

Computational Optimal Transport – Project report:
Optimal Transport and Entropic methods for solving
variational Mean-Field Games

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1 General setting: variational mean-field games

A mean-field game [8, 9] is a strategic decision-making problem with a very large, continuously-distributed number of interacting agents inside a state space: the overall theory developed by Lasry and Lions can be used as a means to model large, computationally intractable games. In the continuous-time setting explored in [9], each agent evolves according to some dynamics and makes choices, but the response to his choices are affected by the states and choices of the numerous other agents – leading to a so-called *differential game* – through a *mean-field* effect.

Several ways of modeling agent cross-interaction exist. More recently, [3] have focused on games where agent interactions take a variational form, allowing to penalize phenomenons such as congestion inside areas of the agent state space.

The (Nash) equilibrium agent-control dynamics can be summarized by the system of coupled nonlinear partial differential equations:

$$-\partial_t u - \frac{1}{2} \Delta u + \frac{1}{2} |\nabla u|^2 = f[\rho_t] \quad (t, x) \in (0, T) \times \Omega \quad (1a)$$

$$\partial_t \rho_t - \frac{1}{2} \Delta \rho_t - \operatorname{div}(\rho_t \nabla u) = 0 \quad (1b)$$

$$\rho_0 \text{ given} \quad (1c)$$

$$u(T, \cdot) = g[\rho_T] \quad (1d)$$

where and $t \mapsto \rho_t$ is a trajectory in the space of measures, and Ω is the standard Euclidean space \mathbb{R}^d . The applications f and g are supposed to be derivatives of some real-valued functionals F and G . For instance, if $G(\mu) = \int_{\Omega} \Psi d\mu(x)$ then its derivative is $g[\mu](x) = \Psi(x)$.

The equations (1a)–(1b) form a coupled system of control (Hamilton-Jacobi-Bellman) and diffusion (forward Kolmogorov) equations.

1.1 The variational problem

The first idea of [3] is to cast the MFG partial differential equations to a variational problem over an appropriate function space. Denote $\mathbb{W}_2(\Omega) = (\mathcal{P}_2(\Omega), \mathcal{W}_2)$ the set of probability measures with finite second moment, equipped with the Wasserstein metric

$$\mathcal{W}_2(\mu, \nu)^2 = \inf_{\gamma \in \Pi(\mu, \nu)} \int |x - y|^2 d\gamma \quad (2)$$

where $\Pi(\mu, \nu) = \{\gamma \in \mathcal{P}_2(\Omega \times \Omega) : P_{\#}^1 \gamma = \mu, P_{\#}^2 \gamma = \nu\}$ is the set of transport plans from μ to ν . Then, $\mathcal{C}([0, T], \mathbb{W}_2(\Omega))$ is the Wiener space of continuous \mathbb{W}_2 -valued trajectories. Benamou, Carlier, and Santambrogio [3] show that the MFG be reformulated to the following variational problem:

$$\inf_{\rho, v} J(\rho, v) = \frac{1}{2} \int_0^T \int_{\Omega} |v_t|^2 d\rho_t(x) dt + \int_0^T F(\rho_t) dt + G(\rho_T) \quad (3a)$$

$$\text{s.t. } \partial_t \rho_t - \frac{1}{2} \Delta \rho_t + \text{div}(\rho_t v) = 0 \quad (3b)$$

$$\rho_0 \in \mathbb{W}_2(\Omega) \quad (3c)$$

where $\rho = (\rho_t)_{t \in [0, T]} \in \mathcal{C}([0, T], \mathbb{W}_2(\Omega))$ is a trajectory in \mathbb{W}_2 and v is a sufficiently regular function on $[0, T] \times \Omega$ (most likely a Sobolev space).

Benamou et al. [4] also introduce the following partial problem:

$$\text{FP}_h(\mu, \nu) = \inf_{\rho, v} \int_0^h \int_{\Omega} |v_t|^2 d\rho_t(x) dt \quad \text{s.t. } \partial_t \rho_t - \frac{1}{2} \Delta \rho_t + \text{div}(\rho_t v), \rho_0 = \mu, \rho_h = \nu \quad (4)$$

It can be used to connect approximations of the solution measure to our MFG problem at discrete times $t_k = kh$, $k = 0, \dots, N$.

This point of view [3] is called *Eulerian*: we minimize over both the velocity v and the time-trajectory of the agents' density ρ . It is not very practical because of the structure of the constraint (a Fokker-Planck equation). Instead, we could minimize over measures in the space of individual agents' trajectories, which is the base of the *Lagrangian* formulation [2, 3] proposed by Benamou, Carlier, and Santambrogio and that we explore in the sequel.

2 Lagrangian dual formulation

2.1 Wiener space and measure

This new point of view involves a change in function spaces. We denote $\mathcal{E} = \mathcal{C}([0, T], \Omega)$ the Wiener space of (agents') trajectories $[0, T] \rightarrow \Omega$. Following [3, 2], we equip it with the Wiener measure (the law of a Wiener process with any starting point x)

$$R = \int_{\Omega} \delta_{x+W} dx$$

where W is a standard Wiener process in \mathbb{R}^d . It is an analogue in the space \mathcal{E} to the usual finite-dimensional Lebesgue measure¹.

Measures $Q \in \mathcal{P}(\mathcal{E})$ can also be seen as trajectories $(Q_t)_{t \in [0, T]} \in \mathcal{C}([0, T], \mathcal{M}(\Omega))$, with

$$Q_t = e_{t\#} Q \in \mathcal{P}(\Omega)$$

the push-forward of Q by the evaluation map $e_t: \xi \in \mathcal{E} \mapsto \xi(t)$. This naturally defines an injection $i: \mathcal{P}(\mathcal{E}) \rightarrow \mathcal{C}([0, T], \mathcal{P}(\Omega))$. We also introduce the more general marginals $Q_{t_1, \dots, t_n} = (e_{t_1}, \dots, e_{t_n})_{\#} Q$ for $0 \leq t_1 < \dots < t_n \leq T$.

¹https://en.wikipedia.org/wiki/Infinite-dimensional_Lebesgue_measure

Marginals of the Wiener measure R . We introduce the heat kernel $G_t(u) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|u|^2}{2t}\right)$. In particular, R_t is the Lebesgue measure \mathcal{L}^d on \mathbb{R}^d , and

$$R_{s,t}(dx, dy) = G_{t-s}(x - y) dx dy. \quad (5)$$

Benamou, Carlier, and Santambrogio [3] and Benamou and Carlier [2] re-cast the Eulerian MFG variational problem (3) into an optimization problem over the set of Borel probability measures (more specifically that associated with the Sobolev subspace H^1 of \mathcal{E}). This new optimization problem is solved in [3] using a finite element method, which is computationally expensive.

2.2 The entropic Lagrangian approach

Instead, Benamou et al. [4] propose using an entropy minimization approach to allow for a more computationally efficient method adapted from the Sinkhorn algorithm [6] developed by Cuturi.

This method, just like the Sinkhorn for OT between histograms (discrete measures), introduces some sort of entropic regularization [4], but this time on the measure over the trajectory space \mathcal{E} . The resulting numerical algorithm becomes a regularization of the Lagrangian from [1].

For all measures Q on \mathcal{E} admitting a density with respect to R , we define the relative entropy

$$H(Q|R) = \int_{\mathcal{E}} \ln\left(\frac{dQ}{dR}\right) dQ(\xi) \quad (6)$$

The entropic Lagrangian version of (3) is the variational problem

$$\inf_{Q \in \mathcal{P}(\mathcal{E})} H(Q|R) + \int_0^T F(Q_t) dt + G(Q_T), \text{ s.t. } Q_0 = \rho_0 \quad (7)$$

Partial transport problem Benamou et al. provide another partial transport problem:

$$S_h(\mu, \nu) = \inf \{H(Q|R) : Q \in \mathcal{P}(\mathcal{C}([0, h], \Omega)), Q_0 = \mu, Q_h = \nu\} \quad (8)$$

This problem can be seen as a continuous OT problem between the two measures μ and ν . Benamou et al. [4] show that it is linked to the partial Eulerian problem (4) as

$$S_h(\mu, \nu) = \text{FP}_h(\mu, \nu) + \text{Ent } \mu.$$

The dimensionality of problem (8) can be greatly simplified; according to [4] we can rewrite it as a static OT problem

$$S_h(\mu, \nu) = \inf \{H(\gamma, R_{0,h}) : \gamma \in \Pi(\mu, \nu)\}. \quad (9)$$

3 Numerical algorithm

Let N be the number of discrete steps for the time discretization of the problem, and $h = T/N$ the time step.

We consider the following multi-marginal OT problem

$$\mathcal{S}(\mu_0, \dots, \mu_N) = \inf_{\gamma \in \Pi(\mu_0, \dots, \mu_N)} H(\gamma|R^N) \quad (10)$$

where $t_k = kh$, $R^N = R_{t_0, \dots, t_N}$ and the marginals $\mu_k \in \mathcal{P}_2(\Omega)$. Then, define

$$\mathcal{U}(\mu_0, \dots, \mu_N) = h \sum_{k=1}^{N-1} F(\mu_k) + G(\mu_N).$$

The discretized entropy minimization problem can be written

$$\inf \{ \mathcal{S}(\mu_0, \dots, \mu_N) + \mathcal{U}(\mu_0, \dots, \mu_N) : \mu_k \in \mathcal{P}_2(\Omega), \mu_0 = \rho_0 \}.$$

Expanding the inf-within-inf leads to the following convex optimization problem:

$$\begin{aligned} \inf_{\gamma \in \mathcal{P}(\Omega^{N+1})} & H(\gamma | R^N) + \iota_{\rho_0}(\mu_0) + \sum_{k=1}^{N-1} F(\mu_k) + G(\mu_N) \\ \text{s.t. } & \mu_k = P_{\#}^k \gamma \end{aligned} \quad (11)$$

where $\iota_{\rho_0}(\mu) = +\infty$ if $\mu \neq \rho_0$ and 0 otherwise is the convex indicatrix of the measure ρ_0 .

Benamou et al. [4] provide the corresponding dual problem involving the convex conjugates and potential functions, by using a multimarginal generalization of a result from Chizat et al. [5]:

$$\sup_u -\iota_{\rho_0}^*(-u_0) - \sum_{k=1}^{N-1} F^*(-u_k) - G^*(-u_N) - \int_{\Omega^{N+1}} (\exp(\oplus_{k=0}^N u_k) - 1) dR^N \quad (12)$$

where the supremum is taken over $u = (u_0, \dots, u_N) \in L^\infty(\Omega)^{N+1}$.

Remark 1 (Some convex conjugates)

In practice, the convex conjugates of the cost functions are difficult to compute. For some of the examples in the paper, we have closed-form conjugates.

- The conjugate of the convex indicatrix ι_ν of any given measure ν is given by $\iota_\nu^*(u) = \langle \nu, u \rangle$.

- The hard congestion constraint $F(\rho) = \begin{cases} 0 & \text{if } \rho \leq \bar{\rho} \\ +\infty & \text{otherwise} \end{cases}$, has convex conjugate

$$F^*(u) = \sup_{\rho \leq \bar{\rho}} \langle \rho, u \rangle = \bar{\rho} \|u\|_{L^\infty(\Omega)}$$

- Obstacle constraints, given by

$$F(\rho) = \int_{\Omega} V(x) d\rho(x)$$

where V is the convex indicatrix of a set of obstacles $\mathcal{O} \subset \Omega$. Its conjugate is given for $u \in L^\infty(\Omega)$ by

$$F^*(u) = \begin{cases} 0 & \text{if } u \leq 0 \text{ on } \Omega \setminus \mathcal{O} \\ +\infty & \text{otherwise} \end{cases}$$

Benamou et al. [4] introduce a Sinkhorn-like iterative algorithm to solve the above dual problem. We rewrite it more explicitly with slightly different notations inspired by [5]

Algorithm 1. Denote for $k = 0, \dots, N$ and $(a_j)_{j \neq k}$

$$\mathcal{I}_k(a_{\neq k})(z_k) = \int_{(x_{\neq k}) \in \Omega^N} \otimes_{j \neq k} a_j dR^N(x_{0:k-1}, z_k, x_{k+1:N})$$

the partial integral operator on the functions $a_j, j \neq k$ with respect to R^N . For convenience we use the shorthand for the iterates

$$\mathcal{I}_k^{(n)} = \mathcal{I}_k \left(\left(e^{u_j^{(n+1)}} \right)_{j < k}, \left(e^{u_j^{(n)}} \right)_{j > k} \right)$$

for the n th iterate.

Then we compute the dual potentials iteratively:

$$\begin{cases} u_0^{(n+1)} = \operatorname{argmax}_{v \in L^\infty} -i_{\rho_0}^*(-v) - \int_{\Omega} e^{v(x_0)} \mathcal{I}_0^{(n)} dx_0 \\ u_k^{(n+1)} = \operatorname{argmax}_{v \in L^\infty} -hF^*(-v) - \int_{\Omega} e^{v(x_k)} \mathcal{I}_k^{(n)} dx_k, & 1 \leq k < N \\ u_N^{(n+1)} = \operatorname{argmax}_{v \in L^\infty} -G^*(-v) - \int_{\Omega} e^{v(x_N)} \mathcal{I}_N^{(n)} dx_N \end{cases} \quad (13)$$

until convergence.

By strong duality, the iterates $u_k^{(n)}$ satisfy

$$a_k^{(n)} \odot \mathcal{I}_k^{(n)} = \operatorname{prox}_{F_k}^{\text{KL}}(\mathcal{I}_k^{(n)})$$

where $a_k^{(n)} = \exp(u_k^{(n)})$ and

$$\operatorname{prox}_F^{\text{KL}}(z) = \operatorname{argmin}_{s \in L^1} F(s) + \text{KL}(s|z).$$

3.1 Full discretization

For full numerical implementation, all measures are replaced by multi-dimensional arrays representing discrete histograms over a fixed grid of points in \mathbb{R}^d of dimensionality $M = N_1 \times \dots \times N_d$. Integration is exchanged with summation.

The iterations are solved using the Python library CVXPY^{2,3}.

4 Examples

4.1 Crowd congestion

Supposing a model where F is the hard congestion functional, the Kantorovitch dual problem is written

$$\sup_u \langle \rho_0, u_0 \rangle - \sum_{k=1}^{N-1} \|u_k\|_{L^\infty(\Omega)} - \int_{\Omega^{N+1}} (\exp(\oplus_{k=0}^N u_k) - 1) dR^N \quad (14)$$

²<https://github.com/cvxgrp/cvxpy>

³Steven Diamond and Stephen Boyd. “CVXPY: A Python-Embedded Modeling Language for Convex Optimization”. In: *Journal of Machine Learning Research* 17.83 (2016), pp. 1–5.

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