ | \\ P(a) \wedge \exists y.\ \neg P(y) \\ | \\ P(a)\ ,\ \exists y.\ \neg P(y) \\ | \\ P(a)\ ,\ \neg P(b) \end{array}$$

☹

Tableau predicativi

(take one)

$$\forall x\,.\,A$$
$$|$$
$$[a/x]\,A\,,\ \forall x\,.\,A$$

a qualunque

$$\neg(\neg\forall x.\ [P(x) \wedge (\exists y.\ \neg P(y))])$$
$$|$$
$$\forall x.\ [P(x) \wedge \exists y.\ \neg P(y)]$$
$$|$$
$$P(a) \wedge \exists y.\ \neg P(y),\ \forall x.\ [P(x) \wedge \exists y.\ \neg P(y)]$$
$$|$$
$$P(a)\ ,\ \exists y.\ \neg P(y),\ \forall x.\ [P(x) \wedge \exists y.\ \neg P(y)]$$
$$|$$
$$P(a)\ ,\ \neg P(b)\ ,\ \forall x.\ [P(x) \wedge \exists y.\ \neg P(y)]$$
$$|$$
$$\Diamond \quad P(a)\ ,\ \neg P(b)\ ,\ P(b) \wedge \exists y.\ \neg P(y)]$$

Tableau predicativi

α

$A \wedge B$	$\neg(A \to B)$	$\neg\neg A$	$\neg(A \vee B)$	$A \leftrightarrow B$
A, B	$A, \neg B$	A	$\neg A, \neg B$	$A \to B, B \to A$

β

$A \vee B$	$A \to B$	$\neg(A \wedge B)$	…	$\neg(A \leftrightarrow B)$
A | B	$\neg A$ | B	$\neg A$ | $\neg B$		$\neg(A \to B), \neg(B \to A)$

γ

$\forall x.\ A$	$\neg\exists x.\ A$
$\forall x.\ A, [a/x]A$	$\neg\exists x.\ A, [a/x]\neg A$

a qualunque

δ

$\exists x.\ A$	$\neg\forall x.\ A$
$[a/x]A$	$[a/x]\neg A$

a nuovo

Tableau predicativi

correttezza e completezza

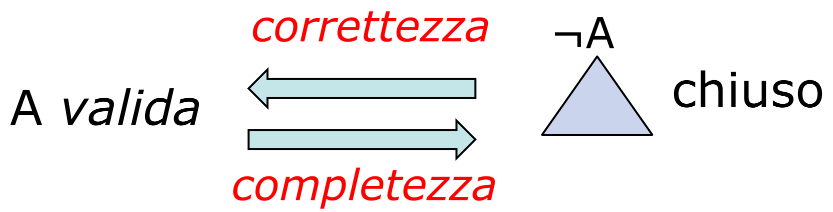


Tableau predicativi

correttezza e completezza

Le seguenti formule sono valide:

$\forall x.\ P(x) \leftrightarrow \neg\exists x.\ \neg P(x)$

$\exists x.\ P(x) \leftrightarrow \neg\forall x.\ \neg P(x)$

$\exists x.\forall y.\ P(x,y) \rightarrow \forall y.\exists x.\ P(x,y)$

$\forall x.\ (P(x) \wedge Q(x)) \leftrightarrow (\forall x.\ P(x) \wedge \forall x.\ Q(x))$

$\forall x.\ (P(x) \vee Q(x)) \leftarrow (\forall x.\ P(x) \vee \forall x.\ Q(x))$

$\exists x.\ (P(x) \vee Q(x)) \leftrightarrow (\exists x.\ P(x) \vee \exists x.\ Q(x))$

$\exists x.\ (P(x) \wedge Q(x)) \rightarrow (\exists x.\ P(x) \wedge \exists x.\ Q(x))$

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Tableau predicativi

correttezza e completezza

$$\neg(\exists x.\forall y.\, P(x,y) \rightarrow \forall y.\exists x.\, P(x,y))$$

$$\exists x.\forall y.\, P(x,y)\ ,\ \neg\forall y.\exists x.\, P(x,y)$$

$$\forall y.\, P(a,y)\ ,\ \neg\forall y.\exists x.\, P(x,y)$$

$$\forall y.\, P(a,y)\ ,\ \neg\exists x.\, P(x,b)$$

$$P(a,b)\ ,\ \neg\exists x.\, P(x,b)$$

$$P(a,b)\ ,\ \neg P(a,b)$$

$$\Diamond$$

◊

That's all, Folks!