SEE609: MATHEMATICAL AND COMPUTATIONAL TOOLS FOR ENGINEERING

Assignment #4

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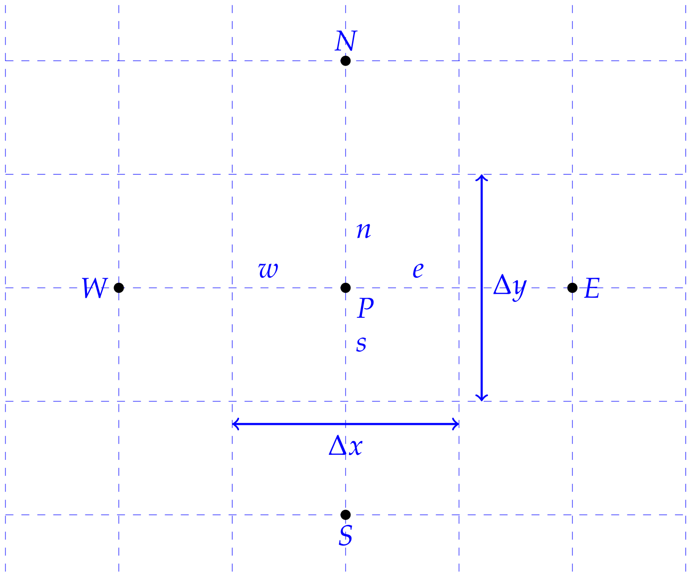
Due date: *November 14th, 2024*

1. **Question 1**

(a) Export this data and make a plot of average viscosity vs $\log(\Delta x)$ Based on the plot, can you comment on the grid independence and the optimum number of elements?

Answer:

1. Problem Setup and Governing Equation



The governing equation in integral form for a scalar field ϕ under a convection-diffusion transport process with a known velocity field is:

$$\int_S \rho\phi\mathbf{v}\cdot\mathbf{n}\,dS = \int_S \Gamma\nabla\phi\cdot\mathbf{n}\,dS + \int_\Omega q_\phi\,d\Omega$$

where:

- $\mathbf{v} = (u, v)$ with $u = x$ and $v = -y$,
- $\Gamma = 0.01$ is the diffusivity,
- $\rho = 1$ (density),
- Source term $q_\phi = xy$

2. Discretization using Finite Volume Method (FVM)

In FVM, the computational domain is divided into a mesh of control volumes (CVs), where the integral form of the transport equation is applied to each CV.

2.1. Mesh Definition. Let N_x and N_y denote the number of control volumes along the x and y directions. The grid spacing is defined as $\Delta x = \frac{1}{N_x}$ and $\Delta y = \frac{1}{N_y}$.

2.2. Discretization of Convection Term. : The convection term

$$\int_S \rho\phi\mathbf{v}\cdot\mathbf{n}\,dS$$

is approximated on the cell faces.

2.3. Discretization of Diffusion Term. The diffusion term $\int_S \Gamma\nabla\phi\cdot\mathbf{n}\,dS$ is also evaluated on the cell faces. Using the central difference approach for the gradient, the diffusion term across a face (e.g., between control volumes P and E) becomes:

$$F_{\text{diff},x} = -\Gamma\frac{\phi_E - \phi_P}{\Delta x}$$

Step-by-Step Discretization for the Interior Cells

i. **Divide the Domain into Control Volumes:**

- Each cell (or CV) has faces centered at the midpoints between adjacent cells.
- The integral form will be applied separately to each CV, with fluxes calculated on each face.

ii. **Flux Balance at Each Cell Face:**

- The left, right, top, and bottom faces of each CV are denoted as e, w, n, and s, respectively.
- We will evaluate the convection and diffusion fluxes at each face and balance them for the cell.

iii. **Convection Flux at Each Face:**

Using the known velocity field $\mathbf{v} = (x, -y)$, the convection flux across each face is:

$$\int_S \rho\phi\mathbf{v} \cdot \mathbf{n}\, dS \approx \sum_{\text{faces}} \rho\phi_f\, \mathbf{v}_f \cdot \mathbf{n}_f\, \Delta S$$

where:

- ϕ_f is the value of ϕ at the cell face.
- $\mathbf{v}_f$ is the velocity at the cell face, with components $u = x$ and $v = -y$ evaluated at the face center.
- ΔS is the area of each face, which depends on the orientation and dimensions of the face in the control volume.

The surface area term ΔS is the product of the corresponding width or height of the control volume. For each face, we define:

- $\Delta S = \Delta y$ for the East and West faces, as these faces are aligned along the y-axis.
- $\Delta S = \Delta x$ for the North and South faces, as these faces are aligned along the x-axis.

For each face:

- **East face (right):** $\rho\phi_e u_e \Delta y \approx \rho\phi_e \cdot x_e \Delta y$
- **West face (left):** $-\rho\phi_w u_w \Delta y \approx -\rho\phi_w \cdot x_w \Delta y$
- **North face (top):** $\rho\phi_n v_n \Delta x \approx -\rho\phi_n \cdot y_n \Delta x$
- **South face (bottom):** $-\rho\phi_s v_s \Delta x \approx \rho\phi_s \cdot y_s \Delta x$

$$\sum_{\text{faces}} \rho\phi_f\, \mathbf{v}_f \cdot \mathbf{n}_f\, \Delta S = (\rho\phi_e \cdot x_e - \rho\phi_w \cdot x_w)\Delta y + (-\rho\phi_n \cdot y_n + \rho\phi_s \cdot y_s)\Delta x$$

Using upwind or central differencing, approximate the face values (ϕ_e, ϕ_w, ϕ_n, ϕ_s) from neighboring cell values.

i. **Diffusion Flux at Each Face:**

The diffusion flux term is given by:

$$\begin{aligned}
\int \Gamma \nabla \phi \cdot \mathbf{n}\, ds &= \Gamma \int \left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} \right) \cdot \mathbf{n}\, ds \\
&= \Gamma \left[\left(\left.\frac{\partial \phi}{\partial x}\right|_e \cdot \Delta y \right) - \left(\left.\frac{\partial \phi}{\partial x}\right|_w \cdot \Delta y \right) \right. \\
&\quad \left. + \left(-\left.\frac{\partial \phi}{\partial y}\right|_n \cdot \Delta x \right) + \left(\left.\frac{\partial \phi}{\partial y}\right|_s \cdot \Delta x \right) \right]
\end{aligned}$$

$$= \Gamma \Delta y \left(\left.\frac{\partial \phi}{\partial x}\right|_e - \left.\frac{\partial \phi}{\partial x}\right|_w \right) + \Gamma \Delta x \left(\left.\frac{\partial \phi}{\partial y}\right|_n - \left.\frac{\partial \phi}{\partial y}\right|_s \right)$$

where:

- ϕ_{neighbor} and ϕ_{current} are the values in the neighboring cell and the current cell, respectively.
- d is the distance between the centroids of the current and neighboring cells.
- ΔS is the area of the face, which as explained earlier, is dependent on the orientation of the face.

The surface area term ΔS remains the same for the diffusion flux:

- $\Delta S = \Delta y$ for the East and West faces.
- $\Delta S = \Delta x$ for the North and South faces.

For each face:

- **East face:** $\Gamma \frac{\phi_E - \phi_P}{\Delta x} \Delta y$
- **West face:** $\Gamma \frac{\phi_P - \phi_W}{\Delta x} \Delta y$
- **North face:** $\Gamma \frac{\phi_N - \phi_P}{\Delta y} \Delta x$
- **South face:** $\Gamma \frac{\phi_P - \phi_S}{\Delta y} \Delta x$

$$\begin{aligned}
\int \Gamma \nabla \phi \cdot \mathbf{n}\, ds \approx \Gamma &\left[\left(\frac{\phi_E - \phi_P}{\Delta x} \Delta y \right) - \left(\frac{\phi_P - \phi_W}{\Delta x} \Delta y \right) \right. \\
&\left. + \left(-\frac{\phi_N - \phi_P}{\Delta y} \Delta x \right) + \left(\frac{\phi_P - \phi_S}{\Delta y} \Delta x \right) \right] \\
= \Gamma &\left[\frac{\Delta y}{\Delta x} (\phi_E - 2\phi_P + \phi_W) - \frac{\Delta x}{\Delta y} (\phi_N - 2\phi_P + \phi_S) \right].
\end{aligned}$$

i. **Source Term Integration:**

The source term integrated over the control volume Ω for cell P is:

$$\int_{\Omega} q_{\phi}\, d\Omega \approx q_{\phi}(x_P, y_P)\,(\Delta x + \frac{\Delta x^2}{2})\,(\Delta y + \frac{\Delta y^2}{2}) = x_P y_P (\Delta x + \frac{\Delta x^2}{2})\,(\Delta y + \frac{\Delta y^2}{2})$$

ii. **Face Locations:** The positions of the faces relative to the cell center are:

- **East face:** $x_e = x_P + \frac{\Delta x}{2}$
- **West face:** $x_w = x_P - \frac{\Delta x}{2}$
- **North face:** $y_n = y_P + \frac{\Delta y}{2}$
- **South face:** $y_s = y_P - \frac{\Delta y}{2}$

iii. Variable Estimation at Faces and Corners

UDS: Approximate face value by its upstream node

$$\phi_e = \begin{cases} \phi_P & \text{if } (v \cdot n)_e > 0, \\ \phi_E & \text{if } (v \cdot n)_e < 0. \end{cases}$$

$$\phi_e = \phi_P, \quad \phi_w = \phi_W, \quad \phi_n = \phi_N, \quad \phi_s = \phi_P$$

$$\int_S \rho\phi \mathbf{v} \cdot \mathbf{n}\, dS \approx \rho \Delta y (\phi_P \cdot x_e - \phi_W \cdot x_w) + \rho \Delta x (\phi_P \cdot y_s - \phi_N \cdot y_n)$$

CDS: Interpolate between adjacent nodes

$$\phi_e = \phi_E \lambda_e + \phi_P (1 - \lambda_e)$$

$$\phi_e = \frac{\phi_E + \phi_P}{2}, \quad \phi_w = \frac{\phi_W + \phi_P}{2}, \quad \phi_n = \frac{\phi_N + \phi_P}{2}, \quad \phi_s = \frac{\phi_S + \phi_P}{2}$$

$$\int_S \rho\phi \mathbf{v} \cdot \mathbf{n}\, dS \approx \rho \Delta y \left(\frac{\phi_E + \phi_P}{2} \cdot x_e - \frac{\phi_P + \phi_W}{2} \cdot x_w \right) + \rho \Delta x \left(\frac{\phi_S + \phi_P}{2} \cdot y_s - \frac{\phi_N + \phi_P}{2} \cdot y_n \right)$$

i. **Final Discretized Equation for an Interior Cell:**
Convection Equation(UDS):

$$\int_S \rho\phi\mathbf{v}\cdot\mathbf{n}\,dS \approx \left(\rho\phi_P\cdot\left(x_P+\frac{\Delta x}{2}\right)-\rho\phi_W\cdot\left(x_P-\frac{\Delta x}{2}\right)\right)\Delta y$$
$$+\left(-\rho\phi_N\cdot\left(y_P+\frac{\Delta y}{2}\right)+\rho\phi_P\cdot\left(y_P-\frac{\Delta y}{2}\right)\right)\Delta x$$

Convection Equation(CDS):

$$\int_S \rho\phi\mathbf{v}\cdot\mathbf{n}\,dS \approx \left(\rho\left(\frac{\phi_E+\phi_P}{2}\right)\cdot\left(x_P+\frac{\Delta x}{2}\right)-\rho\left(\frac{\phi_W+\phi_P}{2}\right)\cdot\left(x_P-\frac{\Delta x}{2}\right)\right)\Delta y$$
$$+\left(-\rho\left(\frac{\phi_N+\phi_P}{2}\right)\cdot\left(y_P+\frac{\Delta y}{2}\right)+\rho\left(\frac{\phi_S+\phi_P}{2}\right)\cdot\left(y_P-\frac{\Delta y}{2}\right)\right)\Delta x$$

Diffusion Equation: Remains same for UDS and CDS

$$\int \Gamma\nabla\phi\cdot\mathbf{n}\,ds \approx \Gamma\left[\frac{\Delta y}{\Delta x}(\phi_E-2\phi_P+\phi_W)-\frac{\Delta x}{\Delta y}(\phi_N-2\phi_P+\phi_S)\right].$$

Source Term Equation:

$$\int_\Omega q_\phi\,d\Omega \approx x_P y_P(\Delta x+\frac{\Delta x^2}{2})\,(\Delta y+\frac{\Delta y^2}{2})$$

Summing up the fluxes and setting up the balance, we get:
Final Discretized Equation for UDS:

$$\left(\rho\phi_P\cdot\left(x_P+\frac{\Delta x}{2}\right)-\rho\phi_W\cdot\left(x_P-\frac{\Delta x}{2}\right)\right)\Delta y$$
$$+\left(-\rho\phi_N\cdot\left(y_P+\frac{\Delta y}{2}\right)+\rho\phi_P\cdot\left(y_P-\frac{\Delta y}{2}\right)\right)\Delta x$$
$$-\Gamma\left[\frac{\Delta y}{\Delta x}(\phi_E-2\phi_P+\phi_W)-\frac{\Delta x}{\Delta y}(\phi_N-2\phi_P+\phi_S)\right]$$
$$=x_P y_P\left(\Delta x+\frac{\Delta x^2}{2}\right)\left(\Delta y+\frac{\Delta y^2}{2}\right)$$

Final Discretized Equation for CDS:

$$\left(\rho\left(\frac{\phi_E+\phi_P}{2}\right)\cdot\left(x_P+\frac{\Delta x}{2}\right)-\rho\left(\frac{\phi_W+\phi_P}{2}\right)\cdot\left(x_P-\frac{\Delta x}{2}\right)\right)\Delta y$$
$$+\left(-\rho\left(\frac{\phi_N+\phi_P}{2}\right)\cdot\left(y_P+\frac{\Delta y}{2}\right)+\rho\left(\frac{\phi_S+\phi_P}{2}\right)\cdot\left(y_P-\frac{\Delta y}{2}\right)\right)\Delta x$$
$$-\Gamma\left[\frac{\Delta y}{\Delta x}(\phi_E-2\phi_P+\phi_W)-\frac{\Delta x}{\Delta y}(\phi_N-2\phi_P+\phi_S)\right]$$
$$=x_P y_P\left(\Delta x+\frac{\Delta x^2}{2}\right)\left(\Delta y+\frac{\Delta y^2}{2}\right)$$

This equation provides the value of ϕ for each interior cell based on convection and diffusion fluxes across its faces, with appropriate inclusion of the surface area ΔS for each face.

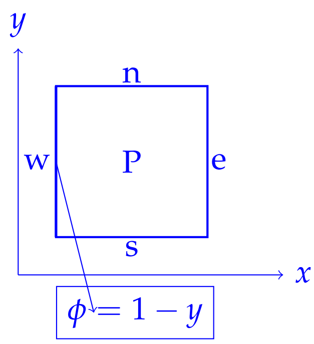
3. Boundary Conditions Implementation

The specific boundary conditions are incorporated directly into the discretized system:

- **Left Wall (**$x = 0$**, Dirichlet)**: $\phi(y) = 1 - y$,

  On the left wall (w), the value of ϕ_w can be directly expressed as:

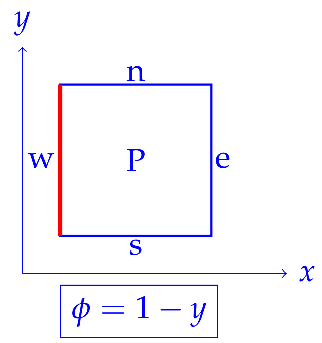
$$\phi_w = 1 - y$$



- **Right Wall (**$x = 1$**, Neumann)**: $\frac{\partial \phi}{\partial x} = 0$,
- **Top Wall (**$y = 1$**, Dirichlet)**: $\phi(x) = 0$,
- **Bottom Wall (**$y = 0$**, Mixed Conditions)**:
  - **Neumann (**$x \in [0, 0.2] \cup [0.8, 1]$**)**: $\frac{\partial \phi}{\partial y} = 0$,
  - **Flux Condition (**$x \in [0.2, 0.8]$**)**: $-\Gamma \frac{\partial \phi}{\partial y} = a$, where a is specified.

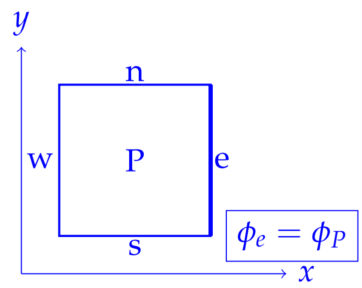
- **Left Wall ($x = 0$, Dirichlet)**: $\phi(y) = 1 - y$,
  On the left wall (w), the value of ϕ_w can be directly expressed as:

$$\phi_w = 1 - y$$



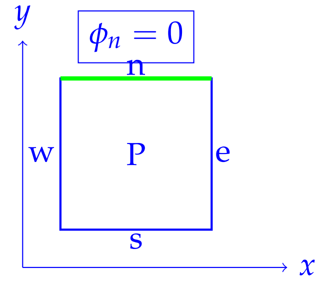
- **Right Wall ($x = 1$, Neumann)**: $\frac{\partial \phi}{\partial x} = 0$,
  On the right wall (e), the Neumann condition implies:

$$\frac{\phi_e - \phi_P}{\Delta x} = 0 \quad \Rightarrow \quad \phi_e = \phi_P$$



- **Top Wall ($y = 1$, Dirichlet)**: $\phi(x) = 0$,
  On the top wall (n), the value of ϕ_n can be directly expressed as:

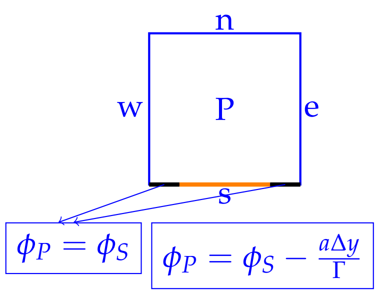
$$\phi_n = 0$$



- **Bottom Wall ($y = 0$, Mixed Conditions)**:
  - **Neumann ($x \in [0, 0.2] \cup [0.8, 1]$)**: $\frac{\partial \phi}{\partial y} = 0$,

    This implies:
$$\frac{\phi_P - \phi_S}{\Delta y} = 0 \quad \Rightarrow \quad \phi_P = \phi_S$$
  - **Flux Condition ($x \in [0.2, 0.8]$)**: $-\Gamma \frac{\partial \phi}{\partial y} = a$,

    This implies:
$$-\Gamma \frac{\phi_P - \phi_S}{\Delta y} = a \quad \Rightarrow \quad \phi_P = \phi_S - \frac{a \Delta y}{\Gamma}$$



Boundary Flux Calculations for the Diffusive Term

The diffusive flux term is given by:

$$\begin{aligned}\int \Gamma \nabla \phi \cdot \mathbf{n}\, ds &= \Gamma \int \left(\frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} \right) \cdot \mathbf{n}\, ds \\ &= \Gamma \left[\left(\left.\frac{\partial \phi}{\partial x}\right|_e \cdot \Delta y \right) - \left(\left.\frac{\partial \phi}{\partial x}\right|_w \cdot \Delta y \right) \right. \\ &\quad \left. + \left(-\left.\frac{\partial \phi}{\partial y}\right|_n \cdot \Delta x \right) + \left(\left.\frac{\partial \phi}{\partial y}\right|_s \cdot \Delta x \right) \right]\end{aligned}$$

$$= \Gamma \Delta y \left(\left.\frac{\partial \phi}{\partial x}\right|_e - \left.\frac{\partial \phi}{\partial x}\right|_w \right) + \Gamma \Delta x \left(\left.\frac{\partial \phi}{\partial y}\right|_n - \left.\frac{\partial \phi}{\partial y}\right|_s \right).$$

Boundary Conditions for the Diffusive Term. The boundary conditions are incorporated by specifying the gradients of ϕ or the value of ϕ at the boundaries.

Boundary Fluxes for Each Cell.

Left Cell ***(*** $x = 0$ ***).*** Dirichlet condition, $\phi_w = 1 - y_w$. The diffusive fluxes are:

- **East face:**

$$\Gamma \frac{\partial \phi}{\partial x}\bigg|_e = \Gamma \frac{\phi_E - \phi_P}{\Delta x}$$

- **West face:**

$$\Gamma \frac{\partial \phi}{\partial x}\bigg|_w = \Gamma \frac{\phi_P - (1 - y_w)}{\Delta x/2}$$

- **North face:**

$$\Gamma \frac{\partial \phi}{\partial y}\bigg|_n = \Gamma \frac{\phi_N - \phi_P}{\Delta y}$$

- **South face:**

$$\Gamma \frac{\partial \phi}{\partial y}\bigg|_s = \Gamma \frac{\phi_P - \phi_S}{\Delta y}$$

The total diffusive flux across the left cell:

$$\begin{aligned}
\int \Gamma \nabla \phi \cdot \mathbf{n}\, ds \approx \Gamma \Bigg[& \left(\frac{\phi_E - \phi_P}{\Delta x} \Delta y \right) - \left(\frac{\phi_P - (1-y)}{\Delta x/2} \Delta y \right) \\
& + \left(-\frac{\phi_N - \phi_P}{\Delta y} \Delta x \right) + \left(\frac{\phi_P - \phi_S}{\Delta y} \Delta x \right) \Bigg] \\
& = \Gamma \left[\frac{\Delta y}{\Delta x} (\phi_E - 3\phi_P + 2(1-y)) - \frac{\Delta x}{\Delta y} (\phi_N - 2\phi_P + \phi_S) \right]. \\
& = \Gamma \frac{\Delta y}{\Delta x} \phi_E - 3\Gamma \frac{\Delta y}{\Delta x} \phi_P + 2\Gamma \frac{\Delta y}{\Delta x} (1-y) - \Gamma \frac{\Delta x}{\Delta y} \phi_N + 2\Gamma \frac{\Delta x}{\Delta y} \phi_P - \Gamma \frac{\Delta x}{\Delta y} \phi_S.
\end{aligned}$$

Right Cell ***(*** $x = 1$ ***).*** Neumann condition, $\phi_e = \phi_P$. The diffusive fluxes are:

- **East face:**

$$\Gamma \frac{\partial \phi}{\partial x}\bigg|_e = 0$$

- **West face:**

$$\Gamma \frac{\partial \phi}{\partial x}\bigg|_w = \Gamma \frac{\phi_P - \phi_W}{\Delta x}$$

- **North face:**

$$\Gamma \frac{\partial \phi}{\partial y}\bigg|_n = \Gamma \frac{\phi_N - \phi_P}{\Delta y} \Delta x$$

- **South face:**

$$\Gamma \frac{\partial \phi}{\partial y}\bigg|_s = \Gamma \frac{\phi_P - \phi_S}{\Delta y} \Delta x$$

The total diffusive flux across the right cell:

$$\begin{aligned}\int \Gamma \nabla\phi \cdot \mathbf{n}\, ds &\approx \Gamma \left[0 - \left(\frac{\phi_P - \phi_W}{\Delta x} \Delta y \right) \right. \\ &\quad \left. + \left(-\frac{\phi_N - \phi_P}{\Delta y} \Delta x \right) + \left(\frac{\phi_P - \phi_S}{\Delta y} \Delta x \right) \right] \\ &= \Gamma \left[\frac{\Delta y}{\Delta x}(-\phi_P + \phi_W) - \frac{\Delta x}{\Delta y}(\phi_N - 2\phi_P + \phi_S) \right]. \\ &= -\Gamma \frac{\Delta y}{\Delta x}\phi_P + \Gamma \frac{\Delta y}{\Delta x}\phi_W - \Gamma \frac{\Delta x}{\Delta y}\phi_N + 2\Gamma \frac{\Delta x}{\Delta y}\phi_P - \Gamma \frac{\Delta x}{\Delta y}\phi_S. \end{aligned}$$

Top Cell **(**$y = 1$**).** Dirichlet condition, $\phi_n = 0$. The diffusive fluxes are:

- **East face:**

$$\Gamma \frac{\partial \phi}{\partial x}\bigg|_e = \Gamma \frac{\phi_E - \phi_P}{\Delta x} \Delta y$$

- **West face:**

$$\Gamma \frac{\partial \phi}{\partial x}\bigg|_w = \Gamma \frac{\phi_P - \phi_W}{\Delta x}$$

- **North face:**

$$\Gamma \frac{\partial \phi}{\partial y}\bigg|_n = -\Gamma \frac{\phi_P}{\Delta y/2}$$

- **South face:**

$$\Gamma \frac{\partial \phi}{\partial y}\bigg|_s = \Gamma \frac{\phi_P - \phi_S}{\Delta y} \Delta x$$

The total diffusive flux across the top cell:

$$\begin{aligned}\int \Gamma \nabla\phi \cdot \mathbf{n}\, ds &\approx \Gamma \left[\left(\frac{\phi_E - \phi_P}{\Delta x} \Delta y \right) - \left(\frac{\phi_P - \phi_W}{\Delta x} \Delta y \right) \right. \\ &\quad \left. + \left(\frac{\phi_P}{\Delta y/2} \Delta x \right) + \left(\frac{\phi_P - \phi_S}{\Delta y} \Delta x \right) \right] \\ &= \Gamma \left[\frac{\Delta y}{\Delta x}(\phi_E - 2\phi_P + \phi_W) - \frac{\Delta x}{\Delta y}(-3\phi_P + \phi_S) \right]. \\ &= \Gamma \frac{\Delta y}{\Delta x}\phi_E - 2\Gamma \frac{\Delta y}{\Delta x}\phi_P + \Gamma \frac{\Delta y}{\Delta x}\phi_W + 3\Gamma \frac{\Delta x}{\Delta y}\phi_P - \Gamma \frac{\Delta x}{\Delta y}\phi_S. \end{aligned}$$

Bottom Cell ***(**$y = 0$**).*** Mixed conditions:

- **Neumann (**$x \in [0, 0.2] \cup [0.8, 1]$**):**

$$\phi_s = \phi_P, \quad \Gamma \frac{\partial \phi}{\partial y}\bigg|_s = 0$$

$$\begin{aligned}
\int \Gamma \nabla \phi \cdot \mathbf{n}\, ds \approx \Gamma \Bigg[& \left(\frac{\phi_E - \phi_P}{\Delta x} \Delta y \right) - \left(\frac{\phi_P - \phi_W}{\Delta x} \Delta y \right) \\
& + \left(- \frac{\phi_N - \phi_P}{\Delta y} \Delta x \right) \Bigg] \\
= \Gamma & \left[\frac{\Delta y}{\Delta x} (\phi_E - 2\phi_P + \phi_W) - \frac{\Delta x}{\Delta y} (\phi_N - \phi_P) \right]. \\
= \Gamma & \frac{\Delta y}{\Delta x} \phi_E - 2\Gamma \frac{\Delta y}{\Delta x} \phi_P + \Gamma \frac{\Delta y}{\Delta x} \phi_W + \Gamma \frac{\Delta x}{\Delta y} \phi_P - \Gamma \frac{\Delta x}{\Delta y} \phi_N.
\end{aligned}$$

- **Flux Condition (**$x \in [0.2, 0.8]$**):**

$$\phi_s = \phi_P - \frac{a \Delta y}{\Gamma}, \quad \Gamma \frac{\partial \phi}{\partial y}\bigg|_s = -a$$

$$\begin{aligned}
\int \Gamma \nabla \phi \cdot \mathbf{n}\, ds \approx \Gamma \Bigg[& \left(\frac{\phi_E - \phi_P}{\Delta x} \Delta y \right) - \left(\frac{\phi_P - \phi_W}{\Delta x} \Delta y \right) \\
& - \left(\frac{\phi_N - \phi_P}{\Delta y} \Delta x \right) + \left(\frac{\phi_P - \phi_S}{\Delta y} \Delta x \right) \Bigg] \\
= \Gamma & \left[\frac{\Delta y}{\Delta x} (\phi_E - 2\phi_P + \phi_W) - \frac{\Delta x}{\Delta y} (\phi_N - \phi_P) \right] - a\Delta x. \\
= \Gamma & \frac{\Delta y}{\Delta x} \phi_E - 2\Gamma \frac{\Delta y}{\Delta x} \phi_P + \Gamma \frac{\Delta y}{\Delta x} \phi_W + \Gamma \frac{\Delta x}{\Delta y} \phi_P - \Gamma \frac{\Delta x}{\Delta y} \phi_N - a\Delta x.
\end{aligned}$$

Boundary Flux Calculations for Convection Term

The convection term is expressed as:

$$\sum_{\text{faces}} \rho\phi_f \, \mathbf{v}_f \cdot \mathbf{n}_f \, \Delta S = (\rho\phi_e \cdot x_e - \rho\phi_w \cdot x_w)\Delta y + (-\rho\phi_n \cdot y_n + \rho\phi_s \cdot y_s)\Delta x$$

The fluxes $\phi_e, \phi_w, \phi_n, \phi_s$ are located on the faces. The boundary fluxes defined by the boundary conditions remain unchanged, while internal fluxes are computed using either the **Upwind Differencing Scheme (UDS)** or **Central Differencing Scheme (CDS)**.

Face Locations. The positions of the faces relative to the cell center are:

- **East face:** $x_e = x_P + \frac{\Delta x}{2}$
- **West face:** $x_w = x_P - \frac{\Delta x}{2}$
- **North face:** $y_n = y_P + \frac{\Delta y}{2}$
- **South face:** $y_s = y_P - \frac{\Delta y}{2}$

Left Boundary ($x = 0$): Dirichlet Condition ($\phi_w = 1 - y$). - Prescribed Flux:

$$\phi_w = 1 - y$$

- UDS for Internal Faces:

$$\phi_e = \phi_P, \quad \phi_n = \phi_N, \quad \phi_s = \phi_P$$

The convection term becomes:

$$\sum_{\text{faces}} \rho\phi_f \, \mathbf{v}_f \cdot \mathbf{n}_f \, \Delta S = \rho\Delta y(\phi_P \cdot x_e - (1 - y) \cdot x_w) + \rho\Delta x(\phi_P \cdot y_s - \phi_N \cdot y_n)$$

- CDS for Internal Faces:

$$\phi_e = \frac{\phi_E + \phi_P}{2}, \quad \phi_n = \frac{\phi_N + \phi_P}{2}, \quad \phi_s = \frac{\phi_S + \phi_P}{2}$$

The convection term becomes:

$$\sum_{\text{faces}} \rho\phi_f \, \mathbf{v}_f \cdot \mathbf{n}_f \, \Delta S = \rho\Delta y \left(\frac{\phi_E + \phi_P}{2} \cdot x_e - (1 - y) \cdot x_w \right) + \rho\Delta x \left(\frac{\phi_S + \phi_P}{2} \cdot y_s - \frac{\phi_N + \phi_P}{2} \cdot y_n \right)$$

Right Boundary ($x = 1$): Neumann Condition ($\frac{\partial \phi}{\partial x} = 0 \rightarrow \phi_e = \phi_P$). - Prescribed Flux:

$$\phi_e = \phi_P$$

- UDS for Internal Faces:

$$\phi_w = \phi_W, \quad \phi_n = \phi_N, \quad \phi_s = \phi_P$$

The convection term becomes:

$$\sum_{\text{faces}} \rho \phi_f \, \mathbf{v}_f \cdot \mathbf{n}_f \, \Delta S = \rho \Delta y (\phi_P \cdot x_e - \phi_W \cdot x_w) + \rho \Delta x (\phi_P \cdot y_s - \phi_N \cdot y_n)$$

- CDS for Internal Faces:

$$\phi_w = \frac{\phi_W + \phi_P}{2}, \quad \phi_n = \frac{\phi_N + \phi_P}{2}, \quad \phi_s = \frac{\phi_S + \phi_P}{2}$$

The convection term becomes:

$$\sum_{\text{faces}} \rho \phi_f \, \mathbf{v}_f \cdot \mathbf{n}_f \, \Delta S = \rho \Delta y \left(\phi_P \cdot x_e - \frac{\phi_W + \phi_P}{2} \cdot x_w \right) + \rho \Delta x \left(\frac{\phi_S + \phi_P}{2} \cdot y_s - \frac{\phi_N + \phi_P}{2} \cdot y_n \right)$$

Top Boundary ($y = 1$): Dirichlet Condition ($\phi_n = 0$). - Prescribed Flux:

$$\phi_n = 0$$

- UDS for Internal Faces:

$$\phi_e = \phi_E, \quad \phi_w = \phi_W, \quad \phi_s = \phi_P$$

The convection term becomes:

$$\sum_{\text{faces}} \rho \phi_f \, \mathbf{v}_f \cdot \mathbf{n}_f \, \Delta S = \rho \Delta y (\phi_E \cdot x_e - \phi_W \cdot x_w) + \rho \Delta x (\phi_P \cdot y_s - 0 \cdot y_n)$$

- CDS for Internal Faces:

$$\phi_e = \frac{\phi_E + \phi_P}{2}, \quad \phi_w = \frac{\phi_W + \phi_P}{2}, \quad \phi_s = \frac{\phi_S + \phi_P}{2}$$

The convection term becomes:

$$\sum_{\text{faces}} \rho \phi_f \, \mathbf{v}_f \cdot \mathbf{n}_f \, \Delta S = \rho \Delta y \left(\frac{\phi_E + \phi_P}{2} \cdot x_e - \frac{\phi_W + \phi_P}{2} \cdot x_w \right) + \rho \Delta x \left(\frac{\phi_S + \phi_P}{2} \cdot y_s - 0 \cdot y_n \right)$$

Bottom Boundary ($y = 0$): Mixed Conditions. 1. **Neumann Condition ($x \in [0, 0.2] \cup [0.8, 1]$): - Prescribed Flux:**

$$\phi_s = \phi_P$$

- UDS for Internal Faces:

$$\phi_e = \phi_P, \quad \phi_w = \phi_W, \quad \phi_n = \phi_N$$

The convection term becomes:

$$\sum_{\text{faces}} \rho \phi_f \mathbf{v}_f \cdot \mathbf{n}_f \, \Delta S = \rho \Delta y (\phi_P \cdot x_e - \phi_W \cdot x_w) + \rho \Delta x (\phi_P \cdot y_s - \phi_N \cdot y_n)$$

- CDS for Internal Faces:

$$\phi_e = \frac{\phi_E + \phi_P}{2}, \quad \phi_w = \frac{\phi_W + \phi_P}{2}, \quad \phi_n = \frac{\phi_N + \phi_P}{2}$$

The convection term becomes:

$$\sum_{\text{faces}} \rho \phi_f \mathbf{v}_f \cdot \mathbf{n}_f \, \Delta S = \rho \Delta y \left(\frac{\phi_E + \phi_P}{2} \cdot x_e - \frac{\phi_W + \phi_P}{2} \cdot x_w \right) + \rho \Delta x \left(\phi_P \cdot y_s - \frac{\phi_N + \phi_P}{2} \cdot y_n \right)$$

2. **Flux Condition ($x \in [0.2, 0.8]$): - Prescribed Flux:**

$$\phi_s = \phi_P - \frac{a \Delta y}{\Gamma}$$

- UDS for Internal Faces:

$$\phi_e = \phi_E, \quad \phi_w = \phi_W, \quad \phi_n = \phi_N$$

The convection term becomes:

$$\sum_{\text{faces}} \rho \phi_f \mathbf{v}_f \cdot \mathbf{n}_f \, \Delta S = \rho \Delta y (\phi_E \cdot x_e - \phi_W \cdot x_w) + \rho \Delta x \left(\left(\phi_P - \frac{a \Delta y}{\Gamma} \right) \cdot y_s - \phi_N \cdot y_n \right)$$

- CDS for Internal Faces:

$$\phi_e = \frac{\phi_E + \phi_P}{2}, \quad \phi_w = \frac{\phi_W + \phi_P}{2}, \quad \phi_n = \frac{\phi_N + \phi_P}{2}$$

The convection term becomes:

$$\sum_{\text{faces}} \rho \phi_f \mathbf{v}_f \cdot \mathbf{n}_f \, \Delta S = \rho \Delta y \left(\frac{\phi_E + \phi_P}{2} \cdot x_e - \frac{\phi_W + \phi_P}{2} \cdot x_w \right) + \rho \Delta x \left(\left(\phi_P - \frac{a \Delta y}{\Gamma} \right) \cdot y_s - \frac{\phi_N + \phi_P}{2}$$

4. Final Coefficents:

Left Boundary:
Final Discretized Equation for UDS:

$$\left(\rho\phi_P \cdot \left(x_P + \frac{\Delta x}{2}\right) - \rho\phi_W \cdot \left(x_P - \frac{\Delta x}{2}\right)\right)\Delta y$$
$$+\left(-\rho\phi_N \cdot \left(y_P + \frac{\Delta y}{2}\right) + \rho\phi_P \cdot \left(y_P - \frac{\Delta y}{2}\right)\right)\Delta x$$
$$-\left(\Gamma\frac{\Delta y}{\Delta x}\phi_E - 3\Gamma\frac{\Delta y}{\Delta x}\phi_P + 2\Gamma\frac{\Delta y}{\Delta x}(1-y) - \Gamma\frac{\Delta x}{\Delta y}\phi_N + 2\Gamma\frac{\Delta x}{\Delta y}\phi_P - \Gamma\frac{\Delta x}{\Delta y}\phi_S\right)$$
$$= x_P y_P \left(\Delta x + \frac{\Delta x^2}{2}\right)\left(\Delta y + \frac{\Delta y^2}{2}\right)$$

Final Discretized Equation for CDS:

$$\left(\rho\left(\frac{\phi_E + \phi_P}{2}\right) \cdot \left(x_P + \frac{\Delta x}{2}\right) - \rho\left(\frac{\phi_W + \phi_P}{2}\right) \cdot \left(x_P - \frac{\Delta x}{2}\right)\right)\Delta y$$
$$+\left(-\rho\left(\frac{\phi_N + \phi_P}{2}\right) \cdot \left(y_P + \frac{\Delta y}{2}\right) + \rho\left(\frac{\phi_S + \phi_P}{2}\right) \cdot \left(y_P - \frac{\Delta y}{2}\right)\right)\Delta x$$
$$-\left(\Gamma\frac{\Delta y}{\Delta x}\phi_E - 3\Gamma\frac{\Delta y}{\Delta x}\phi_P + 2\Gamma\frac{\Delta y}{\Delta x}(1-y) - \Gamma\frac{\Delta x}{\Delta y}\phi_N + 2\Gamma\frac{\Delta x}{\Delta y}\phi_P - \Gamma\frac{\Delta x}{\Delta y}\phi_S\right)$$
$$= x_P y_P \left(\Delta x + \frac{\Delta x^2}{2}\right)\left(\Delta y + \frac{\Delta y^2}{2}\right)$$

Right Boundary:
Final Discretized Equation for UDS:

$$\left(\rho\phi_P \cdot \left(x_P + \frac{\Delta x}{2}\right) - \rho\phi_W \cdot \left(x_P - \frac{\Delta x}{2}\right)\right)\Delta y$$
$$+\left(-\rho\phi_N \cdot \left(y_P + \frac{\Delta y}{2}\right) + \rho\phi_P \cdot \left(y_P - \frac{\Delta y}{2}\right)\right)\Delta x$$
$$-\left(\Gamma\frac{\Delta y}{\Delta x}\phi_P + \Gamma\frac{\Delta y}{\Delta x}\phi_W - \Gamma\frac{\Delta x}{\Delta y}\phi_N + 2\Gamma\frac{\Delta x}{\Delta y}\phi_P - \Gamma\frac{\Delta x}{\Delta y}\phi_S\right)$$
$$= x_P y_P \left(\Delta x + \frac{\Delta x^2}{2}\right)\left(\Delta y + \frac{\Delta y^2}{2}\right)$$

Final Discretized Equation for CDS:

$$\left(\rho\left(\frac{\phi_E + \phi_P}{2}\right) \cdot \left(x_P + \frac{\Delta x}{2}\right) - \rho\left(\frac{\phi_W + \phi_P}{2}\right) \cdot \left(x_P - \frac{\Delta x}{2}\right)\right)\Delta y$$
$$+\left(-\rho\left(\frac{\phi_N + \phi_P}{2}\right) \cdot \left(y_P + \frac{\Delta y}{2}\right) + \rho\left(\frac{\phi_S + \phi_P}{2}\right) \cdot \left(y_P - \frac{\Delta y}{2}\right)\right)\Delta x$$
$$-\left(\Gamma\frac{\Delta y}{\Delta x}\phi_P + \Gamma\frac{\Delta y}{\Delta x}\phi_W - \Gamma\frac{\Delta x}{\Delta y}\phi_N + 2\Gamma\frac{\Delta x}{\Delta y}\phi_P - \Gamma\frac{\Delta x}{\Delta y}\phi_S\right)$$
$$= x_P y_P \left(\Delta x + \frac{\Delta x^2}{2}\right)\left(\Delta y + \frac{\Delta y^2}{2}\right)$$

Bottom Boundary:
Final Discretized Equation for UDS:

$$\begin{aligned}
&\left(\rho\phi_P \cdot \left(x_P + \frac{\Delta x}{2}\right) - \rho\phi_W \cdot \left(x_P - \frac{\Delta x}{2}\right)\right)\Delta y \\
&\quad + \left(-\rho\phi_N \cdot \left(y_P + \frac{\Delta y}{2}\right) + \rho\phi_P \cdot \left(y_P - \frac{\Delta y}{2}\right)\right)\Delta x \\
&\quad - \left(\Gamma\frac{\Delta y}{\Delta x}\phi_E - 2\Gamma\frac{\Delta y}{\Delta x}\phi_P + \Gamma\frac{\Delta y}{\Delta x}\phi_W + \Gamma\frac{\Delta x}{\Delta y}\phi_P - \Gamma\frac{\Delta x}{\Delta y}\phi_N\right) \\
&= x_P y_P \left(\Delta x + \frac{\Delta x^2}{2}\right)\left(\Delta y + \frac{\Delta y^2}{2}\right)
\end{aligned}$$

Final Discretized Equation for CDS:

$$\begin{aligned}
&\left(\rho\left(\frac{\phi_E + \phi_P}{2}\right) \cdot \left(x_P + \frac{\Delta x}{2}\right) - \rho\left(\frac{\phi_W + \phi_P}{2}\right) \cdot \left(x_P - \frac{\Delta x}{2}\right)\right)\Delta y \\
&\quad + \left(-\rho\left(\frac{\phi_N + \phi_P}{2}\right) \cdot \left(y_P + \frac{\Delta y}{2}\right) + \rho\left(\frac{\phi_S + \phi_P}{2}\right) \cdot \left(y_P - \frac{\Delta y}{2}\right)\right)\Delta x \\
&\quad - \left(\Gamma\frac{\Delta y}{\Delta x}\phi_E - 2\Gamma\frac{\Delta y}{\Delta x}\phi_P + \Gamma\frac{\Delta y}{\Delta x}\phi_W + \Gamma\frac{\Delta x}{\Delta y}\phi_P - \Gamma\frac{\Delta x}{\Delta y}\phi_N\right) \\
&= x_P y_P \left(\Delta x + \frac{\Delta x^2}{2}\right)\left(\Delta y + \frac{\Delta y^2}{2}\right)
\end{aligned}$$

Top Boundary:
Final Discretized Equation for UDS:

$$\begin{aligned}
&\left(\rho\phi_P \cdot \left(x_P + \frac{\Delta x}{2}\right) - \rho\phi_W \cdot \left(x_P - \frac{\Delta x}{2}\right)\right)\Delta y \\
&\quad + \left(-\rho\phi_N \cdot \left(y_P + \frac{\Delta y}{2}\right) + \rho\phi_P \cdot \left(y_P - \frac{\Delta y}{2}\right)\right)\Delta x \\
&\quad - \left(\Gamma\frac{\Delta y}{\Delta x}\phi_E - 2\Gamma\frac{\Delta y}{\Delta x}\phi_P + \Gamma\frac{\Delta y}{\Delta x}\phi_W + 3\Gamma\frac{\Delta x}{\Delta y}\phi_P - \Gamma\frac{\Delta x}{\Delta y}\phi_S\right) \\
&= x_P y_P \left(\Delta x + \frac{\Delta x^2}{2}\right)\left(\Delta y + \frac{\Delta y^2}{2}\right)
\end{aligned}$$

Final Discretized Equation for CDS:

$$\begin{aligned}
&\left(\rho\left(\frac{\phi_E + \phi_P}{2}\right) \cdot \left(x_P + \frac{\Delta x}{2}\right) - \rho\left(\frac{\phi_W + \phi_P}{2}\right) \cdot \left(x_P - \frac{\Delta x}{2}\right)\right)\Delta y \\
&\quad + \left(-\rho\left(\frac{\phi_N + \phi_P}{2}\right) \cdot \left(y_P + \frac{\Delta y}{2}\right) + \rho\left(\frac{\phi_S + \phi_P}{2}\right) \cdot \left(y_P - \frac{\Delta y}{2}\right)\right)\Delta x \\
&\quad - \left(\Gamma\frac{\Delta y}{\Delta x}\phi_E - 2\Gamma\frac{\Delta y}{\Delta x}\phi_P + \Gamma\frac{\Delta y}{\Delta x}\phi_W + 3\Gamma\frac{\Delta x}{\Delta y}\phi_P - \Gamma\frac{\Delta x}{\Delta y}\phi_S\right) \\
&= x_P y_P \left(\Delta x + \frac{\Delta x^2}{2}\right)\left(\Delta y + \frac{\Delta y^2}{2}\right)
\end{aligned}$$

Discretized Equations for Boundary Cells. The discretization process will use a finite-volume approach, with each boundary condition integrated individually:
a) Left Wall ($x = 0$, Dirichlet Condition). For cells along the left boundary, set:

$$\phi(i, j) = 1 - j \cdot \Delta y$$

This Dirichlet boundary condition directly assigns values to cells along $x = 0$ without needing further flux calculations.
b) Right Wall ($x = 1$, Neumann Condition). For cells along $x = 1$, use a zero-gradient condition:

$$\frac{\phi(N_x, j) - \phi(N_x - 1, j)}{\Delta x} = 0 \Rightarrow \phi(N_x, j) = \phi(N_x - 1, j)$$

This enforces $\frac{\partial \phi}{\partial x} = 0$ by setting the boundary cell value equal to its adjacent interior cell.
c) Top Wall ($y = 1$, Dirichlet Condition). For cells along the top wall, set:

$$\phi(i, N_y) = 0$$

This assigns values directly to ensure $\phi = 0$ along $y = 1$.
d) Bottom Wall ($y = 0$, Mixed Neumann and Flux Condition).

- **For $x \in [0, 0.2] \cup [0.8, 1]$ (Neumann condition)**:

$$\frac{\phi(i, 1) - \phi(i, 0)}{\Delta y} = 0 \Rightarrow \phi(i, 0) = \phi(i, 1)$$

- **For $x \in [0.2, 0.8]$ (Flux condition)**:

$$-\Gamma \frac{\phi(i, 1) - \phi(i, 0)}{\Delta y} = a \Rightarrow \phi(i, 0) = \phi(i, 1) + \frac{a \cdot \Delta y}{\Gamma}$$

These equations set up the values of ϕ at each boundary cell based on the given boundary conditions, ensuring consistent application across the domain's edges.

5. Matrix Formulation

Once the equation is discretized, it can be rearranged into a linear system $A\phi = b$, where:

- A is a sparse matrix encoding coefficients from convection, diffusion, and boundary terms,
- ϕ is the vector of unknowns,
- b includes source terms and contributions from boundary conditions.

6. Solution Procedure and Analysis

i. **Grid Independence Study**: Refine the mesh and evaluate the average flux on the left boundary. Plot the flux against grid size to assess convergence for both UDS and CDS schemes.

ii. **Exact Flux using Richardson Extrapolation**: Apply Richardson extrapolation to improve accuracy in the computed flux values.