Time Series Analysis

Lecture Notes

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Preface

These lecture notes provide an exposition for the theory and practice of Time Series Analysis on a formal level. Run of the mill forecasting and projection courses serve a general business audience for surface level analyses. Formal study of Time Series Analysis requires many concepts in Probability Theory and Mathematical Statistics. The author suggests reading Time Series Econometrics[1] by Justin Eloriaga as a quick guide before starting.

Time Series Analysis has many applications, from Economics to Finance, from Stochastic to Dynamic processes. Time series may be seen as random variables defined on integer values. They may serve as discrete approximations to stochastic processes, which take continuous time as their domains. One may also discretize differential equations as time series; here, they are known as difference equations.

These notes begin with exposition on difference equations and lag operators. The theory of ARIMA processes and derived concepts build upon these. There also exists exposition on estimation methods. These notes initially focus on linear models and stationary processes. Nonlinear models follow once we realize how prevalent nonstationary processes are.

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[1]Eloriaga, J. (2020). Time series econometrics. Retrieved from: `https://justineloriaga.files.wordpress.com/2020/11/eloriaga_appliedtimeseries_lecturenotes.pdf`.

1.0 Difference Equations

1.1 First-Order Difference Equations

Suppose a random variable $\{y\}$ which takes values at date t. We denote each value by y_t. We use dynamic equations to relate each value of $\{y\}$ at date t to another random variable w_t and to values of $\{y\}$ in previous dates.

$$y_t = \phi y_{t-1} + w_t \tag{1.1}$$

This equation is first-order since it relates y_t to only the immediately preceding date, or the first lag. We note that equation 1.1 expresses y_t as a linear function of y_{t-1} and w_t.

1.1.1 Solving by Recursive Substitution

Suppose that equation 1.1 governs the behavior of $\{y\}$ for all dates. If we know the value

Date	Equation
0	$y_0 = \phi y_{-1} + w_0$
1	$y_1 = \phi y_0 + w_1$
$\vdots$	$\vdots$
t	$y_t = \phi y_{t-1} + w_t$

of $\{y\}$ at date $t = -1$ and the value of $\{w\}$ for all dates $t = 0, 1, 2, \ldots$, then we can simulate a dynamic system to find all values of $\{y\}$. For example, if y_{-1} and w_0 are known, we can calculate y_0. Given y_0 and w_1, we can calculate y_1.

$$y_1 = \phi y_0 + w_1 = \phi(\phi y_{-1} + w_0) + w_1 = \phi^2 y_{-1} + \phi w_0 + w_1 \tag{1.2}$$

We call this procedure recursive substitution.

1.1.2 Dynamic Multipliers

Equation 1.2 expresses y_t as a linear function of y_{-1} and historical values of $\{w\}$. From this equation, we find that calculating the effects of w_0 on y_t becomes easier.

1 Difference Equations

If w_0 changes with y_{t-1}, but $w_1, w_2, \ldots, w_t$ are unaffected, then

$$\frac{\partial y_t}{\partial w_0} = \phi^t \tag{1.3}$$

We arrive at the same calculation if our dynamic simulation began with date t, taking y_{t-1} as given. We find y_{t+j} to be a function of y_{t-1} and $w_t, w_{t+1}, \ldots, w_{t+j}$:

$$y_{t+j} = \phi^{j+1} y_{t-1} + \sum_{n=0}^{j} \phi^{j-n} w_{t+n}. \tag{1.4}$$

We find a similar effect of changes in w_t on y_{t+j}:

$$\frac{\partial y_{t+j}}{\partial w_t} = \phi^j \tag{1.5}$$

so that dynamic multipliers in the form of 1.5 depend only on j, or how many dates separate a disturbance input w_t from observed output values y_{t+j}. Note that dynamic multipliers **do not** depend on the date t.

Different values of ϕ in equation 1.1 can produce different dynamic responses of $\{y\}$ to $\{w\}$. For $0 < \phi < 1$, the dynamic multiplier 1.5 geometrically vanishes. For $-1 < \phi < 0$, 1.5 alternates in sign for different dates—increases in w_t cause y_t to increase, y_{t+1} to decrease, and so on. The absolute value of 1.5 once again vanishes geometrically.