STATISTICS AND NUMERICAL METHODS

UNIT- I

Population:

A population consists of collection of individual units, which may be person's or experimental outcomes, whose characteristics are to be studied.

Sample:

A sample is proportion of the population that is studied to learn about the characteristics of the population.

Random sample:

A random sample is one in which each item of a population has an equal chance of being selected.

Sampling:

The process of drawing a sample from a population is called sampling.

Sample size:

The number of items selected in a sample is called the sample size and it is denoted by 'n'. If $n \geq 30$, the sample is called large sample and if $n \leq 30$, it is called small sample

Sampling distribution:

Consider all possible samples of size' n' drawn from a given population at random. We calculate mean values of these samples.

If we group these different means according to their frequencies, the frequency distribution so formed is called sampling distribution.

The statistic is itself a random variate. Its probability distribution is often called sampling distribution.

All possible samples of given size are taken from the population and for each sample, the statistic is calculated. The values of the statistic form its sampling distribution.

Standard error:

The standard deviation of the sampling distribution is called the standard error.

Notation:

Pop. mean = μ; Pop. S.D = σ; P - Pop. proportion sample mean = $\bar{x}$; sample S.D = s; P' = sample Proportion

Note

Statistic	S.E (Standard Error)
$\bar{x}$	$\dfrac{\sigma}{\sqrt{n}}$
$p_1' - p_2'$ (Difference of sample proportions)	$\sqrt{pq\left(\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)}$
$\bar{x}_1 - \bar{x}_2$ (Difference of sample means)	$\sqrt{\dfrac{\sigma_1^2}{n_1}+\dfrac{\sigma_2^2}{n_2}}$
p' (Sample proportion)	$\sqrt{\dfrac{pq}{n}}$

Null Hypothesis (H_0)

The hypothesis tested for possible rejection under the assumption that it is true is usually called null hypothesis. The null hypothesis is a hypothesis which reflects no change or no difference. It is usually denoted by H_0

Alternative Hypothesis (H_1)

The Alternative hypothesis is the statement which reflects the situation anticipated to be correct if the null hypothesis is wrong. It is usually denoted by H_1.

For example:

If $H_0 : \mu_1 = \mu_2$ (There is no diff' bet' the means) then the formulated alternative hypothesis is

$$H_1 : \mu_1 \neq \mu_2$$

ie., either $H_1 : \mu_1 < \mu_2$ (or) $\mu_1 > \mu_2$

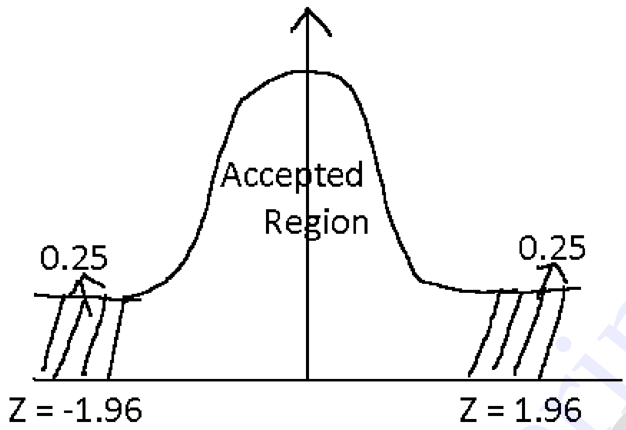
Level of significance

It is the probability level below which the null hypothesis is rejected. Generally, 5% and 1% level of significance are used.

Critical Region (or) Region of Rejection

The critical region of a test of statistical hypothesis is that region of the normal curve which corresponds to the rejection of null hypothesis.

The shaded portion in the following figure is the critical region which corresponds to 5% LOS



Critical values (or) significant values

The sample values of the statistic beyond which the null hypothesis will be rejected are called critical values or significant values

Types of test	*Level of significance* 1%	5%	10%
Two tailed test	2.58	1.96	1.645
One tailed test	2.33	1.645	1.28

Two tailed test and one-tailed tests:

When two tails of the sampling distribution of the normal curve are used, the relevant test is called two tailed test.

The alternative hypothesis $H_1 : \mu_1 \neq \mu_2$ is taken in two tailed test for $H_0 : \mu_1 = \mu_2$

When only one tail of the sampling distribution of the normal curve is used, the test is described as one tail test $H_1 : \mu_1 < \mu_2$ (or) $\mu_1 > \mu_2$

$$\left.\begin{array}{l} H_0 = \mu_1 = \mu_2 \\ H_1 = \mu_1 \neq \mu_2 \end{array}\right\} \text{two tailed test}$$

Type I and type II Error

Type I Error : Rejection of null hypothesis when it is correct

Type II Error : Acceptance of null hypothesis when it is wrong

Procedure for testing Hypothesis:

1. Formulate H_0 and H_1
2. Choose the level of significance α
3. Compute the test statistic Z, using the data available in the problem
4. Pick out the critical value at α % level say Z_α
5. Draw conclusion: If $|Z| < Z_\alpha$, accept H_0 at $\alpha\%$ level. Otherwise reject H_0 at $\alpha\%$ level

Test of Hypothesis (Large Sample Tests)

Large sample tests (Test based in Normal distribution.)

Type - I: (Test of significance of single mean)

Let $\{x_1, x_2, \ldots x_n\}$ be a sample of size $(n \geq 30)$ taken from a population with mean μ and S.D σ. Let $\bar{x}$ be the sample mean. Assume that the population is Normal.

To test whether the difference between Population mean μ and sample mean $\bar{x}$ is significant or not and this sample comes from the normal population whose mean is μ or not.

$H_0 : \mu =$ a specified value

$H_1 : \mu \neq$ a specified value

we choose $\alpha = 0.05(5\%)$ (or) $0.01(1\%)$ as the Level of significance

the test statistic is

$$Z = \frac{\bar{x}-\mu}{S.E(\bar{x})} = \frac{\bar{x}-\mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0.1) \text{ for large n.}$$

Note:

1. If σ is not known, for large n, S.E $\bar{x} = \frac{s}{\sqrt{n}}$ where 's' is the sample S.D

<u>Problems:</u>

1. A sample of 900 members is found to have a mean 3.5cm. Can it reasonably regarded as a simple sample from a large population whose mean is 3.38 and a standard deviation 2.4cm?

Solution:

We formulate the null hypothesis that the sample is drawn from population whose mean is 3.38cm.

$i.e., H_0 : \mu = 3.38$

$H_1 : \mu \neq 3.38$

Hence it is a two-tailed test

Level of significance $\alpha = 0.05$

$$\text{Test statistic } Z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$$

Given $\bar{x} = 3.5,\ \mu = 3.38,\ n = 900,\ \sigma = 2.4$

$$\therefore\ Z = \frac{3.5 - 3.32}{\frac{2.4}{\sqrt{900}}} = 1.5$$

Critical value:

At 5% level, the tabulated value of Z is 1.96

Conclusion:

Since |Z| = 1.5 < 1.96, H_0 is accepted at 5% level of significance

i.e., the sample comes from a population with mean 3.38cm

2. A manufacturer claims that his synthetic fishing line has a mean breaking strength of 8kg and S.D 0.5kg. Can we accept his claim if a random sample of 50 lines yield a mean breaking of 7.8kg. Use 1% level of significance.

Solution:

We formulate $H_0 : \mu = 8$

$H_1 : \mu \neq 8$

L.O.S $\alpha = 0.01$

Test statistic $Z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$

Given $\bar{x} = 7.8,\ \mu = 8,\ \mathrm{n} = 50,\ \sigma = 0.5$

$\therefore\ Z = \dfrac{7.8 - 8}{\frac{0.5}{\sqrt{50}}} = -2.828$

$\therefore\ |Z| = 2.828$

Critical value:

At 1% level of significance the table of Z = 2.58

Conclusion:

Since $|Z| > 2.58,\ H_0$ is rejected at 1% level

i.e., the manufacturer's claim is not accepted

3. A random sample of 200 Employee's at a large corporation showed their average age to be 42.8 years, with a S.D of 6.8 years. Test the hypothesis $H_0 : \mu = 40$ versus $H_1 : \mu > 40$ at $\alpha = 0.01$ level of significance.

Solution:

We set up $H_0 : \mu = 40$

$H_1 : \mu \neq 40$

It is one tailed test.

L.O.S $\alpha = 0.01$

Test statistic $Z = \dfrac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}}$

Given $\bar{x} = 42.8,\ \mu = 40,\ \mathrm{n} = 200,\ \sigma = 6.89$

$$\therefore\ Z = \frac{42.8 - 40}{\frac{6.89}{\sqrt{200}}} = 5.747$$

Critical value:

For one-tail test, the table value of Z at 1% level = 2.33

Conclusion:

Since $|Z| = 5.747 > 2.33$, H_0 is rejected at 1% level.

i.e., The hypothesis $\mu = 40$ is accepted at this level.

Type - II:

Test of significance of difference of two means

Consider two samples of sizes n_1 and n_2 taken from two different populations with population means μ_1 and μ_1 and S.D's σ_1 and σ_2

Let $\overline{x_1}$ and $\overline{x_2}$ be the sample means and S_1 and S_2 be the S.D's of the samples

The formulated null and alternative hypothesis is,

$$H_0 : \mu_1 = \mu_2$$

$$H_1 : \mu_1 \neq \mu_2$$

The test statistic 'Z' is defined by

$$Z = \frac{\overline{x_1} - \overline{x_2}}{S.E(\overline{x_1} - \overline{x_2})}$$

$$\text{ie., } Z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{\sigma_1^{\ 2}}{n_1} + \frac{\sigma_2^{\ 2}}{n_2}}} \sim N(0,1)$$

We use the los is $\alpha = 0.05\ (or)\ 0.01$

If $|Z| < Z_\alpha$, H_0 is accepted at $\alpha\ \%Los$

otherwise, H_0 is rejected at $\alpha\ \%Los$

Note:

In many situations, we do not know the S.D's of the populations (or) population from which the samples are drawn.

In such cases, we can subs the S.D's are of samples S_1 *and* S_2 in place of σ_1 *and* σ_2

$$\therefore \text{ The test statistic Z } = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{{s_1}^2}{n_1} + \frac{{s_2}^2}{n_2}}}$$

Problems

The mean of two sample large samples of 1000 and 200 members are 67.5 inches and 68 inches respectively. Can the samples be regard as drawn from the population of standard deviation of 2.5 inches? Test at 5% Los

Solution

we set up $H_0 : \mu_1 = \mu_2$

ie., the samples are drawn from the sample population

H_1: $\mu_1 \neq \mu_2$

$$\text{The test statistic Z } = \frac{\overline{x_1} - \overline{x_2}}{\sigma\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

Given $\overline{x_1} = 67.5; \quad n_1 = 1000$

$\overline{x_2} = 68; \quad n_2 = 2000; \quad \sigma = 2.5$

$$\therefore \quad Z = \frac{67.5 - 68}{2.5\sqrt{\frac{1}{1000} + \frac{1}{2000}}} = -5.164$$

$\therefore$ $|Z| = 5.164$

We choose the Los $\alpha = 0.05$

Critical value:

The table values of Z at 5% Los is Z = 1.96

Conclusion:

Since |Z| > 1.96, H_0 is rejected at 5% Los.

$\therefore$ The sample cannot be regards as drawn from the same population.

2. Samples of students were drawn from two universities and from the weights is kilogram. The means and S.D's are calculated. Test the significance of the difference between the means of two samples

	Mean	S.D	Sample Size
University A	55	10	400
University B	57	15	100

Solution:

we set up $H_0 : \mu_1 = \mu_2$

ie., there is no significant difference between the sample means

H_1: $\mu_1 \neq \mu_2$; $\alpha = 0.05$

The test statistic $Z = \dfrac{\overline{x_1} - \overline{x_2}}{\sqrt{\dfrac{s_1^2}{n_1} + \dfrac{s_2^2}{n_2}}}$

Given $\overline{x_1} = 55$; $s_1 = 10$; $n_1 = 400$

$\overline{x_2} = 57$; $s_2 = 15$; $n_2 = 100$

$$\therefore \quad Z = \frac{55 - 57}{\sqrt{\dfrac{10^2}{400} + \dfrac{15^2}{100}}} = -1.265$$

$\therefore$ |Z| = 1.265

Critical value:

The table values of Z at 5% Los is Z = 1.96

Conclusion:

Since |Z| < 1.96, H_0 is accepted at 5% Los. We conclude that the difference between the means is not significant.

3. The average hourly wage of a sample of 150 workers is plant A was Rs. 2.56 with a S.D of Rs.1.08. The average wage of a sample of 200 workers in plant B was Rs. 2.87 with a S.D of Rs. 1.28. Can an applicant safety assume that the hourly wages paid by plant B are greater than those paid by plant A?

Solution:

Let x_1 and x_2 denote the hourly wages paid to workers in plant A and plant B respectively.

We set up $H_0 : \mu_1 < \mu_2$ (Plant Bnot greater than Plant A)

$H_1 : \mu_1 < \mu_2$ (one tailed test)

$\alpha = 0.05$

$$Z = \frac{\overline{x_1} - \overline{x_2}}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$$

Given $\overline{x_1} = 2.56;$ $s_1 = 1.08;$ $n_1 = 150$

$\overline{x_2} = 2.87;$ $s_2 = 1.28;$ $n_2 = 200$

$$\therefore \quad Z = \frac{2.56 - 2.87}{\sqrt{\frac{(1.08)^2}{150} + \frac{(1.28)^2}{200}}} = -2.453$$

$\therefore$ |Z| = 2.453

Critical value:

The table values of Z at 5% in case of one-tailed test is Z = 1.645

Conclusion:

Since $|Z| > 1.643$, H_0 is rejected at 5% Los.

$\therefore$ The hourly wage paid by Plant B are greater than those paid by Plant A

4. A sample of size 30 from a normal population yielded 80 and variance 150. A sample of size 40 from a second normal population yielded the sample mean 71 and variance 200. Test $H_0 : \mu_1 - \mu_2 = 2$. Versus $H_1 : \mu_1 > \mu_2 = 2$

Solution:

$H_0 : \mu_1 - \mu_2 = 2.$

ie., the diff 'bet the means of two population is 2

Versus $H_1 : \mu_1 > \mu_2 = 2$ (one tailed)

$$\text{Test Statistic Z} = \frac{\overline{x_1} - \overline{x_2} \quad \mu_1 - \mu_2}{\sqrt{\frac{s_1^{\ 2}}{n_1} + \frac{s_2^{\ 2}}{n_2}}}$$

$$\therefore \ Z = \frac{(80-71)-2}{\sqrt{\frac{150}{30} + \frac{200}{40}}} = 2.215$$

Critical value:

For one tail test, at 5% Los the table value of z = 1.645

Conclusion:

Since $|Z| > 1.645$, H_0 is rejected.

$\therefore$ The formulated null hypothesis $H_0 : \mu_1 - \mu_2 = 2$ is wrong

5. A buyer of electric bulbs purchases 400 bulbs; 200 bulbs of each brand. Upon testing these bulbs be found that brand A has an average of 1225 hrs with a S.D of 42 hrs. where as brand B had a mean life of 1265 hrs with a S.D of 60 hrs. Can the buyer be certain that brand B is Superior than brand A in quality?

Solution:

$H_0 : \mu_1 = \mu_2$;

ie., the two brands of bulbs do not differ in quality

ie., they have the same mean life

H_1: $\mu_1 < \mu_2$ (one tailed)

L.o.s : $\alpha = 0.05$

Test Statistic $Z = \dfrac{\overline{x_1} - \overline{x_2}}{S.E\ \overline{x_1} - \overline{x_2}} = \dfrac{\overline{x_1} - \overline{x_2}}{\sqrt{\dfrac{s_1^{\ 2}}{n_1} + \dfrac{s_2^{\ 2}}{n_2}}}$

Here, $\overline{x_1} = 1225$; $s_1 = 42$; $n_1 = 200$

$\overline{x_2} = 1265$; $s_2 = 60$; $n_2 = 200$

$$\therefore\ Z = \frac{1225 - 1265}{\sqrt{\dfrac{(42)^2}{200} + \dfrac{(60)^2}{200}}} = \frac{-40}{5.18} = -7.72$$

$\Rightarrow\ |Z| = 7.72$

Critical value:

The critical value of Z at 5% Los Z = 1.645.

Conclusion:

Since |z| < 1.645 H_0 is rejected.

$\therefore$ The brand B is superior to brand A in equality.

Type - III:

Test of significance of single proportion:

If ‘x’ is the number of times possessing a certain attribute in a sample of n items, The sample proportion $p^{'} = \frac{x}{n}$

$p^{'}$: sample porportion;

p: population proportion.

The hypothesis $H_0 : p = p^{'}$

ie., p has a specified value

Alternative hyp: $H_1 : p \neq p^{'}$

Test statistic $Z = \frac{p^{'} - p}{\sqrt{\frac{pq}{n}}}$

Since the sample is large $Z \sim N(0,1)$

Problems

1. A coin is tossed 400 times and it turns up head 216 times. Discuss whether the coin may be regarded as unbiased one.

Solution

we set up H_0 : coin is unbiased

ie., $p = \frac{1}{2} \Rightarrow q = 1 - p = \frac{1}{2}$

H_1 : coin is biased

$\alpha = 0.05$

Test statistic $Z = \frac{p^{'} - p}{\sqrt{\frac{pq}{n}}}$

Here $p^{'} = \frac{216}{400}; n = 400$

$$\therefore \quad Z = \frac{0.54 - 0.5}{\sqrt{\frac{1}{600}}} = 1.6$$

Table value of $Z = 1.96$

Conclusion:

Since $|z| < 1.96$, H_0 is accepted at 5% Los

Hence the coin may be regarded as unbiased

2. In a city of sample of 500 people, 280 are tea drinkers and the rest are coffee drinkers. Can we assume that both coffee and tea are equally popular in this city at 5% Los.

Solution:

we set up $H_0 : p = \frac{1}{2}$

ie., the coffee and tea are equally popular

$H_1 : p \neq \frac{1}{2}$

Test statistic $Z = \dfrac{p' - p}{\sqrt{\frac{pq}{n}}}$

Here $p' = \frac{280}{500} = 0.56; n = 500;$ p=0.5

$\Rightarrow$ q = 1-p = 0.5

$$\therefore \quad Z = \frac{0.56 - 0.5}{\sqrt{\frac{0.5 \times 0.5}{500}}} = 2.68$$

Conclusion:

Since $|z| > 1.96$, H_0 is rejected at 5% level

Both type of drinkers are not popular at 5% Los.

3. A manfacturing company claims that atleast 95% of its products supplied confirm to the specifications out of a sample of 200 prodcuts, 18 are defective. Test the claim at 5% Los.

Solution

we set up H_0 : The proportion of the products confirming to specification is 95%

ie., p = 0.95

H_1 : p < 0.95 (one tailed test)

$$Z = \frac{p' - p}{\sqrt{\frac{pq}{n}}}$$

Here $p' = \frac{200-18}{200} = 0.91; n = 200$

$p = 0.95 \Rightarrow q = 1 - p = 0.05$

$$\therefore \; Z = \frac{0.91 - 0.95}{\sqrt{\frac{0.95 \times 0.05}{200}}} = -2.595 \Rightarrow |Z| = 2.595$$

Critical value : at 5%Los $Z_\alpha = 1.645$

Conclusion:

$|z| = 2.595 > 1.645$, H_0 is rejected at 5% Los(Level of significance)

4. A manfacturer claims that only 4% of his products supplied by him are defective. Sample of 600 products contained 36 defectives. Test the claim of the manufactrer.

Solution:

we set up H_0 : p = 0.04

H_1 : p > 0.04 (one tailed test)

Test Statistic $Z = \frac{p' - p}{\sqrt{\frac{pq}{n}}}$

Here p =0.04 $\Rightarrow$ q = 1-p = 0.96

$p' = \frac{36}{600} = 0.06; \; n = 500$

$$\therefore \; Z = \frac{0.06 - 0.04}{\sqrt{\frac{0.04 \times 0.96}{600}}} = 2.5$$

Critical value :

The table value of Z =1.645 at 5%L.o.s

Conclusion:

$|Z| = 2.5 > 1.645$, H_0 is rejected

$\therefore$ Manufacturer's claim is not acceptable

Type - IV: Test of significance for Difference of proportion of success in two samples:

To test the significance of the difference between the sample proportions p_1' and p_2'.

We formulate the null hypothesis H_0: $p_1 = p_2$

ie., the population proportions are equal

The alternative hypothesis is H_1: $p_1 \neq p_2$

The standard error of $p_1^{'} - p_2^{'} = \sqrt{pq\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}$

Where $p = \frac{x_1+x_2}{n_1+n_2} = \frac{n_1p_1^{'}+n_2p_2^{'}}{n_1+n_2}$

The test statistic is $Z = \frac{p_1^{'} - p_2^{'}}{\sqrt{pq\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}} \sim N(0,1)$

Problems:

1. If a sample of 300 units of a manufactured product 65 units were found to be defective and in another sample of 200 units, there were 35 defectives. Is there significant difference in the proportion of defectives in the samples at 5% Los.

Solution:

H_0: $p_1 = p_2$ (ie., There is no significant difference in the proportion defectives in the samples)

The alternative hypothesis H_1: $p_1 \neq p_2$

Los: $\alpha = 0.05$

The test statistic is $Z = \frac{p_1^{'} - p_2^{'}}{\sqrt{pq\left(\frac{1}{n_1}+\frac{1}{n_2}\right)}}$

$p_1^{'} = \frac{65}{300} = 0.22$; $p_2^{'} = 0.175$

$p = \frac{100}{500} = \frac{1}{5} \Rightarrow q = \frac{4}{5}$

$$\sqrt{pq\left(\frac{1}{n_1}+\frac{1}{n_2}\right)} = \sqrt{\frac{4}{25}\left(\frac{1}{300}+\frac{1}{200}\right)} = 0.0365$$

$$\therefore\ Z = \frac{0.22 - 0.175}{0.0365} = 1.233$$

Critical value :

The table value of Z at 5% Level =1.96

Conclusion:

$|Z| < 1.96$, H_0 is accepted at 5% Los.

$\therefore$ The difference in the porportion of defectives in the samples is not significant

2. A machine puts out 16 imperfect articles in a sample of 500. After the machine is over-hauled in puts out 3 imperfect articles in a batch of 100. Has the machine improved?

Solution:

H_0: Machine has not been improved

ie., $H_0 :\ p_1 = p_2$

The alternative hypothesis H_1: $p_1\ >\ p_2$(one-tailed)

Los: $\alpha = 0.05$

The test statistic is $Z = \dfrac{p_1' - p_2'}{\sqrt{pq\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$

Here $p_1' = \dfrac{16}{500} = 0.032;\ p_2'$=0.03

$n_1 = 500;\ n_2$ =100

$p = \dfrac{19}{600}$ and $q = \dfrac{581}{600}$

$$\therefore\ Z = \frac{0.032 - 0.03}{\sqrt{\frac{19}{600} \times \frac{581}{600}\left(\frac{1}{500} + \frac{1}{100}\right)}} = 0.104$$

$|Z| = 0.104$

Critical value :

The table value of Z for one tailed test Z = 1.645 at 5 Los

Conclusion:

$|Z| < 1.645$, H_0 is accepted at 5% Los.

The Machine has not improved due to overhaulding.

3. Before an increase in excise duty on tea, 800 perons out of a sample of 1000 persons were found to be tea drinkers. After an increse is excise duty. 800 people were tea drinkers in a sample of 1200 people. Test whether there is a significant decrease in the consumption of tea after the increase in excise duty at 5% Los

Solution:

H_0: the proportion of tea drinkers before and after the increase in excise duty are equal

ie., $p_1 = p_2$

$H_1: p_1 > p_2$

Los: $\alpha = 0.05$

The test statistic is $Z = \dfrac{p_1' - p_2'}{\sqrt{pq\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}}$

Here x_1=800; x_2=800; n_1= 1000;

n_2=1200; $p_1' = \dfrac{800}{1000} = 0.8$; $p_2' = \dfrac{800}{1200} = 0.67$

$$p = \frac{x_1 + x_2}{n_1 + n_2} = \frac{1600}{2200} = \frac{8}{11} \Rightarrow q = \frac{3}{11}$$

$$\sqrt{pq\left(\frac{1}{n_1} + \frac{1}{n_2}\right)} = \sqrt{\frac{24}{121}\ 0.001 + 0.0008} = 0.0189$$

$$\therefore\ Z = \frac{0.13}{0.0189} = 6.88 \Rightarrow\ |Z|\ =\ 6.88$$

Critical value: At 5% Los 1.645

Conclusion:

$|Z| > 1.645$, H_0 is rejected.

$\therefore$ There is a significance decrease in the consumption of tea due to increase in excise duty.

Type - V: (Test of significance for the difference of S.D's of two large samples)

Let S_1 and S_2 be the S.D's of two indepedent samples of sizes n_1 and n_2 respectively

The null hypothesis H_0: $\mu_1 = \mu_2$;

ie., the sample S.D's do not differ significantly.

The Alternative Hypothesis H_1: $\mu_1 \neq \mu_2$

the test statistic is $Z = \dfrac{S_1 - S_2}{S.E(S_1 - S_2)} \sim N(0,1)$ for large 'n'

ie., If σ_1 and σ_2 are known,

$$Z = \frac{S_1 - S_2}{\sqrt{\dfrac{\sigma_1{}^2}{2n_1} + \dfrac{\sigma_2{}^2}{2n_2}}} \sim N(0,1)$$

(or)If σ_1 and σ_2 are not known,

$$Z = \frac{S_1 - S_2}{\sqrt{\dfrac{S_1{}^2}{2n_1} + \dfrac{S_2{}^2}{2n_2}}}$$

If $|Z| > Z_\alpha$, H_0 is rejected at $\alpha\%$ level, otherwise H_0 is accepted

Problems:

1. The sample of sizes 1000 and 800 gave the following results

	Mean	S.D
Sample I	17.5	2.5
Sample II	18	2.7

Assuming that the samples are indepedent, test whether the two samples may be regarded as drawn from the universe with same S.D's at 1% Level.

Solution:

We set up $H_0 : \sigma_1 = \sigma_2$;

ie., two samples maybe regarded as drawn from the universe with same S.D's

$H_1 : \sigma_1 \neq \sigma_2$

Test statistic $Z = \dfrac{S_1 - S_2}{\sqrt{\dfrac{S_1^{\,2}}{2n_1} + \dfrac{S_2^{\,2}}{2n_2}}}$

Here $n_1 = 1000$; $n_2 = 800$; $S_1 = 2.5$; $S_2 = 2.7$

$$\therefore \; Z = \frac{2.5 - 2.7}{\sqrt{\dfrac{(2.5)^2}{2000} + \dfrac{(2.7)^2}{1600}}} = \frac{-0.2}{\sqrt{0.3125 + 0.455625}}$$

$$\Rightarrow \; |Z| = 2.282$$

Critical value :

At 1% Los, the tabulated value is 2.58

Conclusion:

Since $|Z| < 2.58$, H_0 is accepted at 1% Los.

$\therefore$ The two samples may be regarded as drawn from the universe with the same S.D's

2. In a survey of incomes of two classes of workers, two random samples gave the following results. Examine whether the differences between (i) the means and (ii) the S.D's are significant.

Sample	Size	Mean annualincome (Rs)	S.D in Rs
I	100	582	24

II	100	546	28

Examine also whether the samples have been drawn from a population with same S.D

Solution:

(i) We set up $H_0 : \mu_1 = \mu_2$;

ie., the difference is not significant

$H_1 : \mu_1 \neq \mu_2$

Here it is two tailed test

$$\text{Test statistic } Z = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}}} = \frac{582 - 546}{\sqrt{\frac{(24)^2}{100} + \frac{(28)^2}{100}}}$$

$$\therefore \; Z = \frac{360}{\sqrt{(24)^2 + (28)^2}} = 9.76$$

$$\Rightarrow \; |Z| = 9.76$$

Critical value :

At 5% Los, the table value of Z is 1.96

Conclusion:

Since $|Z| > 1.96$, H_0 is rejected at 5% Los.

$\therefore$ There is a significant difference in the means in the two samples.

(ii) $H_0 : \sigma_1 = \sigma_2$

$H_1 : \sigma_1 \neq \sigma_2$

Here it is two tailed test

Los: $\alpha = 0.05$

$$\text{Test statistic } Z = \frac{S_1 - S_2}{\sqrt{\frac{S_1^2}{2n_1} + \frac{S_2^2}{2n_2}}} = \frac{24 - 28}{\sqrt{\frac{(24)^2}{200} + \frac{(28)^2}{200}}}$$

$$\therefore \quad Z = \frac{-40}{\sqrt{288+392}} = -1.53$$

$$\Rightarrow \quad |Z| = 1.53$$

Critical value :

At 5% Los, the table value of Z is 1.96

Conclusion:

Since $|Z| < 1.96$, H_0 is accepted at 5% Los.

$\therefore$ The difference between the sample S.D's is not significant.

Hence we conclude that the two samples have been drawn from population with the same S.D's

3. Two machines A and B produced 200 and 250 items on the average per day with a S.D of 20 and 25 items reply on the basis of records of 50 day's production. Can you regard both machine's equally efficient at 1% Los.

Solution:

(i) $H_0 : \sigma_1 = \sigma_2$; ie., the two machines aer equally efficient

$H_1 : \sigma_1 \neq \sigma_2$

Los: $\alpha = 0.05$

Test statistic $Z = \dfrac{S_1 - S_2}{\sqrt{\dfrac{S_1^{\,2}}{n_1} + \dfrac{S_2^{\,2}}{n_2}}}$

n_1=200 $\times$50; S_2= 25

n_2=250 $\times$50; $S_1 = 20$

$$\therefore \quad Z = \frac{(20-25)\times\sqrt{50}}{\sqrt{\frac{400}{400}+\frac{625}{500}}} = \frac{-5\sqrt{50}}{\sqrt{1+1.25}} = -23.57$$

$$\Rightarrow \quad |Z| = 23.57$$

Critical value :

At 1% Los, the table value of Z is 2.58

Conclusion:

Since $|Z| > 2.58$, H_0 is rejected at 1% Los.

We conclude that the both machines are not equally efficient at 1% Los

Small sample Tests (t - Test):

Definition:

Consider a random sample $\{x_1, x_2, \ldots, x_n\}$ of size 'n' drawn from a Normal population with mean μ and variance σ^2.

$$\text{Sample mean } \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

The unbiased estimate of the pop.variance σ^2 is denoted as s^2.

$$s^2 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n-1}$$

The student's t-statistic is defined as $t = \frac{|\bar{x} - \mu|}{s}\sqrt{n}$, Where n = sample size

The degree of freedon of this statistic

$V = n - 1$

Type I:

To test the significance of a single mean (For small samples)

$$\text{Test Statistic } t = \frac{\bar{x} - \mu}{\frac{S.D}{\sqrt{n-1}}} = \frac{\bar{x} - \mu}{\frac{S}{\sqrt{n}}}$$

s = sample S.D and

$$S^2 = \frac{1}{n-1}\sum(x - \bar{x})^2 \quad \text{(or) } S = \sqrt{\frac{ns^2}{n-1}}$$

If the computed value of t is greater than the critical value t_α, H_o is rejected

(or) if $|t| < t_\alpha$, the null hypothesis H_o is accepted at α level.

1. A machinist is making engine parts with axle diameter of 0.700 inch. A random sample of 10 parts shows a mean diamter of 0.742 inch with a S.D of 0.40. Test whether the work is meeting the specification at 5% Los

Solution:

Given that n =10; $\bar{x} = 0.742$ inches

$\mu = 0.700$ inches $\qquad$ $S = \sqrt{\dfrac{ns^2}{n-1}} = \sqrt{\dfrac{10\times(0.40)^2}{9}} = 0.4216$

s = 0.40 inches $\qquad$ $S = 0.42$

Null hypothesis H_0: the product is confirming to specification ie., there is no significant difference between $\bar{x}$ and μ

H_0: $\mu = 0.700$ inches

H_1: $\mu \neq 0.700$ inches

Test Statistic $t = \dfrac{|\bar{x}-\mu|}{s}\sqrt{n}$ =0.316

degrees of freedom = n-1 = 9

Table value of t at 5% level = 2.26

$\therefore$ the product is meeting the specification.

2. Ten individuals are chosen at random from a population and their heights are found to be in inches 63, 63, 66, 67, 68, 69, 70, 70, 71, 71. In the light of this data, discuss the suggestion that the mean height in the universe is 66 inches.

Solution:

x :	63	63	66	67	68	69	70	70	71	71
$(x-\bar{x})^2$:	23.04	23.04	3.24	0.64	0.04	1.44	4.84	4.84	10.24	10.24

$\therefore \sum x = 678$ and $\sum (x - \bar{x})^2 = 81.6$

$$\bar{x} = \frac{\sum x}{n} = \frac{678}{10} = 67.8$$

$$S = \sqrt{\frac{\sum (x - \bar{x})^2}{9}} = \sqrt{\frac{81.6}{9}} = 3.011$$

Let $H_0 : \mu = 66$ the mean and height if the universe is 66 inches

and $H_1 : \mu \neq 66$

Los $\alpha = 0.05$

Test Statistic $t = \frac{|\bar{x} - \mu|}{s}\sqrt{n} = \frac{67.8 - 66}{3.011}\sqrt{10} = 1.89$

Table value of t for 9 d.f at 5% Los is $t_0 = 2.2$

Since $|t| < t_0, H_0$ is accepted at 5% level.

$\therefore$ The mean height of universe of 66 is accepted.

Type II: (Test of significance of difference of mean)

Test Statistic $t = \dfrac{\bar{x}_1 - \bar{x}_2}{S\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$

Where $S^2 = \dfrac{n_1 s_1^2 + n_2 s_2^2}{n_1 + n_2 - 2}$ (or)

$$S^2 = \frac{\sum (x_1 - \bar{x}_1)^2 + \sum (x_2 - \bar{x}_2)^2}{n_1 + n_2 - 2}$$

The number of degrees of freedom $= V = n_1 + n_2 - 2$

The calculated value of t is less than the table value of t for d.f $= n_1 + n_2 - 2$, H_0 is accepted

Otherwise H_0 is rejected at the selected Los

1. Two independent samples from normal pop's with equal variances gave the following results

Sample Size Mean S.D

1	16	23.4	2.5
2	12	24.9	2.8

Test for the equations of means.

Solution:

(i) We set up $H_0 : \mu_1 = \mu_2$; ie., there is no significant difference between their means

$H_1 : \mu_1 \neq \mu_2$

Los: $\alpha = 0.05$

Test Statistic $t = \dfrac{\overline{x_1} - \overline{x_2}}{S\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$ Where $S^2 = \dfrac{n_1 s_1^{\,2} + n_2 s_2^{\,2}}{n_1 + n_2 - 2}$

Given $\overline{x_1} = 23.4$; $n_1 = 16$; $s_1 = 2.5$

$\overline{x_2} = 24.9$; $n_2 = 12$; $s_2 = 2.8$

$$S^2 = \frac{n_1 s_1^{\,2} + n_2 s_2^{\,2}}{n_1 + n_2 - 2} = \frac{16(2.5)^2 + 12(2.8)^2}{16 + 12 - 2}$$

$$= \frac{100 + 94.08}{26} = 7.465$$

$S = 2.732$

$$\therefore \ t = \frac{23.4 - 24.9}{2.732\sqrt{\dfrac{1}{16} + \dfrac{1}{12}}} = -1.438$$

$\therefore \ |t| = 1.438$

Number of degrees of freedom $= n_1 + n_2 - 2 = 26$

Critical value :

The table value of t for 26 d.f at 5% Los is

$t_{0.05} = 2.056$

Conclusion:

Since the calculated value of t is less than table value of t,

H_0 is accepted at 5% Los.

$\therefore$ There is no significant difference between their means

2. Two independent samples of 8 and 7 items respectively had the following values

Sample I :	9	13	11	11	15	9	12	14
Sample II :	10	12	10	14	9	8	10	

Is the difference between the means of the samples significant?

Solution:

We set up $H_0 : \mu_1 = \mu_2$

$H_1 : \mu_1 \neq \mu_2$

Hence it is a two tailed test

Los: $\alpha = 0.05$

Test Statistic $t = \dfrac{\overline{x}_1 - \overline{x}_2}{S\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}}$

Where $S^2 = \dfrac{\sum (x_1 - \overline{x}_1)^2 + \sum (x_2 - \overline{x}_2)^2}{n_1 + n_2 - 2}$

Computation of t:

x_1	$d_1 = (x_1 - \overline{x_1})$ $d_1 = x_1 - 11.75$	$d_1^{\,2} = (x_1 - \overline{x_1})^2$	x_2	$d_2 = (x_2 - \overline{x_2})$ $d_2 = x_2 - 10.43$	$d_2^{\,2} = (x_2 - \overline{x_2})^2$
9	-2.75	7.5625	10	-0.43	0.1849
13	1.25	1.5625	12	1.57	2.4649
11	-0.75	0.5625	10	-0.43	0.1849
11	-0.75	0.5625	14	3.57	12.7449
15	3.25	10.5625	9	-1.43	2.0449
9	-2.75	7.5625	8	-2.43	5.9049
12	0.25	0.0625	10	-0.43	0.1849
14	2.25	5.0625			
	$\sum d_1 = 3.5$	$\sum d_1^{\,2} = 33.5$		$\sum d_2 = -0.01$	$\sum d_2^{\,2} = 23.7143$

$$\overline{x_1} = 11 + \frac{6}{8} = 11.75$$

$$\overline{x_2} = 10 + \frac{3}{7} = 10.43$$

$$\sum (x_1 - \overline{x_1})^2 = \sum d_1^{\,2} - \frac{\sum d_1^{\;2}}{n_1} = 38 - \frac{36}{8} = 33.5$$

$$\sum (x_2 - \overline{x_2})^2 = \sum d_2^{\,2} - \frac{\sum d_2^{\;2}}{n_2} = 25 - \frac{9}{7} = 33.5$$

$$\therefore S^2 = \frac{33.5 + 23.71}{8 + 7 - 2} \Rightarrow S = 2.097$$

$$\therefore t = \frac{\overline{x_1} - \overline{x_2}}{S\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$$

$$= \frac{11.75 - 10.43}{2.097\sqrt{\frac{1}{8} + \frac{1}{7}}}$$

$$t = 1.218$$

$$d.f = 8 + 7 - 2 = 13$$

Critical value:

The table value of t for 13 d.f at 5% level is 2.16

Conclusion:

Since $|t| < 2.16$, H_0 is accepted

$\therefore$ There is no significant difference between the means of the two samples.

Type III:

Testing of significance of the difference in means paired data.

When the two samples are of the same sizes and the data are paired

the test statistic is $t = \dfrac{\bar{d}}{\frac{S}{\sqrt{n}}}$

Where $\bar{d} =$ mean of differences

and $S = \sqrt{\dfrac{\sum (d - \bar{d})^2}{n-1}}$

Degrees of freedom = n-1

1. Elevan school boys were given a test in painting. They were given a month's further tution and a second test of equal difficulty was held at the end of the month. Do the marks give evidance that the students have beneifit by extra coaching?

Boys:	1	2	3	4	5	6	7	8	9	10	11
First Test (marks)	25	23	19	22	21	19	22	21	25	18	20
Second test (marks)	26	22	22	19	23	21	24	24	25	22	18

Solution:

$H_0 : \mu =$ the student have not been benefited by extra coaching.

ie., The mean of the difference between the marks of the two tests is zero

ie., $H_0 : \bar{d} = 0$

$H_1 : \bar{d} > 0$

Los: $\alpha = 0.05$ (or) 5%

the test statistic is $t = \dfrac{\bar{d}}{\frac{S}{\sqrt{n}}}$

S. No	1	2	3	4	5	6	7	8	9	10	11
d = x - y	-1	1	-3	3	-2	-2	-2	-3	0	-4	2
$d - \bar{d}$	0	2	-2	4	-1	-1	-1	-2	1	-3	3
$(d - \bar{d})^2$	0	4	4	16	1	1	1	4	1	9	9

$$\sum d = -11; \quad \bar{d} = \frac{\sum d}{n} = \frac{-11}{11} = -1$$

$$\sum (d - \bar{d})^2 = 50$$

$$S = \sqrt{\frac{\sum (d-\bar{d})^2}{n-1}} = \sqrt{\frac{50}{10}} = \sqrt{5} = 2.236$$

$$\therefore t = \frac{\bar{d}}{\frac{S}{\sqrt{n}}} = \frac{-1}{\frac{2.236}{\sqrt{11}}}$$

$$\therefore \ |t| = \frac{1}{0.625} = 1.48$$

No. of d.f = 11-1 = 10

Critical value:

At 5% Los, the table value of t at 10 degree freedom is 1.812

Conclusion:

$|t| < 1.812$, H_0 is accepted at 5% Los.

$\therefore$ The students have not been benefited by extra-coaching.

2. The scores of 10 candidates prior and after training are given below,

Prior	:	84	48	36	37	54	69	83	96	90	65
After	:	90	58	56	49	62	81	84	86	84	75

Is the training effective?

Solution:

We set up H_0 : the training is not effective

ie., $H_0 : \bar{d} = 0$

$H_1 : \bar{d} > 0$

the test statistic is $t = \frac{\bar{d}}{\frac{S}{\sqrt{n}}}$

S. No	1	2	3	4	5	6	7	8	9	10
d = x - y	-6	-10	-20	-12	-8	-12	-1	10	6	-10
$d - \bar{d}$	0.3	-3.7	-13.7	-5.7	-1.7	-5.7	5.3	16.3	12.3	-3.7
$(d - \bar{d})^2$	0.09	13.69	187.69	32.49	2.89	32.49	28.09	265.69	151.29	13.69

$$\sum d = -63; \quad \bar{d} = \frac{\sum d}{n} = \frac{-63}{10} = -6.3$$

$$\sum \left(d - \bar{d}\right)^2 = 728.1$$

$$S = \sqrt{\frac{\sum (d-\bar{d})^2}{n-1}} = \sqrt{\frac{728.1}{9}} = \sqrt{80.9} = 8.994$$

$$S = 8.994$$

$$\therefore t = \frac{-6.3}{\frac{8.994}{\sqrt{10}}} = \frac{-6.3}{2.844} = -2.21$$

$$|t| = 2.21$$

Degrees of freedom V= n-1 = 10-1 = 9

Critical value:

At 5% Los, the table value of t at 9 degree freedom is 2.262

Conclusion:

$|t| < 2.262$, H_0 is accepted at 5% Los.

$\therefore$ There is no effective in the training.

Variance Ratio Test (or) F-test for equality of variances

This test is used to test the significance of two or more sample estimates of population variance

The F-statistic is defined as a ratio of unbiased estimate of population variance

$$F = \frac{S_1^2}{S_2^2}; \quad \text{Where } S_1^2 = \frac{\sum \left(x_1 - \bar{x}_1\right)^2}{n_1 - 1}$$

$$S_2^2 = \frac{\sum \left(x_2 - \bar{x}_2\right)^2}{n_2 - 1}$$

$\therefore$ The distribution of $F = \dfrac{S_1^2}{S_2^2}$ $\quad S_1^2 > S_2^2$ is given by the following p.d.f

If S_1^2 and S_2^2 are the variances of two sample of sizes n_1 and n_2 respectively, the estimate of the population variances based on these samples are respectively

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1}; \qquad S_2^2 = \frac{n_2 s_2^2}{n_2 - 1}$$

$$\text{d.f } V_1 = n_1 - 1 \;\&V_2 = n_2 - 1$$

While defining the statistic F, the large oftwo variances is always placed in the numerator and smaller in the denominator

Test of significance for equality of population variances

Consider two independent R, samples $x_1, x_2, ...x_{n_1}$ & $y_1, y_2, ...y_{n_2}$ from normal populations

The hypothesis to be tested is

"The population variances are same".

we set up: $H_0 : \sigma_1^2 = \sigma_2^2$

& $H_1 : \sigma_1^2 \neq \sigma_2^2$

The test statistic $\quad F = \dfrac{S_1^2}{S_2^2} \quad S_1^2 > S_2^2$

$$S_1^2 = \frac{1}{n_1 - 1}\sum_{i=1}^{n} \left(x_i - \bar{x}\right)^2 \text{ and } S_1^2 = \frac{1}{n_1 - 1}\sum_{j=1}^{n} \left(y_j - \bar{y}\right)^2$$

F distribution with d.f $V_1 = n_1 - 1 \;\&V_2 = n_2 - 1$

Problems:

1. It is known that the mean diameters o rivets produced by two firms A and B are practically the same but the standard deviations may differ.
   For 22 rivets produced by A, the S.D is 2.9 m, while for 16 rivets manufactured by B, the S.D is 3.8 m. Test whether the products of A have the same variability as those of B
   Solution:
   $H_0 : \sigma_1^2 = \sigma_2^2$

ie., variability for the two types of products are same.

Los: $\alpha = 0.05$ (or) 5%

The test statistic $F = \frac{S_1^2}{S_2^2}$ $S_1^2 > S_2^2$

Given, $n_1 = 22;$ $n_2 = 16$

$S_1 = 2.9;$ $S_2 = 3.8$

$$S_1^2 = \frac{n_1 s_1^2}{n_1 - 1} = \frac{22(2.9)^2}{22-1} = 8.81$$

$$S_2^2 = \frac{n_2 s_2^2}{n_2 - 1} = \frac{16(3.8)^2}{16-1} = 15.40$$

$$F = \frac{S_2^2}{S_1^2} \quad S_2^2 > S_1^2$$

$$= \frac{15.40}{8.81}$$

$F = 1.748$

Number of degrees of freedom are $V_1 = 16 - 1 = 15$

$V_2 = 22 - 1 = 21$

Critical value:

At 5% Los, the table value of F at d.f (15,21) is F = 2.18

Conclusion:

$F < 2.18$, H_0 is accepted at 5% Los.

$\therefore$ Variability for two types of products may be same.

2. Two random samples of sizes 8 and 11, drawn from two normal populations are characterized as follows

	Size	Sum of observations	Sum of square of observations
Sample I	8	9.6	61.52
Sample II	11	16.5	73.26

You are to decide if the two populations can be taken to have the same variance.

Solution:

Let x and y be the observations of two samples

we set up: $H_0: \sigma_1^2 = \sigma_2^2$

$\& \; H_1 : \sigma_1^{\,2} \neq \sigma_2^{\,2}$

For sample I

$$s_1^{\,2} = \frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2$$

$$= \frac{61.52}{8} - \left(\frac{9.6}{8}\right)^2$$

$$= 7.69 - (1.2)^2 = 7.69 - 1.44$$

$$s_1^{\,2} = 6.25$$

For sample II

$$s_2^{\,2} = \frac{\sum y^2}{n} - \left(\frac{\sum y}{n}\right)^2$$

$$= \frac{73.26}{11} - \left(\frac{16.5}{11}\right)^2$$

$$= 6.66 - (1.5)^2 = 6.66 - 2.25$$

$$s_2^{\,2} = 4.41$$

$$S_1^{\,2} = \frac{n_1 s_1^{\,2}}{n_1 - 1} = \frac{8(6.25)}{7} = 7.143$$

$$S_2^{\,2} = \frac{n_2 s_2^{\,2}}{n_2 - 1} = \frac{11(4.41)}{10} = 4.851$$

$$F = \frac{S_2^{\,2}}{S_1^{\,2}} \quad S_2^{\,2} > S_1^{\,2}$$

$$= \frac{7.143}{4.851} = 1.472$$

$$F = 1.472$$

Number of degrees of freedom are $V_1 = n_1 - 1 = 8 - 1 = 7$

$V_2 = n_2 - 1 = 11 - 1 = 10$

Critical value:

The table value of F for (7,10) d.f at 5% Los is 3.14

Conclusion:

Since $|F| < 3.14$, H_0 is accepted at 5% level

$\therefore$ Variances of two populations may be same.

Variability for two types of products may be same.

Chi-Square Test

Definition

If O_i (i = 1, 2, ... n) are set of observed (experimental) frequencies and $E_i\,(i = 1, 2, \ldots n)$

are the corresponding set of expected frequncies, then the statistic

χ^2 is defined as

$$\chi^2 = \sum_{i=1}^{n} \frac{\left(O_i - E_i\right)^2}{E_i}$$

The degree of freedom is $v = n - 1$

For fitting Binomial distribution $v = n - 1$

For fitting Poisson distribution $v = n - 2$

For fitting Normal distribution $v = n - 3$

Chi-square Test of Goodness of fit

If the calculated value of χ^2 is less than the table value at a specified Los.
The fit is considered to be good
otherwise the fit is considered to be poor.

Conditions for applying χ^2 Test

For the validity of chi-square test of "goodness of fit" between theory and experiment following

Conditions must be satisfied.

(i) The sample of observations should be independent
(ii) Constraints on the cell frequncies. If any, should be linear.
(iii) N, the total frequency should be reasonably large, say greater than 50.
(iv) N_0 theoretical cell frequency should be less than 5, If any theoretical cell frequency less than 5, the for application χ^2 test
It is pooled with the preceeding or succeeding frequency so that the pooled frequency is greater than 5 and finally adjust for the d.f lost in pooling.

Problems

1. The following table gives the number of aircraft accident that occured during the various days of the week. Test whether the acidents are uniformly distributed over the week.

Days	Mon	Tue	Wed	Thu	Fri	Sat	Total
No.of accidents	14	18	12	11	15	14	84

Solution:

We set up H_0 : The accidents are uniformly distributed over the week

Los $\alpha = 0.05$

Test Statistic $\chi^2 = \sum_{i=1}^{n} \frac{\left(O_i - E_i\right)^2}{E_i}$

Under the nul hypothesis,

The expected frequency of the on each day = $\frac{84}{6} = 14$

O_i :	14	18	12	11	15	14
E_i :	14	14	14	14	14	14

$$\chi^2 = \frac{\left(14-14\right)^2}{14} + \frac{\left(18-14\right)^2}{14} + \frac{\left(12-14\right)^2}{14} + \frac{\left(11-14\right)^2}{14} + \frac{\left(15-14\right)^2}{14} + \frac{\left(14-14\right)^2}{14}$$

$= 1.143 + 0.286 + 0.643 + 0.071$

$= 2.143$

Number of degrees of freedom $V = n - 1 = 7 - 1 = 6$

Critical value:

The tablulated value of χ^2 at 5% for 6 d.f is 12.59

Conclusion:

Since $\chi^2 < 12.59$, we accept the null hypothesis

$\therefore$ We conclude that the accidents are uniformly distributed over the week.

2. The theory predicts the population of beans in the four groups A, B , C and D should be

9:3:3:1. In an experiment among 1600 beans, the number in the four groups were 882, 313, 287 and 118. Does the experimental result support the theory?

Solution:

We set up the null hypothesis

H_0 : The theory fits well into the experiment

ie., the experimental results supports the theory

Total Number of beans = 1600

Divide these beans in the ratio 9:3:3:1

To calculate the expected frequencies

$$E(882) = \frac{9}{16} \times 1600 = 900$$

$$E(313) = \frac{3}{16} \times 1600 = 300$$

$$E(287) = \frac{3}{16} \times 1600 = 300$$

$$E(118) = \frac{1}{16} \times 1600 = 100$$

O_i : 882 313 287 118

E_i : 900 300 300 100

Test Statistic $\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$

$$\chi^2 = \frac{(882-900)^2}{900} + \frac{(313-300)^2}{300} + \frac{(287-300)^2}{300} + \frac{(118-100)^2}{100}$$

$$= 0.36 + 0.563 + 0.563 + 3.24$$

$$\chi^2 = 4.726$$

Critical Value:

The table value of χ^2 at 5% for 3 d.f is 7.815

Conclusion:

Since $\chi^2 < 7.815$, H_0 is accepted at 5% Los.

$\therefore$ We conclude that there is a very good correspondent between theory and experiment

3. 4 coins were tossed 160 times and the following results were obtained.

No. of heads :	0	1	2	3	4	
Frequency :	19	50	52	30	9	160
	0	50	104	90	36	280

Test the goodness of fit with the help of χ^2 on the assumption that the coins are unbiased

Solution:

We set up, the null hypothesis, the coins are unbiased:

The probability if getting the success of heads is $p = \frac{1}{2}$

$$q = 1 - p = \frac{1}{2}$$

When 4 coins are tossed, the probability of getting 'r' heads is given by,

$P(x = r) = n_{C_r}\ p^r q^{n-r}$; $r = 0, 1, 2, 3, 4$

$$= 4_{C_r} \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{4-r}$$

$$= 4_{C_r} \left(\frac{1}{2}\right)^4$$

$\therefore$ $P(x = r) = 4_{C_r} \dfrac{1}{16}$ $\quad r = 0, 1, 2, 3, 4$

The expected frequencies of getting 0, 1, 2, 3, 4 heads are given by $1604_{C_r} \dfrac{1}{16}$

$= 104_{C_r}$, $r = 0, 1, 2, 3, 4$

$= 10, 40, 60, 40, 10$

O_i	:	19	50	52	30	9
E_i	:	10	40	60	40	10
		26	48	43	26	12

Test Statistic $\chi^2 = \sum_{i=1}^{n} \frac{O_i - E_i^{\ 2}}{E_i}$

$$\chi^2 = \frac{19-10^{\ 2}}{10} + \frac{50-40^{\ 2}}{40} + \frac{52-60^{\ 2}}{60} + \frac{30-40^{\ 2}}{40} + \frac{9-10^{\ 2}}{40}$$

$= 8.1 + 2.5 + 1.067 + 2.5 + 0.1$

$\chi^2 = 14.267$

D.f V= n-1=5-1=4

Critical value:

The table value of χ^2 for 4 d.f at 5% Los is 9.488

Conclusion:

Since $\chi^2 > 9.488$, H_0 is rejected at 5% Los

$\therefore$ The coins are biased

4. The follwoing table shows the distribution of goals in a football match

No. of goals :	0	1	2	3	4	5	6	7
No. of mistakes :	95	158	108	63	40	9	5	2

Fit a poisson distribution and test the goodness of fit.

Solution:

Fitting of poisson distribution

x :	0	1	2	3	4	5	6	7
f :	95	158	108	63	40	9	5	2

$\sum fx = 812$ and $\sum f = 480$

$$\therefore \quad \bar{x} = \lambda = \frac{\sum fx}{\sum f} = \frac{812}{480} = 1.7$$

$\therefore$ The expected frequencies are computed by

$$= 480 \times \frac{e^{-1.7}(1.7)^r}{r!} \qquad r = 0, 1, 2, 3, 4, 5, 6, 7$$

$= 88, 150, 126, 72, 30, 10, 3, 1$

We set up H_0 : The fit is good

$$\text{Test Statistic } \chi^2 = \sum_{i=1}^{n} \frac{\left(O_i - E_i\right)^2}{E_i}$$

$$\begin{array}{lllllllll} O_i & : & 95 & 158 & 108 & 63 & 40 & \underbrace{9 \quad 5 \quad 2}_{16} \\ E_i & : & 88 & 150 & 126 & 72 & 30 & \underbrace{10 \quad 3 \quad 1}_{14} \end{array}$$

$$\chi^2 = \frac{\left(O-E\right)^2}{E} = \frac{\left(95-88\right)^2}{88} + \frac{\left(158-150\right)^2}{150} + \frac{\left(108-126\right)^2}{126} + \frac{\left(40-30\right)^2}{30} + \frac{\left(16-14\right)^2}{14} + \frac{\left(63-72\right)^2}{72}$$

$= 0.56 + 0.43 + 2.57 + 3.33 + 1.12 + 0.29$

$\chi^2 = 8.30$

Number of degrees of freedom V= n-2 = 6 - 2 =4

Critical value:

The table value of χ^2 at 5% Los for 4 d.f is 9.483

Conclusion:

Since $\chi^2 < 9.483$, H_0 is accepted at 5% Los.

$\therefore$ The fit is good

5. Apply the χ^2 test of goodness of fit to the follwoing data

O_i	:	1	5	20	28	42	22	15	5	2
E_i	:	1	6	18	25	40	25	18	6	1

Solution:

H_0 : The fit is good

$\alpha = 0.05$ (or) 5%

$$O_i : \underbrace{1 \quad 5}_{6} \quad 20 \quad 28 \quad 42 \quad 22 \quad 15 \quad \underbrace{5 \quad 2}_{7}$$

$$E_i : \underbrace{1 \quad 6}_{7} \quad 18 \quad 25 \quad 40 \quad 25 \quad 18 \quad \underbrace{6 \quad 1}_{7}$$

n = 7

Test Statistic $\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$

$$\chi^2 = \frac{(6-7)^2}{7} + \frac{(20-18)^2}{18} + \frac{(28-25)^2}{25} + \frac{(42-40)^2}{40} + \frac{(22-25)^2}{25} + \frac{(15-18)^2}{18} + \frac{(7-7)^2}{7}$$

$= 0.143 + 0.222 + 0.36 + 0.1 + 0.36 + 0.5 + 0$

$\chi^2 = 1.685$

d.f V= n-1 = 7 - 1 = 6

Critical value:

At 5% Los, the table value of χ^2 for 6 d.f is 12.592

Conclusion:

Since $\chi^2 < 12.592$, H_0 is accepted at 5% Los.

$\therefore$ The fit is good

6. The follwoing table shows the number of electricity failures in a town for a period of 180 days

Failures :	0	1	2	3	4	5	6	7
No. of days :	12	39	47	40	20	17	3	2

Use χ^2, examine whether the data are poisson distributed.

Solution:

Fitting of poisson distribution

x :	0	1	2	3	4	5	6	7
f :	12	39	47	40	20	17	3	2
fx :	0	39	94	120	80	85	18	14

$\sum fx = 450$ and $\sum f = 180$

$$\therefore \quad \bar{x} = \lambda = \frac{\sum fx}{\sum f} = \frac{450}{180} = 2.5$$

$\therefore$ The expected frequencies are computed by

$$= 180 \times \frac{e^{-2.5}(2.5)^r}{r!} \qquad r = 0, 1, 2, 3, 4, 5, 6, 7$$

E_i = 15, 37, 46, 38, 24, 12, 5, 2

r = 0 1 2 3 4 5 6 7

We set up H_0 : The fit is good

O_i : 12 39 47 40 20 17 $\underbrace{3 \quad 2}_{5}$

E_i : 15 37 46 38 24 12 $\underbrace{5 \quad 2}_{7}$

$\therefore$ Test Statistic χ^2

$$\chi^2 = \frac{(12-15)^2}{15} + \frac{(39-37)^2}{37} + \frac{(47-46)^2}{46} + \frac{(40-38)^2}{38} + \frac{(20-24)^2}{24} + \frac{(17-12)^2}{12} + \frac{(5-7)^2}{7}$$

$= 0.6 + 0.108 + 0.022 + 0.105 + 0.667 + 2.083 + 0.5 + 1$

$\chi^2 = 4.156$

d.f V = n-1 = 7 - 1 = 6

Critical value:

At 5% Los, the table value of χ^2 for 6 d.f is 12.592

Conclusion:

Since $\chi^2 < 12.592$, H_0 is accepted at 5% Los.

$\therefore$ The fit is good

Test for Independence of Attributes

Attribute A

		A_1	A_2................	A_j..................	A_t	Total
Attribute	B_1	O_{11}	O_{12}................	O_{1j}..................	O_{1t}	(B_1)
B	B_2					
	.					
	.					
	B_i	O_{i1}	O_{i2}................	O_{ij}..................	O_{it}	(B_i)
	.	.				
	.	.				

B_s	O_{s1}	O_{s2}................O_{sj}.................. O_{st}	(B_s)
Total	(A_1)	(A_2)................(A_i).................(A_t)	N

Attribute A

		A_1	A_2................A_j.................. A_t	Total
Attribute B	B_1	O_{11}	O_{12}................O_{1j}.................. O_{1t}	(B_1)
	B_2			
	.			
	.			
	.			
	B_i	O_{i1}	O_{i2}................O_{ij}.................. O_{it}	(B_i)
	.	.		
	.	.		
	.	.		
	B_s	O_{s1}	O_{s2}................O_{sj}.................. O_{st}	(B_s)
Total		(A_1)	(A_2)................(A_i).................(A_t)	N

Such a table is called (s $\times$ t) consistency table

Here, N $\rightarrow$ Total Frequency

O_{ij} $\rightarrow$ Observed frequency of $(i, j)^{th}$ cell

The expected frequency e_{ij} obtained by the rule

$$e_{ij} = \frac{\text{row total } B_i \quad \text{Column total } A_j}{N} \quad \text{Where } i = 1, 2, 3 \text{ .... } s$$

$$j = 1, 2 \text{ .... } t$$

Degrees of freedom associated with s $\times$ t consistency table = (s - 1) $\times$ (t - 1)

Chi-square table for 2×2 consistency table

In a 2×2 consistency table where in the frequencies are $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the value of χ^2 is

$$\chi^2 = \frac{(a+b+c+d)(ad-bc)^2}{(a+b)(a+c)(c+d)(b+d)}$$

Problems :

1. An opinion poll was conducted to find the reaction to a proposed civic reform in 100 members of each of the two political parties theinformation is tabulated below

	Favorable	**Unfavorable**	**Indifferent**
Party A	**40**	**30**	**30**
Party B	**42**	**28**	**30**

Test for Indepedence of reduction with the party affiliations.

Solution:

We set up H_0 : Reactions and party affiliations are independent

The expected frequencies are calculated by

				Total
	40	30	30	100
	42	28	30	100
Total	82	58	60	200

	Favorable	Unfavorable	Indifferent
Party A	$\frac{82\times100}{200}=41$	$\frac{58\times100}{200}=29$	$\frac{60\times100}{200}=30$
Party B	$\frac{82\times100}{200}=41$	$\frac{58\times100}{200}=29$	$\frac{60\times100}{200}=30$

$\therefore$ Test Statistic χ^2

$$\chi^2=\sum_{i=1}^{n}\frac{\left(O_i-E_i\right)^2}{E_i}$$

O_i : 40 30 30 42 28 30

E_i : 41 29 30 41 29 30

$$\chi^2 = \frac{40-41\ ^2}{41} + \frac{30-29\ ^2}{29} + \frac{30-30\ ^2}{30} + \frac{42-41\ ^2}{41} + \frac{28-29\ ^2}{29} + \frac{30-30\ ^2}{30}$$

$= 0.024 + 0.024 + 0.034 + 0.034$

$\chi^2 = 0.116$

Number of degrees of freedom =(2-1)(3-1) = 2

Critical value:

At 5% Los, the table value of χ^2 for 2 d.f is 5.99

Conclusion:

Since $\chi^2 < 5.99$, H_0 is accepted at 5% Los.

$\therefore$ The independence of reactions with the party affiliations may be correct.

2. In a locality 100 persons were randomly selected and asked about their educational achievements. The results are given below.

	Education		
	Middle	**High School**	**College**
Male	**10**	**15**	**25**
Female	**25**	**10**	**15**

Can you say that education depends on sex?

3. The following table gives the classification of 100 workers according to sex and the nature of work. Test whether nature of work is independent of the sec of the worker.

		Skilled	Unskilled	Total
	Male	40	20	60
Sex	Female	10	30	40
	Total	50	50	

Solution:

H_0 : Nature of work is independent of the sex of the worker

Under H_0, the expected frequencies are

$$E(40) = \frac{60\times 50}{100} = 30; \qquad E(20) = \frac{60\times 50}{100} = 30$$

$$E(10) = \frac{40\times 50}{100} = 20; \qquad E(30) = \frac{40\times 50}{100} = 20$$

$\therefore$ Test Statistic χ^2

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

O_i : 40 20 10 30

E_i : 30 30 20 20

$$\chi^2 = \frac{(40-30)^2}{30} + \frac{(20-30)^2}{30} + \frac{(10-20)^2}{20} + \frac{(30-20)^2}{20}$$

$= 3.333 + 3.333 + 5 + 5$

$\chi^2 = 16.67$

Number of degrees of freedom =(2-1)(2-1) = 1

Critical value:

The table value of χ^2 at 5% Los, for 1 d.f is 3.841

Conclusion:

Since $\chi^2 > 3.841$, H_0 is rejected at 5% Los.

$\therefore$ We conclude that the nature of work is dependent on sex of the worker.

4. From the following data, test whether there is any association between intelligency and economics conditions

		Intelligences				
		Excellent	**Good**	**Medium**	**Dull**	**Total**
Economic	**Good**	**48**	**200**	**150**	**80**	**478**
Conditions	**Not Good**	**52**	**180**	**190**	**100**	**522**
	Total	**100**	**380**	**340**	**180**	**1000**

Solution:

H_0 : There is no association between intelligency and economic conditions.

Los : $\alpha = 0.05$ (or) 5%

Under H_0, the expected frequencies are obtained as follows

$$E(48) = \frac{100 \times 478}{1000} = 47.8; \qquad E(52) = \frac{100 \times 522}{1000} = 52.2$$

$$E(200) = \frac{380 \times 478}{1000} = 181.64; \quad E(180) = \frac{380 \times 522}{1000} = 198.36$$

$$E(150) = \frac{478 \times 340}{1000} = 162.52; \quad E(190) = \frac{340 \times 522}{1000} = 177.48$$

$$E(80) = \frac{180 \times 478}{1000} = 86.04; \qquad E(100) = \frac{180 \times 522}{1000} = 93.96$$

O_i	:	48	200	150	80	52	180	190	100
E_i	:	47.8	181.64	162.52	86.04	52.2	198.36	177.48	93.96

$\therefore$ Test Statistic χ^2

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_i - E_i)^2}{E_i}$$

$$= \frac{(48-47.8)^2}{47.8} + \frac{(150-162.52)^2}{162.52} + \frac{(52-52.2)^2}{52.2} + \frac{(190-177.48)^2}{177.48} + \frac{(200-181.64)^2}{181.64}$$

$$+ \frac{(80-86.04)^2}{86.04} + \frac{(180-198.36)^2}{198.36} + \frac{(100-93.96)^2}{93.96}$$

$= 0.0008 + 0.9645 + 0.0008 + 0.8832+1.8558+0.4240+1.6994+0.3883$

$$\chi^2 = 6.2168$$

Number of degrees of freedom $= (s-1)(t-1) = (2-1)(4-1) = 3$

Critical value:

The table value of χ^2 at 5% Los for 3 d.f is 7.815

Conclusion:

Since $\chi^2 < 7.815$, H_0 is accepted at 5% Los.

$\therefore$ We conclude that there is no association between intelligency and economic conditions

5. From the following data, test the hypothesis that the flower color is independent of flatness of leaf

	Flat leaves	**Curved leaves**	**Total**
White Flowers	**99**	**36**	**135**
Red Flowers	**20**	**5**	**25**
Total	**119**	**41**	**160**

Solution:

We set up: H_0 : flower color is independent of flatness of leaf. Los α=0.05 (or) 5%

The given probelm is a 2 x 2 consistency table

$\therefore$ we use the formula to find χ^2 is

$$\chi^2 = \frac{(a+b+c+d)(ad-bc)^2}{(a+b)(a+c)(c+d)(b+d)}$$

Here, a = 99; b = 36; e = 20; d = 5

$$\chi^2 = \frac{160(495-720)^2}{(135)(119)(25)(41)} = \frac{160(50625)}{16,466,625}$$

$$\chi^2 = 0.4919$$

Number of degrees of freedom = (s-1)(t-1) = (2-1)(2-1) = 1

Critical value:

The table value of χ^2 at 5% Los for 1 d.f is 3.841

Conclusion:

Since $\chi^2 < 3.841$, H_0 is accepted at 5% Los.

$\therefore$ Flower colour is independent of flatness of leaf.

Test for single variance

Chi-square test for population variance

In this method, we set up the null hypothesis $H_0 : \sigma^2 = \sigma_0^2$ (with a specified variance)

The test statistic $\chi^2 = \dfrac{ns^2}{\sigma^2}$

Where n = sample size

s = sample variance

σ = population variance

Note:

* If the sample size n is large (>30)

The test statistic $z = \sqrt{2\chi^2} - \sqrt{2n-1} \sim N(0,1)$

We use the usual normal test.

1. A random sample of size 9 from a normal population have the following values 72, 68, 74, 77, 61, 63, 63, 73, 71. Test the hypothesis that the population variance is 36.

Solution:

Null hypothesis $H_0 : \sigma^2 = 36$

Alternative hypothesis $H_1 : \sigma^2 \neq 36$

Los α : 0.05 (or) 5%

$\therefore$ The test statistic $\chi^2 = \dfrac{ns^2}{\sigma^2}$

x : 72 68 74 77 61 63 63 73 71

$\sum x = 622; \quad \bar{x} = \frac{\sum x}{n} = \frac{622}{9} = 69.11$

$x - \bar{x}$: 2.9 -1.1 4.9 7.9 -8.1 -6.1 -6.1 3.9 1.9

$(x - \bar{x})^2$: 8.41 1.21 24.01 62.41 65.61 37.21 37.21 15.21 3.61

$\sum (x - \bar{x})^2 = 254.89$

$\chi^2 = \frac{ns^2}{\sigma^2} = \frac{254.89}{36} = 7.08$

d. f n-1 = 9-1 = 8

Critical value:

The table value of χ^2 for 8 d.f at 5% Los is 15.51

Conclusion:

Since $\chi^2 < 15.51$, H_0 is accepted at 5% Los.

$\therefore$ We conclude that the hypothesis of population variance is 36 is accepted

2. Test the hypothesis that $\sigma = 10$, given that s = 15 for a random sample of size 50 from a normal population

Solution:

Null hypothesis $H_0: \sigma = 10$

Alternative hypothesis $H_1: \sigma \neq 36$

We are given n = 50; s = 15

$\chi^2 = \frac{ns^2}{\sigma^2} = \frac{50 \times 225}{100} = 112.5$

Since 'n' is large (n > 30, the test statistic $z = \sqrt{2\chi^2} - \sqrt{2n-1}$

$= \sqrt{225} - \sqrt{99} = 15 - 9.95$

$$z = 5.05$$

This statistic z follows N(0,1)

Critical value:

At 5% Los, the table value of z is 3

Conclusion:

Since $|z| > 3$, H_0 is rejected.

$\therefore$ We conclude that $\sigma \neq 10$

3. The standard deviation of the distribution of times taken by 12 workers for performing a Job is 11 sec. Can it be taken 1 as a sample from a popualation whose S.D is 10 sec.

Solution:

Let $H_0 : \sigma = 10$

ie., the population standard deviation $\sigma = 10$

$H_1 : \sigma \neq 10$

Los α : 0.05 (or) 5% Los

Given n = 12; s = 11

$\therefore$ The test statistic is

$$\chi^2 = \frac{ns^2}{\sigma^2}$$

$$= \frac{12 \times 121}{100} = 14.52$$

$$\chi^2 = 14.52$$

Degrees of freedom = n - 1 = 12 - 1 = 11

Critical value:

The table value of χ^2 for 11 d.f at 5%Los is 19.675.

Conclusion:

Since $\chi^2 < 19.675$, H_0 is accepted at 5% level

$\therefore$ The S.D of the time element is 10 sec is supported.

ie., the population standard deviation $\sigma = 10$

UNIT-III

Solution Of Equation And Eigen Value Problems Algebraic Equation.

If f(x) is a polynomial, then the equation f(x) = 0 is called an algebraic equation.

Ex: $x^3 + 4x + 3 = 0$; $x^2 - 4 = 0$

Transcendental equation:

Equation which involves transcendental of functions like $\sin x, \cos x, \tan x, \log x$, e^x etc., are called transcendental equation

Ex: $e^x + 2 = 0$, $\log x - 4\cos x = 12$

Location of Roots:

The following results helps us to locate the interval in which the roots of $f(x) = 0$

"If $f(x)$ is a continuous function in the interval (a,b) and if $f(a)$ and $f(b)$ have opposite signs, then the equation $f(x) = 0$ has at least one real root lying in the interval (a,b) ".

The following methods are used for solving algebraic and transcendental equations.

(i) Fixed point iteration $X = g(x)$ method (or) method of successive approximation

Newton's Method (or Newton – Raphson methods)

Formula, $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, $n = 0, 1, 2....$

If we choose the initial approximation x_o close to the root then we get the root of the equation very quickly.

If the initial approximation to the root is not given then we can find any two values of x say a and b such that $f(a)$ & $f(b)$ are of opposite sign.

If $|f(a)| < |f(b)|$ then 'a' is taken as the first approximate to the root.

Newton's method is also referred to as the method of tangents

Condition for convergent is $|f(x).f''(x)| < |f'(x)|^2$

1. Find the +ve root of $x^4 - x = 10$ correct to three decimal places using Newton Raphson Method.

Solution:

Let $f(x) = x^4 - x - 10$

$$f'(x) = 4x^3 - 1$$

$$f(1) = 1 - 1 - 10 = -ve$$

$$f(2) = 2^4 - 2 - 10 = 16 - 2 - 10 = 4(+ve)$$

$\therefore$ A root lies between 1 and 2

Take $x_0 = 2$

Formula $$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{f(2)}{f'(2)}$$

$$= 2 - \left[\frac{2^4 - 2 - 10}{4(2)^3 - 1}\right]$$

$$= 2 - \left[\frac{2^4 - 2 - 10}{4(2)^3 - 1}\right]$$

$$= 2 - \left[\frac{4}{31}\right]$$

$$= 1.8709$$

$$x_1 = 1.871 \text{(Correct to 3 decimal places)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.871 - \frac{f(1.871)}{f'(1.871)}$$

$$= 1.871 - \left[\frac{(1.871)^4 - 1.871 - 10}{4(1.871)^3 - 1}\right]$$

$$=1.871-\left[\frac{0.3835}{25.199}\right]$$

$$x_2=1.856$$

$$x_3=x_2-\frac{f(x_2)}{f'(x_2)}$$

$$=1.856-\frac{f(1.856)}{f'(1.856)}$$

$$=1.856-\left[\frac{(1.856)^4-1.856-10}{4(1.856)^3-1}\right]$$

$$=1.856-\left[\frac{0.010}{24.574}\right]$$

$$x_3=1.856$$

The better approximate root is 1.856

2. Find the real +ve root of $3x-\cos x-1=0$ by Newton's method to 6 decimal place

Solution:

Let $f\ \ x\ =3\text{x}-\cos x-1$

$$f(\text{x})=0-1-1=-2(-\text{ve})$$

$$f(1)=3-\cos 1-1=2-\cos 1$$

$$=1.459698\ (+\text{ve})$$

$\therefore$ The root lies between 0 and 1

$$|f(0)|\ >\ |f(1)|$$

Hence the root is nearer to 1

$$x_0=0.6$$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}$$

$$=0.6-\frac{f(0.6)}{f'(0.6)}$$

$$=0.6-\left[\frac{3(0.6)-\cos(0.6)-1}{3+\sin(0.6)}\right]$$

$$=0.6-(-0.607101)$$

$$x_1 \ =0.607108$$

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}$$

$$=0.607108-\left(\frac{3(0.607108)-\cos(0.607108)-1}{3+\sin\ \ 0.607108}\right)$$

$$=0.607108-(0.000006)$$

$$x_2=0.607102$$

$$x_3=x_2-\frac{f(x_2)}{f'(x_2)}$$

$$x_3=0.607102-\frac{f(0.607102)}{f'(0.607102)}$$

$$=0.607102-\left(\frac{3(0.607102)-\cos(0.607102)-1}{3+\sin\ \ 0.607102}\right)$$

$$=0.607102-(0.0000004)$$

$$x_3=0.607102$$

Here $x_2=x_3=0.607102$

$\therefore$ The root is 0.607102 correct to six decimal.

3. Solve by Newton's method, a root of $e^x-4x=0$

Solution:

Let $f \ \ \mathrm{x} \ =e^x\text{-}4\mathrm{x}$

$$f^{'}(\mathrm{x})=e^x\text{-}4$$

$$f(0)=e^0\text{-}4(0)=1(+\mathrm{ve})$$

$$f(1)=e^1\text{-}4(0)=\text{-}1.2817(\text{-ve})$$

One root lies between 0 and 1

$$|f(0)| < |f(1)|$$

Hence the root is nearer to 0

Let $x_0 = 0.3$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.3 - \frac{f(0.3)}{f'(0.3)}$$

$$= 0.3 - \left[\frac{e^{0.3} - 4(0.3)}{e^{0.3} - 4}\right]$$

$$= 0.3 - \left[\frac{0.1499}{-2.650}\right]$$

$$x_1 = 0.3566$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.3566 - \frac{f(0.3566)}{f'(0.3566)}$$

$$= 0.3566 - \left[\frac{e^{0.3566} - 4(0.3566)}{e^{0.3566} - 4}\right]$$

$$x_2 = 0.3574$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.3574 - \frac{f(0.3574)}{f'(0.3574)}$$

$$= 0.3574 - \left[\frac{e^{0.3574} - 4(0.3574)}{e^{0.3574} - 4}\right]$$

$$x_3 = 0.3574$$

Here, $x_2 = x_3 = 0.3574$

Hence the root is 0.3574

4. Find a root of $x\log_{10} x - 1.2 = 0$ by N.R method correct to three or four decimal places

Solution:

Let $f(x) = x\log_{10} x - 1.2$

$$= \log_{10} + \log_{10} x$$

$$f'(x) = \log_{10} e + \log_{10} x$$

$$f(1) = \log_{10} 1 - 1.2 = -1.2 = -ve$$

$$f(2) = 2\log_{10} 2 - 1.2 = -0.598 = -ve$$

$$f(3) = 3\log_{10} 3 - 1.2 = -0.231 = +ve$$

$$|f(2)| > |f(3)|$$

$\therefore$ a root lies between 2 and 3 and all it is nearer to 3.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2.7 - \frac{f(2.7)}{f'(2.7)}$$

$$= 2.7 - \left[\frac{2.7\log_{10} 2.7 - 1.2}{\log_{10} e + \log_{10} 2.7}\right]$$

$$= 2.7 - \left[\frac{-0.035}{0.867}\right]$$

$$= 2.7 + \left[\frac{0.035}{0.867}\right]$$

$$x_1 = 2.740$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.740 - \left[\frac{(2.74)\log_{10} 2.74 - 1.2}{\log_{10} e + \log_{10} 2.74}\right]$$

$$= 2.74 - \left[\frac{-0.0006}{0.872}\right]$$

$x_2 = 2.740$

$\therefore$ The revised root is 2.740

1. Find an iterative formula to find the reciprocal of a given number N and hence find the value $\frac{1}{19}$

Solution:

Let $x = \frac{1}{N} \Rightarrow N = \frac{1}{x} \Rightarrow x - \frac{1}{N} = 0 \Rightarrow N = \frac{1}{x}$

$$f(x) = \frac{1}{x} - N$$

$$f'(x) = \frac{-1}{x^2}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left(\frac{\frac{1}{x_n} - N}{\frac{-1}{x_n^{\ 2}}} \right)$$

$$= x_n + x_n^{\ 2}\left(\frac{1}{x_n} - N \right)$$

$$x_{n+1} = 2x_n - Nx_n^{\ 2}$$

$x_{n+1} = x_n(2 - Nx_n)$ is the iterative formula to find $\frac{1}{19}$,take N=19 further

$\frac{1}{20} = 0.05$ take $x_0 = 0.05$.

$x_1 = 0.0525$

$x_2 = 0.0526$

$x_3 = 0.0526$

The reciprocal of 19. Correct to 4 decimal place is 0.0526.

2. Show that the iteration formula for finding the square root of N is $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n} \right)$ and hence find the value of $\sqrt{15}$

Solution:

Let $x = \sqrt{N}$

$$x^2 = N \Rightarrow x^2 - N = 0$$

$$f(x) = x^2 - N$$

$$f'(x) = 2x$$

$$\text{Newton's fomula is } x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{(x_n^{\,2} - N)}{2x_n}$$

$$= \frac{2x_n^{\,2} - x_n^{\,2} + N}{2x_n}$$

$$= \frac{x_n^{\,2} + N}{2x_n}$$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right) \quad ; n = 0, 1, 2, \ldots$$

To find $\sqrt{15}$:

The initial approximation is $x_0 = 3.5$

Taking N = 15, the successive approximations are

$$x_1 = \frac{1}{2}\left(3.5 + \frac{15}{3.5}\right) \Rightarrow x_1 = 3.893$$

$$x_2 = \frac{1}{2}\left(3.893 + \frac{15}{3.893}\right) \Rightarrow x_2 = 3.873$$

$$x_3 = \frac{1}{2}\left(3.873 + \frac{15}{3.873}\right) \Rightarrow x_3 = 3.873$$

$\therefore$ The square root of 15, correct to 3 decimal places is 3.873

Solution Of Linear Algebraic Equation:

There are two types of methods to solve simultaneous linear algebraic equations.

1. Direct Method

   i) Gauss Elimination Method

   ii) Gauss Jordon Method

2. Indirect Method (Or) Iterative Method

   i) Gauss Seidel Method

   ii) Gauss Jacobi Method

Solution Of Linear System By Direct Method:

Type – I

Gauss Elimination Method:

Gauss Elimination Method is a direct method which consists of transforming the given system of simultaneous equations to an equivalent upper triangular system. From this system the required solution can be obtained by the method of each substitution.

Working Rules:

Consider the system of eqns. $AX = B$

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

Step: 1

From the augmented matrix

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Step 2:

Reduce to upper triangular matrix

$$\text{ie.,}\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ 0 & a_{22} & a_{23} & b_2 \\ 0 & 0 & a_{33} & b_3 \end{array}\right]$$

Here, $a_{11}, a_{22}, a_{33} \neq 0$. This element is also called pivot element

Step 3:

By using back substitutions we get the values of x, y, & z

Note:

1. The first equation is called the pivotal equation.
2. The leading coefficient a_{11} is called the first pivot element or key element.

Problems based on gauss elimination method

1. Solve the following system $2x+y=3, 7x-3y=4$ by G.E Method

Solution:

Given, $2x+y=3$

$$7x-3y=4$$

This equation is of the form $AX=B$

The augmented matrix is

$$A,B=\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 7 & -3 & 4 \end{array}\right]$$

$$\sim\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 0 & -13 & -13 \end{array}\right] \quad R_2 \Rightarrow 2R_2 - 7R_1$$

$$2x+y=3$$

$$-13y=-13$$

By back substitution, from From $2 \Rightarrow -13y=-13y$

$$y=1$$

subs. $y=1$ in (1) we get

$$2x+1=3$$

$$2x=2$$

$$x=1$$

Hence the solution is

$$x=1$$

$$y=1$$

2. Solve by Gauss-Elimination method the equation

$$2x+y+4z=12$$

$$8x - 3y + 2z = 20$$

$$4x + 11y - z = 33$$

Solution:

Given system is equivalent to,

$$\begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 20 \\ 33 \end{bmatrix}$$

The augmented matrix is

$$(A,B) = \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 8 & -3 & 2 & 20 \\ 4 & 11 & -1 & 33 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & -7 & -14 & -28 \\ 0 & 9 & -9 & 9 \end{array}\right] \quad R_2 \Rightarrow R_2 - 4R_1, R_3 \Rightarrow R_3 - 2R_1$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & -1 & 1 \end{array}\right] \quad R_2 \Rightarrow \frac{R_2}{-7}, R_3 \Rightarrow \frac{R_3}{9},$$

$$\sim \left[\begin{array}{ccc|c} 2 & 1 & 4 & 12 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -3 & -3 \end{array}\right] \quad R_3 \Rightarrow R_3 - R_2$$

By Back Substitution method

$$-3z = -3$$

$$z = 1$$

$$y + 2z = 4$$

$$y + 2 = 4$$

$$y = 2$$

$$2x + y + 4z = 12$$

$$2x + 2 + 4 = 12$$

$$x = 3$$

Hence the solution

$x = 3 \quad ; \quad y = 2 \quad ; \quad z = 1$

2. Using gauss – Elimination method, solve the system

3.15x - 1.96y + 3.85z = 12.95

2.13x + 5.12y - 2.89z = -8.61

5.92x + 3.05 + 2.15z = 6.88

Solution:

Given system is equivalent is

$$\begin{bmatrix} 3.15 & -1.96 & 3.65 \\ 2.13 & 5.12 & -2.89 \\ 5.92 & 3.05 & 2.15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12.95 \\ -8.61 \\ 6.88 \end{bmatrix}$$

$$AX = B$$

The augmented matrix is

$$(A, B) = \left[\begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 2.13 & 5.12 & -2.89 & -8.61 \\ 5.92 & 3.05 & 2.15 & 6.88 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & 20.3028 & -17.304 & -54.705 \\ 0 & 21.2107 & -16.0195 & -54.992 \end{array}\right] \quad R_2 \Rightarrow 3.15R_2 - 2, R_3 \Rightarrow 3.15R_3 - 5.92$$

$$\sim \left[\begin{array}{ccc|c} 3.15 & -1.96 & 3.85 & 12.95 \\ 0 & 20.3028 & -17.304 & -54.705 \\ 0 & 0 & 41.7892 & 43.8398 \end{array}\right] \quad R_3 \Rightarrow 20.302R_3 - 21.21071$$

by back substitution method

$$41.7892\, z = 43.8398$$

$$z = 1.049$$

$$20.3028\, y - 17.304 z = -54.705$$

$$20.3028 y - 17.304(1.049) = -54.705$$

$$y = -1.800$$

$$3.15x - 1.96(-1.8) + 3.85\,(1.049) = 12.95$$

$$x = 1.709$$

Hence the solution is

$x = 1.709$

$y = -1.800$

$z = 1.049$

Homework

$1. 4x - 3y = 11, 3x + 2y = 4 \, (x = 2, y = -1)$

$2. x - y + z = 1, -3x + 2y - 3z = -6, 2x - 5y + 4z = 5 \, (x = -2, y = z = 6)$

$3. 10x - 2y + 3z = 23, 2x + 10y - 5z = -33, 3x - 4y + 10z = 41 \; (x = 1, y = -2, z = 3)$

TYPE-II

Gauss Jordon Method

This method is a modification of the gauss elimination method. In this method we reduce the augmented matrix into a diagonal matrix (or) unit matrix.

Working Rule:

Consider the system of eqns. $AX = B$

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$$

$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3$$

1) Solve the following system by G.E Method

$$5x_1 + x_2 + x_3 + x_4 = 4$$

$$x_1 + 7x_2 + x_3 + x_4 = 12$$

$$x_1 + x_2 + 6x_3 + x_4 = -5$$

$$x_1 + x_2 + x_3 + 4x_4 = -6$$

Solution:

The given system is of the form is

$$AX = B$$

$\therefore$ The augment matrix of the given system is

$$(A/B) = \begin{bmatrix} 1 & 1 & 1 & 4 & -6 \\ 1 & 7 & 1 & 1 & 12 \\ 1 & 1 & 6 & 1 & -5 \\ 5 & 1 & 1 & 1 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 & -6 \\ 0 & 6 & 0 & -3 & 13 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & -4 & -4 & -19 & 34 \end{bmatrix} \quad R_2 \Rightarrow R_2 - \frac{1}{1}R_1, R_3 \Rightarrow R_3 - \frac{1}{1}R_1, R_4 \Rightarrow R_4 - \frac{5}{1}R_1$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 & -6 \\ 0 & 6 & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & 0 & -4 & -21 & 46 \end{bmatrix} \quad R_4 \Rightarrow R_4 - \frac{-4}{6}R_2$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 4 & -6 \\ 0 & 6 & 0 & -3 & 18 \\ 0 & 0 & 5 & -3 & 1 \\ 0 & 0 & 0 & \frac{-117}{5} & \frac{234}{5} \end{bmatrix} \quad R_4 \Rightarrow R_4 - \left(\frac{-4}{5}\right)R_3$$

By back substitution method

$$x_1 + x_2 + x_3 + 4x_4 = -6$$

$$6x_2 + 0x_3 - 3x_4 = 18$$

$$5x_3 - 3x_4 = 1$$

$$\frac{-117x_4}{5} = \frac{234}{5}$$

$$x_4 = -2$$

$$5x_3 + 6 = 1$$

$$5x_3 = -5 \Rightarrow x_3 = -1$$

$$6x_2 + 6 = 18$$

$$6x_2 = 12 \Rightarrow x_2 = 2$$

$$x_1 + 2 - 1 - 8 = -6$$

$$x_1 = -6 + 9 - 2$$

$$x_1 = 1$$

$\therefore$ The Solution is

$x_1 = 1$

$x_2 = 2$

$x_3 = -1$

$x_4 = -2$

Gauss Jordon Method

1. Solve by gauss Jordan method

   $3x + 4y + 5z = 18;\ 2x - y + 8z = 13; 5x - 2y + 7z = 20$

Solution:

The given is of the form is $AX = B$

The augmented matrix of the given system is

$$(A/B) = \left[\begin{array}{ccc|c} 3 & 4 & 5 & 18 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & -3 & 5 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{array}\right] R_1 \Rightarrow R_1 - R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & -3 & 5 \\ 0 & -11 & 14 & 3 \\ 5 & -27 & 22 & -5 \end{array}\right] R_2 \Rightarrow R_2 - \frac{2}{1}R_1,\ R_3 \Rightarrow R_3 - \frac{5}{1}R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 5 & -3 & 5 \\ 0 & 1 & \frac{-14}{11} & \frac{-3}{11} \\ 5 & -27 & 22 & -5 \end{array}\right] R_2 \Rightarrow \frac{R_2}{-11}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{37}{11} & \frac{70}{11} \\ 0 & 1 & \frac{-14}{11} & \frac{-3}{11} \\ 0 & 0 & \frac{-136}{11} & \frac{-136}{11} \end{array}\right] R_1 \Rightarrow R_1 - \frac{5R_2}{1}, R_3 \Rightarrow R_3 + \frac{27R_2}{1},$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & \frac{37}{11} & \frac{70}{11} \\ 0 & 1 & \frac{-14}{11} & \frac{-3}{11} \\ 0 & 0 & 1 & 1 \end{array}\right] R_3 \Rightarrow \frac{R_3}{\frac{-136}{11}}$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad R_1 \Rightarrow R_1 - \frac{37R_3}{11}, R_2 \Rightarrow R_2 + \frac{14R_3}{11},$$

$\therefore$ The Solution is $x = 3, \quad y = 1, \quad z = 1$

2. Solve by gauss Jordan method

$$2x_1 - 7x_2 + 4x_3 = 9;\ x_1 + 9x_2 - 6x_3 = 1\ ;\ -3x_1 + 8x_2 + 5x_3 = 6$$

Solution:

The given is of the form is $AX = B$

The augmented matrix of the given system is

$$(A/B) = \left[\begin{array}{ccc|c} 2 & -7 & 4 & 9 \\ 1 & 9 & -6 & 1 \\ -3 & 8 & 5 & 6 \end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c} 1 & 9 & -6 & 1 \\ 2 & -7 & 4 & 9 \\ -3 & 8 & 5 & 6 \end{array}\right] R_1 \Leftrightarrow R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & 9 & -6 & 1 \\ 0 & -25 & 16 & 7 \\ 0 & 35 & -13 & 9 \end{array}\right] R_2 \Rightarrow R_2 - \frac{2}{1}R_1,\ R_3 \Rightarrow R_3 - \frac{-3}{1}R_1$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & \frac{-6}{25} & \frac{88}{25} \\ 0 & -25 & 16 & 7 \\ 0 & 0 & \frac{47}{5} & \frac{94}{5} \end{array}\right] R_1 \Rightarrow R_1 + \frac{9}{25}R_2 \quad , R_3 \Rightarrow R_3 + \frac{35}{25}R_2$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{100}{25} \\ 0 & -25 & 0 & -25 \\ 0 & 0 & \frac{47}{5} & \frac{94}{5} \end{array}\right] R_1 = R_1 + \frac{6}{25}\times\frac{5}{47}R_3 \,,\quad R_2 \Rightarrow R_2 - 16\times\frac{5}{47}R_3$$

$$\sim\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] R_2 \Rightarrow \frac{R_2}{-25} \,,\; R_3 \Rightarrow \frac{R_3}{\frac{47}{5}}$$

$\therefore$ The solution is

$$x = 4 \;;\quad y = 1 \;;\quad z = 2.$$

or

$$x_1 = 3 \;;\quad x_2 = 1 \;;\quad x_3 = 2.$$

Gauss Jacobi Method

If x^{0}, y^{0}, z^{0} are the initial values of x, y, z respectively, then the first iteration value are

$$x^{1} = \frac{1}{a_1}(d_1 - b_1 y^{0} - c_1 z^{0})$$

$$y^{1} = \frac{1}{b_2}(d_2 - a_2 x^{0} - c_2 z^{0})$$

$$z^{1} = \frac{1}{c_3}(d_3 - a_3 x^{0} - b_3 y^{0})$$

Again using first Iteration value following System the Second Iteration Value are

$$x^{2} = \frac{1}{a_1}\; d_1 - b_1 y^{1} - c_1 z^{1}$$

$$y^{2} = \frac{1}{b_2}\; d_2 - a_2 x^{1} - c_2 z^{1}$$

$$z^{2} = \frac{1}{c_3}\; d_3 - a_3 x^{1} - b_3 y^{1}$$

Proceeding in the same way if the r^{th} iterates are $x^{(r)}, y^{(r)}, z^{(r)}$ then the iteration for this method is

$$x^{r+1} = \frac{1}{a_1}\ d_1 - b_1 y^{r} - c_1 z^{r}$$

$$y^{r+1} = \frac{1}{b_1}\ d_2 - a_2 x^{r} - c_2 z^{r}$$

$$x^{r+1} = \frac{1}{a_1}\ d_3 - a_3 x^{r} - b_3 y^{r}$$

The iteration is stopped when the values x, y, z respectively with the desired degree of accuracy .

1. Solve the Following System by Gauss-Jacobi Method

   10x-5y-2z=3;   4x-10y+3z=-3;   x+6y+10z=-3

   Solution:

$$A = \begin{bmatrix} <10> & -5 & -2 \\ 4 & <-10> & 3 \\ 1 & 6 & <10> \end{bmatrix} \text{ is Diagonally dominant}$$

   Since

   |10|>|-5|+|-2|;

   |-10|>|4|+|3|;

   |10|>|1|+|6|

   By Gauss Jacobi Iteration Process is

$$x=\frac{1}{10}\ 3+5y+2z$$

$$y=\frac{1}{10}\ 3+4x+3z$$

$$z=\frac{-1}{10}\ 3+x+6y$$

Initial Values of $[x^{0}, y^{0}, z^{0}] = (0,0,0)$ the Successive iteration value are

Iteration	$x = \frac{1}{10}(3+5y+2z)$	$y = \frac{1}{10}(3+4x+3z)$	$z = \frac{1}{10}(3+x+6y)$
1	0.3	0.3	-0.3
2	0.39	0.33	-0.57

3	0.363	0.303	-0.537
4	0.3441	0.2841	-0.5181
5	0.3384	0.2822	-0.5048
6	0.3401	0.2839	-0.5032
7	0.3413	0.2851	-0.5044
8	0.3416	0.2852	-0.5052
9	0.3416	0.2851	-0.5052

$\therefore$ The Solution is x = 0.342; y = 0.285; z = - 0.505

2. Solve By Gauss Jacobi method the following System

$$28x+4y-z=32; x+3y+10z=24; 2x+17y+4z=35$$

Solution:

Rearranging the Given System as

$$28x+4y-z=32$$

$$x+3y+10z=24$$

$$2x+17y+4z=35$$

$\therefore$ The Coefficient matrix is A= $\begin{bmatrix} <28> & 4 & -1 \\ 2 & <17> & 4 \\ 1 & 3 & <10> \end{bmatrix}$ is diagonally dominant

Solve for x , y, z we have

$$x=\frac{1}{28}[32-4y+z]$$

$$y=\frac{1}{17}\ 35-2x-4z$$

$$z=\frac{1}{10}\ 24-x-3y$$

Initial Iteration Values Of (x, y, z) = (0, 0, 0)

The Successive iteration Values are

Iteration	$x=\frac{1}{28}(32-4y+z)$	$y=\frac{1}{17}(35-2y+4z)$	$z=\frac{1}{10}(24-x-3y)$
1	1.143	2.059	2.400
2	0.934	1.360	1.668
3	1.008	1.556	1.899
4	0.988	1.493	1.852
5	0.995	1.512	1.853
6	0.993	1.506	1.847
7	0.994	1.507	1.849
8	0.994	1.507	1.849

$\therefore$ The Solution is x = 0.994 ; y = 1.507; z = 1.849;

3. Solve the following Equation using Jacobi's Iteration Method

$$20x+y-2z=17; 3x+20y-z=-18; 2x-3y+20z=25$$

Solution:

The Coefficient matrix is

$$A=\begin{bmatrix}20 & 1 & -2\\ 3 & 20 & -1\\ 2 & -3 & 20\end{bmatrix} \text{ is Diagonally Dominant}$$

Solve for x, y, z we have

$$x=\frac{1}{20}[17-y+2z]$$

$$y=\frac{1}{20}[-18-3x+z]$$

$$z=\frac{1}{20}\ 25-2x+3y$$

Initial Iteration Values are (x, y, z) = (0, 0, 0)

The successive values are

Iteration	$x=\frac{1}{20}[17-y+2z]$	$y=\frac{1}{20}[-18-3x+z]$	$z=\frac{1}{20}[25-2x+3y]$
1	0.850	-0.900	1.250
2	1.020	-0.965	1.030
3	1.00125	-1.0015	1.00325
4	1.000425	-1.00025	0.99965
5	0.9999	-1.00008	0.9999
6	0.9999	-0.9999	0.9999
7	1.000	-1.000	1.000

$\therefore$ The Solution is X=1 ; Y= -1 ; Z=1

Indirect method (or) Iteration Method

Gauss Seidel method

Consider the System of Equation

$$a_1x+b_1y+c_1z=d_1$$

$$a_2x+b_2y+c_2z=d_2 \quad \rightarrow (1)$$

$$a_3x+b_3y+c_3z=d_3$$

Let us assume that,

$$|a_1|>|b_1|+|c_1|$$

$$|b_2|>|a_2|+|c_2|$$

$$|c_3|>|a_3|+|b_3|$$

then iteration method can be used for the system solve the values of x ,y ,z in terms of the other variables

$$x=\frac{1}{a_1}(d_1-b_1y-c_1z)$$

$$y=\frac{1}{b_2}(d_2-a_2x-c_2z)$$

$$z=\frac{1}{c_3}(d_3-a_3x-b_3y)$$

If the initial value y^0, z^0 for y and z and x get x^1,

$$x^1=\frac{1}{a_1}(d_1-b_1y^0-c_1z^0)$$

Again substitute x^1, z^0 for x and z we get

$$y^1=\frac{1}{b_2}(d_2-a_2x^1-c_2z^0)$$

Again Substitute x^1, y^1 for x and y we get

$$z^1=\frac{1}{c_3}(d_3-a_3x^{(1}-b_3y^1)$$ And so on.

NOTE:

1. The Sufficient Condition for the Convergence of Gauss Seidel Method

The methods of iteration will convergence if in each equation of the given System, the Absolute value of the largest coefficient is greater than the sum of the absolute value of all the remaining coefficient.

2. Gauss Seidel method equations only if the coefficient matrix is diagonally dominant .

Problems based on gauss Seidal method

1. Solve the following System by gauss Seidal method $10x-5y-2z=3; 4x-10y+3z=-3; x+6y+10z=-3$

Solution:

Given, $10x-5y-2z=3$ ------ (1)

$4x-10y+3z=-3$ ----- (2)

$x+6y+10z=-3$ ------- (3)

Take, $|10|>|-5|+|-2| \Rightarrow |10|>|5|+|2|$

$|-10|>|4|+|3| \Rightarrow |10|>|4|+|3|$

$|10|>|1|+|6| \Rightarrow |10|>|1|+|6|$

$\therefore$ the given equation is diagonally dominant

From (**1**) $10x - 5y - 2z = 3$

$$10x = 3 + 5y + 2z$$

$$x = \frac{1}{10}(3 + 5y + 2z)$$

From (**2**) $4x - 10y + 3z = -3$

$$-10y = -3 - 4x - 3z$$

$$y = \frac{1}{10}(3 + 4x + 3z)$$

From (**3**) $x + 6y + 10z = -3$

$$10z = -3 - x - 6y$$

$$z = \frac{1}{10}(-3 - x - 6y)$$

Let y = 0, z = 0 be the initial value

Iteration	X	Y	Z
1	0.3	0.42	-0.582
2	0.3936	0.28284	-0.509064
3	0.3396	0.2831	-0.5038
4	0.3408	0.2852	-0.5052
5	0.3416	0.2851	-0.5052
6	0.3415	0.2850	-0.5052
7	0.3415	0.2850	-0.5052

Here 6^{th} *and* 7^{th} iteration value of x, y, z is same

$\therefore$ x = 0.3415; y = 0.2850; z = -0.5052

2. Solve the following equations by gauss seidal method

$$4x + 2y + z = 14; x + 5y - z = 10; x + y + 8z = 20$$

Solution:

Given:

$$4x+2y+z=14 \rightarrow 1$$

$$x+5y-z=10 \rightarrow 2$$

$$x+y+8z=20 \rightarrow 3$$

Take,

|4| > |3|+|1|

|5|>|1|+|-1|

|8|>|1|+|1|

$\therefore$ the given equation are diagonally dominant

From (1) $\Rightarrow$ $4x+2y+z=14$

$$4x=14-2y-z$$

$$x=\frac{1}{4}(14-2y-z)$$

From (2) $\Rightarrow$ $x+5y-z=10$

$$5y=10-x+z$$

$$y=\frac{1}{5}(10-x+z)$$

From (3) $\Rightarrow$ $x+y+8z=20$

$$z=\frac{1}{8}(20-x-y)$$

Let y = 0, z = 0 be the initial value

$$x = \frac{1}{4}(14-2y-z)$$

$$y = \frac{1}{5}(10-x+z)$$

$$z = \frac{1}{8}(20-x-y)$$

Iteration	x	Y	z
1	3.5	1.3	1.9
2	2.375	1.905	1.965
3	2.056	1.982	1.995
4	2.010	1.997	1.997
5	2.002	1.999	2
6	2.001	2	2
7	2	2	2
8	2	2	2

Here the 7^{th} *and* 8^{th} iteration Value is

Hence, x = 2, y = 2, z = 2

3. Solve the following system of the equation by gauss seidal method

$$27x+6y-z=85; x+y+54z=110; 6x+15y+2z=72$$

Solution:

Given,

$$27x+6y-z=85 \rightarrow 1$$

$$x+y+54z=110 \rightarrow 2$$

$$6x+15y+2z=72 \rightarrow 3$$

Take |27| >|6|+|-1|

|1|>|1|+|54|

|2|>|6|+|15|

The given equations are not diagonally dominant now we interacting equation 2 & equation 3

$$27x+6y-z=85 \rightarrow 4$$

$$6x+15y+2z=72 \rightarrow 5$$

$$x+y+54z=110 \rightarrow 6$$

Take |27| >|6|+|-1|

|15|>|6|+|2|

|54|>|1|+|1|

$\therefore$ the given equation are diagonally dominant

From (4) $\Rightarrow 27x = 85 - 6y + z$

$$x = \frac{1}{27}(85 - 6y + z)$$

From (5) $\Rightarrow 15y = 72 - 6x - 2z$

$$y = \frac{1}{15}(72 - 6x - 2z)$$

From (6) $\Rightarrow 54z = 110 - x - y$

$$z = \frac{1}{54}(110 - x - y)$$

Let y = 0, z = 0 be the initial value

Iteration	x = $\frac{1}{27}(85 - 6y - z)$	y = $\frac{1}{15}(72 - 6x - 2z)$	z = $\frac{1}{54}(110 - x - y)$
1	3.1481	3.5408	1.9132
2	2.4322	3.5720	1.9258
3	2.4257	3.5729	1.9260
4	2.4255	3.5730	1.9260
5	2.4255	3.5730	1.9260

Here 4^{th} *and* 5^{th} iteration values are equal

$\therefore$ x = 2.4255

y = 3.5730

z = 1.9260

Home Work:

1) 8x-3y+2z = 20, 4x+11y-z = 33, 6x+3y+12z = 35 (x = 3.0168; y = 1.9859; z = 0.9118)

2.) 28x+4y-z = 32, 2x+17y+4z = 35, x+3y+10z = 24(x = 0.9936; y = 1.5069, z = 1.8486)

3) 20x+y-2z = 17,3x+20y-z = -18; 2x-3y+20z = 25(x = 1; y = -1, z = 1)

Inverse of Matrix By Gauss Jordan method

Let A be a square non-singular matrix of order 3

Consider the inverse of a matrix

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

We know that,

The unit matrix of order 3 is

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

the augmented matrix,

$$[A, I] = \left[\begin{array}{ccc|ccc} a_1 & b_1 & c_1 & 1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 1 \end{array}\right]$$

By using Gauss Jordan method gives

$$[I, A^{-1}] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & a_1 & b_1 & c_1 \\ 0 & 1 & 0 & a_2 & b_2 & c_2 \\ 0 & 0 & 1 & a_3 & b_3 & c_3 \end{array}\right]$$

Hence [A.I] ~[I, A^{-1}]

1. Find the Inverse Of A = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$ by using gauss Jordan method

Solution:

Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

$$\text{Let } [A, I] = \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 3 & -3 & 0 & 1 & 0 \\ -2 & -4 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & -6 & -1 & 0 & 0 \\ 0 & -2 & 2 & 2 & 0 & 1 \end{array}\right] R_2 \Rightarrow R_2 - R_1, \quad R_3 \Rightarrow R_3 + R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 12 & 3 & -1 & 0 \\ 0 & 2 & -6 & -1 & 1 & 0 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right] R_1 \Rightarrow 2R_1 - R_2, \quad R_3 \Rightarrow R_3 + R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 6 & 2 & 3 \\ 0 & -8 & 0 & 10 & 2 & 6 \\ 0 & 0 & -4 & 1 & 1 & 1 \end{array}\right] R_1 \Rightarrow R_1 + 3R_3, \; R_2 \Rightarrow 6R_3 - 4R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{-5}{4} & \frac{-1}{4} & \frac{-3}{4} \\ 0 & 0 & 1 & \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \end{array}\right] R_1 \Rightarrow \frac{R_1}{2}, R_2 \Rightarrow \frac{R_2}{-8}, \; R_3 \Rightarrow \frac{R_3}{-4}$$

$$\sim [I, A^{-1}]$$

$$A^{-1} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ \frac{-5}{4} & \frac{-1}{4} & \frac{-3}{4} \\ \frac{-1}{4} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix}$$

$$\text{Hence,} \quad A^{-1} = \frac{1}{4}\begin{bmatrix} 12 & 4 & 3 \\ -5 & -1 & -3 \\ -1 & -1 & -1 \end{bmatrix}$$

2. Find the Inverse of $A = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}$ using Gauss Jordan method

Solution:

Given $A = \begin{bmatrix} 4 & 1 & 2 \\ 2 & 3 & -1 \\ 1 & -2 & 2 \end{bmatrix}$

Let $[A, I] = \left[\begin{array}{ccc|ccc} 4 & 1 & 2 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 1 \end{array}\right]$

$$\sim \left[\begin{array}{ccc|ccc} 4 & 1 & 2 & 1 & 0 & 0 \\ 0 & 5 & -4 & -1 & 2 & 0 \\ 0 & -9 & 6 & -1 & 0 & 4 \end{array}\right] R_2 \Rightarrow 2R_2 - R_1, R_3 \Rightarrow 4R_3 - R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 20 & 0 & 14 & 6 & -2 & 0 \\ 0 & 5 & -4 & -1 & 2 & 0 \\ 0 & 0 & -6 & -14 & 18 & 20 \end{array}\right] R_1 \Rightarrow 5R_1 - R_2 , R_3 \Rightarrow 5R_3 + 9R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 120 & 0 & 0 & -160 & 240 & 280 \\ 0 & 30 & 0 & 50 & -60 & -80 \\ 0 & 0 & -6 & -14 & 18 & 20 \end{array}\right] R_1 \Rightarrow 6R_1 + 14R_3 \;\; R_2 \Rightarrow 6R_2 - 4R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-4}{3} & 2 & \frac{7}{3} \\ 0 & 1 & 0 & \frac{5}{3} & -2 & \frac{-8}{3} \\ 0 & 0 & 1 & \frac{7}{3} & -3 & \frac{-10}{3} \end{array}\right] R_1 \Rightarrow \frac{R_1}{120} \;, R_2 \Rightarrow \frac{R_2}{30} \;, R_3 \Rightarrow \frac{R_3}{-6}$$

$\sim [I, A^{-1}]$

Hence $A^{-1} = \begin{bmatrix} \frac{-4}{3} & 2 & \frac{7}{3} \\ \frac{5}{3} & -2 & \frac{-8}{3} \\ \frac{7}{3} & -3 & \frac{-10}{3} \end{bmatrix}$

3. Find the Inverse of $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & -1 & -4 \end{bmatrix}$ Using Gauss Jordan method

Solution:

Given: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & -1 & -4 \end{bmatrix}$

Let $[A, I] = \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 \\ 3 & -1 & -4 & 0 & 0 & 1 \end{array}\right]$

Inter Changing R_1 *and* R_2 *i.e* $R_1 \Leftrightarrow R_2$

$$[A, I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 3 & -1 & -4 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & -7 & -4 & 0 & -3 & 1 \end{array}\right] R_3 \Rightarrow R_3 - 3R_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -2 & -2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 3 & 7 & -3 & 1 \end{array}\right] R_1 \Rightarrow R_1 - R_2 \;, R_3 \Rightarrow R_3 + 7R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 3 & 0 & 0 & 8 & -3 & 2 \\ 0 & 3 & 0 & -4 & 3 & -1 \\ 0 & 0 & 3 & 7 & -3 & 1 \end{array}\right] R_1 \Rightarrow 3R_1 + 2R_2 \;, R_2 \Rightarrow 3R_2 - R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{8}{3} & -1 & \frac{2}{3} \\ 0 & 1 & 0 & \frac{-4}{3} & 1 & \frac{-1}{3} \\ 0 & 0 & 1 & \frac{7}{3} & -1 & \frac{-1}{3} \end{array}\right] R_1 \Rightarrow \frac{R_1}{3} , R_2 \Rightarrow \frac{R_2}{3} \;, R_3 \Rightarrow \frac{R_3}{3}$$

$$\sim [I, A^{-1}]$$

Hence $A^{-1} = \begin{bmatrix} \frac{8}{3} & -1 & \frac{2}{3} \\ \frac{-4}{3} & 1 & \frac{-1}{3} \\ \frac{7}{3} & -1 & \frac{1}{3} \end{bmatrix}$

Home Work

1. $A = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$ $\qquad A^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

2. $A = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 3 & 5 \end{bmatrix}$ $\qquad A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ -9 & 7 & 4 \\ 5 & -4 & -2 \end{bmatrix}$ $\left\lfloor \therefore AA^{-1} = I \right\rfloor$

<u>Eigen value of a matrix by power method</u>

Power method is used to determine numerical largest Eigen value and corresponding eigenvector of a matrix A.

Let A be an $n\times$ n matrix and let $\lambda_{1,}\lambda_2......\lambda_n$ be the distinct Eigenvalue of A, so that $|\lambda_1|>|\lambda_2|>||\lambda_3|>......|\lambda_n|$

Let $x_{1,}x_2............x_n$ be the corresponding eigenvector.

ie, $Ax_i=\lambda_i x_i$

Note

1. λ_1 is dominant $|\lambda_1|>|\lambda_2|>||\lambda_3|>......|\lambda_n|$

2. Sum of the Eigen values of a matrix is equal to the sum of the main diagonal elements of the matrix.

3. To find the numerically smallest eigenvalue of A, obtain the dominant eigenvalue λ_1 of A and then find $B=A-\lambda_1 I$ and find the dominant eigenvalue of B.

Then the smallest eigenvalue of A is equal to the dominant eigenvalue of $B+\lambda_1$

ie, to find $B=A-\lambda_1 I$

to find $A=B+\lambda_1$

4. The smallest eigenvalue of $A=\frac{1}{\lambda}$ and the corresponding eigenvector is X.

Problems based on power method

1. Find the eigenvalue of $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ by power method and hence find the other eigenvalue also

Let $X_1=\begin{bmatrix}1\\ 0\end{bmatrix}$ be an arbitrary initial eigenvector

$$AX_1=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}1\\ 3\end{bmatrix}=3\begin{bmatrix}0.3333\\ 1\end{bmatrix}=3X_2$$

$$AX_2=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}=\begin{bmatrix}0.3333\\ 1\end{bmatrix}=\begin{bmatrix}2.3333\\ 7.3333\end{bmatrix}=7.3333\begin{bmatrix}0.3182\\ 1\end{bmatrix}$$

$$=7.3333\,X_3$$

$$AX_3=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}0.3182\\ 1\end{bmatrix}=\begin{bmatrix}2.3182\\ 4.9546\end{bmatrix}=4.9546\begin{bmatrix}0.4679\\ 1\end{bmatrix}$$

$$=4.9546X_4$$

$$AX_4=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}0.4679\\ 1\end{bmatrix}=\begin{bmatrix}2.4679\\ 5.4037\end{bmatrix}=5.4037\begin{bmatrix}0.4567\\ 1\end{bmatrix}$$

$$=5.4037\,X_5$$

$$AX_5=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}0.4567\\ 1\end{bmatrix}=\begin{bmatrix}2.4567\\ 5.3701\end{bmatrix}=5.3701\begin{bmatrix}0.4575\\ 1\end{bmatrix}$$

$$=5.3701X_6$$

$$AX_6=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}0.4575\\ 1\end{bmatrix}=\begin{bmatrix}2.4575\\ 5.3725\end{bmatrix}=5.3725\begin{bmatrix}0.4574\\ 1\end{bmatrix}$$

$$=5.3725X_7$$

$$AX_7=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}0.4574\\1\end{bmatrix}=\begin{bmatrix}2.4574\\5.3722\end{bmatrix}=5.3722\begin{bmatrix}0.4574\\1\end{bmatrix}$$

$$=5.3722X_8$$

$$AX_8=\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\begin{bmatrix}0.4574\\1\end{bmatrix}=\begin{bmatrix}2.4574\\5.3722\end{bmatrix}=5.3722\begin{bmatrix}0.4574\\1\end{bmatrix}$$

$$=5.3722X_9$$

$\therefore$ The dominant eigenvalue $\lambda_1=5.3722$ and corresponding eigenvalue is $\begin{bmatrix}0.4574\\1\end{bmatrix}$

To find λ_2 :

Sum of the eigenvalue = sum of the main diagonal element of A

$$\lambda_1+\lambda_2=1+4$$

$$5.3722+\lambda_2=5$$

$$\lambda_2 = 5-5.3722$$

$$\lambda_2 = -0.3722$$

Hence the Eigen values are

$$\lambda_1=5.3722,\ \lambda_2=-0.3722$$

2. Find the numerically Largest eigenvalue of $A=\begin{bmatrix}25 & 1 & 2\\1 & 3 & 0\\2 & 0 & -4\end{bmatrix}$ and the corresponding eigenvector

Solution:

Given $A=\begin{bmatrix}25 & 1 & 2\\1 & 3 & 0\\2 & 0 & -4\end{bmatrix}$

Let $X_1=\begin{pmatrix}1\\0\\0\end{pmatrix}$ be an arbitrary initial eigenvector

$$AX_1 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix} = 25\begin{bmatrix} 1 \\ 0.0400 \\ 0.0800 \end{bmatrix} = 25X_2$$

$$AX_2 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.0400 \\ 0.0800 \end{bmatrix} = \begin{bmatrix} 25.2000 \\ 1.1200 \\ 1.6800 \end{bmatrix} = \begin{bmatrix} 25.2000 \\ 1.1200 \\ 1.6800 \end{bmatrix} = 25.2\begin{bmatrix} 1 \\ 0.0444 \\ 0.0667 \end{bmatrix}$$
$$= 25.2X_3$$

$$AX_3 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.0444 \\ 0.0667 \end{bmatrix} = \begin{bmatrix} 25.1778 \\ 1.1332 \\ 1.7332 \end{bmatrix} = 25.1778\begin{bmatrix} 1 \\ 0.0450 \\ 0.0688 \end{bmatrix}$$
$$= 25.1778X_4$$

$$AX_4 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.0450 \\ 0.0688 \end{bmatrix} = \begin{bmatrix} 25.1820 \\ 1.1350 \\ 1.7248 \end{bmatrix} = 25.1826\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix}$$
$$= 25.1826x_5$$

$$AX_5 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix} = \begin{bmatrix} 25.1821 \\ 1.1353 \\ 1.7260 \end{bmatrix} = 25.1821\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix}$$
$$= 25.1821X_6$$

$$AX_6 = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix} = \begin{bmatrix} 25.1821 \\ 1.1353 \\ 1.7260 \end{bmatrix} = 25.1821\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix}$$
$$= 25.1821X_7$$

$\therefore$ The dominant eigenvalue $\lambda_1 = 25.1821$ and the eigenvalue is $\begin{bmatrix} 1 \\ 0.0451 \\ 0.0685 \end{bmatrix}$

3. Find the dominant eigenvalue and corresponding eigenvector of $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Find also the least taken root and hence the third eigenvalue

Solution:

Given $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Let $X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be the initial eigenvector

$$AX_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} . \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1X_2$$

$$AX_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} . \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = 7X_3$$

$$AX_3 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4286 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.5716 \\ 1.8572 \\ 0 \end{bmatrix} = 3.5716 \begin{bmatrix} 1 \\ 0.5200 \\ 0 \end{bmatrix}$$
$$= 3.5716X_4$$

$$AX_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5200 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.1200 \\ 2.0400 \\ 0 \end{bmatrix} = 4.1200 \begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix}$$
$$= 4.1200X_5$$

$$AX_5 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9706 \\ 1.9902 \\ 0 \end{bmatrix} = 3.9706 \begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix}$$
$$= 3.9706X_6$$

$$AX_6 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.0072 \\ 2.0024 \\ 0 \end{bmatrix} = 4.0072 \begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix}$$
$$= 4.0072X_7$$

$$AX_7=\begin{bmatrix}1&6&1\\1&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}1\\0.4997\\0\end{bmatrix}=\begin{bmatrix}3.9982\\1.9994\\0\end{bmatrix}=3.9982\begin{bmatrix}1\\0.500\\0\end{bmatrix}$$

$$=3.9982X_8$$

$$AX_8=\begin{bmatrix}1&6&1\\1&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}1\\0.5001\\0\end{bmatrix}=\begin{bmatrix}4.0006\\2.0002\\0\end{bmatrix}=4.0006\begin{bmatrix}1\\0.5000\\0\end{bmatrix}$$

$$=4.0006X_9$$

$$AX_9=\begin{bmatrix}1&6&1\\1&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}1\\0.5000\\0\end{bmatrix}=\begin{bmatrix}4.0006\\2.0002\\0\end{bmatrix}=4\begin{bmatrix}1\\0.5000\\0\end{bmatrix}=4X_{10}$$

$$AX_{10}=\begin{bmatrix}1&6&1\\1&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}1\\0.5000\\0\end{bmatrix}=\begin{bmatrix}4\\2\\0\end{bmatrix}=4\begin{bmatrix}1\\0.5000\\0\end{bmatrix}=4X_{11}$$

$\therefore$ The dominant eigenvalue $\lambda_1 = 4$ and corresponding eigenvector i s $\begin{bmatrix}1\\0.5\\0\end{bmatrix}$

To find the least latent root (or) smallest root smallest eigenvalue of A = λ_1 and eigenvalue of B where eigenvalue of $B=A-\lambda_1 I$

$$B=\begin{bmatrix}1&6&1\\1&2&0\\0&0&3\end{bmatrix}-4\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$B=\begin{bmatrix}-3&6&1\\1&-2&0\\0&0&-1\end{bmatrix}$$

Let $Y_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$ be an arbitrary initial Eigen vector

$$BY_1=\begin{bmatrix}-3&6&1\\1&-2&0\\0&0&-1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}-3\\1\\0\end{bmatrix}=-3\begin{bmatrix}1\\-0.3333\\0\end{bmatrix}$$

$$=-3Y_2$$

$$BY_2=\begin{bmatrix}-3 & 6 & 1\\ 1 & -2 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\ -0.3333\\ 0\end{bmatrix}=\begin{bmatrix}-4.9998\\ 1.6666\\ 0\end{bmatrix}$$

$$= -4.9998\begin{bmatrix}1\\ -0.3333\\ 0\end{bmatrix}$$

$$= -4.9998Y_3$$

$$BY_3=\begin{bmatrix}-3 & 6 & 1\\ 1 & -2 & 0\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}1\\ -0.3333\\ 0\end{bmatrix}=\begin{bmatrix}-4.9998\\ 1.6666\\ 0\end{bmatrix} = -4.9998\begin{bmatrix}1\\ -0.3333\\ 0\end{bmatrix}$$

$$=-4.9998Y_4$$

$\therefore$ The dominant eigenvalue B = -4.998 and the corresponding eigenvector is $\begin{bmatrix}1\\ -0.3333\\ 0\end{bmatrix}$

$\therefore$ Smallest eigenvector of A = λ_1 + Eigen value of B

$$= 4 - 4.49998 \text{ (or) } 5$$

$$= -0.9998 \text{ (or) } 1$$

$$\lambda_2 = -1$$

1. Use the power method to find the dominant eigenvalue and the corresponding

   Eigenvector of the matrix $A=\begin{bmatrix}9 & 1 & 8\\ 7 & 4 & 1\\ 1 & 7 & 9\end{bmatrix}$

   Solution:

   Let $X_0= 1,1,1^T$ then

$$AX_0=\begin{bmatrix}9 & 1 & 8\\ 7 & 4 & 1\\ 1 & 7 & 9\end{bmatrix}\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}=\begin{bmatrix}18\\ 12\\ 17\end{bmatrix}=18\begin{bmatrix}1\\ 0.67\\ 0.94\end{bmatrix}=\lambda_1X_1$$

$$AX_1=\begin{bmatrix}9&1&8\\7&4&1\\1&7&9\end{bmatrix}\begin{bmatrix}1\\0.67\\0.94\end{bmatrix}=\begin{bmatrix}17.19\\10.62\\14.15\end{bmatrix}=17.19\begin{bmatrix}1\\0.62\\0.82\end{bmatrix}=\lambda_2X_2$$

$$AX_2=\begin{bmatrix}9&1&8\\7&4&1\\1&7&9\end{bmatrix}\begin{bmatrix}1\\0.62\\0.82\end{bmatrix}=\begin{bmatrix}16.18\\10.30\\12.72\end{bmatrix}=16.18\begin{bmatrix}1\\0.64\\0.79\end{bmatrix}=\lambda_3X_3$$

$$AX_3=\begin{bmatrix}9&1&8\\7&4&1\\1&7&9\end{bmatrix}\begin{bmatrix}1\\0.64\\0.79\end{bmatrix}=\begin{bmatrix}15.96\\10.35\\12.59\end{bmatrix}=15.96\begin{bmatrix}1\\0.65\\0.79\end{bmatrix}=\lambda_4X_4$$

$$AX_4=\begin{bmatrix}9&1&8\\7&4&1\\1&7&9\end{bmatrix}\begin{bmatrix}1\\0.65\\0.79\end{bmatrix}=\begin{bmatrix}15.96\\10.39\\12.66\end{bmatrix}=15.96\begin{bmatrix}1\\0.65\\0.79\end{bmatrix}=\lambda_5X_5$$

Hence the dominant eigenvalue is 15.97 and the corresponding Eigen vector is $1,0.65,0.79^{T}$

2. Use the power method to find the dominant eigenvalue and the corresponding eigenvector of

Matrix $A=\begin{bmatrix}5&0&1\\0&-2&0\\1&0&5\end{bmatrix}$

Solution:

Let $X_0= 1,1,1^{T}$ be an arbitrary initial eigenvector

$$AX_0=\begin{bmatrix}5&0&1\\0&-2&0\\1&0&5\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}6\\-2\\6\end{bmatrix}=6\begin{bmatrix}1\\-0.33\\1\end{bmatrix}=\lambda_1X_1$$

$$AX_1=\begin{bmatrix}5&0&1\\0&-2&0\\1&0&5\end{bmatrix}\begin{bmatrix}1\\-0.33\\1\end{bmatrix}=\begin{bmatrix}6\\+0.66\\6\end{bmatrix}=6\begin{bmatrix}1\\0.11\\1\end{bmatrix}=\lambda_2X_2$$

$$AX_2=\begin{bmatrix}5&0&1\\0&-2&0\\1&0&5\end{bmatrix}\begin{bmatrix}1\\-0.11\\1\end{bmatrix}=\begin{bmatrix}6\\-0.22\\6\end{bmatrix}=6\begin{bmatrix}1\\-0.04\\1\end{bmatrix}=\lambda_3X_3$$

$$AX_3=\begin{bmatrix}5&0&1\\0&-2&0\\1&0&5\end{bmatrix}\begin{bmatrix}1\\-0.04\\1\end{bmatrix}=\begin{bmatrix}6\\0.08\\6\end{bmatrix}=6\begin{bmatrix}1\\0.01\\1\end{bmatrix}=\lambda_4X_4$$

$$AX_4=\begin{bmatrix}5&0&1\\0&-2&0\\1&0&5\end{bmatrix}\begin{bmatrix}1\\0.01\\1\end{bmatrix}=\begin{bmatrix}6\\-0.02\\6\end{bmatrix}=6\begin{bmatrix}1\\-0.003\\1\end{bmatrix}=\lambda_5X_5$$

$$AX_5=\begin{bmatrix}5&0&1\\0&-2&0\\1&0&5\end{bmatrix}\begin{bmatrix}1\\-0.003\\1\end{bmatrix}=\begin{bmatrix}6\\0.006\\6\end{bmatrix}=6\begin{bmatrix}1\\0.001\\1\end{bmatrix}=\lambda_6X_6$$

$$AX_6=\begin{bmatrix}5&0&1\\0&-2&0\\1&0&5\end{bmatrix}\begin{bmatrix}1\\0.001\\1\end{bmatrix}=\begin{bmatrix}6\\-0.002\\6\end{bmatrix}=\begin{bmatrix}1\\0\\1\end{bmatrix}=\lambda_7X_7$$

$$AX_7=\begin{bmatrix}5&0&1\\0&-2&0\\1&0&5\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}6\\0\\6\end{bmatrix}=6\begin{bmatrix}1\\0\\1\end{bmatrix}$$

Hence the dominant eigenvalue is 6 and the corresponding eigenvector is $1,0,1^{T}$

3. Find the smallest eigenvalue and the corresponding eigenvector of $A=\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}$ by Power method

Solution:

Given $A=\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&2\end{bmatrix}$

$$|A| = 4$$

To find the inverse of A

$$A^{-1} = B = \begin{bmatrix} 0.75 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 0.75 \end{bmatrix}$$

To find the largest eigenvalue and the corresponding eigenvector of B by power method

Let $X_0 = (1,1,1)^T$ be an arbitrary eigenvector

$$BX_0 = \begin{bmatrix} 0.75 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 0.75 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 2 \\ 1.85 \end{bmatrix} = 2 \begin{bmatrix} 0.75 \\ 1 \\ 0.75 \end{bmatrix} = \lambda_1 X_1$$

$$BX_1 = \begin{bmatrix} 0.75 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 0.75 \end{bmatrix} \begin{bmatrix} 0.75 \\ 1 \\ 0.75 \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.75 \\ 1.25 \end{bmatrix} = 1.75 \begin{bmatrix} 0.714 \\ 1 \\ 0.714 \end{bmatrix} = \lambda_2 X_2$$

$$BX_2 = \begin{bmatrix} 0.75 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 0.75 \end{bmatrix} \begin{bmatrix} 0.714 \\ 1 \\ 0.714 \end{bmatrix} = \begin{bmatrix} 1.214 \\ 1.714 \\ 1.214 \end{bmatrix} = 1.714 \begin{bmatrix} 0.708 \\ 1 \\ 0.708 \end{bmatrix} = \lambda_3 X_3$$

$$BX_3 = \begin{bmatrix} 0.75 & 0.5 & 0.25 \\ 0.5 & 1 & 0.5 \\ 0.25 & 0.5 & 0.75 \end{bmatrix} \begin{bmatrix} 0.708 \\ 1 \\ 0.708 \end{bmatrix} = \begin{bmatrix} 1.208 \\ 1.708 \\ 1.208 \end{bmatrix} = 1.708 \begin{bmatrix} 0.707 \\ 1 \\ 0.707 \end{bmatrix} = \lambda_4 X_4$$

$$BX_4 = \begin{bmatrix} 1.207 \\ 1.707 \\ 1.207 \end{bmatrix} = 1.707 \begin{bmatrix} 0.707 \\ 1 \\ 0.707 \end{bmatrix} = \lambda_5 X_5$$

$$BX_5 = \begin{bmatrix} 1.207 \\ 1.707 \\ 1.207 \end{bmatrix} = 1.707 \begin{bmatrix} 0.707 \\ 1 \\ 0.707 \end{bmatrix} = \lambda_6 X_6$$

$\therefore$ The largest eigenvalue of B is 1.707 and the corresponding eigenvector is $(0.707,1,0.707)^T$

The smallest eigenvalue of A is $\frac{1}{\lambda} = \frac{1}{1.707} = 0.586$ and the corresponding eigenvector is $(0.707,1,0.707)^T$

4. Find the dominant eigenvalue and the corresponding eigenvector of $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.Find also the other two Eigen values.

Solution:

Given, $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Let $X_0 = \left(1,1,1\right)^T$ be an initial arbitrary eigenvector then

$$AX_0 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 3 \end{bmatrix} = 8\begin{bmatrix} 1 \\ 0.375 \\ 0.375 \end{bmatrix} = \lambda_1 X_1$$

$$AX_1 = \begin{bmatrix} 3.625 \\ 1.750 \\ 1.123 \end{bmatrix} = 3.625\begin{bmatrix} 1 \\ 0.483 \\ 0.310 \end{bmatrix} = \lambda_2 X_2$$

$$AX_2 = \begin{bmatrix} 4.208 \\ 1.966 \\ 0.930 \end{bmatrix} = 4.208\begin{bmatrix} 1 \\ 0.467 \\ 0.221 \end{bmatrix} = \lambda_3 X_3$$

$$AX_3 = \begin{bmatrix} 4.023 \\ 1.934 \\ 0.663 \end{bmatrix} = 4.023\begin{bmatrix} 1 \\ 0.481 \\ 0.165 \end{bmatrix} = \lambda_4 X_4$$

$$AX_4 = \begin{bmatrix} 4.051 \\ 1.962 \\ 0.495 \end{bmatrix} = 4.051\begin{bmatrix} 1 \\ 0.484 \\ 0.122 \end{bmatrix} = \lambda_5 X_5$$

$$AX_5 = \begin{bmatrix} 4.026 \\ 1.968 \\ 0.366 \end{bmatrix} = 4.026\begin{bmatrix} 1 \\ 0.489 \\ 0.091 \end{bmatrix} = \lambda_6 X_6$$

$$AX_6 = \begin{bmatrix} 4.025 \\ 1.978 \\ 0.273 \end{bmatrix} = 4.025\begin{bmatrix} 1 \\ 0.491 \\ 0.068 \end{bmatrix} = \lambda_7 X_7$$

$$AX_7=\begin{bmatrix}4.014\\1.982\\0.204\end{bmatrix}=4.014\begin{bmatrix}1\\0.494\\0.051\end{bmatrix}=\lambda_8X_8$$

$$AX_8=\begin{bmatrix}4.015\\1.988\\0.153\end{bmatrix}=4.015\begin{bmatrix}1\\0.495\\0.038\end{bmatrix}=\lambda_9X_9$$

$$AX_9=\begin{bmatrix}4.008\\1.990\\0.114\end{bmatrix}=4.008\begin{bmatrix}1\\0.497\\0.028\end{bmatrix}=\lambda_{10}X_{10}$$

$$AX_{10}=\begin{bmatrix}4.010\\1.994\\0.084\end{bmatrix}=4.010\begin{bmatrix}1\\0.497\\0.021\end{bmatrix}=\lambda_{11}X_{11}$$

$$AX_{11}=\begin{bmatrix}4.003\\1.994\\0.063\end{bmatrix}=4.003\begin{bmatrix}1\\0.498\\0.016\end{bmatrix}=\lambda_{12}X_{12}$$

$$AX_{12}=\begin{bmatrix}4.004\\1.996\\0.048\end{bmatrix}=4.004\begin{bmatrix}1\\0.499\\0.012\end{bmatrix}=\lambda_{13}X_{13}$$

$$AX_{13}=\begin{bmatrix}4.006\\1.998\\0.036\end{bmatrix}=4.006\begin{bmatrix}1\\0.499\\0.009\end{bmatrix}=\lambda_{14}X_{14}$$

$$AX_{14}=\begin{bmatrix}4.003\\1.998\\0.027\end{bmatrix}=4.003\begin{bmatrix}1\\0.499\\0.007\end{bmatrix}=\lambda_{15}X_{15}$$

$$AX_{15}=\begin{bmatrix}4.001\\1.998\\0.021\end{bmatrix}=4.001\begin{bmatrix}1\\0.499\\0.005\end{bmatrix}=\lambda_{16}X_{16}$$

$$AX_{16}=\begin{bmatrix}3.999\\1.998\\0.015\end{bmatrix}=3.999\begin{bmatrix}1\\0.5\\0.004\end{bmatrix}=\lambda_{17}X_{17}$$

$$AX_{17}=\begin{bmatrix}4.004\\2.000\\0.012\end{bmatrix}=4.004\begin{bmatrix}1\\0.5\\0.003\end{bmatrix}=\lambda_{18}X_{18}$$

$$AX_{18}=\begin{bmatrix}4.003\\2.000\\0.009\end{bmatrix}=4.003\begin{bmatrix}1\\0.5\\0.002\end{bmatrix}=\lambda_{19}X_{19}$$

Sum of the eigenvalue = sum of the main diagonal of A

$$\lambda_1+\lambda_2+\lambda_3=1+2+3$$

$$4-1+\lambda_3=6$$

$$\lambda_3=6-4+1$$

$$\lambda_3=3$$

Hence the Eigen values are 4,3,-1

Home Work:

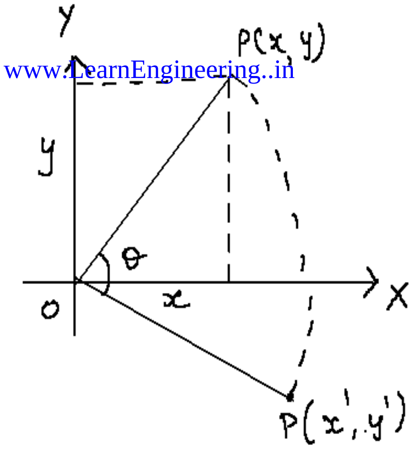
1) Find the numerically largest eigenvalue of $\begin{bmatrix}4&1\\1&3\end{bmatrix}$ and the corresponding eigenvector.

2) Find the dominant eigenvalue and the corresponding eigenvector of $A=\begin{bmatrix}5&0&1\\0&-2&0\\1&0&5\end{bmatrix}$ .Find also the least Latent root and hence the third eigenvalue.

Eigen value of a matrix by Jacobi Method for Symmetric Matrix

Rotation Matrix:

If $P(x,y)$ is any point in the XY plane and if OP is rotated in the clockwise direction through an angle θ, then the new position of $P(x',y')$ is given by



$$x' = x\cos\theta - y\sin\theta$$

$$y' = x\sin\theta + y\cos\theta$$

$$\text{ie.,} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\text{Where} \quad P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Hence P is called a rotation Matrix in the x y Plane. Here P is also an orthogonal matrix since $PP^T = 1$.

Eigen Values of $2\times$ 2 matrix by Jacobi Method:

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ be a symmetric matrix of order 2. Where $a_{12} = a_{21}$.

Step 1:

Assume the most general orthogonal Rotation Matrix of order 2 is $P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Step 2:

To find the θ value using $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{12}}{a_{11} - a_{22}}\right)$ if $a_{11} \neq a_{22}$

Step 3:

Write down $P = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ using the value of θ

Step 4:

Get $D=P^TAP$

The diagonal of D are the Eigen values the columns of p are the corresponding Eigen vectors.

1. Using Jacobi method, find the Eigen values and eigenvector of $A=\begin{bmatrix}4 & 1\\ 1 & 4\end{bmatrix}$

Solution:

Given, $A=\begin{bmatrix}4 & 1\\ 1 & 4\end{bmatrix}$

Here, $a_{11}=a_{22}=4, a_{12}=a_{21}=1>0$

The rotation matrix is $P=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$

$$\text{Here, } \theta=\frac{1}{2}\tan^{-1}\left(\frac{2a_{12}}{a_{11}-a_{22}}\right)$$

$$=\frac{1}{2}\tan^{-1}\left(\frac{2(1)}{4-4}\right)=\frac{1}{2}\tan^{-1}\left(\frac{2}{0}\right)$$

$$=\frac{1}{2}\tan^{-1}\ \infty$$

$$=\frac{1}{2}.\frac{\pi}{2}$$

$$\theta=\frac{\pi}{4}$$

$$\therefore \text{Rotation matrix p}=\begin{pmatrix}\cos\frac{\pi}{4} & -\sin\frac{\pi}{4}\\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4}\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$D=P^TAP=\left[\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}4 & 1\\ 1 & 4\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\right]$$

$$=\left[\begin{pmatrix} \frac{4}{\sqrt{2}}+\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}+\frac{4}{\sqrt{2}} \\ \frac{-4}{\sqrt{2}}+\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}}+\frac{4}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\right]$$

$$=\begin{pmatrix} \frac{5}{\sqrt{2}} & \frac{5}{\sqrt{2}} \\ \frac{-3}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$=\begin{pmatrix} \frac{5}{2}+\frac{5}{2} & \frac{-5}{2}+\frac{5}{2} \\ \frac{-3}{2}+\frac{3}{2} & \frac{3}{2}+\frac{3}{2} \end{pmatrix}$$

$$D = P^T AP = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$$

The Eigen values are 5 , 3 and eigenvectors column of the P matrix $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

2. Using Jacobi method, find the Eigen values and eigenvectors of A $= \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

Solution:

Given $A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$

Here, $a_{13} = a_{31} = 1$, $a_{11} = a_{33} = 5$, $a_{12} = 0$

Here, $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{13}}{a_{11}-a_{33}}\right)$

$$= \frac{1}{2}\tan^{-1}\left(\frac{2\times 2}{5-5}\right)$$

$$= \frac{1}{2}\tan^{-1}\left(\frac{4}{0}\right)$$

$$= \frac{1}{2}\tan^{-1} \infty$$

$$=\frac{1}{2}\frac{\pi}{2}$$

$$\theta=\frac{\pi}{4}$$

The rotational matrix is P = $\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & c\,os\,\theta\end{pmatrix}$

$$P=\begin{pmatrix}\cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta\end{pmatrix}=\begin{pmatrix}\cos\frac{\pi}{4} & 0 & -\sin\frac{\pi}{4} \\ 0 & 1 & 0 \\ \sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4}\end{pmatrix}$$

$$P=\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}$$

$$p^T=\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}$$

To Find D=P^TAP

$$D=\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{bmatrix}$$

$$D=\begin{bmatrix}6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4\end{bmatrix}$$

Hence the eigenvalue of the given matrix are 6,-2, 4

$\lambda_1=6,\ \lambda_2=-2,\ \lambda_3=4$ and the Corresponding eigenvector are the columns of P are

$$X_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}; \quad X_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}; \quad X_3 = \begin{bmatrix} \frac{-1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

3. Find the Eigen Values and Eigen Vector Of the Matrix A = $\begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$

Solution:

$$\text{Given } A = \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{bmatrix}$$

Here the Largest off-Diagonal element is

$$a_{13}=a_{31}=2 \text{ and } a_{11}=a_{33}=1$$

$$\text{Take } \theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{13}}{a_{11}-a_{33}}\right)$$

$$=\frac{1}{2}\tan^{-1}\left(\frac{2\ 2}{1-1}\right)$$

$$=\frac{1}{2}\tan^{-1}\infty \quad =\frac{1}{2}\frac{\pi}{2}$$

$$\theta=\frac{\pi}{4}$$

$$\text{The Rotation Matrix } P=\begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$P = \begin{bmatrix} \cos\frac{\pi}{4} & 0 & -\sin\frac{\pi}{4} \\ 0 & 1 & 0 \\ \sin\frac{\pi}{4} & 0 & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$P^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

To Find $D=P^T AP$

$$D = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 3 & \sqrt{2} \\ 2 & \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$$= \begin{bmatrix} \frac{3}{\sqrt{2}} & 2 & \frac{3}{\sqrt{2}} \\ \sqrt{2} & 3 & \sqrt{2} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$D = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Again reduce the largest off-diagonal element $a_{12}=a_{21}=2$ in D into zero

Consider the rotation matrix:

$$P_1 = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{here } a_{11} = a_{22} = 3$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2a_{12}}{a_{11}-a_{22}}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2\ 2}{3-3}\right)$$

$$\theta = \frac{\pi}{4}$$

$$P_1 = \begin{pmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} & 0 \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$P_1^T = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$D_1 = P_1^T D P_1 = \left[\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$\begin{bmatrix} \frac{5}{\sqrt{2}} & \frac{5}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

After two rotations, A is reduced to diagonal matrix D_1 .Hence the Eigen values of A are 5, 1,-1.

$$\text{Now } P_2 = PP_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Hence the corresponding eigenvectors are

$$x_1 = \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}; \quad x_2 = \begin{bmatrix} -1 \\ \sqrt{2} \\ -1 \end{bmatrix}; \quad x_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

H.W :

1. $\begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$

2. $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

UNIT-II

DESIGN OF EXPERIMENTS

Analysis of variance:

The technique of analysis of variance is referred to as ANOVA. A table showing the source of variance, the sum of squares, degrees of freedom, mean squares(variance)and the formula for the "F ratio is known as ANOVA table"

The technique of analysis if variance can be classified as

(i) One way classification(CRD)
(ii) Two way classification(RBD)
(iii) Three way classification(LSD)

One way classification:

In one way classification the data are classified on the basic of one criterion

The following steps are involved in one criterion of classification

(i) The null hypothesis is

$$H_o : \mu_1 = \mu_2 = ... = \mu_k$$

$$H_1 : \mu_1 \neq \mu_2 \neq .... \neq \mu_k$$

(ii) Calculation of total variation

Total sum of squares $V = \sum_i \sum_j x_{ij}{}^2 - \frac{G^2}{N}$

Where $G = \sum_i \sum_j x_{ij}$ (Grand total)

$\frac{G^2}{N}$ =correction formula

(iii) Sum of squares between the variates

$$V_1 = \sum_i \left[\frac{T_i^2}{n_i} \right] - \frac{G^2}{N}$$ With k-1 degree of freedom

(iv) Sum of squares within samples

$$V_2 = V - V_1$$

then the ratio $\dfrac{\dfrac{V_1}{K-1}}{\dfrac{V_2}{N-K}}$ follows F-distribution with degrees of freedom. Choosing the ratio which is greater than one, we employ the F-test

If we calculated F < table value $F_{0.05}$, the null hypothesis is accepted.

ANOVA Table for one way classification

Source of variation	Sum of square	Degrees of freedom	Mean square	Variance ratio
Between classes	V_1	K-1	$\dfrac{V_1}{K-1}$	$\dfrac{\dfrac{V_1}{K-1}}{\dfrac{V_2}{N-K}}$ (or)
Within classes	V_2	$N-K$	$\dfrac{V_2}{N-K}$	$\dfrac{\dfrac{V_2}{N-K}}{\dfrac{V_1}{K-1}}$
	V	$N-1$		

1. To test the significance of the variation of the retail prices of a certain commodity in the four principal plates A,B,C &D, seven shops were chosen at random in each city and the prices observed were as follows (prices in paise)

A	82	79	73	69	69	63	61
B	84	82	80	79	76	68	62
C	88	84	80	68	68	66	66
D	79	77	76	74	72	68	64

Do the data indicate that the prices in the four cities are significantly different?

Solution:

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$$

$$H_1 : \mu_1 \neq \mu_2 \neq \mu_3 \neq \mu_4$$

i.e., the prices of commodity in the four cities are same.

we take the origin at $x = 80$ and the calculation are done as follows.

Calculation of ANOVA (use new values)

Cities K=4	Shop(n = 7) 1	2	3	4	5	6	7	T_i	$\frac{T_i^2}{n}$	$\sum x^2$
A	2	-1	-7	-11	-11	-17	-19	-64	585.14	946
B	4	2	0	-1	-4	-12	-18	-29	120.14	505
C	8	4	0	-12	-12	-14	-14	-40	228.57	760
D	-1	-3	-4	-6	-8	-12	-16	-50	357.14	526
	$\frac{G^2}{N}=1196.03$							$G=-183$	$\frac{\sum T_i^2}{n}=1290.9$	$\sum_i \sum_j x_{ij}^2 = 2737$

Total sum of squares $V=\sum_i \sum_j x_{ij}^2 - \frac{G^2}{N}$

$$=2737-1196.03$$

$$V=1540.97$$

Sum of squares between cities

$$V_1 = \sum \frac{T_i^2}{n} - \frac{G^2}{N}$$

$$= 1290.9 - 1196.03$$

$$V_1 = 94.87$$

Sum of squares within cities

$$V_2 = V\text{-}V_1 = 1540.97 - 94.87$$

$$V_2 = 1446.1$$

ANOVA Table:

Source of variation	Sum of square of deviation	Degrees of f	Mean square	F
Between cities	$V_1 = 94.87$	K-1=4-1=3	$\frac{V_1}{K-1} = \frac{94.87}{3}$ $= 31.62$	$= \frac{60.25}{31.62}$ $= 1.90$
Within cities	$V_2 = 1446.1$	N-K=28-4=24	$\frac{V_2}{N-K} = \frac{1446.1}{24}$ $= 60.25$	
Total	V=1540.97	N-1=27		

Number of degrees of freedom = (N - K, K - 1) = (24,3)

Critical value:

The table value of F for (24, 3) degree of freedom at 5% Los is 8.64

Conclusion:

Since F< 8.64, H_o is accepted at 5% Los

$\therefore$ The prices of commodity in the four cities are same

2. Fill up the following Analysis of variance table

Source of variation	Degrees of freedom	Sum of squares	Mean squares	F ratio
Treatments	-	-	117	-
Error	-	704	-	
Total	16	938		

Solution:

From the given table we have,

$$V_2 = 704; V = 938$$

degree of freedom (total) N - 1 = 16 => N = 17

$$\text{mean squares } \frac{V_1}{K-1} = 117$$

$$\text{We Know that } \quad V_2 = V - V_1$$

$$=> \quad V_1 = V - V_2$$

$$= 938 - 704$$

$$\boxed{V_1 = 234}$$

$$\frac{V_1}{K-1} = 117$$

$$=> \frac{234}{K-1} = 117 => \frac{234}{K-1} = K-1$$

K – 1 = 2

degree of freedom (K-1) = 2

=>K=3

Next, N-K = 17-3 = 4

$$\frac{V_2}{N\text{-}K} = \frac{938}{14} = 50.29$$

Source of variation	Degrees of freedom	Sum of squares	Mean squares	F ratio
Treatments	K-1=3-1=2	$V_1 = 234$	$\frac{V_1}{K\text{-}1} = 117$	$\frac{117}{50.29}$ $= 2.327$
Error	N-K=17-3=14	$V_2 = 704$	$\frac{V_2}{N\text{-}K} = 50.29$	
Total	16	V = 938		

3. The following are the number of mistakes made in 5 successive days of 4 technicians working in a photographic laboratory

Technicians I	Technicians II	Technicians III	Technicians IV
6	14	10	9
14	9	12	12
10	12	7	8
8	10	15	10
11	14	11	11

Test at the 1% Los whether the difference among the 4 samples means can be attributed to chance

Solution:

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$$

ie., There is no differences among the 4 samples mean

$$H_1 : \mu_1 \neq \mu_2 \neq \mu_3 \neq \mu_4$$

We take the origin at 12 and the calculation are done as follows

Calculation of ANOVA (NEW Values)

Technicians K = 4	Days(5)					T_i	$\frac{T_i^2}{n}$	$\sum x^2$
	1	2	3	4	5			
I	-6	2	-2	-4	-1	-11	24.2	61
II	2	-3	0	-2	2	-1	0.2	21
III	-2	0	-5	3	-1	-5	5	39
IV	-3	0	-4	-2	-1	-10	20	30
Total	$\frac{G^2}{N}=\frac{(-27)^2}{20}=36.45$					G=-27	49.4	151

Total sum of squares:

$$V=\sum_i \sum_j x_{ij}^2 - \frac{G^2}{N}$$

$$=151-36.45$$

$$V=114.55$$

Sum of squares b/w cities:

$$V_1=\sum \frac{T_i^2}{n}-\frac{G^2}{N}$$

$$=49.4-36.45$$

$$V_1=12.95$$

Sum of squares within cities:

$$V_2=V\text{-}V_1=114.55-12.95$$

$$V_2=101.6$$

Source of variation	Sum of squares of deviation	Degrees of freedom	Mean squares	F ratio
B/W Technicians Within Technicians	$V_1 = 12.95$ $V_2 = 101.6$	K-1=4-1=3 N-K=20-4=16	$\frac{V_1}{K-1} = \frac{12.95}{3} = 4.31$ $\frac{V_2}{N-K} = \frac{101.6}{16} = 6.35$	$= \frac{6.35}{4.31}$
Total	V=114.55	N-1=19		$=1.473$

Degrees of freedom ((N - K, K - 1) = (16,3)

Critical value:

The table value of 'F' for (16,3) degree of freedom at 1% Los is 5.29

Conclusion:

Since F < 5.29, H_0 accepted at 1% level

$\therefore$ There is no difference among the four sample means.

4. The following table shows the lives in hours of four batches of electric lamps.

Batches	Lives in hours							
1	1610	1610	1650	1680	1700	1720	1800	
2	1580	1640	1640	1700	1750			
3	1460	1550	1600	1620	1640	1660	1740	1820
4	1510	1520	1530	1570	1600	1680		

Perform an analysis of the variance on these data and show that a significant test does not reject their homogeneity

Solution:

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4$$

I.e., the means of the lives of the four brands are homogeneous.

$$H_1 : \mu_1 \neq \mu_2 \neq \mu_3 \neq \mu_4$$

We take the origin $x_{ij} = \dfrac{\text{old}x_{ij} - 1700}{10}$

Calculation of ANOVA

Brand K=4	Lives								T_i	$\frac{T_i^2}{n}$	$\sum_{ij} x_{ij}$
	1	2	3	4	5	6	7	8			
1	-9	-9	-5	-2	0	2	10	-	-13	24.143	295
2	-12	-6	-6	0	5	-	-	-	-19	72.2	241
3	-24	-15	-10	-8	-6	-4	4	12	-51	325.125	1177
4	-19	-18	-17	-13	-10	-2	-	-	-79	1040.167	1247
Total	$\frac{G^2}{N} = \frac{(-162)^2}{26} = 1009.38$								G=-162	=1461.635	2960

$$N = n_1 + n_2 + n_3 + n_4 = 7 + 5 + 8 + 6 = 26$$

Total sum of squares:

$$V = \sum_i \sum_j (x_{ij})^2 - \frac{G^2}{N}$$

$$= 2960 - 1009.38$$

$$V = 1950.62$$

Total sum of squares b/w brands:

$$V_1 = \sum \frac{T_i^2}{n} - \frac{G^2}{N}$$

$$= 1461.635 - 1009.38$$

$$V_1 = 452.255$$

Sum of squares within brands:

$$V_2 = V - V_1$$

$$= 1950.62 - 452.255$$

$$V_2 = 1498.365$$

ANOVA Table:

Source of variation	Sum of squares	Degrees of freedom	Mean squares	F ratio
B/W Brands	$V_1 = 452.255$	K-1=4-1=3	$\frac{V_1}{\text{K-1}} = \frac{452.255}{3} = 150.75$	$= \frac{150.75}{68.11}$ $= 2.21$
Within Brands	$V_2 = 1498.365$	N-K=26-4=22	$\frac{V_2}{\text{N-K}} = \frac{1498.365}{22} = 68.11$	
Total	$1950.62 = V$	N-1=25		

Degrees of freedom (3, 22) =3.05

Critical value:

The table value of 'F' for (3,22) d.f at 5% Los is 3.05

Conclusion:

Since $F<3.05, H_0$ is accepted at 5% level

$\therefore$ The means of the lives of the four brands are homogeneous.

ie., the lives of the four brands of lamps do not differ significantly.

Two way classification:

In two way classification the data are classified on the basis of two criterions

The following steps are involved in two criterion of classification

(i) The null hypothesis

H_{01} and H_{02} framed

We compute the estimates of variance as follows

(ii) $G=\sum_i \sum_j x_{ij} =$ Grand total of $K \times n$ Observations

(iii) S : Total sum of squares $\sum \sum x_{ij}^2 - \frac{G^2}{N}$

(iv) S_1 :Sum of squares b/w rows (class-B) $= \frac{1}{K}\sum_{j=1}^{n} R_j^{\,2} - \frac{G^2}{N}$

(v) S_2 :Sum of squares b/w (classes A) $= \frac{1}{n}\sum_{i=1}^{K} C_i^{\,2} - \frac{G^2}{N}$

S_3 : Sum of squares due to error (or) Residual sum of squares

(vi) Errors (or) Residual $S_3 = S - S_1 - S_2$

(vii) The degrees of freedoms of

$S_1 = n\text{-}1$; $S_2 = k\text{-}1$; $S_3 = (n\text{-}1)(k\text{-}1)$

S=nk-1

ANOVA Table for two way classification

Source of variation	Sum of squares	Degrees of freedom	Mean squares	F ratio
B/W 'B' classes(rows)	S_1	n-1	$\frac{S_1}{n\text{-}1} = Q_B$	$F_1 = \frac{Q_B}{Q_{AB}}$ $d.f = [(n\text{-}1)(k\text{-}1)(n\text{-}1)]$
B/W 'A' classes(column)	S_2	k-1	$\frac{S_2}{k\text{-}1} = Q_A$	$F_2 = \frac{Q_A}{Q_{AB}}$
Residual (or) error	S_3	(n-1)(k-1)	$\frac{S_3}{(n\text{-}1)(k\text{-}1)} = Q_{AB}$	$d.f = [(k\text{-}1),(k\text{-}1)(n\text{-}1)]$
Total	S	nk-1	-	-

Advantages of R.B.D:

The chief advantages of R.B.D are as follows

(i) This design is more efficient or more accurate than CRD. This is because of reduction of experimental error.

(ii) The analysis of the design is simple and even with missing observations, it is not much complicated

(iii) It is Quite flexible, any number of treatments and any number of replication may be used

(iv) It is easily adaptable as in agricultural experiment it can be accommodated well in a rectangular, squares(or)in a field of any shape

(v) It provides a method of eliminating or reducing the long term effects.

(vi) This is the most popular design with experiments in view of its simplicity, flexibility and validity. No other has been used so frequently as the R.B.D

Disadvantages:

(i) The number of treatments is very large, than the side of the blocks will increase and this may introduce heterogeneity within blocks.

(ii) If the interactions are large, the experiments may yield misleading results.

1. The following data represent the number of units of production per day turned out by four randomly chosen operators using three milling machines

		Machines. M_1	M_2	M_3
Operators	1	150	151	156
	2	147	159	155
	3	141	146	153
	4	154	152	159

Perform analysis of variance and test the hypothesis

(i) That the machines are not significantly different

(ii) That the operators are not significantly different at 5% level

Solution:

H_{01} : There is no significantly difference bet machine and

H_{02} : There is no significantly a difference b/w operator

We take the origin 155 and the calculations are done as follows.

Calculation of ANOVA (using new values)

Operators	Machines M1	M2	M3	Row total R_j	$\sum_j x_{ij}^2$
1	-5	-4	1	-8	42
2	-8	4	0	-4	80
3	-14	-9	-2	-25	281
4	-1	-3	4	0	26
Column total C_i	-28	-12	3	$\boxed{-37}$	429
$\sum_i x_{ij}^2$	286	122	21	429	

Here N=12 ; G=-37

$$\text{Correction factor } \frac{G^2}{N} = \frac{(-37)^2}{12} = 114.08$$

Total sum of squares:

$$S=\sum_{i}\sum_{j}x_{ij}^{2}-\frac{G^{2}}{N}$$

$$=429-114.08$$

$$=314.92$$

Sum of squares between operators:

$$S_1=\sum_{j}\frac{R_j^{2}}{n_j}-\frac{G^{2}}{N}$$

$$=\frac{1}{3}[(-8)^2+(-4)^2+(-25)^2]-114.08$$

$$=235-114.08$$

$$=120.92$$

Sum of squares between machines:

$$S_2=\sum_{i}\left(\frac{C_i^{2}}{n_i}\right)-\frac{G^{2}}{N}$$

$$=\frac{1}{4}\left[(-28)^2+(-12)^2+(3)^2\right]-114.08$$

$$=234.25-114.08$$

$$S_2=120.17$$

Residual sum of squares:

$$S_3=S\text{-}S_1-S_2$$

$$=314.92-120.92-120.17$$

$$=73.83$$

AVOVA Table for two way classification

Source of variation	Sum of squares	Degrees of freedom	Mean sum squares	F ratio
B/W operators	120.92	n-1=4-1=3	$Q_B = \frac{S_1}{n-1} = 40.31$	
B/W machines	120.17	k-1=3-1=2	$Q_A = \frac{S_2}{k\text{-}1}$ $= 60.09$	$\frac{40.31}{12.305} = 1.49$ $(3,6)$
Residual	73.83	(n-1)(k-1)=6	$Q_{AB} = \frac{S_3}{(k\text{-}1)(n\text{-}1)}$ $= 12.305$	$\frac{60.09}{12.305} = 4.88$ $(2,6)$
Total	314.92	nk-1=11		

Degrees of freedom $V_1 = 2; V_2 = 6$ (machines)

Degrees of freedom $V_1 = 3; V_2 = 6$ (operators)

Critical value:

(i) Machines

The table value of 'F' for (2,6) d.f at 5% Los is 5.14

(ii) Operators

The table value of 'F' for (3,6) d,f at 5% Los is 4.76

Conclusion:

(i) Operators

Since F< 4.76 , H_{02} is accepted at 5% level

$\therefore$ The operators are not significantly different

(ii) For Machines

Since F< 5.14, H_{01} is accepted at 5% level

$\therefore$ The machines are not significantly different

2. An experiment was designed to study then performance of four different detergents, the following "whiteness" readings were obtained with specially designed equipment for 12 loads of washing distributed over three different models of washing machines.

Machines / Detergents	1	2	3	Total
A	45	43	51	139
B	47	46	52	145
C	48	50	55	153
D	42	37	49	128
Total	182	176	207	565

Looking on the detergents as treatment and the machines as blocks, obtain the appropriate analysis of variance table and test at 0.01 level of Significance whether there are differences in the detergents (or) in the washing machines

Solution:

H_{01} : There is no significant different b/w detergent

H_{02} : There is no significant different b/w washing machine

We take the origin is 50 and the calculation are done as follows.

Calculation of ANOVA (using new values)

Detergents	Washing machines			Row total R_j	$\sum_j x_{ij}^2$
	M1	M2	M3		
A	-5	-7	1	-11	75
B	-3	-4	2	-5	29
C	-2	0	5	3	29
D	-8	-13	-1	-22	234
Column total C_i	-18	-24	7	$\boxed{-35}$	367
$\sum_i x_{ij}^2$	102	234	31	367	

Here N=12; G=-35

$$\text{Correction factor } \frac{G^2}{N} = \frac{(-35)^2}{12} = 102.08$$

$$\text{Total sum of squares: } S=\sum_i \sum_j x_{ij}^2 - \frac{G^2}{N}$$

$$= 367 - 102.08$$

$$S = 264.92$$

Sum of squares b/w detergents: $S_1 = \sum_j \frac{R_j^2}{h_j} - \frac{G^2}{N}$

$$= \frac{1}{3}\left[(-11)^2 + (-5)^2 + (3)^2 + (-22)^2\right] - 102.08$$

$$= 213 - 102.08$$

$$S_1 = 110.92$$

Sum of squares between machines

$$S_2 = \sum_i \left(\frac{C_i^2}{n_i}\right) - \frac{G^2}{N}$$

$$= \frac{1}{4}\left((-18)^2 + (-24)^2 + (7)^2\right) - 102.08$$

$$= 237.25 - 102.08$$

$$S_2 = 135.17$$

Residual sum of squares $S_3 = S\text{-}S_1 - S_2$

$$= 264.92 - 110.92 - 135.17$$

$$S_3 = 18.83$$

ANOVA table for two way classification:

Source of variation	Sum of squares	Degrees of freedom	Mean squares	F ratio
B/W detergents	$S_1 = 110.92$	n-1=4-1=3	$Q_B = \frac{S_1}{n-1} = \frac{110.92}{3} = 36.97$	$\frac{Q_B}{Q_{AB}} = \frac{36.97}{3.14} = 11.77$
B/W machines	$S_2 = 135.17$	k-1=3-1=2	$Q_A = \frac{S_2}{k-1} = \frac{135.17}{2} = 67.59$	
Residual (or) Error	$S_3 = 18.83$	(n-1)(k-1)=6	$Q_{AB} = \frac{S_3}{(n-1)(k-1)} = \frac{18.83}{6} = 3.14$	$\frac{Q_A}{Q_{AB}} = \frac{67.59}{3.14} = 21.52$
Total	S=264.92	nk-1=11		

Degrees of freedom $V_1 = 2; V_2 = 6$ (machines)

Degrees of freedom $V_1 = 3; V_2 = 6$ (detergents)

Critical value:

(i) Detergents:

The table value of F for (3,6) degree of freedom at 1% Los is 9.78

(ii)Machines

The table value of F for (2,6) degree of freedom at 1% Los is 10.92

Conclusion:

(i) For detergents

Since F>9.78, H_{01} is rejected at 5% level

$\therefore$ The detergents are significantly different

(ii) For machines

Since F>10.92, H_{02} is rejected at 5% level

$\therefore$ The machines are significantly different

3. To study the performance of three detergents and three different water temperatures the following whiteness readings were obtained with specially designed equipment.

Water temp	Detergents A	Detergents B	Detergents C
Cold Water	57	55	67
Worm Water	49	52	68
Hot Water	54	46	58

Solution:

We set the null hypothesis

H_{01} : There is no significant different in the three varieties of detergents

H_{02} : There is no significant different in the water temperatures

We choose the origin at x=50

Water temp	Detergents A	Detergents B	Detergents C	Row total R_j	$\sum_j x_{ij}^2$
Cold Water	7	5	17	29	363
Worm Water	-1	2	18	19	329
Hot Water	4	-4	8	8	96
Column total C_i	10	3	43	$\boxed{56}$	788
$\sum_i x_{ij}^2$	66	45	677	788	

Total sum of squares:

$$S=\sum_j \sum_i x_{ij}^2 - \frac{G^2}{N}$$

$$= 788 - \frac{(56)^2}{9} = 788 - 348.44$$

S=439.56

Sum of squares between detergents:

$$S_1 = \sum_i \frac{C_i^{\,2}}{n_i} - \frac{G^2}{N}$$

$$= \frac{1}{3}\left[(10)^2 + (3)^2 + (43)^2\right] - 348.44$$

$$= 652.67 - 348.44$$

$$S_1 = 304.23$$

Sum of squares b/w temperatures:

$$S_2 = \sum_j \frac{R_j^{\,2}}{n_j} - \frac{G^2}{N}$$

$$= \frac{1}{3}[1266] - 348.44$$

$$= 422 - 348.44$$

$$S_2 = 73.56$$

Error sum of squares:

$$S_3 = S\text{-}S_1 - S_2$$

$$= 439.56 - 304.23 - 73.56$$

$$S_3 = 61.77$$

ANOVA Table:

Source of variation	Sum of squares	Degrees of freedom	Mean squares	F ratio
B/W detergents	304.23	2	$\frac{304.23}{2} = 152.11$	$\frac{152.11}{15.445} = 9.848$ $(2,4)$
B/W temperatures	73.55	2	$\frac{73.56}{2} = 36.78$	$\frac{36.78}{15.445} = 2.381$
Error	61.79	4	15.445	
Total	439.56	8		

Degrees of freedom (2,4) and (2,4)

Critical value:

The table value of F for (2,4) d.f at 5% Los is 6.94

Conclusion:

(i) For detergents:

Since $F > 9.85, H_{01}$ is rejected at 5% Los

$\therefore$ There is a significant different between the three varieties detergents,

(iii) For water temperature

Since $F<6.94, H_{02}$ is accepted at 5% Level

$\therefore$ There is no significant different in the water temperatures.

4. Four experiments determine the moisture content of samples of a powder, each man taking a sample from each of six consignments. These assignments are

Observer	Consignment					
	1	2	3	4	5	6
1	9	10	9	10	11	11
2	12	11	9	11	10	10
3	11	10	10	12	11	10
4	12	13	11	14	12	10

Perform an analysis if variance on these data and discuss whether there is any significant different b/w consignments (or) b/w observers.

Solution:

We formulate the hypothesis

H_{02} : There is no significant different b/w observer

H_{02} : There is no significant different b/w consignment

We take origin at x=11 and the calculations are done are as follows

Calculation ANOVA:

Observer	consignments						Rowtotal R_j	$\sum_j x_{ij}^2$
	1	2	3	4	5	6		
1	-2	-1	-2	-1	0	0	-6	10
2	1	0	-2	0	-1	-1	-3	7
3	0	-1	-1	1	0	-1	-2	4
4	1	2	0	3	1	-1	6	16
Column total C_i	0	0	-5	3	0	-3	$\boxed{-5}$	37
$\sum_j x_{ij}^2$	6	6	9	11	2	3	37	

Total sum of squares $= \sum_j \sum_i x_{ij}^2 - \frac{G^2}{N}$

$$S=37-\frac{(-5)^2}{24}=35.96$$

Sum of squares b/w observers $= \sum \frac{(R_j)^2}{n_j} - \frac{G^2}{N}$

$$S_1 = \frac{1}{6}\left[(-6)^2+(-3)^2+(-2)^2+(6)^2\right]-\frac{25}{24}$$

$$S_1 = 13.13$$

Sum of squares b/w consignments $= \sum \left(\frac{C_i^{\,2}}{n_i}\right) - \frac{G^2}{N}$

$$S_2 = \frac{1}{4}\left[(0+0+25+9+9)\right]-\frac{25}{24}$$

$$S_2 = 9.71$$

Error sum of squares $S_3 = S - S_1 - S_2$

$$= 35.96 - 13.13 - 9.71$$

$$S_3 = 13.12$$

Source of variation	Sum of squares	Degrees of freedom	Mean squares	'F' ratio
B/W Consignments	$S_1 = 9.71$	n-1=5	$\frac{9.71}{5} = 1.94$	$\frac{1.94}{0.87} = 2.23$ (5,15)
B/W observers	$S_2 = 13.13$	k-1=3	$\frac{13.13}{3} = 4.38$	$\frac{4.38}{0.87} = 5.03$ (3,15)
Error	$S_3 = 13.12$	(n-1)(k-1)=15	$\frac{13.12}{15} = 0.87$	
Total	$S = 35.96$	nk-1=23		

Critical value:

(i) For consignments ,
The table value of 'F' for (5, 15) d.f at 5% Los is 2.90

(ii) For observers:
The table value of F for (3, 15) d,f at Los 3.29

Conclusion:

(i) For observers
Since F>3.29, H_{01} is rejected
Hence these is a difference between observers is significant

(ii) For consignment:
Since F<2.33, H_{02} is accepted
$\therefore\therefore$ There is no significant different b/w the consignments

LATIN SQUARES DESIGN:

A Latin squares is a squares arrangement of m-rows and m-columns such that each symbol appearly once and only once in each row and column.

In randomized block design the randomization is done within blocks the units in each block being relatively similar in L.S.D there are two restrictions

(i) The number of rows and columns are equal
(ii) Each treatment occurs once and only once in each row and column.

This design is a three way classification model analysis of variance

The following steps are involved in Latin square design

Correction factor $=\frac{G^2}{N}$; G -> Grand total

$$\text{S.S b/w rows}=S_a=\sum_{i=1}^{m}\frac{S_i^2}{m}-C.F \quad \text{(S.S means Sum of Squares)}$$

$$\text{S.S b/w Columns}=S_b=\sum_{j=1}^{m}\frac{S_j^2}{m}-\frac{G^2}{N}|C.F$$

$$\text{S.S b/w Varieties}=S_c=\sum_{i=1}^{m}\frac{V_i^2}{m}-C.F$$

$$\left.\begin{matrix}\textit{Total sum of}\\ \textit{squares}\end{matrix}\right\} S=\sum_j\sum_i x_{ij}^2-C.F$$

$$\text{and } S_d = S\text{-}S_a - S_b - S_c$$

Here S_i=sum of i^{th} row

S_j=sum of j^{th} column

V_i=sum of i^{th} variety

ANOVA Table:

Source of variation	Sum of squares	Degrees of freedom	Mean squares	'F' ratio
B/W Rows	S_a	m-1	$\frac{S_a}{m\text{-}1} = R$	$\frac{R}{E}$ $\left[(m\text{-}1),(m\text{-}1)(m\text{-}2)\right]$
B/W Columns	S_b	m-1	$\frac{S_b}{m\text{-}1} = C$	$\frac{C}{E}$ $\left[(m\text{-}1),(m\text{-}1)(m\text{-}2)\right]$
B/W varieties	S_c	m-1	$\frac{S_c}{m\text{-}1} = V$	$\frac{V}{E}$ $\left[(m\text{-}1),(m\text{-}1)(m\text{-}2)\right]$
Error	S_d	(m-1)(m-2)	$\frac{S_d}{(m\text{-}1)(m\text{-}2)} = E$	
Total	S	$m^2 - 1$		

Comparison of LSD and RBD

(i) In LSD, the number of rows and number of columns are equal and hence the number of replication is equal to the number of treatments there is no such restriction in RBD

(ii) L.S.D is suitable for the case when the number of treatments is b/w 5 and 12 where as R.B.D can be used for any number of treatments and replications

(iii) The main advantage of L.S.D is that it removes the variations b/w rows and columns from that within the rows resulting in the reduction of experiment error to a large extent

(iv) The RBD can be performed equally on rectangular of square plots but for LSD, a mose (or) less a squares field is required due to (iii) LSD is preferred over RBD

Note: A 2×2 Latin Square Design is not possible. The degree of freedom for error in a $m\times m$ Latin squares design is (m-1)(m-2)

For m=2 the degree of freedom is 'o' and hence comparisons are not possible.

Hence a 2×2 LSD is not possible.

1. The following is the LSD layout of a design when 4 varieties of seeds are being tested set up the analysis of variance table and state four conclusion

A 105	B 95	C 125	D 115
C 115	D 125	A 105	B 105
D 115	C 95	B 105	A 115
B 95	A 135	D 95	C 115

Solution:

H: There is no significant difference

we take the origin as $u_{ij} = \dfrac{x_{ij} - 100}{5}$ and the calculations are done as follows

Varieties	Values				V_i
A	1	1	3	7	12
B	-1	1	-1	-1	0
C	5	3	-1	3	10
D	3	5	3	-1	10

Columns / Rows	C_1	C_2	C_3	C_4	Row total R_j	$\sum_i x_{ij}^2$
R_1	1	-1	5	3	8	36
R_2	3	5	1	1	10	36
R_3	3	-1	1	3	6	20
R_4	-1	7	-1	3	8	60
Columns total C_i	6	10	6	10	G=32	152
$\sum_j x_{ij}^2$	20	76	28	28	152	

G=32 N=16; $\sum_j \sum_i x_{ij}^2 = 152$

$$\text{C.F} = \frac{G^2}{N} = \frac{(+32)^2}{16} = 64$$

Total sum of squares $=\sum_{j}\sum_{i} x_{ij}^{2}-\frac{G^{2}}{N}$

$$=152-\frac{(32)^{2}}{16}$$

$$=152-64$$

$$S=88$$

Sum of squares b/w rows $=\frac{1}{4}\left[8^{2}+10^{2}+6^{2}+8^{2}\right]-64$

$$=66-64$$

$$S_{a}=2$$

Sum of squares b/w columns $=\frac{1}{4}\left[6^{2}+10^{2}+6^{2}+10^{2}\right]-64$

$$S_{b}=68-64$$

$$S_{b}=4$$

Sum of squares b/w Varieties $=\frac{1}{4}\left[12^{2}+0^{2}+10^{2}+10^{2}\right]-64$

$$=86-64$$

$$S_{c}=22$$

Error sum of squares $S_{d}=S-S_{a}-S_{b}-S_{c}$

$$=88-2-4-22$$

$$S_{d}=60$$

ANOVA Table:

Source of variation	Sum of squares	Degrees of freedom	Mean sum of squares	'F' ratio
B/W rows	$S_a = 2$	m-1=4-1=3	$\frac{S_a}{\text{m-1}} = \frac{2}{3} = 0.67$	$\frac{0.67}{10} = 0.067$
B/W columns	$S_b = 4$	m-1=4-1=3	$\frac{S_b}{\text{m-1}} = \frac{4}{3} = 1.33$	$\frac{1.33}{10} = 0.133$
B/W varieties	$S_c = 22$	m-1=3	$\frac{S_c}{\text{m-1}} = \frac{22}{3} = 7.33$	$\frac{7.33}{10} = 0.733$
Error	$S_d = 60$	(m-1)(m-2) =3×2=6	$\frac{S_d}{\text{(m-1)(m-2)}} = 10$	-
Total	$S = 88$	$m^2 - 1 = 15$	-	-

Number of degrees of freedom $V_1 = 3\,;\, V_2 = 6$

Critical value:

The table value of F for (3, 6) d.f at 5% Los is 4.76

Conclusion:

Since F<4.76, for all the case.

$\therefore$ There is no significant difference for the varieties

2. Analyse the variance in the following Latin squares of fields (in keys) of paddy where A,B,C,D denote the difference methods of calculation

D122	A121	C123	B122
B124	C123	A122	D125
A120	B119	D120	C121
C122	D123	B121	A122

Examine whether the different methods of cultivation have given significantly different fields.

Solution:

Re arrange the table in order

A121	A122	A120	A122
B122	B124	B119	B121
C123	C123	C121	C122
D122	D125	D120	D123

We take the origin 122 and the table is

Letter	Values				V_i total
A	-1	0	-2	0	-3
B	0	2	-3	-1	-2
C	1	1	-1	0	1
D	0	3	-2	1	2

Calculation of LSD:

Columns / Rows	1	2	3	4	Row total	$\sum_j x_{ij}^2$
1	0	-1	1	0	0	2
2	2	1	0	3	6	14
3	-2	-3	-2	-1	-8	18
4	0	1	-1	0	0	2
Columns total	0	-2	-2	2	$\boxed{-2}$	36
$\sum_i x_{ij}^2$	8	12	6	10	36	

Here N=16; G=-2

Correction factor $= \frac{G^2}{N} = \frac{4}{16} = 0.25$

Total sum of squares $S=\sum_i \sum_j x_{ij}^2 - \frac{G^2}{N}$

$= 36 - 0.25$

$S=35.75$

Sum of squares b/w rows $S_a = \sum_{i=1}^{m} \frac{S_i^2}{m} - \frac{G^2}{N}$

$= \frac{1}{4}\left[(6)^2 + (-8)^2\right] - 0.25$

$= 25 - 0.25$

$S_a = 24.75$

Sum of squares b/w columns $S_b = \sum_{j=1}^{m} \frac{S_j^{\,2}}{m} - \frac{G^2}{N}$

$$= \frac{1}{4}\left[(0)^2 + (-2)^2 + (-2)^2 + (2)^2\right] - 0.25$$

$$S_b = 2.75$$

Sum of squares b/w varieties $S_c = \sum_{i=1}^{m} \frac{V_i^{\,2}}{m} - \frac{G^2}{N}$

$$= \frac{1}{4}\left[(-3)^2 + (-2)^2 + (1)^2 + (2)^2\right] - 0.25$$

$$= 4.5 - 0.25$$

$$S_c = 4.25$$

Error (or) Residual $S_d = S\text{-}S_a - S_b - S_c$

$$= 35.75 - 24.75 - 2.75 - 4.25$$

$$S_d = 4$$

LSD Table:

Source of variation	Sum of squares	Degrees of freedom	Mean sum of squares	'F' ratio
B/W rows	$S_a = 24.75$	m-1=3	$\frac{S_a}{m\text{-}1} = \frac{24.75}{3} = 8.25$	$\frac{8.25}{0.67} = 12.31$
B/W columns	$S_b = 2.75$	3	$\frac{S_b}{m\text{-}1} = \frac{2.75}{3} = 0.92$	$\frac{0.92}{0.67} = 1.37$
B/W varieties	$S_c = 4.25$	3	$\frac{S_c}{m\text{-}1} = \frac{4.25}{3} = 1.42$	$\frac{1.42}{0.67} = 2.12$
Error (or) Residual	$S_d = 4.0$	6=(m-1)(m-2)	$\frac{S_d}{(m\text{-}1)(m\text{-}2)} = 0.67$	
Total	$S = 35.75$	$m^2 - 1 = 8$		

Critical value:

The value of ‘F’ for (3,6) d.f at 5% Los is 4.76

Conclusion:

Since F<4.76, we accept the null hypothesis

$\therefore$ The difference between the methods of cultivation is not significant.

3. The following data resulted from an experiment to compare three burners A,B, and C,A Latin squares design was used as the tests were made on 3 engines and were spread over 3 days.

	Engine 1	Engine 2	Engine 3
Day 1	A 16	B 17	C 20
Day 2	B16	C 21	A 15
Day 3	C15	A 12	B 13

Test the hypothesis that there is no diff between the burners

Solution:

We take the origin x=15 and the calculation are done as follows

Re arrangement of given table is

A 16	B 17	C 20
A 15	B 16	C 21
A 12	B 13	C 15

Varieties	Values			V_i
A	1	0	-3	-2
B	2	1	-2	1
C	5	6	0	11

Calculation of LSD

Columns/ Rows	C_1	C_2	C_3	Row total	$\sum_j x_{ij}^2$
R_1	1	2	5	8	30
R_2	1	6	0	7	37
R_3	0	-3	-2	-5	13
Column total	2	5	3	10	80
$\sum_i x_{ij}^2$	2	49	29	80	

Here N=9; G=10

Correction Factor $=\frac{G^2}{N}=\frac{(10)^2}{9}=11.11$

Total sum of squares $S=\sum_j\sum_i x_{ij}^2-C.F$

$=80-11.11$

$S=68.89$

Sum of squares b/w Rows $S_a=\sum_{i=1}^{m}\frac{S_i^2}{m}-C.F$

$=\frac{1}{3}[8^2+7^2+(-5)^2]-11.11$

$=46-11.11$

$S_a=34.89$

Sum of squares b/w columns $S_b=\sum_{j=1}^{m}\frac{S_j^2}{m}-C.F$

$=\frac{1}{3}[(2)^2+(5)^2+(3)^2]-11.11$

$=1.56$

Sum of squares b/w varieties $S_c=\sum_{i=1}^{m}\frac{V_i^2}{m}-C.F$

$=\frac{1}{3}[(-2)^2+1^2+11^2]-11.11$

$S_c=30.89$

Error (or) Residual $S_d=S\text{-}S_a-S_b-S_c$

$=68.89-34.89-1.56-30.89$

$S_d=1.55$

Source of variation	Sum of squares	Degrees of freedom	Mean sum of squares	'F' ratio
B/W rows	$S_a = 34.89$	m-1=2	$\frac{S_a}{m-1} = \frac{34.89}{2}$ $= 17.445$	$\frac{17.445}{0.775} = 22.5$
B/W columns	$S_b = 1.56$	m-1=2	$\frac{S_b}{m-1} = \frac{1.56}{2}$ $= 0.78$	$\frac{0.78}{0.775} = 1.01$
B/W varieties	$S_c = 30.89$	m-1=2	$\frac{S_c}{m-1} = \frac{30.89}{2}$ $= 15.445$	$\frac{15.445}{0.775} = 19.93$
Error (or) Residual	$S_d = 1.55$	(m-1)(m-2)	S_d(m-1)(m-2) $= \frac{1.55}{2}$ $= 0.775$	
Total	$S = 68.89$	$m^2 - 1 = 8$		

Critical value:

The value of 'F' for (2,8) d.f at 5% Los is 4.46

Conclusion:

Since F> the table value for the burners

$\therefore$ There is a significant difference between the burners

and also F> tabulated F for columns the difference b/w the engines is not significant.

Homework:

1. Analyse the variance in the following LS:

B 20	C 17	D 25	A 34
A 23	D 21	C 15	B 24
D 24	A 26	B 21	C 19
C 26	B 23	A 27	D 22

\

2. Analyse the variance in the following LS:

A 8	C 18	B 9
C 9	B 18	A 16
B 11	A 10	C 20

Factorial Experiments

Definition 1:

A factorial experiment in which each of m factors at ‘S’ is called a symmetrical factorial experiment and is often known as S^m factorial design

Definition 2:

2^m - Factorial experiments means a symmetrical factorial experiments where each of the m-factors is at two levels

2^2 -a factorial experiment means a symmetrical experiment where each of the factors is at two levels

Note:

If the numbers of level of the different factors are equal the experiments is called as a symmetrical factorial experiment.

Uses advantages of factorial experiments:

(i) Factorial designs are widely used in experiments involving several factors where it is necessary
(ii) F.D allow effects of a factor to be estimated at several levels of the others, giving conclusions that are valid over a range of experimental conditions
(iii) The F.D are more efficient than one factor at a time experiments.
(iv) In F.D individual factorial effect is estimated with precision, as whole of the experiment is devoted to it.
(v) Factorial designs from the basis of other designs of considerable practical value.
(vi) F.D are widely used in research work. These design are used to apply the results over a wide range of conditions

2^2 **-Factorial experiment:**

A factorial design with two factors, each at two levels is called a 2^2 factorial design

Yates's notation:

The two factors are denoted by the letters A and B the letters 'a' and 'b' denote one of the two levels of each of the corresponding factors and this will be called the second level.

The first level of A and B is generally expressed by the absence of the corresponding letter in the treatment combinations. The four treatment combinations can be enumerated as follows.

Symbols used:

a_0b_0(or)1:Factors A and B both at first level

a_1a_0(or)a:A at second level and B at first level

a_0a_1 (or) b :A at first level and B at second level

a_1a_1(or) ab : A and B both second levels.

Yates's method of computing factorial effect totals

For the calculation of various factorial effect total for 2^2 -factorial experiments the following table is need

Treatment combination	Total yield from all replicates	(3)	(4)	Effect Totals
'1'	[1]	[1]+[a]	[1]+[a]+[b]+[ab]	Grand total
a	[a]	[b]+[ab]	[ab]-[b]+[a]-[1]	[A]
b	[b]	[a]-[1]	[ab]+[b]-[a]-[1]	[B]
ab	[ab]	[ab]-[b]	[ab]-[b]-[a]+[1]	[AB]

2^2 -factorial experiment conducted in a CRD

Let x_{ij} = j^{th} observation of i^{th} treatment combinations i=1, 2, 3, 4; j=1,2….(say)

i.e., $x_1=[1]; x_2=[a]; x_3=[b]; x_4=[ab]$

Where

x_i=total of i^{th} treatment combination .

G=$\sum_i \sum_f x_{ij}$ grand total

n=4r=Total number of observations

$$TSS=\sum_j \sum_i x_{ij}^2 - \frac{G^2}{4r}$$

1. The follwoing table gives the plan and yields of a 2^2 – factorial experiment conducted in CRD

Analyse the design and give your comments

(1) 20	a 28	a 24	b 10
ab 23	b 11	ab 22	(1) 17
a 24	b 15	ab 21	(1) 19

Solution:

Arrange the observation as in one-way classification, we proceed as follows

Treatment Combination				Total
(1)	20	17	19	56
a	28	24	24	76
b	10	11	15	36
ab	23	22	21	66
Total			G=	234

Correction Formula = $\frac{G^2}{2^2 \times r} = \frac{234^2}{4 \times 3} = 4563$

$$\sum_j \sum_i x_{ij}^2 = 20^2 + 17^2 + 19^2 + 28^2 + 24^2 + 24^2 + 10^2 + 11^2 + 15^2 + 23^2 + 22^2 + 21^2$$

$$\sum_j \sum_i x_{ij}^2 = 4886$$

$$TSS = \sum_j \sum_i x_{ij}^2 - \frac{G^2}{4r} = 4886 - 4563 = 323$$

The values of SSA, SSB and SSAB are obtained by yate's method

Treatment combination	Total (2)	(3)	(4)	Divisor (5)	Sum of squares (6)
1	[1]	[1]+[a]	[1]+[a]+[b]+[ab]=[M]	-	-
a	[a]	[b]+[ab]	[ab]-[b]+[a]-[1]=[A]	4r	$[A]^2/4r$=SSA
b	[b]	[a]-[1]	[ab]+[b]-[a]-[1]=[B]	4r	$[B]^2/4r$=SSB
ab	[ab]	[ab]-[b]	[ab]-[b]-[a]+[1]=[AB]	4r	$[AB]^2/4r$=SSAB

SSE = TSS-(SSA+SSB+SSAB)

The analysis of variance table for 2^2 factorial design conducted in CRD

Source of variation	d.f	S.S	M.S.S	F
A	1	SSA	MSSA	$\frac{MSSA}{MSSE}$
B	1	SSB	MSSB	$\frac{MSSB}{MSSE}$
AB	1	SSAB	MSSAB	$\frac{MSSAB}{MSSE}$
Error	3(r-1)	SSE	MSSE	-
Total	4r-1	TSS	-	-

To obtain the sum of squares SSA, SSB, SSAB use yate's method:

Treatment/ combination	Total response	(3)	(4)	Divisor (5)	S.S (6)
(1)	56	56+76=132	132+102=234	4r=12	Grand total
a	76	36+66=102	20+30=50	12	$\frac{50^2}{12} = 208.33$
b	36	76-56=20	102-132=-30	12	$\frac{(-30)^2}{12} = 75$
ab	66	66-36=30	30-20=10	12	$\frac{(10)^2}{12} = 8.33$
				Total	291.66

SSE=TSS-(SSA+SSB+SSAB)

=323-291.66

SSE=31.34

Analysis of variance table:

Source of variation	d.f	S.S	M.S.S	F	$F_{0.01}(1,6)$
A	1	208.33	208.33	53.15	13.75
B	1	75	75	19.13	
AB	1	8.33	8.33	2.09	
Error	3(r-1)=6	31.34	3.92		
Total	4r-1=11	323			

Critical value:

The table value of for (1,6) d.f at 1% Los is 13.75

Conclusion:

Since F> tabulated value of 'F' for the main effect A and B, we conclude that the main effects A and B both are significantly different at 1% Los

UNIT –IV

Interpolation, Numerical Differentiation And Numerical Integration

Interpolation with unequal intervals:

Lagrangian polynomials

Lagrange's interpolation formula:

Let y=f(x) be a function which takes the values $y_0, y_1, \ldots\ldots, y_n$ corresponding to $x = x_0, x_1, \ldots., x_n$.

Then, Lagrange's interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2).......(x-x_n)}{(x_0-x_1)(x_0-x_2).......(x_0-x_n)} y_0$$

$$+\frac{(x-x_0)(x-x_2).......(x-x_n)}{(x_1-x_0)(x_1-x_2).......(x_1-x_n)} y_1$$

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$$+\frac{(x-x_0)(x-x_1).......(x-x_n)}{(x_n-x_0)(x_n-x_1).......(x_n-x_{n-1})} y_n$$

This called the Lagrange's formula for interpolation

1. Using Lagrange's formula fit a polynomial to the data:

x	0	1	2
y	7	5	15

Solution:

Let $y = f(x)$

W.k.t Lagrange's interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

Let

$x_0 = 0; \quad x_1 = 1; \quad x_2 = 2;$

$y_0 = 7; \quad y_1 = \; 5; \; y_2 = 15;$

$$y=\frac{(x-1)(x-2)}{(-1-1)(-1-2)}(7)+\frac{(x+1)(x-2)}{(1+1)(1-2)}(5)+\frac{(x+1)(x-1)}{(2+1)(2-1)}(15)$$

$$=\frac{7}{6}\ x^2-3x-2\ -\frac{5}{2}\ x^2-x-2\ +5\ x^2-1$$

$$=\frac{7}{6}x^2-\frac{7}{6}3x-\frac{7}{\cancel{6}3}\times\cancel{2}-\frac{5}{2}x^2+\frac{5}{2}x+\cancel{\frac{5}{2}}(\cancel{2})+5x^2$$

$$=\left(\frac{7}{6}-\frac{5}{2}+5\right)x^2+\left(\frac{-7}{2}+\frac{5}{2}\right)x+\left(\frac{7}{5}\right)$$

$$=\left(\frac{22}{6}\right)x^2-x+\frac{7}{3}$$

$$y=\frac{1}{3}\ 11x^2-3x+7$$

2. Find the polynomial $y=f(x)$ by using Lagrange's formula and hence find f(3) for,

x	0	1	2	5
f(x)	2	3	12	147

Solution:

W.k.t Lagrange's interpolation formula is

$$y=f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_2$$

$$=\frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2)+\frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3)$$

$$+\frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12)+\frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

$$=\frac{(x-1)(x-2)(x-5)}{(-10)}(2)+\frac{(x-0)(x-2)(x-5)}{(4)}(3)$$

$$+\frac{(x-0)(x-1)(x-5)}{-6}(12)+\frac{(x-0)(x-1)(x-2)}{60}(147)$$

$$y = f(3)$$

$$= \frac{(3-1)(3-2)(3-5)}{(-10)}(2) + \frac{(3-0)(3-2)(3-5)}{(4)}(3) + + \frac{(3-0)(3-1)(3-5)}{-6}(12) + \frac{(3-0)(3-1)(3-2)}{60}(147)$$

$$= \frac{2(1)(-2)}{-10}(2) + \frac{3(1)(-2)}{4}(3) + \frac{3(2)(-2)}{(-6)}(12) + \frac{3(2)(1)}{60}(147)$$

$$= \frac{8}{10} - \frac{18}{4} + 24 + \frac{147}{10}$$

y = 35

3. Using Lagrange's interpolation, calculate the profit in the year 2000 from the following data

year	1997	1999	2001	2002
Profits in Lakhs of Rs	43	65	159	248

Solution:

W.k.t Lagrange's interpolation formula is

$$y = f(x) = \frac{(x-1999)(x-2001)(x-2002)}{(1997-1999)(1997-2001)(1997-2002)}(43)$$

$$+ \frac{(x-1997)(x-2001)(x-2002)}{(1999-1997)(1999-2001)(1999-2002)}(65)$$

$$+ \frac{(x-1997)(x-1999)(x-2001)}{(2002-1997)(2002-1999)(2002-2001)}(248)$$

$$y = f(2000) = \frac{(2000-1999)(2000-2001)(2000-2002)}{(-2)(-4)(-5)}(43)$$

$$+ \frac{(2000-1997)(2000-2001)(2000-2002)}{(2)(-2)(-3)}(65)$$

$$+ \frac{(2000-1997)(2000-1999)(2000-2002)}{(4)(2)(-1)}(159)$$

$$+ \frac{(2000-1997)(2000-1999)(2000-2001)}{(5)(3)(1)}(248)$$

$$= \frac{(1)(-1)(-2)}{(-2)(-4)(-5)}(43) + \frac{(3)(-1)(-2)}{(2)(-2)(-3)}(65) + \frac{(3)(1)(-2)}{(4)(2)(-1)}(159) + \frac{(3)(1)(-1)}{(5)(3)(1)}(248)$$

$$= \frac{-43}{20} + \frac{65}{2} + \frac{477}{4} - \frac{248}{5}$$

$$y = 100$$

4.Using Lagrange's interpolation formula find y(10) given that y(5) = 12,y(6)=13,y(9)=14,y(11)=16

Solution:

Given

x	5	6	9	11
$y = f(x)$	12	13	14	16

W.K.T Lagrange's interpolation formula is

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Let

$$x_0 = 5; \quad x_1 = 6; \quad x_2 = 9; \quad x_3 = 11$$

$$y_0 = 12; y_1 = 13; y_2 = 14; y_3 = 16$$

$$y = f(x) = \frac{(x-6)(x-9)(x-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(x-5)(x-9)(x-11)}{(6-5)(6-9)(6-11)}(13)$$

$$+ \frac{(x-5)(x-6)(x-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(x-5)(x-6)(x-9)}{(11-5)(11-6)(11-9)}(16) \backslash$$

put x = 10,

$$y(10) = f(10) = \frac{(4)(1)(-1)}{(-1)(-4)(-6)}(12) + \frac{(5)(1)(-1)}{(1)(-3)(-5)}(13)$$

$$+\frac{(5)(4)(-1)}{(4)(3)(-2)}(14)+\frac{(5)(4)(1)}{(6)(5)(2)}(16)$$

y = 14.667

5. Find the missing term in the following table using Lagrange's interpolation

x	0	1	2	3	4
y	1	3	9	-	81

W.K.T Lagrange's interpolation formula is

$$y=f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

Let

$$x_0=0;\quad x_1=1;\quad x_2=2;\quad x_3=4$$

$$y_0=1; y_1=3; y_2=9; y_3=81$$

$$y=f(3)=\frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)}(1)+\frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)}(3)$$

$$+\frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)}(9)+\frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)}(81)$$

Let $x=3$,

$$\text{y = f(3)}=\frac{(3-1)(3-2)(3-4)}{(-1)(-2)(-4)}(1)+\frac{(3-0)(3-2)(3-4)}{(1)(-1)(-3)}(3)$$

$$+\frac{(3-0)(3-1)(3-4)}{(2)(1)(-2)}(9)+\frac{(3-0)(3-1)(3-2)}{(4)(3)(2)}(81)$$

$$=\frac{2(1)(-1)}{(-1)(-2)(-4)}(1)+\frac{3(1)(-1)}{(1)(-1)(3)}(3)+\frac{3(2)(-1)}{(2)(1)(-2)}(9)+\frac{(3)(2)(1)}{(4)(3)(2)}(81)$$

$$=\frac{-2}{-8}-3+\frac{27}{2}+\frac{81}{4}$$

$$= \frac{1}{4} - 3 + \frac{27}{2} + \frac{81}{4}$$

$$y = 31$$

1. Apply Lagrange's formula to find f(5),given that f(1) = 2,f(2) = 4,f(3) = 8 and f(7) = 128

Solution:

Given the data

$$x_0 = 1; x_1 = 2; x_2 = 3; x_3 = 7$$

$$y_0 = 2; y_1 = 4; y_2 = 8; y_3 = 128$$

By Lagrange's interpolation formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Substituting x=5 and the given data

$$y = f(5) = \frac{3(2)(-2)}{(-1)(-2)(-6)}(2) + \frac{4(2)(-2)}{1(-1)(-5)}(4) + \frac{4(3)(-2)}{(2)(1)(-4)}(8) + \frac{4(3)(2)}{6(5)(4)}(128)$$

$$= 2 + (-12.800) + 24 + 25.6$$

$$= 51.6 - 12.800$$

$$f(5) = 38.800$$

2. Find a polynomial of degrees 3 fitting the following data

x	-1	0	2	3
y	-2	-1	1	4

Soln:
Given data:

$$x_0 = -1; x_1 = 0; x_2 = 2; x_3 = 3$$

$$y_0 = -2; y_1 = -1; y_2 = 1; y_3 = 4$$

By Lagrange's interpolation formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

Substituting the given data,

$$y = f(x) = \frac{x(x-2)(x-3)}{(-1)(-3)(-12)}(-2) + \frac{(x+1)(x-2)(x-3)}{1\times(-2)(-3)}(-1)$$

$$+ \frac{(x+1)x(x-3)}{3\times 2\times(-1)}(1) + \frac{(x+1)x(x-2)}{4\times 3\times 1}(4)$$

$$= \frac{1}{6}\ x^3 - 5x^2 + 6x \ - \frac{1}{6}\ x^3 - 4x^2 + x + 6 \ - \frac{1}{6}\ x^3 - 2x^2 - 3x \ + \frac{1}{3}\ x^3 - x^2 - 2x$$

$$= \frac{1}{6}\left[\cancel{x^3} - 5x^2 + 6x - x^3 + 4x^2 - x - 6 - \cancel{x^3} + 2x^2 + 3x + 2x^3 - 2x^x - 4x\right]$$

$$\text{y} = \frac{1}{6}\left[x^3 - x^2 + 4x - 6\right]$$

3.Given $u_0 = 6;\ u_1 = 9;\ u_3 = 33;\ u_7 = -15$. Find u_2

Solution:

Given the data

$$x_0 = 0; x_1 = 1; x_2 = 3; x_3 = 7$$

$$y_o = 6; y_1 = 9; y_2 = 33; y_3 = -15$$

Where y = u(x)

To find y when x = 2

By Lagrange's interpolation formula

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$= \frac{(x-1)(x-3)(x-7)}{(-1)(-3)(-7)}6 + \frac{(x)(x-3)(x-7)}{(1)(-2)(-6)}9 + \frac{(x)(x-1)(x-7)}{(3)(2)(-4)}33 + \frac{x(x-1)(x-3)}{(7)(6)(4)}(-15)$$

put x = 2

$$y = u(x) = \frac{(1)(-1)(-5)}{-21}6 + \frac{2(-1)(-5)}{12}9 + \frac{2(1)(-5)}{-24}33 + \frac{2(1)(-1)}{168}(-15)$$

= 7. 5 -1.429+13.750+0.179

y = y(x) = 2;

$$\therefore u_2 = 20$$

Inverse Lagrangian:

1. Apply lagrange's formula inversely to obtain the root of the equation f(x) = 0 given that f(0) = -4;f(1) = 1;f(3)=29;f(4)=52.

Solution:

Given that

$$x_0 = 0; \quad x_1 = 1; \quad x_2 = 3; \quad x_3 = 4$$

$$y_o = -4; y_1 = 1; \quad y_2 = 29; \quad y_3 = 52$$

To find x such that f(x) = 0

Applying lagrange's interpolation formula inversely we get

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}x_1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}x_3$$

Using the given data and subs y = 0 we've

$$x = \frac{(-1)(-29)(-52)}{(-5)(-23)(-56)}(0) + \frac{4(-29)(-52)}{5\times(-28)(-51)}(1) + \frac{4(-1)(-52)}{33(28)(-51)}(1) + \frac{4(-1)(-29)}{(56)(51)(23)}(4)$$

$$= 0.845 - 0.029 + 0.007$$

$$\boxed{x = 0.823}$$

Homework:

1. By LF Given $y_0 = -12; y_1 = 0; y_3 = 6; y_4 = 12. Find\ y_2$

   Ans: 4
2. Fit a polynomial of minimum order to the data

x	0	1	3	4
y	-4	1	29	52

<u>Inverse Lagrange's Interpolation;</u>

The process of finding a value of x for the corresponding value of y is called inverse interpolation

The inverse of Lagrange's interpolation formula is

$$x = \frac{(y-y_1)(y-y_2)......(y-y_n)}{(y_0-y_1)(y_0-y_2)......(y_0-y_n)}x_0 + \frac{(y-y_0)(y-y_2)......(y-y_n)}{(y_1-y_0)(y_1-y_2)......(y_1-y_n)}x_1$$

$$+.................. + \frac{(y-y_0)(y-y_1)......(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)......(y_n-y_{n-1})}x_n$$

1. Find the age corresponding to the annuity value 13.6 given the table:

Age(x)	30	35	40	45	50
Annuity value (y)	15.9	14.9	14.1	13.3	12.5

Solution:

W.K.T the inverse Lagrange's interpolation formula is

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)(y_0-y_4)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)(y-y_4)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)(y_1-y_4)}x_1$$

$$+\frac{(y-y_0)(y-y_1)(y-y_3)(y-y_4)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)(y_2-y_4)}x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_4)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)(y_3-y_4)}x_3$$

$$+\frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3)}{(y_4-y_0)(y_4-y_1)(y_4-y_2)(y_4-y_4)}x_4 + ...$$

Let y = 13.6 $x = \frac{(13.6-14.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(15.9-14.9)(15.9-14.1)(15.9-13.3)(15.9-12.5)}(30)$

$$+\frac{(13.6-15.9)(13.6-14.1)(13.6-13.3)(13.6-12.5)}{(14.9-15.9)(14.9-14.1)(14.9-13.3)(14.9-12.5)}(35)$$

$$+\frac{(13.6-15.9)(13.6-14.9)(13.6-13.3)(13.6-12.5)}{(14.1-15.9)(14.1-14.9)(14.1-13.3)(14.1-12.5)}(40)$$

$$+\frac{(13.6-15.9)(13.6-14.9)(13.6-14.1)(13.6-13.3)}{(12.5-15.9)(12.5-14.9)(12.5-14.1)(12.5-13.3)}(50)$$

x = 43

2. Find the value of θ given $f(\theta)=0.3887$ where

$$f(\theta)=\int_0^\theta \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$$ Using the table

θ	21°	23°	25°
$f(\theta)$	0.3706	0.4068	0.4433

Solution:

Given: $f(\theta)$=0.3887

$$\theta=\frac{(0.3887-0.4068)(0.3887-0.4433)}{(0.3706-0.4068)(0.3706-0.4433)}(21)$$

$$+\frac{(0.3887-0.3706)(0.3887-0.4433)}{(0.4068-0.3706)(0.4068-0.4433)}(23)$$

$$+\frac{(0.3887-0.3706)(0.3887-0.4068)}{(0.4433-0.3706)(0.4433-0.4068)}(25)$$

= 7.8858+ 17.2027 – 3.0865

$\theta = 22.0020$

$\therefore$ The value of θ such that $f(\theta)$=0.3887 is $\theta = 22.0020$.

Homework:

1. Using Lagrange's interpolation find y (2) from the following data;

x	0	1	3	4	5
y	0	1	81	256	625

(Ans:16)

2. Using Lagrange's interpolation find f(4) given that f(0) = 2,f(1) = 3,f(2) = 12,f(15) = 3587

3. The following table gives the value of the problem integral $f(x)=\frac{2}{\sqrt{\Pi}}\int_0^x e^{-x^2}dx$ corresponding to certain values of x. For what value of x is this integration equal to 0.5

x	0.46	0.47	0.48	0.49
y	0.4846555	0.4937452	0.5027498	0.5116683

<u>Divided differences (unequal intervals)</u>

Let the function $y=f(x)$ take the values $f(x_0),f(x_1),f(x_2).....f(x_n)$ corresponding to the value $x_0,x_1,x_2,.....,x_n$ of the argument x where $x_1-x_0,x_2-x_1,.......,x_n-x_{n-1}$, need not necessarily be equal

The first divided difference of f(x) for the arguments $x_0,x_1,$ is

$$f(x_0,x_1)=\frac{f(x_1)-f(x_0)}{x_1-x_0}$$

$$\text{'''}ly,\ f(x_1,x_2)=\frac{f(x_2)-f(x_1)}{x_2-x_1} \qquad \text{and so on}$$

The second divided difference of f(x) for the three arguments x_0,x_1,x_3 is defined as

$$f(x_0,x_1,x_2)=\frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}$$

$$f(x_0,x_1,x_2)=\frac{f(x_2,x_3)-f(x_1,x_2)}{x_3-x_1} \quad \text{and so on}$$

Divided Difference Table:

Argument X	Entry f(x)	First D.D $\Delta_1 f(x)$	Second D.D $\Delta_1^2 f(x)$	Third D.D $\Delta_1^3 f(x)$
x_0	$f(x_0)$			
x_1	$f(x_1)$	$f(x_0,x_1)$	$f(x_0,x_1,x_2)$	
x_2	$f(x_2)$	$f(x_1,x_2)$	$f(x_1,x_2,x_3)$	$f(x_0,x_1,x_2,x_4)$
x_3	$f(x_3)$	$f(x_2,x_3)$	$f(x_2,x_3,x_4)$	$f(x_1,x_2,x_3,x_4)$
x_4	$f(x_4)$	$f(x_3,x_4)$		

Properties of divided differences:

1. The divided difference are symmetrical in all their arguments ie., the value of any difference is independent of the order of the arguments
2. The operator $'\Delta'$ is linear
3. The n^th divided differences of a polynomial of the n^th degree are constant.
4. **** The divided difference of the product of a constant and a function is equal to the product of the constant and the divided difference of the function.

Problems based on Divided Differences:

1. If $f(x)=\frac{1}{x}$. find the divided difference of f(a,b,c,d) (or) show that $\underset{bcd}{\Delta}^3\left(\frac{1}{a}\right)=-\frac{1}{abcd}$

   Solution:

Given

$$f(x)=\frac{1}{x}$$

$$f(a)=\frac{1}{a};f(b)=\frac{1}{b}$$

$$f(a,b)=\frac{f(b)-f(a)}{b-a}=\frac{\frac{1}{b}-\frac{1}{a}}{b-a}$$

$$=\frac{\frac{a-b}{ab}}{-(a-b)}$$

$$=\frac{-(a-b)}{ab}\times\frac{1}{(a-b)}$$

$$f(a,b)=\frac{-1}{ab}$$

$$f(a,b,c)=\frac{f(b,c)-f(a,b)}{c-a}$$

$$= \frac{\frac{1}{-bc} - \left(\frac{-1}{ab}\right)}{c-a} = \frac{\frac{-1}{bc} + \frac{1}{ab}}{(c-a)}$$

$$= \frac{\cancel{(-a+c)}}{abc} \times \frac{1}{\cancel{(c-a)}} = \frac{1}{abc}$$

$$f(a,b,c) = \frac{1}{abc}$$

$$f(a,b,c,d) = \frac{f(b,c,d) - f(a,b,c)}{d-a}$$

$$= \frac{\frac{1}{bcd} - \frac{1}{abc}}{d-a} = \frac{(a-d)}{abcd} x \frac{1}{-(a-d)}$$

$$f(a,b,c,d) = \frac{-1}{abcd}$$

$$ie., \Delta^3_{\text{bcd}}\left(\frac{1}{\text{a}}\right) = \frac{-1}{\text{abcd}}$$

2. If $f(x) = \frac{1}{x^2}$ find the divided difference f(a, b) and f(a, b, c)(or) prove that

$$f(a,b,c) = \frac{ac+bc+ca}{a^2b^2c^2}$$

Solution:

Given: $f(x) = \frac{1}{x^2}$

$$f(a) = \frac{1}{a^2}; f(b) = \frac{1}{b^2}$$

$$f(a,b) = \frac{f(b) - f(a)}{b-a} = \frac{\frac{1}{b^2} - \frac{1}{a^2}}{b-a} = \frac{a^2 - b^2}{a^2b^2} \times \frac{1}{-(a-b)}$$

$$= \frac{\cancel{(a-b)}(a+b)}{-(a^2b^2)\cancel{(a-b)}}$$

$$f(a,b) = \frac{-(a+b)}{a^2b^2}$$

$$f(a,b,c) = \frac{f(b,c) - f(a,b)}{(c-a)}$$

$$= \frac{\frac{-(b+c)}{b^2c^2} + \frac{a+b}{a^2b^2}}{(c-a)}$$

$$= \frac{1}{(c-a)}\left[\frac{-a^2b - a^2c^2 + c^2a + c^2b}{a^2b^2c^2}\right]$$

$$= \frac{1}{(c-a)}\left[\frac{ac(c-a) + b(c^2 - a^2)}{a^2b^2c^2}\right]$$

$$= \frac{\cancel{(c-a)}}{\cancel{(c-a)}}\left[\frac{ac + b(c+a)}{a^2b^2c^2}\right] = \frac{ac + bc + ab}{a^2b^2c^2}$$

$$\therefore f(a,b,c) = \frac{ab + bc + ca}{a^2b^2c^2}$$

3. From the divided difference table for the following data:

x	5	15	22
y	7	36	160

Solution:
The divided difference table is

x	$y = f(x)$	$\Delta_1 f(x)$	$\Delta_1^{\ 2} f(x)$
5	7		
		$\frac{36-7}{15-5} = 2.9$	$\frac{17.71-2.9}{22-5}$ $= 0.87$
15	36		
		$\frac{160-36}{22-15} 17.71$	
22	160		

4. Obtain the divided difference table for the following data

x	-1	0	2	3
y	-8	3	1	12

Solution:
The divided difference table is,

x	$y = f(x)$	$\Delta_1 f(x)$	$\Delta_1^{\ 2} f(x)$	$\Delta_1^{\ 3} f(x)$
-1	-8			
		$\frac{3+8}{0+1} = 11$		
0	3		$\frac{-1-11}{2+1} = -4$	
		$\frac{1-3}{2-0} = -1$		$\frac{4+4}{3+1} = 2$
2	1		$\frac{11+1}{3-0} = 4$	
		$\frac{12-1}{3-2} = 11$		
3	12			

5. Find the divided difference of $f(x)=x^3+x+2$ for the arguments 1,3,6,11

Solution:

Given, $f(x)=x^3+x+2$

When

x = 1, f(1) = 4

x = 3, f(3) = 82

x = 6, f(6) = 224

x =11, f(11) = 1344

The divided difference table is

x	$y=f(x)$	$\Delta_1 f(x)$	$\Delta_1^2 f(x)$	$\Delta_1^3 f(x)$
1	4	$\frac{32-4}{3-1}=14$		
3	32	$\frac{224-32}{6-3}=64$	$\frac{64-14}{6-1}=10$	
6	224	$\frac{1344-224}{11-6}=224$	$\frac{224-64}{11-3}=20$	$\frac{20-10}{11-1}=1$
11	1344			

Homework:

1. Find the divided difference table for the following data

(i)

x	1	1	4	5
f(x)	8	11	78	123

(Ans: 0.168)

(ii)

x	1	2	4	7	12
f(x)	22	30	82	106	206

(Ans: 0.19)

2. Find the 3rd divided difference with arguments 2, 4,9,10 of the function $f(x)=x^3-2x$.

Ans:f(2,4,9,10)=1.

<u>Newton's Divided difference formula (or) Newton's interpolation formula for unequal intervals:</u>

Let $y = f(x)$ take values $f(x_0), f(x_1).....f(x_n)$ corresponding to the arguments $x_0, x_1,....., x_n$ then

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) + ... + (x-x_0)(x-x_1).....(x-x_{n-1})f(x_0,x_1,...,x_n)$$

<u>Problems base on divided difference:</u>

1. Use Newton's divided difference formula fit a polynomial to the data:

x	-1	0	2	3
y	-8	3	1	12

And hence find y when x=1.

Solution:

The divided difference table is,

x	$y = f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
-1	-8			
		$\frac{3+8}{0+1} = 11$		
0	3		$\frac{-1-11}{2+1} = -4$	
		$\frac{1-3}{2-0} = -1$		$\frac{4+4}{3+1} = 2$
2	1		$\frac{11+1}{3-0} = 4$	
		$\frac{12-1}{3-2} = 11$		
3	12			

The Newton's divided difference formula is,

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2)$$

$$+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2x_3) -----> 1$$

Here $x_0 = -1, x_1 = 0, x_2 = 2, x_3 = 3$

$$f(x_0) = -8; f(x_0,x_1) = 11; f(x_0,x_1,x_2) = -4; f(x_0,x_1,x_2,x_3) = 2$$

$$1 => f(x) = -8 + (x+1)11 + (x+1)(x-0)(-4) + (x+1)(x-0)(x-2)(2)$$

$$f(x) = 2x^3 - 6x^2 + 3x + 3$$

When $x = 1, f(1) = 2 - 6 + 3 + 3 = 2$

Hence $x = 1, y = 2$

2. Find the Newton's divided difference formula find the missing value from the table

x	1	2	4	5	6
y	14	15	5	-	9

Solution:

The divided difference table is

x	$y=f(x)$	$\Delta_1 f(x)$	$\Delta_1^2 f(x)$	$\Delta_1^3 f(x)$
1	14			
		$\frac{15-14}{2-1}=1$		
2	15		$\frac{-5-1}{4-1}=-2$	
		$\frac{5-15}{4-2}=-5$		$\frac{1.75+2}{6-1}=0.75$
4	5		$\frac{2+5}{6-2}=1.75$	
		$\frac{9-5}{6-4}=2$		
6	9			

Newton's divided difference formula is

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)$$

$$+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2x_3)----> 1$$

here $x_0=1, x_1=2, x_2=4$

$f(x_0)=14, f(x_0,x_1)=1, f(x_0,x_1,x_2)=-2, f(x_0,x_1,x_2,x_3)=0.75$

Here x = 5

$1 \Rightarrow f(5)=14+(5-1)(1)+(5-1)(5-2)(-2)+(5-1)(5-2)(5-4)(0.75)$

$f(5)=3$

3. Find the polynomial equation $y=f(x)$ passing through (-1,3),(0,-6),(3,39),(6,822) and (7,1611) by using divided difference formula,

Solution:

Given

x	-1	0	3	6	7
f(x)	3	-6	39	822	1611

The divided difference table is ,

x	$f(x)$	$\Delta_1 f(x)$	$\Delta_1^2 f(x)$	$\Delta_1^3 f(x)$	$\Delta_1^4 f(x)$

-1	3	$\frac{-6-3}{0+1}=-9$	$\frac{15+9}{3+1}=6$	$\frac{41-6}{6+1}=5$	$\frac{13-5}{7+1}=1$
0	-6	$\frac{39+6}{3-0}=15$	$\frac{261-15}{6-0}=41$	$\frac{132-41}{7-0}=13$	
3	39	$\frac{822-39}{6-0}=261$	$\frac{789-261}{7-3}=132$		
6	822	$\frac{1611-822}{7-6}=789$			
7	161				

Newton's divided difference formula is,

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)$$
$$+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)$$
$$+(x-x_0)(x-x_1)(x-x_2)(x-x_3)f(x_0,x_1,x_2,x_3,x_4)----->1$$

Here $x_0=-1, x_1=0, x_2=3, x_3=6, x_4=7$

$$f(x_0)=3,\ f(x_0,x_1)=-9,\ f(x_0,x_1,x_2)=6,\ f(x_0,x_1,x_2,x_3)=5,\ f(x_0,x_1,x_2,x_3,x_4)=1$$
$$1 \Rightarrow f(x)=3+(x+1)(-9)+(x+1)x(6)+(x+1)x(x-3)5+(x+1)x(x-3)(x-6)(1)$$
$$=3-9x-9+6x^2+6x+5x^3-10x^2-15x+(x^2-2x-3)(x^2-6x)$$
$$=-6-18x-4x^2+5x^3+x^4-6x^3-2x^3+12x^2-3x^2+18x$$
$$f(x)=x^4-3x^3+5x^2-6$$

<u>Interpolation with equal Intervals :</u>

<u>Newton's forward Interpolation formula:</u>

Let $y=f(x)$ be a function which takes the values $y_0, y_1, \ldots\ldots, y_n$ corresponding to the values $x_0, x_1, \ldots\ldots, x_n$ Where the values of x are equally spaced

Ie., $x_i = x_0 + ih\,;\, i=0,1,2,\ldots\ldots,n$

Suppose, to find the values of y when $x = x_0 + ph$

where p is any real number

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots.$$

Where $p=\frac{x-x_0}{h}$

This formula is called Newton's Gregory interpolation forward interpolation formula.

Note:

(1) If y_x is a polynomial in x of degrees n then the formula becomes

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ... + \frac{p(p-1)(p-2)......(p-n-1)}{n!}\Delta^n y_0$$

(2) This formula is used to interpolate the value y near $x = x_0$ (beginning of the given date) and for extrapolation of the values of 'y' a short distance backward from y_0.

Newton's backward interpolation formula:

This formula is used for interpolating a value of y for a given x near the end of a table of values. Let $y_0, y_1, ..., y_n$ be the values of $y = f(x)$ for $x = x_0, x_1, ..., x_n.$

Where $x_i = x_0 + ih \quad i = 0, 1, 2, 3, ..., n$

To find y when $x = x_n + ph$ where p is any number (p is negative in this case)

$$y_p = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + ...$$

$$\text{where } p = \frac{x - x_n}{h}$$

This formula is called Newton's backward interpolation formula

Error in Newton's Interpolation formula:

(1) The error caused in using Newton's forward formula for interpolation is given by,

$$\text{Error} = \frac{p(p-1)(p-2).....(p-n)}{(n+1)!}\Delta^{n+1} y_{(c)}$$

$$\text{where } x_0 < c < y_n \text{ and } p = \frac{x - x_0}{h}$$

(2) The error in using Newton's backward formula in the form

$$\text{Error} = \frac{p(p+1)(p+2).....(p+n)}{(n+1)!} h^{n+1} y^{n+1}{}_{(c)}$$

$$\text{where } x_0 < c < y_n \text{ and } x - x_n = ph$$

Problems

1. Using Newton's forward interpolation formula, five f(1.02) from the following data.

x	1.0	1.1	1.2	1.3	1.4

f(x)	0.841	0.891	0.932	0.964	0.985

Solution:

Forward difference table

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
1.0	0.841	0.050	–0.009	0
1.1	0.891	0.041	–0.009	–0.002
1.2	0.932	0.032	–0.011	
1.3	0.964	0.021		
1.4	0.985			

$y_0 = 0.841\,;\ \Delta y_0 = 0.050\,;\ \Delta^2 y_0 = -0.009$

Here, $x_0 = 1.0$ *Let* $x = 1.2$

$$p = \frac{x - x_0}{h} \Rightarrow p = 0.2$$

By Newton's forward interpolation formula,

$$\therefore y(1.2) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \dots..$$

$$= 0.841 + (0.2)(0.050) + \frac{(0.2)(-0.8)}{2}(-0.009)$$

$$\therefore y(1.2) = 0.852$$

2. Using Newton's forward interpolation formula find f(1.5) from the following data:

x	0	1	2	3	4
f(x)	858.3	869.6	880.9	892.3	903.6

Solution:

Difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	858.3				
		11.3			
1	869.6		0		
		11.3		0.1	
2	880.9		0.1		-0.3
		11.4		-0.2	
3	892.3		-0.1		
		11.3			
4	903.6				

Here, $x_0 = 1$; $y_0 = 869.6$; $\Delta y_0 = 11.3$; $\Delta^2 y_0 = 0.1$; $\Delta^3 y_0 = -0.2$

To find y for x=1.5

$$p = \frac{x - x_0}{h} \Rightarrow p = \frac{0.5}{1} = 0.5$$

By Newton's forward interpolation formula:

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + ... + \frac{p(p-1)(p-2)......(p-n-1)}{n!}\Delta^n y_0$$

$$y(1.5) = 869.6 + (0.5)(11.3) + \frac{(0.5)(0.5)(0.1)}{2} - \frac{(0.5)(0.5)(1.5)}{6}(0.2)$$

$$y(1.5) = 875.2$$

3. Using Newton's backward interpolation formula find y when x=27 from the following data,

x	10	15	20	25	30
y	35.4	32.2	29.1	26.0	23.1

Solution:

Backward difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
10	35.4				
		-3.2			
15	32.2		0.1		
		-3.1		-0.1	
20	29.1		0		0.3
		-3.1		0.2	
25	26.0		0.2		
		-2.9			
30	23.1				

Here,

$$x_n = 30,\ y_n = 23.1,\ \nabla y_n = -2.9,$$
$$\nabla^2 y_n = 0.2,\ \nabla^3 y_n = 0.2,\ \nabla^4 y_n = 0.3$$

By Newton's backward interpolation formula

$$y(x_n + ph) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + ...$$

Here $x = 27$; $p = \frac{x - x_n}{h} = \frac{27-30}{5} = -0.6$

$$\therefore y(27) = 23.1 + (2.9)(0.6) + \frac{(-0.6)(0.4)}{2}(0.2)$$

$$+ \frac{(-0.6)(0.4)(1.4)}{6}(0.2) + \frac{(-0.6)(0.4)(1.4)(2.4)}{24}(0.2)$$

$$= 24.7947$$

$$\therefore y(27) \approx 24.8$$

2. Using Newton's backward formula, find from the following data

x	1.00	1.25	1.50	1.75	2.00
e^{-x}	0.3679	0.2865	0.2231	0.1738	0.1353

Solution:

Backward difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
1.00	0.3679				
		–0.0814			
1.25	0.2865		0.0180		
		–0.0634		–0.0039	
1.50	0.2231		0.141		0.0006
		–0.0493		–0.0033	
1.75	0.1735		0.0108		
		–0.0385			
2.00	0.1353				

Here, $x_n = 2$; $y_n = 0.1353$ from the difference table ,

$$\nabla y_n = -0.0385 ; \nabla^2 y_n = 0.0108 ; \nabla^3 y_n = -0.0033 ; \nabla^4 y_n = 0.0006.$$

Let $x = 1.9$; $p = \frac{x - x_n}{h} = \frac{1.9-2}{0.25} = -0.4$

By Newton's backward interpolation formula

$$y(x_n + ph) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + ...$$

$$y(1.9) = 0.1353 + (-0.4)(-0.0385) + \frac{(0.4)(0.6)}{2}(0.010) + \frac{(-0.4)(0.6)(1.6)}{6}(-0.0033)$$

$$\therefore e^{-1.9} = 0.1496$$

3. Find $\tan(0.26)$ from the following values of tans for $0.10 \le x \le 0.30$

x	0.10	0.15	0.20	0.25	0.30
$\tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

Solution:

Difference table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
0.10	0.1003				
		0.0508			
0.15	0.1511		0.0008		
		0.0516		0.0002	
0.20	0.2027		0.0010		0.0002
		0.0526		0.0004	
0.25	0.2553		0.0014		
		0.0540			
0.30	0.3093				

Here $x_n = 0.30$; $y_n = 0.3093$

From the difference table

$$\nabla y_n = 0.0540;\ \nabla^2 y_n = 0.0014;\ \nabla^3 y_n = 0.0004;\ \nabla^4 y_n = 0.0002;$$

$$Let\ x = 0.26$$

$$P = \frac{x - x_n}{h} = \frac{0.26 - 0.30}{0.05} = -0.8$$

By Newton's backward difference formula

$$\tan(0.26) = 0.3093 - (0.8)(0.0540) + \frac{(-0.8)}{2}(0.0014) + \frac{(-0.8)(0.2)(1.2)}{6}(0.0004) + \frac{(-0.8)(0.2)(1.2)}{24}(0.0002)$$

$$= 0.2662$$

$$\tan(0.26) = 0.2662$$

4. From the following table , find the number of student whose obtained less than 45 marks

Marks	30-40	40-50	50-60	60-70	70-80
No of students	30	42	51	35	31

Solution:

Difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
below 40	30				
		42			
below 50	72		9		
		51		−25	
below 60	123		−16		37
		35		12	
below 70	158		−4		
		31			
below 80	189				

Here $x_0 = 40$; $y_0 = 30$; $\Delta y_0 = 41$; $\Delta^2 y_0 = 10$; $\Delta^3 y_0 = -25$; $\Delta^4 y_0 = 37$

$$Let\ x = 45; \quad p = \frac{x - x_0}{h} = \frac{45-40}{10} = \frac{5}{10} = 0.5$$

By Newton's forward difference formula,

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 ....$$

$$y(45) = 30 + (0.5)42 + \frac{(0.5)(-0.5)}{2}(9) + \frac{(0.5)(-0.5)(-1.5)}{6}(-25) + \frac{(0.5)(-0.5)(-1.5)(-2.5)}{24}37$$

$$= 30 + 21.00 - 1.13 - 1.56 - 1.45$$

$$y(45) = 46.86$$

$\therefore$ Number of students with less than 45 marks

$= 47$

5. From the following table values of x and f(x),determine (i) f(0.23) and (ii) f(0.29)

x	0.20	0.22	0.24	0.26	0.28	0.30
f(x)	1.6596	1.6698	1.6804	1.6912	1.7024	1.7139

Solution:

The difference table is,

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.20	1.6596				
		0.0106			
0.22	1.6698		0.0002		
		0.0108		0.0002	
0.24	1.6804		0.0004		−0.0003
		0.0112		−0.0001	
0.26	1.6912		0.0003		
		0.0115			
0.28	1.7024				
0.30	1.7139				

(i) To find f(0.23)

Take $x = 0.23$ and $x_0 = 0.22$

From the difference table,

$$y_0 = 1.6698; \Delta y_0 = 0.0106; \Delta^2 y_0 = 0.0002; \Delta^3 y_0 = 0.0002; \Delta^4 y_0 = -0.0003$$

$$p = \frac{x - x_0}{h} = \frac{0.23 - 0.22}{0.02} = 0.5$$

By Newton's forward interpolation formula

$$y_p = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \ldots.$$

$$f(0.23) = 1.6698 + (0.5)(0.0106) + \frac{0.5(-0.5)}{2}(0.0002)$$

$$+\frac{0.5(-0.5)(-1.5)}{6}(0.0002) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24}(-0.0003)$$

$$= 1.6698 + 0.01 - 0.00003 + 0.00001 + 0.000001$$

$$\boxed{f(0.23) = 1.6797}$$

(ii). To find f(0.29)

Take $x_n = 0.30 \; and \; x = 0.29$

$$p = \frac{x - x_n}{h} = -0.5$$

$$y_n = 1.7139; \nabla y_n = 0.0115; \nabla^2 y_n = 0.0003; \nabla^3 y_n = -0.0001$$

$$y(x_n + ph) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \ldots..$$

$$y(0.29) = 1.7139 + (-0.5)(0.0115) + \frac{(-0.5)(0.5)}{2}(0.0003)$$

$$+\frac{(-0.5)(0.5)(1.5)}{6}(-0.0001) + \frac{(-0.5)(0.5)(1.5)(2.5)}{24}(-0.0003)$$

$$= 1.7139 - 0.00575 - 0.000004 + 0.00001 + 0.00001$$

$$\boxed{f(0.29) = 1.7081}$$

Numerical Differentiation and integration:

Differentiation using interpolation formula:

Derivatives using divided difference:

The divided difference formula is

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2)$$

$$+(x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \ldots\ldots$$

First fit a polynomial for the given data using Newton's divided difference interpolation formula and compute the derivative for a given x.

Problems based on divided differences:

1. Find $f'(5) and f''(5)$ using the following the data

x	0	2	3	4	7	9
F(x)	4	26	58	112	466	922

Solution:

Since the values of x are not equally spaced we shall use Newton's divided difference formula:

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	4				
		$\frac{26-4}{2-0}=11$			
2	26		$\frac{32-11}{3-0}=7$		
		$\frac{58-26}{3-2}=32$		$\frac{11-7}{4-0}=1$	
3	58		$\frac{54-32}{4-2}=11$		$\frac{1-1}{1-0}=1$
		$\frac{112-58}{4-3}=54$		$\frac{16-11}{7-2}=1$	
4	112		$\frac{118-54}{7-3}=16$		$\frac{1-1}{9-2}=0$
		$\frac{466-112}{7-4}=118$		$\frac{22-16}{9-3}=1$	
7	466		$\frac{22-16}{9-4}=22$		
		$\frac{922-466}{9-7}=228$			
9	922				

Let $x_0 = 0; x_1 = 2; x_2 = 3; x_3 = 4; x_4 = 7; x_5 = 9$

$f(x_0) = 4;\ f(x_0, x_1) = 11;\ f(x_0, x_1, x_2) = 7;\ f(x_0, x_1, x_2, x_3) = 1$

w.k.t the Newton's divided diff formula is,

$$f(x)=f(x_0)+(x-x_0)f(x_0,x_1)+(x-x_0)(x-x_1)f(x_0,x_1,x_2)$$
$$+(x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)+......$$
$$=4+(x-0)11+(x-0)(x-2)7+(x-0)(x-2)(x-3)(1)$$
$$=4+11x+(x^2-2x)7+(x^3-5x^2+6x)$$
$$=4+11x+7x^2-14x+x^3-5x^2+6x$$
$$f(x)=x^3+2x^2+3x+4----------->(1)$$

Differentiate with respect to x

$$f'(x)=3x^2+4x+3 \quad ------>(2)$$
$$f'(5)=3(5)^2+4(5)+3=98$$
$$f''(x)=6x+4$$
$$f''(5)=6(5)+4=34$$
$$\therefore f'(5)=98; f''(x)=34$$

Homework:

1. Find the values of $f''(3)$ using divided difference Given the data:

x	0	1	4	5
$f(x)$	8	11	68	123

Newton's forward difference Formula:

Let y=f(x) be a function taking the values $y_0, y_1, ..., y_n$ corresponding to $x_0, x_1, ..., x_n$ of the independent variable x.

Let the values of x be at equidistant intervals of size h

Then, if x is non- tabular value

$$f'(x)\frac{dy}{dx}=\frac{1}{h}\left[\frac{\Delta y_0}{1!}+\frac{(2u-1)}{2!}\Delta^2 y_0+\frac{(3u^2-6u)+2)}{3!}\Delta^3 y_0 + \frac{(4u^3-18u^2+22u-6)}{4!}\Delta^4 y_0+......\right]$$

where $u=\dfrac{x-x_0}{h}$

Similarly,

$$f''(x)=\frac{d^2y}{dx^2}=\frac{1}{h^2}\left(\Delta^2 y_0+(u-1)\Delta^3 y_0+\frac{(6u^2-18u+11)}{12}\Delta^4 y_0+......\right)$$

$$f'''(x)=\frac{d^3y}{dx^3}=\frac{1}{h^3}\left(\Delta^3 y_0+\frac{12u-18}{12}\Delta^4 y_0+......\right)$$

For the tabular value at $x=x_0$

$$f'(x_0)=\left(\frac{dy}{dx}\right)_{x=x_0}=\frac{1}{h}\left(\Delta y_0-\frac{1}{2}\Delta^2 y_0+\frac{1}{3}\Delta^3 y_0-\frac{1}{4}\Delta^4 y_0+.......\right)$$

$$f''(x_0)=\left(\frac{d^2y}{dx^2}\right)_{x=x_0}=\frac{1}{h^2}\left(\Delta^2 y_0-\Delta^3 y_0+\frac{11}{12}\Delta^4 y_0+.......\right)$$

$$f'''(x_0)=\left(\frac{d^3y}{dx^3}\right)_{x=x_0}=\frac{1}{h^3}\left(\Delta^3 y_0-\frac{3}{2}\Delta^4 y_0+.......\right)$$

Newton's backward difference formula:

Let y=f(x) be a function taking the values $y_0, y_1, ..., y_n$ corresponding to $x_0, x_1, ..., x_n$ of the independent variable x. Let the values of x be at equidistant intervals of size h

If x is non- tabular value:

Then

$$f'(x)=\frac{dy}{dx}=\frac{1}{h}\left(\begin{array}{l}\frac{\nabla y_n}{1!}+\frac{2v+1}{2!}\nabla^2 y_n+\frac{(3v^2+6v+2)}{3!}\nabla^3 y_n\\+\frac{(4v^3+18v^2+22v+6)}{4!}\nabla^4 y_n+.....\end{array}\right)$$

$$\text{where } v=\frac{x-x_n}{h}$$

Similarly,

$$f''(x)=\frac{d^2y}{dx^2}=\frac{1}{h^2}\left(\nabla^2 y_n+(v+1)\nabla^3 y_n+\frac{(6v^2+18v+11)}{12}\nabla^4 y_n+......\right)$$

$$f'''(x)=\frac{d^3y}{dx^3}=\frac{1}{h^3}\left(\nabla^3 y_n+\frac{(12v+18)}{12}\nabla^4 y_n+......\right)$$

For the Tabluar value at $x=x_n$

$$f'(x_n)=\left(\frac{dy}{dx}\right)_{x=x_n}=\frac{1}{h}\left(\nabla y_n-\frac{1}{2}\nabla^2 y_n+\frac{1}{3}\nabla^3 y_n+......\right)$$

$$f''(x_n)=\left(\frac{d^2y}{dx^2}\right)_{x=x_n}=\frac{1}{h^2}\left(\nabla^2 y_n+\nabla^3 y_n+\frac{11}{12}\nabla^4 y_n+......\right)$$

$$f'''(x_n) = \left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left(\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + ...\right)$$

Note:

1. Numerical differentiation can be used only when the difference of some order are constant.

Problem based on Newton's forward and backward

1. Find $f'(3)$ *and* $f''(3)$ for the following data

x	3	3.2	3.4	3.6	3.8	4
$f(x)$	-14	-10.032	-5.296	-0.256	6.672	14

Solution:

Difference Table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
3	-14					
		3.968				
3.2	-10.032		0.768			
		4.736		-0.464		
3.4	-5.296		0.304		2.048	
		5.04		1.584		-5.12
3.6	-0.256		1.888		-3.072	
		6.928		-1.488		
3.8	6.672		0.4			
		7.328				
4	14					

To find $f'(3)$:

Here h=0.2

By Newton's forward difference Formula is :

$$f'(x_0) = \left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \frac{1}{5}\Delta^5 y_0 + ...\right)$$

$$f'(3) = \left(\frac{dy}{dx}\right)_{x=3} = \frac{1}{0.2}\left(3.968 - \frac{1}{2}(0.768) + \frac{1}{3}(-0.464) - \frac{1}{4}(2.048) + \frac{1}{5}(-5.12)\right)$$

$$= \frac{1}{0.2}(3.968 - 0.384 - 0.1547 - 0.512 - 1.024)$$

$$f'(3) = 9.4665$$

To find $f''(3)$:

We know that

$$f''(x_0)=\left(\frac{d^2y}{dx^2}\right)_{x=x_0}=\frac{1}{h^2}\left(\Delta^2y_0-\Delta^3y_0+\frac{11}{12}\Delta^4y_0-\frac{5}{6}\Delta^5y_0+\ldots\ldots\right)$$

$$f''(3)=\left(\frac{d^2y}{dx^2}\right)_{x=3}=\frac{1}{(0.2)^2}(0.768-(-0.464)+\frac{11}{12}(2.048)-\frac{5}{6}(-5.12))$$

$$=\frac{1}{0.04}(0.768+0.464+1.8773+4.267)$$

$$f''(3)=184.4075$$

2. The following data gives the velocity of a particle for 20 seconds at an interval of 5 second Find the initial acceleration using the entire data

Time(sec)	0	5	10	15	20
Velocity(m/sec)	0	3	14	69	228

Solution:

Difference table is dependent on time t.ie.,

t	v	Δv	$\Delta^2 v$	$\Delta^3 v$	$\Delta^4 v$
0	0				
		3			
5	3		8		
		11		36	
10	14		44		24
		55		60	
15	69		104		
		159			
20	228				

For initial acceleration ie., $\frac{dv}{dt}$ at t=0; Here h=5

$\therefore$ We use forward formula

$$f'(t_0)=\left(\frac{dv}{dt}\right)_{t=t_0}=\frac{1}{h}\left(\Delta v_0-\frac{1}{2}\Delta^2v_0+\frac{1}{3}\Delta^3v_0-\frac{1}{4}\Delta^4v_0+\ldots\ldots\right)$$

$$f'(0)=\left(\frac{dv}{dt}\right)_{t=0}=\frac{1}{5}\left(3-\frac{1}{2}(8)+\frac{1}{3}(36)-\frac{1}{4}(24)\right)$$

$$=\frac{1}{5}(3-4+12-6)$$

$$=\frac{1}{5}(5)$$

$$f'(0)=1$$

3. Find $f'(4)$ *and* $f''(4), f'''(4)$ *for* the following data

x	0	1	2	3	4
y	1	2.718	7.381	20.086	54.598

Solution:

Difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1	1.718	2.945	5.097	8.668
1	2.718	4.663	8.042	13.765	
2	7.381	12.705	21.807		
3	20.086	34.512			
4	54.598				

To Find $f'(4)$:

The Newton's backward difference formula:

$$f'(x_n)=\left(\frac{dy}{dx}\right)_{x=x_n}=\frac{1}{h}\left(\nabla y_n+\frac{1}{2}\nabla^2 y_n+\frac{1}{3}\nabla^3 y_n-\frac{1}{4}\Delta^4 y_n+\ldots\ldots\right)$$

$$f'(4)=\left(\frac{dy}{dx}\right)_{x=4}=\frac{1}{1}\left((34.512)+\frac{1}{2}(21.807)+\frac{1}{3}(13.765)+\frac{1}{4}(8.668)\right)$$

$$=34.512+10.9035+4.58+2.167$$

$$f'(4)=52.1705$$

To find $f''(4)$:

$$f''(x_n)=\left(\frac{d^2y}{dx^2}\right)_{x=x_n}=\frac{1}{h^2}\left(\nabla^2 y_n+\nabla^3 y_n+\frac{11}{12}\Delta^4 y_n+\frac{5}{6}\Delta^5 y_n\ldots\ldots\right)$$

$$f''(4)=\left(\frac{d^2y}{dx^2}\right)_{x=4}=\frac{1}{1}\left(21.807+13.765+\frac{11}{12}(8.668)\right)$$

$$=21.807+13.765+7.9457$$

$$f''(4)=43.5177$$

To find $f'''(4)$:

$$f'''(x_n)=\left(\frac{d^3y}{dx^3}\right)_{x=x_n}=\frac{1}{h^3}\left(\nabla^3 y_n+\frac{3}{2}\Delta^4 y_n+\ldots\ldots\right)$$

$$f'''(4)=\left(\frac{d^3y}{dx^3}\right)_{x=4}=\frac{1}{1}\left(13.765+\frac{3}{2}(8.668)\right)$$

$$=13.765+13.002$$

$$f'''(4)=26.767$$

4. Find the 1st and 2nd derivatives of f(x) at the point x=1.5 and x=4

x	1.5	2	2.5	3	3.5	4
y	3.375	7	13.625	24	38.875	59

Solution:

Difference table:

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.5	3.375				
		3.625			
2	7		3		
		6.625		0.750	
2.5	13.625		3.750		0
		10.375		0.750	
3	24		4.500		0
		14.875		0.750	
3.5	38.875		5.250		
		20.125			
4	59				

To find x = 1.5
Here h = 0.5

By Newton's forward difference Formula is :

$$f'(x_0)=\left(\frac{dy}{dx}\right)_{x=x_0}=\frac{1}{h}\left(\Delta y_0-\frac{1}{2}\Delta^2 y_0+\frac{1}{3}\Delta^3 y_0-\frac{1}{4}\Delta^4 y_0+\frac{1}{5}\Delta^5 y_0+\ldots\ldots\right)$$

$$f'(1.5)=\left(\frac{dy}{dx}\right)_{x=1.5}=\frac{1}{0.5}\left(3.625-\frac{1}{2}(3)+\frac{1}{3}(0.75)\right)$$

$$f'(1.5)=4.75$$

$$f''(x_0)=\left(\frac{d^2y}{dx^2}\right)_{x=x_0}=\frac{1}{h^2}\left(\Delta^2 y_0-\Delta^3 y_0+\frac{11}{12}\Delta^4 y_0-\frac{5}{6}\Delta^5 y_0+\ldots\ldots\right)$$

$$f''(1.5)=\left(\frac{d^2y}{dx^2}\right)_{x=1.5}=\frac{1}{(0.5)^2}(3-0.75)$$

$$f''(1.5)=9$$

To find x = 4

The Newton's backward difference formula is,

$$f'(x_n)=\left(\frac{dy}{dx}\right)_{x=x_n}=\frac{1}{h}\left(\nabla y_n+\frac{1}{2}\nabla^2 y_n+\frac{1}{3}\nabla^3 y_n-\frac{1}{4}\Delta^4 y_n+\ldots\ldots\right)$$

$$f'(4) = \left(\frac{dy}{dx}\right)_{x=4} = \frac{1}{0.5}\left(20.125 + \frac{1}{2}(5.250) + \frac{1}{3}(0.75)\right)$$

$$= \frac{1}{0.5}\ 20.125 + 2.625 + 0.25$$

$$f'(4) = 46$$

$$f''(x_n) = \left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\ \nabla^2 y_n + \nabla^3 y_n + \ldots\ldots$$

$$f''(4) = \left(\frac{d^2y}{dx^2}\right)_{x=4} = \frac{1}{(0.5)^2}(5.250 + 0.75)$$

$$f''(4) = 24$$

5. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at x=51 from the following data

x	50	60	70	80	90
y	19.96	36.65	58.81	77.21	94.61

Solution:

Given: x=51

$x_0 = 50$ $\qquad h = 60 - 50 = 10$

$$u = \frac{x - x_0}{h} = \frac{51 - 50}{10} = 0.1$$

At x=51 , u=0.1

Difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
50	19.96				
		16.69			
60	36.65		5.47		
		22.16		−9.23	
70	58.81		−3.76		11.99
		18.40		2.76	
80	77.21		−1.00		
		17.40			
90	94.61				

w.k.t the Newton's forward difference Formula is :

$$f'(x) = \left(\frac{dy}{dx}\right)_{x = x_0} = \left(\frac{dy}{dx}\right)_{u = 0.1} = \frac{1}{h}\left(\Delta y_0 + \frac{(2u-1)}{2!}\Delta^2 y_0 + \frac{(3u^2 - 6u + 2)}{3!}\Delta^3 y_0 + \frac{(4u^3 - 18u^2 + 22u - 6)}{4!}\Delta^4 y_0 + \ldots\ldots\right)$$

$$f'(51)=\left(\frac{dy}{dx}\right)_{u=0.1}=\frac{1}{10}\left(\begin{array}{l}16.69+\frac{(0.2-1)}{2}(5.47)+\frac{3(0.1)^2-6(0.1)+2)+2}{6}(-9.23)\\+\frac{(4(0.02)^3-18(0.1)^2+22(0.1)-6)}{24}(11.99)+......\end{array}\right)$$

$$=\frac{1}{10}(16.69-2.188-2.1998-1.9863)$$

$$f'(51)=1.0316$$

$$f''(x)=\left(\frac{d^2y}{dx^2}\right)_{u=0.1}=\frac{1}{h^2}\left(\Delta^2 y_0+(u-1)\Delta^3 y_0+\frac{(6u^2-18u+11)}{12}\Delta^4 y_0+......\right)$$

$$f''(51)=\frac{1}{100}\left[5.47+(0.1-1)(-9.23)+\frac{6(0.1)-18(0.1)+11}{12}(11)\right]$$

$$=\frac{1}{100}\ 5.47+8.307+9.2523$$

$$f''(51)=0.2303$$

$$f'(51)=1.0316$$

$$f''(51)=0.2303$$

Numerical differentiation (using Stirling's formula)

Derivatives at points near the middle of the data.

If the derivatives of $y=f(x)$ required at a point $x=x_0+ph$ near the middle of the data.

$$\frac{dy}{dx}=\frac{1}{h}\left[\begin{array}{l}(\Delta y_0+\Delta y_{y-1})+p\Delta^2 y_{-1}+\frac{3p^2-1}{12}(\Delta^3 y_{-1}+\Delta^3 y_{-2})+\frac{1}{12}(2p^3-p)\Delta^4 y_{-1}\\+\frac{5p^4-15p^2+4}{240}(\Delta^5 y_{-2}+\Delta^5 y_{-3})+.....\end{array}\right]$$

$$\frac{d^2y}{dx^2}=\frac{1}{h^2}\left[\Delta^2 y_{-1}+\frac{p}{2}(\Delta^3 y_{-1}+\Delta^3 y_{-2})+\frac{6p^2-1}{12}\Delta^4 y_{-1}+...\right]$$

Where $p=\frac{x-x_0}{h}$

In particular at $x=x_0$ (at the center of the data)

$$\left(\frac{dy}{dx}\right)_{x=x_0}=\frac{1}{h}\left[\frac{1}{2}\ \Delta y_0+\Delta y_{-1}\ -\frac{1}{12}\ \Delta^3 y_{-1}+\Delta^3 y_{-2}\ +\frac{1}{60}\ \Delta^5 y_{-2}+\Delta^5 y_{-3}\ +....\right]$$

$$\left(\frac{d^2y}{dx^2}\right)=\frac{1}{h^2}\left(\Delta^2 y_{-1}+\frac{1}{12}\Delta^4 y_{-2}+.....\right)$$

We can also use other central difference formula such as Bessel's Formula.

1. Find the first and second derivatives of the Function tabulated below at x=0.6

x	0.4	0.5	0.6	0.7	0.8
y	1.5836	1.7974	2.0442	2.3275	2.6511

Solution:

Since x = 0.6 is in the middle of the data, we will use Stirling's formula

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.4	1.5836				
		0.2138			
0.5	1.7974				
		0.2468	0.0330	0.0035	
0.6	2.0442				0.0003
		0.2833	0.0365	0.0038	
0.7	2.3275				
		0.3236	0.0403		
0.8	2.6511				

h=0.1; y_0=0.6

By stirlings Formula

$$\left(\frac{dy}{dx}\right)_{x=x_0}=\frac{1}{h}\left[\frac{1}{2}\left(\Delta y_0+\Delta y_{-1}-\frac{1}{12}\ \Delta^3 y_{-1}+\Delta^3 y_{-2}\ +...\right)\right]$$

$$=\frac{1}{0.1}\left[\frac{1}{2}\ 0.2833+0.2468\ -\frac{1}{12}\ 0.0038+0.0035\ \right]$$

$$=10\ 0.2651-0.0006\ =10(0.2645)$$

$$\left(\frac{dy}{dx}\right)_{x=0.6}=2.645$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0}=\frac{1}{h^2}\left(\Delta^2 y_{-1}-\frac{1}{12}\Delta^4 y_{-2}+.....\right)$$

$$=\frac{1}{0.01}\left[0.0365-\frac{1}{12}\ 0.0003\ \right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=0.6} = 3.6475$$

2. Given the following table of values of x and

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
y	1.0000	1.0247	1.0486	1.0723	1.0954	1.1180	1.1401

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at x=1.15.

Solution:

Since x=1.15 is in the middle of the table we will use stirlings formula

Difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.00	1.0000						
		0.0247					
1.05	1.0247		−0.0006				
		0.0241		0			
1.10	1.0488		−0.0006		0.0002		
		0.0235		0.0002		−0.0005	
1.15	1.0723		−0.0004		−0.0003		0.0009
		0.0231		−0.0001		0.0004	
1.20	1.0954		−0.0005		0.0001		
		0.0226		0			
1.25	1.1180		−0.0005				
		0.0221					
1.30	1.1401						

h=0.05; y_0=1.15

By stirlings Formula,

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{1}{2}\ \Delta y_0+\Delta y_{-1}\ -\frac{1}{12}\ \Delta^3 y_{-1}+\Delta^3 y_{-2}\ +\frac{1}{60}\ \Delta^5 y_{-2}+\Delta^5 y_{-3}\ +...\right]$$

$$=\frac{1}{0.05}\left[\frac{1}{2}\ 0.0231+0.0235\ -\frac{1}{12}\ -0.0001+0.0002\ +\frac{1}{60}\ 0.0004+0.0005\ \right]$$

=20(0.0233-0.00001-0)

=20(0.02329)

$$\left(\frac{dy}{dx}\right)_{x=1.15} = 0.46580$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left(\Delta^2 y_{-1} + \frac{1}{12}\Delta^4 y_{-2} + \dots\right)$$

$$= \frac{1}{0.0025}\left(-0.0004 + \frac{1}{12}\ -0.0003\right)$$

$$= \frac{1}{0.0025}\ -0.0004-0.00003 = \frac{1}{0.0025}\ -0.00043$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.15} = -\,0.1720$$

Bessel's formula:

$$y\ x_0 + ph\ = \frac{1}{2}\ y_0 + y_1\ + \left(p - \frac{1}{2}\right)\Delta y_0 + \frac{p\ \ p-1}{4}\ \Delta^2 y_{-1} + \Delta^2 y_0\ + \frac{p\left(p-\frac{1}{2}\right)\ p-1}{6}\Delta^3 y_{-1}$$

$$+ \frac{p+1\ \ p\ \ p-1\ \ \ p-2}{48}\ \Delta^4 y_{-2} + \Delta^4 y_{-1}\ + \dots$$

Then

$$y'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{4}\ \Delta^2 y_{-1} + \Delta^2 y_0\ + \frac{3p^2-3p+\frac{1}{2}}{6}\Delta^3 y_{-1} + \dots\right]$$

1. Obtain the value of f'(7.50) from the following table:

x	7.47	7.48	7.49	7.50	7.51	7.52	7.53
Y	0.193	0.195	0.198	0.201	0.203	0.206	0.208

Solution:

Since x=7.50 is in the middle of the table using Bessel's interpolation formula

Difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
7.47	0.193						
		0.002					
7.48	0.195		0.001	0.0010			
		0.003					
7.49	0.198		0	−0.001	0		
		0.003				0.003	
7.50	0.201		−0.001	0.001	0.003		-0.010
		0.002				−0.007	
7.51	0.203		0.001	−0.001	−0.004		
		0.003					
7.52	0.206		−0.001				
		0.002					
7.53	0.208						

Here h=0.01; y_0=7.50

$$y\left(x_0+ph\right)=\frac{1}{2}\left(y_0+y_1\right)+\left(p-\frac{1}{2}\right)\Delta y_0+\frac{p\left(p-1\right)}{4}\left(\Delta^2 y_{-1}+\Delta^2 y_0\right)+\frac{p\left(p-\frac{1}{2}\right)\left(p-1\right)}{6}\Delta^3 y_{-1}$$

$$+\frac{\left(p+1\right)p\left(p-1\right)\left(p-2\right)}{48}\left(\Delta^4 y_{-2}+\Delta^4 y_{-1}\right)+\ldots$$

Then $$y'(x)=\frac{1}{h}\left[\Delta y_0+\frac{2p-1}{4}\left(\Delta^2 y_{-1}+\Delta^2 y_0\right)+\frac{3p^2-3p+\frac{1}{2}}{6}\Delta^3 y_{-1}+\frac{1}{24}\left(\Delta^3 y_{-2}+\Delta^4 y_{-1}\right)+\ldots\right]$$

For P = 0

$$\left(\frac{dy}{dx}\right)_{x=7.50}=\frac{1}{0.01}\left(0.002-\frac{1}{4}\left(0.001-0.001\right)+\frac{1}{12}\left(0.002\right)+\frac{1}{24}\left(-0.001-0.004\right)\right)$$

$$\left(\frac{dy}{dx}\right)_{x=7.50}=0.21646$$

2. Find the Value of Sec 31° from the following data

θ (deg) :	31	32	33	34
$\tan\theta$:	0.6008	0.6249	0.6494	0.6745

Solution:

Let x = 0; y= $\tan\theta$

Here $x_0=31$

$$\frac{d}{d\theta}(tan\,\theta\,) = \sec^2\theta \Rightarrow \frac{dy}{dx} = \sec^2 x$$

$$\text{To find } \left(\frac{dy}{dx}\right)_{x\ =x_0} = \sec^2 31$$

Difference table:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
31	0.6008			
		0.0241		
32	0.6249		0.0004	
		0.0245		0.0002
33	0.6494		0.0006	
		0.0251		
34	0.6745			

Here h = 32-31=1

Newton's forward difference table is

$$f^{'}(x) = \left(\frac{dy}{dx}\right)_{x=x_0} = \sec^2\theta = \frac{1}{h}(\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 + ...)$$

$$\text{radians } i = \frac{\pi}{180}\text{radians} = 0.0$$

$$\text{f}^{'}(31) = \sec^2 31 = (0.0241 - \frac{1}{2}(0.004) + \frac{1}{3}(0.0002)).$$

$$sec^2 31 = 1.3695$$

$$\sec\ 31 = 1.1703$$

Numerical Integration by Trapezoidal Rule and Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rules **:**

Numerical Integration:

The process of computing the value of a definite Integral $\int_{x_0}^{x_n} \text{f(x)dx}$ from a set of values $\text{x}_{i,}\text{y}_i$, $\text{i} = 0,1,2.........,\text{n}$ then where $\text{x}_0 = \text{a},\ \text{x}_\text{n} = \text{b}$ of the function y = f(x) is called numerical integration.

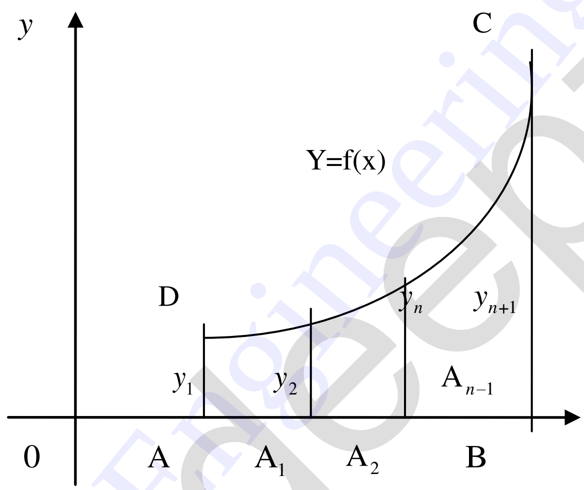
Trapezoidal Rule:

The trapezoidal Rule to evaluate $\int_a^b f\ \ x\ \ dx$ .Let DC be the curve y = f(x) and DA, CB be the terminal ordinates.

Let OA= a and OB = b then

AB = OB - OA = b-a

Divide AB into n equal parts $AA_{1,}AA_2......A_{n-1}$, B so that each part = $\frac{b-a}{n}$ =h



Draw the ordinates through $AA_{1,}AA_2......A_{n-1}$,B and let then be called $y_{1,}y_{2,....}y_{n,}y_{n+1}$ respectively then

$$\int_{x_0}^{x_n} f(x)dx = \int_a^b f(x)dx = \frac{h}{2}\ \ A+2B \qquad ----->1$$

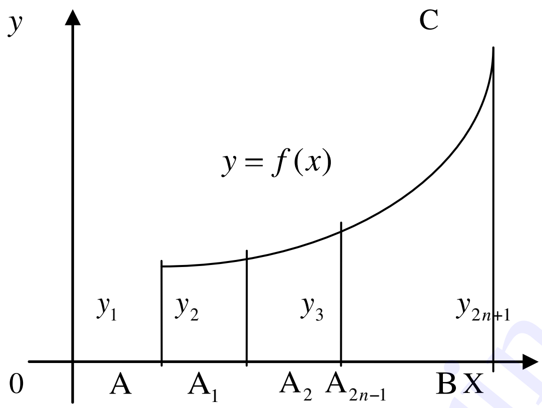
Where $A=y_1+y_{n+1}=$ sum of the first and last ordinates

$B=y_2+y_3+......+y_n=$ Sum of the remaining ordinates

Then equation (1) is called the Trapezoidal rule.

Simpson's rule or Simpson's one-third Rule:

Let DC be the curve y = f(x) and DC, CB be the terminal ordinates. Let OA = a and OB = b. Divide AB into even number of equal parts equal to h.



Let $x_1, x_2, \ldots, x_{2n+1}$ be the points $A, A_1, \ldots B$ and $y_1, y_2, \ldots, y_{2n+1}$ be the corresponding ordinates then

$$\int_{x_0}^{x_n} f(x)\,dx = \int_a^b f(x)\,dx = \frac{h}{3}(A+4B+2C) \quad \text{------> 1}$$

Where $A = y_1 + y_{2n+1} =$ Sum of the first and last ordinates

$B = y_2 + y_4 + \ldots.. + y_{2n}$ =Sum of the even ordinate

$c = y_3 + y_{5+\ldots..} + y_{2n-1}$ =Sum of the remaining ordinates.

Then equation (1) is known as Simpson's rule or Simpson's one-third Rule

Simpsons Three Eight Rule:

Let DC be the curve $y = f(x)$ and DA, CB be the terminal ordinates. Let OA = a and OB = b.

Divide AB into a number 'n' of equal parts of size h. Then $\frac{b-a}{n} = h$ .Let 'n' be a multiple of 3

Then $$\int_a^b y\,dx = \frac{3h}{8}\left[(y_1 + y_{n+1}) + 3(y_2 + y_5 + \ldots. y_n) + 2(y_4 + y_7 + \ldots.. y_{n-2})\right]$$

$$\int_a^b y\,dx = \frac{3h}{8}(A+3B+2c)$$

This is known as Simpson's Three Eight Rule

Truncation error in Trapezoidal Rule:

$$\text{Error} = |E| < \frac{nh^3}{12} M$$

Where M is the maximum value,

$$|E| < \frac{(b-a)\, h^2}{12} M, \text{ where } n = \frac{b-a}{h}$$

Error in the trapezoidal rule is of the order h^2 .

Truncation Error in Simpson's Formula:

$$\text{Error} = |E| < \frac{nh^5}{90} M$$

Where M is the maximum value

$$|E| < \frac{(b-a)\, h^4}{180} M \qquad \text{where} \quad n = \frac{b-a}{2h}$$

Error in the Simpson's formula is of the order h^4

Formula:

Rule	Degree of y(x)	No of intervals n	Error	Order
Trapezoidal rule	One	Any	$\|E\| < \frac{(b-a)h^2}{12} M$	h^2
Simpson's 1/3 Rule	Two	Even(or)Add	$\|E\| < \frac{(b-a)h^4}{180} M$	h^4
Simpson's 3/8 rule	Three	Multiple of 3	$\|E\| < \frac{3}{8} h^5$	h^5

Note:

1. The trapezoidal rule y(x) is a linear function of the rule is the simplest one but it is least accurate.
2. In Simpson's one- third Rule: y(x) is a polynomial of degree two. In this 'n' the number of interval must be even ie., the number of ordinates must be odd.
3. Simpson's one third rule approximates the area of two adjacent strips by the area under a quadratic parabola.
4. In Simpson's three eighth rule, y(x) is polynomial of degree 3.In this 'n' the number of intervals is a multiple of 3.

Problems:

1. Evaluate $\int_{-3}^{3} x^4 dx$ by using (i) Trapezoidal Rule (ii) Simpson's Rule. Verify your results by actual integration

Solution:

Given

$$\int_{-3}^{3} x^4 dx = \int_{a}^{b} y(x) dx$$

Here $y(x) = x^4$

Length of the interval $= b - a = 3 + 3 = 6$.

we divide '6' equal intervals $h = \frac{6}{6} = 1$

we form the below table:

x	-3	-2	-1	0	1	2	3
$y = f(x)$	81	16	1	0	1	16	81

(i) Trapezoidal Rule:

$$\int_{a}^{b} f(x)dx = \frac{h}{2}(A + 2B)$$

$$= \frac{h}{2}\begin{bmatrix} (\text{sum of first and last ordinates}) \\ +2(\text{sum of the remaining ordinates}) \end{bmatrix}$$

$$= \frac{1}{2}\ (81+81)+2(16+1+0+1+16)$$

$$\int_{-3}^{3} f(x)dx = 115$$

(ii)By simpson's one-third rule.

$$\int_{a}^{b} f(x)dx = \frac{h}{3}(A + 4B + 2C)$$

$$=\frac{1}{3}\left[\begin{pmatrix}\text{(sum of the first and last ordinates)}\\ +4\text{(sum of the even ordinates)}\\ +2\text{(sum of the remaining ordinates)}\end{pmatrix}\right]$$

$$=\frac{1}{3}\ (81+81)+4(16+0+16)+2(1+1)\ =\frac{1}{3}\ 162+128+2$$

$$\int_{-3}^{3} f(x)dx=98$$

(iii) by simpson's three-eighth rule:

since n=6 (multiple of three)

$$\int_{a}^{b} f(x)dx=\frac{3h}{8}\ (y_1+y_{n+1})+3(y_2+y_5+.......y_n)+2(y_4+y_7+.....+y_n)$$

$$=\frac{3}{8}\left[\ 81+81\ \ +3\ \ 16+1+1+16\ \ +2\ \ 0\ \right]$$

$$\int_{-3}^{3} f(x)dx=99$$

iv By actual integration :

$$\int_{-3}^{3} x^4 dx = 2\int_{0}^{3} x^4 dx = 2\left[\frac{x^5}{5}\right]_0^3 = \frac{2\times(3)^5}{5}$$

$$=\frac{2\times 243}{5}=97.2$$

$$\int_{-3}^{3} x^4 dx = 97.2$$

2. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule with h=0 Hence obtain an approximate value of π. . Can you use other formula in this case.

Solution:

$$\text{Given}: \int_0^1 \frac{dx}{1+x^2} = \int_a^b y(x)dx$$

$$\text{Let } y(x) = \frac{1}{1+x^2}$$

$$\text{length of the interval} = b-a = 1-0 = 1$$

$$\text{Taking h} = 0.2$$

x	0	0.2	0.4	0.6	0.8	1.0
$y = \dfrac{1}{1+x^2}$	1	0.96154	0.86207	0.73529	0.60976	0.50000

(i) By trapezoidal rule :

$$\int_a^b f(x)dx = \frac{h}{2}(\text{A}+2\text{B})$$

$$= \frac{0.2}{2}\left[\ 1+0.5\ \ +2\ \ 0.96154+0.86207+0.73529+0.60976\ \right]$$

$$= \ 0.1 \ \ 1.5+6.33732$$

$$\int_0^1 \frac{dx}{1+x^2} = 0.783732$$

(ii) By Actual integration:

$$\int_0^1 \frac{dx}{1+x^2} = \ \tan^{-1} x \ \Big|_0^1 = \pi/4 \qquad\qquad Here\ \pi = 3.1416$$

$$= 0.7854$$

3.Evalute the integral $\text{I} = \int_4^{5.2} \log_e x dx$ using trapezoidal, simpson's rules :

Solution:

given:

$$I=\int_{4}^{5.2}\log e^{x}dx=\int_{a}^{b}y(x)dx$$

$$Let\ y(x)=\log_e x$$

Here, $b-a=5.2-4=1.2$

We divide the interval into 6 equal parts

$$h=\frac{1.2}{6}=0.2$$

We form the table

x	4	4.2	4.4	4.6	4.8	5.0	5.2
$\log_e x$	1.3862944	1.4350845	1.48160451	1.5260563	1.5686159	1.6094379	1.6486586

(i) By Trapezoidal Rule :

$$\int_{a}^{b}f(x)dx=\frac{h}{2}\ \mathrm{A}+2\mathrm{B}$$

$$\int_{4}^{5.2}\log e^{x}dx=\frac{0.2}{2}\left[\ 1.3862944+1.6486586\ \ +2\begin{pmatrix}1.4350845+1.48160451+1.5260563\\ 1.5686159+1.6094379\end{pmatrix}\right]$$

$$=1.82765512$$

(ii) By simpson's rule:

$$\int_{a}^{b}f(x)dx=\frac{h}{3}(A+4B+2C)$$

$$\int_{4}^{5.2}\log x\,dx=\frac{0.2}{3}\begin{bmatrix}1.3862944+1.6486586\\ +4\ \ 1.4350845+1.5260563\\ +2\ \ 1.4816045+1.5686159\end{bmatrix}$$

$$=1.82784724$$

(iii) By simpson's three eighths rule:

$$\int_a^b f(x)dx = \frac{3h}{8}\left[A + 3B + 2C \right]$$

$$= \frac{3(0.2)}{8}\left[\begin{array}{l} (1.3862944 + 1.6486586) + 3(1.4350845 + 1.4816045 \\ + 1.5686159 + 1.6094379) + 2(1.5260563) \end{array} \right]$$

$$\int_4^{5.2} \log x \, \mathrm{dx} = 1.82784705$$

4.Evaluate $I = \int_0^6 \frac{1}{1+x}dx$ using (i) Trapezoidal rule (ii) Simpson's rule

Also find the acutal integration

Solution:

Given : $\int_0^6 \frac{1}{1+x}dx$

$$let\ y = \frac{1}{1+x}; \quad h = \frac{6-0}{6} = 1$$

x	0	1	2	3	4	5	6
$y = \frac{1}{1+x}$	1	1/2	1/3	1/4	1/5	1/6	1/7

(i) By Trapezoidal Rule:

$$\int_a^b f(x)dx = \frac{h}{2}(A + 2B)$$

$$\int_0^6 \frac{1}{1+x}dx = \frac{1}{2}\left[\left(1+\frac{1}{7}\right) + 2\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}\right)\right]$$

$$= 2.02142857$$

(ii) By simpson's one-third Rule:

$$\int_a^b f(x)dx = \frac{h}{3}(A + 4B + 2C)$$

$$\int_0^6 \frac{1}{1+x}dx = \frac{1}{3}\left[\left(1+\frac{1}{7}\right) + 4\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right) + 2\left(\frac{1}{3}+\frac{1}{5}\right)\right]$$

$$=\frac{1}{3}\left(1+\frac{1}{7}+\frac{16}{15}+\frac{22}{6}\right)$$
$$=1.95873016$$

(iii) By simpson's three-eighths Rule:

$$\int_a^b f(x)dx=\frac{3h}{8}\quad A+3B+2C$$

$$=\frac{3}{8}\left[\left(1+\frac{1}{7}\right)+3\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}\right)+2\left(\frac{1}{4}\right)\right]$$

$$\int_0^6\frac{1}{1+x}dx=1.96607143$$

(iv)By Actual integration :

$$\int_0^6\frac{1}{1+x}dx=\ \log(1+x)\ _0^6=\ \log(1+6)-\log(1+0)$$

$$=\log_e 7$$

$$=1.94591015$$

5.By dividing the range into ten equal parts, evaluate $\int_0^{\pi}\sin x\,dx$. Trapezoidal

and simpson's rule verify actual integration:

solution:

Given: $\int_0^{\pi}\sin x\,dx$

$$Range=b-a=\pi-0=\pi$$

$$Hence\,h=\frac{\pi}{10}$$

x	0	$\pi/0$	$2\pi/10$	$3\pi/10$	$4\pi/10$	$5\pi/10$	$6\pi/10$	$7\pi/10$	$8\pi/10$	$9\pi/10$	π
$y=\sin x$	0	0.3090	0.5878	0.8090	0.9511	1.0	0.9511	0.8090	0.5878	0.3090	0

(i) By Trapezoidal rule

$$\int_a^b f(x)dx = \frac{h}{2} \quad A+2B$$

$$= \frac{\pi}{2\times 10}\left[\quad 0+0 \quad +2\begin{pmatrix} 0.3090+0.5878+0.8090+0.9511+1.0 \\ +0.9511+0.8090+0.5878+0.3090 \end{pmatrix}\right]$$

$$\int_0^\pi \sin x\,dx = 1.9843$$

(ii) By simpson's rule :

$$\int_a^b f(x)dx = \frac{h}{3} \quad A+4B+2C$$

$$\int_0^\pi \sin x\,dx = \frac{1}{3} \quad \pi/10 \left[\begin{array}{l} 0+0 \quad +4 \quad 0.3090+0.8090+1+0.8090+0.3090 \\ +2 \quad 0.5878+0.9511+0.5878 \end{array}\right]$$

$$= 2.00091$$

$$= 2$$

(iii)By Actual integration :

$$\int_0^\pi \sin x\,dx = \quad -\cos x \;{}_0^\pi = - \quad \cos\pi - \cos\theta$$

$$= -(-2)$$

$$\int_0^\pi \sin x\,dx = 2$$

6.Evaluate $\int_0^5 \frac{dx}{4x+5}$ by simpson's one-third rule and hence find the value of $\log e^5 (n=10)$

Solution :

Given : $\int_0^5 \frac{dx}{4x+5}$

Here, $\quad y(x) = \frac{1}{4x+5}$

$$h = \frac{5-0}{10} = \frac{1}{2}$$

x	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
y	0.2	0.1429	0.1111	0.0909	0.0769	0.0667	0.0588	0.0526	0.0476	0.0434	0.04

By simpson's one-third rule :

$$\int_0^5 \frac{dx}{4x+5} = \frac{h}{3}\ (y_0 + y_n) + 2(y_2 + y_4 + y_6 + ...) + 4(y_1 + y_3 + y_5 + ...)$$

$$= \frac{1}{2\times 3}\begin{bmatrix}(0.2+0.04)+2(0.111+0.0769+0.0588+0.0476)\\ +4(0.1429+0.0909+0.0667+0.0526+0.0434)\end{bmatrix}$$

$$= \frac{1}{6}\ 10.24 + 2(0.2944) + 4(0.3964)$$

$$= \frac{1}{6}(2.4148)$$

$$\int_0^5 \frac{dx}{4x+5} = 0.4025 ---> 1$$

by actual integration:

$$\int_0^5 \frac{dx}{4x+5} = \left[\frac{\log(4x+5)}{4}\right]_0^5 = \frac{1}{4}\ \log 25 - \log 25$$

$$= \frac{1}{4}\log 5 ---> 2$$

From 1 & 2

$$\frac{1}{4}\log 5 = 0.4025$$

$$\log 5 = 1.61$$

Homework:

1.Evaluate $\int_0^6 \frac{dx}{1+x^2}\, by (i) T.R\, (ii)\, S.R$, also find the actual integration

$$T.R = 1.41079950,\ S.R = 1.36617433, S.R = 1.35708188, A.I = 1.40564765$$

2.Evaluate $\int_1^2 \frac{\sin x}{x}dx$ taking 6 intervals

3.Evaluate $\int_0^2 e^x dx$ taking 6 intervals

UNIT-V

Numerical Solution of Ordinary DE

<u>Taylor's Series</u>

Type :1

Solution of the first order ODE

Formula:

$\frac{dy}{dx} = y' = f(x, y)$ with initial condition $y(x_0) = y_0$

$$y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \dots\dots\dots$$

Problems:

1. Solve $y' = x + y, y(0) = 1$ by Taylor's Series method . find the value of y at $x = 0.1; x = 0.2$

Solution:

Given, $y' = x + y$

$y(0) = 1 \quad x_1 = 0.1 \,\&\, x_2 = 0.2$

$y(x_0) = y_0$

$x_0 = 0; y_0 = 1$

By Taylor's Series ,

$$y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h^3}{3!} y_n''' + \dots\dots$$

Put n=0,

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0'' + \dots\dots\dots\dots$$

$y' = x + y \qquad y_0' = y'_{x_0, y_0} = 1 \qquad y(x_0) = y_0, y(x_1) = y_1, y(x_2) = y_2$

$y'' = 1 + y' \qquad y_0'' = 2 \qquad x_1 = x_0 + h, \quad x_2 = x_1 + h$

$$y''' = y'' \qquad y_0''' = 2$$

$$y_1 = 1 + \frac{0.1}{1!}(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \ldots\ldots\ldots\ldots\ldots$$

$$y_1 = 1.1103$$

Now, $x_1 = 0.1,\ y_1 = 1.1103, h = 0.1$

$$\therefore y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \ldots\ldots\ldots\ldots$$

$$y_1' = (x + y)_{x_1, y_1} = 1.2103$$

$$y_1'' = (1 + y')_{x_1, y_1} = 2.2103$$

$$y_1''' = y'(x_1, y_1) = 2.2103$$

$$y_2 = 1.1103 + \frac{0.1}{1!}(1.2103) + \frac{(0.1)^2}{2!}(2.2103) + \frac{(0.1)^3}{3!}(2.2103)$$

$$y_2 = 1.243$$

Hence the solution is ,

$$y_1 = y(0.1) = 1.1103$$

$$y_2 = y(0.2) = 1.243$$

2. Using Taylor's series find y at $x=0.1$ given that $\frac{dy}{dx} = x^2 - y, y(0) = 1,$ correct to 4 decimal points using Taylor's series find y at $x=0.1, y(0) = 1$

Solution:

$$y' = x^2 y - 1$$

$$y(0) = 1; \quad x_1 = 0.1$$

$$y(x_0) = y_0$$

$\Rightarrow x_0 = 0; \quad y_0 = 1$

$$y_{n+1} = y_n + hy_n' + \frac{h^2}{2!} y_n'' + \ldots\ldots\ldots \quad \text{-------} \ (1)$$

Here, $h = 0.1$

and $y' = x^2 y - 1$ $\qquad y_0' = y'_{(x_0, y_0)} = -1$

$y'' = x^2 y' + 2xy$ $\qquad y_0'' = y''_{(x_0, y_0)} = 0$

$y''' = x^2 y'' + 2xy' + 2y + 2xy'$ $\qquad y_0''' = y'''_{(x_0, y_0)} = 2$

(1) becomes,

$y_1 = y\ 0.1$

$$= 1 + \ 0.1 \ \ -1 \ + \frac{0.1^{\ 2}}{2!} \ 0 \ + \frac{2 \ 0.1^{\ 3}}{3!}$$

$y_1 = 0.9003$

$$\frac{dy}{dx} = x^2 - y$$

$y' = x^2 - y$

$y\ 0\ = 1, \quad x_1 = 0.1$

$x_0 = 0; y_0 = 1$

By Taylor's series,

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \ldots..$$

Here $h = 0.1$

$y' = x^2 - y$ $\qquad y_0' = y'_{\ x_0, y_0} = -1$

$y'' = 2x - y'$ $\qquad y'' = -1$

$$y''' = 2 - y'' \qquad y''' = 1$$

(1) becomes

$$y_1 = y(0.1)$$

$$y_1 = 1 + (0.1)(-1) + \frac{(0.2)^2}{2}(-1) + \frac{(0.1)^3}{3!}(1)$$

$$y_1 = 1 - 0.1 + 0.005 + 0.00016$$

$$= -0.0948$$

3. Find the Taylor's series with three terms for the initial value problem $\frac{dy}{dx} = x^3 + y;\ y(1) = 1$

Solution:

Given $y' = x^3 + y$ *and* $x_0 = 1, y_0 = 1$

The Taylor's formula for y_1 is

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots\dots\dots (1)$$

$$y' = x^3 + y \qquad y_0' = 2$$

$$y'' = 3x^2 + y' \qquad y_0'' = 3 + 2 = 5$$

$$y''' = 6x + y'' \qquad y_0''' = 6 + 5 = 11$$

$$(1) \Rightarrow$$

$$y_1 = 1 + \frac{(x - x_0)}{1!}(2) + \frac{(x - x_0)^2}{2!}(5) + \frac{(x - x_0)}{3!}(11) + \dots\dots\dots\dots\dots\dots\dots$$

$$= 1 + \frac{(x - x_0)}{1}(2) + \frac{(x - x_0)^2}{2}(5) + \frac{(x - x_0)}{3}(11) + \dots\dots\dots\dots\dots\dots\dots$$

$y_1 = 1 + 2(x - x_0) + \frac{5}{2}(x - x_0)^2 + \frac{11}{6}(x - x_0)^3 + \dots\dots\dots\dots\dots$ is the required Taylor series.

Type-II

Simultaneous first order ODE

The simultaneous first order differential equations of the type

$\frac{dy}{dx} = f\ (x, y, z)\ ; \frac{dz}{dx} = g\ (x, y, z)$ with initial conditions $y\ (x_0) = y_0 and\ z\ (x_0) = z_0$ and can be solved by using taylor's series method.

1. Solve $\frac{dx}{dt} = yt + 1, \frac{dy}{dt} = -xt; x = 0, y = 1\ at\ t = 0$
   $find\ (x, y)\ at\ t = 0.2$

Solution:

We use 2 Taylor's series formula,

$$x_1 = x_0 + hx_0' + \frac{h^2}{2!} x_0'' + \ldots\ldots\ldots\ldots\ldots\ (1)$$

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \ldots\ldots\ldots\ldots\ldots\ (2)$$

Given,

$x' = yt + 1 \qquad y' = -xt$

$x(0) = 0 \qquad y(0) = 1$

$t_0 = 0;\ x_0 = 0 \qquad t_0 = 0;\ x_0 = 1$

$x' = yt + 1$	$y' = -xt$
$x'' = y't + y$	$y'' = -x't - x$
$x''' = y''t + y' + y'$	$y''' = -x''t - x' - x'$
$= y''t + 2y'$	$y''' = -x''t - 2x'$
$x_0' = (x')_{t_0, x_0, y_0} = 1$	$y_0' = (y')_{t_0, x_0, y_0} = 0$
$x_0'' = (x'')_{t_0, x_0, y_0} = 1$	$y_0'' = 0$

$x_0''' = 0$ $\qquad\qquad$ $y_0''' = -2$

$\therefore \quad 1 \ \& \ 2 \ \ Becomes$

$$x_1 = (0.2)(1) + \frac{(0.2)^2}{2!}(1) = 0.22$$

$$y_1 = 1 + (0.2)(0) + \frac{(0.2)^3}{3!}(-2) = 0.9973.$$

2. Solve the following simultaneous differential equations using taylor's series method of the fourth order for x=0.1 and 0.2

$$\frac{dy}{dx} = xz + 1; \frac{dz}{dx} = -xy \ , y(0) = 0 \, and \ z(0) = 1$$

Solution:

Given $\quad y' = xz+1$; $\qquad\qquad$ $z' = -xy$

$y(0) = 0 \qquad\qquad z(0) = 1$

$x_0 = 0; y_0 = 0 \qquad\qquad y_0 = 0; z_0 = 1; h = 0.1$

$y' = xz + 1 \qquad\qquad |z' = -xy$

$y'' = xz' + z \qquad\qquad z'' = -xy' - y$

$y''' = xz'' + z' + z' \qquad\qquad |z''' = -xy'' - y' - y'$

$y''' = xz'' + 2z' \qquad\qquad z''' = -xy'' - 2y'$

$y^{iv} = xz''' + z'' + 2z'' \qquad\qquad |z^{iv} = -xy''' - y'' - 2y''$

$= xz''' + 3z'' \qquad\qquad |= -xy''' - 3y''$

$y_0' = 1 \qquad\qquad z_0' = 0$

$y_0'' = 1 \qquad\qquad z_0'' = 0$

$y_0''' = 0 \qquad\qquad z_0''' = -2$

$y_0^{iv} = 0 \qquad\qquad z_0^{iv} = -3$

To find $y(0.1)$ and $z(0.1)$

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \ldots\ldots\ldots\ldots.$$

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \frac{h^4}{4!}z_0^{iv} + \ldots\ldots\ldots\ldots\ldots\ldots$$

$$y_1 = 0.1 + \frac{(0.1)^2}{2!}(1) = 0.105$$

$$z_1 = 1 - \frac{2(0.1)^3}{6} - \frac{3(0.1)^4}{24} \Rightarrow z_1 = 0.99966$$

$$y(0.2) = 0.2 + \frac{(0.2)^2}{2}(1)$$

$$y(0.2) = 0.22$$

$$z(0.2) = 1 - \frac{2(0.2)^3}{6} - \frac{3(0.2)^4}{4!}$$

$$z(0.2) = 0.99714$$

Type-III

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = f(x) \to (1)$$

$$Put \quad y' = z \to (2)$$

$$\frac{dy}{dx} = z$$

$$\therefore (1) \; becomes$$

$$\frac{dz}{dx} + pz + qy = f(x)$$

$$\Rightarrow z' = f(x) - pz - qy$$

$$z' = f(x, y, z) \to (3)$$

Thus we get 2 first order equation given by 2 & 3

$y\ x_0 = y_0; y'\ x_0 = y_0'; z\ x_0 = z_{0.}$

1.Find the value of y 1.1 and y 1.2 from $\frac{d^2y}{dx^2} + y^2\frac{dy}{dx} = x^3$

$y(1) = 1\ y'(1) = 1$ *by using taylor's series method*

Solution:

$$\frac{d^2y}{dx^2} + y^2\frac{dy}{dx} = x^3 \text{ ----- (1)} \qquad y(1) = 1; y'(1) = 1$$

Put $y' = z$ ------ (2)

$y'' = z'$

$\therefore$ (1) becomes

$z' + y^2 z = x^3$

$z' = x^3 - y^2 z$ -------- (3)

Given initial conditions are,

$y\ 1 = 1;\ y'(1) = 1 \Rightarrow z(1) = 1$

$x_0 = 1; y_0 = 1; z_0 = 1; h_0 = 0.1$

By Taylor's series

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \text{..........(4)}$$

$$z_1 = z_0 + \frac{h}{1!}z_0' + \frac{h^2}{2!}z_0'' + \text{..........(5)}$$

Now , $y' = z \qquad \Rightarrow y_0' = z_0 = 1$

$y'' = z' \qquad y_0'' = 1 - 1\ \ 1 = 0$

$y'' = x^3 - y^2 z$

$$y''' = z'' = 3x^2 - y^2 z' - 2zyy' \qquad y_0''' = 3 - 2 = 1$$

$$y^{iv} = z''' = 6x - y^2 z'' - 2z'yy' - 2zy'^2 - 2z'yy' - 2zy$$

$$z''' = y_0^{iv} = 6 - 1 - 2 = 3$$

$$Now, \ z_0' = 0$$

$$z_0'' = 1$$

$$z_0''' = 3$$

$$\therefore \ 4 \ \ and \ \ 5 \ \ becomes$$

$$y\ 1.1 = y_1 = 1 + \frac{0.1}{1!}\ 1 + \frac{0.1}{3!}^3\ 1 = 1.1002$$

$$z(1.1) = z_1 = 1 + \frac{0.1}{1!}(0) + \frac{(0.1)^2}{2!}(1) + \frac{(0.1)^3}{3!}(3)$$

$$= 1.005$$

To find y 1.2

$$y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + ...(6)$$

Here $h = 0.1; \quad y_1 = 1.002; \quad z_1 = 1.005$

Here $z_1' = (z')_1$

$$= (x^3 - y^2 z)_1 = x_1^3 - y_1^2 z_1$$

$$x_1 = 1.1; \qquad y_1 = 1.1002; \qquad z_1 = 1.005$$

$$z_1' = (1.1)^3 - (1.1002)^2 (1.005)$$

$$= 3x_1^2 - y_1^2 z_1' - 2z_1 y_1 y_1'$$

$$= 3(1.1)^2 - (1.1002)^2 (0.1145) - 2(1.005)(1.1002)(1.005)$$

$$z_1'' = 1.2676$$

$$\therefore\ y(1.2) = y_2 = 1.1002 + \frac{0.1}{1!}(1.005) + \frac{(0.1)^2}{2!}(0.1145) + \frac{(0.1)^3}{3!}(1.2676)$$

$$y(1.2) = 1.2015$$

Euler Method

$$y_{n+1} = y_n + hf(x_n, y_n);\ \ n = 0,1,.........$$

Where $x_i = x_0 + ih; \quad i = 0,1,2.............$

and h is a step size

Modified Euler Method

$$y_{n+1} = y_n + h\left[f\left(x_n + \frac{h}{2},\ y_n + \frac{h}{2}(f(x_n, y_n)\right)\right]$$

Where $y_{n+1} = y(x_n + h)$ and h is a step size.

Problems on Euler Method

1. Using Euler method compute 'y' in the range $0 \le x \le 0.5$ if y satisfies $\frac{dy}{dx} = 3x + y^2,\ y(0) = 1$

Solution:

Given, $\frac{dy}{dx} = 3x + y^2$

$f(x, y) = 3x + y^2$

$x_0 = 0;\ y_0 = 1$

By Euler's method

$y_{n+1} = y_n + h\, f(x_n, y_n), \qquad n = 0,1,2..........$

Let $h = 0.1$

$$\begin{aligned} y(0.1) = y_1 &= y_0 + (0.1) f(x_0, y_0) \\ &= 1 + (0.1)\ f(0,1) \\ &= 1 + (0.1)(1) \\ y_1 &= 1.1 \end{aligned}$$

$$\begin{aligned} y(0.2) = y_2 &= y_1 + (0.1) f(x_1, y_1) \\ &= 1.1 + (0.1)(0.3 + 1.21) \end{aligned}$$

$y(0.2) = 1.251$

$y(0.3) = y_3 = y_2 + (0.1) f(x_2, y_2)$

$$=1.251+(0.5)f(0.2,1.251)$$

$$y_3=1.4675$$

$$y(0.4)=y_4=y_3+(0.1)f(x_3,y_3)$$

$$=1.4675+0.1\,f(0.3,1.4675)$$

$$y_4=1.7728$$

$$y(0.5)=y_5=y_4+(0.1)f(x_4,y_4)$$

$$=1.7728+(0.1)\;f(0.4,1.7728)$$

$$y_5=2.2071$$

2.Using Euler method solve $y^{'}=x+y+xy,\; y(0)=1$ compute y(1.0) with h=0.2

Solution:

Given, $y'=x+y+xy$

$$f(x,y)=x+y+xy$$

$$h=0.2$$

$$y(0)=1 \;=> x_0=0;\; y_0=1$$

Let $\mathrm{x}_i=x_0+ih$

$$i=1,2,3,4,5$$

$$x_1=0+0.2(1)$$

$$=0.2$$

$$x_2=0.4;\quad x_3=0.6;\quad x_4=0.8;\quad x_5=1$$

$$y_{n+1}=y_n+h\mathrm{f}(x_n,y_n)$$

$$y(0.2)=y_1=y_0+h\mathrm{f}(x_0,y_0)$$

$$y_1=1.2$$

$$y(0.4)=y_2=y_1+h\;f(x_1,y_1)$$

$$=1.2+(0.2)\;f(0.2,1.2)$$

$$y_2=1.528$$

$$y(0.6) = y_3 = y_2 + h\, f(x_2, y_2)$$

$$= 1.528 + (0.2)\, f(0.4, 1.528)$$

$$y_3 = 2.0358$$

$$y(0.8) = y_3 + h\, f(x_3, y_3)$$

$$= 2.0358 + (0.2)\, f(0.6, 2.0358)$$

$$y_4 = 2.8072$$

$$y(1.0) = y_5 = y_4 + h\, f(x_4, y_4)$$

$$y_5 = 3.9778$$

Home work:

1. Using Euler's method find y(0.2)&y(0.4) from $\frac{dy}{dx} = x + y,\ y(0) = 1$ with h=0.2

2. Solve $y' = y - \frac{2x}{y},\ y(0) = 1$, find y(0.1)&y(0.2) by Euler's method

3. Solve $y' = 1 + xy,\ y(2) = 0$, find y(2.2) with stepwise 0.1

4. Solve $y' = y \sin x + \cos x,\ y(0) = 0$ find y(0.2)

1. Compute y at x = 0.25 by modified Euler method $y' = 2xy \quad y(0) = 1$

   Solution:

$$\text{Given, } y' = 2xy$$

$$y(0) = 1; \quad h = 0.25$$

$$x_0 = 0; \ y_0 = 1$$

$$y_{n+1} = y_n + h\, f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} y_n' \right)$$

Put n = 0

$$y_1 = y_0 + h\, f\left(0 + \frac{0.25}{2}, 1 + \frac{0.25}{2} 2x_0 y_0 \right)$$

$$= 1 + 0.25 f(0.125, 1)$$

$$= 1 + 0.25(2(0.125)(1))$$

$$y(0.25) = 1.0625$$

2. Solve the equation $\frac{dy}{dx} = 1 - y$ given y(0)=0 using modified Euler method find the values x=0.1,0.2,0.3

Solution:

$$y' = 1 - y$$

$$x_0 = 0; \quad y_0 = 0; \quad h = 0.1$$

Modified Euler method

$$y_{n+1} = y_n + h\, f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} y_n'\right)$$

$$y_1 = y_0 + hf\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} y_0'\right)$$

$$= 0 + 0.1f\left(0 + \frac{0.1}{2}, 0 + \frac{0.1}{2}(1 - y_0)\right)$$

$$= 0.1\, f\,(0.05, 0.05(1))$$

$$= 0.1\, f\,(0.05, 0.05)$$

$$= 0.1(1 - 0.05)$$

$$y(0.1) = 0.095$$

Put n= 1,

$$y_2 = y_1 + hf\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} y_1'\right)$$

$$= 0.095 + 0.1f\left(0.1 + \frac{0.1}{2}, 0.095 + \frac{0.1}{2}(1 - 0.095)\right)$$

$$= 0.095 + 0.1\, f(0.150, 0.140)$$

$$= 0.095 + 0.1(1 - 0.1403)$$

$$y_2 = y(0.2) = 0.1810$$

Put n=2

$$y_3 = y_2 + hf\left(x_2 + \frac{h}{2}, y_2 + \frac{h}{2} y_3^{'}\right)$$

$$= 0.1810 + (0.1)\ f\left(0.2 + \frac{0.1}{2}, 0.1810 + \frac{0.1}{2}(1 - 0.1810)\right)$$

$$= 0.1810 + (0.1)\ f(0.250, 0.222)$$

$$= 0.1810 + (0.1)(1 - 0.222)$$

$$y_3 = 0.259$$

1.Using MEM find y 0.2 , y 0.1 given $\frac{dy}{dx} = x^2 + y^2, y(0) = 1$

2.Given $\frac{dy}{dx} = x + y^2, y(0) = 1$ find an approximate value of y at x=0.5 by EME, h=0.1

Am y(0.5)=2.2070

1.Given that $\frac{dy}{dx} = \log(x + y)$ with y = 1 when x = 1 using MEM, find y for x = 0.2 and x = 0.5 correct to four decimal places.

Solution:

Given f(x,y) = log(x+y), $x_0 = 0$, $y_0 = 1$ to find y(0.2)

Take h = 0.2 $x_1 = x_0 + h = 0 + 0.2 = 0.2$

By modified Euler method

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} y_n^{'}\right)$$

$$y_1 = y_0 + hf\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} y_0^{'}\right)$$

$$= 1 + (0.2) f\left(0 + \frac{0.2}{2}, 1 + \frac{0.1}{2}\log(x_0 + y_0)\right)$$

$$=1+(0.2)f(0.1,0)$$

$$=1+(0.2)(-1.000)$$

$$\boxed{y_1 = 0.8000}$$

To find y 0.5

$$x_2 = x_1 + h = 0.2 + h \;=>\; 0.5 - 0.2 = 0.3$$

$$x_2 = 0.5$$

$$y_2 = y_1 + hf\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} y_1^{'}\right)$$

$$= 0.800 + (0.3) f\left(0.2 + \frac{0.3}{2}, 0.8 + \frac{0.3}{2} \log(x_1 + y_1)\right)$$

$$= 0.8 + 0.3 f(0.35, 0.8)$$

$$= 0.8 + 0.3(0.0607)$$

$$= 0.8 + 0.01821$$

$$\boxed{y_2 = 0.81821}$$

2.Given $\frac{dy}{dx} = -xy^2$, $y(0) = 2$. using Euler's modified method, find y(0.2) in two steps of 0.1 each

Solution:

Let $x_1 = x_0 + h$ Here $x_0 = 0$; $y_0 = 1$

$$h = 0.1$$

$$\frac{dy}{dx} = y^{'} = -xy^2$$

By modified Euler method

$$y_{n+1} = y_n + hf\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} y_n^{'}\right)$$

$$y_1 = y_0 + hf\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}y_0^{'}\right)$$

$$= 2 + (0.1)f\left(0 + \frac{0.1}{2}, 2 + \frac{0.1}{2}(0)\right)$$

$$= 2 + (0.1)f(0.05, 2)$$

$$= 2 - 0.02$$

$$\boxed{y_1 = 1.98}$$

To find y(0.2)

$$y_2 = y_1 + hf\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2}y_1^{'}\right)$$

$$= 1.98 + (0.1)f(0.1 + 0.05, 1.98 + 0.05(-0.392))$$

$$= 1.98 + (0.1)f(0.15, 1.9604)$$

$$= 1.98$$

$$\boxed{y_2 = 1.9224}$$

Fourth-order Runge - kutta method (RK method)

$$k_1 = hf(x_o, y_0)$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$k_4 = hf(x_0 + h, y_0 + k_3)$$
$$\Delta y = \frac{1}{6}\ k_1 + 2k_2 + 2k_3 + k_4$$
$$y_1 = y_0 + \Delta y$$

The second iteration we replace (x_0, y_0) to (x_1, y_1) and we proceed the second iteration.

Problems:

1.Apply the 4^{th} order RKmethod find y(0.2) given that $y^{'} = x + y,\ y(0) = 1.$

Solution:

Given $y^{'} = x + y$

$y(0) = 1, \ x_0 = 0, \quad y_0 = 1$

$h = 0.2$

$k_1 = hf(x_0, y_0)$

$= 0.2 f(0.1)$

$= 0.2(1+0) = 0.2$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.2 f\left(0 + \frac{0.2}{2}, 1 + \frac{0.24}{2}\right)$$

$= 0.2(0.1, 1.1)$

$k_2 = 0.24$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= (0.2) + \left(0 + \frac{0.2}{2}, 1 + \frac{0.24}{2}\right)$$

$= 0.2 f(0.1, 1.12)$

$k_3 = 0.244$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.2(0 + 0.2, 1 + 0.244)$

$k_4 = 0.2888$

$$\Delta y = \frac{1}{6}(0.2 + 2(0.24) + 2(0.244) + 0.2888)$$

$\Delta y = 0.2428$

$$y_1 = y_0 + \Delta y = 1 + 0.2428$$

$$y_1 = 1.2428$$

2. Compute $y(0.2)$ Given that $\frac{dy}{dx} + y + xy^2 = 0,\ y(0) = 1$ by taking h=0.1 using RK method of 4^{th} order.

Solution:

$$\frac{dy}{dx} = -y - xy^2 = -(y + xy^2)$$

$$y(0) = 1 \quad x_0 = 0 \quad y_0 = 1$$

$$h = 0.1$$

$$k_1 = hf(x_0, y_0)$$

$$= 0.1 f(0,1)$$

$$= 0.1(-1(1+0))$$

$$= 0.1(-1) = -0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1 f\left(0 + \frac{0.1}{2}, 1 + \frac{(-0.1)}{2}\right)$$

$$= 0.1 f(0.05, 0.95)$$

$$= 0.1(-(0.95 + (0.5)(0.95)^2)$$

$$= 0.1(-0.9951)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1 f\left(0 + \frac{0.1}{2}, 1 + \frac{(-0.0995)}{2}\right)$$

$$= -0.0995$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.1f(0 + 0.1, 1 - 0.0995)$$

$$= 0.1f(0.1, 0.9005)$$

$$k_4 = -0.0982$$

$$\Delta y = \frac{1}{6}(-0.1 - 2(0.0995) - 2(0.0995) - 0.0982$$

$$= \frac{-0.5962}{6}$$

$$\Delta y = -0.0994$$

$$y_1 = y_n + \Delta_y = 1 - 0.0994 = 0.9006$$

Second iteration

$$\mathrm{k}_1 = hf(x_1, y_1)$$

$$\frac{dy}{dx} = -y - yx^2, \ = 0.1\mathrm{f}(0.1,\ 0.9006)$$

$$= 0.1(-0.9006 - (0.9006)^2(0.1))$$

$$k_1 = -0.0982$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= 0.1f\left(0.1 + \frac{0.1}{2}, 0.9006 + \frac{(-0.0982)}{2}\right)$$

$$= 0.1f(0.15, 0.8515)$$

$$= 0.1(-0.8515 - (0.8515)^2(0.15))$$

$$= 0.1(-0.9603)$$

$$= -0.0960$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= 0.1f\left(0.1 + \frac{0.1}{2}, 0.9006 - \frac{0.0960}{2}\right)$$

$$= 0.1f(0.15, 0.8526)$$

$$= 0.1(-0.8526 - (0.8526)^2(0.15))$$

$$= 0.1(-0.9616)$$

$$= -0.0962$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= 0.1f(0.1 + 0.1, 0.9006 - 0.0962)$$

$$= 0.1f(0.2, 0.8044)$$

$$= 0.1(-0.8044 - (0.2)(0.8044)^2)$$

$$k_4 = -0.0934$$

$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(-0.0982 + 2(-0.0960) + 2(-0.0962) - 0.0934)$$

$$\Delta y = -0.0960$$

$$y_2 = y_1 + \Delta y$$

$$= 0.9006 - 0.0960$$

$$y_2 = 0.8046$$

Home work:

1.Using RK method of 4^{th} order $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ Given y(0) = 1,x = 0.2 & x = 0.4

Ans:$y_1 = 1.1960$;

$y_2 = 1.3753$

2.Using RKmethod of 4^{th} order determine correct to 3 decimal place the value

of y at x=0.1,0.2 of y satisfies the equation $\frac{dy}{dx} = x^2 y + x, y(0) = 1$

Ans:y(0.1)=1.003

y(0.2)=1.02

RK method for simultaneous first order differential equation

To solve the simultaneous equations $\frac{dy}{dx} = f_1(x, y, z)$ and $\frac{dz}{dx} = f_2(x, y, z)$ with the intial conditions $\mathrm{y(x_0)} = y_0, z(x_0) = z_0$

Then the first order RK method is

$$\mathrm{k}_1 = hf_1(x_0, y_0, z_0)$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$$

and $\Delta \mathrm{y} = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$l_1 = hf_2(x_0, y_0, z_0)$$

$$l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$l_4 = hf_2 \quad x_0 + h, y_0 + k_3, z_0 + l_3$$

and $\Delta z = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

Hence $y_1 = y_0 + \Delta y$ and $z_1 = z_0 + \Delta z$

Where $h = x_1 - x_0 \Rightarrow x_1 = x_0 + h$

Problems based on R-K method for simultaneous first ODE.

1.Find y(0.1),z(0.1) from the system of equation $\frac{dy}{dx} = x + z, \frac{dz}{dx} = x - y^2$ given y(0) = 2, z(0) = 1

using R-K Method of fourth order

Solution:

Given $\frac{dy}{dx} = x + z$ $\qquad$ $\frac{dz}{dx} = x - y^2$

$\Rightarrow y' = x + z$ $\qquad$ $\Rightarrow z' = x - y^2$

$\Rightarrow f_1(x, y, z) = x + z$ $\qquad$ $\Rightarrow f_2(x, y, z) = x - y^2$

and y(0)=2 $\qquad$ z(0)=1

Here, $x_0 = 0$; $y_0 = 2$ $\qquad$ Here, $x_0 = 0, z_0 = 1$

where $\quad h = 0.1$

First iteration:

Let $x_0 = 0$, $y_0 = 2$, $z_0 = 1$ and h=0.1

By R-K Method.

$k_1 = hf_1(x_0, y_0, z_0)$

$= hf_1(0, 2, 1)$

$= (0.1)(x_0 + z_0)$

$= (0.1)(0 + 1)$

$k_1 = 0.1$

$l_1 = hf_2(x_0, y_0, z_0)$

$= h(x_0 - y_0{}^2)$

$= (0.1)(0 - 4)$

$= (0.1)(-4)$

$l_2 = -0.4$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= hf_1\left(0 + \frac{0.1}{2}, 2 + \frac{0.1}{2}, 1 - \frac{0.4}{2}\right)$$

$$= hf_1(0.05, 2.05, 0.8)$$

$$= (0.1)(0.05 + 0.8)$$

$$k_2 = 0.085$$

$$l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$= hf_2\left(\pi 0 + \frac{0.1}{2}, 2 + \frac{0.1}{2}, 1 - \frac{0.8}{2}\right)$$

$$= hf_2(0.05, 2.05, 0.8)$$

$$= (0.1)(0.05 - (2.05)^2)$$

$$l2 = -0.4153$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= (0.1)\, f_1\left(0 + \frac{0.1}{2}, 2 + \frac{0.085}{2}, 1 - \frac{0.4153}{2}\right)$$

$$= (0.1)\, f_1(0.05, 2.0425, 0.7924)$$

$$k_3 = (0.1)(0.05 + 0.7924)$$

$$= (0.1)(0.05 + 0.7924)$$

$$k_3 = 0.0842$$

$$l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$= (0.1)\, f_2(0.05, 2.0425, 0.7924)$$

$$l_3 = (0.1)(0.05 - (2.0425)^2)$$

$$l_3 = -0.14122$$

$$k_4 = hf_1(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= (0.1)\, f_1(0 + 0.1, 2 + 0.0842, 1 - 0.4122)$$

$$= (0.1)\, f(0.1, 2.0842, 0.5878)$$

$$k_4 = 0.0688$$

$$l_4 = hf_2(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$= (0.1)\, f(0.1, 2.0842, 0.5878)$$

$$= (0.1)(0.1 - (0.5878)^2)$$

$$l_4 = -0.4244$$

$$\Delta y = \frac{1}{6}\;\; k_1 + 2k_2 + 2k_3 + k_4$$

$$= \frac{1}{6}(0.1 + 2(0.085) + 2(0.0842) + 0.0688)$$

$$\Delta z = \frac{1}{6}\;\; l_1 + 2l_2 + 2l_3 + l_4$$

$$= \frac{1}{6}(-0.4 - 2(0.4153) - 2(0.4125) - 0.4244)$$

$\Delta y = 0.0845$ $\Delta z = -0.4132$

Hence, $y_1 = y_0 + \Delta y$ $z_1 = z_0 + \Delta z$

$y(x_1) = 2 + 0.0845$ $z(x_1) = 1 - 0.4132$

$y(0.1) = 2.0845$ $z(0.1) = 0.5868$

Home work:

1.Using RK method tabulate the solution of the system $\frac{dy}{dx} = x + z, \frac{dz}{dx} = x - y, y = 0, z = 1$ when x=0 at internal of h=0.1 from x=0.0 to x=0.2

Runge-Kutta method for second order differential equations

To solve the second order differential equation $y'' = f(x, y, y')$ with the initial condition $y(x_0) = y_0, y'(x_0) = y_0'$

Let $y' = z$ *and* $y'' = z'$

Hence the differential equation reduces to

$$\frac{dy}{dx} = y' = z = f_1(x, y, z)$$
$$\frac{dz}{dx} = z' = y'' = f_2(x, y, z)$$

With the initial condition $y(x_0) = y_0, z(x_0) = z_0$

Problems

i. Given $y'' + xy' + y = 0, y(0) = 1, y'(0) = 0$ find the value of $y(0.1)$ *and* $y'(0.1)$ by using R-K method of 4th order.

Solution:

Given,

$$y'' + xy' + y = 0$$

let $y' = z$ and $y'' = z'$

The given equation can be written as,

$$z' + xz + y = 0$$

$$y' = z \qquad \text{and} \qquad z' = -\left(xz + y\right)$$

$$f_1\left(x, y, z\right) = z \qquad f_2\left(x, y, z\right) = -\left(xz + y\right)$$

The initial condition,

$$y\left(0\right) = 1, \qquad y'\left(0\right) = 0 \Rightarrow z\left(0\right) = 0$$

$$x_0 = 0,\ y_0 = 1 \qquad x_0 = 0,\ z_0 = 0$$

By Runge – Kutta method.

$k_1 = hf_1\left(x_0, y_0, z_0\right)$	$l_1 = hf_2\left(x_0, y_0, z_0\right)$
$= hf_1\left(0,1,0\right)$	$= hf_2\left(0,1,0\right)$
$= \left(0.1\right)\left(0\right)$	$= \left(0.1\right)\left(-0-1\right)$
$k_1 = 0$	$l_1 = -0.1$
$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$	$l_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$
$= hf_1\left(0 + \frac{0.1}{2}, 1 + \frac{0}{2}, 0 - \frac{0.1}{2}\right)$	$= \left(0.1\right) f_2\left(0.05, 1, -0.05\right)$
$= \left(0.1\right) f_1\left(0.05, 1, -0.05\right)$	$= \left(0.1\right)\left(\left(0.05\right)\left(-0.05\right) - 1\right)$
$k_2 = -0.0050$;	$l_2 = -0.0998$
$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_1}{2}\right)$	$l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$
$= hf_1\left(0 + \frac{0.1}{2}, 1 - \frac{0.0050}{2}, 0 - \frac{0.0998}{2}\right)$	$= hf_2\left(0.05, 0.9975, -0.499\right)$
$= \left(0.1\right) f_2\left(0.05, 0.9975, -0.0499\right)$	$= \left(0.1\right)\left(-\left(0.05\right)\left(-0.0499\right) - 0.9975\right)$
$k_3 = -0.0050$	$l_3 = -0.0995$

$k_4 = hf_1\ \ x_0+h, y_0+k_3, z_0+l_3$

$=hf_1\ \ 0.051, 1-0.0050, 0-0.0995$

$=hf_1\ \ 0.05, 0.9950, -0.0995$

$=k_4=-0.01$

$$\Delta y = \frac{1}{6}\ \ k_1+2k_2+2k_3+k_4$$

$$=\ 0-\ \ 2\times 0.0050\ \ -2\ \ 0.0050\ \ -0.01$$

$\Delta y=-0.005$

Hence $y_1=y_0+\Delta y$

$y\ \ x_1\ \ =1-0.005$

$y\ \ 0.1\ \ =0.995$

$l_4=hf_2\ \ x_0+h, y_0+k_3, z_0+l_3$

$=hf_2\ \ 0.05, 0.9950, 0.0995$

$=\ 0.1\ \ -\ 0.05\ \ -0.0995\ \ -0.9950$

$l_4=-0.099$

$$\Delta z = \frac{1}{6}\ \ l_1+2l_2+2l_3+l_4$$

$$=\frac{1}{6}\ \ -0.1-2\ \ 0.0998\ \ -2\ \ 0.099\ \ -0.09920$$

$\Delta z=-0.099$

$z_1=z_0+\Delta z$

$z\ \ x_1\ \ =0-0.099$

$z\ \ 0.1\ \ =-0.099$

$\Rightarrow y'\ \ 0.1\ \ =-0.099$

$$\therefore y\ \ 0.1\ \ =0.995, y'\ \ 0.1\ \ =-0.099$$

2.Find $y\ \ 0.1\ \ , y(0.2)$ *given* $y''-x^2y'-2xy=1$ with ic $y\ \ 0\ \ =1, y'\ \ 0\ \ =0.$

Solution:

$$y''-x^2y'-2xy=1 \rightarrow\ 1$$

$Let\ y' = z = f_1\ \ x, y, z$

$1\ \Rightarrow z' - x^2 z - 2xy = 1$

$z' = 1 + x^2 z + 2xy$

$z' = f_2\ \ x, y, z\ \ = 1 + x^2 z + 2xy$

$y\ \ 0\ \ = 1, y'\ \ 0\ \ = 0; z\ \ 0\ \ = 0$

$y_0 = 1; x_0 = 0; z_0 = 0$

$k_1 = hf_1\ \ x_0, y_0, z_0$	$l_1 = hf_2\ \ x_0, y_0, z_0$
$= 0.1\ \ f_1\ \ 0,1,0$	$= 0.1\ \ f_2\ \ 0,1,0$
$k_1 = 0$	$l_1 = 0$
$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$	$l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$
$= 0.1\ \ f_1\left(0 + \frac{0.1}{2}, 1 + \frac{0}{2}, 0 + \frac{0.1}{2}\right)$	$= 0.1\ \ f_2\left(0 + \frac{0.1}{2}, 1 + \frac{0}{2}, 0 + \frac{0.1}{2}\right)$
$= 0.1 f_1\ \ 0.05, 1, 0.05$	$= 0.1 f_2\ \ 0.05, 1, 0.05$
$k_2 = 0.05$	$l_2 = 0.1100$
$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$	$l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$
$= 0.1\ \ f_1\left(0.05, 1 + \frac{0.005}{2}, \frac{0.1100}{2}\right)$	$= 0.1\ \ f_2\ \ 0.05, 1.0025, 0.0550$
$= 0.1\ \ f_1\ \ 0.05, 1.0025, 0.0550$	$= (0.1)(1 + 0.0001 + 0.1003)$
$k_3 = 0.0055$	$l_3 = 0.1100$
$k_4 = hf_1\ \ x_0 + h, y_0 + k_3, z_0 + l_3$	$l_4 = hf_2\ \ x_0 + h, y_0 + k_3, z_0 + l_3$

$= 0.1\ f_1\ 0.1, 0.0055, 0.1100$ | $= 0.1\ \ 1 + 0.0011 + 0.2011$

$= 0.0110$ | $= 0.1202$

$\Delta y = \frac{1}{6}\ k_1 + 2k_2 + 2k_3 + k_4$ | $\Delta z = \frac{1}{6}\ k_1 + 2k_2 + 2k_3 + k_4$

$= \frac{1}{6}\ 0 + 2\ 0.0050\ + 2\ 0.0055\ + 0.0110$ | $= \frac{1}{6}\ 0.6602$

$= \frac{1}{6}\ 0.0320\ = 0.0053$ | $\Delta z = 0.1100$

$\Delta y = 0.0053$ $\qquad$ $z_1 = z_0 + \Delta z$

$y_1 = y_0 + \Delta y$ $\qquad$ $= 0 + 0.1100$

$= 1 + 0.0053$ $\qquad$ $z_1 = 0.1100$

Second Iteration

$k_1 = hf_1\ \ x_1, y_1, z_1$ | $l_1 = hf_2\ \ x_1, y_1, z_1$

$= 0.1\ f_1\ 0.1, 1.0053, 0.1100$ | $= 0.1\ f_2\ 0.1, 1.0053, 0.110$

$= 0.1(0.1100)$ | $= (0.1))(1.202)$

$k_1 = 0.01100$ | $l_1 = 0.1202$

$k_2 = hf_1\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right)$ | $l_2 = hf_2\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}, z_1 + \frac{l_1}{2}\right)$

$= (0.1) f_1\left(0.1 + \frac{0.1}{2}, 1.0053 + \frac{0.01100}{2}, 0.1100 + \frac{0.1202}{2}\right)$ | $= 0.1\ f_2\ 0.15, 1.0107, 0.1701$

$=(0.1) f_1\ \ 0.15, 1.0107, 0.1701$ | $= 0.1\ \ 1.3070$

$k_2 = 0.0170$ | $l_2\ = 0.13070$

$k_3 = hf_1\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2}\right)$ | $l_3 = hf_2\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}, z_1 + \frac{l_2}{2}\right)$

$= 0.1 f_1 \left(0.15, 1.0138, 0.1754\right)$ | $= 0.1 f_2 \left(0.15, 1.0138, 0.1754\right)$

$k_3 = \left(0.1\right)\left(0.1754\right) = 0.01751$ | $l_3 = 0.1308$

$\text{k}_4 = hf_1\left(x_1 + h, y_1 + k_3, z_1 + l_3\right)$ | $l_4 = hf_2\left(x_1 + h, y_1 + k_3, z_1 + l_3\right)$

$= \left(0.1\right) f_1\left(0.2, 1.0228, 0.2408\right)$ | $= \left(0.1\right) f_2\left(0.2, 1.0228, 0.2408\right)$

$k_4 = 0.0241$ | $l_4 = 0.1419$

$$\Delta \text{y} = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \qquad \Delta z = \frac{1}{6}\left(l_1 + 2l_2 + 2l_3 + l_4\right)$$

$$= 0.0174 \qquad = \frac{1}{6}\left(0.1309\right)$$

$$y(0.2) = y_2 = y_1 + \Delta y \qquad z_2 = z_1 + \Delta z = 0.1100 + 0.1309$$

$$\text{y}\left(0.2\right) = 1.0227 \qquad z_2 = 0.2409$$

Hence the solution is

$y_1 = y\left(0.1\right) = 1.0053$

$y_2 = y\left(0.2\right) = 1.0227$

$z_1 = z(0.1) = 0.1100$

$z_2 = z(0.2) = 0.2409$

Homework:

1. Solve $\text{y}'' - x(y')^2 + y^2 = 0$ using RK method for x = 0.2 given y(0) = 1, y'(0)= 0 taking h = 0.2
Ans: 0.9801

Predictor Corrector Method

- Milne's predictor- Corrector Method
- Adam's Bash forth predictor – corrector method

These two methods are called multistep-method

i. <u>Milne's</u> <u>predictor</u>- <u>Corrector</u> <u>Method:</u>

Predictor:

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\ 2y_{n-2}^{'} - y_{n-1}^{'} + 2y_{n}^{'}$$

Corrector Method:

$$y_{n+1,C} = y_{n-1} + \frac{h}{3}\ y_{n-1}{}^{'} + 4y_{n}{}^{'} + y_{n+1}{}^{'}$$

1.Find y(2) if y(x) is the solution of $\frac{dy}{dx} = \frac{1}{2}(x+y)$ given y(0) = 2, y(0.5) = 2.636,y(1) = 3.595, & y(1.5) = 4.968

Solution:

$$\frac{dy}{dx} = y^{'} = \frac{1}{2}(x+y)$$

Given, y(0)=2, y(0.5)=2.636,y(1)=3.595, &y(1.5)=4.968

$x_0 = 0;\ x_1 = 0.5;\quad x_2 = 1;\ \ x_3 = 1.5$

$y_0 = 2;\ y_1 = 2.636\ \ y_2 = 3.595\ \ y_3 = 4.968$

$$y^{'} = \frac{1}{2}(x+y)$$

$$y_0^{'} = \frac{1}{2}(x_0 + y_0) = \frac{1}{2}(0+2) = 1$$

$$y_1^{'} = \frac{1}{2}(x_1 + y_1) = \frac{1}{2}(0.5 + 2.636)$$

$$= 1.5680$$

$$y_2^{'} = \frac{1}{2}(1.0 + 3.595) = 2.298$$

$$y_3^{'} = \frac{1}{2}(1.5 + 4.968) = 3.234$$

By Corrector formula,

$$y_{n+1,C} = y_{n-1} + \frac{h}{3}\ \ y_{n-1}' + 4y_n' + y_{n+1}'$$

put n=3

$$y_{4,C} = y_2 + \frac{0.5}{3}(y_2' + 4y_3' + y_4')$$

$$y_{4,C} = 6.873$$

By Predictor formula,

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\ \ 2y_{n-2}' - y_{n-1}' + 2y_n'$$

Put n=3

$$y_{n+1,P} = y_0 + \frac{4(0.5)}{3}(2y_1' - y_2' + 2y_3')$$

$$= 2 + \frac{4(0.5)}{3}(2(1.5680) - (2.298) + 2(3.234))$$

$$y_{4,P} = 6.871$$

2. y (4.4) given $5xy' + y^2 - 2 = 0$,given $y(4) = 1, y(4.1) = 1.0049, y(4.2) = 1.0097, y(4.3) = 1.0143$

Solution:

$$5xy' + y^2 - 2 = 0$$

$$5xy' = -y^2 + 2 = 2 - y^2$$

$$y' = \frac{2 - y^2}{5x}$$

$$x_0 = 4;\ x_1 = 4.1;\ x_2 = 4.2;\ x_3 = 4.3$$

$$y_0 = 1;\ y_1 = 1.0049;\ y_2 = 1.0097;\ y_3 = 1.0413$$

$$y_1' = \frac{2 - y_1^{\ 2}}{5x_1} = \frac{2 - (1.0049)^2}{5(4.1)}$$

$$y_1' = 0.0483$$

$$y_2' = \frac{2 - y_2^{\,2}}{5x_2} = \frac{2-(1.0097)^2}{5(4.2)}$$

$$y_2' = 0.0467$$

$$y_3' = \frac{2 - y_3^{\,2}}{5x_3} = \frac{2-(1.043)^2}{5(4.3)}$$

$$y_3' = 0.0452$$

By Predictor formula,

$$y_{n+1,p} = y_{n-3} + \frac{4h}{3}\ \ 2y_{n-2}' - y_{n-1}' + 2y_n'$$

when $\quad n = 3, h = 0.1$

$$y_{4,P} = y_0 + \frac{4(0.1)}{3}(2y_1' - y_2' + 2y_3')$$

$$= 1 + \frac{4(0.1)}{3}(2(0.0483) - (0.0467) + 2(0.0452))$$

$$y_{4,P} = 1.0817$$

$$y_4' = \frac{2 - y_4^{\,2}}{5x_4} = \frac{2-(1.0187)^2}{5(4.4)} = 0.0437$$

By corrector formula,

$$y_{n+1,C} = y_{n-1} + \frac{h}{3}\ \ y_{n-1}' + 4y_n' + y_{n+1}'$$

when $n = 3$

$$y_{4,c} = y_2 + \frac{(0.1)}{3}\ \ y_2' + 4y_3' + y_4'$$

$$= (1.0049) + \frac{0.1}{3}(0.0467 + 4(0.0452) + 0.0437)$$

$$y_{4,c} = 1.0187$$

Homework:

1.Find y(0.4) given $y' = y - \frac{2x}{y}$; $y(0) = 1$, $y(0.1) = 1.095$; $y(0.2) = 1.1841$, $y(0.3) = 1.2662$

Using Milne's M ethod

Ans: 1.3428

<u>Adam's</u> <u>Bash forth</u> <u>Method</u>

Adam's predictor formula,

$$y_{n+1,p} = y_n + \frac{h}{24}\ 55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'$$

Adam's corrector formula,

$$y_{n+1,c} = y_n + \frac{h}{24}(9y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}')$$

Problems:

1. Using Adam's method find y(0.4) given
   $\frac{dy}{dx} = \frac{1}{2xy}$; $y(0) = 1,\ y(0.1) = 1.01,\ y(0.2) = 1.022,\ y(0.3) = 1.023$

Solution:

Given $\frac{dy}{dx} = \frac{1}{2xy}$

$$x_0 = 0;\ x_1 = 0.1;\ x_2 = 0.2;\ x_3 = 0.3;\ x_4 = 0.4$$

$$y_0 = 1;\ y_1 = 1.01;\ y_2 = 1.022;\ y_3 = 1.023;\ y_4 = ?$$

$$y_0^1 = \frac{1}{2x_0y_0} = 0$$

$$y_1^1 = \frac{1}{2x_1y_1} = 0.0505$$

$$y_2^1 = \frac{1}{2x_2y_2} = 0.1022$$

$$y_3^1 = \frac{1}{2x_3y_3} = 0.1535$$

Predictor formula,

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'\right]$$

When n=3

$$y_{4,p} = y_3 + \frac{0.1}{24}\left[55y_3' - 59y_{2-1}' + 37y_1' - 9y_0'\right]$$

$$= 1.023 + \frac{0.1}{24}\left[55(0.1535) - 59(0.1022) + 37(0.0505) - 9(0)\right]$$

$$y_4 = 1.0408$$

$$y_4' = \frac{1}{2x_4y_4} = \frac{1}{2(0.4)(1.0408)}$$

$$y_4' = 0.2082$$

Corrector formula,

$$y_{n+1,c} = y_n + \frac{h}{24}(9y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}')$$

When n=3

$$y_{4,c} = y_3 + \frac{0.1}{24}(9y_4' + 19y_3' - 5y_2' + y_1')$$

$$= 1.023 + \frac{0.1}{24}\left[9(0.2082) + 19(0.1535) - 5(0.1022) + 0.050\right]$$

$$y_{4,c} = 1.0410$$

2.Determine the value of $y(0.4)$ using adam's find $y(0.1)$, $y(0.2)$, $y(0.3)$ from $\frac{dy}{dx}$=xy+y^2 using RK method $y(0)=1$

Solution:

$$\frac{dy}{dx} = xy + y^2$$

$$y(0) = 1$$

$$x_0 = 0; y_0 = 1$$

$$f(x, y) = y' = xy + y^2$$

$$k_1 = hf\ \ x_0, y_0\ \ = (0.1) f(0,1) = 0.1 f(x_0 y_0 + y_0{}^2)$$

$$k_1 = 0.1$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1) f(0.05, 1.05)$$

$$k_2 = 0.1155$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= (0.1) f(0.05, 1.0578)$$

$$k_3 = 0.1172$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$k_4 = 0.1360$$

$$\Delta_y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = \frac{1}{6}(0.1014)$$

$$= 0.1169$$

$$y_1 = y(0.1) = y_0 + \Delta_y = 1 + 0.1169 = 1.1169$$

Second iteration:

$$k_1 = hf(x_1, y_1)$$

$$= (0.1) f(0.1, 1.1169)$$

$$k_1 = 0.1359$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1)f(0.15, 1.1849)$$

$$k_2 = 0.1582$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= (0.1)f\left(0.1 + \frac{0.1}{2}, 1.1169 + \frac{0.1582}{2}\right)$$

$$k_3 = (0.1)f(0.15, 1.196) = 0.1610$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= 0.1f(0.2, 1.2779)$$

$$k_4 = 0.1889$$

$$\Delta_y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.9630)$$

$$\Delta_y = 0.1605$$

$$y_2 = y(0.2) = y_1 + \Delta_y$$

$$= 1.1169 + 0.1605$$

$$y_2 = 1.2779$$

Third iteration:

$$k_1 = hf(x_2, y_2)$$

$$= (0.1)f(0.2, 1.2774)$$

$$k_1 = 0.1887$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1) f(0.25, 1.3718)$$

$$k_2 = 0.2225$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= (0.1) f(0.25, 1.3887)$$

$$k_3 = 0.2276$$

$$k_4 = hf(x_1 + h, y_1 + k_3)$$

$$= 0.1 f(0.3, 1.505)$$

$$k_4 = 0.2717$$

$$\Delta_y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(1.3606)$$

$$\Delta y = 0.2268$$

$$y_3 = y(0.3) = y_2 + \Delta y$$

$$y_3 = 1.5042$$

$$x_0 = 0; x_1 = 0.1; x_2 = 0.2; x_3 = 0.3; x_4 = 0.4$$

$$y_0 = 1; y_1 = 1.1169; y_2 = 1.2774; y_3 = 1.5042; y_4 = ?$$

$y_0' = x_0 y_0 + y_0^2$	$y_1' = x_1 y_1 + y_1^2$	$y_2' = x_2 y_2 + y_2^2$
$= (0)(1) + (1)^2$	$= (0.1)(1.1169) + (1.1169)^2$	$= (0.2)(1.2774) + (1.2774)^2$
$y_0' = 1$	$y_1' = 1.3592$	$y_2' = 1.8872$

$$y_3' = x_3 y_3 + y_3^2$$

$$= (0.3)(1.5042) + (1.5042)^2$$

$$y_3' = 2.7139$$

Adam's predictor formula,

$$y_{n+1,p} = y_n + \frac{h}{24}\left(55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'\right)$$

When n=3

$$y_{4,p} = y_3 + \frac{0.1}{24}\left(55y_3' - 59y_2' + 37y_1' - 9y_0'\right)$$

$$y_{4,p} = 1.8342$$

$$y_4' = x_4 y_4 + y_4^2 = (0.4)(1.8342) + (1.8342)$$

$$y_4' = 4.0980$$

Corrector formula,

$$y_{n+1,c} = y_n + \frac{h}{24}(9y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}')$$

When n=3

$$y_{4,c} = y_3 + \frac{0.1}{24}(9y_4' + 19y_3' - 5y_2' + y_1')$$

$$y_{4,c} = 1.8391$$

Homework:

1.Given $\frac{dy}{dx} = x^2(1+y);\ y(1) = 1;\ y(1.1) = 1.233;\ y(1.2) = 1.548;\ y(1.3) = 1.979$

Evaluate by adam's bash forth mtd.Ans : $pre - 2.5871,\ corr - 2.5773$

2.Find $y(0.4)$ given $y' = y - \frac{2x}{y}, y(0)=1, y(0.1)=1.0959, y(0.2)=1.1841, y(0.3)=1.2662$

using adam's method

Finite difference methods for solving second order equation:

The solution of a differential equation of second order of the form $F(x, y, y', y'')=0$ contains two arbitrary constants.

These constants are determined by means of two conditions. The conditions on y or y' or their combination are prescribed at two different values of x are called boundary conditions.

The differential equation together with the boundary conditions is called a boundary value problem

Finite difference approximations to derivatives

First derivative approximations:

$$y'(x) = \frac{y(x+h)-y(x)}{h} + 0(h)$$

$$y'(x) = \frac{y(x)-y(x-h)}{h} + 0(h)$$

$$y'(x) = \frac{y(x+h)-y(x-h)}{2h} + 0(h)^2 \qquad (central\ \ diff)$$

Second Derivative Approximation:

$$y''(x) = \frac{y(x-h)-2y(x)+y(x+h)}{h^2} + 0(h^2)$$

Third Approximation:

$$y'''(x) = \frac{1}{2h^3}\ y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}$$

Fourth Derivative Approximation:

$$y^{iv}(x) = \frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{h^4}$$

<u>Solution of ODE of second order:</u>

The second order ODE is given by

$$F(x,y,y',y'')=0$$

with b.c, x=a and x=b

Divide the interval into 'n' equal parts

$$h=\frac{b-a}{n}$$

$$x_i = x_0 + ih \,;\, i = 0,1,2,......,n \text{ where } x_0 = a \;\&\; x_n = b$$

$$y_i = y(x_i) \,;\, y'(x_i) = y_i' \,;\, y''(x_i) = y_i''$$

The finite difference approximations to the detrivatives are given by,

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2h} \text{ and}$$

$$y_i'' = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} \quad i = 1, 2, ...n-1$$

$$y_0 = y(a) \text{ and } y_n = y(b)$$

1. Solve $xy'' + y = 0, y(1); y(2) = 2$ with $h = 0.5$ *and* $h = 0.25$ by using finite difference method:

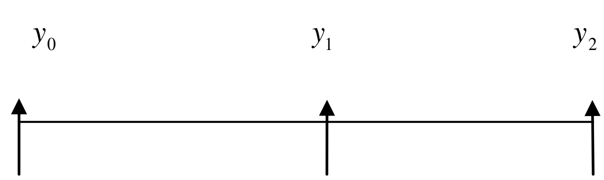
Solution:

Given

$$xy'' + y = 0 \,;\, y(1) = 1;\, y(2) = 2 \quad \text{------ (1)}$$

$$\Rightarrow y_0 = 1 \text{ and } y_2 = 2$$

$(i)\ h = 0.5$



x_0 1 $\qquad$ x_1 1.5 $\qquad$ x_2 2

Here , $x_0 = 1 ; x_1 = 1.5 ; x_2 = 2$

By boundary conditions ; $y_0 = 1; y_2 = 2$

To find y_1 *for* $x_1 = 1.5$

By finite difference approximation

$$y_i^{"} = \frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} \; ; i = 1, 2, ..., n-1$$

$(1) \Rightarrow x_i y_i^{"} + y_i = 0$

$$\Rightarrow x_i \left(\frac{y_{i-1} - 2y_i + y_{i+1}}{h^2} \right) + y_i = 0$$

$$\frac{x_i}{h^2}(y_{i-1} - 2y_i + y_{i+1}) + y_i = 0 ----------> 2$$

put $i = 1 ; h = 0.5$

$$\frac{x_1}{0.25} \;\; y_0 - 2y_1 + y_2 \;\; + y_1 = 0$$

$$4x_1 \;\; y_0 - 2y_1 + y_2 \;\; + y_1 = 0$$

$$4(1.5) \;\; 1 - 2y_1 + 2 \;\; + y_1 = 0$$

$$6 - 12y_1 + 12 + y_1 = 0$$

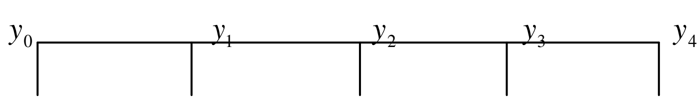
$$18 - 11y_1 = 0$$

$$18 = 11y_1$$

$$\Rightarrow y_1 = 1.6364$$

$$\therefore y(1.5) = 1.6364$$

(ii) $h = 0.25$



x_0	x_1	x_2	x_3	x_4
1	1.25	1.5	1.75	2

$$\therefore \quad y_0 = 1; \quad y_4 = 2$$

$$x_0 = 1 ; x_1 = 1.25 ; x_2 = 1.5 ; x_3 = 1.75 ; x_4 = 2$$

$$(2) \Rightarrow \frac{x_i}{h^2}\left(y_{i-1} - 2y_i + y_{i+1}\right) + y_i = 0; i = 1, 2, 3, \ldots .n-1$$

$$i = 1, 2, 3$$

$$put\ h = 0.25\ \ \&\ i = 1, 2, 3$$

$$16x_i\left(y_{i-1} - 2y_i + y_{i+1}\right) + y_i = 0$$

$$put\ i = 1,$$

$$16x_1\left(y_0 - 2y_1 + y_1\right) + y_1 = 0$$

$$16(1.25)(1 - 2y_1 + y_2) + y_1 = 0$$

$$20 - 40y_1 + 20y_2 + y_1 = 0$$

$$-39y_1 + 20y_2 = -20$$

$$39y_1 - 20y_2 = 20 ----------->(3)$$

$$put\ i = 2,$$

$$16x_2\left(y_1 - 2y_2 + y_3\right) + y_2 = 0$$

$$16(1.5)(y_1 - 2y_2 + y_3) + y_2 = 0$$

$$24y_1 - 48y_2 + 24y_3 + y_2 = 0$$

$$\Rightarrow 24y_1 - 47y_2 + 24y_3 = 0 ------->4$$

$$put\ i = 3,$$

$$16x_3\left(y_2 - 2y_3 + y_4\right) + y_3 = 0$$

$$16(1.75)(y_2 - 2y_3 + y_4) + y_3 = 0$$

$$28y_2 - 56y_3 + 56 + y_3 = 0$$

$$28y_2 - 55y_3 = -56 ------------->5$$

Solve equation (3),(4)&(5) by gauss elimination method

$$A,B = \begin{bmatrix} 39 & -20 & 0 & 20 \\ 24 & -47 & 24 & 0 \\ 0 & 28 & -55 & -56 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -0.513 & 0 & 0.513 \\ 1 & -1.958 & 1 & 0 \\ 0 & 1 & -1.964 & -2 \end{bmatrix} \quad R_1^{'} \Rightarrow \frac{R_1}{39}, R_2^{'} \Rightarrow \frac{R_2}{24}, R_3^{'} \Rightarrow \frac{R_3}{28}$$

$$\sim \begin{bmatrix} 1 & -0.513 & 0 & 0.513 \\ 0 & -1.445 & 1 & -0.513 \\ 0 & 1 & -1.964 & -2 \end{bmatrix} \quad R_2^{'} = R_2 - R_1,\ R_3^{'} = R_3$$

$$\sim \begin{bmatrix} 1 & -0.513 & 0 & 0.513 \\ 0 & -1.445 & 1 & -0.513 \\ 0 & 0 & -1.838 & -3.403 \end{bmatrix} \quad R_3^{'} = 1.445R_3 + R_2$$

By back substitution method:

$$\cancel{-}1.838y_3 = \cancel{-}3.403$$

$$=> y_3 = 1.851$$

$$-1.445y_2 + y_3 = -0.513$$

$$=> y_2 = 1.636$$

$$y_1 - 0.513y_2 = 0.513$$

$$=> y_1 = 1.352$$

$$\therefore\ y(1.25) = 1.352\ ;\ y(1.5) = 1.636\ ;\ y(1.75) = 1.851$$

2. Determine the value of 'y'at the pivotal points of the interval (0,1) if 'y' satisfies the boundary value problem

$$y^{iv}+81y=81x^2, y(0)=y(1)=y^{'}(0)=y^{''}(1)=0 \quad \text{(take n = 3)}$$

Solution:

Given, $y^{iv}+81y=81x^2,$

$$(a,b)=(0,1)$$

$$y(0)=y(1)=y^{'}(0)=y^{''}(1)=0$$

$$h=\frac{b-a}{n}=\frac{1-0}{3}=\frac{1}{3}$$

y_0	y_1	y_2	y_3
x_0 0	x_1 $\frac{1}{3}$	x_2 $\frac{2}{3}$	x_3 1

$$x_0=0; x_1=\frac{1}{3}; x_2=\frac{2}{3}; x_3=1$$

$$y_0=0; y_3=0; y_0^{''}=0; y_3^{''}=0$$

To find y_1 and y_2

By using central difference approximation, the differential equation becomes,

$$\frac{1}{h^4}\ y_{i+2}-4y_{i+1}+6y_i-4y_{i-1}+y_{i-2}\ +81y_i=81x_i^2$$

$put\ h=\frac{1}{3}$

$$\frac{1}{(1/3)^4}\ y_{i+2}-4y_{i+1}+6y_i-4y_{i-1}+y_{i-2}\ +81y_i=81x_i^2$$

$$81\ y_{i+2}-4y_{i+1}+6y_i-4y_{i-1}+y_{i-2}\ +81y_i=81x_i^2$$

$\div 81$ and put $i=1$ and $i=2$

$$y_3-4y_2+6y_1-4y_0+y_{-1}\ +y_1=x_1^2=\frac{1}{9}$$

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 + y_2 = \frac{4}{9}$$

$$y_3 - 4y_2 + 7y_1 - 4y_0 + y_{-1} = \frac{1}{9}$$

$$-4y_2 + 7y_1 + y_{-1} = \frac{1}{9} \quad -----> 1$$

$$y_4 + 7y_2 - 4y_1 = \frac{4}{9} -------> 2$$

$$y_i^{"} = \frac{1}{h^2}(y_{i+1} - 2y_i + y_{i-1})$$

$i = 0,$

$$y_0^{"} = 9(y_1 - 2y_0 + y_{i-1}) -----> 3$$

$$y_0 = 0, y_0^{"} = 0, y_{-1} = -y_1$$

$$i = 3, \; y_3^{"} = 9(y_4 - 2y_3 + y_2) -----> 4$$

$$y_3 = 0; \;\; y_3^{"} = 0; \;\; y_4 = -y_2$$

Using (3) and (4) in eqn (1) & (2)

$$-4y_2 + 6y_1 = \frac{1}{9} ----> 5$$

$$6y_2 - 4y_1 = \frac{4}{9} ----> 6$$

$$(5)\times(6) \qquad -24y_2 + 36y_1 = \frac{6}{9}$$

$$(6)\times(4) \qquad 24y_2 - 16y_1 = \frac{16}{9}$$

$$20y_1 = \frac{22}{9}$$

$$y_1 = 0.1222$$

$$5 \Rightarrow \quad -4y_2 + 0.7332 = \frac{1}{9}$$

$$\cancel{-}4y_2 = \cancel{-}0.6221$$

$$y_2 = 0.1555$$

$$\therefore y\left(\frac{1}{3}\right) = 0.1222\,;\; y\left(\frac{2}{3}\right) = 0.1556$$

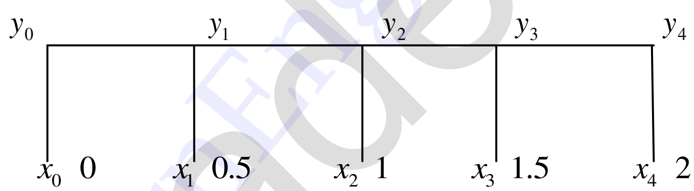
3. Solve the boundary value problem $(x^3+1)y''(x) + x^2 y'(x) - 4xy = 2$ subject to $y(0)=0,\; y(2)=4$ taking $n=4$ and choosing $h=0.5$

Solution:

Given, $(x^3+1)y''(x) + x^2 y'(x) - 4xy = 2 ----> 1$

$$y(0)=0;\; y(2)=4$$

$$\Rightarrow y_0 = 0;\; y_4 = 2\,;\; h = 0.5 \;\&\; n = 4$$



$$x_0 = 0; x_1 = 0.5; x_2 = 1; x_3 = 1.5; x_4 = 2$$

$$y_0 = 0; y_4 = 2$$

to find y_1, y_2, y_3 :

The finite difference approximates,

$$y_i''(x) = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}\,;\; y'(x) = \frac{y_{i+1} - y_{i-1}}{2h}$$

put $h = \frac{1}{2}$

$$(1) \Rightarrow 4(x_i^3+1)\; y_{i+1} - 2y_i + y_{i-1} \; + x_i^2(y_{i+1} - y_{i-1}) - 4x_i y_i = 2 ---> (2)$$

$\text{put i} = 1, 2, 3$

$(2) \Rightarrow \quad 4(x_1^3 + 1)(y_2 - 2y_1 + y_0) + x_1^2(y_2 - y_0) - 4x_1 y_1 = 2$

$4(1.125)(y_2 - 2y_1) + 0.25y_2 - 2y_1 = 2$

$4.5y_2 - 9y_1 + 0.25y_2 - 2y_1 = 2$

$4.75y_2 - 11y_1 = 2 \quad --> (3)$

$4(2)(y_3 - 2y_2 + y_1) + y_3 - y_1 - 4y_2 = 2$

$8y_3 - 16y_2 + 8y_1 + y_3 - 4y_2 - y_1 = 2$

$7y_1 - 20y_2 + 9y_3 = 2 \; ----> (4)$

$4(4375)(y_4 - 2y_3 + y_2) + 2.25y_4 - 6y_3 - 2.25y_2 = 2$

$35 - 35y_3 + 17.5y_2 + 2.25y_4 - 6y_3 - 2.25y_2 = 2$

$15.25y_2 - 41y_3 = 37.5 ----> (5)$

Solving the equations (3),(4) and (5) by gauss elimination method , we get

$$y(0.5) = 0.25,\ y(1) = 1 \text{ and } y(1.75) = 2.25$$

Homework:

1. Solve the BVP $y'' + xy = 1, y(0) = 0, y'(1) = 1$ with $n=2$ taking $h = \frac{1}{2}$

2. Solve the BVP $x^2 y''(x) + xy'(x) + (x^2 - 3)y(x) = 0$ given $y(1) = 0$, $y(2) = 2$ and take $h = 0.25$ *and* $n = 4$

$Ans: \; y_1 = 0.6044; y_2 = 1.1304; y_3 = 1.5973$