MVA – Probabilistic Graphical Models

Homework 2

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1 K-means and the EM algorithm

We consider a mixture model of K components for a dataset (X_i) where $X_i \in \mathbb{R}^d$. We denote $Z_i \in \{1, \dots, K\}$ the latent hidden label. Each component occurs with probability $p_k = \mathbb{P}(Z_i = k)$ and is distributed as

$$X_i \sim \mathcal{N}(\mu_k, D_k)$$

i.e. $p(x|k) = \frac{1}{((2\pi)^d |D_k|)^{1/2}} \exp(-\frac{1}{2}(x-\mu_k)^T D_k^{-1}(x-\mu_k))$.

The data log-likelihood under parameters $\Theta = ((p_1, \mu_1, D_1), \dots, (p_K, \mu_K, D_K))$ is

$$\mathcal{L}(X_1, \dots, X_n; \Theta) = \sum_{i=1}^n \log\left(\sum_{k=1}^K p_k p(X_i | k; \mu_k, D_k)\right)$$

We seek to compute the MLE

$$\widehat{\Theta} \in \operatorname*{argmax}_{\Theta} \mathcal{L}(X_1, \dots, X_n; \Theta)$$

This optimization problem is intractable when using straightforward methods.

The EM algorithm goes as follows:

- *Expectation* Compute the posterior probability of the latent variables Z_i:

$$q_{k,i}^{(t)} = p(Z_i = k | X_i; \Theta^{(t)}) = \frac{p_k^{(t)} p(X_i | Z_i = k; \Theta^{(t)})}{\sum_{\ell=1}^K p_\ell^{(t)} p(X_i | Z_i = \ell; \Theta^{(t)})}$$

  and denote $w_k^{(t)} = \sum_{i=1}^n q_{k,i}^{(t)}$ – we then have $\sum_{k=1}^K w_k^{(t)} = n$.

- *Maximization* Update the parameters $\Theta^{(t)}$ by maximizing the lower bound objective:

$$\max_{\Theta} \mathcal{J}(q^{(t)}, \Theta) = \sum_{i=1}^n \left(\sum_{k=1}^K q_{k,i}^{(t)} \left(\log p_k - \frac{d}{2}\log(2\pi) - \frac{1}{2}\log|D_k| - \frac{1}{2}(X_i - \mu_k)^T D_k^{-1}(X_i - \mu_k)\right)\right)$$

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subject to $\sum_{k=1}^{K} p_k = 1$ (associated to a multiplier ν). The KKT conditions give the null gradient condition:

$$\frac{1}{p_k} w_k^{(t)} - \nu = 0 \tag{1a}$$

$$-\sum_{i=1}^{n} q_{k,i}^{(t)} D_k^{-1}(\mu_k - X_i) = 0 \tag{1b}$$

$$-\frac{1}{2} \sum_{i=1}^{n} q_{k,i}^{(t)} (D_k^{-1} - D_k^{-2} \operatorname{diag}((X_i - \mu_k))^2) = 0 \tag{1c}$$

Which leads to the updates:

$$p_k = \frac{1}{n} w_k^{(t)} \tag{2a}$$

$$\mu_k = \frac{1}{w_k^{(t)}} \sum_{i=1}^{n} q_{k,i}^{(t)} X_i \tag{2b}$$

$$D_k = \frac{1}{w_k^{(t)}} \sum_{i=1}^{n} q_{k,i}^{(t)} \operatorname{diag}((X_i - \mu_k))^2 \tag{2c}$$