Trees

Introduction

Agenda

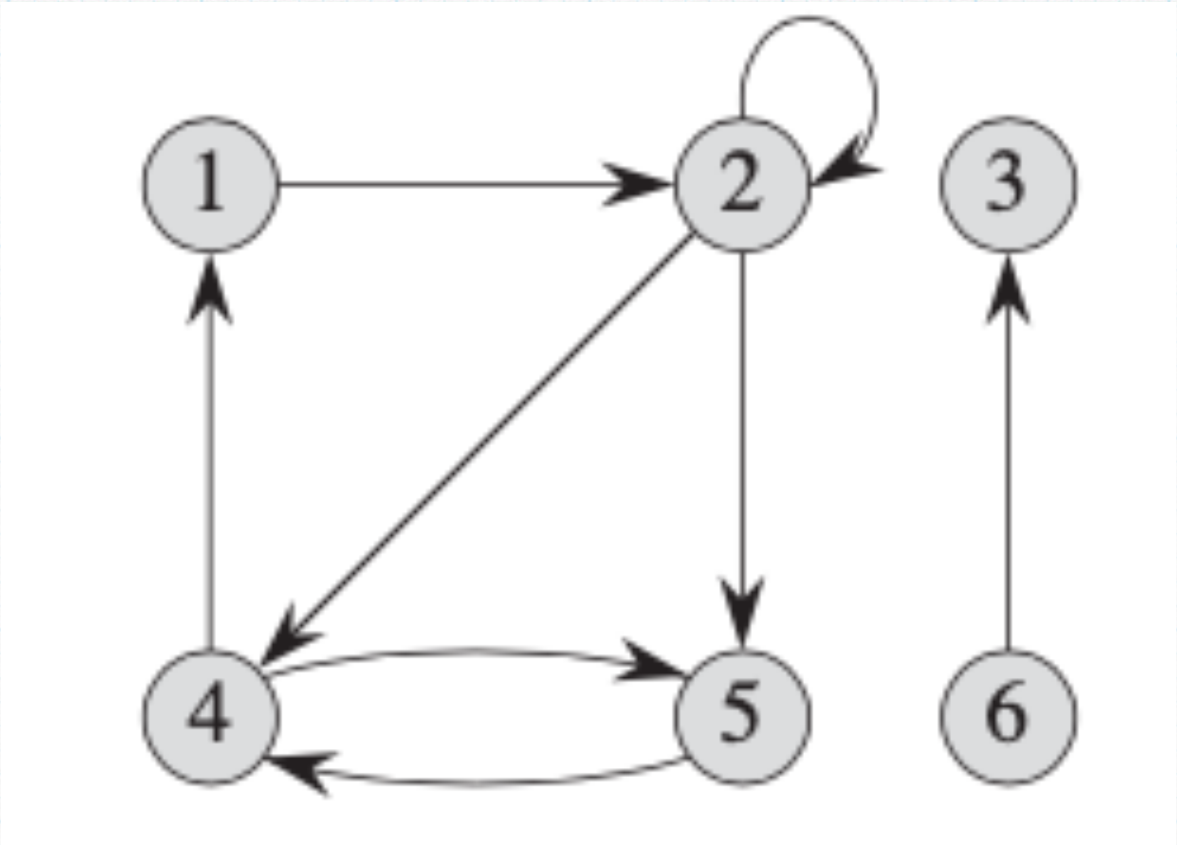
* Definitions - Path, cycle, subgraph, induced subgraph
* Properties of Trees
* Definitions & Examples: Ordered trees, Unordered trees
* Full Binary trees

Free Trees

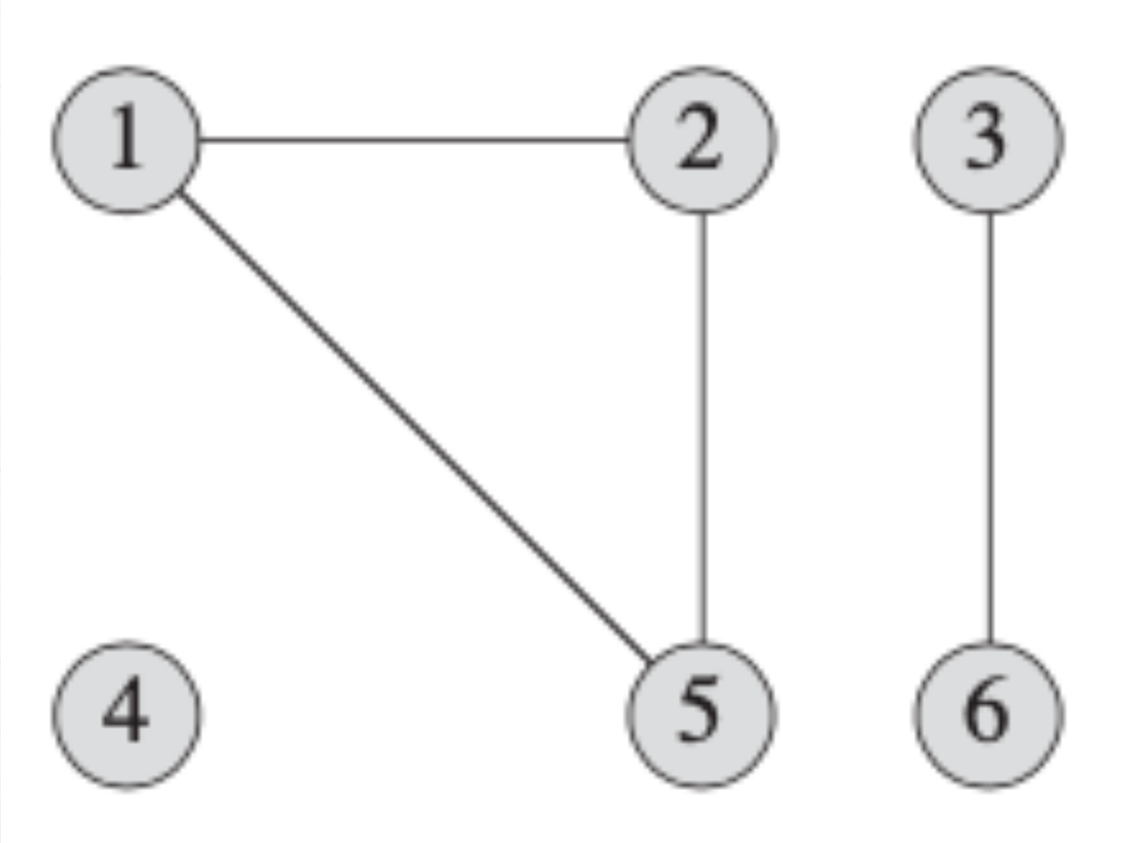
* A Free tree is a connected, ______, _______ graph

Graphs

* Two kinds of Graphs
* ________Graph

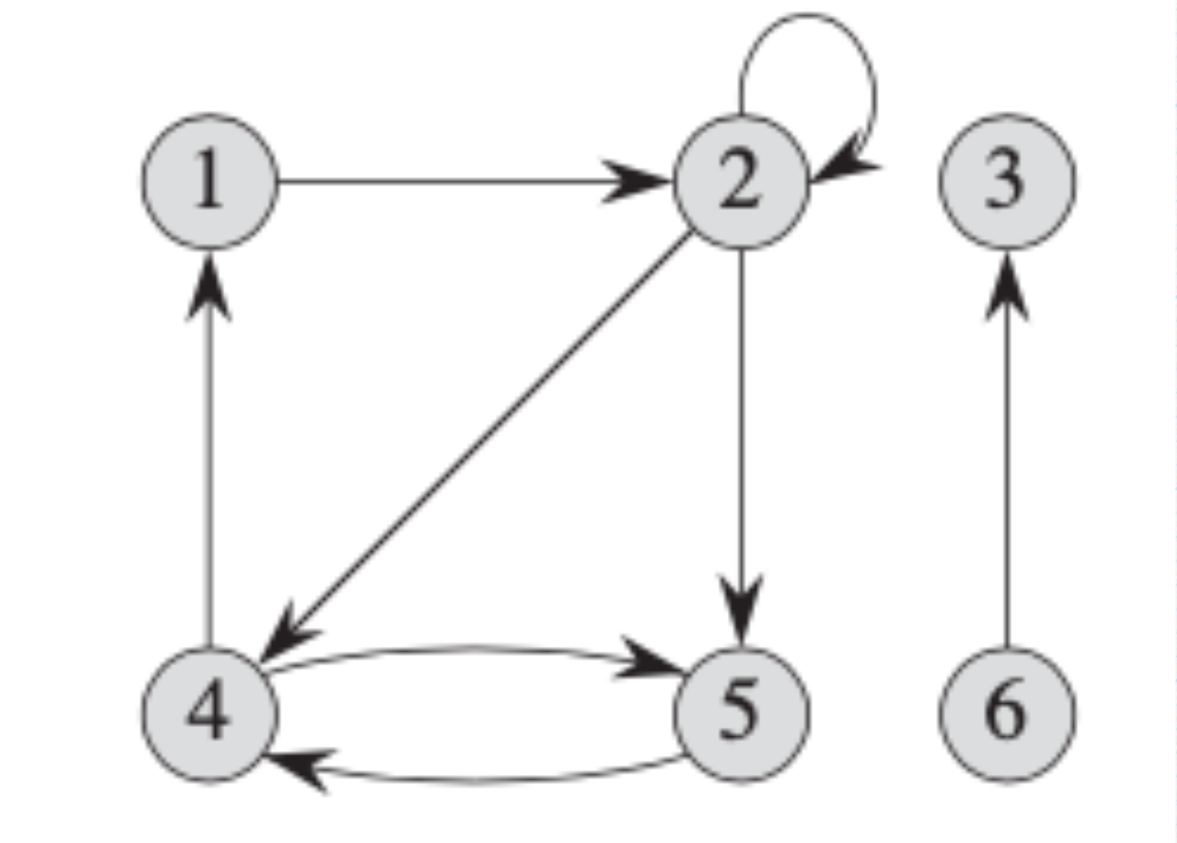


* _________ Graph



Directed Graph or Digraph

A **directed graph** (or digraph) G is a pair (V, E) where V is a finite set and E is a binary relation on V .



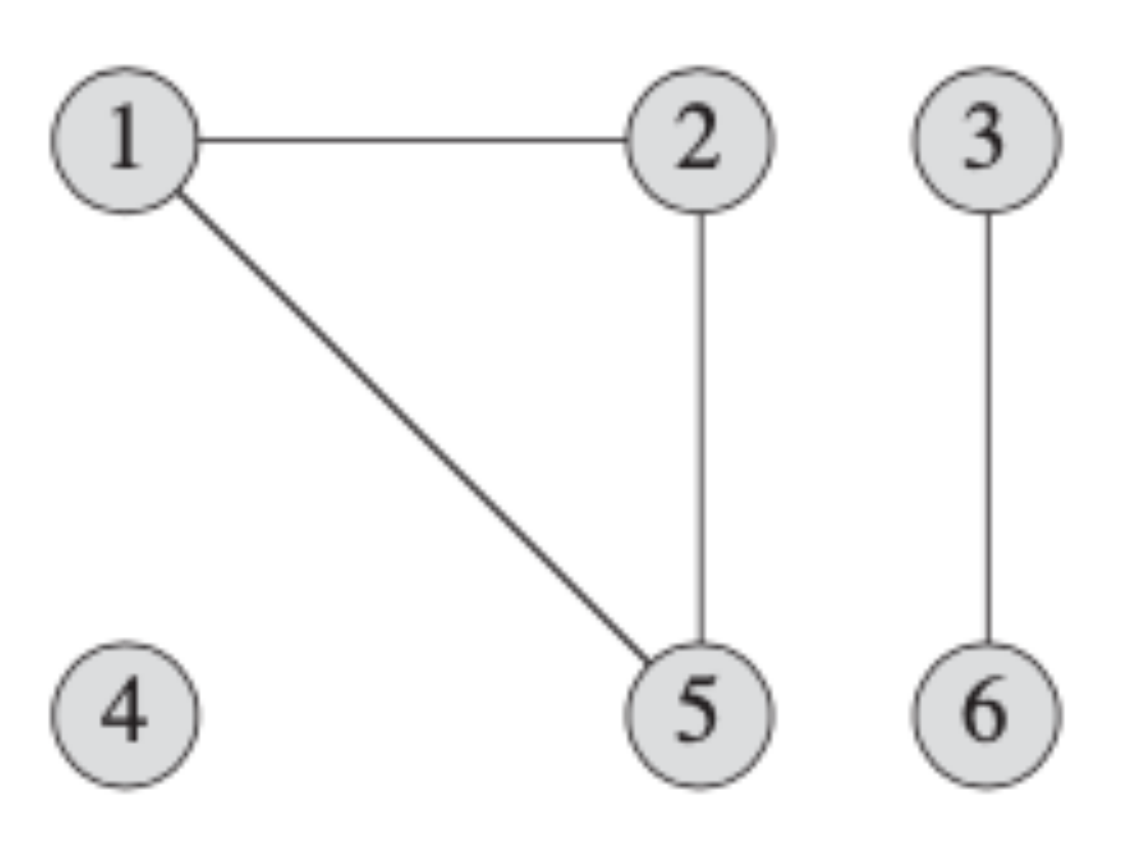
Self-loops: edges from a vertex to itself.

Undirected Graph

- In an **undirected graph** $G = (V, E)$, the edge set E consists of $\{u, v\}$ **unordered pairs of vertices,** rather than ordered pairs.

- An edge is a set $\{u, v\}$, where $u, v \in V$ and $u \neq v$

- Pictorial representation of an undirected graph vertex set $\{1,2,3,4,5,6\}$



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Subgraph & Induced subgraph

- A graph G' = (V', E') is a **subgraph of G = (V, E)** if V' $\subseteq$ V and E' $\subseteq$ E.
- Given a set V' $\subseteq$ V, the subgraph of G induced by V' is the graph G' = (V', E') where E' = { (u,v) $\in$ E : u,v in V' }
- The **induced subgraph** of Fig.(a) induced by the vertex set {1, 2, 3, 6 } is shown in Fig.(b)

Fig.(a)

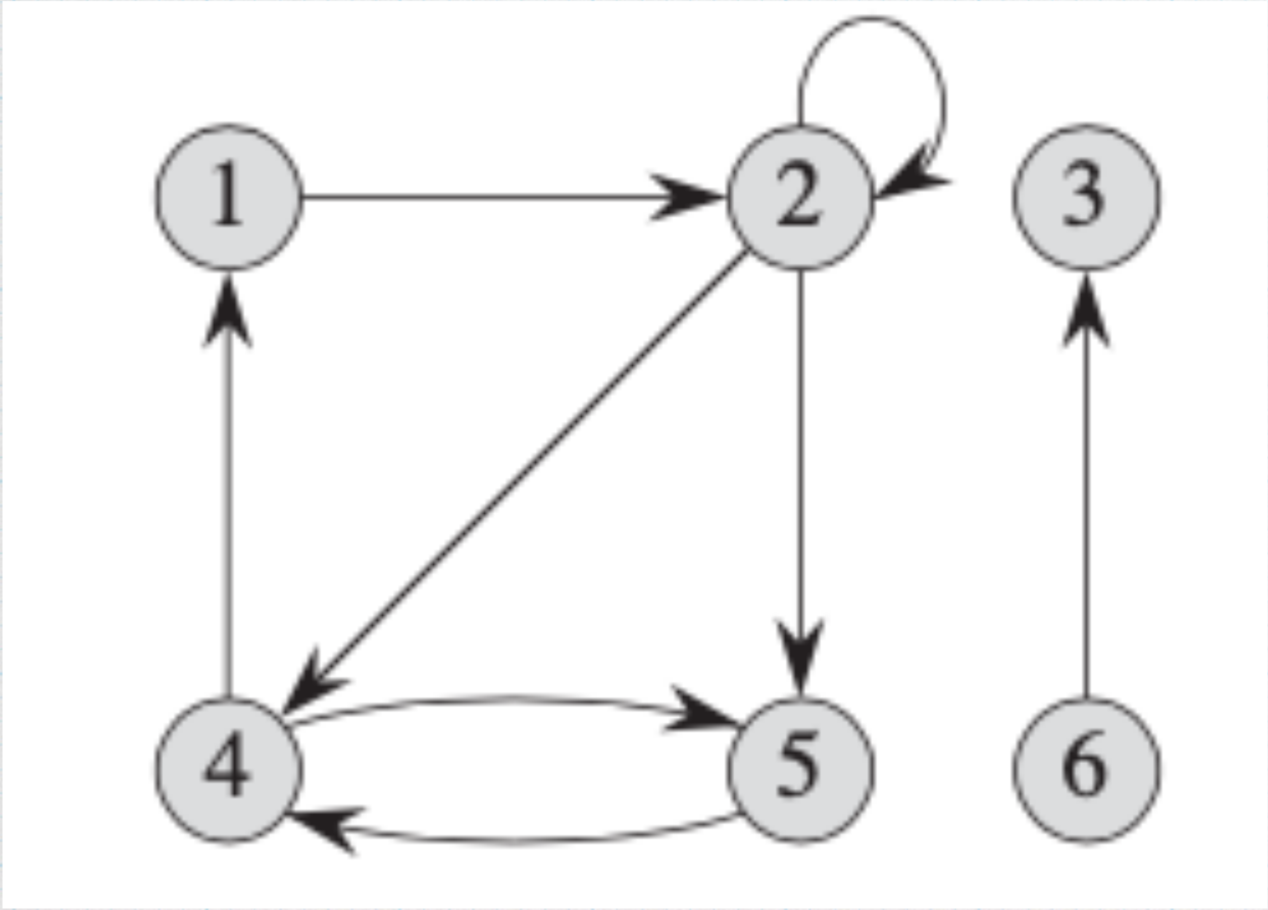
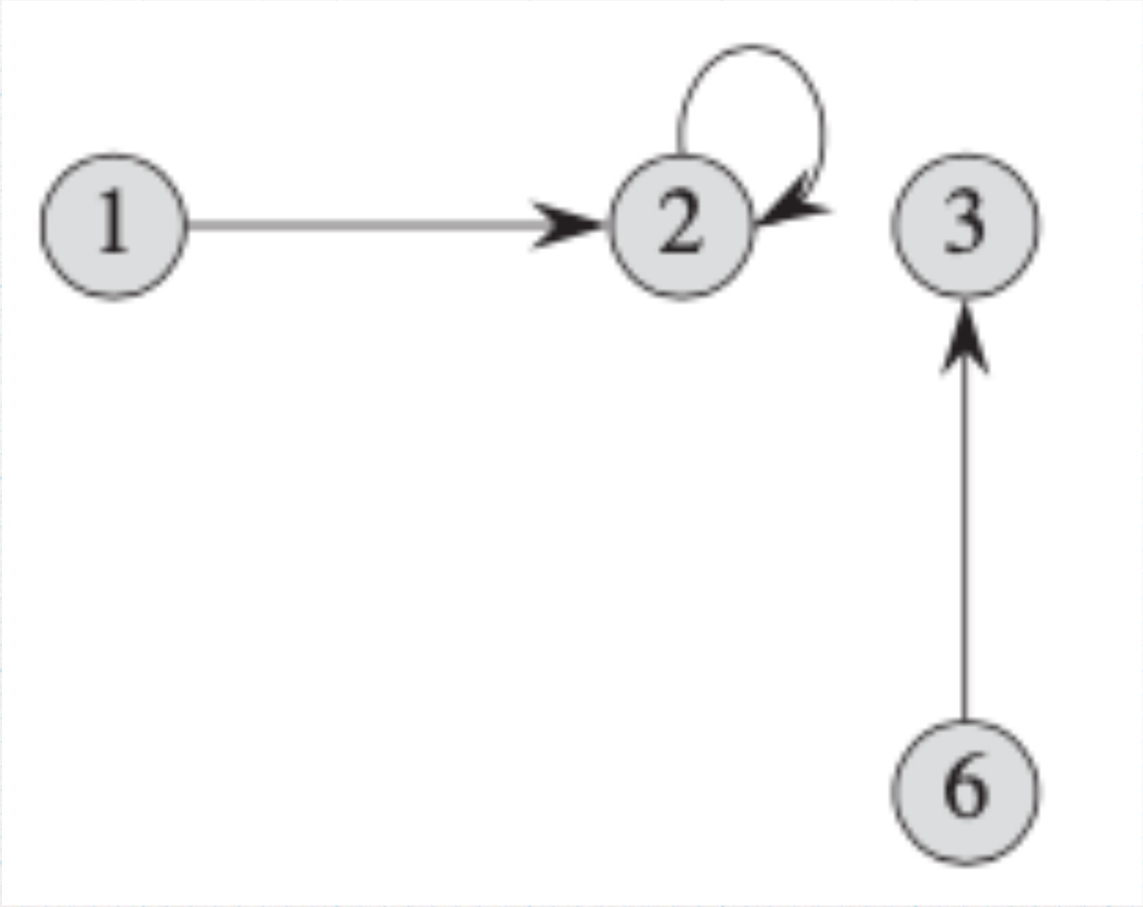


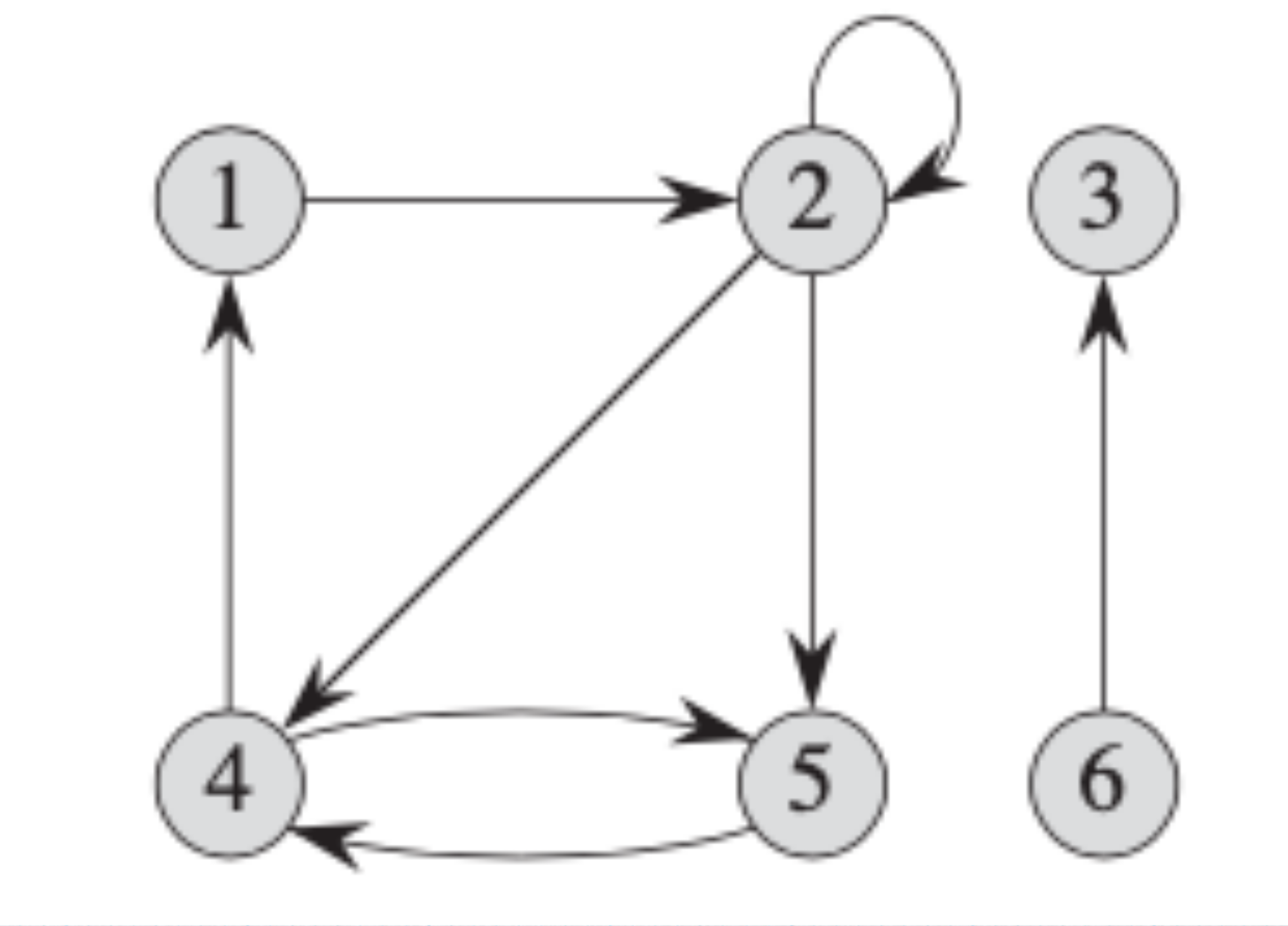
Fig.(b)



- <u>Ex:</u> What is the subgraph of Fig.(a) with V'= { 2, 4, 5} and E' = { (2, 4), (4, 5) } ?

Path

- A **path of length** k from a vertex u to a vertex u' in a graph $G = (V, E)$ is a sequence $< v_0, v_1, v_2, \ldots, v_k >$ of vertices such that $u = v_0$ and $u' = v_k$ and $(v_{i-1}, v_i) \in E$ for $i = 1,2,\ldots,k$

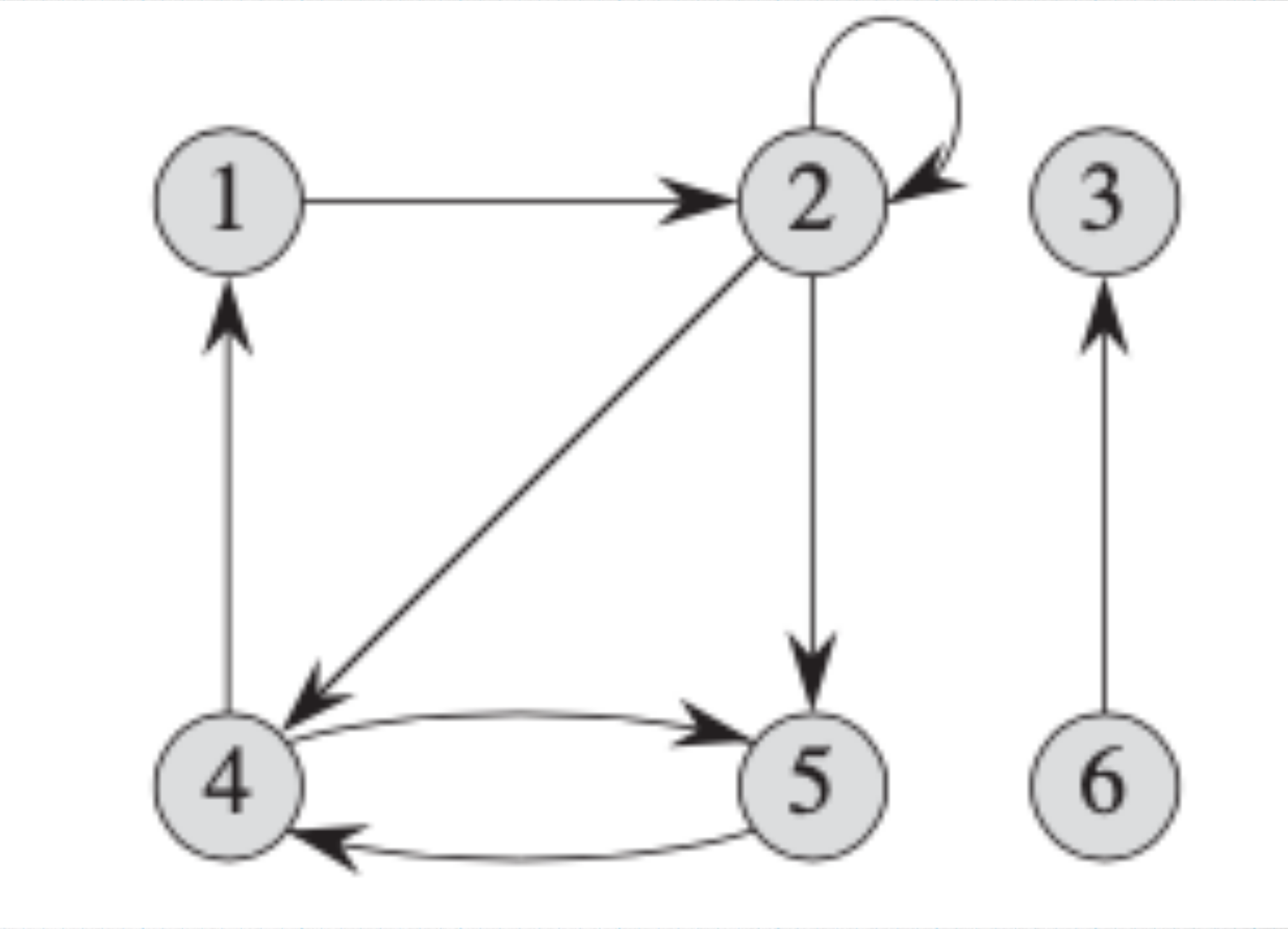


- **Length of the path :** number of edges in the path.

- The path contains the vertices $v_0, v_1, v_2, \ldots, v_k$ and the edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$

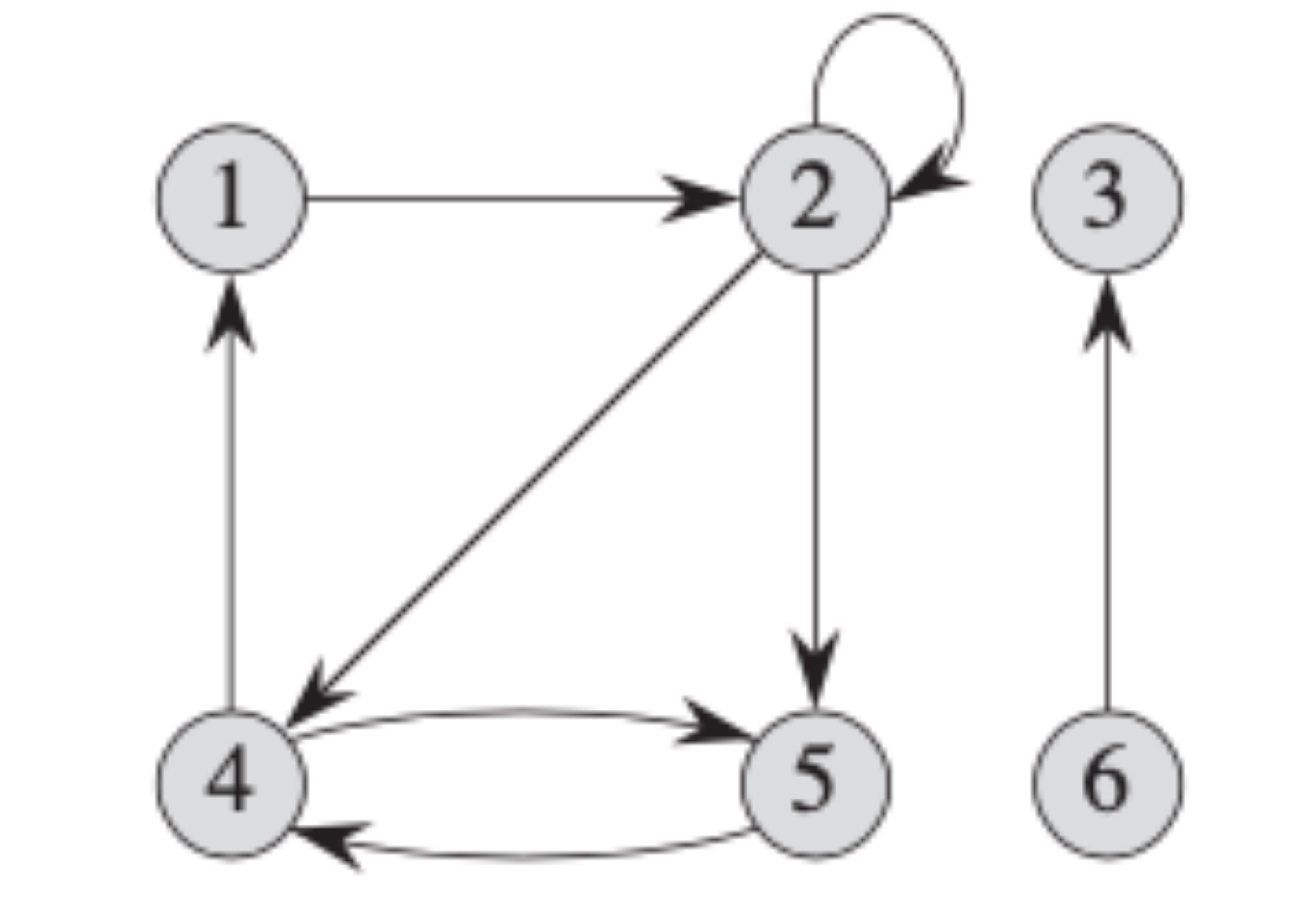
- There is always a **0-length path** from u to u.

Simple Path



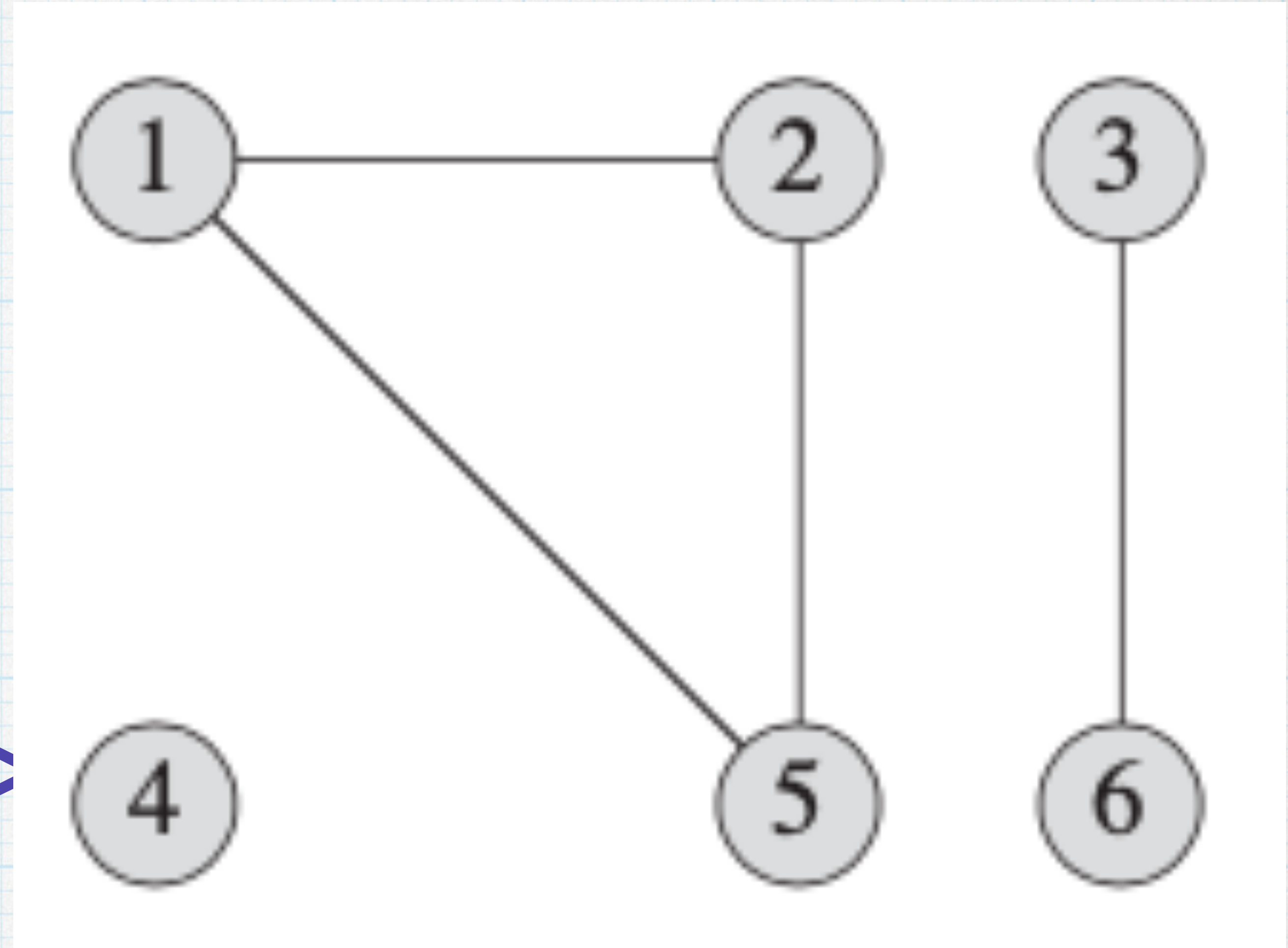
- If there is a **path p** from u to u', we say that u' is **reachable** from u via p
- A **path is simple** if all vertices in the path are **distinct.**
- In this Figure, the path <1,2,5,4> is a **simple path of length 3.**
- The path <2,5,4,5> is **not simple.**

Cycle (Directed Graph)



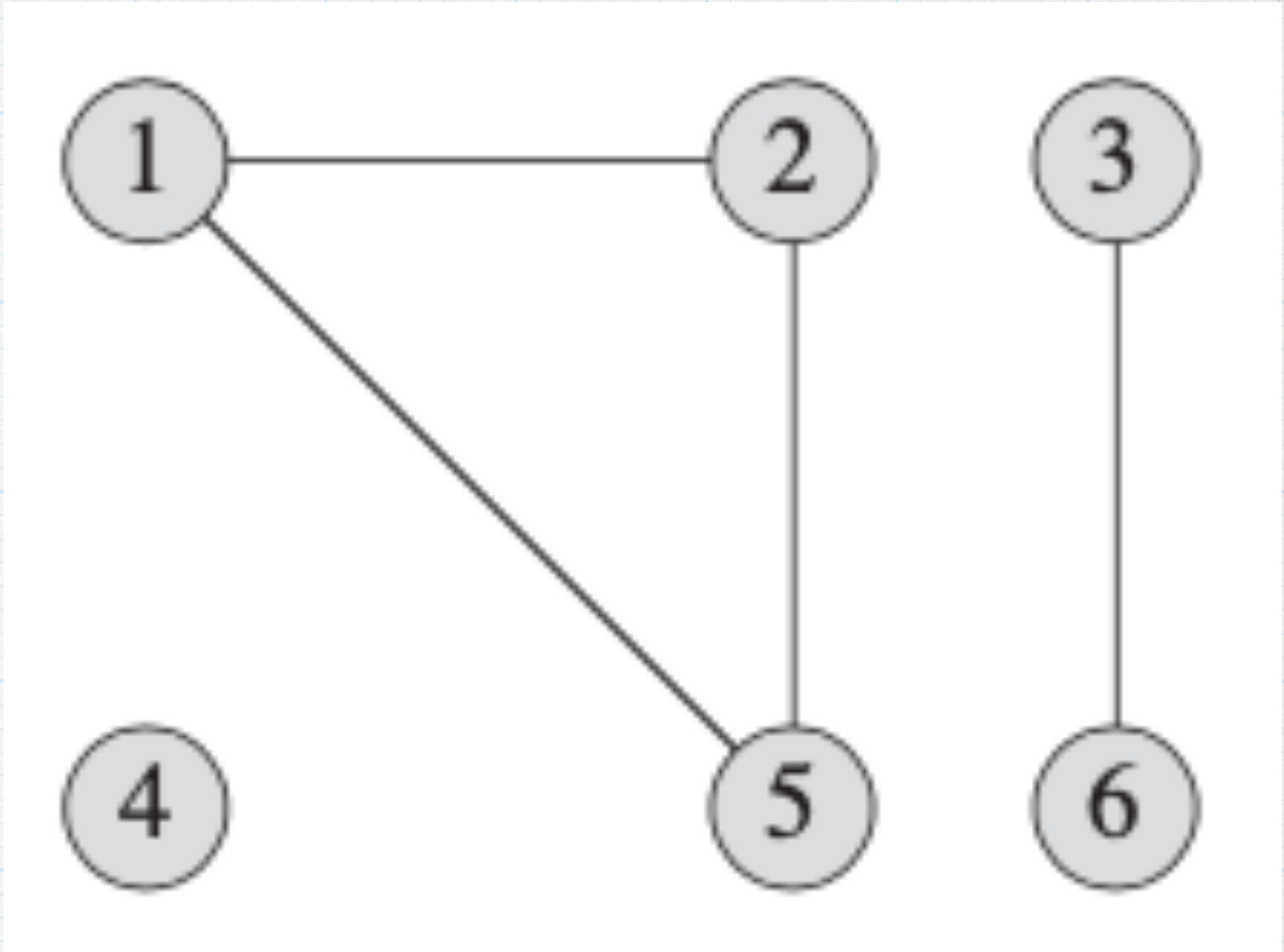
- **Directed graph**, a path $< v_0, v_1, v_2, \ldots, v_k >$ forms a **cycle** if $v_0 = v_k$ and the path contains at least one edge.
- **Simple cycle:** vertices are distinct except v_0, v_k
- A **self-loop** is a cycle of length 1.
- **Simple Cycle**: the path <1,2,4,1> forms the same cycle as the paths <2,4,1,2> and <4,1,2,4>.
- Cycle <1,2,4,5,4,1> is **not a simple cycle**.
- The cycle <2,2> formed by the edge (2,2) is a self-loop.
- A directed graph with **no self-loops** is simple graph

Simple Cycle (Undirected Graph)



- In an undirected graph, a path $< v_0, v_1, v_2, \ldots, v_k >$ forms a **cycle** if k $\geq$ 3 and $v_0 = v_k$
- Cycle is **simple** if $v_0, v_1, v_2, \ldots, v_k$ are distinct.
- Figure: the path <1,2,5,1> is a **simple cycle**.
- A graph with no cycles is **acyclic.**

Connected Graph



- An undirected graph is **connected** if every vertex is **reachable** from all other vertices.
- The ***connected components*** of a graph are the equivalence classes of vertices under the “is reachable from” relation.
- **Example:** Three components
- Every vertex in {1,2,5} is reachable from every other vertex in {1,2,5}
- An undirected graph is **connected** if it has exactly one connected component.
- A ***complete graph*** is an undirected graph in which **every pair of vertices is adjacent.**

Forest

* A **Free tree** is a **connected, acyclic, undirected graph**
* Disconnected **acyclic** undirected graph ________

Properties of Trees

Let $G = (V, E)$ be an undirected graph. The following statements are equivalent.

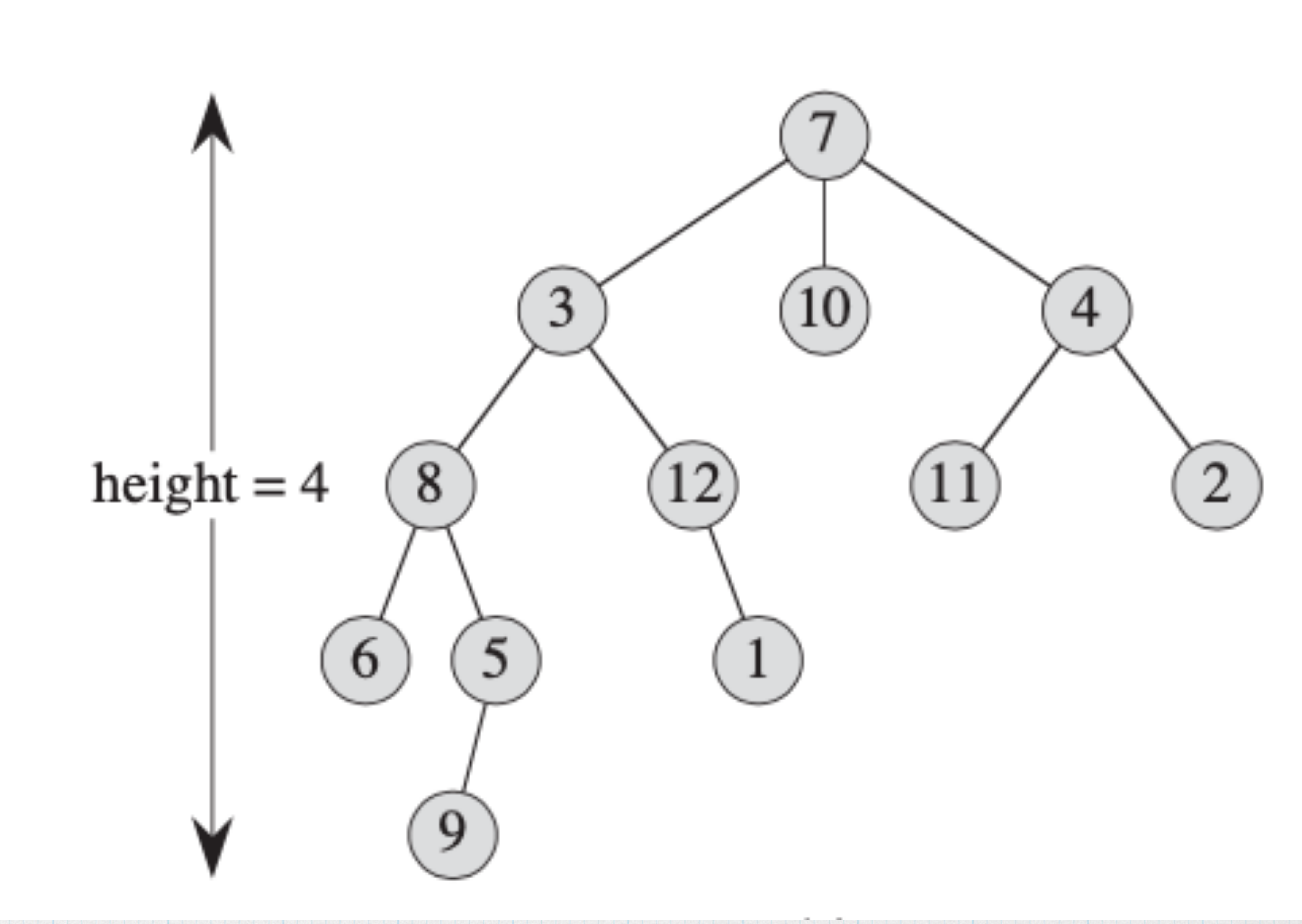
1. G is a free tree.
2. Any two vertices in G are connected by a unique simple path.
3. G is connected, but if any edge is removed from E, the resulting graph is disconnected.
4. G is connected, and $|E| = |V| - 1$.
5. G is acyclic, and $|E| = |V| - 1$.
6. G is acyclic, but if any edge is added to E, the resulting graph contains a cycle.

Rooted Trees

- A **rooted tree** is a free tree in which one of the vertices is distinguished from the others.

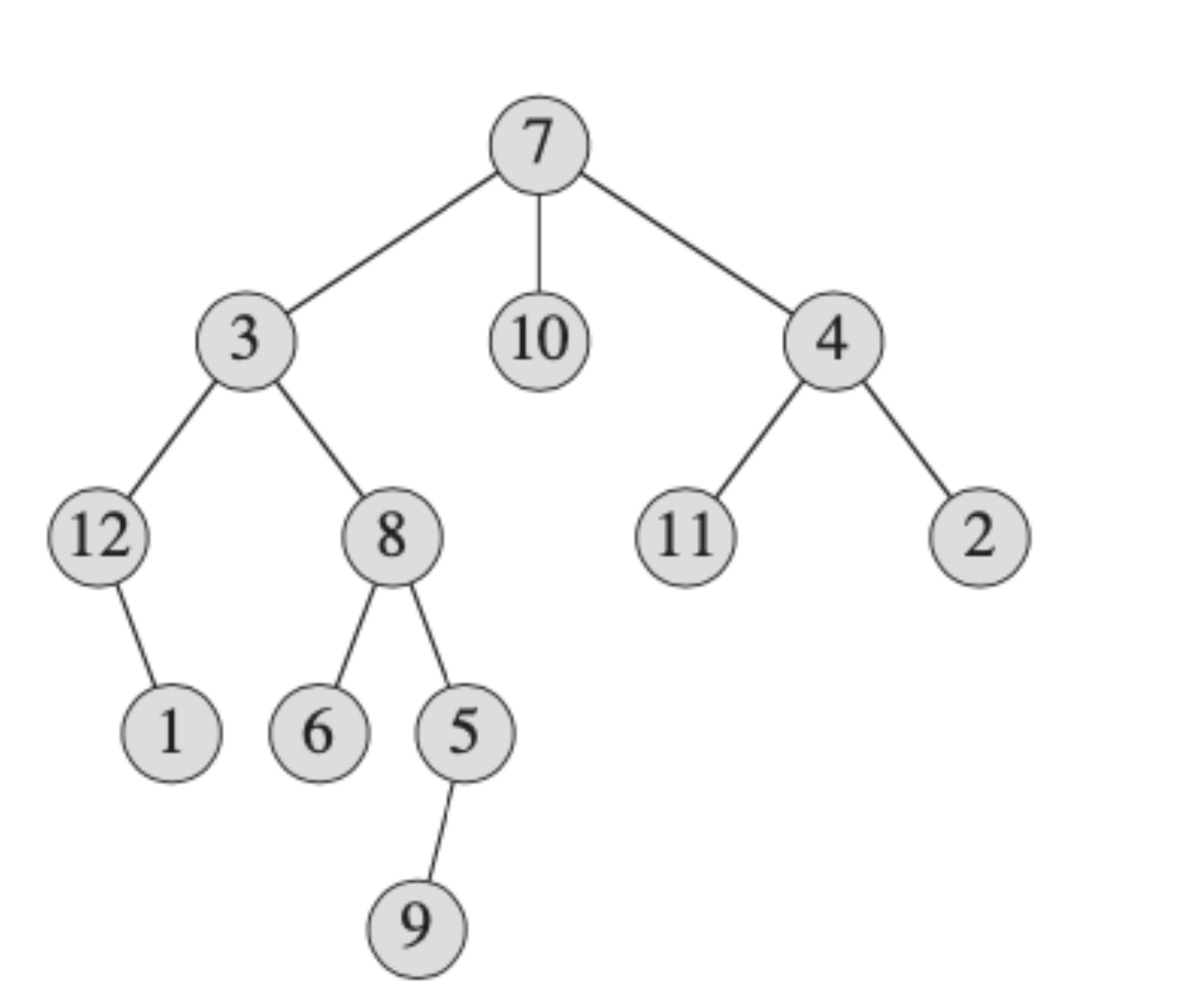
- Distinguished vertex the **root** of the tree.

- **Vertex** of a rooted tree as a **node of the tree**.



- Figure shows a rooted tree on a **set of 12 nodes with root 7.**

Key terms - Rooted Trees



Consider a node x in a rooted tree T with root r.

Any node y on the unique simple path from r to x an ***ancestor*** of x.

If y is an ancestor of x, then x is a ***descendant*** of y.

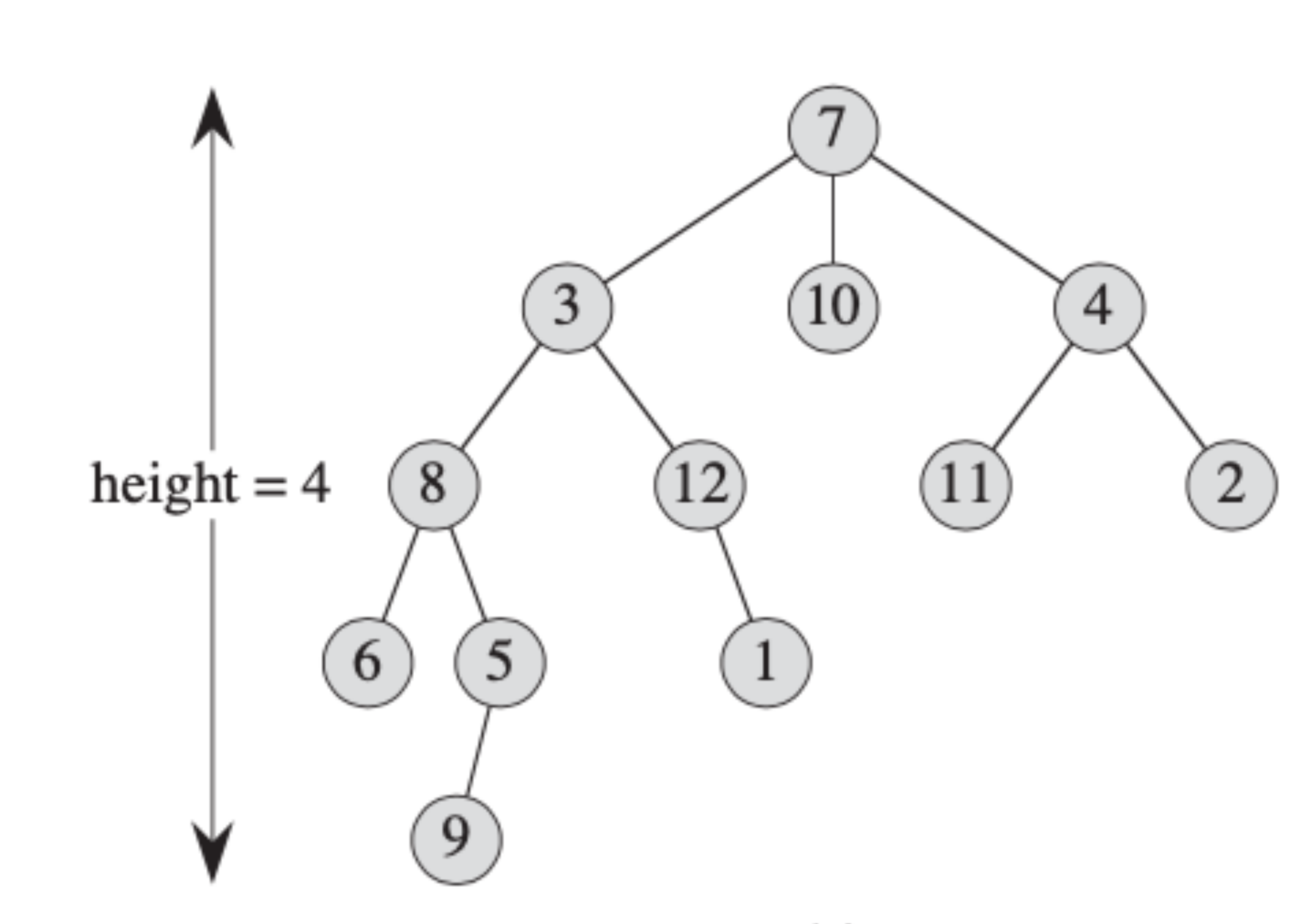
Note: Every node is both an ancestor and a descendant itself.

If y is an ancestor of x and x not equal to y, then y is a ***proper ancestor*** of x and x is a ***proper descendant*** of y.

The ***subtree rooted at*** x is the tree induced by descendants of x, rooted at x.

For example, the subtree rooted at node 8 in the figure above contains nodes 8, 6, 5, and 9.

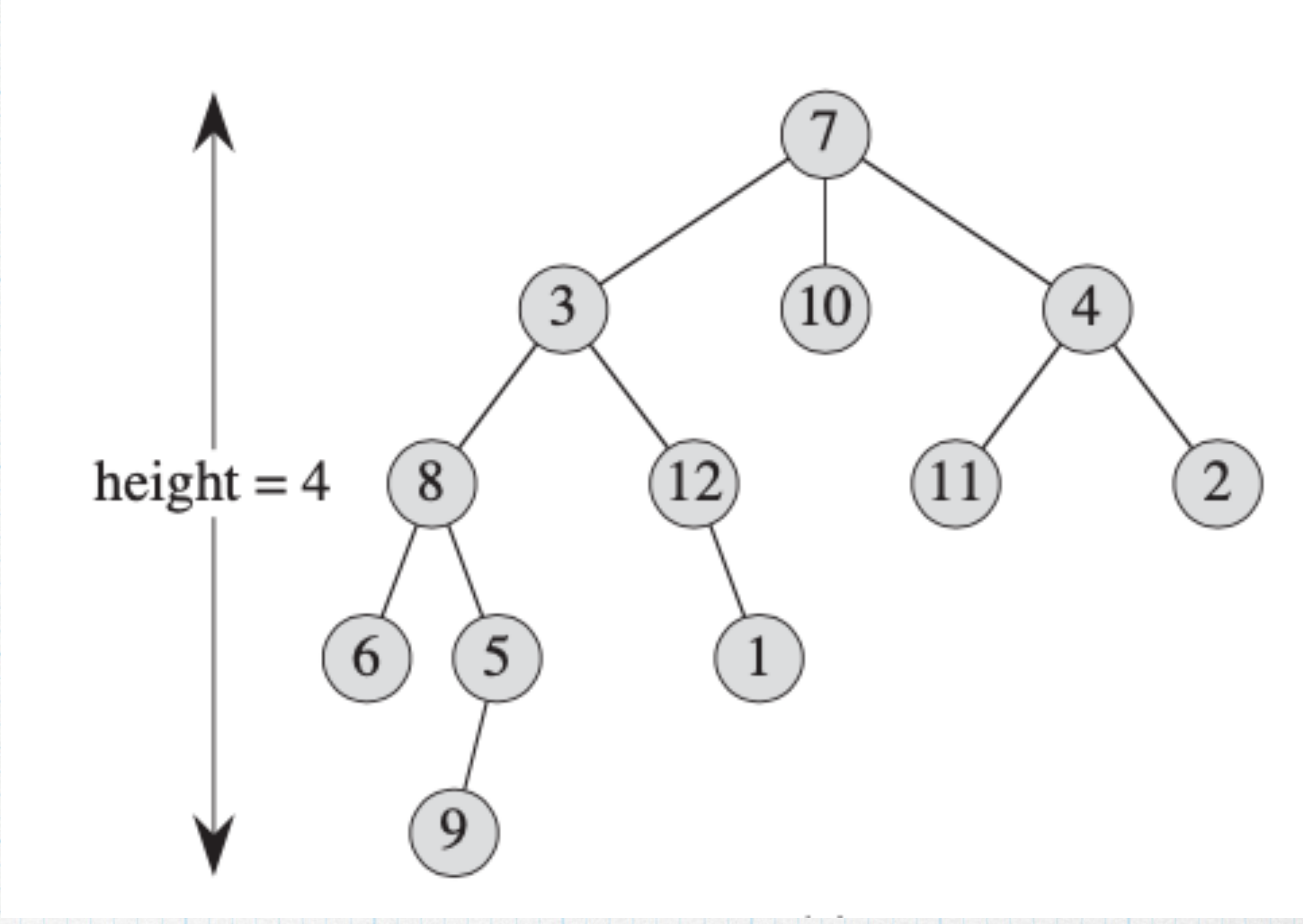
Key terms - Rooted Trees



- If the last edge on the simple path from the root r of a tree T to a node x is (y,x), then y is the *parent* of x, and x is a *child* of y
- If two nodes have the same parent, they are *siblings*.
- A node with no children is a *leaf* or *external node*. A nonleaf node is an *internal node.*

<u>Ex:</u> Node with no parents ?, Node with no children ?

Degree of a node



- The number of children of a node x in a rooted tree T equals the ***degree*** of x.

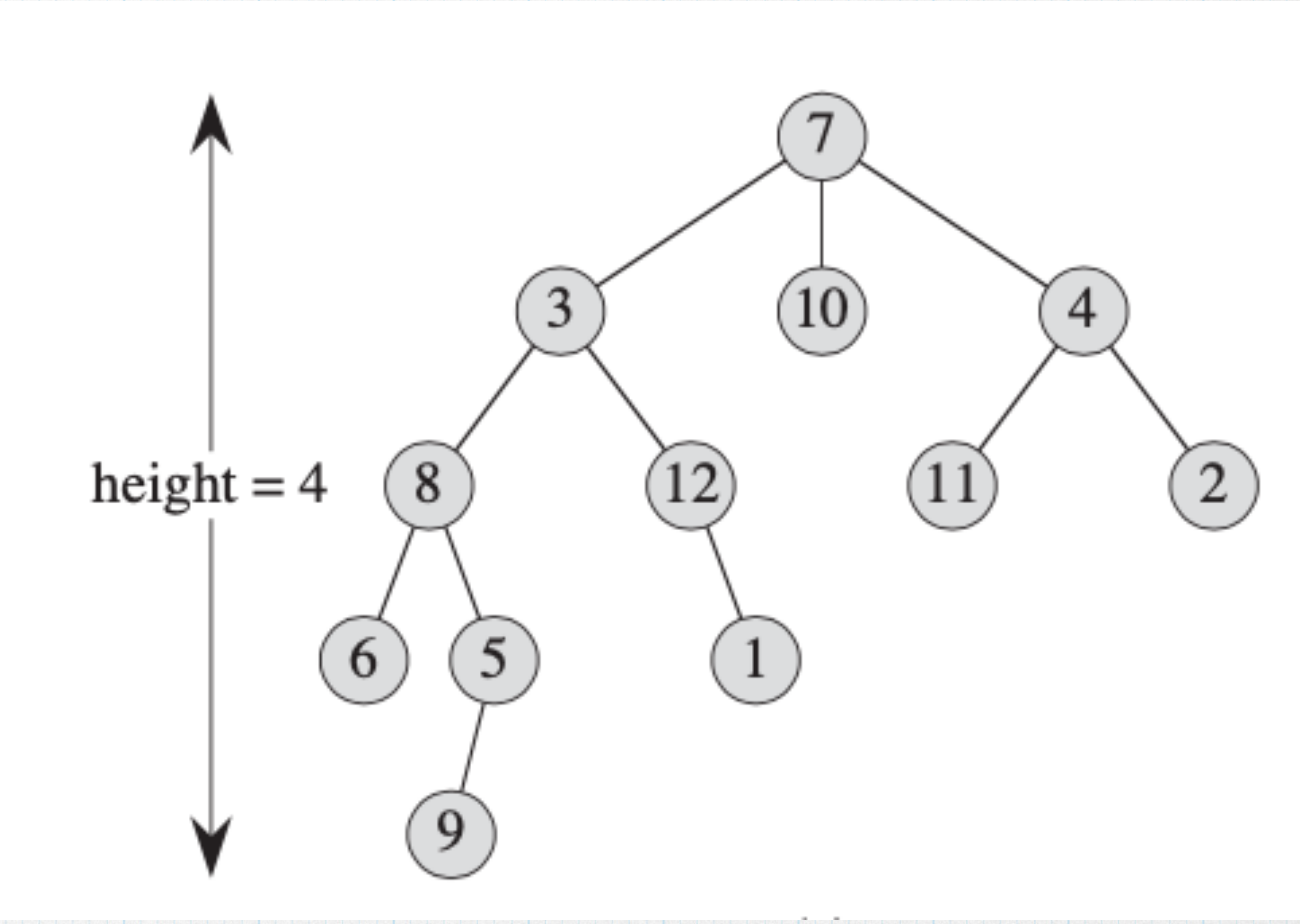
- Note that the degree of a node depends on whether we consider T to be a **rooted tree or a free tree.**

- The **degree** of a vertex in a **free tree** is, as in any **undirected graph**, the number of adjacent vertices.

- In a **rooted tree -** the degree is the number of children
  - the **parent of a node does not count** toward its degree.

Depth, Height and Level

- The length of the simple path from the root r to a node x is the **depth** of x in T .



- A **level** of a tree consists of all nodes at the same depth.

- The **height** of a node in a tree is the **number of edges** on the longest simple downward path from **the node to a leaf,** and the height of a tree is the height of its root.

- The **height of a tree** is also equal to the **largest depth of any node in the tree.**

Recap of Few terms….

- The **number of children of a node** x in a rooted tree T equals the **degree** of x

Example:

- Degree of node 3:

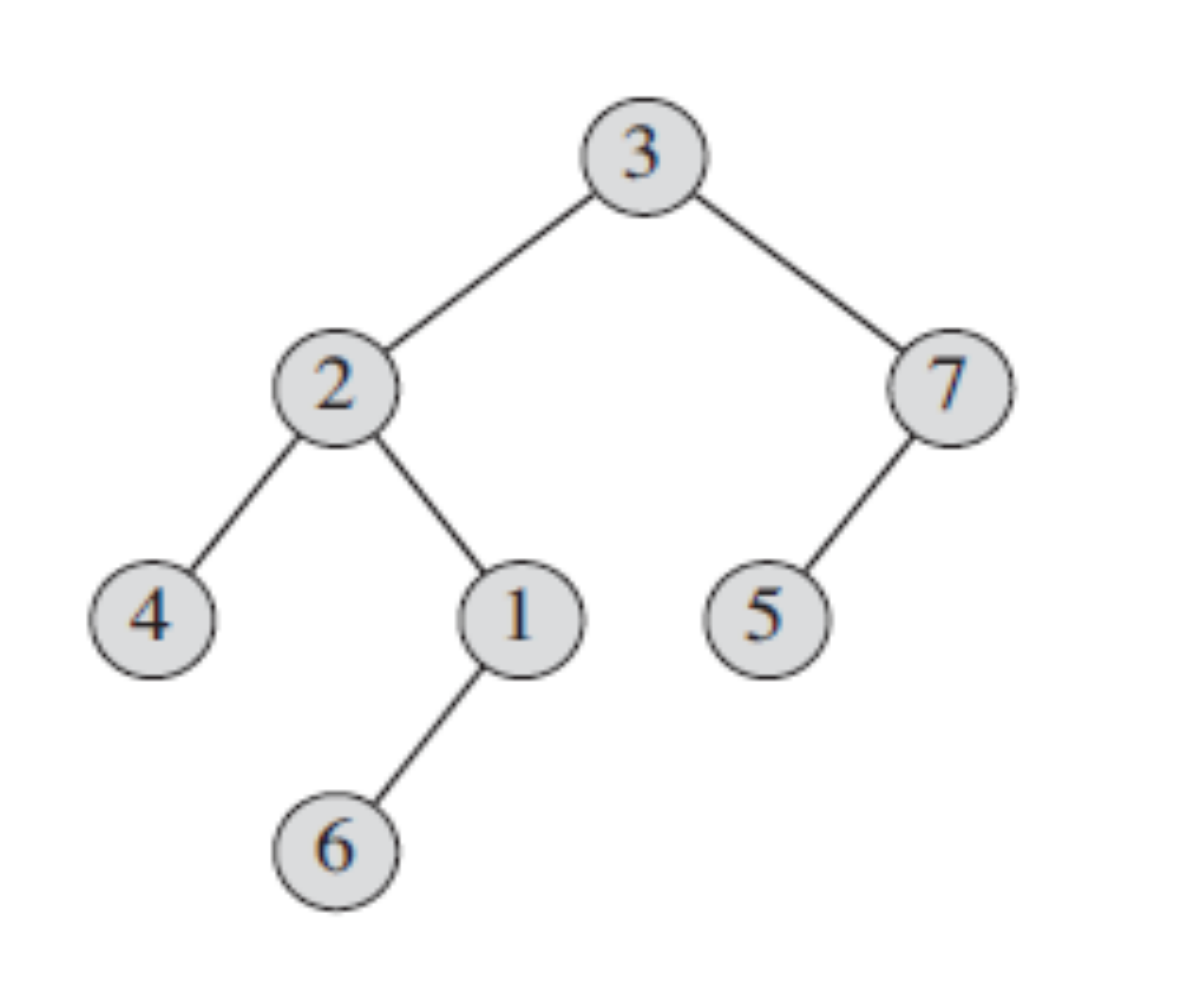
  2

- Degree of node 7:

  1

- Degree of node 6:

  0



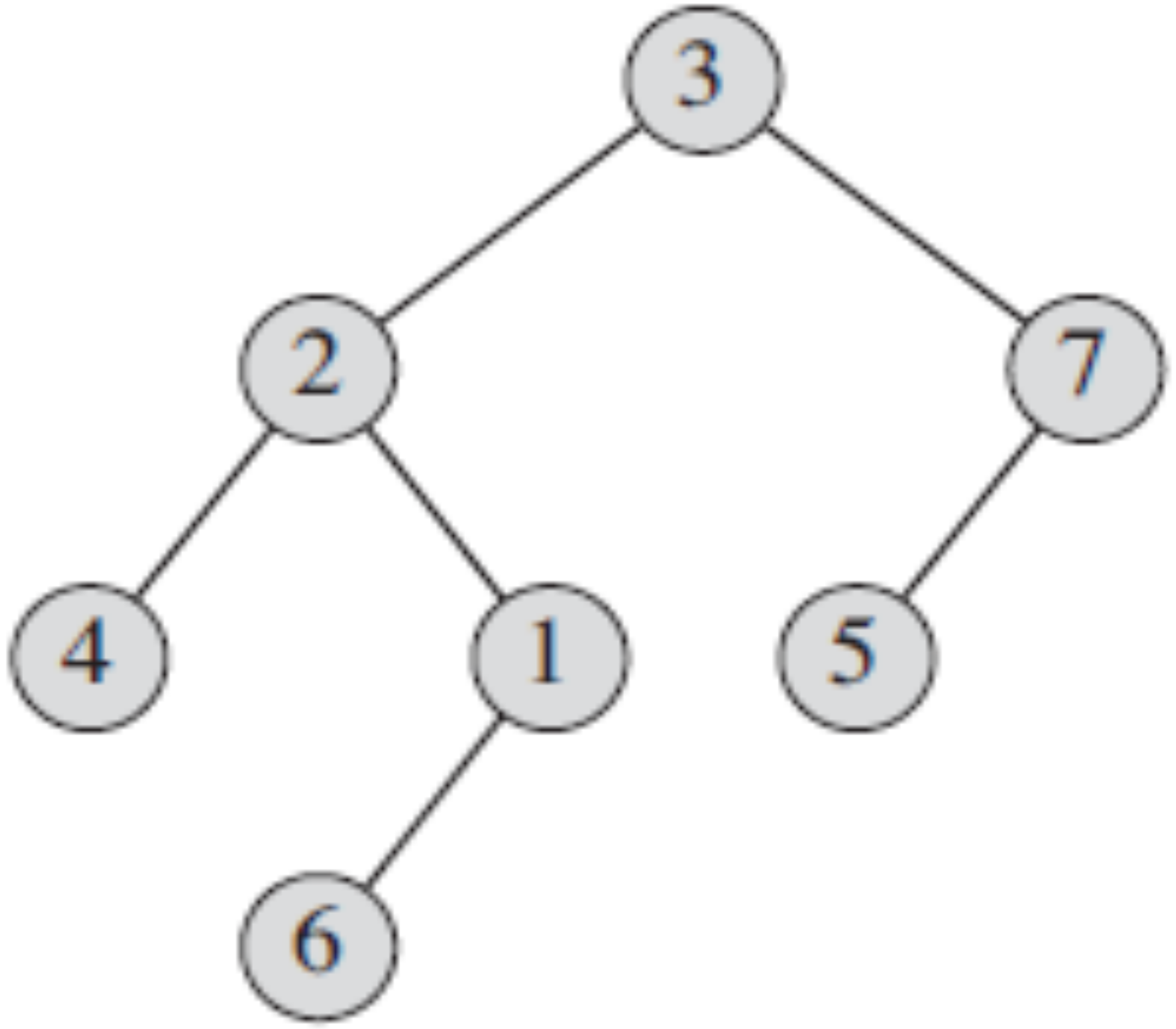
* The **length of the simple path** from the root r to a node x is the ***depth* of x** in T .
  * Eg: Depth of node 6:

    3
  * Depth of nodes 1, 4, 5:

    2
  * Depth of nodes 2 and 7:

    1
  * Depth of node 3:

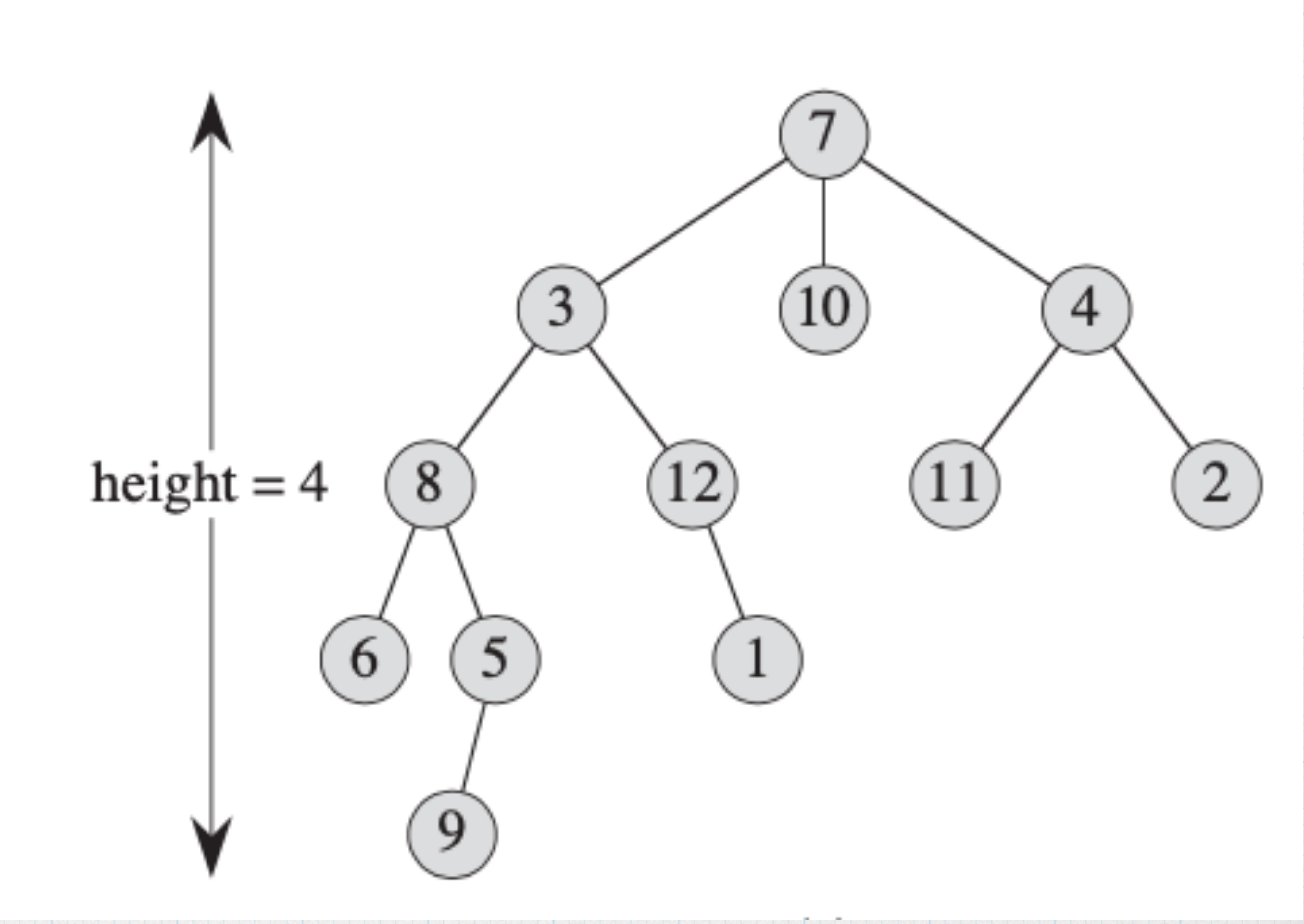
    0
* A ***level* of a tree** consists of all nodes at the same depth.
  * **Nodes at level 2:** 4, 1 and 5
* **Ex:** What are the nodes at level 0,1 and 3 in the above tree?



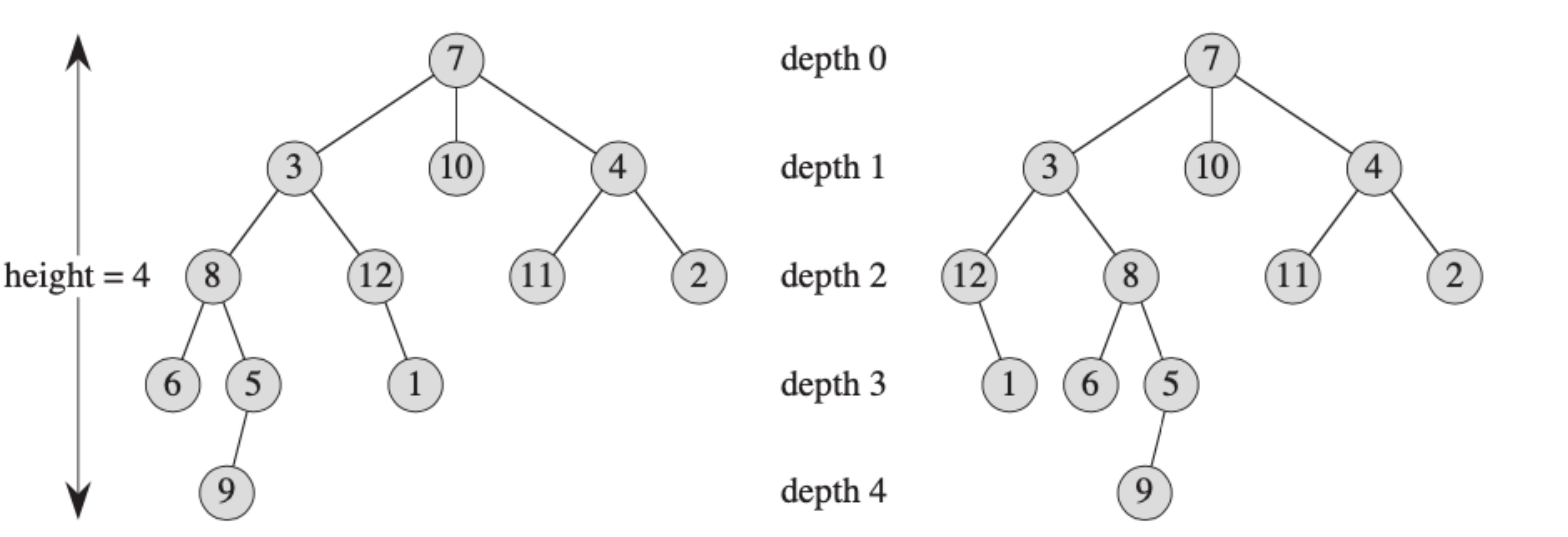
Ordered Trees

- An ***ordered tree*** is a rooted tree in which the children of each node are ordered.

- That is, if a node has **k children**, then there is a **first child, a second child, . . . , and a kth child.**



Ordered and Unordered trees



(a) A **rooted tree with height 4.** If the tree is ordered, the **relative left-to-right order of the children** of a node matters; otherwise it doesn't.

(b) As a rooted tree, it is identical to the tree in (a), but as an ordered tree it is different, since the **children of node 3 appear in a different order.**

Binary Trees

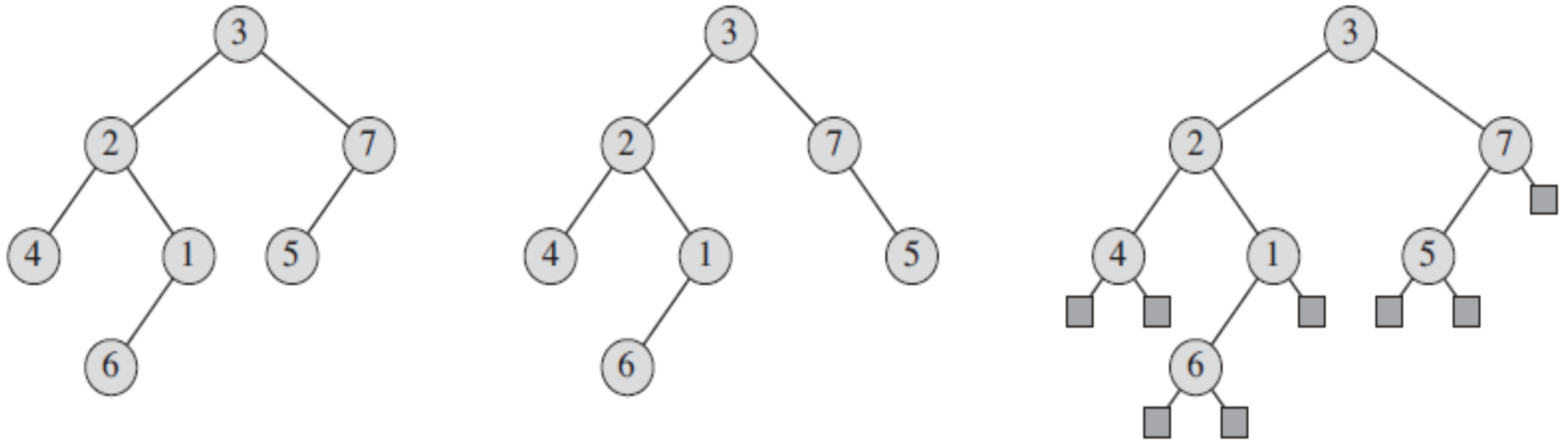
- Binary tree: A binary tree is defined recursively.

- A binary tree T is a structure defined on finite set of nodes that either
  - Contains no nodes (the *empty tree* or *null tree)* denoted NIL or
  - Composed of three disjoint set of nodes:
    - a root node
    - a binary tree called its left subtree
    - a binary tree called its right subtree
- *Note:* Left child, Right child, absent or missing child

Binary Tree Example

- Binary tree - **Not simply an ordered tree** in which each node has **degree at most 2.**
- Example: If a node has just one child, **the position of the child**, whether it is the **left child or the right** child matters.
- In **Ordered tree** there is *no distinguishing a sole child as being either left or right.*
- Fig (a) binary tree differs from Fig (b) - due to position of one node.
- As ordered trees,

two trees are identical.



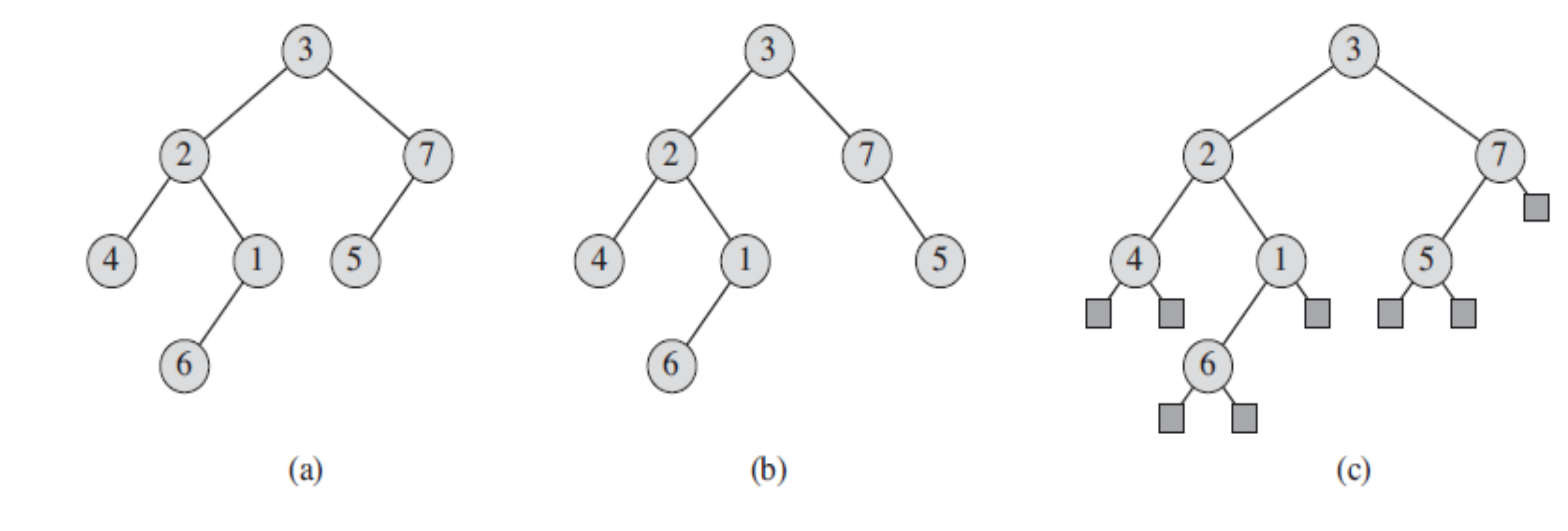
(a) (b) (c)

Full Binary Tree

- **Replace each missing child** in the binary tree with a **node having no children.**
- These **leaf nodes** are drawn as **squares** in the figure.
- ***Full binary tree -*** each node is either a leaf or has degree exactly 2.
- **No degree-1 nodes**
- Order of the children of a node preserves the position information.

Fib (c):

The binary tree in (a) represented by the internal nodes of a full binary tree: an ordered tree in which each internal node has degree 2. The leaves in the tree are shown as squares.



Binary tree - Examples

In the following example,

root node:

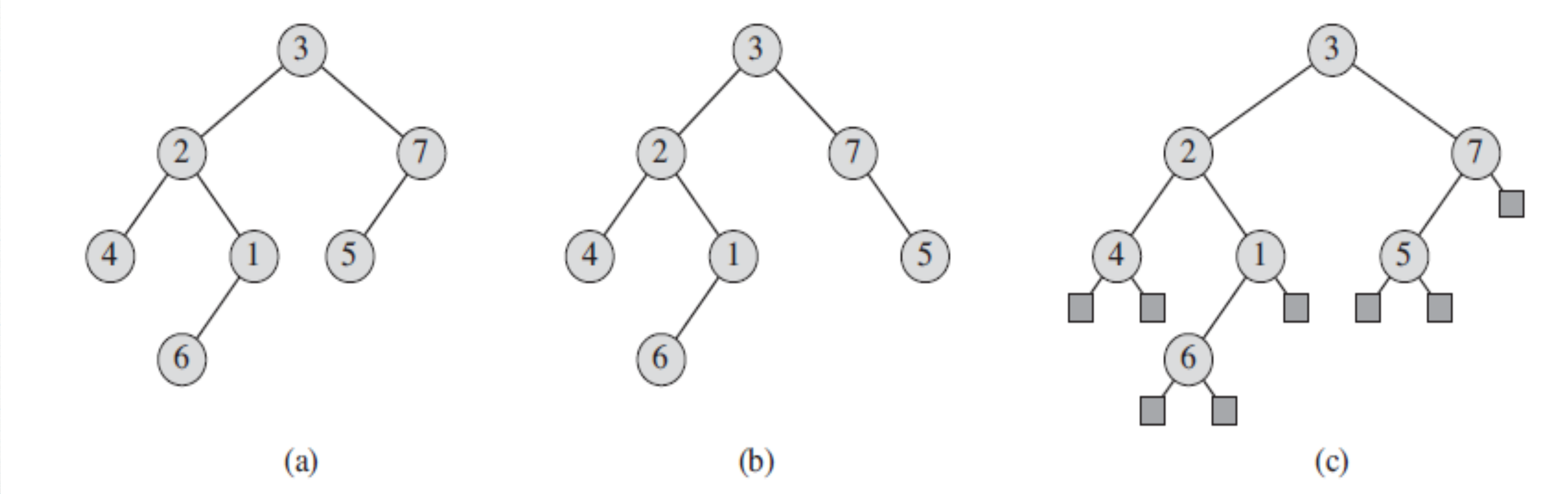
Node 3

left subtree:

Nodes 2,4,1 and 6 together form Left subtree

right subtree:

Nodes 7 and 5 together form Right subtree



Reference

* CLRS Book