Computing signature from the Bunch-Kaufman factorization

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Let $A \in \mathbb{R}^{n\times n}$ be a symmetric real matrix. Our goal is to compute the **signature of** A[1], which is the tuple $(\sigma_1, \dots, \sigma_n)$, $\sigma_i \in \{-1, 0, +1\}$ of the signs of the eigenvalues of A, given a factorization of it.

The case of the classical $LDL^\top$ factorization with L unit-triangular and D a diagonal matrix is easily treated. Sylvester's law of inertia leads to A and D having the same signatures. Furthermore, the signature of D can be obtained by direct inspection as

$$\sigma = (\operatorname{sign} d_1, \dots, \operatorname{sign} d_n), D = \begin{bmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{bmatrix}. \tag{1}$$

Now, the Bunch-Kaufman factorization of the real symmetric matrix A is given by

$$A = LBL^\top \tag{2}$$

where L is unit-triangular, and B is a real symmetric block-diagonal matrix with p blocks which are either 1×1 or 2×2:

$$B = \begin{bmatrix} B_1 & & \\ & \ddots & \\ & & B_p \end{bmatrix} \tag{3}$$

Once again, by Sylvester's law of inertia, the signatures of A and the block-diagonal matrix B are the same, thus we only need the signature of B.

Denote $\mathcal{I} = \{1 \leq i \leq p : B_i \in \mathbb{R}^{1\times 1}\}$. The **characteristic polynomial** of B factorizes easily due to its structure:

$$\chi_B(X) \stackrel{\text{def}}{=} \det(XI_n - B) = \prod_{i=1}^{p} \chi_{B_i}(X) = \prod_{i\in\mathcal{I}} (X - B_i) \prod_{i\notin\mathcal{I}} \chi_{B_i}(X) \tag{4}$$

[1] `https://en.wikipedia.org/wiki/Metric_signature`

The case of the 1×1 submatrices is trivial – their signatures are given by the sign of their sole element. The B_i, $i \notin \mathcal{I}$, are 2×2 matrices and thus their characteristic polynomials are given by

$$\chi_{B_i}(X) = X^2 - \mathrm{tr}(B_i)X + \det(B_i). \tag{5}$$

By spectral theorem applies to B_i (which is real symmetric), this polynomial has two real roots $\lambda_{1,2}$. Moreover,

$$\mathrm{tr}\, B_i = \lambda_1 + \lambda_2,\ \det(B_i) = \lambda_1 \lambda_2. \tag{6}$$

Computing the signature can thus be done by disjunction:

- if $\det(B_i) = 0$, then say $\lambda_1 = 0$ and $\lambda_2 = \mathrm{tr}\, B_i$ (without loss of generality) and the signature of B_i is $(\mathrm{sign}(\mathrm{tr}\, B_i), 0)$;
- if $\det(B_i) > 0$, then both eigenvalues have the same (nonzero) sign, which will be the sign of $\mathrm{tr}\, B_i$;
- if $\det(B_i) < 0$, then the eigenvalues have opposite (nonzero) signs, the signature will be $(-, +)$.