Real and Complex Analysis

Lecture Notes

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Contents

Preface

These lecture notes serve as an introduction to undergraduate Real and Complex Analysis. Mathematics undergraduates usually take this course in their first year, focusing on the Real Field, sequences, and series. Advanced Economics undergraduates would need the same topics as preparation for graduate studies. Mathematics undergraduates then go on to the Riemann Integral and other topics throughout their second year. One would usually need to take a course on Set Theory and Formal Logic before learning Analysis. The author recommends Book of Proof by Richard Hammack and For all x by PD Magnus, Cathal Woods, and J Robert Loftis for those self-studying these notes.

A Mathematics undergraduate usually encounters Real and Complex Analysis as one's first formal study of familiar tools like sequences, Calculus and complex numbers. Proof-based Mathematics becomes the norm, with decreasing focus on computational and algorithmic exercises so common in lower education. This transition helps a student become rigorous—no more clinging onto intuitive facts or formulas or algorithms, more focus on trial and error, solutions and insight than answers. Only in Mathematics can proving a problem to be unsolvable be cause for celebration—consistency, sense, and rigor matter more than world-changing insights. Physicists prefer the familiarity of the physical world. Engineers prefer tangible sense for stability. Economists prefer splitting hairs on special cases than finding general principles and theorems. Computer Scientists wish to endlessly verify whether heuristics and algorithms work to solve problems. Mathematicians, however, seek escape from reality by performing useless work with symbols—the highest form of creative and artistic expression known to man.

1.0   Real Numbers

1.1   Set-Theoretic Preliminaries

We call a collection S of elements a set. The expression $x \in S$ means that the object x is an element of S. Likewise, $x \notin S$ means that x is not an element of S. As a general rule, we denote sets by big letters and their elements by small letters.

We may take some elements of S as their own set T. We call the set T a subset of S, which we denote by $T \subset S$.

> **Definition 1.1.1.** A subset T of S, denoted by $T \subseteq S$, satisfies:
>
> $$\forall x \in T (x \in S).$$

We may also find that two sets may be subsets of each other. One may show that

$$A \subseteq B \wedge B \subseteq A \implies A = B.$$

A proper subset $T \subset S$ satisfies

$$\exists x \in S (x \notin T)$$

so that S contains elements that are not elements of T.

Operations on sets A and B comprise unions and intersections.

> **Definition 1.1.2.** A union of sets A or B satisfies
>
> $$A \cup B = \{x : x \in A \vee x \in B\}$$
>
> while an intersection of sets A and B satisfies
>
> $$A \cap B = \{x : x \in A \wedge x \in B\}.$$

1.2   Real Number Axioms

Mathematical Analysis studies concepts involving the real number system $\mathbb{R}$. Instead of real numbers themselves, we focus on their properties as undefined objects satisfying certain axioms.

1 Real Numbers

Axiom 1.1. (Addition). The sum $(x + y) \in \mathbb{R}$ of $x, y \in \mathbb{R}$ satisfies

1. $x + y = y + x$ (commutativity)

2. $\forall z \in \mathbb{R}[(x + y) + z = x + (y + z)]$ (associativity)

3. $\exists 0 \in \mathbb{R} \forall x \in \mathbb{R}(x + 0 = x)$ (zero element)

4. $\forall x \in \mathbb{R} \exists y \in \mathbb{R}(x + y = 0)$ (negative element)

Axiom 1.2. (Multiplication). The product $xy \in \mathbb{R}$ satisfies

1. $\forall x, y \in \mathbb{R}(xy = yx)$ (commutativity)

2. $\forall z \in \mathbb{R}[(xy)z = x(yz)]$ (associativity)

3. $\exists 1 \in \mathbb{R}, 1 \neq 0, \forall x \in \mathbb{R}(1x = x)$ (unitary element)

4. $\forall x \in \mathbb{R}, x \neq 0, \exists y \in \mathbb{R}(xy = 1)$ (reciprocal element)

5. $x(y + z) = xy + xz$ (distrbutive property)

A set of objects x, y, z satisfying these axioms is called a field.

Axiom 1.3. For all $x, y \in \mathbb{R}(x \geq y \lor x \leq y)$ with properties

1. $\forall x \in \mathbb{R}(x \leq x)$ (reflexivity)

2. $(x \leq y) \land (y \leq x) \implies x = y$ (antisymmetry)

3. $(x \leq y) \land (y \leq z) \implies x \leq z$ (transitivity)

4. $\forall z \in \mathbb{R}(x \leq y \implies x + z \leq y + z)$

5. $0 \leq x \land 0 \leq y \implies 0 \leq xy$

We call a set whose elements satisfy 1–3 a partially ordered set. The set $\mathbb{R}$ is a totally ordered set since it satisfies $x \geq y \lor x \leq y$ (connexity).

Before our last axiom, we need an important definition.

Definition 1.2.1. A set $E \subset \mathbb{R}$ is bounded above when

$$\exists z \in \mathbb{R} \forall x \in E(x \leq z).$$

We denote that is bounded above by $E \leq z$. We call the element z an upper bound of the set E.

Axiom 1.4. (Axiom of Completeness). For elements $z \in \mathbb{R}$, a set $E \leq z$ satisfies the axiom of completeness when

$$\forall E \subset \mathbb{R}, \exists z_0 \in \mathbb{R}(z_0 \leq z).$$

We call z_0 the supremum of E and denote it by $z_0 = \sup[E]$.

1.3 Consequences of Axioms

1.3.1 Addition

Theorem 1.3.1. The element $0 \in \mathbb{R}$ is unique.

Proof. Assume any two zero elements, 0_i and 0_j. By Addition Axioms 1 and 3:

$$0_i = 0_i + 0_j = 0_j + 0_i = 0_j.$$

$\square$

Theorem 1.3.2. For all $x \in \mathbb{R}$, the negative element is unique.

Proof. Suppose $x \in \mathbb{R}$ has any two negative elements y_i and y_j. By Addition Axioms 1, 2, 3, and 4:

$$y_i = y_i + 0 = y_i + (x + y_j) = (y_i + x) + y_j = 0 + y_j = y_j.$$

We denote the unique negative element of $x \in \mathbb{R}$ by $-x \in \mathbb{R}$. We call the sum $x + (-y)$ the difference of x and y, and denote it by $x - y$. $\square$

Theorem 1.3.3. The equation $a + x = b$ has a unique solution $x = b - a$.

Proof. Add the negative element $-a$ to both sides of the equality:

$$a + x + -a = x = b - a.$$

Verify whether this is the solution:

$$a + x = a + (b - a) = a + b - a = b + 0 = b.$$

$\square$

1.3.2   Multiplication

Theorem 1.3.4. The set $\mathbb{R}$ contains a unique unit element 1.

Proof. Suppose two unit elements 1_i and 1_j. Then by Multiplication Axioms 1 and 3

$$1_i = 1_i 1_j = 1_j 1_i = 1_j.$$

$\square$

Theorem 1.3.5. Every element $x \in \mathbb{R}$ has a unique reciprocal.

Proof. Suppose x has two reciprocals y_1 and y_2. Then

$$y_1 = y_1 1 = y_1(xy_2) = (y_1 x)y_2 = 1y_2 = y_2.$$

$\square$

Definition 1.3.1. Numbers $1, 2 = 1+1, 3+2+1, \ldots$ are called the Natural Numbers, denoted $\mathbb{N}$. The set $\mathbb{N}$ may be defined as the smallest set containing 1 and $n+1$ whenever $n \in \mathbb{N}$.

One may need to show sometimes that a numerical set A—a set of all $n \in \mathbb{N}$ for which some property T holds—has all natural numbers. The method of mathematical induction verifies whether

$$(1 \in A) \wedge (n \in A \implies n+1 \in A)$$

The set of integers holds all $n \in \mathbb{N}$, reciprocals $-n$, and the number 0. We denote the set of integers by $\mathbb{Z}$. Mathematicians also differ in convention whether $0 \in \mathbb{N}$ or not. To remove ambiguity, one may use $\mathbb{Z}^+$ instead of our definition of $\mathbb{N}$. Additionally, suppose a number $m \in \mathbb{Z}$. We call the number $2m$ an even number, and $2m+1$ and odd number.

If we get all quotients of form $\frac{m}{n}, m, n \in \mathbb{Z}$ and $n \neq 0$, we get the set of rational numbers $\mathbb{Q} \subset \mathbb{R}$. All other numbers are irrationals.

Theorem 1.3.6. The equation $ax = b, a \neq 0$ has the unique solution $\frac{b}{a} \in \mathbb{R}$.

Proof. Multiply both sides by $\frac{1}{a}$:

$$\frac{1}{a}ax = \frac{1}{a}b = 1x = x = \frac{b}{a}$$

Verify the solution:

$$ax = b = a(\frac{b}{a}) = a\frac{1}{a}b = 1b = b.$$

$\square$

Define $x^n = \prod_n x, n \in \mathbb{N}$. We then have $x^n x^m = \prod_n x \prod_m x = x^{n+m}$. Additionally, $(x^n)^m = \prod_m (\prod_n x) = x^{mn}, m, n = 1, 2, \ldots$. We also define $x^0 = 1$, $x^{-m} = \frac{1}{x^m}, x \neq 0$.

Theorem 1.3.7. If $x \in \mathbb{R}$, then $0x = 0$.

Proof. Using the definition of x^2,

$$0x = (x - x)x = x^2 - x^2 = 0.$$

Alternatively, by Addition Axiom 3 and Multiplication Axiom 3

$$0x + 1x = (1 + 0)x = 1x = x \implies 0x = x - x = 0.$$

$\square$

From the above theorem, we have the following useful corollary.

Corollary 1.3.8. If $xy = 0$ and $x \neq 0$, then

$$y = \frac{1}{x}xy = \frac{1}{x}(xy) = \frac{1}{x}0 = 0.$$

Thus if a product vanishes, then so does one of its factors.

Theorem 1.3.9. If $(u \neq 0) \wedge (v \neq 0)$, then $\frac{x}{u} + \frac{y}{v} = \frac{xv+yu}{vu}$.

Proof. Multiply the lefthand side by $\frac{vu}{vu} = 1$

$$\frac{vu}{vu}\left(\frac{x}{u} + \frac{y}{v}\right) = \frac{vu}{vu}\frac{x}{u} + \frac{vu}{vu}\frac{y}{v} = \frac{xv}{vu} + \frac{yu}{vu} = \frac{xv + yu}{vu}.$$

Alternatively, one may notice that

$$\frac{xv + yu}{vu} = \frac{1}{vu}(xv + yu) = \frac{xv}{vu} + \frac{yu}{vu} = \frac{x}{u} + \frac{y}{v}.$$

$\square$

Theorem 1.3.10. If $x \in \mathbb{R}$, then $-x = (-1)x$.

Proof. By Multiplication Axiom 5,

$$0 = x - x = x(1 - 1) = x[1 + (-1)] = x + (-1)x.$$

$\square$

Definition 1.3.2. The factorial of $n \in \mathbb{N}$, denoted by $n!$ is the term f_n of the sequence $f_0 = 1, f_n = n f_{n-1}, n \geq 1$.

Definition 1.3.3. For $n, k \in \mathbb{N}$ with $n \geq k$, the binomial coefficient $\binom{n}{k} \in \mathbb{N}$ is given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Lemma 1.3.11. For natural numbers n and k with $n \geq k$,

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$$

Proof. The proof is purely computational and is left as an exercise, if the reader pleases. $\square$

Theorem 1.3.12. (Binomial Theorem). For nonzero $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Proof. For the base step, let $n = 1$. Then,

$$(a+b)^1 = a + b = \sum_{k=0}^{1} \binom{1}{k} a^{1-k} b^k = \binom{1}{1} a^{1-1} b^1 + \binom{1}{0} a^{1-0} b^0 = a + b$$

For the induction step, assume that the theorem holds for n. For the case of $n+1$ we use the preceding lemma:

$$(a+b)^{n+1} = (a+b)^n (a+b)^1 = (a+b) \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \sum_{k=0}^{1}$$

$$= a \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k + b \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1}$$

$$= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=0}^{n-1} \binom{n}{k} a^{n-k} b^{k+1} + b^{n+1}$$

$$= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} a^{n+1-k} b^k + \sum_{k=1}^{n} \binom{n}{k-1} a^{n+1-k} b^k + b^{n+1}$$

$$= a^{n+1} + \sum_{k=1}^{n} \binom{n}{k} \binom{n}{k-1} a^{n+1-k} b^k + b^{n+1}$$

$$= a^{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} a^{n+1-k} b^k + b^{n+1}$$

$$= \binom{n+1}{0} a^{n+1} b^0 + \sum_{k=1}^{n} \binom{n+1}{k} a^{n+1-k} b^k + \binom{n+1}{n+1} a^0 b^{k+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n+1-k} b^k$$

$\square$

1.3.3   Consequences of Order Axioms

Lemma 1.3.13. If $x \leq y \wedge y \leq z \wedge x = z$, then $x = y = z$.

Proof.
$$y \leq z = x \implies y \leq x$$

since $x \leq y$, $x = y = z$. □

Lemma 1.3.14. If $x < y \wedge y \leq z$, then $x < z$. Similarly, if $x \leq y \wedge y < z$, then $x < z$.

Proof. If $y < z$, then by Order Axiom 3, $x < z$. If $y = z$, then by hypothesis $x < y = z \iff x < z$. One may show the second result through a similar way. □

Theorem 1.3.15. The following inequalities are equivalent:

- $x \leq y$
- $0 \leq y - x$
- $x - y \leq 0$

Proof. We take advantage of Order Axiom 4:

$$x \leq y \implies x - x \leq y - x \iff 0 \leq y - x$$
$$0 \leq y - x \implies 0 - y \leq y - x + (-y) \iff -y \leq -x$$
$$-y + x \leq -x + x \implies x - y \leq -x + x \implies x - y \leq 0.$$

□

Lemma 1.3.16. For all $z \in \mathbb{R}$, $x < y \implies x + z, y + z$

Proof.
$$x < y \implies x \leq y \implies x + z \leq y + z.$$

It must be that $x + z < y + z$ since $x + z = y + z \implies x = y$, a contradiction. □

Theorem 1.3.17. For all $i \in \mathbb{N}$, if $x_i \leq y_i$ then

$$\sum_{i=1}^{n} x_i \leq \sum_{i=1}^{n} y_i.$$

The inequality becomes strict when $x_j < y_j$ for at least one pair x_j, y_j.

Proof. By Order Axiom 3:

$$\sum_{i=1}^{n} x_i \leq y_1 + \sum_{i=2}^{n} x_i \leq y_1 + y_2 + \sum_{i=3}^{n} x_i \leq \sum_{i=1}^{n} y_i.$$

Using Lemma 1.3.16,

$$x_j < y_j \implies \sum_{i=1}^{n} x_i < \sum_{i=1}^{n} y_i$$

for at least one pair x_j, y_j. $\qquad\square$

One can note that $x_1 \leq 0, \ldots, x_n \leq 0 \implies s = \sum_{i=1}^{n} x_i \leq 0$. If $x_j < 0$, then $s < 0$ for any j.

Theorem 1.3.18. The following inequalities are equivalent:

- $x < y$
- $0 < x - y$
- $x - y < 0$

Proof. One may apply Lemma 1.3.16 to Theorem 1.3.15 to obtain this theorem. $\qquad\square$

Definition 1.3.4. A number $x \in \mathbb{R}$ is nonnegative if $x \geq 0$, positive if $x > 0$, nonpositive if $x \leq 0$, and negative if $x < 0$.
The number 0 is both nonnegative and nonpositive.

Definition 1.3.5. Suppose $x \leq y$ for $x, y \in \mathbb{R}$. We call x the minimum of numbers x and y. We denote this fact by $\min\{x, y\} = x$. Similarly, we call y the maximum, denoted by $\max\{x, y\} = y$.

One can use induction to define $\min\{x_1, \ldots, x_n\}$ and $\max\{x_1, \ldots, x_n\}$:

$$\max\{x_1, \ldots, x_n\} = \max\{\max\{x_1, \ldots, x_{n-1}\}, x_n\}$$

Definition 1.3.6. The number $|x| = \max\{x, -x\}$ is called the absolute value, or modulus, of x.
We have that $x > 0 \implies |x| = x$ and $x < 0 \implies |x| = -x$. Also,

$$\forall x > 0(|x| > 0 \land |x| = |-x|).$$

Theorem 1.3.19. If $a > 0$, then the following are equivalent:

- $|x| \leq a$
- $x \leq a$
- $-x \leq a$

Proof. By definition of absolute value,

$$|x| \leq a \iff \max\{x, -x\} \leq a.$$

It follows that if $0 < x$, then $x \leq a$. Otherwise, if $x < 0$, then $-x \leq a$. Either way, both $x \leq a$ and $-x \leq a$. $\square$

By Theorem 1.3.15, $-x \leq a \iff -a \leq x$, so that $-a \leq x \leq a$.

Theorem 1.3.20. (Triangle inequality). For all $x, y \in \mathbb{R}$, we have that $|x + y| \leq |x| + |y|$

Proof. Let $x, y > 0$ or $x, y < 0$. Then, $|x + y| = \max\{(x + y), -(x + y)\}$.
If $x \geq 0$ and $y < 0$, then

$$x + y \leq x \leq x + |y| = |x| + |y|$$
$$-x - y \leq -y \leq -y + |x| = |x| + |y|$$

so that $|x + y| = \max\{(x + y), -(x + y)\} \leq |x| + |y|$. $\square$

One may use induction to show that

$$\left| \sum_{i=1}^{n} x_i \right| \leq \sum_{i=1}^{n} |x_i|.$$

Lemma 1.3.21. If $x > 0$ and $y > 0$, then $xy > 0$.

Proof. This lemma follows from Axiom 5 and Corollary 1.3.8:

$$xy = 0 \wedge x \neq 0 \implies y = 1y = \frac{1}{x}xy = \frac{1}{x}(xy) = \frac{1}{x}0 = 0.$$

$\square$

Theorem 1.3.22. If $x \leq y$ and $0 < z$, then $xz \leq yz$

Proof. By Axiom 4,

$$yz - xz = (y - x)z \geq 0.$$

$\square$

One may note that by Corollary 1.3.8, $x < z \wedge z > 0 \implies xz < yz$. Additionally,

$0 < x < 1 \implies x^2 < x$ and $1 < x \implies x < x^2$. One can also show that

$$0 < x \le y \wedge 0 < z \le u \implies xz \le yz \le yu.$$

We note that one may multiply inequalities under these conditions. In general, if $0 < x < y$, then for all $n \in \mathbb{N}$, we have that $x^n < y^n$.

Theorem 1.3.23. If $x \le 0$ and $0 \le y$, then $xy \le 0$. Similarly, if $x \le 0$ and $y \le 0$, then $0 \le xy$.

Proof. Since $x \le 0$, we have that $0 \le -x$. It follows from Axiom 5 and Theorem 1.3.10 that $0 \le -xy \iff xy \le 0$.

Similarly, $x \le 0$ and $y \le 0$ implies that $0 \le -x$ and $0 \le -y$, so that $0 \le (-x)(-y) = -1(x) - 1(y) = -1(-1)xy = 1xy = xy$. $\square$

In particular, one may note that $x^2 = xx > 0 \forall x \ne 0$. One consequence of this fact is that $1 = 1(1) > 0$. Also, by Lemma 1.3.16, $2 = 1 + 1 > 0$, $3 = 2 + 1 > 2$, etc.

Theorem 1.3.24. For all $x, y \in \mathbb{R}$, we have that $|xy| = |x||y|$.

Proof. By definition of absolute value, $|xy| = \max\{xy, -xy\}$. Let $x > 0$ and $y > 0$ or $x < 0$ and $y < 0$. Then, $\max\{xy, -xy\} = |x||y|$ by Theorem 1.3.23. Suppose without loss of generality that $x > 0$ and $y < 0$. Then, $xy < 0$. However, we now have that $\max\{xy, -xy\} = (-x)y = |x||y|$. If $x = 0$ or $y = 0$, then the absolute value is 0 by Corollary 1.3.8. $\square$

Theorem 1.3.25. If $x > 0$, then $\frac{1}{x} > 0$. Moreover, if $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$.

Proof. The quotient $\frac{1}{x}$ cannot be 0 since it is undefined for that value. Let $\frac{1}{x} < 0$. Then we would have $1 < 0$, which is impossible for $1 = 1^2 > 0$. Moreover, one can multiply the inequality $0 < x < y$ by $\frac{1}{xy}$ to get $0 < x\frac{1}{xy} < y\frac{1}{xy} \iff 0 < \frac{1}{y} < \frac{1}{x}$. $\square$

Theorem 1.3.26. Let $r \in \mathbb{R}$ and $r > 0$. For all r, if $z \ge 0$ and $z < r$, then $z = 0$.

Proof. If $z > 0$, then then $z > z \ge 0$, which is impossible by conexity. $\square$

1.3.4   Axiom of Completeness

Definition 1.3.7. A set $E \subset \mathbb{R}$ is bounded below if $\exists z \in \mathbb{R} \forall x \in E(z \leq x)$. We denote that E is bounded below by $z \leq E$.

All numbers z that satisfy this property are called lower bounds of E.

Proposition 1.3.27. If a set E is bounded above, then the set $-E = \{-x | x \in E\}$ is bounded below.

Proof. By definition of upper bounds, there exists $z \in \mathbb{R}$ such that for all $x \in E$, $x \leq z$. Taking their negative elements, one can find that for all $-x \in -E$, $-z \leq -x$. The number $-z \in \mathbb{R}$ easily satisfies the definition of a lower bound. $\square$

One may also show that if E is bounded below by z, then $-E$ is bounded above by $-z$ in a similar way.

Theorem 1.3.28. Every set $E \subset \mathbb{R}$ bounded below has a greatest lower bound $-\sup(-E)$.

Proof. For $-E = \{-x | x \in E\}$, we find using Proposition 1.3.27 that $-E$ is bounded above. By Axiom 1.4, there exists $\sup(-E)$. Taking the inequality $-x \leq \sup(-E)$, we find that $x \geq -\sup(-E)$, satisfying E having a greatest lower bound. $\square$

We denote the greatest lower bound of E as $\inf(E)$, and call it the infimum.

Theorem 1.3.29. For $E \subseteq F$, if E and F are bounded above, then $\sup(E) \leq \sup(F)$. Likewise, if E and F are bounded below, then $\inf(E) \geq \inf(F)$.

Proof. Let $E = F$. They then have the same supremum. Let $E \subset F$. There exists at least one $z \in F$ such that $z \notin E$. If $z > \sup(E)$, then $\sup(E) < \sup(F)$. If $z < \sup(E)$, then $\sup(E) = \sup(F)$. One may use a similar way to prove the lower bounded case. $\square$

Theorem 1.3.30. For arbitrary $x \in E, y \in F$, if $x \leq y$, then E is bounded above, F is bounded below, and $\sup(E) \leq \inf(F)$.

Proof. An arbitrary $x \in E$ immediately satisfies the definition for a lower bound since for all $y \in F$, $x \leq y$. Likewise holds for arbitrary $y \in F$ being an upper bound since for all $x \in E$, $y \geq x$. It immediately follows that $\sup(E) \leq F$ such that $\sup(E) \leq \inf(F)$. $\square$

Theorem 1.3.31. For all $x \in \mathbb{R}$ and $n \in \mathbb{Z}, x, n > 0$, there exists a unique $y > 0$ such that $y^n = x$.

Proof. Let A be the set of all $z > 0$ such that $z^n \leq x$. Then A is bounded above, by 1 if $x \leq 1$ and by x if $1 \leq x$. Let $y = \sup(A)$. We will show that $y^n = x$.

Suppose that $y^n < x$ and let $x - y^n = \varepsilon$. For all h such that $0 < h \leq 1$, and by Theorem 1.3.12:

$$(y + h)^n = y^n + ny^{n-1}h + \frac{n(n-1)}{1 \cdot 2}y^{n-2}h^2 + \ldots$$
$$= y^n + h(ny^{n-1} + \frac{n(n-1)}{1 \cdot 2}y^{n-2}h + \ldots)$$
$$\leq y^n + h(ny^{n-1} + \frac{n(n-1)}{1 \cdot 2}y^{n-2} + \ldots) = y^n + h[(1+y)^n - y^n]$$

Set $h < \frac{\varepsilon}{(1+y)^n - y^n}$ so that we have $(y + h)^n \leq y^n + \varepsilon = x$. However, this contradicts the fact that $y = \sup(A)$. Therefore, $y^n \geq x$.

Suppose that $y^n > x$. Let $\varepsilon = y^n - x$. For all h such that $0 < h \leq 1$, and by Theorem 1.3.12:

$$(y - h)^n = y^n - ny^{n-1}h + \frac{n(n-1)}{1 \cdot 2}y^{n-2}h^2 + \ldots$$
$$= y^n - h(ny^{n-1} + \frac{n(n-1)}{1 \cdot 2}y^{n-2}h + \ldots)$$
$$\geq y^n - h(ny^{n-1} + \frac{n(n-1)}{1 \cdot 2}y^{n-2} + \ldots) = y^n - h[(1+y)^n - y^n]$$

Set $h < \frac{\varepsilon}{(1+y)^n - y^n}$ so that we have $(y - h)^n \geq y^n - \varepsilon = x$. However, this contradicts the fact that $y = \sup(A)$. Therefore, $y^n = x$.

We call y the nth root of x. We denote the nth root of x by $\sqrt[n]{x}$, or by $x^{\frac{1}{n}}$. The nth root is unique since $y_1 < y_2 \implies y_1^n < y_2^n \iff x < x$, which is impossible. $\square$

Theorem 1.3.32. For all $x > 0$ and $y > 0$, $(xy)^{\frac{1}{n}} = x^{\frac{1}{n}}y^{\frac{1}{n}}$

Proof. Note that

$$xy = x^{\frac{n}{n}}y^{\frac{n}{n}} = (xy)^{\frac{n}{n}} \iff (x^{\frac{1}{n}}y^{\frac{1}{n}})^n = \left[(xy)^{\frac{1}{n}}\right]^n \iff (x^{\frac{1}{n}}y^{\frac{1}{n}}) = \left[(xy)^{\frac{1}{n}}\right].$$

$\square$

Simlarly, one can show that $\sqrt[m]{\sqrt[n]{x}} = \sqrt[mn]{x}$.

Suppose an even interger n. Then, $(-x)^n = (-1)^n x^n = x^n > 0 \forall x \neq 0$. The equation $y^n = x > 0$ has both real solutions $y_1 = \sqrt[n]{x}$ and $y_2 = -\sqrt[n]{x}$, while the equation $y^n = x < 0$ has no real solutions.

Suppose an odd integer n. Then, $y^n = x > 0$ has a unique real solution $y = \sqrt[n]{x}$, while

$y^n = x < 0$ has a unique real solution $y = -\sqrt[n]{x}$.

By Theorem 1.3.32, formulas like the quadratic and cubic equations hold for all real numbers.

Definition 1.3.8. A set $E \subseteq \mathbb{R}$ is bounded from both sides or bounded if it has both upper and lower bounds. It follows that both $\inf(E)$ and $\sup(E)$ exist.

One important class of bounded sets follows.

Definition 1.3.9. The set of all $x \in \mathbb{R}$ such that $a < x < b$ is called an open interval, with left-hand and right-hand endpoints a and b, respectively. If $a \leq x \leq b$, then we call the set a closed interval, with identical endpoints.

One may see that $\sup[a, b] = \sup(a, b)$, and that $\inf[a, b] = \inf(a, b)$. Both 'half-closed' and 'half-open' intervals $[a, b)$ and $(a, b]$ are defined analogously.

1.4   Archimedean Property and its Consequences

Theorem 1.4.1. (Archimedean Property). For all real $x \geq 0$ and $y > 0$, there exists $n \in \mathbb{Z}$ such that $(n-1)x \leq y < nx$.

Proof. Suppose that for all $p \in \mathbb{Z}$, we have that $px \leq y$. Define the set $A = \{px | px \leq y\}$, and see that y forms an upper bound for A. Let $\sup(A) = d \leq y$. Take the difference $d - x < d$. We can set p_0 such that $d - x < p_0 x$. However, $d - x < p_0 x \iff d < p_0 x + x \iff d < (p_0 + 1)x$ with $(p_0 + 1)x \in A$, forming a contradiction for d being an upper bound.

One may set from pairs $\{(p_i - 1), p_i\} = \{(n + 1), n\}$ so that $(n - 1)x \leq y < nx$. $\square$

If we set $x = 1$, we find that for all $y \in \mathbb{R}$ there exists $n \in \mathbb{Z}$ such that $n - 1 \leq y \leq n$. We call n the integral part of y, denoted $[y]$, and $y - [y]$ the fractional part y, denoted (y). Also, $\forall y \in \mathbb{R}[y \leq [y] + (y)]$.

Theorem 1.4.2. For all $x \in \mathbb{R}$ and $y \in \mathbb{R}$, $x, y > 0$, there exists $n \in \mathbb{Z}$ such that $x^{n-1} \leq y < x^n$.

Proof. Suppose that for all $p \in \mathbb{Z}$, we have that $x^{p-1} < y$. Define the set $A = \{x^p | x^p \leq y\}$, and see that y forms an upper bound for A. Let $\sup(A) = d \leq y$. Take the quotients $\frac{d}{x} < d$. We can set p_0 such that $\frac{d}{x} < x^{p_0}$. However, $\frac{d}{x} < x^{p_0} \iff d < x x^{p_0} \iff d < x^{p_0+1}$ with $x^{p_0+1} \in A$, forming a contradiction for d being an upper bound.

One may set from pairs $\{(p_i - 1), p_i\} = \{(n+1), n\}$ so that $x^{n-1} \leq y < x^n$. $\qquad\square$

Theorem 1.4.3. For all $x, y > 0$, there exists an integer $n > 0$ such that $\frac{y}{n} < x$.

Proof. By the Archimedean Property, there exists $n \in \mathbb{Z}$ such that $y < xn$. Set $n > \frac{y}{x} > 0$. Multiply both sides by $\frac{x}{n}$ to get $\frac{y}{n} < x$. $\qquad\square$

It follows that for all $y > 0$, $\inf\{\frac{y}{n} | n \in \mathbb{Z}^+\} = 0$.

Corollary 1.4.4. The following systems of half-open intervals for $y > 0$ has an empty intersection:

$$\ldots \subseteq (0, \frac{y}{n}] \subseteq \ldots \subseteq (0, \frac{y}{2}] \subseteq (0, y] \tag{1.1}$$

$$\ldots \subseteq (a, a + \frac{y}{n}] \subseteq \ldots \subseteq (a, a + \frac{y}{2}] \subseteq (a, a + y] \tag{1.2}$$

$$\ldots \subseteq (a - \frac{y}{n}, a] \subseteq \ldots \subseteq (a - \frac{y}{2}, a] \subseteq (a - y, a] \tag{1.3}$$

Proof. Suppose systems 1.2 and 1.3 had common elements c and d, respectively. That would mean that system 1.1 would have commmon points $c - a$ and $a - d$. However, by Theorem 1.4.1, system 1.1 has no common points. $\qquad\square$

Theorem 1.4.5. For all open intervals (a, b), there exists an element $z \in \mathbb{Q}$ such that $a < z < b$.

Proof. Take the difference $b - a = h$. By Theorem 1.4.1, there exists $n \in \mathbb{N}$ for $\frac{1}{h}$ such that $(n-1) \leq \frac{1}{h} < n \iff \frac{1}{n} < h$. By that same theorem, there exists $m \in \mathbb{N}$ for $\frac{1}{n}$ such that $\frac{m}{n} \leq a < \frac{m+1}{n}$. Add $\frac{1}{n}$ to both sides and subtract a to obtain $\frac{m}{n} + \frac{1}{n} - a \leq \frac{1}{n} + a - a \iff \frac{m+1}{n} - a \leq \frac{1}{n} < h = b - a \iff \frac{m+1}{n} < b$, showing that there exists $z = \frac{m+q}{n}$ so that $a < z < b$. $\qquad\square$

One can show that infinitely many rational elements exist in (a, b) by applying the preceding theorem to the interval $(\frac{m+1}{n}, b)$, and so on.

Theorem 1.4.6. For all $c \in \mathbb{R}$, let N_c be the set of all s such that $s \leq c$, and P_c the set of all r such that $c \leq r$. Then, $\sup(N_c) = c = \inf(P_c)$.

Proof. The set N_c is bounded above by c, so that it has a supremum. Denote $\sup(N_c) = a$. By definition, $a \leq c$. Suppose that $a < c$. By Theorem 1.4.5, there exists a rational element p such that $p \in (a, c)$. Since $a = \sup(N_c)$, however, $p \leq a$, creating a contradiction. Therefore, $a = c = \sup(N_c)$. $\qquad\square$