## Computer Problem Set 2

## Simulation of the Brownian motion

The present problem set is attached to Chapter 4 of the lectures notes. All implementations should be run with the value T=1. For a positive integer n, we denote  $\Delta T:=2^{-n}T$ ,  $t_i^n:=i$   $\Delta T$ ,  $i=0,\ldots,2^n$ . Our objective is to simulate a discretization of a Brownian motion W, and to study some properties.

- 1. Forward simulation of  $\{W_{t_1^n}, \ldots, W_{t_n^n}\}$ .
  - (a) Justify that  $W_{t_i^n} = W_{t_{i-1}^n} + Z_i \sqrt{\Delta T}$  where  $(Z_i)_{1 \leq i \leq 2^n}$  is an iid family of  $\mathcal{N}(0,1)$  random variables.
  - (b) Draw a sample of 1000 copies of the discretized Brownian motion  $\{W_{t_1^n}, \ldots, W_{t_n^n}\}$ .
  - (c) Compute the corresponding sample mean and variance of  $W_T$ , and the sample covariance of  $(W_T, W_{T/2})$ . Comment the results by varying the value of n.
- 2. Backward simulation of  $\{W_{t_1^n}, \ldots, W_{t_n^n}\}$ .
  - (a) For  $0 \leq s_1 < s_2$ , we recall that the pair  $(W_{s_1}, W_{s_2})$  is a centered Gaussian vector with variance matrix  $\begin{pmatrix} s_1 & s_1 \\ s_1 & s_2 \end{pmatrix}$ , and we verify therefore that  $W_{s_1}|W_{s_2}$  is also Gaussian with characteristics

$$\mathbb{E}\big[W_{s_1}|W_{s_2}\big] = \frac{s_1}{s_2}W_{s_2} \quad \text{and} \quad \mathbb{V}ar\big[W_{s_1}|W_{s_2}\big] = s_1\Big(1 - \frac{s_1}{s_2}\Big).$$

With  $\bar{s}:=\frac{s_1+s_2}{2}$ , justify that  $W_{\bar{s}}|(W_{s_1}=x_1,W_{s_2}=x_2)$  has a Gaussian distribution with conditional mean  $\bar{x}:=\frac{x_1+x_2}{2}$  and conditional variance  $\frac{s_2-s_1}{4}$ .

- (b) Justify that the conditional distribution of  $W_{\bar{s}}|(W_{s_1} = x_1, W_{s_2} = x_2, (W_u)_{u \notin [s_1, s_2]})$  is  $\mathcal{N}(\bar{x}, \frac{s_2 s_1}{4})$ .
- (c) Use the last property to simulate backward the discretized Brownian motion: start by drawing copies of  $W_T$ , then  $W_{T/2} = W_{t_1^1}$ , then  $W_{T/4} = W_{t_1^2}$  and  $W_{3T/4} = W_{t_2^2}$ , etc...
- (d) Compute the corresponding sample mean and variance of  $W_T$ , and the sample covariance of  $(W_T, W_{T/2})$ . Comment the results by varying the value of n.
- 3. Using successively the forward and backward simulated samples, compute an approximation of  $QV^n(W)_T$ , the quadratic variation of the Brownian motion along the partition  $(t_i^n)_i$ . Provide two graphs displaying the departure from the limit T as a function of  $n \in \{10, \ldots, 20\}$ .