Computational Optimal Transport – Project report:

Optimal Transport and Entropic methods for solving variational Mean-Field Games

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1 General setting: variational mean-field games

A mean-field game [8, 9] is a strategic decision-making problem with a very large, continuously-distributed number of interacting agents inside a state space: the overall theory developed by Lasry and Lions can be used as a means to model large, computationally intractable games. In the continuous-time setting explored in [9], each agent evolves according to some dynamics and makes choices, but the response to his choices are affected by the states and choices of the numerous other agents – leading to a so-called *differential game* – through a *mean-field* effect.

Several ways of modeling agent cross-interaction exist. More recently, [3] have focused on games where agent interactions take a variational form, allowing to penalize phenomenons such as congestion inside areas of the agent state space.

The (Nash) equilibrium agent-control dynamics can be summarized by the system of coupled nonlinear partial differential equations:

$$-\partial_t u - \frac{1}{2}\Delta u + \frac{1}{2}|\nabla u|^2 = f[\rho_t] \quad (t,x) \in (0,T)\times\Omega \tag{1a}$$

$$\partial_t \rho_t - \frac{1}{2}\Delta\rho_t - \operatorname{div}(\rho_t \nabla u) = 0 \tag{1b}$$

$$\rho_0 \text{ given} \tag{1c}$$

$$u(T,\cdot) = g[\rho_T] \tag{1d}$$

where and $t \mapsto \rho_t$ is a trajectory in the space of measures, and Ω is the standard Euclidean space $\mathbb{R}^d$. The applications f and g are supposed to be derivatives of some real-valued functionals F and G. For instance, if $G(\mu) = \int_\Omega \Psi\, d\mu(x)$ then its derivative is $g[\mu](x) = \Psi(x)$.

The equations (1a)–(1b) form a coupled system of control (Hamilton-Jacobi-Bellman) and diffusion (forward Kolmogorov) equations.

1.1 The variational problem

The first idea of [3] is to cast the MFG partial differential equations to a variational problem over an appropriate function space. Denote $\mathbb{W}_2(\Omega) = (\mathcal{P}_2(\Omega), \mathcal{W}_2)$ the set of probability measures with finite second moment, equipped with the Wasserstein metric

$$\mathcal{W}_2(\mu,\nu)^2 = \inf_{\gamma\in\Pi(\mu,\nu)} \int |x-y|^2 d\gamma \tag{2}$$

where $\Pi(\mu,\nu) = \{\gamma \in \mathcal{P}_2(\Omega\times\Omega) : P^1_{\#}\gamma = \mu,\ P^2_{\#}\gamma = \nu\}$ is the set of transport plants from μ to ν. Then, $\mathcal{C}([0,T],\mathbb{W}_2(\Omega))$ is the Wiener space of continuous $\mathbb{W}_2$-valued trajectories. Benamou, Carlier, and Santambrogio [3] show that the MFG be reformulated to the following variational problem:

$$\inf_{\rho,v} J(\rho,v) = \frac{1}{2}\int_0^T \int_\Omega |v_t|^2\, d\rho_t(x)\, dt + \int_0^T F(\rho_t)\, dt + G(\rho_T) \tag{3a}$$

$$\text{s.t. } \partial_t\rho_t - \frac{1}{2}\Delta\rho_t + \operatorname{div}(\rho_t v) = 0 \tag{3b}$$

$$\rho_0 \in \mathbb{W}_2(\Omega) \tag{3c}$$

where $\rho = (\rho_t)_{t\in[0,T]} \in \mathcal{C}([0,T],\mathbb{W}_2(\Omega))$ is a trajectory in $\mathbb{W}_2$ and v is a sufficiently regular function on $[0,T]\times\Omega$ (most likely a Sobolev space).

Benamou et al. [4] also introduce the following partial problem:

$$\mathrm{FP}_h(\mu,\nu) = \inf_{\rho,v}\int_0^h \int_\Omega |v_t|^2 d\rho_t(x)\, dt \quad \text{s.t. } \partial_t\rho_t - \frac{1}{2}\Delta\rho_t + \operatorname{div}(\rho_t v),\ \rho_0 = \mu,\ \rho_h = \nu \tag{4}$$

It can be used to connect approximations of the solution measure to our MFG problem at discrete times $t_k = kh$, $k = 0,\dots,N$.

This point of view [3] is called *Eulerian*: we minimize over both the velocity v and the time-trajectory of the agents' density ρ. It is not very practical because of the structure of the constraint (a Fokker-Planck equation). Instead, we could minimize over measures in the space of individual agents' trajectories, which is the base of the *Lagrangian* formulation [2, 3] proposed by Benamou, Carlier, and Santambrogio and that we explore in the sequel.

2 Lagrangian dual formulation

2.1 Wiener space and measure

This new point of view involves a change in function spaces. We denote $\mathcal{X} = \mathcal{C}([0,T],\Omega)$ the Wiener space of (agents') trajectories $[0,T]\to\Omega$. Following [3, 2], we equip it with the Wiener measure (the law of a Wiener process with any starting point x)

$$R = \int_\Omega \delta_{x+W}\, dx$$

where W is a standard Wiener process in $\mathbb{R}^d$. It is an analogue in the space $\mathcal{X}$ to the usual finite-dimensional Lebesgue measure[1].

Measures $Q\in\mathcal{P}(\mathcal{X})$ can also be seen as trajectories $(Q_t)_{t\in[0,T]} \in \mathcal{C}([0,T],\mathcal{P}(\Omega))$, with

$$Q_t = e_{t\#}Q \in \mathcal{P}(\Omega)$$

the push-forward of Q by the evaluation map $e_t\colon \xi\in\mathcal{X} \longmapsto \xi(t)$. This naturally defines an injection $\underline{i}\colon \mathcal{P}(\mathcal{X}) \to \mathcal{C}([0,T],\mathcal{P}(\Omega))$. We also introduce the more general marginals $Q_{t_1,\dots,t_n} = (e_{t_1},\dots,e_{t_n})_{\#}Q$ for $0\le t_1 < \cdots < t_N \le T$.

[1] https://en.wikipedia.org/wiki/Infinite-dimensional_Lebesgue_measure

Marginals of the Wiener measure $\boldsymbol{R}$. We introduce the heat kernel $G_t(u) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|u|^2}{2t}\right)$. In particular, R_t is the Lebesgue measure $\mathcal{L}^d$ on $\mathbb{R}^d$, and

$$R_{s,t}(dx, dy) = G_{t-s}(x-y)\, dx\, dy. \tag{5}$$

Benamou, Carlier, and Santambrogio [3] and Benamou and Carlier [2] then re-cast the Eulerian variational game (3) into a so-called Lagrangian optimization problem over the set of Borel probability measures (more specifically that associated with the Sobolev subspace H^1 of $\mathcal{X}$). This new problem is solved in [3] using a finite element method, which is computationally expensive.

2.2 The entropic Lagrangian approach

Instead, Benamou et al. [4] propose using an entropy minimization approach to allow for a more computationally efficient method adapted from the Sinkhorn algorithm [6] developed by Cuturi.

This method, just like the Sinkhorn for OT between histograms (discrete measures), introduces some sort of entropic regularization [4], but this time on the measure over the trajectory space $\mathcal{X}$. The resulting numerical algorithm becomes a regularization of the Lagrangian from [3, 2].

The entropic Lagrangian variational problem is

$$\inf_{Q \in \mathcal{P}(\mathcal{X})} \mathrm{KL}(Q|R) + \int_0^T F(Q_t)\, dt + G(Q_T),\ \text{s.t. } Q_0 = \rho_0 \tag{6}$$

Partial transport problem Benamou et al. provide another partial transport problem:

$$S_h(\mu,\nu) = \inf \left\{ \mathrm{KL}(Q|R) : Q \in \mathcal{P}(\mathcal{C}([0,h],\Omega)),\ Q_0 = \mu,\ Q_h = \nu \right\} \tag{7}$$

This problem can be seen as a continuous OT problem between the two measures μ and ν. Benamou et al. [4] show that it is linked to the partial Eulerian problem (4) as

$$S_h(\mu,\nu) = \mathrm{FP}_h(\mu,\nu) + \mathrm{Ent}\,\mu.$$

The dimensionality of problem (7) can be greatly simplified; according to [4] we can rewrite it as a static OT problem

$$S_h(\mu,\nu) = \inf \left\{ \mathrm{KL}(\gamma|R_{0,h}) : \gamma \in \Pi(\mu,\nu) \right\}. \tag{8}$$

3 Numerical algorithm

Let N be the number of discrete steps for the time discretization of the problem, and $h = T/N$ the time step.

We consider the following multi-marginal OT problem

$$\mathcal{S}(\mu_0, \dots, \mu_N) = \inf_{\gamma \in \Pi(\mu_0,\dots,\mu_N)} H(\gamma|R^N) \tag{9}$$

where $t_k = kh$, $R^N = R_{t_0,\dots,t_N}$ and the marginals $\mu_k \in \mathcal{P}_2(\Omega)$. Then, define

$$\mathcal{U}(\mu_0,\dots,\mu_N) = h \sum_{k=1}^{N-1} F(\mu_k) + G(\mu_N).$$

Thus, the discretized entropy minimization problem can be written as

$$\inf \left\{ \mathcal{S}(\mu_0, \dots, \mu_N) + \mathcal{U}(\mu_0, \dots, \mu_N) : \mu_k \in \mathcal{P}_2(\Omega),\ \mu_0 = \rho_0 \right\}.$$

Expanding the inf-within-inf leads to the following convex optimization problem:

$$\begin{aligned} &\inf_{\gamma \in \mathcal{P}(\Omega^{N+1})} H(\gamma | R^N) + \imath_{\rho_0}(\mu_0) + \sum_{k=1}^{N-1} F(\mu_k) + G(\mu_N) \\ &\text{s.t. } \mu_k = P^k_{\#}\gamma \end{aligned} \tag{10}$$

where $\imath_{\rho_0}(\mu) = +\infty$ if $\mu \neq \rho_0$ and 0 otherwise is the convex indicatrix of the measure ρ_0. This is a generalized multimarginal optimal transport problem.

Benamou et al. [4] provide the corresponding dual problem involving the convex conjugates and potential functions, by using a multimarginal generalization of a result from Chizat et al. [5]:

$$\sup_u -\imath^*_{\rho_0}(-u_0) - \sum_{k=1}^{N-1} F^*(-u_k) - G^*(-u_N) - \int_{\Omega^{N+1}} \left(\exp\left(\oplus_{k=0}^N u_k \right) - 1 \right) dR^N \tag{11}$$

where the supremum is taken over $u = (u_0, \dots, u_N) \in L^\infty(\Omega)^{N+1}$.

Benamou et al. [4] introduce a Sinkhorn-like iterative algorithm to solve the above dual problem. We rewrite it more explicitly with slightly different notations inspired by [5]

Algorithm 1

Denote for $k = 0, \dots, N$ and $(a_j)_{j \neq k}$

$$\mathcal{I}_k((a_j)_{j\neq k})(z_k) = \int_{\Omega^N} \prod_{j \neq k} a_j(x_j)\, dR^N(x_{0:k-1}, z_k, x_{k+1:N})$$

the integral of functions a_j, $j \neq k$ with respect to R^N and variables x_j, $j \neq k$. For convenience we use the shorthand

$$\mathcal{I}_k^{(n)} = \mathcal{I}_k \left(\left(e^{u_j^{(n+1)}} \right)_{j<k}, \left(e^{u_j^{(n)}} \right)_{j>k} \right)$$

for the nth iterate.

Then we compute the dual potentials iteratively:

$$\begin{cases} u_0^{(n+1)} = \operatorname*{argmax}_{v \in L^\infty} \int_\Omega (1 - e^{-v(x_0)}) \mathcal{I}_0^{(n)}\, dx_0 - \imath^*_{\rho_0}(v) \\ u_k^{(n+1)} = \operatorname*{argmax}_{v \in L^\infty} \int_\Omega (1 - e^{-v(x_k)}) \mathcal{I}_k^{(n)}\, dx_k - hF^*(v), \quad 1 \leq k < N \\ u_N^{(n+1)} = \operatorname*{argmax}_{v \in L^\infty} \int_\Omega (1 - e^{-v(x_N)}) \mathcal{I}_N^{(n)}\, dx_N - G^*(v) \end{cases} \tag{12}$$

until convergence.

Using duality, we find that the iterates $u_k^{(n)}$ satisfy

$$a_k^{(n)} = \exp(-u_k^{(n)}) = \frac{\operatorname{prox}^{\mathrm{KL}}_{F_k}(\mathcal{I}_k^{(n)})}{\mathcal{I}_k^{(n)}} \tag{13}$$

where

$$\operatorname{prox}^{\mathrm{KL}}_F(z) = \operatorname*{argmin}_s F(s) + \mathrm{KL}(s|z).$$

Remark 1 (***Some convex conjugates***)

In practice, the convex conjugates of the cost functions are difficult to compute. For some of the examples in the paper, we have closed-form conjugates.

- *The conjugate of the convex indicatrix $\imath_\nu$ of any measure ν is given by $\imath_\nu^*(u) = \langle u, \nu \rangle$.*
- *The hard congestion constraint*

$$C(\rho) = \begin{cases} 0 & \text{if } \rho \leq \bar{m} \\ +\infty & \text{otherwise} \end{cases}$$

has convex conjugate (on the domain $\rho \geq 0$)

$$C^*(u) = \sup_{\rho \leq \bar{m}} \langle u, \rho \rangle = \langle u^+, \bar{m}\mathbb{1} \rangle$$

- *Obstacle constraints, given by*

$$F(\rho) = \int_\Omega V(x)\, d\rho(x) = \begin{cases} 0 & \text{if } \rho = 0 \text{ on } \mathscr{O} \\ +\infty & \text{otherwise} \end{cases} = \imath_0(\mathbb{1}_{\mathscr{O}}\rho)$$

where V is the convex indicatrix of the complement $\Omega \backslash \mathscr{O}$ of the obstacles. Its conjugate is given by

$$F^*(u) = \begin{cases} 0 & \text{if } u \leq 0 \text{ on } \Omega \backslash \mathscr{O} \\ +\infty & \text{otherwise} \end{cases}$$

3.1 Full discretization

For full numerical implementation, all measures are replaced by multi-dimensional arrays representing discrete histograms over a fixed grid of points in $\mathbb{R}^d$ of dimensionality $M = N_1 \times \cdots \times N_d$. Integration is exchanged with summation.

In the general case, the KL-projections in the Sinkhorn iterations can be solved using the Python library CVXPY[2,3]. Some can be computed explicitly.

Remark 2

The KL-projection under the hard congestion constraint of a measure $\beta \in \mathbb{R}^M$ is given by

$$\mathrm{prox}_C^{\mathrm{KL}}(\beta) = \min(\beta, \overline{m}) \tag{14}$$

where the minimum is taken element-wise.

If we also add the obstacle constraint on a set $\mathscr{O}$ of points in the grid, then the proximal operator reads

$$\mathrm{prox}_F^{\mathrm{KL}}(\beta) = \min(\beta, \bar{m}\mathbb{1}_{\Omega\backslash\mathscr{O}}). \tag{15}$$

[2] https://github.com/cvxgrp/cvxpy

[3] Steven Diamond and Stephen Boyd. "CVXPY: A Python-Embedded Modeling Language for Convex Optimization". In: *Journal of Machine Learning Research* 17.83 (2016), pp. 1–5.

4 Examples

4.1 Toy 1D model

We suppose that $\Omega = [0, 1]$ and

$$f[\mu](x) = -16\left(x - \frac{1}{2}\right)^2 - 0.1\min(5, \mu(x))$$

suggesting the agents are attracted to the midpoint $x = 1/2$ of the unit segment but would like

4.2 Crowd congestion

References

[1] Yves Achdou, Fabio Camilli, and Italo Capuzzo-Dolcetta. "Mean Field Games: Convergence of a Finite Difference Method". In: *SIAM Journal on Numerical Analysis* 51 (2013), pp. 2585–2612. DOI: 10.1137/120882421. URL: https://hal.archives-ouvertes.fr/hal-01456506.

[2] Jean-David Benamou and Guillaume Carlier. "Augmented Lagrangian Methods for Transport Optimization, Mean Field Games and Degenerate Elliptic Equations". In: *Journal of Optimization Theory and Applications* 167 (Mar. 2015). DOI: 10.1007/s10957-015-0725-9.

[3] Jean-David Benamou, Guillaume Carlier, and Filippo Santambrogio. "Variational Mean Field Games". working paper or preprint. Mar. 2016. URL: https://hal.archives-ouvertes.fr/hal-01295299.

[4] Jean-David Benamou et al. *An entropy minimization approach to second-order variational mean-field games*. 2018. arXiv: 1807.09078 [math.OC].

[5] Lenaic Chizat et al. *Scaling Algorithms for Unbalanced Transport Problems*. 2016. arXiv: 1607.05816 [math.OC].

[6] Marco Cuturi. "Sinkhorn Distances: Lightspeed Computation of Optimal Transport". In: *Advances in Neural Information Processing Systems 26*. Ed. by C. J. C. Burges et al. Curran Associates, Inc., 2013, pp. 2292–2300. URL: http://papers.nips.cc/paper/4927-sinkhorn-distances-lightspeed-computation-of-optimal-transport.pdf.

[7] Steven Diamond and Stephen Boyd. "CVXPY: A Python-Embedded Modeling Language for Convex Optimization". In: *Journal of Machine Learning Research* 17.83 (2016), pp. 1–5.

[8] Jean-Michel Lasry and Pierre-Louis Lions. "Jeux à champ moyen. I – Le cas stationnaire". In: *Comptes Rendus Mathématique* 343.9 (2006), pp. 619–625. ISSN: 1631-073X. DOI: https://doi.org/10.1016/j.crma.2006.09.019. URL: http://www.sciencedirect.com/science/article/pii/S1631073X06003682.

[9] Jean-Michel Lasry and Pierre-Louis Lions. "Jeux à champ moyen. II – Horizon fini et contrôle optimal". In: *Comptes Rendus Mathematique* 343.10 (2006), pp. 679–684. ISSN: 1631-073X. DOI: https://doi.org/10.1016/j.crma.2006.09.018. URL: http://www.sciencedirect.com/science/article/pii/S1631073X06003670.