AI1110
Assignment 10

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Outline

Problem Statement

Question:

Let x and y be independent gamma random variables with parameters $(\alpha_1, \beta) and (\alpha_2, \beta) respectively$

(a) Determine the p.d.f.s of the random variables x + y and $\frac{x}{x+y}$

(b) Show that x + y and $\frac{x}{y}$ are independent random variables

(c) Show that x + y and $\frac{x}{x+y}$ are independent gamma and beta random variables, respectively.

Solution

(a) let z = x + y and w = $\frac{x}{x+y}$

f$_{xy}(x,y) = f_x(x)f_y(y) = \frac{1}{\alpha^{\alpha_1+\alpha_2}\Gamma_{(\alpha_1)}\Gamma_{(\alpha_2)}}x^{\alpha_1-1}y^{\alpha_2-1}e^{\frac{-(x+y)}{\beta}} x > 0, y > 0$

note that $0 < z < 1$, since x and y are non-negative random variables

$$F_z(Z) = P(z \leq Z) = P\left(\frac{x}{x+y} \leq z\right) = P\left(x \leq y\frac{z}{1-z}\right) \tag{1}$$

$$= \int_0^{\infty}\int_0^{\frac{yz}{1-z}} f_{xy}(x,y)dx.dy \tag{2}$$

differentiation with respect to z gives

$$f_z(Z) = \int_0^{\infty} \frac{y}{(1-z)^2} f_{xy}\left(\frac{yz}{1-z}, y\right)dy \tag{3}$$

Solution

$$= \int_0^\infty \frac{y}{(1-z)^2} \frac{1}{\alpha^{\alpha_1+\alpha_2}\Gamma_{(\alpha_1)}\Gamma_{(\alpha_2)}} \left(\frac{yz}{1-z}\right)^{\alpha_1-1} y^{\alpha_2-1} e^{\frac{-y}{(1-z)\alpha}} dy \tag{4}$$

$$= \frac{1}{\alpha^{\alpha_1+\alpha_2}\Gamma_{(\alpha_1)}\Gamma_{(\alpha_2)}} \frac{z^{\alpha_1-1}}{(1-z)^{\alpha_1+1}} \int_0^\infty y^{\alpha_1+\alpha_2-1} e^{\frac{-y}{\alpha(1-z)}} dy \tag{5}$$

$$= \frac{z^{\alpha_1-1}(1-z)^{\alpha_2-1}}{\Gamma_{(\alpha_1)}\Gamma_{(\alpha_2)}} \int_0^\infty u^{\alpha_1+\alpha_2-1} e^{-u} du \tag{6}$$

$$= \frac{\Gamma_{(\alpha_1+\alpha_2)}}{\Gamma_{(\alpha_1)}\Gamma_{(\alpha_2)}} z^{\alpha_1-1}(1-z)^{\alpha_2-1} \tag{7}$$

$$= \begin{cases} \frac{1}{\beta(\alpha_1,\alpha_2)} z^{\alpha_1-1}(1-z)^{\alpha_2-1} & 0 < z < 1 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

Solution

(b) let z = x + y and w = $\frac{x}{y}$
also, let $x_1 = \frac{zw}{1+w}, y_1 = \frac{z}{1+w}$

$$J = \begin{vmatrix} 1 & 1 \\ \frac{1}{y} & \frac{-x}{y^2} \end{vmatrix} = -\frac{x+y}{y^2} = -\frac{(1+w)^2}{z} \tag{9}$$

$$f_{zw}(z,w) = \frac{1}{\alpha^{\alpha_1+\alpha_2}\Gamma_{(\alpha_1)}\Gamma_{(\alpha_2)}} \frac{z}{(1+w)^2} \left(\frac{zw}{1+w}\right)^{\alpha_1-1} \left(\frac{z}{1+w}\right)^{\alpha_2-1} e^{-\frac{z}{\alpha}} \tag{10}$$

$$= \frac{1}{\alpha^{\alpha_1+\alpha_2}} \frac{z^{\alpha_1+\alpha_2-1}}{\Gamma_{(\alpha_1)}\Gamma_{(\alpha_2)}} e^{-\frac{z}{\alpha}} \frac{w^{\alpha_1-1}}{(1+w)^{\alpha_1+\alpha_2}} \tag{11}$$

$$= \left(\frac{z^{\alpha_1+\alpha_2-1}}{\alpha^{\alpha_1+\alpha_2}\Gamma_{(\alpha_1+\alpha_2)}} e^{-\frac{z}{\alpha}}\right)\left(\frac{\Gamma_{(\alpha_1+\alpha_2)}}{\Gamma_{(\alpha_1)}\Gamma_{(\alpha_2)}} \frac{w^{\alpha_1-1}}{(1+w)^{\alpha_1+\alpha_2}}\right) \tag{12}$$

$$= f_z(Z)f_w(W) \tag{13}$$

Thus Z and W are independent random variables

Solution

(c) let z = x + y and w = $\frac{x}{x+y}$
also, let $x_1 = zw, y_1 = z - x_1 = z(1-w)$

$$J = \begin{vmatrix} 1 & 1 \\ \frac{y}{(x+y)^2} & \frac{-x}{(x+y)^2} \end{vmatrix} = \frac{1}{x+y} = \frac{1}{z} \tag{14}$$

$$f_{zw}(z,w) = \frac{z}{\alpha^{\alpha_1+\alpha_2}\Gamma_{(\alpha_1)}\Gamma_{(\alpha_2)}}(zw)^{\alpha_1-1}\{z(1-w)\}^{\alpha_2-1} \tag{15}$$

$$f_{zw}(z,w) = \left(\frac{z}{\alpha^{\alpha_1+\alpha_2}\Gamma_{(\alpha+\beta)}}e^{\frac{-z}{\alpha}}\right)\left(\frac{\Gamma_{(\alpha_1+\alpha_2)}}{\Gamma_{(\alpha_1)}\Gamma_{(\alpha_2)}}w^{\alpha_1-1}(1-w)^{\alpha_2-1}\right) \tag{16}$$

$$= f_Z(z)f_W(w) \tag{17}$$

Thus Z and W are independent random variables.