Dimensionality Reduction

Question 1: Note: In this question, all columns will be written in their transposed form, as rows, to make the typography simpler. Matrix M has three rows and three columns, and the columns form an orthonormal basis. One of the columns is [2/7,3/7,6/7], and another is [6/7, 2/7, -3/7]. Let the third column be [x,y,z]. Since the length of the vector [x,y,z] must be 1, there is a constraint that $x^2+y^2+z^2 = 1$. However, there are other constraints, and these other constraints can be used to deduce facts about the ratios among x, y, and z. Compute these ratios.

1 ans. let C_1 be $[2/7, 3/7, 6/7]$, C_2 be $[6/7, 2/7, -3/7]$ and C_3 be $[x, y, z]$

The dot Product of any two columns must be zero

$C_1 . C_2 = (2/7 * 6/7) + (3/7 * 2/7) + (6/7 * -3/7) = 0$

$C_2 . C_3 = (6/7 * x) + (2/7 * y) + (-3/7 * z) = 0$

$6x + 2y - 3z = 0 \rightarrow Eq(1).$

$C_3 . C_1 = (x * 2/7) + (y * 3/7) + (z * 6/7) = 0$

$2x + 3y + 6z = 0 \rightarrow Eq(2).$

$2 * Eq1 + Eq2 \rightarrow 12x + 4y - 6z + 2x + 3y + 6z = 0.$

$14x + 7y = 0 \rightarrow y = -2x$

$3 * Eq2 - Eq1 \rightarrow 6x + 9y + 18z - 6x - 2y + 3z = 0.$

$7y + 21z = 0 \Rightarrow y = -3z.$

$x : y : z = -2 : 1 : 3$

Question 2: Find the eigenvalues and eigenvectors of the following matrix:

2	3
3	10

You should assume the first component of an eigenvector is 1. Then, find out One eigenvalue and One eigenvector.

(2) ans

let the given matrix be $A = \begin{matrix} 2 & 3 \\ 3 & 10 \end{matrix}$ and the eigen vector be of the form $\begin{matrix} 1 \\ e \end{matrix}$

$Ax = \lambda x \rightarrow \begin{matrix} 2 & 3 \\ 3 & 10 \end{matrix} * \begin{matrix} 1 \\ e \end{matrix} = \lambda * \begin{matrix} 1 \\ e \end{matrix} \rightarrow 2+3e = \lambda,\ 3+10e = \lambda e$

$\Rightarrow 3 + 10e = (2+3e)e$

$3e^2 - 8e + 3 = 0 \rightarrow e = 3, -1/3$

The eigen vectors are $\begin{matrix} 1 \\ 3 \end{matrix}$ and $\begin{matrix} 1 \\ -1/3 \end{matrix}$.

The eigen values are $2+3e = \lambda \rightarrow \lambda = 2+3*3$

$\lambda = 2+3(-1/3) = 1$

Question 3: Suppose [1,3,4,5,7] is an eigenvector of some matrix. What is the unit eigenvector in the same direction? Find out the components of the unit eigenvector.

(3) ans. Given eigen vector of some matrix be M = [1, 3, 4, 5, 7]

To get the unit eigen vector of given matrix.

Sum of the squares $= 1^2+3^2+4^2+5^2+7^2 = 100$. and its square root is 10.

Unit Eigen vector = [1/10, 3/10, 4/10, 5/10, 7/10]

Question 4: Suppose we have three points in a two dimensional space: (1,1), (2,2), and (3,4). We want to perform PCA on these points, so we construct a 2-by-2 matrix, call it N, whose eigenvectors are the directions that best represent these three points. Construct the matrix N and identify, its elements.

(4) ans.

The given three Point in a 2-D space are (1,1), (2,2), (3,4)
we should construct a matrix whose rows correspond to Points and colums respond to dimensions of the space.

Then the matrix will be $M = \begin{matrix} 1 & 1 \\ 2 & 2 \\ 3 & 4 \end{matrix}$

$$M^T M = \begin{matrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{matrix} * \begin{matrix} 1 & 1 \\ 2 & 2 \\ 3 & 4 \end{matrix} = \begin{matrix} 14 & 17 \\ 17 & 21 \end{matrix}.$$

Question 5: Consider the diagonal matrix M =

1	0	0
0	2	0
0	0	0

Compute its Moore-Penrose pseudoinverse.

(5) ans Moore-Penrose Pseudoinverse means the matrix having diagonal elements replaced by 1 and divided by corresponding elements of given matrix and other elements will be zero

Moore-Penrose Pseudoinverse of given matrix is

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{matrix}$$

Question 6: When we perform a CUR dcomposition of a matrix, we select rows and columns by using a particular probability distribution for the rows and another for the columns. Here is a matrix that we wish to decompose:

1	2	3
4	5	6
7	8	9
10	11	12

Calculate the probability distribution for the rows.

(6) ans

Probability with which we choose row

$$= \frac{\text{Sum of squares of element in the rows}}{\text{Sum of squares of elements in the matrix}}$$

Sum of squares of elements in matrix

$$= 12 * 13 * 25/6 = 3900/6 = 650$$

$$P(R_1) = \frac{1^2+2^2+3^2}{650} = 14/650 = 0.02 \qquad P(R_2) = \frac{4^2+5^2+6^2}{650} = 77/650 = 0.12$$

$$P(R_3) = \frac{7^2+8^2+9^2}{650} = 194/650 = 0.298 \qquad P(R_4) = \frac{10^2+11^2+12^2}{650} = \frac{365}{650} = 0.56$$