Computer Problem Set 6

Profit and Loss of the Black-Scholes hedging

The present problem set refers to Section 9.2.6 of the lectures notes. Consider a European call option, with strike $K > 0$ and maturity $T > 0$. Then, given a volatility parameter $\Sigma > 0$, the Profit and Loss induced by the Black-Scholes hedging is defined by

$$P\&L_T(\Sigma) := X_T^{\Delta^{BS}} - (S_T - K)^+ \quad \text{where} \quad \Delta^{BS} := \mathbf{N}\left(\mathbf{d}_+(S_0, K, \Sigma^2 T)\right),$$

and $\mathbf{d}_+(s,k,v) := \frac{\ln(s/k)}{\sqrt{v}} + \frac{1}{2}\sqrt{v}$. *Assume that the underlying risky asset price is defined by the following stochastic volatility model:*

$$S_t := S_0 e^{-\frac{1}{2}\int_0^t \sigma_u^2 du + \int_0^t \sigma_u dW_u^{(1)}}, \quad d\sigma_t = \lambda(c - \sigma_t)dt + \gamma dW_t^{(2)},$$

where $W = (W^{(1)}, W^{(2)})$ *is a Brownian motion in* $\mathbb{R}^2$, *and* $S_0, \sigma_0, \lambda, c, \gamma$ *are given parameters. We recall that an explicit expression for the Ornstein-Uhlenbeck process* σ *is available by applying Itô's formula to* $\sigma_t e^{\lambda t}$, *see Section 8.1. In this context, it is then shown that the Profit and Loss reduces to*

$$P\&L_T(\Sigma) \quad = \quad \frac{1}{2}\int_0^T e^{r(T-u)}(\Sigma^2 - \sigma_u^2) S_u^2 \Gamma^{BS}(u, S_u, \Sigma) du,$$

where $\Gamma^{BS}(t,s,\Sigma) := \partial^2_{ss}\mathrm{BS}(t,s,\Sigma)$.

1. build a program which produces a sample of $N = 1000$ copies of the discrete path $\{\sigma_{t_i^n}, i = 0, \ldots, n\}$, $t_i^n := iT/n$.

2. build a program which produces a sample of $N = 1000$ copies $\{S_{t_i^n}^n, i = 0, \ldots, n\}$ of an appropriate discretization of S.

3. Using the parameters values $S_0 = 100$, $T = 1$, $\sigma_0 = 0.4$, $\lambda = 2$, $c = 0.4$, $\gamma = 0.3$, and $r = 0.02$, provide $N = 1000$ copies of an appropriate discretization of $\mathrm{P\&L}_T(\Sigma)$ for $\Sigma = \sigma_0$, $K \in \{100 + j, j = 5, \ldots, 150\}$, and $n \in \{50, 60, \ldots, 100\}$. Compute the corresponding sample mean and variance.