Computational Optimal Transport – Project report:

Optimal Transport and Entropic methods for solving variational Mean-Field Games

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1 General setting: variational mean-field games

A mean-field game [6, 7] is a strategic decision-making problem with a very large, continuously-distributed number of interacting agents inside a state space: the overall theory developed by Lasry and Lions can be used as a means to model large, computationally intractable games. In the continuous-time setting explored in [7], each agent evolves according to some dynamics and makes choices, but the response to his choices are affected by the states and choices of the numerous other agents – leading to a so-called *differential game* – through a *mean-field* effect.

Several ways of modeling agent cross-interaction exist. More recently, [3] have focused on games where agent interactions take a variational form, allowing to penalize phenomenons such as congestion inside areas of the agent state space.

The (Nash) equilibrium agent-control dynamics can be summarized by the system of coupled nonlinear partial differential equations:

$$-\partial_t u - \frac{1}{2}\Delta u + \frac{1}{2}|\nabla u|^2 = f[\rho_t] \quad (t,x) \in (0,T)\times\Omega \tag{1a}$$

$$\partial_t \rho_t - \frac{1}{2}\Delta \rho_t - \operatorname{div}(\rho_t \nabla u) = 0 \tag{1b}$$

$$\rho_0 \text{ given} \tag{1c}$$

$$u(T,\cdot) = g[\rho_T] \tag{1d}$$

where and $t \mapsto \rho_t$ is a trajectory in the space of measures, and Ω is the standard Euclidean space $\mathbb{R}^d$. The applications f and g are supposed to be derivatives of some real-valued functionals F and G. For instance, if $G(\mu) = \int_\Omega \Psi\, d\mu(x)$ then its derivative is $g[\mu](x) = \Psi(x)$.

The equations (1a)–(1b) form a coupled system of control (Hamilton-Jacobi-Bellman) and diffusion (forward Kolmogorov) equations.

1.1 The variational problem

Denote $\mathbb{W}_2(\Omega) = (\mathcal{P}_2(\Omega), \mathcal{W}_2)$ the set of probability measures with finite second moment, equipped with the Wasserstein metric

$$\mathcal{W}_2(\mu,\nu)^2 = \inf_{\gamma\in\Pi(\mu,\nu)} \int |x-y|^2 d\gamma \tag{2}$$

where $\Pi(\mu, \nu) = \{\gamma \in \mathcal{P}_2(\Omega \times \Omega) : P^1_\sharp \gamma = \mu,\ P^2_\sharp \gamma = \nu\}$ is the set of transport plants from μ to ν. Then, $\mathcal{C}([0,T], \mathbb{W}_2(\Omega))$ is the Wiener space of continuous $\mathbb{W}_2$-valued trajectories.

It has been shown [3] that the previous system of MFG PDEs can be reformulated to a variational problem:

$$\inf_{\rho,v} J(\rho, v) = \frac{1}{2}\int_0^T \int_\Omega |v_t|^2 \, d\rho_t(x)\, dt + \int_0^T F(\rho_t)\, dt + G(\rho_T) \tag{3a}$$

$$\text{s.t. } \partial_t \rho_t - \frac{1}{2}\Delta\rho_t + \operatorname{div}(\rho_t v) = 0 \tag{3b}$$

$$\rho_0 \in \mathbb{W}_2(\Omega) \tag{3c}$$

where the unknowns (ρ, v) live in appropriate functional spaces: $\rho = (\rho_t)_{t\in[0,T]} \in \mathcal{C}([0,T], \mathbb{W}_2(\Omega))$ is a trajectory in $\mathbb{W}_2$ and v is a sufficiently regular function on $[0,T] \times \Omega$ (most likely a Sobolev space).

Benamou et al. [4] also introduce the following partial problem:

$$\mathrm{FP}_h(\mu, \nu) = \inf_{\rho,v} \int_0^h \int_\Omega |v_t|^2 d\rho_t(x)\, dt \quad \text{s.t. } \partial_t\rho_t - \frac{1}{2}\Delta\rho_t + \operatorname{div}(\rho_t v),\ \rho_0 = \mu,\ \rho_h = \nu \tag{4}$$

It can be used to connect approximations of the solution measure to our MFG problem at discrete times $t_k = kh$, $k = 0, \dots, N$.

This point of view [3] is called *Eulerian*: we minimize over both the velocity v and the time-trajectory of the agents' density ρ. Instead, we could minimize over measures in the space of individual agents' trajectories, which is the base of the *Lagrangian* formulation [2, 3] proposed by Benamou, Carlier, and Santambrogio and that we explore in the sequel.

2 Lagrangian dual formulation

This new point of view involves a change in function spaces. We denote $\mathscr{E} = \mathcal{C}([0,T], \Omega)$ the Wiener space of (agents') trajectories $[0,T] \to \Omega$. Following [3, 2], we equip it with the Wiener measure (the law of a Wiener process with any starting point x)

$$R = \int_\Omega \delta_{x+W}\, dx$$

where W is a standard Wiener process in $\mathbb{R}^d$. It is an analogue in the space $\mathscr{E}$ to the usual finite-dimensional Lebesgue measure[1].

Measures $Q \in \mathcal{P}(\mathscr{E})$ can also be seen as trajectories $(Q_t)_{t\in[0,T]} \in \mathcal{C}([0,T], \mathcal{M}(\Omega))$, with

$$Q_t = e_{t\sharp} Q \in \mathcal{P}(\Omega)$$

the push-forward of Q by the evaluation map $e_t \coloneq w \in \mathscr{E} \longmapsto w(t)$. This naturally defines an injection $i \colon \mathcal{P}(\mathscr{E}) \to \mathcal{C}([0,T], \mathcal{P}(\Omega))$. We also introduce the more general marginals $Q_{t_1,\dots,t_n} = (e_{t_1}, \dots, e_{t_n})_\sharp Q$ for $0 \le t_1 < \dots < t_N \le T$.

[1] https://en.wikipedia.org/wiki/Infinite-dimensional_Lebesgue_measure

Marginals of the Wiener measure $\boldsymbol{R}$. We introduce the kernel $G_t(u) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|u|^2}{2t}\right)$. In particular, R_t is the Lebesgue measure $\mathcal{L}^d$ on $\mathbb{R}^d$, and

$$R_{s,t}(dx, dy) = G_{t-s}(x-y)\, dx\, dy. \tag{5}$$

Benamou, Carlier, and Santambrogio [3] and Benamou and Carlier [2] re-cast the Eulerian MFG variational problem (3) into an optimization problem over the set of Borel probability measures (more specifically that associated with the Sobolev subspace H^1 of $\mathscr{E}$). This new optimization problem is solved in [3] using a finite element method, which is computationally expensive.

2.1 The entropic Lagrangian approach

Instead, Benamou et al. [4] propose using an entropy minimization approach to allow for a more computationally efficient method adapted from the Sinkhorn algorithm [5] developed by Cuturi.

This method, just like the Sinkhorn for OT between histograms (discrete measures), introduces some sort of entropic regularization [4], but this time on the measure over the trajectory space $\mathscr{E}$. The resulting numerical algorithm becomes a regularization of the Lagrangian from [1].

For all measures Q on $\mathscr{E}$ admitting a density with respect to R, we define the relative entropy

$$H(Q|R) = \int_{\mathscr{E}} \ln\left(\frac{dQ}{dR}\right) dQ(w) \tag{6}$$

Partial transport problem Benamou et al. provide another partial transport problem:

$$S_h(\mu,\nu) = \inf\left\{H(Q|R) : Q \in \mathcal{P}(\mathcal{C}([0,h],\Omega)),\ Q_0 = \mu,\ Q_h = \nu\right\} \tag{7}$$

This problem can be seen as a continuous OT problem between the two measures μ and ν. Benamou et al. [4] show that it is linked to the partial Eulerian problem (4) as

$$S_h(\mu,\nu) = \mathrm{FP}_h(\mu,\nu) + \mathrm{Ent}\,\mu.$$

The dimensionality of problem (7) can be greatly simplified; according to [4] we can rewrite it as a static OT problem

$$S_h(\mu,\nu) = \inf\left\{H(\gamma, R_{0,h}) : \gamma \in \Pi(\mu,\nu)\right\}. \tag{8}$$

3 Numerical algorithm

Let N be the number of discrete steps for the time discretization of the problem, and $h = T/N$ the time step.

We consider the following multi-marginal OT problem

$$\mathcal{S}(\mu_0,\dots,\mu_N) = \inf_{\gamma\in\Pi(\mu_0,\dots,\mu_N)} H(\gamma|R^N) \tag{9}$$

where $t_k = kh$, $R^N = R_{t_0,\dots,t_N}$ and the marginals $\mu_k \in \mathcal{P}_2(\Omega)$. Then, define

$$\mathcal{U}(\mu_0,\dots,\mu_N) = h\sum_{k=1}^{N-1} F(\mu_k) + G(\mu_N).$$

The discretized entropy minimization problem can the be written

$$\inf \left\{ \mathcal{S}(\mu_0, \dots, \mu_N) + \mathcal{U}(\mu_0, \dots, \mu_N) : \mu_k \in \mathcal{P}_2(\Omega),\ \mu_0 = \rho_0 \right\}.$$

Expanding the inf-within-inf leads to the following convex optimization problem:

$$\begin{aligned} &\inf_{\gamma \in \mathcal{P}(\Omega^{N+1})} H(\gamma | R^N) + \tilde{F}(\mu_0) \sum_{k=1}^{N-1} F(\mu_k) + G(\mu_N) \\ &\text{s.t. } \mu_k = P^k_\sharp \gamma \end{aligned} \tag{10}$$

where $\tilde{F}(\mu) = I_{\rho_0}(\mu) = +\infty$ if $\mu \neq \rho_0$ and 0 otherwise.

Benamou et al. [4] provide the corresponding dual problem involving the convex conjugates and potential functions:

$$\sup_u -\tilde{F}^*(-u_0) - \sum_{k=1}^{N-1} F^*(-u_k) - G^*(u_N) - \int_{\Omega^{N+1}} \left(\exp\left(\oplus_{k=0}^N u_k \right) - 1 \right) \, dR^N \tag{11}$$

where the supremum is taken over $u = (u_0, \dots, u_N)$ with $u_k \in L^\infty(\Omega)$.

Remark 1 (Some convex conjugates)**.** *In our case, we have that the conjugate of the convex indicatrix I_{ρ_1} of any given measure ρ_1 is given by $I^*_{\rho_1}(u) = \langle \rho_0, u \rangle$.*

The hard congestion constraint $F(\rho) = \begin{cases} 0 & \text{if } \rho \leq \bar{\rho} \\ +\infty & \text{otherwise} \end{cases}$, has convex conjugate

$$F^*(u) = \sup_{\rho \leq \bar{\rho}} \langle \rho, u \rangle = \bar{\rho} \|u\|_{L^\infty(\Omega)}.$$

Obstacle constraints given by

$$F(\rho) = \int_\Omega V(x) \, d\rho(x)$$

where V is the convex indicatrix of a set of obstacles $\mathscr{O} \in \Omega$. Its conjugate is given for $u \in L^\infty(\Omega)$ by

$$F^*(u) = \begin{cases} 0 & \text{if } u \leq 0 \text{ on } \Omega \backslash \mathscr{O} \\ +\infty & \text{otherwise} \end{cases}$$

4 Examples

4.1 Crowd congestion

Supposing a model where F, the Kantorovitch dual problem is written

$$\sup_u \langle \rho_0, u_0 \rangle - \sum_{k=1}^{N-1} \|u_k\|_{L^\infty(\Omega)} - \int_{\Omega^{N+1}} \left(\exp(\oplus_{k=0}^N u_k) - 1 \right) \, dR^N \tag{12}$$

References

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