Rikitake - Supplementary material

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This short guide is the supplementary material for the Rikitake software. This material is meant to give the user an insight of the theoretical background behind this project and to present the algorithms that *Rikitake* uses.

1 The Rikitake Geodynamo

Paleomagnetism shows evidences that Earth's magnetic field at surface has been a dipole roughly directed as Earth's rotation axis for most of the time, even though its polarity appears to change in time without any clear periodicity. This magnetic field is generated by the motion of the liquid metals (mostly iron) present in Earth's outer core: the system behaves like a self-excited dynamo. Iron is in fact a good conductor at the typical core conditions. This suggests that depicting the motion of this fluid allows us to depict the evolution of Earth's magnetic field. However, due to the low viscosity of the outer core, the fluid motion is highly turbolent, so that simpler models have been searched. In 1958 Rikitake[1] proposed his model of self-excited dynamo. This system shows a chaotic behavior and non-periodic inversions of the polarity.

The system

The model of the Rikitake geodynamo consists in a set of three non-linear coupled equations. The physical details of the system can be easily found in [1] or [2]. For our purposes, the non-dimensional form of the equations will be considered:

$$\frac{dX_1}{d\tau} = -\mu X_1 + Y_1 X_2 \tag{1a}$$

$$\frac{dX_2}{d\tau} = -\mu X_2 + (Y_1 - A) X_1 \tag{1b}$$

$$\frac{dY_1}{d\tau} = 1 - X_1 X_2 \tag{1c}$$

$$Y_2 = Y_1 - A \tag{1d}$$

where the two parameters μ and A appear. For simplicity, we can define a third parameter k via the relation $A = \mu(k^2 - k^{-2})$. Y_2. As can be seen in eq. (1d) is just a shift of Y_1, so it is not of interest for our purposes. The *Rikitake* program will not calculate this, but its value will still be present in the output files.

The steady states of the system can be easily obtained equating the equations

to 0, leading to:

$$X_1 = \pm k \tag{2}$$
$$X_2 = \pm k^{-1} \tag{3}$$
$$Y_1 = \mu k^2 \tag{4}$$
$$Y_2 = \mu k^{-2} \tag{5}$$

These stationary points are unstable for all parameter values.

Integration

The integration has been performed using a Runge-Kutta algorithm of order IV. For a general system of equation in the form

$$\frac{d\mathbf{x}}{dt} = \mathbf{\Phi}(\mathbf{x}(t), t)$$

a Runge-Kutta integration algorithm of order m is given by:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta t \sum_{j=1}^{m} A_j \mathbf{K}_j \tag{6}$$
$$\mathbf{K}_1 = \mathbf{\Phi}(\mathbf{x}_k, t_k) \tag{7}$$
$$\mathbf{K}_j = \mathbf{\Phi}(\mathbf{x}_k + \beta_j \Delta t \mathbf{K}_{j-1}, t + \alpha_j \Delta t) \tag{8}$$

where the coefficients A_j, β_j and α_j are determined imposing the global error to be in the order of $O(\Delta t^m)$. The IV order RK method used by *Rikitake* is given by

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \frac{\Delta t}{6}(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + \mathbf{K}_4) \tag{9}$$
$$\mathbf{K}_1 = \mathbf{\Phi}(\mathbf{x}_k, t_k) \tag{10}$$
$$\mathbf{K}_2 = \mathbf{\Phi}(\mathbf{x}_k + \frac{1}{2}\Delta t \mathbf{K}_1) \tag{11}$$
$$\mathbf{K}_3 = \mathbf{\Phi}(\mathbf{x}_k + \frac{1}{2}\Delta t \mathbf{K}_2) \tag{12}$$
$$\mathbf{K}_4 = \mathbf{\Phi}(\mathbf{x}_k + \Delta t \mathbf{K}_3) \tag{13}$$

So considering the equations (1), *Rikitake* performs the following calculations:

$$X^1_{k+1} = X^1_k + \frac{\Delta t}{6}(K^1_1 + 2K^1_2 + 2K^1_3 + K^1_3)$$
$$X^2_{k+1} = X^2_k + \frac{\Delta t}{6}(K^2_1 + 2K^2_2 + 2K^2_3 + K^2_3)$$
$$Y^1_{k+1} = Y^1_k + \frac{\Delta t}{6}(K^Y_1 + 2K^Y_2 + 2K^Y_3 + K^Y_3)$$

$$K^1_1 = -\mu X^1_k + Y^1_k X^2_k$$
$$K^2_1 = -\mu X^2_k + (Y^1_k - A) X^1_k$$
$$K^Y_1 = 1 - X^1_k X^2_k$$

$$
\begin{aligned}
K_2^1 &= -\mu(X_k^1 + \frac{\Delta t}{2}K_1^1) + (Y_k^1 + \frac{\Delta t}{2}K_1^Y)(X_k^2 + \frac{\Delta t}{2}K_1^2) \\
K_2^2 &= -\mu(X_k^2 + \frac{\Delta t}{2}K_1^2) + (Y_k^1 + \frac{\Delta t}{2}K_1^Y - A)(X_k^1 + \frac{\Delta t}{2}K_1^1) \\
K_2^Y &= 1 - (X_k^1 + \frac{\Delta t}{2}K_1^1)(X_k^2 + \frac{\Delta t}{2}K_1^2)
\end{aligned}
$$

$$
\begin{aligned}
K_3^1 &= -\mu(X_k^1 + \frac{\Delta t}{2}K_2^1) + (Y_k^1 + \frac{\Delta t}{2}K_2^Y)(X_k^2 + \frac{\Delta t}{2}K_2^2) \\
K_3^2 &= -\mu(X_k^2 + \frac{\Delta t}{2}K_2^2) + (Y_k^1 + \frac{\Delta t}{2}K_2^Y - A)(X_k^1 + \frac{\Delta t}{2}K_2^1) \\
K_3^Y &= 1 - (X_k^1 + \frac{\Delta t}{2}K_2^1)(X_k^2 + \frac{\Delta t}{2}K_2^2)
\end{aligned}
$$

$$
\begin{aligned}
K_4^1 &= -\mu(X_k^1 + \Delta t K_3^1) + (Y_k^1 + \Delta t K_3^Y)(X_k^2 + \Delta t K_3^2) \\
K_4^2 &= -\mu(X_k^2 + \Delta t K_3^2) + (Y_k^1 + \Delta t K_3^Y - A)(X_k^1 + \Delta t K_3^1) \\
K_4^Y &= 1 - (X_k^1 + \Delta t K_3^1)(X_k^2 + \Delta t K_3^2)
\end{aligned}
$$

While the values of parameters μ and k are properties of the system, the value of the time step Δt highly impacts the results of the simulation. After repeated trials it is found that the value $\Delta t = 2^{-6}$ grants a good "smoothness" of the solution, that is, the distance between two subsequent points is acceptably small. Smaller values certainly will improve the precision of the integration, at the cost of increasing the computation time if we wish to reach a particular instant T. It is also better to use $\Delta t = 2^n$, with n integer, in order to limit the floating point round-off error.

Estimate of the greatest Lyapunov exponent

Let us consider a system of differential equations in the form $\dot{\mathbf{x}} = \mathbf{\Phi}(\mathbf{x})$ two solutions $\mathbf{x}(t)$ and $\mathbf{y}(t)$ with initial conditions, respectively, $\mathbf{x}(0) = \mathbf{x}_0$ and $\mathbf{y}(0) = \mathbf{y}_0$. We can define $\boldsymbol{\eta}(t) = \mathbf{x}(t) - \mathbf{y}(t)$, with $\boldsymbol{\eta}_0 = \mathbf{x}_0 - \mathbf{y}_0$ the perturbation between the two solutions. It is possible to show that $\boldsymbol{\eta}(t)$ has an exponential behavior, determined by the exponents $\lambda_1, \ldots, \lambda_d$, with d dimension of the phase space. These exponents are called *Lyapunov exponents*. The evolution of the dynamics of the system is mostly determined by $\lambda = \max(\lambda_j)$ (with $j = 1, ..., d$), the maximum Lyapunov exponent. In particular if $\lambda > 0$ the distance between the two solutions grows exponentially, indicating chaotic behavior. This exponent is defined by the relation

$$
\lambda = \max_{\boldsymbol{\eta}_0} \lim_{t \to \infty} \frac{1}{t} \ln \frac{\|\boldsymbol{\eta}(t)\|}{\|\boldsymbol{\eta}_0\|} \tag{14}
$$

Numerical estimate

In order to numerically estimate λ from the solution obtained via the way explained before, *Rikitake* uses an approximation of eqn. (14). The first step consists in the integration of the solutions $\mathbf{x}(t)$ and $\mathbf{y}(t)$. First we choose the

values of the parameters μ and k and a point $\mathbf{x}_0$, and integrate for N steps in order to obtain samples $\mathbf{x}(t_k)$, with $k = 1, \ldots, N$.

Then *Rikitake* needs a perturbed solution $\mathbf{y}(t_k)$, so we must choose an initial perturbation $\boldsymbol{\eta}_0$. Being G_{ij} the jacobian matrix of $\boldsymbol{\Phi}(\mathbf{x})$

$$G_{ij} = \begin{pmatrix} -\mu & y_1 & x_2 \\ y_a - A & -\mu & -AX_1 \\ -X_2 & -X_1 & 0 \end{pmatrix} \tag{15}$$

Rikitake calculates its eigenvectors and eigenvalues (using numpy.linalg.eig function) in the point $\mathbf{x}_0$. Let $\mathbf{v}$ be the eigenvector relative to the maximum eigenvalue of G_{ij}. Then, $\mathbf{v}$ indicates the direction of the maximum local variation of $\boldsymbol{\Phi}(\mathbf{x})$, that is a good direction to look for rapidly diverging orbits. So the chosen initial perturbation is $\boldsymbol{\eta}_0 = b\mathbf{v}$, with $b = 2^{-49}$([1]).

This is the way *Rikitake* calculates the perturbed initial state when performing the *in-run* input procedure. *Rikitake* then integrates the system with the same values of μ and k as the former integration and with the initial condition $\mathbf{y}_0 = \mathbf{x}_0 + b\mathbf{v}$.

With the two solutions $\mathbf{x}_k$ and $\mathbf{y}_k$ *Rikitake* computes

$$\boldsymbol{\eta}_k = \mathbf{x}_k - \mathbf{y}_k \tag{16}$$

This allows us to calculate a succession of approximate Lyapunov exponent λ_k that approximates the real Lyapunov exponent λ as $N \to \infty$:

$$\lambda_N = \frac{1}{N} \sum_{k=1}^{N} \ln \frac{\|\boldsymbol{\eta}_k\|}{\|\boldsymbol{\eta}_0\|} \tag{17}$$

Rikitake creates a series of values λ_N with $N = 1, 2, \ldots, N_s$, where N_s is the input parameter N_steps. Obviously, *Rikitake* can never approach infinity: the "final result" of the Rikitake routine is the mean of the last 100 λ_N of the series

References

[1] Tsuneji Rikitake. “Oscillations of a system of disk dynamos”. In: *Proc. Camb. Phil. Soc.* 54 (1958).

[2] Donald L. Turcotte. *Fractals and chaos in geology and geophysics*. New York: Cambridge Universisty Press, 1997.

[1] We want the two initial conditions $\mathbf{x}_0$ and $\mathbf{y}_0$ to be as close as possible, so we need b to be as small as possible. Several tries showed that the estimate of the Lyapunov exponent gets larger as b gets smaller. After several simulation it is found out that the smallest value of b that grants meaningful results is $b = 2^{-49}$. This is not a "magic number", as can depend on the particular simulation and on the calculator architecture, and feel free to modify it!