

## Computer Problem Set 9 Two-factor gaussian yield curve

Let  $B = (B^1, B^2)$  be a Brownian motion in  $\mathbb{R}^2$  under the risk-neutral measure  $\mathbb{Q}$ , and  $x_i, \lambda_i, \theta_i, \sigma_i$  be some positive parameters. We assume that the instantaneous interest rate is defined by

$$r_t := X_t^{(1)} + X_t^{(2)} \text{ where } dX_t^{(i)} = \lambda_i(\theta_i - X_t^{(i)})dt + \sigma_i dB_t^i, \quad X_0^{(i)} = x_i, \quad i = 1, 2.$$

The no-arbitrage price of the  $T$ -maturity zero-coupon bond is given by  $P_0(T) := \mathbb{E}^{\mathbb{Q}}[e^{-\int_0^T r_t dt}]$ , inducing the yields to maturity  $R_0(T) := -\frac{\ln P_0(T)}{T}$ :

$$R_0(T) = \rho_1(T) + \rho_2(T), \quad \text{for all } T \geq 0,$$

$$\text{with } \rho_i(T) = \theta_i + (x_i - \theta_i) \frac{\Lambda_i(T)}{T} - \frac{\sigma_i^2}{2T} \int_0^T \Lambda_i(t)^2 dt, \quad \text{and } \Lambda_i(t) := \frac{1 - e^{-\lambda_i t}}{\lambda_i}.$$

1. Fix some maturity  $T > 0$ , a number of time steps  $n$ , and set  $h := T/n$ ,  $t_j = jh$ ,  $j = 0, \dots, n$ . By Itô's formula, we have  $X_{t_j}^{(i)} = \theta_i + (X_{t_{j-1}}^{(i)} - \theta_i)e^{-\lambda_i h} + \sigma \int_{t_{j-1}}^{t_j} e^{-\lambda_i(t_j-s)} dB_s^i$ ,  $j = 1, \dots, n$ ,  $i = 1, 2$ .
  - (a) Build a program which simulates  $N$  trajectories of the instantaneous rate  $\{r_{t_j}, j = 0, \dots, n\}$ .
  - (b) Build a program which computes a Monte Carlo approximation  $\hat{P}_0^{n,N}(t_j)$  of the zero-coupon bond approximate value  $P_0^n(t_j) := \mathbb{E}^{\mathbb{Q}}[e^{-\sum_{k=1}^j r_{t_k}}]$ , and the corresponding Monte Carlo approximation of the yields to maturity  $\hat{R}_0^{n,N}(t_j)$ .
  - (c) Consider the parameters values  $\lambda_1 = 1$ ,  $\lambda_2 = 0.1$ ,  $h := 0.01$ ,  $T := 15$ ,  $N = 10^4$ , and  $\theta_i = 0.05$ ,  $\sigma_i = 0.05$ ,  $x_i = 0.02$ ,  $i = 1, 2$ . Plot the yield curve  $\{R_0(t_j), j = 0, \dots, n\}$ , and compare to the corresponding Monte Carlo approximation  $\{\hat{R}_0^{n,N}(t_j), j = 0, \dots, n\}$ .
2. Under the parameters values of Question 1c, comment the effect of  $x_i$  by
  - (a) fixing  $x_2$  and varying  $x_1$  in  $\{0.01, 0.02, 0.05\}$ ,
  - (b) fixing  $x_1$  and varying  $x_2$  in  $\{0.01, 0.02, 0.05\}$ .
3. Under the parameters values of Question 1c, comment the effect of  $\theta_i$  by
  - (a) fixing  $\theta_2$  and varying  $\theta_1$  in  $\{0.01, 0.05, 0.1\}$ ,
  - (b) fixing  $\theta_1$  and varying  $\theta_2$  in  $\{0.01, 0.05, 0.1\}$ .
4. Under the parameters values of Question 1c, comment the effect of  $\sigma_i$  by
  - (a) fixing  $\sigma_2$  and varying  $\sigma_1$  in  $\{0.01, 0.05, 0.1\}$ ,
  - (b) fixing  $\sigma_1$  and varying  $\sigma_2$  in  $\{0.01, 0.05, 0.1\}$ .