*Computational Optimal Transport*

# Regularized Optimal Transport methods for solving variational Mean-Field Games

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## 1 Quadratic Mean-field games

A mean-field game [8, 9] is a strategic decision-making problem with a very large, continuously-distributed number of interacting agents inside a state space: the overall theory developed by Lasry and Lions can be used as a means to model large, computationally intractable games. In the continuous-time setting explored in [9], each agent evolves according to some dynamics – leading to a so-called *differential game* – with the response to his choices depends on other agents' states and actions through a *mean-field* effect.

The general setup of a MFG has every agent penalize a running cost on the control and mean-field interaction, as well as a terminal cost on the its final position and the overall final distribution of agents (see [9]). The framework of [3, 4] focuses on games with agent dynamics $dX_t = \alpha_t dt + dW_t$ and quadratic running cost on the control $\alpha$

$$\int_0^T L(\alpha_t) = \int_0^T \frac{\alpha_t^2}{2}$$

The (Nash) equilibrium agent-control dynamics can be summarized by the system of coupled nonlinear partial differential equations:

$$-\partial_t u - \frac{1}{2}\Delta u + \frac{1}{2}|\nabla u|^2 = f[\rho_t], \quad (t,x) \in (0,T) \times \Omega \tag{1.1a}$$

$$\partial_t \rho_t - \frac{1}{2}\Delta \rho_t - \operatorname{div}(\rho_t \nabla u) = 0 \tag{1.1b}$$

$$\rho_0 \text{ given} \tag{1.1c}$$

$$u(T,\cdot) = g[\rho_T] \tag{1.1d}$$

where and $t \mapsto \rho_t$ is a trajectory in the space of measures, and $\Omega$ a subset of the Euclidean space $\mathbb{R}^d$. The applications $f[\mu]$ and $g[\mu]$ are supposed to be derivatives of some real-valued functionals $F$ and $G$ on the space of measures. For instance, if $G(\mu) = \int_\Omega \Psi \, d\mu(x)$ then its derivative is $g[\mu](x) = \Psi(x)$.

Equations (1.1a) and (1.1b) form a coupled system of control (Hamilton-Jacobi-Bellman) and diffusion (Fokker-Planck) partial differential equations. They can be solved in some cases using finite-difference methods (see Achdou, Camilli, and Capuzzo-Dolcetta [1]).

# 2 Variational formulations for the quadratic MFG

The first idea of [3] is to cast the MFG partial differential equations to a variational problem over an appropriate function space. Denote $\mathbb{W}_2(\Omega) = (\mathcal{P}_2(\Omega), \mathcal{W}_2)$ the set of probability measures with finite second moment, equipped with the Wasserstein metric

$$\mathcal{W}_2(\mu,\nu)^2 = \inf_{\gamma\in\Pi(\mu,\nu)} \int |x-y|^2 d\gamma \tag{2.1}$$

where $\Pi(\mu,\nu) = \{\gamma \in \mathcal{P}_2(\Omega\times\Omega) : P^1_{\#}\gamma = \mu,\ P^2_{\#}\gamma = \nu\}$ is the set of transport plants from $\mu$ to $\nu$. Then, $\mathcal{C}([0,T], \mathbb{W}_2(\Omega))$ is the Wiener space of continuous $\mathbb{W}_2$-valued trajectories. Benamou, Carlier, and Santambrogio [3] show that the MFG can be reformulated to the following variational problem:

$$\begin{aligned} &\inf_{\rho,v} J(\rho,v) = \frac{1}{2}\int_0^T \int_\Omega |v_t|^2\, d\rho_t(x)\, dt + \int_0^T F(\rho_t)\, dt + G(\rho_T) \\ &\text{s.t. } \partial_t\rho_t - \frac{1}{2}\Delta\rho_t + \operatorname{div}(\rho_t v) = 0 \\ &\rho_0 \in \mathbb{W}_2(\Omega) \end{aligned} \tag{2.2}$$

where $\rho = (\rho_t)_{t\in[0,T]} \in \mathcal{C}([0,T], \mathbb{W}_2(\Omega))$ is a trajectory in $\mathbb{W}_2$ and $v$ is a sufficiently regular function on $[0,T]\times\Omega$, most likely lying in a Sobolev space – see [3] for further discussion on regularity.

This point of view [3] is called *Eulerian*: we minimize over both the velocity $v$ and the time-trajectory of the agents' density $\rho$. This can be solved by introducing Lagrange multipliers, exploiting duality, and using a finite element method, as shown in [3].

Benamou, Carlier, and Santambrogio [3] and Benamou et al. [4] also introduce a *Lagrangian* point of view, which allows to use tools from optimal transport theory: the variational problem is changed to optimize over the space of probability distributions on the space of agent trajectories.

## 2.1 Lagrangian formulation

**Wiener space and measure.** This new point of view involves a change in function spaces. We denote $\mathcal{X} = \mathcal{C}([0,T],\Omega)$ the Wiener space of (agents') trajectories $[0,T]\to\Omega$. Following [3, 2], we equip it with the Wiener measure (the law of a Wiener process with any starting point $x$)

$$R = \int_\Omega \delta_{x+W}\, dx$$

where $W$ is a standard Wiener process in $\mathbb{R}^d$. It is an analogue in the space $\mathcal{X}$ to the usual finite-dimensional Lebesgue measure[1].

Measures $Q \in \mathcal{P}(\mathcal{X})$ can also be seen as trajectories $(Q_t)_{t\in[0,T]}$ in $\mathcal{P}(\Omega)$ with

$$Q_t = e_{t\#}Q \in \mathcal{P}(\Omega)$$

the push-forward of $Q$ by the evaluation map $e_t \colon \xi \in \mathcal{X} \longmapsto \xi(t)$. This naturally defines an injection $\underline{i} \colon \mathcal{P}(\mathcal{X}) \to \mathcal{C}([0,T], \mathcal{P}(\Omega))$. We also introduce the more general marginals $Q_{t_1,\dots,t_n} = (e_{t_1}, \dots, e_{t_n})_{\#}Q$ for $0 \le t_1 < \dots < t_N \le T$.

**Marginals of the Wiener measure.** In particular, $R_t$ is the Lebesgue measure $\mathcal{L}^d$ on $\mathbb{R}^d$, and we have that the final marginals have densities:

$$R_{s,t}(x,y) = P_{t-s}(y-x). \tag{2.3}$$

where $P_t$ is the standard $d$-dimensional heat kernel:

$$P_t(u) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|u|^2}{2t}\right) \tag{2.4}$$

The $N$-marginals are given by

$$R_{t_1,\dots,t_N}(x_1,\dots,x_n) = \prod_{i=1}^{N-1} P_h(x_{i+1} - x_i) \tag{2.5}$$

Benamou, Carlier, and Santambrogio [3] and Benamou and Carlier [2] then re-cast the Eulerian variational game (2.2) into a Lagrangian optimization problem over the set of Borel probability measures. This new problem is also solved in [3] using a finite element method.

**Integration** Partial integration with respect to the 2-marginal measure $R_{0,h}$ is actually convolution with respect to the heat kernel $P_h$:

$$\int_\Omega u(x) R_{0,h}(x,y)\,dx = \int_\Omega u(x) P_h(y-x)\,dx = (u * P_h)(y)$$

The effect of integration against the $N$-marginal can then be deduced by induction.

## 2.2 Entropic Lagrangian

Instead of using finite element methods, Benamou et al. [4] propose using an entropy minimization approach to allow for a more computationally efficient method adapted from the Sinkhorn algorithm introduced by Cuturi [6]. This method introduces entropic regularization in the problem, but this time on the measure over the trajectory space

[1] https://en.wikipedia.org/wiki/Infinite-dimensional_Lebesgue_measure

$\mathcal{X}$. The resulting numerical algorithm becomes a regularization of the Lagrangian from [3, 2].

The entropic Lagrangian variational problem is

$$\inf_{Q\in\mathcal{P}(\mathcal{X})} H(Q|R) + \int_0^T F(Q_t)\,dt + G(Q_T) \quad \text{s.t. } Q_0 = \rho_0 \tag{2.6}$$

Intuitively, this is the same as fixing the marginals $\rho_t$, finding the optimal bridge $Q$ between them that has minimal entropy relative to the Wiener measure, and then optimizing over the $\rho_t$.

### 2.3 Viscosity and the deterministic limit

We change the MFG problem to one following the agent dynamics $dX_t = \alpha_t dt + \sigma dW_t$ with a diffusion coefficient $\sigma$. The MFG equilibrium equations become

$$\begin{aligned} -\partial_t u - \frac{\sigma^2}{2}\Delta u + \frac{1}{2}|\nabla u|^2 &= f[\rho_t] \\ \partial_t \rho - \frac{\sigma^2}{2}\Delta\rho - \operatorname{div}(\rho\nabla u) &= 0 \end{aligned} \tag{2.7}$$

This can be used to approximate first-order MFGs by setting a low viscosity parameter $\sigma$. Denoting $\varepsilon = \sigma^2$, the entropic variational problem (2.6) becomes

$$\inf_{Q\in\mathcal{P}(\mathcal{X})} \varepsilon H(Q|R_\varepsilon) + \int_0^T F(Q_t)\,dt + G(Q_T) \quad \text{s.t. } Q_0 = \rho_0 \tag{2.8}$$

where $R_\varepsilon$ is the Wiener measure associated with Wiener processes scaled by $\varepsilon$.

# 3 Numerical algorithm

### 3.1 Time discretization

Let $N$ be the number of discrete steps for the time discretization of the problem, and $h = T/N$ the time step.

Benamou et al. [4] propose a discretization of (2.6) obtained by connecting the marginals through a multimarginal OT problem:

$$\mathcal{S}(\mu_0, \dots, \mu_N) = \inf_{\gamma\in\Pi(\mu_0,\dots,\mu_N)} H(\gamma|R^N) \tag{3.1}$$

where $t_k = kh$, $R^N = R_{t_0,\dots,t_N}$ and the marginals $\mu_k \in \mathcal{P}_2(\Omega)$. Then, define

$$\mathcal{U}(\mu_0, \dots, \mu_N) = h\sum_{k=1}^{N-1} F(\mu_k) + G(\mu_N). \tag{3.2}$$

Thus, the discretized entropy minimization problem can be written as

$$\inf \left\{ \mathcal{S}(\mu_0, \ldots, \mu_N) + \mathcal{U}(\mu_0, \ldots, \mu_N) : \mu_k \in \mathcal{P}_2(\Omega),\ \mu_0 = \rho_0 \right\}.$$

Expanding the inf-within-inf leads to the following convex optimization problem:

$$\begin{aligned} &\inf_{\gamma \in \mathcal{P}(\Omega^{N+1})} \mathrm{KL}(\gamma | R^N) + \imath_{\rho_0}(\mu_0) + \sum_{k=1}^{N-1} F(\mu_k) + G(\mu_N) \\ &\text{s.t. } \mu_k = P^k_\# \gamma \end{aligned} \tag{3.3}$$

where $\imath_{\rho_0}(\mu) = +\infty$ if $\mu \neq \rho_0$ and 0 otherwise is the convex indicatrix of the measure $\rho_0$. This is a generalized multimarginal optimal transport problem.

Benamou et al. [4] provide the corresponding dual problem involving the convex conjugates and potential functions, by using a multimarginal generalization of a result from Chizat et al. [5]:

$$\sup_u \int_{\Omega^{N+1}} \left(1 - \exp\left(\oplus_{k=0}^N u_k\right)\right) dR^N - \imath^*_{\rho_0}(-u_0) - \sum_{k=1}^{N-1} F^*(-u_k) - G^*(-u_N) \tag{3.4}$$

where the supremum is taken over $u = (u_0, \ldots, u_N) \in L^\infty(\Omega)^{N+1}$.

Benamou et al. [4] introduce a Sinkhorn-like iterative algorithm to solve the above dual problem. We rewrite it more explicitly with slightly different notations inspired by [5].

**Algorithm 1.** *Denote for $k = 0, \ldots, N$ and $(a_j)_{j \neq k}$*

$$\mathcal{I}_k((a_j)_{j\neq k})(\tilde{x}_k) = \int_{\Omega^N} \prod_{j \neq k} a_j(x_j) R^N(dx_{0:k-1}, \tilde{x}_k, dx_{k+1:N})$$

*the partial integral of the $a_j, j \neq k$ with respect to $R^N$ without variable $x_k$. For convenience we use the shorthand*

$$\mathcal{I}_k^{(n)} = \mathcal{I}_k\left(\left(a_j^{(n+1)}\right)_{j<k}, \left(a_j^{(n)}\right)_{j>k}\right)$$

*for the $n$th iterate where we denote $a_j = \exp(-u_j)$.*

*Then we compute the dual potentials iteratively:*

$$\begin{cases} u_0^{(n+1)} = \operatorname*{argmax}_{v \in L^\infty} \displaystyle\int_\Omega (1 - e^{-v(x_0)}) \mathcal{I}_0^{(n)}\, dx_0 - \imath^*_{\rho_0}(v) \\ u_k^{(n+1)} = \operatorname*{argmax}_{v \in L^\infty} \displaystyle\int_\Omega (1 - e^{-v(x_k)}) \mathcal{I}_k^{(n)}\, dx_k - hF^*(v), \quad 1 \leq k < N \\ u_N^{(n+1)} = \operatorname*{argmax}_{v \in L^\infty} \displaystyle\int_\Omega (1 - e^{-v(x_N)}) \mathcal{I}_N^{(n)}\, dx_N - G^*(v) \end{cases} \tag{3.5}$$

*until convergence.*

*Using duality, we find that the iterates $u_k^{(n)}$ satisfy*

$$a_k^{(n)} = \exp\left(-u_k^{(n)}\right) = \frac{\operatorname{prox}^{\mathrm{KL}}_{F_k}(\mathcal{I}_k^{(n)})}{\mathcal{I}_k^{(n)}} \tag{3.6}$$

*where*

$$\operatorname{prox}^{\mathrm{KL}}_F(z) = \operatorname*{argmin}_s F(s) + \mathrm{KL}(s|z)$$

*is the KL-proximal operator.*

**Remark 1** (Some convex conjugates)**.** In practice, the convex conjugates of the cost functions are difficult to compute. For some of the examples in the paper, we have closed-form conjugates.

- The conjugate of the convex indicatrix $\imath_\nu$ of any measure $\nu$ is given by $\imath_\nu^*(u) = \langle u, \nu\rangle$.
- The hard congestion constraint

  $$C(\rho) = \begin{cases} 0 & \text{if } \rho \leq \bar{m} \\ +\infty & \text{otherwise} \end{cases}$$

  has convex conjugate (on the domain $\rho \geq 0$)

  $$C^*(u) = \sup_{\rho\leq\bar{m}} \langle u, \rho\rangle = \langle u^+, \bar{m}\mathbb{1}\rangle$$

- Obstacle constraints, given by

  $$F(\rho) = \int_\Omega V(x)\,d\rho(x) = \begin{cases} 0 & \text{if } \rho = 0 \text{ on } \mathscr{O} \\ +\infty & \text{otherwise} \end{cases} = \imath_0(\mathbb{1}_{\mathscr{O}}\rho)$$

  where $V$ is the convex indicatrix of the complement $\Omega\backslash\mathscr{O}$ of the obstacles. Its conjugate is given by

  $$F^*(u) = \begin{cases} 0 & \text{if } u \leq 0 \text{ on } \Omega\backslash\mathscr{O} \\ +\infty & \text{otherwise} \end{cases}$$

### 3.2 Spatial discretization

For full numerical implementation, all measures are replaced by multi-dimensional arrays representing discrete histograms over a fixed grid of points $x_i$ in $\mathbb{R}^d$ of size $M = N_1 \times \cdots \times N_d$, and we naturally exchange convolution with the heat kernel for discrete convolution. This will be the main computational issue.

**Projections.** In the general case, the KL-projections in the Sinkhorn iterations can be solved using the Python library CVXPY[2,3]. Some can be computed explicitly.

**Proposition 1.** *The KL-projection on the hard congestion constraint of a measure $\beta \in \mathbb{R}^M$ is given by*

$$\mathrm{prox}_C^{\mathrm{KL}}(\beta) = \min(\beta, \overline{m}) \tag{3.7}$$

*where the minimum is taken element-wise.*

*The KL-projection on the obstacle constraint is*

$$\mathrm{prox}^{\mathrm{KL}}(\beta) = \beta \mathbb{1}_{\Omega\setminus\mathscr{O}} \tag{3.8}$$

*If we also add the obstacle constraint on a set $\mathscr{O}$ of points in the grid, then the proximal operator reads*

$$\mathrm{prox}_F^{\mathrm{KL}}(\beta) = \min(\beta, \bar{m})\mathbb{1}_{\Omega\setminus\mathscr{O}}. \tag{3.9}$$

**Efficient convolution.** Denote $\mathbf{R} \in \mathbb{R}^{M^N}$ the discretized measure $R^N$. Summation of multiple vectors $u_0, \dots, u_N$ with respect to $\mathbf{R}$ is the following tensor contraction

$$\mathbf{R}[u_0, \dots, u_N] = \sum_{i_0,\dots,i_N} \mathbf{R}_{i_0,\dots,i_N} \prod_{k=0}^{N} u_{i_k}$$

The multi-marginal kernel $\mathbf{R}$ can be factorized as $\mathbf{R}_{i_0,\dots,i_N} = \prod_{k=0}^{N-1} \mathbf{P}_{i_k,i_{k+1}}$ where $\mathbf{P}$ is the discrete heat kernel on $\mathbb{R}^M$. This allows us to write the partial convolution $\mathcal{I}_k$ (leaving the $k$th component out) as

$$\mathcal{I}_k = \mathbf{R}[(u_j)_{j\neq k}] = \mathbf{A}_{k-1} \odot \mathbf{B}_{k+1} \tag{3.10}$$

where $\mathbf{A}_k = \mathbf{P}^T(a_k \odot \mathbf{P}^T(a_{k-1} \odot \cdots))$ and $\mathbf{B}_k = \mathbf{P}(a_k \odot \mathbf{P}(a_{k+1} \odot \cdots))$.

# 4 Examples

## 4.1 Transport with a soft target

### 4.1.1 Two-marginal case

We start with a very simplified approximation of the crowd displacement problem on $\Omega = [0,1]^2$, with only the first step (with initial agent distribution) and final step decided by the terminal penalty function $G$.

We set $G$ to be the obstacle constraint related to a subset $\mathscr{O}$ of $\Omega$ as well as a potential $\Psi(x) = d(x, \mathscr{A})^\beta$ for some $\beta > 0$, related to the distance to a target subset $\mathscr{A}$ (see fig. 1):

$$G(\mu) = \int_\Omega \Psi \, d\mu + \imath_0(\mu \mathbb{1}_{\mathscr{O}})$$

[2] https://github.com/cvxgrp/cvxpy

[3] Steven Diamond and Stephen Boyd. "CVXPY: A Python-Embedded Modeling Language for Convex Optimization". In: *Journal of Machine Learning Research* 17.83 (2016), pp. 1–5.

Thus, the agents engage in a one-round mean-field game where they are only concerned with moving to regions with lower potential $\Psi$ – as close as possible to $\mathscr{A}$ – whilst obeying physical constraints related to the obstacles.

The discretized MFG problem with viscosity parameter $\varepsilon = \sigma^2$ can be written as the following transport problem:

$$\begin{aligned} &\inf_{\gamma} \langle \Psi, \gamma^T \mathbb{1} \rangle + \varepsilon \, \mathrm{KL}(\gamma | \mathbf{R}) \\ &\text{s.t. } \gamma \mathbb{1} = \rho_0, \quad \gamma^T \mathbb{1} \odot \mathbb{1}_{\mathscr{O}} = 0 \end{aligned} \tag{4.1}$$

In this two-marginal case, the matrix $\mathbf{R} = \mathbf{P}$ is the discretization of the heat kernel $P_h$ as discussed in section 3.2:

$$\mathbf{R}_{i,j} = P_{h\varepsilon}(x_j - x_i)$$

for all grid indices $i, j$.

In this problem we want to observe the optimal target distribution $\rho_1^* = (\gamma^*)^T \mathbb{1}$ the agents reach at the final time $t = 1$.

**Proposition 2.** *Problem (4.1) can be solved in closed form: the Lagrange multiplier $u_0^*$ for the marginal law constraint satisfies*

$$a_0^* = e^{-u_0^*/\varepsilon} = \frac{\rho_0}{\mathbf{R} a_1^*}$$

*where $a_1^* = e^{-\Psi/\varepsilon} \odot \mathbb{1}_{\Omega \setminus \mathscr{O}}$, and the optimal coupling is*

$$\gamma^* = \mathbf{R} \odot (a_0^* \otimes a_1^*)$$

*It satisfies, as expected, that $\gamma_{i,j}^* = 0$ for all $j \in \mathscr{O}$. The final distribution of agents is*

$$\rho_1 = \gamma^T \mathbb{1} = a_1^* \odot \mathbf{R} a_0^*$$

**Numerical experiment** We ran a numerical experiment by implementing the solutions given by proposition 2 to the discrete MFG (4.1). fig. 2a provides a representation of both . We also checked the results when removing the constraints on the obstacles (essentially setting $\mathscr{O}$), and when lowering the viscosity parameter $\sigma = \sqrt{\varepsilon}$ (see figs. 2b and 2c).

Since the kernel $\mathbf{R}$ on the domain is separable, the convolution can be sped up [10, ch. 4].

**Remark 2** (A "smarter" (more realistic) potential for crowd dynamics)**.** The results shown fig. 2 are satisfactory for the given potential $\Psi$ – as expected the agents try to stay near the low-potential regions. However, for modeling of crowd dynamics they would be deeply nonphysical because the potential is inadequate. In a room evacuation scenario, for instance, agents would seek to minimize the time-to-exit: the literature shows this leads to the Eikonal equation, a kind of Hamilton-Jacobi PDE. We computed the adequate potential shown fig. 3a using the Fast Sweeping method [11], as well as the discrete MFG fig. 3b.

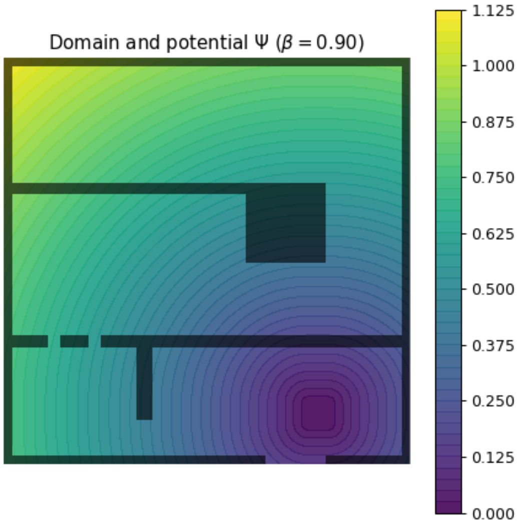



Figure 1: Computational domain of the game $\Omega$ with set of obstacles $\mathscr{O}$ (transparent grey), and contour of the potential function $\Psi(x) = d(x, \mathscr{A})^\beta$.

### 4.1.2 Three time steps

We now go up to three marginals $(\rho_0, \rho_1, \rho_2)$. We assign to the single intermediate marginal $\rho_1$ the congestion constraint $\rho_1 \leq \bar{m}$ and the obstacle constraint. The primal problem then reads

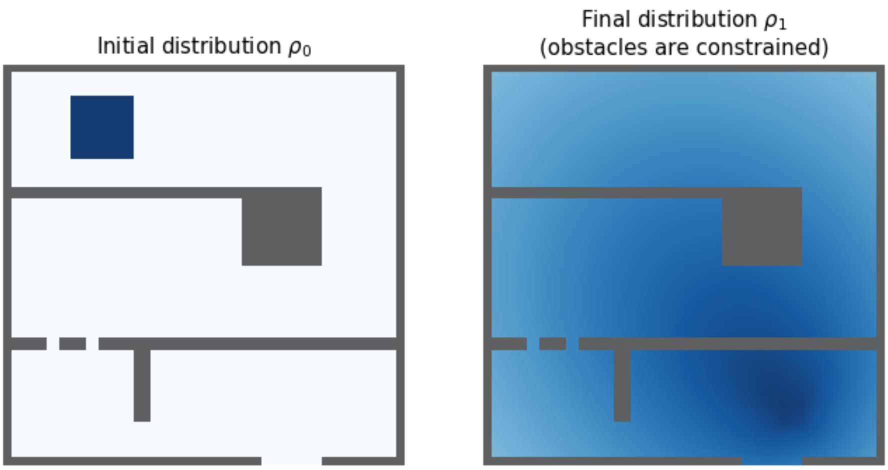
$$\begin{aligned} &\inf_{\gamma,\rho_1,\rho_2} \langle \Psi, \rho_2 \rangle + \varepsilon\, \mathrm{KL}(\gamma|\mathbf{R}) \\ \text{s.t. } & P^k_\# \gamma = \rho_k,\ k = 0,1,2 \\ & \rho_1 \leq \bar{m} \\ & \rho_1 \odot \mathbb{1}_{\mathscr{O}} = 0 \\ & \rho_2 \odot \mathbb{1}_{\mathscr{O}} = 0 \end{aligned} \tag{4.2}$$

Contraction/convolution with the kernel $\mathbf{R}$ in this case satisfies

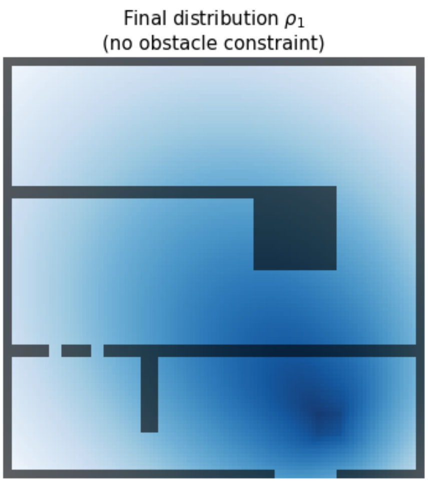
$$\begin{aligned} \mathbf{R}[\cdot, v, w] &= \mathbf{P}(v \odot \mathbf{P}w) \\ \mathbf{R}[u, \cdot, w] &= (\mathbf{P}^T u) \odot (\mathbf{P}w) \\ \mathbf{R}[u, v, \cdot] &= \mathbf{P}^T (v \odot \mathbf{P}^T u) \end{aligned}$$

**Proposition 3.** *The Lagrange multipliers $u_i^*$ at the optimum satisfy the fixed-point conditions:*

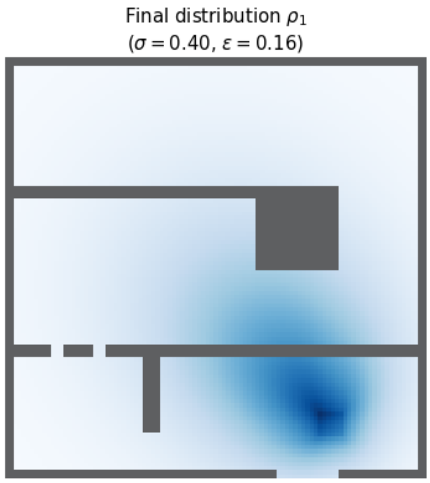
$$\begin{aligned} a_0^* &= \frac{\rho_0}{\mathbf{R}[\,\cdot\,, a_1^*, a_2^*]} \\ a_1^* &= \min\left( \frac{\bar{m}}{\mathbf{R}[a_0^*, \,\cdot\,, a_2^*]}, 1 \right) \end{aligned}$$




(a) Result of the "fuzzy" transport problem with enforcement of the obstacle constraints and viscosity parameter $\sigma = 1$.

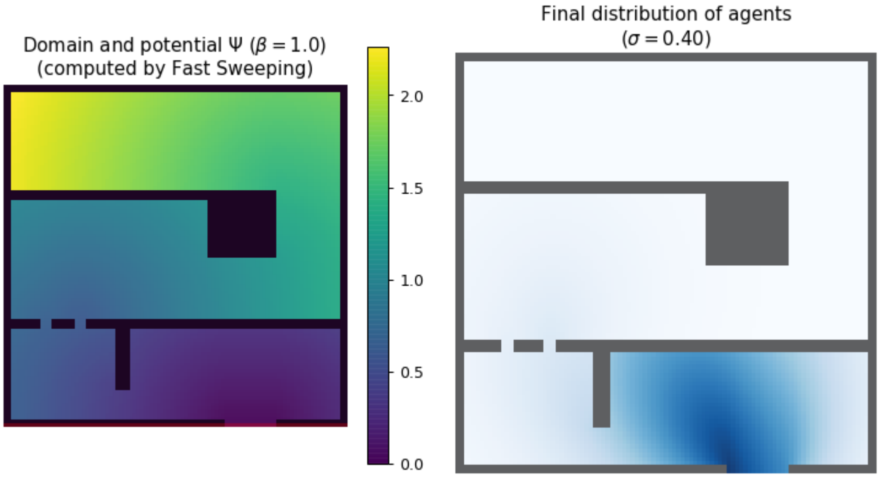



(b) Final distribution $\rho_1^*$ without enforcing the obstacles. The mass of the distribution "bleeds" through the obstacles.




(c) Obstacles are constrained as in fig. 2a, but with viscosity parameter $\sigma = 0.4$.

Figure 2: Marginal distributions of the solution of two-step MFG or "fuzzy transport" problem (4.1), with a few variations.




(a) Domain and potential associated with the fastest path distance.

(b) Optimal terminal distribution $\rho_1^*$ of the discrete MFG with the potential from fig. 3a.

Figure 3: Setup and solution for the discrete MFG using the time-to-exit potential discussed in remark 2.

*where* $a_i^* = \exp(-u_i^*)$, $a_1^*$ *is supported on* $\Omega \backslash \mathscr{O}$ *and* $a_2^* = e^{-\Psi/\varepsilon} \odot \mathbb{1}_{\Omega \backslash \mathscr{O}}$, *and we denote* $\mathbf{R}[\cdot,\cdot,\cdot]$ *the appropriate tensor contraction by* $\mathbf{R}$.

*The marginals are obtained as*

$$\rho_1^* = a_1^* \odot \mathbf{R}[a_0^*, \cdot, a_2^*] = a_1^* \odot (\mathbf{P}^T a_0^*) \odot (\mathbf{P} a_2^*)$$

*and*

$$\rho_2^* = a_2^* \odot \mathbf{R}[a_0^*, a_1^*, \cdot] = a_2^* \odot \mathbf{P}^T (a_1^* \odot \mathbf{P}^T a_0^*)$$

The fixed point can then computed using an iterative algorithm à la generalized Sinkhorn, just as in the Algorithm 1 suggested by [4].

The issue of computational efficiency is more pronounced here than before due to the tensor product and need for multiple iterations until convergence.

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