Computational Optimal Transport: Final Project

A regularized Optimal Transport formulation for variational Mean-Field Games

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Abstract

Mean-field games (MFG) are strategic decision-making problems designed to approximate complex large-scale, many-agent differential games using partial differential equations and study their Nash equilibria. This gives access to the convenient theoretical tools of differential equations. In recent years, work has been done on finding variational formulations for MFGs so they can be written as convex optimization problems and eventually be connected to the theory of optimal transport [1, 2]. In particular, a recent paper by Benamou et al. [3] explores a class of games that can be written as minimal-entropy problems over a Wiener space, with an efficient numerical algorithm in tow.

1 Quadratic Mean-Field Games

1.1 Control problem and Nash equilibrium

A mean-field game [4, 5] is a strategic decision-making problem with a very large, continuously-distributed number of interacting agents inside a state space: the overall theory developed by Lasry and Lions can be used as a means to model large, computationally intractable games. Each agent's actions get a feedback response that depends on other agents' states and actions through a *mean-field* effect. In the time-evolving setting, every agent obeys to some dynamics and his actions are modeled by a dynamic control problem [5].

The general setup of a dynamic MFG has every agent penalize a running cost on the control, aspects of the trajectory (namely the mean-field interactions with other agents), as well as a terminal cost on the its final position and the overall final distribution of agents [5]. The framework of [1, 3] focuses on games with stochastic agent dynamics

$$dX_t = \alpha_t dt + \sigma dW_t \tag{1.1}$$

with viscosity σ and control α and a quadratic running cost on the control $L(\alpha_t) = |\alpha_t|^2/2$. Mean-field interaction penalties are given by L^2-valued functionals f and g. A representative

agent's objective is to minimize the overall penalty

$$\inf_{\alpha} J(\alpha) = \mathbb{E}\left[\int_0^T \frac{1}{2}|\alpha_t|^2 + f(X_t, \rho_t)\, dt + g(X_T, \rho_T)\right] \tag{1.2}$$

subject to eq. (1.1), and where ρ_t is the overall distribution of agents at time t. In the usual dynamic programming framework, we consider Markov **closed-loop feedback form** controls

$$\alpha = \varphi(t, X_t). \tag{1.3}$$

In the quadratic cost framework the feedback is the gradient of the value function $\varphi = \nabla u$.

The Nash equilibrium agent-control dynamics can be summarized by the partial differential equations:

$$-\partial_t u - \frac{\sigma^2}{2}\Delta u + \frac{1}{2}|\nabla u|^2 = f(x, \rho_t), \quad (t, x) \in (0, T) \times \Omega \tag{1.4a}$$

$$\partial_t \rho_t - \frac{\sigma^2}{2}\Delta \rho_t - \operatorname{div}(\rho_t \nabla u) = 0 \tag{1.4b}$$

$$u(T, \cdot) = g(x, \rho_T) \tag{1.4c}$$

where $t \mapsto \rho_t$ is a trajectory in the space of measures and ρ_0 is given, and $\Omega \subset \mathbb{R}^d$. The applications $\mu \mapsto f(\cdot, \mu)$ and $\mu \mapsto g(\cdot, \mu)$ are supposed to be derivatives of some real-valued functionals F and G on the space of measures. For instance, in the case considered by Benamou et al. [1], the running cost functional is a function of space $f(x, \mu) = \Psi(x)$, which has antiderivative $F(\mu) = \int_\Omega \Psi(x)\, d\mu(x)$ in the space of measures.

Equations (1.4a) and (1.4b) form a coupled system of control (Hamilton-Jacobi-Bellman) and diffusion (Fokker-Planck) partial differential equations. They can be solved in some cases using finite-difference methods (see Achdou et al. [6]).

1.2 Variational formulation

The idea of [1] is to cast the MFG partial differential equations to a variational problem over an appropriate function space. Denote $\mathbb{W}_2(\Omega) = (\mathcal{P}_2(\Omega), \mathcal{W}_2)$ the set of probability measures with finite second moment, equipped with the Wasserstein metric $\mathcal{W}_2$. [1] show that the MFG can be reformulated to the following variational problem:

$$\begin{aligned} \inf_{\rho, v} J(\rho, v) &= \frac{1}{2}\int_0^T \int_\Omega |v_t|^2\, d\rho_t(x)\, dt + \int_0^T F(\rho_t)\, dt + G(\rho_T) \\ \text{s.t. } \partial_t \rho_t - \frac{\sigma^2}{2}\Delta \rho_t &+ \operatorname{div}(\rho_t v) = 0 \end{aligned} \tag{1.5}$$

where $(\rho_t)_{t\in[0,T]} \in \mathcal{C}([0, T], \mathbb{W}_2(\Omega))$ is a trajectory in $\mathbb{W}_2$ and v is a function on $[0, T] \times \Omega$ most likely lying in a Sobolev space.

This point of view is called *Eulerian*: we minimize over both the velocity v and the agent density trajectory (ρ_t). This nonconvex problem is solved by Benamou et al. [1] and Benamou and Carlier [2] using an augmented saddle-point approximation.

1.3 Lagrangian formulation

Benamou et al. [1, 3] introduce a *Lagrangian* point of view: the variational problem is changed to a **weighted energy minimization problem on the space of trajectories**. The energy of all paths is aggregated using a measure on the space of trajectories, and we seek the optimal measure: this allows the use of transportation theory to formulate the problem.

1.3.1 The Wiener space of trajectories

This new point of view involves a change in function spaces. We denote $\mathcal{X} = \mathcal{C}([0,T],\Omega)$ the Wiener space of agents' trajectories $[0,T] \to \Omega$. The space $\mathcal{X}$ is equipped with the **Wiener measure**. In the unitary viscosity case ($\sigma = 1$), the Wiener measure R on $\mathcal{X}$ is defined as follows: the measure of a set of paths $\mathcal{A} \subset \mathcal{X}$ is

$$R(\mathcal{A}) = \int_{\mathbb{R}^d} \mathbb{P}(x + W \in \mathcal{A})\, dx$$

where W is the standard Brownian motion (starting at 0). In the non-unitary viscosity case ($\sigma \neq 1$), we introduce $\varepsilon = \sigma^2$ and the Wiener measure R_ε associated with Wiener processes scaled by σ. It is an analogue in the space $\mathcal{X}$ to the usual finite-dimensional Lebesgue measure[1]. [1, 2] This construction is needed because it allows us to talk about notions and **density** and **entropy of a probability measure on** $\mathcal{X}$ with respect to the Wiener measure, which we will use to define the energy to minimize.

> **Remark 1.** This was not necessary in the framework of [1] because the energy $\mathcal{E}(Q)$ used in that article was the expected kinetic energy of a trajectory ξ distributed under $Q \in \mathcal{P}(\mathcal{X})$.

Measures $Q \in \mathcal{P}(\mathcal{X})$ can also be seen as trajectories $(Q_t)_{t\in[0,T]}$ in $\mathcal{P}(\Omega)$ with

$$Q_t = e_{t\#}Q \in \mathcal{P}(\Omega)$$

the push-forward of Q by the evaluation map $e_t \colon x \in \mathcal{X} \longmapsto x(t)$. This defines a natural injection $\underline{i} \colon \mathcal{P}(\mathcal{X}) \to \mathcal{C}([0,T],\mathcal{P}(\Omega))$ from the probability measures on trajectory space to the trajectory space of probabilities on Ω. We also introduce the more general marginals $Q_{t_1,\dots,t_n} = (e_{t_1},\dots,e_{t_n})_\# Q$ for $0 \leq t_1 < \cdots < t_N \leq T$.

Marginals of the Wiener measure. Benamou et al. [3] provide the following results on the Wiener measure when $\Omega = \mathbb{R}^d$.

- The single marginals R_t are the Lebesgue measure $\mathcal{L}^d$ on $\mathbb{R}^d$.
- The 2-marginals have densities on $\mathbb{R}^d \times \mathbb{R}^d$:

$$R_{s,t}(x,y) = P_{t-s}(y-x). \tag{1.6}$$

[1] https://en.wikipedia.org/wiki/Infinite-dimensional_Lebesgue_measure

where P_t is the standard d-dimensional heat kernel:

$$P_t(u) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|u|^2}{2t}\right) \tag{1.7}$$

- The N-marginals are given by

$$R_{t_1,\dots,t_N}(x_1,\dots,x_n) = \prod_{i=1}^{N-1} P_h(x_{i+1} - x_i) \tag{1.8}$$

The last property is especially **important** for computational reasons, as we will see later.

> **Remark 2** (Heat kernel)**.** As Benamou et al. [3, p. 5] remark, the heat kernel (1.7) is defined for the Laplacian operator $\frac{1}{2}\Delta$, which is the generator of the standard Brownian motion W.

Integration. Partial integration with respect to the 2-marginal measure $R_{0,h}$ is actually convolution with respect to the heat kernel P_h:

$$\int_\Omega u(x) R_{0,h}(x,y)\,dx = \int_\Omega u(x) P_h(y-x)\,dx = (u * P_h)(y)$$

The effect of integration against the N-marginal can then be deduced by induction.

1.3.2 Energy objective

Instead of using finite element methods to solve for a kinetic objective as in [1], Benamou et al. [3] propose introducing an entropic objective to allow for a more computationally efficient numerical method adapted from the Sinkhorn algorithm introduced by Cuturi [7].

This method introduces entropic regularization in the problem. As shown in the initial paper [3], the resulting variational problem and associated numerical scheme become regularizations of the problem from [1, 2]. More precisely, the energy to minimize is the entropy with respect to the Wiener measure:

$$\mathcal{E}(Q) = \varepsilon H(Q|R_\varepsilon) = \begin{cases} \varepsilon \int_{\mathcal{X}} \log(dQ/dR_\varepsilon)\,dQ & \text{if } Q \ll R_\varepsilon \\ +\infty & \text{otherwise} \end{cases} \tag{1.9}$$

The associated Lagrangian variational problem is

$$\inf_{Q \in \mathcal{P}(\mathcal{X})} \varepsilon H(Q|R_\varepsilon) + \int_0^T F(Q_t)\,dt + G(Q_T) \quad \text{s.t. } Q_0 = \rho_0 \tag{1.10}$$

Intuitively, this is the same as fixing the marginals ρ_t, finding the entropy-optimal bridge Q^* between them that has minimal entropy relative to the Wiener measure, and then optimizing over the ρ_t.

2 Numerical algorithm

2.1 Time discretization

Let N be the number of discrete steps for the time discretization of the problem, and $h = T/N$ the time step.

Benamou et al. [3] propose a discretization of (1.10) obtained by connecting the marginals through a multimarginal OT problem:

$$\mathcal{S}(\mu_0, \dots, \mu_N) = \inf_{\gamma \in \Pi(\mu_0, \dots, \mu_N)} H(\gamma | R^N) \tag{2.1}$$

where $t_k = kh$, $R^N = R_{t_0, \dots, t_N}$ and the marginals $\mu_k \in \mathcal{P}_2(\Omega)$. Then, define

$$\mathcal{U}(\mu_0, \dots, \mu_N) = h \sum_{k=1}^{N-1} F(\mu_k) + G(\mu_N). \tag{2.2}$$

Thus, the discretized entropy minimization problem can be written as

$$\inf \left\{ \mathcal{S}(\mu_0, \dots, \mu_N) + \mathcal{U}(\mu_0, \dots, \mu_N) : \mu_k \in \mathcal{P}_2(\Omega),\ \mu_0 = \rho_0 \right\}.$$

Expanding the inf-within-inf leads to the following convex optimization problem:

$$\begin{aligned} &\inf_{\gamma \in \mathcal{P}(\Omega^{N+1})} H(\gamma | R^N) + \imath_{\rho_0}(\mu_0) + \sum_{k=1}^{N-1} F(\mu_k) + G(\mu_N) \\ &\text{s.t. } \mu_k = P^k_{\#} \gamma \end{aligned} \tag{2.3}$$

where $\imath_{\rho_0}(\mu) = +\infty$ if $\mu \neq \rho_0$ and 0 otherwise is the convex indicatrix of the measure ρ_0. This is a generalized multimarginal optimal transport problem.

Benamou et al. [3] provide the corresponding dual problem involving the convex conjugates and potential functions, by using a multimarginal generalization of a result from Chizat et al. [8]:

$$\sup_u \int_{\Omega^{N+1}} \left(1 - \exp\left(\oplus_{k=0}^N u_k\right)\right) dR^N - \imath^*_{\rho_0}(-u_0) - \sum_{k=1}^{N-1} F^*(-u_k) - G^*(-u_N) \tag{2.4}$$

where the supremum is taken over $u = (u_0, \dots, u_N) \in L^\infty(\Omega)^{N+1}$.

Benamou et al. [3] introduce a Sinkhorn-like iterative algorithm to solve the above dual problem. We rewrite it more explicitly with slightly different notations inspired by [8].

Proposition 1. *Denote for $k = 0, \dots, N$ and $(a_j)_{j \neq k}$*

$$\mathcal{I}_k((a_j)_{j \neq k})(\tilde{x}_k) = \int_{\Omega^N} \prod_{j \neq k} a_j(x_j) R^N(dx_{0:k-1}, \tilde{x}_k, dx_{k+1:N})$$

the partial integral of the $a_j, j \neq k$ with respect to R^N without variable x_k. For conve-

nience we use the shorthand

$$\mathcal{I}_k^{(n)} = \mathcal{I}_k\left(\left(a_j^{(n+1)}\right)_{j<k}, \left(a_j^{(n)}\right)_{j>k}\right)$$

for the nth iterate where we denote $a_j = \exp(u_j)$.

Then we compute the dual potentials iteratively:

$$\begin{cases} u_0^{(n+1)} = \underset{v\in L^\infty}{\operatorname{argmax}} \displaystyle\int_\Omega (1 - e^{v(x_0)})\mathcal{I}_0^{(n)}\, dx_0 - \imath^*_{\rho_0}(-v) \\ u_k^{(n+1)} = \underset{v\in L^\infty}{\operatorname{argmax}} \displaystyle\int_\Omega (1 - e^{v(x_k)})\mathcal{I}_k^{(n)}\, dx_k - hF^*(-v), \quad 1 \le k < N \\ u_N^{(n+1)} = \underset{v\in L^\infty}{\operatorname{argmax}} \displaystyle\int_\Omega (1 - e^{v(x_N)})\mathcal{I}_N^{(n)}\, dx_N - G^*(-v) \end{cases} \tag{2.5}$$

until convergence.

Using duality, we find that the iterates $u_k^{(n)}$ *satisfy*

$$a_k^{(n)} = \exp\left(u_k^{(n)}\right) = \frac{\operatorname{prox}_{F_k}^H(\mathcal{I}_k^{(n)})}{\mathcal{I}_k^{(n)}} \tag{2.6}$$

where

$$\operatorname{prox}_F^H(z) = \underset{s}{\operatorname{argmin}}\, F(s) + H(s|z)$$

is the KL-proximal operator.

Remark 3 (Some convex conjugates)**.** In practice, the convex conjugates of the cost functions are difficult to compute. For some of the examples in the paper, we have closed-form conjugates.

- The conjugate of the convex indicatrix $\imath_\nu$ of any measure ν is given by $\imath_\nu^*(u) = \langle u, \nu\rangle$.
- The hard congestion constraint

$$C(\rho) = \begin{cases} 0 & \text{if } \rho \le \bar{m} \\ +\infty & \text{otherwise} \end{cases}$$

has convex conjugate (on the domain $\rho \ge 0$)

$$C^*(u) = \sup_{\rho\le\bar{m}} \langle u, \rho\rangle = \langle u^+, \bar{m}\mathbb{1}\rangle$$

- Obstacle constraints, given by

$$F(\rho) = \int_\Omega V(x)\, d\rho(x) = \begin{cases} 0 & \text{if } \rho = 0 \text{ on } \mathscr{O} \\ +\infty & \text{otherwise} \end{cases} = \imath_0(\mathbb{1}_{\mathscr{O}}\rho)$$

  where V is the convex indicatrix of the complement $\Omega\backslash\mathscr{O}$ of the obstacles. Its conjugate is given by

$$F^*(u) = \begin{cases} 0 & \text{if } u \leq 0 \text{ on } \Omega\backslash\mathscr{O} \\ +\infty & \text{otherwise} \end{cases}$$

2.2 Spatial discretization

For full numerical implementation, all measures are replaced by multi-dimensional arrays representing discrete histograms over a fixed grid of points x_i in $\mathbb{R}^d$ of size $M = N_1 \times \cdots \times N_d$.

Integration with respect to the marginalized Wiener measure R^N is the main computational bottleneck.

Denote $\mathbf{R} \in \mathbb{R}^{M^N}$ the discretized measure R^N. Integration of multiple vectors $a_0, \ldots, a_N \in \mathbb{R}^M$ with respect to $\mathbf{R}$ is the following tensor contraction

$$\mathbf{R}[a_0, \ldots, a_N] = \sum_{i_0,\ldots,i_N} \mathbf{R}_{i_0,\ldots,i_N} \prod_{k=0}^{N} a_{i_k}$$

A naive implementation would compute the sum in exponential time $\mathcal{O}(NM^N)$: this is a well known problem is computational statistics and graphical models, and an efficient way of dealing with it is exploiting the structure of the kernel $\mathbf{R}$.

Proposition 2 (Efficient convolution)**.** *The kernel* $\mathbf{R}$ *can be factorized as* $\mathbf{R}_{i_0,\ldots,i_N} = \prod_{k=0}^{N-1} \mathbf{P}_{i_k,i_{k+1}}$ *where* $\mathbf{P}$ *is the discrete heat kernel on* $\mathbb{R}^M$*. The partial convolution* $\mathcal{I}_k$ *(leaving the* k*th component out) can now be written as*

$$\mathcal{I}_k = \mathbf{R}[(a_j)_{j\neq k}] = \mathbf{A}_{k-1} \odot \mathbf{B}_{k+1} \tag{2.7}$$

where $\mathbf{A}_k = \mathbf{P}^T(a_k \odot \mathbf{P}^T(a_{k-1} \odot \cdots))$ *and* $\mathbf{B}_k = \mathbf{P}(a_k \odot \mathbf{P}(a_{k+1} \odot \cdots))$.

Due to the structure of the kernel, the marginals $\mu_0, \ldots, \mu_N$ only communicate as in an undirected chain. This leads to the usual message-passing algorithm: see Algorithm 1.

The computational complexity of this algorithm depends on how efficiently we can compute the convolution $\mathbf{P}u$. The naive matrix product performs in time $\mathcal{O}(M^3)$ and the overall algorithm is $\mathcal{O}(NM^3)$ which can still be very high. For separable kernels, decomposing the convolution can net considerable speedups [9, p. 74].

Algorithm 1: Efficient computation of the integral $\mathcal{I}_k$.

Input: Base heat kernel $\mathbf{P}$, index k, vectors $(a_j)_{j\neq k}$

$\mathbf{A} \leftarrow \mathbb{1}$;
for $i = 0$ *to* $k-1$ **do**
  $\mathbf{A} \leftarrow \mathbf{P}^T(a_i \odot \mathbf{A})$;
$\mathbf{B} \leftarrow \mathbb{1}$;
for $i = N$ *down to* $k+1$ **do**
  $\mathbf{B} \leftarrow \mathbf{P}(a_i \odot \mathbf{B})$;
return $\mathbf{A} \odot \mathbf{B}$;

Projections. In the general case, the KL-proximal operators in the Sinkhorn iterations can be solved using the Python library CVXPY[2,3]. Some can be computed explicitly:

Proposition 3. *The KL-projection on the hard congestion constraint of a measure $\beta \in \mathbb{R}^M$ is given by*

$$\mathrm{proj}^H_{\mathcal{C}}(\beta) = \min(\beta, \overline{m}) \tag{2.8}$$

where the minimum is taken element-wise.

The KL-projection on the obstacle constraint is

$$\mathrm{proj}^H_{\mathcal{C}}(\beta) = \beta \mathbb{1}_{\Omega\setminus\mathscr{O}} \tag{2.9}$$

If we also add the obstacle constraint on a set $\mathscr{O}$ of points in the grid, then the projector is

$$\mathrm{proj}^H_{\mathcal{C}}(\beta) = \min(\beta, \bar{m})\mathbb{1}_{\Omega\setminus\mathscr{O}}. \tag{2.10}$$

Under a system of constraints with hard congestion and obstacles, a linear penalty function $G(\alpha) = \langle \Psi, \alpha\rangle$ has proximal operator

$$\mathrm{prox}^H_{G,\mathcal{C}}(\beta) = \min(\beta \odot e^{-\Psi/\varepsilon}, \bar{m})\mathbb{1}_{\Omega\setminus\mathscr{O}} \tag{2.11}$$

3 Examples

3.1 Two-marginal case

We start with a very simplified approximation of the crowd displacement problem on $\Omega = [0,1]^2$, with only the first step (with initial agent distribution) and final step decided by the terminal penalty function G.

[2] https://github.com/cvxgrp/cvxpy

[3] Steven Diamond and Stephen Boyd. "CVXPY: A Python-Embedded Modeling Language for Convex Optimization". In: *Journal of Machine Learning Research* 17.83 (2016), pp. 1–5.

We set G to be the obstacle constraint related to a subset $\mathscr{O}$ of Ω as well as a potential $\Psi(x) = d(x, \mathscr{A})^\beta$ for some $\beta > 0$, related to the distance to a target subset $\mathscr{A}$ (see fig. 1):

$$G(\mu) = \int_\Omega \Psi \, d\mu + \imath_0(\mu \mathbb{1}_{\mathscr{O}})$$

Thus, the agents engage in a one-round mean-field game where they are only concerned with moving to regions with lower potential Ψ – as close as possible to $\mathscr{A}$ – whilst obeying physical constraints related to the obstacles.

The discretized MFG problem with viscosity parameter $\varepsilon = \sigma^2$ can be written as the following transport problem:

$$\begin{aligned} &\inf_\gamma \langle \Psi, \gamma^T \mathbb{1} \rangle + \varepsilon \, \mathrm{KL}(\gamma | \mathbf{R}) \\ &\text{s.t. } \gamma \mathbb{1} = \rho_0, \quad \gamma^T \mathbb{1} \odot \mathbb{1}_{\mathscr{O}} = 0 \end{aligned} \tag{3.1}$$

In this two-marginal case, the Gibbs kernel $\mathbf{R} = \mathbf{P}$ is the discretization of the heat kernel as discussed in section 2.2:

$$\mathbf{R}_{i,j} = P_{h\varepsilon}(x_j - x_i)$$

for all grid indices i, j.

In this problem we want to observe the optimal target distribution $\rho_1^* = (\gamma^*)^T \mathbb{1}$ the agents reach at the final time $t = 1$.

Proposition 4. *Problem (3.1) can be solved in closed form: the Lagrange multiplier u_0^* for the marginal law constraint satisfies*

$$a_0^* = e^{u_0^*/\varepsilon} = \frac{\rho_0}{\mathbf{R} a_1^*}$$

where $a_1^* = e^{-\Psi/\varepsilon} \odot \mathbb{1}_{\Omega \setminus \mathscr{O}}$, and the optimal coupling is

$$\gamma^* = \mathbf{R} \odot (a_0^* \otimes a_1^*)$$

It satisfies, as expected, that $\gamma^*_{i,j} = 0$ for all $j \in \mathscr{O}$. The final distribution of agents is

$$\rho_1 = \gamma^T \mathbb{1} = a_1^* \odot \mathbf{R} a_0^*$$

Numerical experiment We ran a numerical experiment by implementing the solutions given by proposition 4 to the discrete MFG (3.1). fig. 2a provides a representation of both. We also checked the results when removing the constraints on the obstacles (essentially setting $\mathscr{O}$), and when lowering the viscosity parameter $\sigma = \sqrt{\varepsilon}$ (see figs. 2b and 2c).

Since the kernel $\mathbf{R}$ on the domain is separable, the convolution can be sped up.

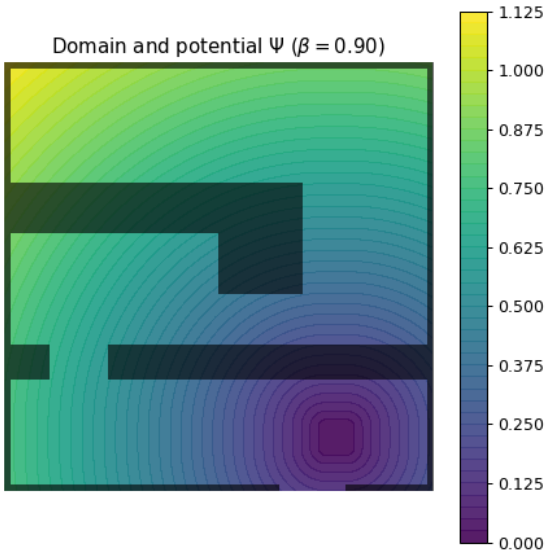


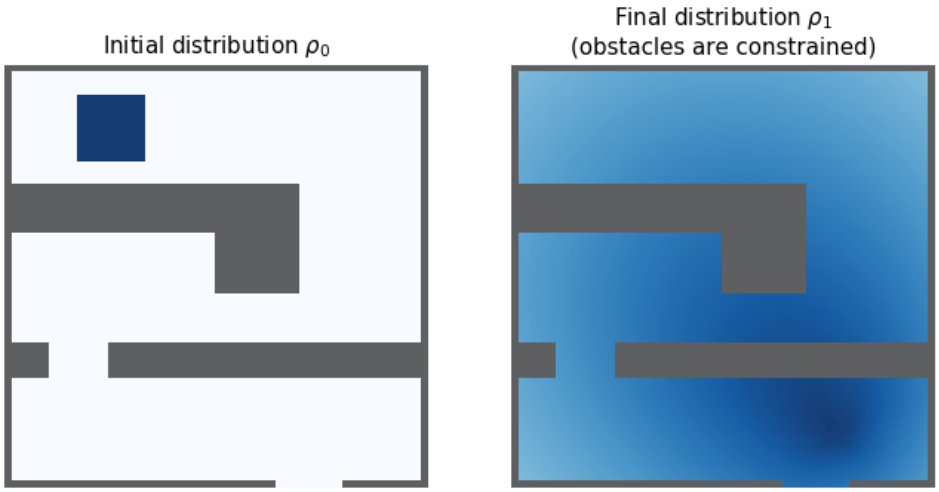
Figure 1: Computational domain of the game Ω with set of obstacles $\mathscr{O}$ (transparent grey), and contour of the potential function $\Psi(x) = d(x, \mathscr{A})^\beta$.

> **Remark 4** (A more realistic potential for crowd dynamics)**.** The results shown fig. 2 are satisfactory for the given potential Ψ – as expected the agents try to stay near the low-potential regions. However, for modeling of crowd dynamics they would be deeply nonphysical because the potential is inadequate. In a room evacuation scenario, for instance, agents would seek to minimize the time-to-exit: the literature shows this leads to the Eikonal equation $|\nabla u(x)| = 1/f(x)$, a kind of Hamilton-Jacobi PDE. We computed the adequate potential shown fig. 3a using the Fast Sweeping method [11]. The solution to the associated discrete 2-step MFG is shown fig. 3b.

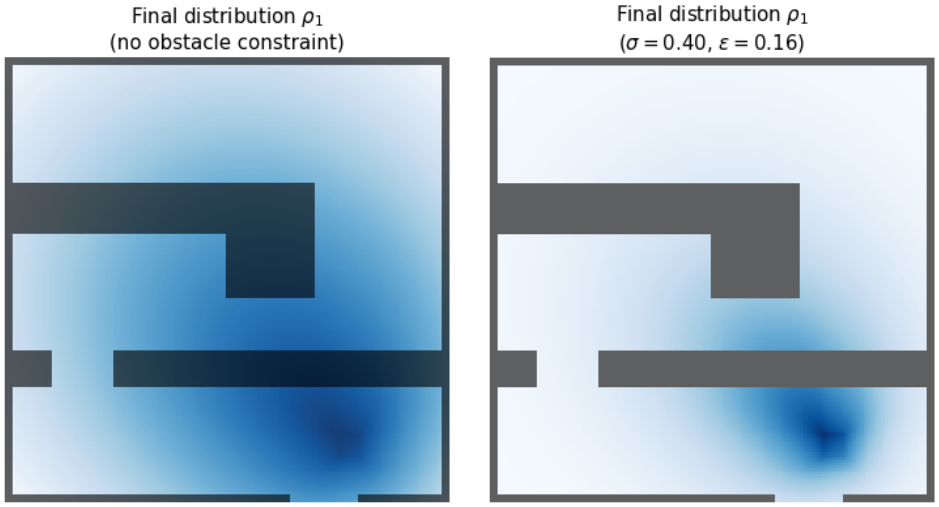
3.2 Three time steps

We now go up to three marginals (ρ_0, ρ_1, ρ_2). We assign to the single intermediate marginal ρ_1 the same constraints: congestion $\rho_1 \leq \bar{m}$ and the obstacles. The primal problem then reads

$$\begin{aligned} &\inf_{\gamma, \rho_1, \rho_2} \langle \Psi, \rho_2 \rangle + \varepsilon \, \mathrm{KL}(\gamma | \mathbf{R}) \\ \text{s.t. } &P^k_\# \gamma = \rho_k, \ k = 0, 1, 2 \\ &\rho_1 \leq \bar{m}, \quad \rho_1 \odot \mathbb{1}_{\mathscr{O}} = 0 \\ &\rho_2 \leq \bar{m}, \quad \rho_2 \odot \mathbb{1}_{\mathscr{O}} = 0 \end{aligned} \tag{3.2}$$



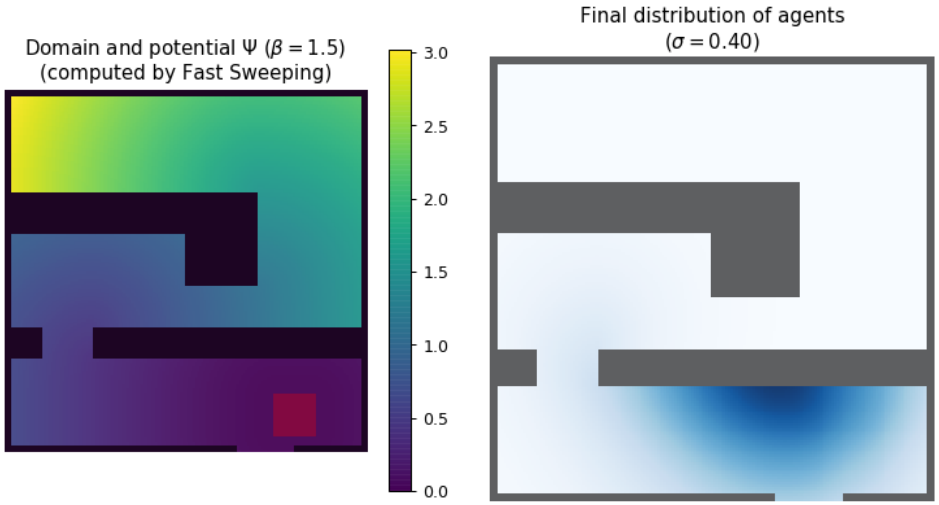
(a) Result of the "fuzzy" transport problem with enforcement of the obstacle constraints and viscosity parameter $\sigma = 1$.



(b) Final distribution ρ_1^* without enforcing the obstacles. The mass of the distribution "bleeds" through the obstacles.

(c) Obstacles are constrained as in fig. 2a, but with viscosity parameter $\sigma = 0.4$.

Figure 2: Numerical solution of the two-step MFG problem (3.1), with a few variations.



(a) Domain and potential associated with the fastest path distance.

(b) Optimal terminal distribution ρ_1^* of the discrete MFG with the potential from fig. 3a.

Figure 3: Setup and solution for the discrete MFG using the time-to-exit potential discussed in remark 4.

Proposition 5. *The Lagrange multipliers u_i^* at the optimum satisfy the fixed-point conditions:*

$$a_0^* = \frac{\rho_0}{\mathbf{R}[\,\cdot\,, a_1^*, a_2^*]}$$
$$a_1^* = \min\left(\frac{\bar{m}}{\mathbf{R}[a_0^*, \,\cdot\,, a_2^*]}, 1\right)$$
$$a_2^* = \min\left(\frac{\bar{m}}{\mathbf{R}[a_0^*, a_1^*, \,\cdot\,]}, e^{-\Psi/\varepsilon}\right)$$

where $a_i^* = \exp(u_i^*)$ are supported on $\Omega \backslash \mathscr{O}$ and we denote $\mathbf{R}[\cdot,\cdot,\cdot]$ the appropriate tensor contraction by $\mathbf{R}$.

The marginals are obtained as

$$\rho_1^* = a_1^* \odot \mathbf{R}[a_0^*, \,\cdot\,, a_2^*] = a_1^* \odot (\mathbf{P}^T a_0^*) \odot (\mathbf{P} a_2^*)$$
$$\rho_2^* = a_2^* \odot \mathbf{R}[a_0^*, a_1^*, \,\cdot\,] = a_2^* \odot \mathbf{P}^T (a_1^* \odot \mathbf{P}^T a_0^*)$$

The fixed point can then computed using an iterative algorithm à la generalized Sinkhorn, just as in the Algorithm 1 suggested by [3].

The issue of computational efficiency is more pronounced here than before due to the tensor product and need for multiple iterations until convergence. The bottleneck of com-

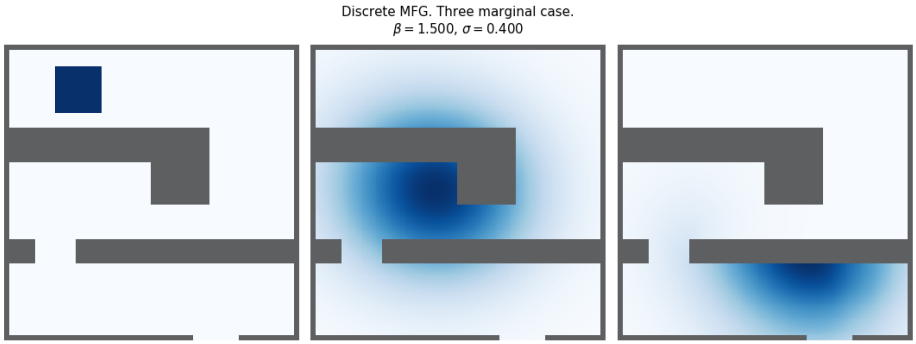


Figure 4: Three time step discrete MFG.

puting the tensor contractions has already been expanded upon in section 2.2. In this simple case, the contractions can be simplified as

$$\begin{aligned}\mathbf{R}[\cdot, v, w] &= \mathbf{P}(v \odot \mathbf{P}w)\\ \mathbf{R}[u, \cdot, w] &= (\mathbf{P}^T u) \odot (\mathbf{P}w)\\ \mathbf{R}[u, v, \cdot] &= \mathbf{P}^T(v \odot \mathbf{P}^T u)\end{aligned}$$

Numerical experiment. An example on the domain Ω from before is given fig. 4. The result is not qualitatively satisfying, because the intermediate step looks as if some of the mass "teleported" through the boundary instead of actually being constrained.

3.3 Full N-marginal case

The setup is given in fig. 5.

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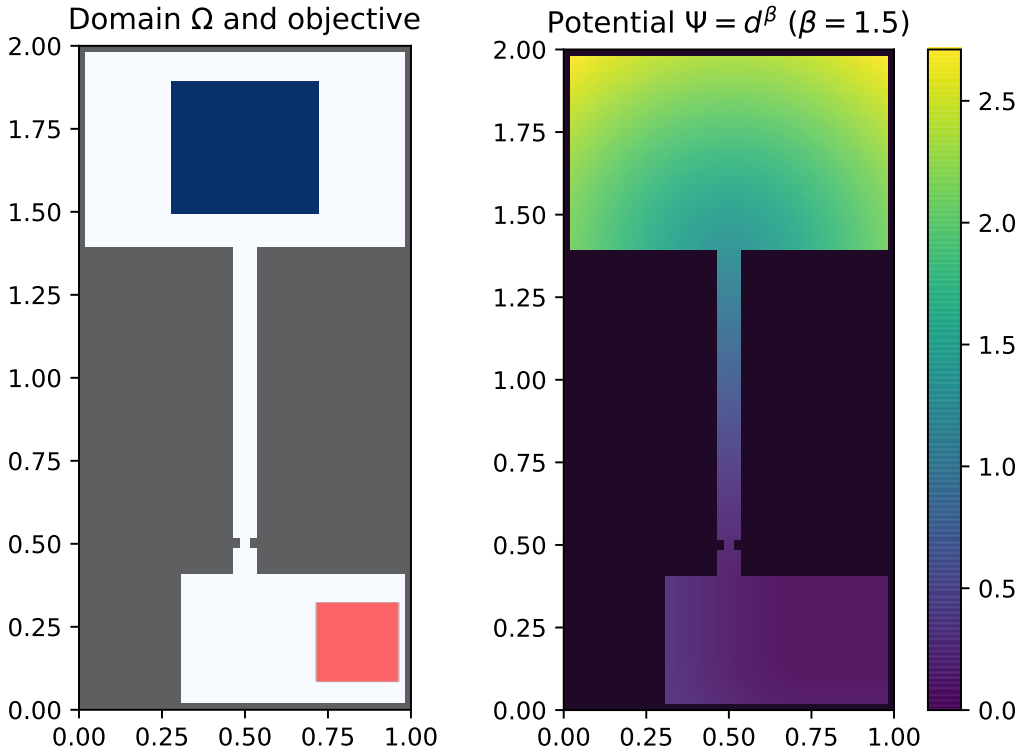

Figure 5: Domain, objective and associated potential for the multi-marginal problem. The initial distribution ρ_0 is in blue, the objective is in red.

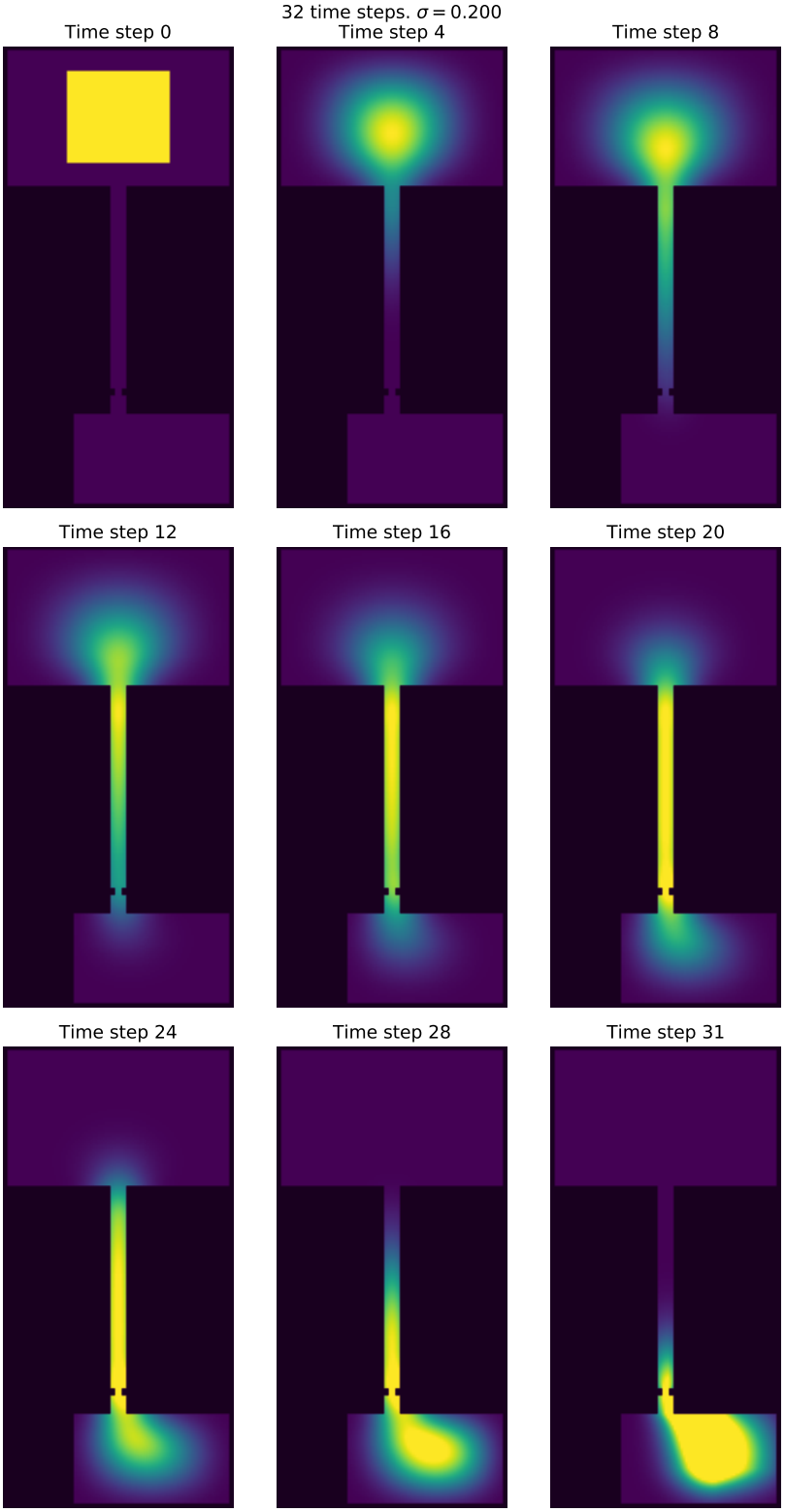

Figure 6: Numerical solution of the MFG.

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