# Chapter 1

# Introduction to Functional Equations

**Learning Objectives**

In this chapter, you will learn to:

- Recall the definitions of function, domain, codomain, injective, surjective, and bijective.
- Understand what a functional equation is and how it differs from algebraic equations.
- Solve basic functional equations and check solutions correctly.
- Apply classical results and use function properties (injectivity, surjectivity, bijectivity, Cauchy's Functional Equations).
- Work through examples and Olympiad-style problems involving functional equations.

## 1.1 Introduction

### What is a Function?

A **function** is a rule that assigns to each element of a set (called the *domain*) exactly one element of another set (called the *codomain*).

We usually write this as

$$f : A \to B,$$

meaning the function $f$ takes inputs from the set $A$ and produces outputs in the set $B$. For each $x \in A$, the output $f(x)$ is a single element of $B$.

**Examples**

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$. The domain and codomain are the real numbers. Each real number $x$ is mapped to its square.
2. $g : \mathbb{Z} \to \mathbb{Z},\ g(n) = n + 1$. The domain and codomain are integers. Each integer is

mapped to the next integer.

3. $h : \mathbb{N} \to \{0, 1\}$, $h(n) = n \bmod 2$. This function assigns 0 to even numbers and 1 to odd numbers.

**Remarks**

- A function can take many different forms: linear (e.g. $f(x) = 2x$), constant (e.g. $f(x) = 5$), quadratic, trigonometric, or even more exotic.
- In Olympiad problems, functions are sometimes not given by a definition or formula. Instead, they are defined by an *equation they must satisfy*. Solving such equations is the subject of **functional equations**.

## 1.2 Injective, Surjective, and Bijective Functions

In many functional equation problems, it is useful to know whether a function is *injective*, *surjective*, or *bijective*. These properties describe how the outputs of the function relate to the inputs.

- **Injective (one-to-one):** A function $f : A \to B$ is injective if each element in the domain of the function has a unique image in the co-domain. A useful consequence of a function $f$ being injective is that:

$$f(x_1) = f(x_2) \implies x_1 = x_2.$$

- **Surjective (onto):** A function $f : A \to B$ is surjective if for every $b \in B$, there exists some $a \in A$ such that $f(a) = b$. This means the function covers the entire codomain.
- **Bijective:** A function is bijective if it is both injective and surjective. This means every element of $B$ is hit exactly once by some element of $A$.

**Examples**

1. $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x$. This is injective (different inputs give different outputs) and surjective (every real number has a preimage), hence bijective.
2. $g : \mathbb{Z} \to \mathbb{Z}$, $g(n) = n^2$. This is not injective (since $g(2) = g(-2)$), and not surjective (negative numbers are never outputs).
3. $h : \mathbb{N} \to \mathbb{N}$, $h(n) = n + 1$. This is injective, but not surjective because 1 has no preimage.

**Exercise**

For each of the following functions, determine whether it is injective, surjective, bijective, or none of the above. Give a short justification for your answer.

1. $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^3$.
2. $g : \mathbb{R} \to \mathbb{R},\ g(x) = |x|$.
3. $h : \mathbb{N} \to \mathbb{N},\ h(n) = n + 5$.
4. $p : \mathbb{R} \to \mathbb{R}_{\geq 0},\ p(x) = x^2$.
5. $q : \mathbb{Z} \to \mathbb{Z},\ q(n) = n - 1$.

## 1.3 What is a Functional Equation?

A **functional equation** is an equation in which the unknown is not a number, but a *function*. The goal is to determine all functions $f$ (with a given domain and codomain) that satisfy the equation for all possible inputs.

This contrasts with ordinary algebraic equations: instead of solving for a value of $x$, we are solving for the entire function $f$.

More often than not, the answer or solution set for a functional equation will be quite straightforward to guess. So, the most difficult part of functional equations is usually proving that the solutions you assert are the **only** ones that satisfy the equation.

### Basic Examples

1. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$f(x+1) = f(x) + 1 \quad \text{for all } x \in \mathbb{Z}.$$

*Solution Ideas:* This condition suggests linearity as the difference between two consecutive values of the function are constant. Indeed, the solutions are

$$f(x) = x + c, \quad \text{for some constant } c \in \mathbb{Z}.$$

2. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x+y) = f(x) + f(y).$$

This is called **Cauchy's Additive Functional Equation** (Think about what *Additive* and *Multiplicative* mean). A consequence of such a property is that $\boxed{f(kx) = kf(x) \forall k \in \mathbb{Q}}$ Over the rationals, the only solutions are

$$f(x) = kx, \quad k \in \mathbb{Q}.$$

(Consider what happens over the Reals. We will discuss this in a later section!)

**Key Insight**

One of the first steps in solving a functional equation is to *guess possible solutions*. Common candidates include

$$f(x) = 0, \quad f(x) = x, \quad f(x) = kx, \quad f(x) = kx + c,$$

and sometimes $f(x) = \pm x$.
Checking these quickly can often reveal the right family of solutions or guide the next steps. Even if the guess is not complete, it gives valuable hints about the structure of the function.

**Example.** Consider the equation

$$f(xy) = f(x) + f(y), \quad f : \mathbb{R}^+ \to \mathbb{R}.$$

Trying $f(x) = \log x$ works immediately, and you can now focus on ruling out other solutions, or finding some others in the process.

## 1.4 Useful Substitutions

The core of solving functional equations, easy or hard, lies in substituting the correct values. Using the fact that the problem statement holds for **ALL** values within which the function is defined is a very powerful idea. With some common and intuitive substitutions, one can gain a lot of information about a function, sometimes enough to complete a problem entirely.

### What does a substitution look like?

Essentially, we are substituting x or y (or whatever variables our functional equation is defined on) for other variables or constants that exist within the functions domain. While writing solutions to functional equations, an optional but seriously recommended step is to define an **assertion**. Consider the following functional equation:

$$f(x^2) - f(y^2) = f(x-y)f(x+y)$$

If we define our assertion $P$ as:

$$P(x, y)$$

Then, instead of writing "setting $x = x_1$ and $y = y_1$," One can simply write:

$$P(x_1, y_1) \implies f(x_1^2) - f(y_1^2) = f(x_1 - y_1)f(x_1 + y_1)$$

.

### Common Substitutions

When approaching a new functional equation, some standard substitutions often lead to quick progress:

- Set $x = 0$, $y = 0$, or both. Often forces $f(0) = 0$ or reveals constants.
- Substitute $x = 1$ or $y = 1$ to reduce the equation to a recurrence-like form.
- **Equal variables:** $(x, x)$, etc. Useful for collapsing symmetric expressions.
- **Opposite variables:** $x = -y$. Helps uncover even/odd properties or cancel terms.
- **Other constants:** Plugging values like $x = 2$, $x = -1$, or small integers to detect patterns.
- Substitute $x = f(y)$ or $y = f(x)$ when the equation involves nested functions.
- **Symmetry forcing:** If the equation is not symmetric, test $(x, y)$ vs. $(y, x)$.
- Supposing $f(a) = b$ for constants $a, b$. For example, supposing $f(0) = c$ for simplicity.

A clever substitution should aim to achieve one of the following (not limited to these!):

1. Reduce the equation to a simpler form by eliminating terms or constants. Example: Setting $x = 0$ in $f(x + y) = f(x) + f(y)$ immediately gives $f(0) = 0$.
2. Expose structural traits of the function such as additivity, parity (even/odd), or injectivity/surjectivity. Example: Setting $x = -y$ in $f(x + y) = f(x) + f(y)$ shows that $f(-x) = -f(x)$, hence $f$ is odd.
3. Guide the problem toward the form of a potential solution (linear, constant, quadratic, etc.). Example: $f(x + 1) = f(x) + 1$ suggests that $f(x)$ grows linearly with $x$ if $f : \mathbb{Z} \to \mathbb{Z}$.

**Example 1**

The following problem illustrates the usefulness of exploiting symmetry using simple substitutions:
Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x) + x\,f(1 - x) = x \quad \text{for all } x \in \mathbb{R}.$$

**Solution.** Let our assertion be $P(x)$. Whenever we are substituting values in our solution, there must be some motivation behind why we are doing so. For some $a$, consider $P(a)$ and $P(1 - a)$ respectively:

$$f(a) + a\,f(1 - a) = a,$$

$$f(1 - a) + (1 - a)\,f(a) = 1 - a.$$

Here we notice that they share a common variable, $f(a)$, and we can reduce our problem to the form $f(x) = \textit{something}$ (This is useful tool!). Solving these two equations for $f(a)$ in terms of $a$ gives:

$$f(a) = \frac{a^2}{1 - a + a^2}.$$

**Verification:**

$$
\begin{aligned}
f(x) + x\,f(1-x) &= \frac{x^2}{1-x+x^2} + \frac{x(1-x)^2}{1-(1-x)+(1-x)^2} \\
&= \frac{x-x^2+x^3}{1-x+x^2} \\
&= x.
\end{aligned}
$$

Hence the only solution is

$$\boxed{f(x) = \frac{x^2}{1-x+x^2}}.$$

**Example 2**

The following problem illustrates the usefulness of exploiting symmetry using simple substitutions:
Find all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(x-y) = f(x) + f(y) - 2xy \quad \text{for all } x \in \mathbb{R}.$$

We claim $f(x) = x^2$. Then

$$f(x-y) = (x-y)^2 = x^2 + y^2 - 2xy = f(x) + f(y) - 2xy,$$

so the claim satisfies the equation.

Let

$$P(x,y): \quad f(x-y) = f(x) + f(y) - 2xy.$$

**From** $P(0,0)$**.**

$$f(0) = f(0) + f(0) - 0 \;\Rightarrow\; f(0) = 0.$$

From $P(a,a)$.

$$f(0) = f(a) + f(a) - 2a^2 \;\Rightarrow\; 0 = 2f(a) - 2a^2 \;\Rightarrow\; f(a) = a^2 \quad (\forall a \in \mathbb{R}).$$

Hence the unique solution is

$$\boxed{f(x) = x^2}.$$

## 1.5 Inverses and Involutions

### Inverses

Let $f\colon X \to Y$ be a function. We say that $f$ has an *inverse* if there exists a function $g\colon Y \to X$ such that

$$g(f(x)) = x \quad \text{for all } x \in X, \qquad f(g(y)) = y \quad \text{for all } y \in Y.$$

Such a function $g$ is unique and is denoted by $f^{-1}$. A function $f$ is invertible (i.e. admits such a $g$) if and only if it is bijective.

- If only $g(f(x)) = x$ holds, then $f$ is injective. (Why?)
- If only $f(g(y)) = y$ holds, then $f$ is surjective. (Why?)

### Involutions

A function $f\colon X \to X$ is called an *involution* if

$$f(f(x)) = x \quad \text{for all } x \in X.$$

Thus an involution is a special case of an invertible function in which $f = f^{-1}$.

**Exercise: Prove that any involution is bijective.**

**Solution.**

*Injectivity.* Suppose $f(a) = f(b)$. Applying $f$ to both sides,

$$f(f(a)) = f(f(b)) \quad \Rightarrow \quad a = b.$$

*Surjectivity.* Let $y \in X$. Take $x = f(y)$. Then

$$f(x) = f(f(y)) = y.$$

Since $f$ is both injective and surjective, it is bijective.

**Strategies for proving, injectivity and surjectivity**

**Strategy for proving injectivity:**

**Example 3**

Example 1 had a "weird" answer that was pretty hard to guess. In Example 3, multiple methods used in functional equations from previous sections are used. (*Hint: Try **correctly** guessing the answer first.*)

Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that:

$$f((f(x))^2 + f(y)) = x\,f(x) + y \qquad (\forall x, y \in \mathbb{R}).$$

*Kyrgyzstan 2012*

**Solution.** Let $P(x, y)$ denote

$$P(x, y) : \quad f((f(x))^2 + f(y)) = x\,f(x) + y.$$

$$P(0, 0) \implies f(f(0)^2 + f(0)) = 0.$$

Let $a = f(0)^2 + f(0)$. Then $f(a) = 0$.

$$P(a, x) \implies f(f(y)) = y.$$

Try and figure out the motivation for the next step:

$$\begin{aligned} P(f(b), y) &\implies f(f(f(b))^2 + f(y)) = f(b)\,f(f(b)) + y \\ &\implies f(b^2 + f(y)) = b\,f(b) + y \qquad (\text{since } f(f(b)) = b). \end{aligned}$$

Also,

$$P(b, y) \implies f((f(b))^2 + f(y)) = b\,f(b) + y.$$

By injectivity,

$$b^2 + f(y) = (f(b))^2 + f(y) \implies (f(b))^2 = b^2 \implies f(b) = \pm b.$$

Is the solution now complete? Not quite. although we know that $f(b) = \pm b$, we don't know whether $f(b) = b$ only or $f(b) = -b$ only for all $b$. For example, $f(b) = |b|$ could still be possible, it is $b$ for $b > 0$ and $-b$ for $b < 0$. This is known as the **Pointwise Trap**.

**Pointwise Trap**

From an FE you sometimes get a *pointwise* trap, e.g.

$$f(f(x)) = x \;\text{ and }\; f(x)^2 = x^2 \;\Rightarrow\; f(x) \in \{x, -x\} \quad (\forall x).$$

The trap is to conclude immediately that either $f(x) = x$ for all $x$ or $f(x) = -x$ for all $x$.
**Why it's wrong:** The choice may depend on $x$. Pointwise information does not

force a global sign.
**A tool for escape:** Pick $a, b$ with $f(a) = a$ and $f(b) = -b$. Plug these into the original FE (or a derived identity) to derive a contradiction. Therefore the sign is constant, yielding the two global solutions.

In example 3, using the technique mentioned above is sufficient:

$$P(a, b) \implies f(a^2 - b) = a^2 + b, \text{ a contradiction.}$$

□

## 1.6 Induction

Induction, like any other genre of question, is a powerful tool that can be used in functional equations, especially when functions are defined within the rational numbers. The following examples will be more of a justification for their usefulness:

**Example 4**

If a function $f : \mathbb{Q} \to \mathbb{Q}$ exists such that:

$$f(x + y) = f(x) + f(y)$$

, then prove that $f(x) = qx$ is a solution for $q \in \mathbb{Q}$

**Proof.**

We write the following observations:

$P(v, w) : f(v + w) = f(v) + f(w)$.

$P(0, 0)$: $f(0) = f(0) + f(0) \implies f(0) = 0$.

$P(v, -v)$: $0 = f(v) + f(-v) \implies f(-v) = -f(v)$.

We now prove the statement for positve integers. By induction on $m \in \mathbb{N}$,

$$f(mv) = mf(v).$$

For negative $m$, write $m = -k$ with $k > 0$:

$$f(mv) = f(-kv) = f((-k)(v)) = (-k)f(v) = mf(v).$$

Thus $f(mv) = mf(v)$ for all $m \in \mathbb{Z}$.

To extend to reciprocals of integers,

Let $n \in \mathbb{N}$. Since $v = n(n^{-1}v)$,

$$f(v) = f\big(n(n^{-1}v)\big) = nf(n^{-1}v) \quad \Rightarrow \quad f(n^{-1}v) = \frac{1}{n}f(v).$$

Finally, for all rational numbers:

Any $q \in \mathbb{Q}$ can be written as $q = \frac{m}{n}$ with $m \in \mathbb{Z}$, $n \in \mathbb{N}$. Then

$$f(qv) = f\Big(\frac{m}{n}v\Big) = f\Big(\frac{1}{n}(mv)\Big) = \frac{1}{n}f(mv) = \frac{1}{n}mf(v) = qf(v).$$

Thus for all $q \in \mathbb{Q}$ and $v \in V$,

$$\boxed{f(qv) = qf(v)}. \qquad \square$$

**Example 5**

Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that:

$$f(1) = 2 \text{ and } f(xy) = f(x).f(y) - f(x+y) + 1 \quad (\forall x, y \in \mathbb{Q}).$$

**Solution**:

**Claim.** We claim that the only solution is $f(x) = x + 1$ for all $x \in \mathbb{Q}$. We will verify this at the end.

**Assertion:** Let

$$P(x, y) : \quad f(xy) = f(x)f(y) - f(x+y) + 1.$$

First, we will prove our claim for the integers.

$P(1, n)$:

$$f(n+1) = f(n) + 1 \quad \text{with} \quad f(1) = 2,$$

hence by induction $f(n) = n + 1$ for every $n \in \mathbb{N}$.

$P(-1, n)$:

$$f(-n) = -f(n-1) + 1 = -\big((n-1)+1\big) + 1 = -n + 1,$$

so $f(z) = z + 1$ for all $z \in \mathbb{Z}$.

$P(0, n)$:

$$f(0) = 1 \cdot n - n = n \quad \Rightarrow \quad f(0) = 1.$$

Next, we will prove our claim for reciprocals of integers

$P(n, \frac{1}{n})$:

$$f(1) = (n+1)f\Big(\frac{1}{n}\Big) - f\Big(n + \frac{1}{n}\Big) + 1.$$

$P(1, m + \frac{1}{n})$:

$$f\Big(m + \frac{1}{n}\Big) = m + f\Big(\frac{1}{n}\Big).$$

Hence, by induction

$$f\Big(\frac{1}{n}\Big) = \frac{1}{n} + 1 \quad \forall n \in \mathbb{N}.$$

Now, we extend our proof to the positive rationals.

$P(m, \frac{1}{n})$:

$$f\Big(\frac{m}{n}\Big) = \frac{m}{n} + 1,$$

so $f(r) = r + 1$ for every positive rational $r$.

Finally, to extend to the negatives:

$P(-1, r)$:

$$f(-r) = -f(r-1) + 1 = -((r-1)+1) + 1 = -r + 1,$$

hence $f(x) = x + 1$ for all $x \in \mathbb{Q}$.

**Verification:** For all $x, y \in \mathbb{Q}$,

$$f(x)f(y) - f(x+y) + 1 = (x+1)(y+1) - (x+y+1) + 1 = xy + 1 = f(xy),$$

so $f(x) = x + 1$ indeed satisfies the equation.

$$\boxed{f(x) = x + 1 \quad \forall x \in \mathbb{Q}}$$

## 1.7 An Application of Even and Odd Functions

**Even and Odd Functions**

A function $f\colon D \to \mathbb{R}$ is called ***even*** if

$$f(-x) = f(x) \quad \text{for all } x \in D,$$

where $D$ is symmetric about 0 (i.e. if $x \in D$ then $-x \in D$), e.g. $f(x) = x^2$.

A function $f\colon D \to \mathbb{R}$ is called ***odd*** if

$$f(-x) = -f(x) \quad \text{for all } x \in D,$$

again assuming $D$ is symmetric about 0, e.g. $f(x) = \sin(x)$

A clear application of even and odd functions is given in the example below:

**Example 6**

Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all real numbers $x$ and $y$. *(USAMO 2002)*

**Solution:** Let

$$P(x, y): \quad f(x^2 - y^2) = x\,f(x) - y\,f(y) \qquad (\forall x, y \in \mathbb{R})$$

be the assertion.

We first make the following observations:

$$P(0,0) \;\Rightarrow\; f(0) = 0. \tag{1.1}$$

$$P(x,0) \;\Rightarrow\; f(x^2) = x\,f(x) \qquad (\forall x). \tag{1}$$

$$P(0,y) \;\Rightarrow\; f(-y^2) = -y\,f(y) \qquad (\forall y). \tag{2}$$

We now claim that $f$ is odd.
Proof:

Let $t \geq 0$ and write $t = s^2$. Then by (1)–(2),

$$f(t) = f(s^2) = s\,f(s), \qquad f(-t) = f(-s^2) = -s\,f(s) = -f(t).$$

If $x < 0$, set $t = -x > 0$. Then $f(-x) = f(t) = -f(-t) = -f(x)$. Hence

$$f(-x) = -f(x) \qquad (\forall x \in \mathbb{R}). \tag{3}$$

We now claim that $f$ is additive.
Using (1) in $P(x, y)$,

$$f(x^2 - y^2) = f(x^2) - f(y^2) \qquad (\forall x, y).$$

Since any $a, b \geq 0$ are squares, we get

$$f(a - b) = f(a) - f(b) \qquad (\forall a, b \geq 0). \tag{4}$$

Replace $b$ by $-b$ in (4) and use (3):

$$f(a + b) = f(a) + f(b) \qquad (\forall a, b \geq 0).$$

For arbitrary $a, b \in \mathbb{R}$, write $a = a_+ - a_-$, $b = b_+ - b_-$ with $a_\pm = \max(\pm a, 0)$, $b_\pm = \max(\pm b, 0)$. Then

$$\begin{aligned} f(a+b) &= f\big((a_+ + b_+) - (a_- + b_-)\big) = f(a_+ + b_+) - f(a_- + b_-) \\ &= \big(f(a_+) + f(b_+)\big) - \big(f(a_-) + f(b_-)\big) = \big(f(a_+) - f(a_-)\big) + \big(f(b_+) - f(b_-)\big) \\ &= f(a) + f(b). \end{aligned}$$

Thus

$$f(a+b) = f(a) + f(b) \qquad (\forall a, b \in \mathbb{R}). \tag{5}$$

Finally, we claim that $f(x) = c\,x \quad (\forall x \in \mathbb{R}), \quad c \in \mathbb{R}$.
Apply (1) with $x = x+1$:

$$f\big((x+1)^2\big) = (x+1)\,f(x+1). \tag{6}$$

By (5),

$$\begin{aligned} f\big((x+1)^2\big) &= f(x^2+2x+1) = f(x^2) + f(2x) + f(1) \\ &= x\,f(x) + 2f(x) + f(1), \end{aligned} \tag{7}$$

using (1) and $f(2x) = 2f(x)$ from (5). Equate (6) and (7) and use $f(x+1) = f(x) + f(1)$ from (5):

$$(x+1)\big(f(x)+f(1)\big) = (x+2)f(x) + f(1) \;\Rightarrow\; f(x) = x\,f(1).$$

Therefore $f(x) = c\,x$ with $c = f(1) \in \mathbb{R}$. A direct check shows these satisfy $P(x, y)$. Hence

$$\boxed{f(x) = c\,x \quad (\forall x \in \mathbb{R}), \quad c \in \mathbb{R}.}$$

## 1.8 Cauchy's Additive Functional Equation

### 1.8.1 Cauchy's Additive Functional Equation in $\mathbb{Q}$

Earlier, in Example 4, we discussed the following problem:

**Example 4**

If a function $f : \mathbb{Q} \to \mathbb{Q}$ exists such that:

$$f(x+y) = f(x) + f(y)$$

, then prove that $f(x) = qx$ is a solution for $q \in \mathbb{Q}$

This is known as **Cauchy's Functional Equation**. Recall that the solution is $f(x) = kx$. Notice that the nature of the proof for this example makes it work only when the domain of the function is $\mathbb{Q}$. We will discuss what happens when the domain is extended to $\mathbb{R}$ in the next section. Here, we will look at an idea of reducing a functional equation to an additive form to use Cauchy's result.

**Example 7**

Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that

$$f(x) + f(t) = f(y) + f(z)$$

for all rational numbers $x < y < z < t$ that form an arithmetic progression. *(USAJMO 2015)*

**Solution:** Let
$$P(x,y,z,t): \quad f(x)+f(t)=f(y)+f(z)$$
whenever $x<y<z<t\in\mathbb{Q}$ form an arithmetic progression.

Let $a\in\mathbb{Q}$ and $d>0$. Then
$$x=a, \quad y=a+d, \quad z=a+2d, \quad t=a+3d.$$
Apply $P(x,y,z,t)$:
$$f(a)+f(a+3d)=f(a+d)+f(a+2d). \tag{1}$$

Similarly, with
$$x=a-d, \quad y=a, \quad z=a+d, \quad t=a+2d,$$
we get
$$f(a-d)+f(a+2d)=f(a)+f(a+d). \tag{2}$$

From (1) and (2),
$$f(a-d)+f(a+3d)=2f(a+d) \quad \text{(by (1) and (2))}.$$
Thus we have for any $a,d\in\mathbb{Q}$:
$$f(a-d)+f(a+3d)=2f(a+d). \tag{3}$$

Now, Let $x=a-d$ and $y=a+3d$. Then
$$\frac{x+y}{2}=a+d.$$
Thus (3) reads
$$f(x)+f(y)=2f\left(\frac{x+y}{2}\right) \qquad (\forall x,y\in\mathbb{Q}). \tag{4}$$

**Our answer will now follow due to a known result that we will also prove below:**

**Jensen's Functional Equation**

We are solving

$$f(x) + f(y) = 2f\left(\frac{x+y}{2}\right) \qquad \text{for all } x, y \in \mathbb{Q}.$$

Rewrite the equation as

$$(f(x) - f(0)) + (f(y) - f(0)) = 2\left(f\left(\frac{x+y}{2}\right) - f(0)\right).$$

Define

$$h(x) = f(x) - f(0) \qquad (\forall x \in \mathbb{Q}),$$

so $h(0) = 0$. Then

$$h(x) + h(y) = 2h\left(\frac{x+y}{2}\right). \tag{1}$$

Setting $(x, y) = (a, 0)$ in (1) gives

$$h(a) = 2h\left(\frac{a}{2}\right) \qquad (\forall a \in \mathbb{Q}).$$

Thus for all $x, y$,

$$h(x) + h(y) = 2h\left(\frac{x+y}{2}\right) = h(x+y).$$

Therefore $h$ is additive:

$$h(x+y) = h(x) + h(y) \qquad (\forall x, y \in \mathbb{Q}). \tag{2}$$

and thus $h(q) = kq$ for all $q \in \mathbb{Q}$, where $k = h(1)$.

Since $f(x) = h(x) + f(0)$, the general solution is

$$\boxed{f(x) = kx + c \qquad (x \in \mathbb{Q}),\ k, c \in \mathbb{Q}}.$$

This family satisfies Jensen's functional equation over $\mathbb{Q}$.

Equation (4) is Jensen's equation on $\mathbb{Q}$:

$$f(x) + f(y) = 2f\left(\frac{x+y}{2}\right),$$

whose general solution over $\mathbb{Q}$ is

$$f(q) = Aq + B.$$

This matches the given conditions through verification and every such linear $f$ satisfies $P(x, y, z, t)$. □

### 1.8.2 Cauchy's Additive Functional Equation in $\mathbb{R}$

#### Introduction

In the previous chapter we solved Cauchy's functional equation over the rationals:

$$f(x+y) = f(x) + f(y) \qquad (\forall x, y \in \mathbb{Q}),$$

and showed all solutions are of the form $f(q) = c\,q$. Over the real numbers, however, additivity alone does not guarantee linearity. In this chapter we introduce additional "tameness" conditions that restore linearity.

#### Definition of Additivity over $\mathbb{R}$

**Definition 2** [Additive Function]**.** A function $f : \mathbb{R} \to \mathbb{R}$ is *additive* if

$$f(x+y) = f(x) + f(y) \qquad \text{for all } x, y \in \mathbb{R}.$$

From this definition it follows immediately that

$$f(0) = 0, \quad f(-x) = -f(x), \quad \text{and} \quad f(qx) = qf(x) \quad (\forall q \in \mathbb{Q}).$$

#### Definitions of Key Conditions

**Definition 3** [Continuity at a Point]**.** A function $f : \mathbb{R} \to \mathbb{R}$ is *continuous at a point* $x_0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$.

**Definition 4** [Boundedness on an Interval]**.** A function $f : \mathbb{R} \to \mathbb{R}$ is *bounded on an interval* $I$ if there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in I$.

**Definition 5** [Monotonicity on an Interval]**.** A function $f : \mathbb{R} \to \mathbb{R}$ is *monotonic* on an interval $I \subseteq \mathbb{R}$ if either

$$x_1 < x_2 \quad \Longrightarrow \quad f(x_1) \leq f(x_2) \quad \text{(non-decreasing)}, \quad \text{or} \quad f(x_1) \geq f(x_2) \quad \text{(non-increasing)}.$$