Question 1 – Unfair Coin

This problem was asked by Facebook.

There is a fair coin (one side heads, one side tails) and an unfair coin (both sides tails). You pick one at random, flip it 5 times, and observe that it comes up as tails all five times. What is the chance that you are flipping the unfair coin?

Solution:

This problem can be solved using Bayes Rule. We are asked to calculate the following quantity: $P(\text{Unfair} \mid \text{TTTTT})$.

We have that

$$
\begin{aligned}
P(\text{Unfair} \mid \text{TTTTT}) &= \frac{P(\text{TTTTT} \mid \text{Unfair}) \cdot P(\text{Unfair})}{P(\text{TTTTT})} \\
&= \frac{P(\text{TTTTT} \mid \text{Unfair}) \cdot P(\text{Unfair})}{P(\text{TTTTT} \mid \text{Unfair}) \cdot P(\text{Unfair}) + P(\text{TTTTT} \mid \text{Fair}) \cdot P(\text{Fair})} \\
&= \frac{1^5 \cdot 0.5}{1^5 \cdot 0.5 + 0.5^5 \cdot 0.5} \\
&= \frac{0.5}{0.5 + 0.015625} \\
&= \boxed{0.9697}
\end{aligned}
$$

Question 2 – Flips until two heads

This problem was asked by Lyft.

What is the expected number of coin flips needed to get two consecutive heads?

Solution:

This is a slighly more complicated version of the classic problem – expected number of coin flips needed to get heads (which is simply the expected value of the geometric random variable with $p = 0.5$)

We can represent this sytem as a Markov chain as follows:

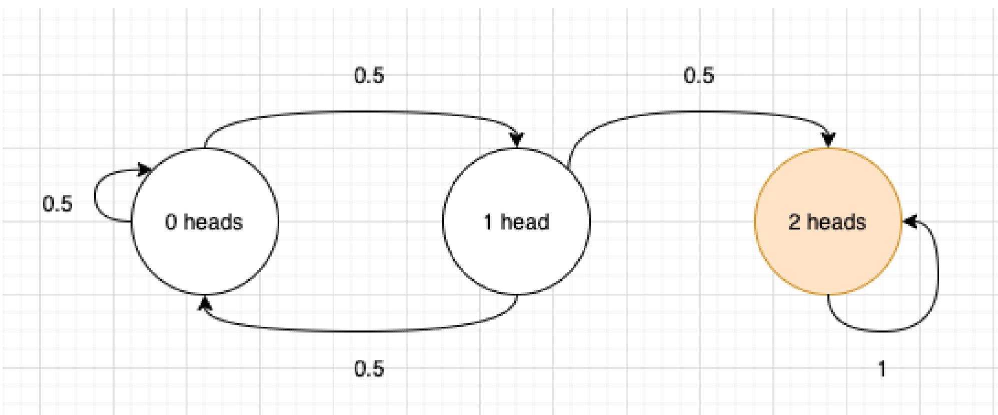


Figure 1: We start with 0 heads. With 0.5 probability, we see 1 head. After we have seen 1 head, if we see another head (which happens with $p = 0.5$), we are in the absorbing state 2. If we see a tails ($p = 0.5$), we go back to state 0.

If we are in state 2, the expected time to see two heads is $E[2] = 0$ as we have already seen two heads.

If we are in state 1, the expected time to see two heads is given by (using the law of total expectation):

$$E[1] = 1 + \frac{1}{2} \cdot E[0] + 12 \cdot E[2] \tag{1}$$
$$= 1 + \frac{1}{2} \cdot E[0] \tag{2}$$

If we are in state 0, the expected time to see two heads is given by:

$$E[0] = 1 + \frac{1}{2} \cdot E[0] + \frac{1}{2} \cdot E[1] \tag{3}$$

Substituting (2) in (3), we have:

$$E[0] = 1 + \frac{1}{2} \cdot E[0] + \frac{1}{2} \cdot \left(1 + \frac{1}{2} \cdot E[0]\right) \tag{4}$$

$$E[0] = \boxed{6} \tag{5}$$

Thus, if we are in state 0, that is when we start the experiment, the expected number of flips to see 2 heads is 6.

Question 3 – Drawing normally

This problem was asked by Quora.

You are drawing from a normally distributed random variable $X \sim \mathcal{N}(0, 1)$ once a day. What is the approximate expected number of days until you get a value of more than 2?

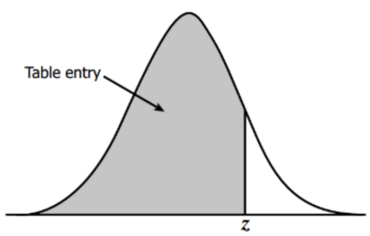
Solution

We can look at this problem as follows. Each day we carry out an experiment in which we draw from a standard unit normal. If the value sampled is greater than 2, then the experiment is successful. We want to know the average number of days in which we can expect a success.

The second part of the problem – average number of days in which we can expect a success sounds like a geometric random variable with parameter $Y \sim \text{Geom}(p)$ All we need to do is to find the value of parameter p and then the mean of the geometric random variable is $\frac{1}{p}$ which will give us the average number of days until we see the first "success" as we have defined it.

Now, to find p, consider the experiment that we perform every day. For it to be a "success", i.e., have a value more extreme than 2, it needs to be 2 standard deviations above the mean. In other words, the $z-$score is 2.0. We can look up the value for a $z-$score of 2.0, (from a $z-$table such as 2) which will give us the probability of getting a value less than 2.0. We can then subtract this from 1 to give us the probability of getting a value more extreme than 2.0.

The value here is 0.9772. Thus the probability we want is $1 - 0.9772 \approx 0.0228$. Thus, we have that $Y \sim \text{Geom}(0.0228)$. Thus, the average number of days that we have to wait are $\frac{1}{0.0228} \approx \boxed{43.86.}$



Find values on the right of the mean in this z-table. Table entries for z represent the area under the bell curve to the left of z. Positive scores in the Z-table correspond to the values which are greater than the mean.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

Figure 2: We need to find the value for $z-$score equal to 2.0. Figure is from here.