Proof by Contradiction for Dummies

A Beginner's Guide to *Reductio ad Absurdum*

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1 General Exposition

One law of logic central to discourse, whether in Mathematics or Philosophy, is the *law of excluded middle*. Put simply, the law states that a proposition is either true or false. Formally, this means that propositions have truth values of either 1 or 0:

P	$\neg P$
1	0
0	1

with no in-between—hence the *excluded middle*. That being so, two contradictory propostions may not both be true:

$$\begin{array}{c} P = \neg Q \\ Q = \neg P \\ \hline \therefore P \vee Q. \end{array}$$

Taking advantage of such properties, we can now perform *proofs by contradiction*, which consist of assuming a false statement to be true, then deriving some sort of *contradiction* from having such:

$$P_1, P_2, \ldots, | \therefore Q \iff P_1, P_2, \ldots, \neg Q | 0.$$

We can prove that direct proofs and proofs by contradiction are equivalent like so:

Statement	Rule of Inference
$1. P$	
$2. P \vee 0$	$1.$, Addition
$3. \neg(\neg P) \vee 0$	$2.$, Double Negation
$4. \neg P \to 0$	$3.$, Material Implication

One can simply reverse these steps to show the backwards conditional.

One classic example of proof by contradiction that those taking their first Real Analysis course shows why the square root of 2 is an irrational number:

Theorem 1. $\sqrt{2} \notin \mathbb{Q}$

Proof. Assume for contradiction that $\sqrt{2} \in \mathbb{Q}$. By definition, we can take integers p and q with no common factors such that $\sqrt{2} = \frac{p}{q}$. Taking the square of both sides, we have $2 = \frac{p^2}{q^2} \iff q^2 = \frac{p^2}{2}$. This result implies that p^2 is an even number with factors 2 and q^2, contradicting our earlier statement that p and q had *no common factors*. (One can show that if two integers have no common factors, then their squares have no common factors as an exercise). □

Another classic proof by contradiction is used to prove that the sum of a rational and an irrational number is itself irrational:

Theorem 2. *The sum of a rational and an irrational number is itself irrational.*

Proof. Assume that a rational c and an irrational d had a rational sum x. Then, $c + d = x$. Move some variables such that $c - x = c + (-x) = d$. Note that the sum of two rational must always be rational: $\frac{p_1}{q_1} + \frac{p_2}{q_2} = \frac{p_1 q_2}{q_1 q_2} + \frac{q_1 p_2}{q_1 q_2} = \frac{q_2 p_1 + q_1 p_2}{q_1 q_2}$, contradicting that d is an irrational number. □

2 Validity and Soundness

Arguments have to fulfill two criteria. *Valid* arguments have proper logical form, yet may not always be correct conditional on assumptions. An argument that uses valid arguments and sound assumptions is called a *sound argument*.

St Thomas of Aquino notably uses this form of argumentation in his works, especially the *Summa Theologiae*. One example:

Objection 1 It seems that the existence of God cannot be demonstrated. For it is an article of faith that God exists. But what is of faith cannot be demonstrated, because a demonstration produces scientific knowledge; whereas faith is of the unseen (Hebrews 11:1). Therefore it cannot be demonstrated that God exists.

Reply to Objection 1

Proof. The existence of God and other like truths about God, which can be known by natural reason, are not articles of faith, but are preambles to the articles; for faith presupposes natural knowledge, even as grace presupposes nature, and perfection supposes something that can be perfected. Nevertheless, there is nothing to prevent a man, who cannot grasp a proof, accepting, as a matter of faith, something which in itself is capable of being scientifically known and demonstrated.

The assumption in Objection 1 was that God's existence was an article of Faith, which is of the unseen. As such, proving God's existence through reason could not be done. However, St Thomas causes this argument to collapse by pointing out that one needs to know that God exists before accepting any article of Faith, causing a contradiction. □

3 Steel-manning an Argument

One often needs to *steelman* an opponent's argument before arguing against it. This may happen if the opponent's argument is written in paragraph form, or is incoherent, or is written informally. For example:

> You said that in Austrian Economics, prices are flexible and respond fast to supply and demand. However, you also admitted that central banks print money which causes inflation, and that this inflation causes price stickiness. Real prices aren't even the type of price flexibility required in Austrian Economics—the originals used nominal prices. You gotta either admit that nominal prices are sticky thanks to Central Banks or that

Central Banks really don't cause nominal price stickiness and Austrian price flexibility still holds.

One can steelman this argument in the following fashion:

1. Austrian Economics postulates nominal price flexibility
2. Central Banks cause nominal price stickiness
3. Statements 1 and 2 contradict each other
4. Either statement 1 is false or statement 2 is false

One can reduce this argument to absurdity with help of the following analogical argument:

1. Cold weather causes cold temperatures
2. Heaters cause warm temperatures
3. Statements 1 and 2 contradict each other
4. Either statement 1 is false or statement 2 is false

As one can note, statement 1 of both arguments is true in the absence of outside intervention. Statement 2 pertains to the intervention itself. The argument falls apart by noting that *no such contradiction between statements 1 and 2 exists* as we have just shown.

Exercise

3.1

Refute these arguments using proof by contradiction.

One may commit evil and advance God's will

1. Evil exists through God's permission to advance His will.
2. God's will is good.
3. Therefore, one may commit evil and advance God's will.

One must not stop or fight against evil to advance God's will

1. Evil exists through God's permission to advance His will.
2. God's will is good.
3. Therefore, one must not stop or fight against evil to advance His will.

3.2

Steelman and disprove the following arguments using proof by contradiction. The first few steps have been given.

Original sin caused a fallen world, therefore all time periods are equally evil. (argument c/o Monica Labbao)

1. Original sin caused a fallen world.
2. ...

One may note that this argument makes the implicit assumption of being evil in one degree casues evil in all degrees.

God's permissive will surpasses imperfect human action, therefore trying to shape economic-political conditions instead of bearing the cross of modernity is evil. One must rather be a corporate slave than a muck farmer. (argument c/o Monica Labbao)

1. God's permissive will surpasses imperfect human action.
2. ...

One may note that this argument makes the implicit dichotomy between drudgerious muck farming and soul-crushing corporate slavery.