Computational Optimal Transport – Project report:

Regularized Optimal Transport methods for solving variational Mean-Field Games

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1 General setting: variational mean-field games

A mean-field game [8, 9] is a strategic decision-making problem with a very large, continuously-distributed number of interacting agents inside a state space: the overall theory developed by Lasry and Lions can be used as a means to model large, computationally intractable games. In the continuous-time setting explored in [9], each agent evolves according to some dynamics – leading to a so-called *differential game* – with the response to his choices depends on other agents' states and actions through a *mean-field* effect.

The general setup of a MFG has every agent penalize a running cost on the control and mean-field interaction, as well as a terminal cost on the its final position and the overall final distribution of agents (see [9]). The framework of [3, 4] focuses on games with agent dynamics $dX_t = \alpha_t dt + dW_t$ where the running cost of the control α is a quadratic function.

The (Nash) equilibrium agent-control dynamics can be summarized by the system of coupled nonlinear partial differential equations:

$$-\partial_t u - \frac{1}{2}\Delta u + \frac{1}{2}|\nabla u|^2 = f[\rho_t] \quad (t,x) \in (0,T)\times\Omega \tag{1.1a}$$

$$\partial_t \rho_t - \frac{1}{2}\Delta \rho_t - \mathrm{div}(\rho_t \nabla u) = 0 \tag{1.1b}$$

$$\rho_0 \text{ given} \tag{1.1c}$$

$$u(T,\cdot) = g[\rho_T] \tag{1.1d}$$

where and $t \mapsto \rho_t$ is a trajectory in the space of measures, and Ω a subset of the Euclidean space $\mathbb{R}^d$. The applications $f[\mu]$ and $g[\mu]$ are supposed to be derivatives of some real-valued functionals F and G on the space of measures. For instance, if $G(\mu) = \int_\Omega \Psi \, d\mu(x)$ then its derivative is $g[\mu](x) = \Psi(x)$.

Equations (1.1a) and (1.1b) form a coupled system of control (Hamilton-Jacobi-Bellman) and diffusion (Fokker-Planck) partial differential equations. They can be solved in some cases using finite-difference methods (see Achdou, Camilli, and Capuzzo-Dolcetta [1]).

2 Variational formulations for the quadratic MFG

The first idea of [3] is to cast the MFG partial differential equations to a variational problem over an appropriate function space. Denote $\mathbb{W}_2(\Omega) = (\mathcal{P}_2(\Omega), \mathcal{W}_2)$ the set of probability measures with finite second moment, equipped with the Wasserstein metric

$$\mathcal{W}_2(\mu,\nu)^2 = \inf_{\gamma\in\Pi(\mu,\nu)} \int |x-y|^2 d\gamma \tag{2.1}$$

where $\Pi(\mu,\nu) = \{\gamma \in \mathcal{P}_2(\Omega\times\Omega) : P^1_{\#}\gamma = \mu,\ P^2_{\#}\gamma = \nu\}$ is the set of transport plants from μ to ν. Then, $\mathcal{C}([0,T],\mathbb{W}_2(\Omega))$ is the Wiener space of continuous $\mathbb{W}_2$-valued trajectories. Benamou, Carlier, and Santambrogio [3] show that the MFG be reformulated to the following variational problem:

$$\begin{aligned} &\inf_{\rho,v} J(\rho,v) = \frac{1}{2}\int_0^T \int_\Omega |v_t|^2\, d\rho_t(x)\, dt + \int_0^T F(\rho_t)\, dt + G(\rho_T) \\ \text{s.t. } &\partial_t\rho_t - \frac{1}{2}\Delta\rho_t + \operatorname{div}(\rho_t v) = 0 \\ &\rho_0 \in \mathbb{W}_2(\Omega) \end{aligned} \tag{2.2}$$

where $\rho = (\rho_t)_{t\in[0,T]} \in \mathcal{C}([0,T],\mathbb{W}_2(\Omega))$ is a trajectory in $\mathbb{W}_2$ and v is a sufficiently regular function on $[0,T]\times\Omega$, most likely lying in a Sobolev space – see [3] for further discussion on regularity.

This point of view [3] is called *Eulerian*: we minimize over both the velocity v and the time-trajectory of the agents' density ρ. This can be solved by introducing Lagrange multipliers, exploiting duality, and using a finite element method, as shown in [3].

Benamou, Carlier, and Santambrogio [3] and Benamou et al. [4] also introduce a *Lagrangian* point of view, which allows to use tools from optimal transport theory: the variational problem is changed to optimize over the space of probability distributions on the space of agent trajectories.

2.1 Lagrangian formulation

2.1.1 Wiener space and measure

This new point of view involves a change in function spaces. We denote $\mathcal{X} = \mathcal{C}([0,T],\Omega)$ the Wiener space of (agents') trajectories $[0,T]\to\Omega$. Following [3, 2], we equip it with the Wiener measure (the law of a Wiener process with any starting point x)

$$R = \int_\Omega \delta_{x+W}\, dx$$

where W is a standard Wiener process in $\mathbb{R}^d$. It is an analogue in the space $\mathcal{X}$ to the usual finite-dimensional Lebesgue measure[1].

Measures $Q\in\mathcal{P}(\mathcal{X})$ can also be seen as trajectories $(Q_t)_{t\in[0,T]} \in \mathcal{C}([0,T],\mathcal{P}(\Omega))$, with

$$Q_t = e_{t\#}Q \in \mathcal{P}(\Omega)$$

the push-forward of Q by the evaluation map $e_t\colon \xi\in\mathcal{X} \longmapsto \xi(t)$. This naturally defines an injection $\underline{i}\colon \mathcal{P}(\mathcal{X}) \to \mathcal{C}([0,T],\mathcal{P}(\Omega))$. We also introduce the more general marginals $Q_{t_1,\dots,t_n} = (e_{t_1},\dots,e_{t_n})_{\#}Q$ for $0\le t_1 < \cdots < t_N \le T$.

[1] https://en.wikipedia.org/wiki/Infinite-dimensional_Lebesgue_measure

Marginals of the Wiener measure $\boldsymbol{R}$. In particular, R_t is the Lebesgue measure $\mathcal{L}^d$ on $\mathbb{R}^d$, and

$$R_{s,t}(dx, dy) = G_{t-s}(x-y)\, dx\, dy. \tag{2.3}$$

Where G_t is the standard heat kernel

$$G_t(u) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|u|^2}{2t}\right)$$

Benamou, Carlier, and Santambrogio [3] and Benamou and Carlier [2] then re-cast the Eulerian variational game (2.2) into a Lagrangian optimization problem over the set of Borel probability measures. This new problem is also solved in [3] using a finite element method, which is computationally expensive.

2.1.2 Entropic Lagrangian

Instead, Benamou et al. [4] propose using an entropy minimization approach to allow for a more computationally efficient method adapted from the Sinkhorn algorithm introduced by Cuturi [6].

This method, just like the Sinkhorn for OT between histograms (discrete measures), introduces some sort of entropic regularization [4], but this time on the measure over the trajectory space $\mathcal{X}$. The resulting numerical algorithm becomes a regularization of the Lagrangian from [3, 2].

The entropic Lagrangian variational problem is

$$\inf_{Q \in \mathcal{P}(\mathcal{X})} H(Q|R) + \int_0^T F(Q_t)\, dt + G(Q_T) \quad \text{s.t. } Q_0 = \rho_0 \tag{2.4}$$

Intuitively, this is the same as fixing the marginals ρ_t, finding the optimal bridge Q between them that has minimal entropy relative to the Wiener measure, and then optimizing over the ρ_t.

2.2 Viscosity and the deterministic limit

We change the MFG problem to one following the agent-level dynamics $dX_t = \alpha_t dt + \sigma dW_t$ with a diffusion coefficient σ. The MFG equilibrium equations become

$$\begin{aligned} -\partial_t u - \frac{\sigma^2}{2}\Delta u + \frac{1}{2}|\nabla u|^2 &= f[\rho_t] \\ \partial_t \rho - \frac{\sigma^2}{2}\Delta \rho - \operatorname{div}(\rho \nabla u) &= 0 \end{aligned} \tag{2.5}$$

This can be used to approximate first-order MFGs by setting a low viscosity parameter σ. Denoting $\varepsilon = \sigma^2$, the entropic variational problem (2.4) becomes

$$\inf_{Q \in \mathcal{P}(\mathcal{X})} \varepsilon H(Q|R_\varepsilon) + \int_0^T F(Q_t)\, dt + G(Q_T) \quad \text{s.t. } Q_0 = \rho_0$$

where R_ε is the Wiener measure associated with Wiener processes scaled by ε.

3 Numerical algorithm

3.1 Time discretization

Let N be the number of discrete steps for the time discretization of the problem, and $h = T/N$ the time step.

Benamou et al. [4] propose a discretization of (2.4) obtained by connecting the marginals through a multimarginal OT problem:

$$\mathcal{S}(\mu_0, \dots, \mu_N) = \inf_{\gamma \in \Pi(\mu_0, \dots, \mu_N)} H(\gamma | R^N) \tag{3.1}$$

where $t_k = kh$, $R^N = R_{t_0, \dots, t_N}$ and the marginals $\mu_k \in \mathcal{P}_2(\Omega)$. Then, define

$$\mathcal{U}(\mu_0, \dots, \mu_N) = h \sum_{k=1}^{N-1} F(\mu_k) + G(\mu_N). \tag{3.2}$$

Thus, the discretized entropy minimization problem can be written as

$$\inf \left\{ \mathcal{S}(\mu_0, \dots, \mu_N) + \mathcal{U}(\mu_0, \dots, \mu_N) : \mu_k \in \mathcal{P}_2(\Omega),\ \mu_0 = \rho_0 \right\}.$$

Expanding the inf-within-inf leads to the following convex optimization problem:

$$\begin{aligned} &\inf_{\gamma \in \mathcal{P}(\Omega^{N+1})} H(\gamma | R^N) + \imath_{\rho_0}(\mu_0) + \sum_{k=1}^{N-1} F(\mu_k) + G(\mu_N) \\ &\text{s.t. } \mu_k = P^k_{\#}\gamma \end{aligned} \tag{3.3}$$

where $\imath_{\rho_0}(\mu) = +\infty$ if $\mu \neq \rho_0$ and 0 otherwise is the convex indicatrix of the measure ρ_0. This is a generalized multimarginal optimal transport problem.

Benamou et al. [4] provide the corresponding dual problem involving the convex conjugates and potential functions, by using a multimarginal generalization of a result from Chizat et al. [5]:

$$\sup_u -\imath^*_{\rho_0}(-u_0) - \sum_{k=1}^{N-1} F^*(-u_k) - G^*(-u_N) - \int_{\Omega^{N+1}} \left(\exp\left(\oplus_{k=0}^N u_k \right) - 1 \right) dR^N \tag{3.4}$$

where the supremum is taken over $u = (u_0, \dots, u_N) \in L^\infty(\Omega)^{N+1}$.

Benamou et al. [4] introduce a Sinkhorn-like iterative algorithm to solve the above dual problem. We rewrite it more explicitly with slightly different notations inspired by [5]

Algorithm 1. *Denote for $k = 0, \dots, N$ and $(a_j)_{j \neq k}$*

$$\mathcal{I}_k((a_j)_{j \neq k})(z_k) = \int_{\Omega^N} \prod_{j \neq k} a_j(x_j)\, dR^N(x_{0:k-1}, z_k, x_{k+1:N})$$

the integral of functions $a_j, j \neq k$ with respect to R^N and variables x_j, $j \neq k$. For convenience we use the shorthand

$$\mathcal{I}_k^{(n)} = \mathcal{I}_k \left(\left(e^{u_j^{(n+1)}} \right)_{j<k}, \left(e^{u_j^{(n)}} \right)_{j>k} \right)$$

for the nth iterate.

Then we compute the dual potentials iteratively:

$$
\begin{cases}
u_0^{(n+1)} = \underset{v\in L^\infty}{\operatorname{argmax}} \displaystyle\int_\Omega (1 - e^{-v(x_0)})\mathcal{I}_0^{(n)}\, dx_0 - \imath_{\rho_0}^*(v) \\
u_k^{(n+1)} = \underset{v\in L^\infty}{\operatorname{argmax}} \displaystyle\int_\Omega (1 - e^{-v(x_k)})\mathcal{I}_k^{(n)}\, dx_k - hF^*(v), \quad 1 \leq k < N \\
u_N^{(n+1)} = \underset{v\in L^\infty}{\operatorname{argmax}} \displaystyle\int_\Omega (1 - e^{-v(x_N)})\mathcal{I}_N^{(n)}\, dx_N - G^*(v)
\end{cases}
\tag{3.5}
$$

until convergence.

Using duality, we find that the iterates $u_k^{(n)}$ satisfy

$$
a_k^{(n)} = \exp(-u_k^{(n)}) = \frac{\operatorname{prox}_{F_k}^{\mathrm{KL}}(\mathcal{I}_k^{(n)})}{\mathcal{I}_k^{(n)}} \tag{3.6}
$$

where

$$
\operatorname{prox}_F^{\mathrm{KL}}(z) = \underset{s}{\operatorname{argmin}}\, F(s) + \mathrm{KL}(s|z).
$$

Remark 1 (Some convex conjugates)**.** In practice, the convex conjugates of the cost functions are difficult to compute. For some of the examples in the paper, we have closed-form conjugates.

- The conjugate of the convex indicatrix $\imath_\nu$ of any measure ν is given by $\imath_\nu^*(u) = \langle u, \nu\rangle$.
- The hard congestion constraint

  $$
  C(\rho) = \begin{cases} 0 & \text{if } \rho \leq \bar{m} \\ +\infty & \text{otherwise} \end{cases}
  $$

  has convex conjugate (on the domain $\rho \geq 0$)

  $$
  C^*(u) = \sup_{\rho\leq\bar{m}} \langle u, \rho\rangle = \langle u^+, \bar{m}\mathbb{1}\rangle
  $$

- Obstacle constraints, given by

  $$
  F(\rho) = \int_\Omega V(x)\, d\rho(x) = \begin{cases} 0 & \text{if } \rho = 0 \text{ on } \mathscr{O} \\ +\infty & \text{otherwise} \end{cases} = \imath_0(\mathbb{1}_{\mathscr{O}}\rho)
  $$

  where V is the convex indicatrix of the complement $\Omega\backslash\mathscr{O}$ of the obstacles. Its conjugate is given by

  $$
  F^*(u) = \begin{cases} 0 & \text{if } u \leq 0 \text{ on } \Omega\backslash\mathscr{O} \\ +\infty & \text{otherwise} \end{cases}
  $$

3.2 Full discretization

For full numerical implementation, all measures are replaced by multi-dimensional arrays representing discrete histograms over a fixed grid of points in $\mathbb{R}^d$ of dimensionality $M = N_1 \times \cdots \times N_d$. Integration is exchanged with summation.

In the general case, the KL-projections in the Sinkhorn iterations can be solved using the Python library CVXPY[2,3]. Some can be computed explicitly.

Proposition 1. *The KL-projection under the hard congestion constraint of a measure $\beta \in \mathbb{R}^M$ is given by*

$$\mathrm{prox}_C^{\mathrm{KL}}(\beta) = \min(\beta, \overline{m}) \tag{3.7}$$

where the minimum is taken element-wise.

If we also add the obstacle constraint on a set $\mathscr{O}$ of points in the grid, then the proximal operator reads

$$\mathrm{prox}_F^{\mathrm{KL}}(\beta) = \min(\beta, \bar{m}\mathbb{1}_{\Omega\setminus\mathscr{O}}). \tag{3.8}$$

4 Examples

4.1 Transport with a soft target

4.1.1 Two-marginal case

We start with a very simplified approximation of the crowd displacement problem on $\Omega = [0,1]^2$, with only the first step (with initial agent distribution) and final step decided by the terminal penalty function G.

We set G to be the obstacle constraint related to a subset $\mathscr{O}$ of Ω as well as a potential $\Psi(x) = d(x, \mathscr{A})^\beta$ for some $\beta > 0$, related to the distance to a target subset $\mathscr{A}$ (see fig. 1):

$$G(\mu) = \int_\Omega \Psi\, d\mu + \imath_0(\mu \mathbb{1}_{\mathscr{O}})$$

Thus, the agents engage in a one-round mean-field game where they are only concerned with moving to regions with lower potential Ψ – as close as possible to $\mathscr{A}$ – whilst obeying physical constraints related to the obstacles.

The discretized MFG problem with viscosity parameter $\varepsilon = \sigma^2$ can be written as the following transport problem:

$$\begin{aligned} &\inf_\gamma\, \langle \Psi, \gamma^T \mathbb{1}\rangle + \varepsilon H(\gamma|R_\varepsilon) \\ &\text{s.t. } \gamma\mathbb{1} = \rho_0, \quad \gamma^T\mathbb{1} \odot \mathbb{1}_{\mathscr{O}} = 0 \end{aligned} \tag{4.1}$$

The interesting aspect of this problem is observing what kind of optimal distribution $\rho_1^* = (\gamma^*)^T\mathbb{1}$ the agents reach.

Proposition 2. *Problem (4.1) can be solved in closed form: the Lagrange multiplier u_0^* for the marginal law constraint satisfies*

$$e^{-u_0^*} = \frac{\rho_0}{R_\varepsilon a_1^*}$$

where $a_1^* = e^{-\Psi/\varepsilon} \odot \mathbb{1}_{\Omega\setminus\mathscr{O}}$, and the optimal coupling is

$$\gamma^* = R_\varepsilon \odot (e^{-u^*} \otimes \hat{\varphi})$$

It satisfies, as expected, that $\gamma_{i,j}^* = 0$ for all $j \in \mathscr{O}$.

[2] https://github.com/cvxgrp/cvxpy

[3] Steven Diamond and Stephen Boyd. "CVXPY: A Python-Embedded Modeling Language for Convex Optimization". In: *Journal of Machine Learning Research* 17.83 (2016), pp. 1–5.

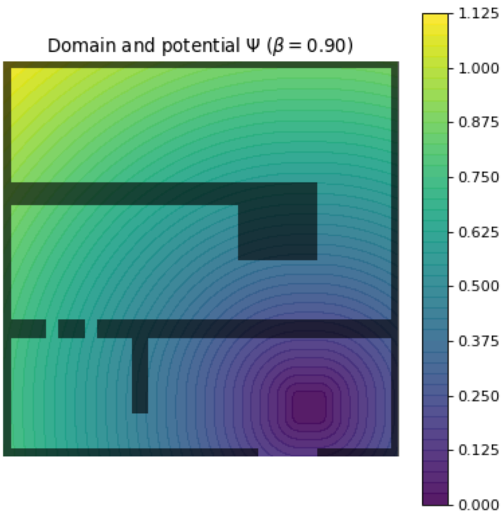


Figure 1: Computational domain of the game Ω with set of obstacles $\mathscr{O}$ (transparent grey), and contour of the potential function $\Psi(x) = d(x, \mathscr{A})^\beta$.

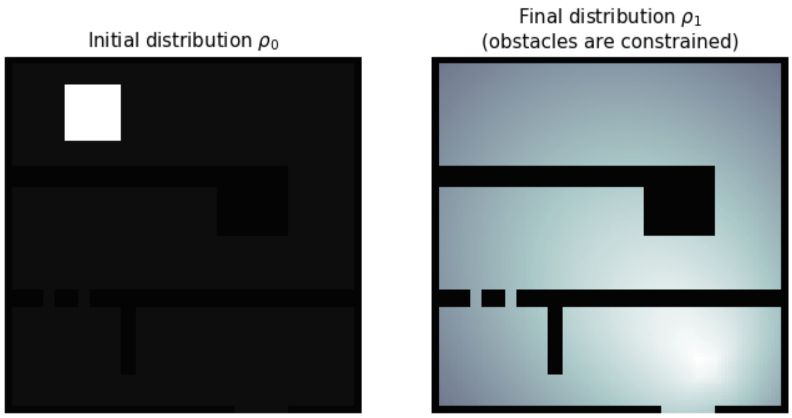
Numerical experiment We ran a numerical experiment by implementing the solutions given by proposition 2 to the discrete MFG (4.1). fig. 2a provides a representation of both . We also checked the results when removing the constraints on the obstacles (essentially setting $\mathscr{O}$), and when lowering the viscosity parameter $\sigma = \sqrt{\varepsilon}$ (see figs. 2b and 2c).

Of course, replacing G by a hard marginal constraint turns the problem into a classical regularized OT problem, and the above proposition leads to usual Sinkhorn iterations on the grid. The matrix-vector product in the Lagrange multiplier potentially becomes a computational bottleneck, but the separability of the heat kernel R_ε allows for fast computation using simple 1D convolutions.

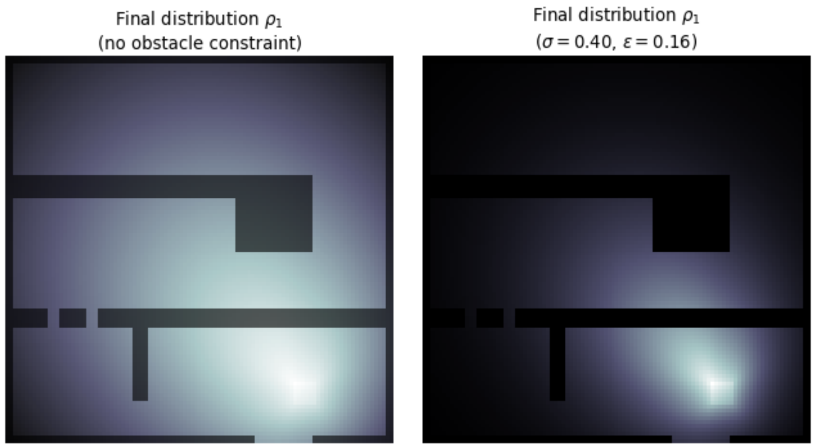
Remark 2 (A "smarter" (more realistic) potential for crowd dynamics)**.** The results shown fig. 2 are satisfactory for the given potential Ψ – as expected the agents try to stay near the low-potential regions. However, for modeling of crowd dynamics they would be deeply nonphysical because the potential is inadequate. In a room evacuation scenario, for instance, agents would seek to minimize the time-to-exit: the literature shows this leads to the Eikonal equation, a kind of Hamilton-Jacobi PDE. We computed the adequate potential shown fig. 3a using the Fast Sweeping method [10], as well as the discrete MFG fig. 3b.

4.1.2 One intermediate time step

We now go up to three marginals (ρ_0, ρ_1, ρ_2). We assign to the single intermediate marginal ρ_1 the congestion constraint $\rho_1 \leq \bar{m}$ and the obstacle constraint. The primal problem then



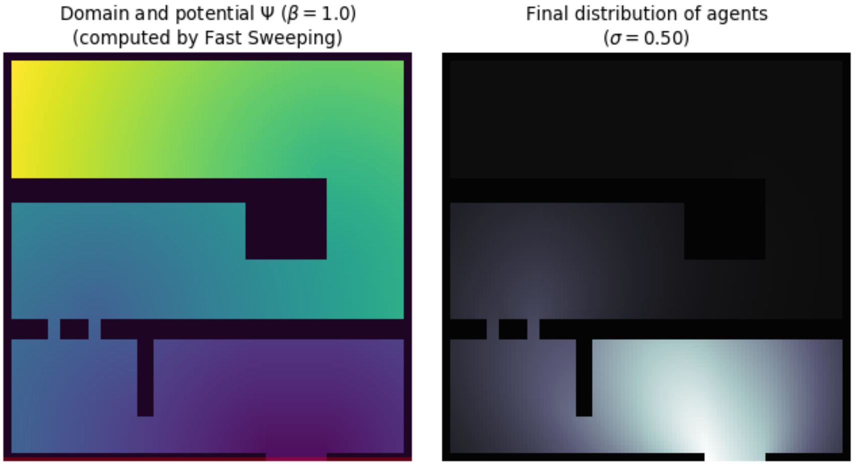
(a) Result of the “fuzzy” transport problem with enforcement of the obstacle constraints and viscosity parameter $\sigma = 1$.



(b) Final distribution ρ_1^* without enforcing the obstacles. The mass of the distribution “bleeds” through the obstacles.

(c) Obstacles are constrained as in fig. 2a, but with a lower viscosity parameter $\sigma = 0.4$.

Figure 2: Marginal distributions of the solution of two-step MFG or “fuzzy transport” problem (4.1), with a few variations.



(a) Domain and potential associated with the fastest path distance.

(b) Optimal terminal distribution ρ_1^* of the discrete MFG with the potential from fig. 3a.

Figure 3: Setup and solution for the discrete MFG using the time-to-exit potential discussed in remark 2.

reads

$$\begin{aligned} &\inf_{\gamma,\rho_1,\rho_2} \langle \Psi, \rho_2 \rangle + \varepsilon H(\gamma | R_\varepsilon) \\ \text{s.t. } & P^k_\# \gamma = \rho_k, \ k = 1, 2 \\ & \rho_1 \leq \bar{m} \\ & \rho_1 \odot \mathbb{1}_{\mathscr{O}} = 0 \\ & \rho_2 \odot \mathbb{1}_{\mathscr{O}} = 0 \end{aligned} \tag{4.2}$$

Proposition 3. *The Lagrange multipliers u_i^* at the optimum satisfy the fixed-point conditions:*

$$\begin{aligned} a_0^* &= \frac{\rho_0}{R_\varepsilon[\cdot, a_1^*, a_2^*]} \\ a_1^* &= \min\left(\frac{\bar{m}}{R_\varepsilon[a_0^*, \cdot, a_1^*]}\right) \end{aligned}$$

where $a_i^* = \exp(-u_i^*)$, a_1^* is supported on $\Omega \backslash \mathscr{O}$ and $a_2^* = e^{-\Psi/\varepsilon} \odot \mathbb{1}_{\Omega\backslash\mathscr{O}}$, and we denote $R_\varepsilon[\cdot,\cdot,\cdot]$ the tensor product by R_ε.

The fixed point can then computed using an iterative algorithm à la generalized Sinkhorn, just as in the Algorithm 1 suggested by [4].

The issue of computational efficiency is more pronounced here than before due to the tensor product and need for multiple iterations until convergence.

References

[1] Yves Achdou, Fabio Camilli, and Italo Capuzzo-Dolcetta. "Mean Field Games: Convergence of a Finite Difference Method". In: *SIAM Journal on Numerical Analysis* 51

(2013), pp. 2585–2612. DOI: 10.1137/120882421. URL: https://hal.archives-ouvertes.fr/hal-01456506.

[2] Jean-David Benamou and Guillaume Carlier. "Augmented Lagrangian Methods for Transport Optimization, Mean Field Games and Degenerate Elliptic Equations". In: *Journal of Optimization Theory and Applications* 167 (Mar. 2015). DOI: 10.1007/s10957-015-0725-9.

[3] Jean-David Benamou, Guillaume Carlier, and Filippo Santambrogio. "Variational Mean Field Games". working paper or preprint. Mar. 2016. URL: https://hal.archives-ouvertes.fr/hal-01295299.

[4] Jean-David Benamou et al. *An entropy minimization approach to second-order variational mean-field games*. 2018. arXiv: 1807.09078 [math.OC].

[5] Lenaic Chizat et al. *Scaling Algorithms for Unbalanced Transport Problems*. 2016. arXiv: 1607.05816 [math.OC].

[6] Marco Cuturi. "Sinkhorn Distances: Lightspeed Computation of Optimal Transport". In: *Advances in Neural Information Processing Systems 26*. Ed. by C. J. C. Burges et al. Curran Associates, Inc., 2013, pp. 2292–2300. URL: http://papers.nips.cc/paper/4927-sinkhorn-distances-lightspeed-computation-of-optimal-transport.pdf.

[7] Steven Diamond and Stephen Boyd. "CVXPY: A Python-Embedded Modeling Language for Convex Optimization". In: *Journal of Machine Learning Research* 17.83 (2016), pp. 1–5.

[8] Jean-Michel Lasry and Pierre-Louis Lions. "Jeux à champ moyen. I – Le cas stationnaire". In: *Comptes Rendus Mathématique* 343.9 (2006), pp. 619–625. ISSN: 1631-073X. DOI: https://doi.org/10.1016/j.crma.2006.09.019. URL: http://www.sciencedirect.com/science/article/pii/S1631073X06003682.

[9] Jean-Michel Lasry and Pierre-Louis Lions. "Jeux à champ moyen. II – Horizon fini et contrôle optimal". In: *Comptes Rendus Mathematique* 343.10 (2006), pp. 679–684. ISSN: 1631-073X. DOI: https://doi.org/10.1016/j.crma.2006.09.018. URL: http://www.sciencedirect.com/science/article/pii/S1631073X06003670.

[10] Hongkai Zhao. "A fast sweeping method for Eikonal equations". In: *Math. Comput.* 74 (2004), pp. 603–627.