Kramers' escape problem for white noise driven switching in ferroelectrics

Madhav Ramesh, Amit Verma, and Arvind Ajoy

Abstract— **A simulation-based study of Kramers' escape problem in the bistable well of a ferroelectric capacitor is presented. This problem deals with the escape of a particle undergoing Brownian motion over an energy barrier. Using this framework, and under the assumption of homogeneous polarisation switching according to the Landau-Ginzburg-Devonshire (LGD) theory, we outline two prospective device applications – the possibility of observing true random number generation (TRNG) and stochastic resonance (SR) in a ferroelectric capacitor. Our simulation results for the former show that by adding white noise and an appropriately tuned voltage pulse to a ferroelectric capacitor, it is possible to facilitate probabilistic switching between its two stable polarisation states. We predict that this could provide the theoretical framework for practical implementations of TRNGs. In addition, we investigate stochastic resonance in a ferroelectric capacitor using linear response theory. We show that the addition of an optimal amount of noise to a weak periodic signal, given as an input to this system, can enhance its detection.**

Index Terms— **Kramers' escape problem, White noise, True random number generation, Stochastic resonance, Ferroelectrics**

I. INTRODUCTION

KRAMERS' escape problem deals with the dynamics of the escape of a Brownian particle from a potential well, over an energy barrier [1]–[3]. Brownian motion [4] is the stochastic process which gives rise to thermal noise in electronic systems, also known as Johnson-Nyquist noise [5], [6]. It is white, implying that its power spectral density is flat as a function of frequency. For the escape problem, the case where the particle is in a strongly damped environment subject to thermal noise is of significant interest. The equations (provided in the next section of this paper) governing the dynamics of the unknown stochastic quantity (in this case, position of the Brownian particle), closely resemble the polarisation dynamics of a ferroelectric in accordance with the Landau-Ginzburg-Devonshire (LGD) theory [7]–[9]. An analogue of the classic escape problem under these conditions, therefore, is one that deals with electric polarisation as the unknown stochastic quantity, rather than position. Since the defining property of a ferroelectric is spontaneous polarisation, it is possible to map

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Kramers' problem to ferroelectrics to understand its utility for engineering applications.

In order to make use of this analogy, a fundamental assumption is that all the domains of the ferroelectric behave homogeneously in the presence of an applied electric field, i.e., we approximate the material as a single domain. Under this assumption, the polarisation switching dynamics is described by the Time Dependent Ginzburg-Landau (TDGL) equation. This is commonly referred to as homogeneous or intrinsic switching, as against extrinsic switching which is largely governed by nucleation and domain growth [10]. Although the extrinsic mechanism is more common, homogeneous switching has been experimentally observed in certain ferroelectrics at scaled thicknesses, such as P(VDF-TrFE) [11], PVDF [12] and $PbTiO_3$ [13].

In this work, we consider an HfO_2-based thin-film, Hafnium Zirconium Oxide ($Hf_{1-x}Zr_xO_2$ or HZO) as the ferroelectric. Applications based on HZO could potentially have commercial significance, owing to this material's compatibility with the existing CMOS processes [14]–[17]. It is still an ongoing debate as to whether switching in HZO is truly intrinsic or whether the domain growth contributes to the behaviour [18]. However, there have been recent reports of intrinsic switching in polycrystalline $Si:HfO_2$ [19] and HZO [20], suggesting that such a mechanism may be possible. More recently, it was shown that in $Y:HfO_2$ capacitors (both epitaxial and polycrystalline), the inhomogeneous switching mechanisms (Kolmogorov Avrami Ishibashi, KAI [21]–[23] and Nucleation Limited Switching, NLS [24]) converge for high electric fields, making switching resemble the homogeneous process [25].

Under the assumption of homogeneous switching, it is possible to use the LGD theory to model the ferroelectric as a single domain and study the escape problem. Etesami et al. [26] simulated the TDGL equation to explain the impact of thermal fluctuations in the bistable potential well, enabling switching between the two states. We present the theory and simulations for two applications that emerge from this formalism: True random number generation (TRNG) and stochastic resonance (SR). Although we are using HZO parameters for our simulations, our analysis is transferrable to any material system with a bistable well for which the LGD theory may be applied.

The generation of true random numbers using electrical circuits involves harnessing a source of entropy in the system [27]. Mulaosmanovic et al. [28], [29] exploit the polarisation fluctuations due to domain wall motion, which is a Poisson process [30], in an HfO_2 based field-effect transistor. Other

such TRNGs have recently been developed using ferroelectric random access memory (FRAM) technology [31]–[33]. However, none of these works exploit the randomness due to thermal fluctuations in the polarisation of a ferroelectric. We predict through our simulations that the addition of thermal noise (using an external resistor, for example) to a ferroelectric capacitor can give true random numbers based on the analysis of the escape problem. This is to be achieved with the help of a voltage pulse with an appropriate magnitude and duration.

Stochastic Resonance is a counter-intuitive phenomenon wherein weak periodic signals given as input to systems which have an inherent threshold or energetic bistability, may be detected with the addition of an optimal amount of noise. In these systems, Signal-to-Noise Ratio (SNR) increases with an increase in noise intensity, within a certain range of noise intensities. It was first studied by Benzi et al. [34] using a climate change model to interpret the atmospheric temperature variations over large time periods. Subsequently, it was discovered in the nervous systems of crayfish and paddlefish as an evolutionary mechanism, resulting in its importance in neurobiological research [35], [36]. Recently, several engineering applications which use this phenomenon have also emerged, such as a visual aid [37], a low power photodetector [38], an Insulator-Metal Transition (IMT) based system for ultra-low power auditory processing [39]. Few papers [40], [41] have presented device applications for weak electronic signal detection. However, none of them incorporate ferroelectrics, which have an inherent bistable well. It is to be noted that SR has been experimentally reported in ferroelectric Triglycine Sulphate (TGS) [42] – however, this work does not present a quantitative comparison of the experimental data with theoretical predictions. Additionally, it has not been motivated as an application of Kramers' problem to detect weak periodic signals.

In this paper, we present an analysis of Kramers' escape problem in a ferroelectric capacitor. We hope to convey its relevance to the devices community through theory and simulations for the two applications mentioned above. This would involve an initial introduction to the problem and the underlying mathematics, followed by comparison of numerical results with analytical results.

II. OVERVIEW OF KRAMERS' ESCAPE PROBLEM

In 1940, Kramers provided the mathematical framework to describe the escape of a particle over a potential barrier, in the context of dynamics of chemical reactions [1]. The dynamics of a classical particle of mass M in a potential $U(x)$, with damping γ, can be summarized as

$$M\ddot{x} = -\frac{\partial U(x)}{\partial x} - \gamma \dot{x} + \eta(t) \tag{1}$$

where x is the position, and $\eta(t)$ is a fluctuating force described by Gaussian-white noise. This is similar to the equation for a spring-mass damper, with an additional fluctuating force. In the strongly damped limit, this equation reduces to Langevin's equation [43],

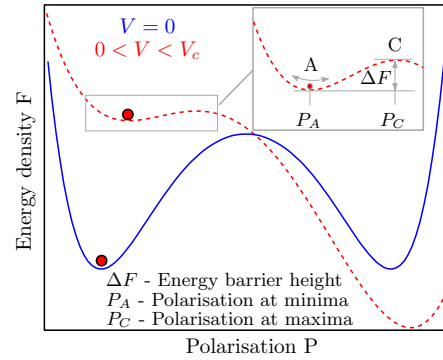$$\gamma \dot{x} = -\frac{\partial U(x)}{\partial x} + \eta(t) \tag{2}$$

Fig. 1. Kramers' problem for a double well potential landscape in a ferroelectric. The blue(solid) line depicts the double well potential when a voltage $V = 0$ is applied across the ferroelectric, while the red(dashed) line depicts the situation under an applied voltage $0 < V < V_c$. V_c is the coercive voltage where the potential barrier disappears. The state of polarisation denoted by the red disc is analogous to a particle trapped in a potential well. Assuming the polarisation is at point A when $V < V_c$, it can still escape the well, over point C, when subject to thermal fluctuations. Kramers time quantifies the average lifetime of system in state A.

since the inertial term may be neglected. This particular equation maps to the equation that governs the spontaneous polarisation dynamics of a ferroelectric.

Consider the double well potential in Figure 1. Given this energy landscape for the ferroelectric, one may observe this is similar to the setup for Kramers' problem with electric polarisation being the stochastic variable (instead of position x). Let the state of the system initially be at point A. We are then interested in the average time and the probability for which the system reaches C and ultimately crosses the barrier. Note that the switching dynamics of the system can be altered by applying a voltage, which changes the relative depths of the two potential wells.

Since thermal noise has already been studied in a ferroelectric [26], we use the same model to define the problem at hand. The free energy density F (in J/m^3) of the ferroelectric is related to the polarisation P (under the assumption of homogeneous switching) and is given by

$$F = \alpha P^2 + \beta P^4 - PE, \tag{3}$$

where E is the electric field and α and β are the Landau coefficients. Then, the TDGL equation is

$$\rho \frac{\partial P}{\partial t} = -\frac{\partial F}{\partial P} + \xi(t), \tag{4}$$

where $\xi(t)$ is the noise (a fluctuating term) and ρ is the resistivity (a dissipative term). Note that this resembles the Langevin equation. Further note that this equation represents a class of Stochastic Differential Equations (SDEs) which model a system that has a fluctuating term along with a dissipative term. Assuming Gaussian white noise, the auto-correlation of the noise is given by

$$\langle \xi(t)\xi(t') \rangle = \frac{2k_B T \rho}{A_F t_F} \delta(t - t'), \tag{5}$$

from the Fluctuation dissipation relation, which relates the fluctuating and dissipative forces in the system. Here, k_B is Boltzmann's constant and T is the temperature of the

system. Further, t_F and A_F represent the thickness and area respectively of the ferroelectric under consideration. These terms play the important role of determining how significant internal noise is in the system. Highly scaled ferroelectrics (i.e. low t_F) exhibit more thermal noise, as is evident from the autocorrelation expression in eq. (5).

In order to define a scale for the noise in the problem, we consider the ==Fokker-Planck equation or FPE [44] to describe== the evolution of the probability density function $w(P,t)$. The FPE for the situation under consideration is given as

$$\frac{\partial}{\partial t}w(P,t) = \frac{1}{\rho}\frac{\partial}{\partial P}\left[w\frac{\partial F}{\partial P} + D\frac{\partial w}{\partial P}\right] \quad (6)$$

where D is the diffusion constant. We exploit the fact that at equilibrium, $w(P,t)$ is a Boltzmann distribution. Setting the probability current $\frac{\partial}{\partial t}w(P,t) = 0$ in eq. (6) at equilibrium yields $D = \frac{k_B T}{t_F A_F}$. Note that the $t_F A_F$ term is the volume and it emerges because the energy density F is in units of J/m^3, or energy per unit volume. ==The noise $\xi(t)$ can then be written in terms of a Brownian motion or Wiener Process $W(t)$== [45]

$$\xi(t) = \sqrt{2\rho D}\frac{dW(t)}{dt}. \quad (7)$$

So far, in eq. (5, 7), there is no notion of external noise being added to the system. We now include an external noise voltage (with root mean squared value V_{noise}) having a power spectral density that is flat over a bandwidth Δf. This external noise is uncorrelated to the internal noise in the system. Hence, eq. (7) can be modified as

$$\xi(t) = \left(\sqrt{2\rho D_{int}} + \sqrt{2\rho D_{ext}}\right)\frac{dW(t)}{dt} \quad (8)$$

where

$$D_{int} = \frac{k_B T}{t_F A_F} \quad (9)$$

$$D_{ext} = \frac{V_{noise}^2}{2R_F}\frac{1}{\Delta f}\frac{1}{t_F A_F}, \text{ with } R_F = \frac{\rho t_F}{A_F} \quad (10)$$

represent the effect of internal and external noise respectively. This definition is similar to the description of a magnetic tunnel junction with stochastic input, as discussed in [46].

We reiterate that the ==SDE eq. (4) is analogous to the classic escape problem under strongly-damped conditions.== Following Metzler and Klafter [47], we can determine the rate of escape of the state of polarisation over the energy barrier. This is known as Kramers rate r_K where

$$r_K = \frac{1}{t_K} = \frac{\sqrt{|F''(P_A)F''(P_C)|}}{2\pi\rho}\exp\left(-\frac{\Delta F}{D_{ext}}\right) \quad (11)$$

under the assumption that $D_{ext} \gg D_{int}$. Refer to Figure 1 for the notations. F'' denotes the second derivative of the free energy density. The reciprocal of r_K is t_K, which is referred to as Kramers time. This particular metric can be interpreted as the average time spent in the well around point A, before a transition is made over point C at the top of the barrier. Note finally that the ratio $D_{ext}/\Delta F$ naturally provides a scale for the noise, since this ratio determines Kramers time.

Parameter	Value
Thickness t_F (nm)	10
α (mF^{-1})	-7.603×10^8
β $(m^5 F^{-1} C^{-2})$	1.204×10^{10}
Resistivity ρ $(\Omega - m)$	30
Temperature T (K)	300
Area A_F (μm^2)	1

III. SIMULATION RESULTS AND DISCUSSION

The ferroelectric parameters and thickness values for the simulations have been determined from the remnant polarisation ($P_r = 17.76 \ \mu C/cm^2$) and coercive electric field ($E_c = 104$ MV/m, corresponding to $V_c = 1.04$ V) reported for $t_F = 10$ nm thick Hf$_{0.5}$Zr$_{0.5}$O$_2$ in [48]. The α and β values were obtained using the following expressions [49]

$$\alpha = \frac{-3\sqrt{3}E_c}{4P_r} \text{ and } \beta = \frac{3\sqrt{3}E_c}{8P_r^3} \quad (12)$$

The value of resistivity ρ is taken from experimental data [15], while the area has been chosen arbitrarily but in the range of existing values in literature. All relevant parameters have been listed in Table I.

We solve eq. (4) using the Euler-Maruyama method, similar to the Euler method that is used to solve ordinary differential equations [50]. In discrete form, with $D_{ext} \gg D_{int}$, we have

$$P[i] = P[i-1] - \frac{\Delta t}{\rho}\cdot\frac{dF}{dP}[i] + \sqrt{\frac{2D_{ext}}{\rho}}\cdot\Delta W[i] \quad (13)$$

where $[i]$ represents the ith time-step. The term $\Delta W[i]$ is obtained by extracting numbers from a normal distribution with mean 0 and variance Δt. For our simulations, we selected a time step $\Delta t = 1$ ns. Note that the time step for Brownian dynamics simulations must be as small as possible, so as to prevent the possibility of divergence for high noise intensities.

Figure 2 presents the results of our simulations for Kramers time t_K, compared with the analytical results predicted by eq.
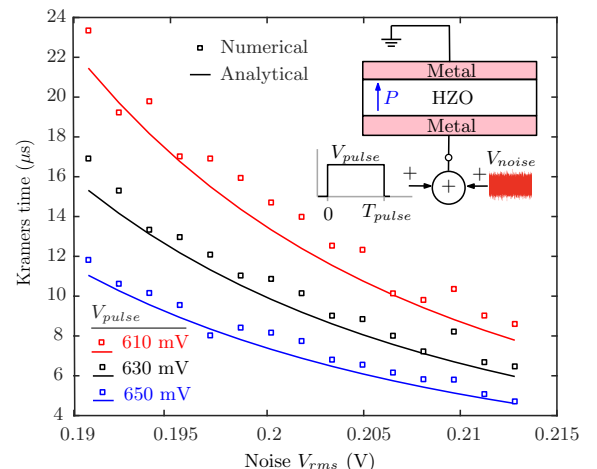

Fig. 2. Verification of Kramers time through numerical simulations. Our results match the ==analytical predictions, based on eq. (11),== very well.

(11). The polarization is initially assumed to be in the left well. A positive voltage pulse with amplitude V_{pulse} is applied such that the double well is asymmetric. A significantly large pulse width $T_{pulse} = 90\mu s$ has been taken (about $4\times$ larger than the highest value of Kramers time) such that a switching event is almost certainly guaranteed. The asymmetry has been chosen such that the average time spent in the left well is finite, whereas the right well yields a very large time approaching infinity. An ensemble of 300 systems have been considered (for each of the V_{rms} and V_{pulse} values), with the average time spent in the well before a switching event occurring being recorded. Based on our simulation setup, switching can never occur from the right well to the left well, ensuring that the numerically calculated average does not include any unwanted reverse switching events. The good match between the analytical and numerical results verifies the working of our numerical solver. This solver will be used to investigate TRNG and SR.

A. True random number generation

The focus of this section is the elucidation of the random number generation scheme and the potential for probabilistic ferroelectric polarisation switching using thermal noise and an appropriately tuned input voltage pulse. It is based on the work that uses a Hafnium Oxide based Field-Effect Transistor [29]. The noise that is exploited in their paper is governed by a Poisson process [28], [30] due to the inherent randomness in domain wall motion. The fundamental difference in our work is that we suggest using an external, and hence controllable thermal noise source, rather than rely on the inherent randomness (due to domain wall motion or thermal noise) of the ferroelectric. This introduces an extra element of tunability aside from the voltage pulse magnitude-duration trade-off to control the probabilistic switching. Here, we focus primarily on a ferroelectric capacitor in order to explain the simplicity in the equations that emerge from Kramers' escape problem.

Survival probability, $p(t)$, is defined as the probability of finding a particle in a well over a period of time [47]. In our analogy, instead of a particle in a well, we track the polarisation state of the ferroelectric. In the strongly damped scenario, it is given by

$$p(t) = e^{-r_K t} \qquad (14)$$

and the switching probability, $S(t)$, is the complement, given by

$$S(t) = 1 - e^{-r_K t} \qquad (15)$$

where r_K is Kramers rate. Now that the theoretical framework has been constructed, the proposal for a practical implementation is briefly explained below, followed by numerical results for the switching probability.

It may be assumed that the left well corresponds to a '0' and the right well to a '1'. To initialise the state of polarisation in the ferroelectric to any one of these, a negative reset pulse value below the coercive voltage $-V_c$ is applied. Subsequently, the optimal voltage value V_{switch} and pulse width T_{pulse} can be selected based on eq. (14). The former sets the value of r_K and the latter enables the time-dependent switching
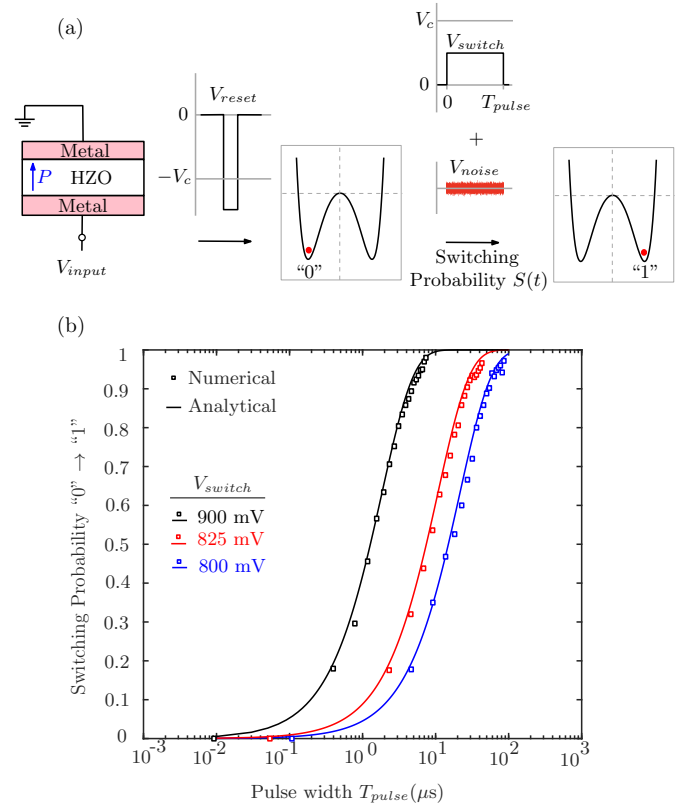

Fig. 3. (a) Schematic showing the use of external noise with a ferroelectric to generate random numbers. A reset pulse sets the system in the "0" state. Subsequently, a voltage pulse (with amplitude V_{switch} and width T_{pulse}) and external noise V_{noise} causes the system to switch from "0" → "1". (b) Switching probability for different input voltages in HZO. Noise is fixed, with $V_{rms} = 126$mV and a bandwidth of 10MHz.

upon which this scheme is based. For a given noise intensity and voltage pulse height, there is a corresponding r_K value. Therefore, it is possible to achieve any probability value for '0' or '1' by varying the pulse width. By integrating the ferroelectric into a device like a transistor, one may realise a true random number generator with tunability. The thermal noise may be given from an external source such as a resistor, and this introduces an extra element of tunability aside from the voltage pulse height and width.

By taking multiple input noise voltages and pulse widths, it is possible to obtain numerical fits for eq. (15). For each of these values, the probability has been found by computing the number of switching events for an ensemble (of size 500 in this work) and dividing by the total number of members of this ensemble. This result is shown in Figure 3. The numerical results match the analytical predictions (eq. (15)) very well. Note that the switching probability can be controlled using V_{switch}. An interesting observation is that this result is similar to that obtained in [28], even though the process governing the stochastic switching is different. This is due to the exponential nature of the probabilities in both cases.

B. Stochastic resonance (SR)

As mentioned in the introduction, the constructive role played by noise in non-linear systems has been exploited to

realise many engineering applications. The incorporation of ferroelectrics in semiconductor devices elicits an investigation of this phenomenon. We study the possibility of observing SR in a ferroelectric capacitor in this section, and provide numerical simulation results to motivate an application for weak periodic signal detection. The relation between SR and Kramers' problem is also provided towards the end of the section.

We are interested in determining the polarisation of the ferroelectric capacitor when a voltage comprising of noise and a weak periodic signal is applied, as shown in Figure 4(a). The response of non-linear systems to stochastic inputs has been of significant interest to the mathematics and engineering communities. There are various approaches to this problem, involving different approximations [51]. Linear Response Theory (LRT) [52], treats the problem in two steps. First, the response of the non-linear system to only the noise, is characterized. Then, the weak periodic input is treated as a perturbation, so that the ensemble-averaged output is a scaled version of the periodic input. The scaling factor (a complex number, with magnitude and phase) depends on the response of the non-linear system to only the noise. This is inferred from Bussgang's theorem [53]. Luchinsky et al. have exploited this idea and demonstrated SR in a circuit with a double-well landscape [52]. Along similar lines, we have investigated SR in ferroelectrics below.

Consider a weak signal $V_{signal}(t) = V_0 \cos(\Omega t)$ given as an input to the ferroelectric as shown in Figure 4(a). A noise voltage, with root mean squared value V_{noise}, corresponding to noise strength D_{ext} is added to the system (eq. (9)). In the framework of LRT, we first look at the system only with noise, but without the weak signal. Let $\chi(\omega; D_{ext})$ represent the electrical susceptibility (dimensionless) of the ferroelectric at a frequency ω with noise strength D_{ext}. For brevity, we drop the reference to D_{ext} in $\chi(\omega; D_{ext})$ unless where required. The susceptibility $\chi(t)$ in time-domain represents the impulse response of the system. Then, treating the weak applied electric field $E(t) = V(t)/t_F = E_0 \cos(\Omega t)$ (V/m) as a perturbation, we write the ensemble-averaged (denoted by $\langle \cdot \rangle$) output polarisation

$$\langle P(t) \rangle = \epsilon_0 E_0 |\chi(\Omega)| \cos(\Omega t + \phi) \quad (16)$$

where

$$\phi = -\tan^{-1}\left[\frac{\Im\{\chi(\Omega)\}}{\Re\{\chi(\Omega)\}} \right]. \quad (17)$$

Here, ϵ_0 is the permittivity of vacuum. $\Re\{(\cdot)\}$ and $\Im\{(\cdot)\}$ represent the real and imaginary components of a complex number. The objective of the analysis that follows is to obtain an expression for $\chi(\omega)$ using tools that are commonly employed in statistical physics and signal processing.

In accordance with Kubo's fluctuation-dissipation theorem [54], we can relate the susceptibility $\chi(t)$ to the autocorrelation $R(t)$ of the polarisation of the unperturbed system (i.e with noise but without sinusoidal input),

$$\chi(t) = -\frac{1}{\epsilon_0 D_{ext}} \frac{dR(t)}{dt} \Theta(t) \quad (18)$$

where $\Theta(t)$ is the Heaviside function. Applying the Weiner-Khinchin theorem to eq. (18), this equation may be expressed in the frequency domain, since power spectral density (PSD) is related to the Fourier transform of the autocorrelation $R(t)$.

$$\chi(\omega) = \frac{-1}{2\pi\varepsilon_0 D_{ext}} \left[j\omega \frac{S_N^0(\omega)}{2} * \left(\frac{1}{j\omega} + \pi\delta(\omega) \right) \right] \quad (19)$$

where $S_N^0(\omega) \equiv S_N^0(\omega; D_{ext})$ is the one-sided PSD of the polarisation switching in the unperturbed system, given by

$$S_N^0(\omega; D_{ext}) = \frac{4D_{ext}\epsilon_0}{\omega} \Im\{\chi(\omega; D_{ext})\} \quad (20)$$

The two-sided power spectral density would be half of this term, since the power is spread over both positive and negative frequencies. Note that $\Im\{\chi(\omega)\}$ is associated with the dissipation in the system. Inverting eq. (20), we can hence determine $\Im\{\chi(\omega)\}$ from the PSD, $S_N^0(\omega; D_{ext})$, of either the experimentally measured, or numerically simulated polarisation of the unperturbed system. Given $\Im\{\chi(\omega)\}$, the real part can be found using the Kramers-Kronig relation [55], [56],

$$\Re\{\chi(\omega)\} = \frac{2}{\pi} \cdot \mathcal{P}\left\{ \int_0^\infty \frac{\omega' \Im\{\chi(\omega')\}}{\omega'^2 - \omega^2} d\omega' \right\} \quad (21)$$

where $\mathcal{P}\{\cdot\}$ denotes Cauchy's principal value integral, to account for the singularity at $\omega' = \omega$. Note that this relation between the real and imaginary parts is similar to the Hilbert transform in signal processing, which is a useful mathematical tool when dealing with causal signals [57]. Thus, we can obtain the complete $\chi(\omega; D_{ext}) = \Re\{\chi(\omega; D_{ext})\} + j\Im\{\chi(\omega; D_{ext})\}$ from the response of the unperturbed system.

Going now to the perturbed system, the ensemble averaged output consists of a periodic signal component along with background noise. We wish to analytically characterize the relative strengths of the periodic signal component and the background noise. From eq. (16), the power in the periodic signal component is given by $\mathcal{P}_{sig}(D_{ext}) = \epsilon_0^2 E_0^2 |\chi(\Omega)|^2/2$. We assume that the PSD of the noise background (around $\omega = \Omega$) in the output of the perturbed system, is identical to the PSD of the corresponding noise background in the unperturbed system. Hence, an analytical estimate of Signal-to-Noise Ratio (SNR) at the output is

$$\text{SNR}_{theory}(D_{ext}) = \frac{\mathcal{P}_{sig}(D_{ext})}{S_N^0(\Omega; D_{ext})} \quad (22)$$

Note that since this is the ratio between a power and a PSD, it has units of dB-rad/s.

We also estimate the SNR numerically by taking the ratio of the signal power within a narrow band $[\Omega - \Delta\omega, \Omega + \Delta\omega]$, to the noise background at $\omega = \Omega$. This is the definition that is often used in the SR literature [51].

$$\text{SNR}_{num}(D_{ext}) = \lim_{\Delta\omega \to 0} \frac{\int_{\Omega-\Delta\omega}^{\Omega+\Delta\omega} S(\omega)d\omega - 2S_N(\Omega)\Delta\omega}{S_N(\Omega)}, \quad (23)$$

where $S(\omega) \equiv S(\omega; D_{ext})$ is the PSD of the perturbed system. We estimate the PSD of the noise background, $S_N(\Omega) \equiv S_N(\Omega; D_{ext})$, by fitting several data points of the noise floor around the signal peak with a straight line, and evaluating the
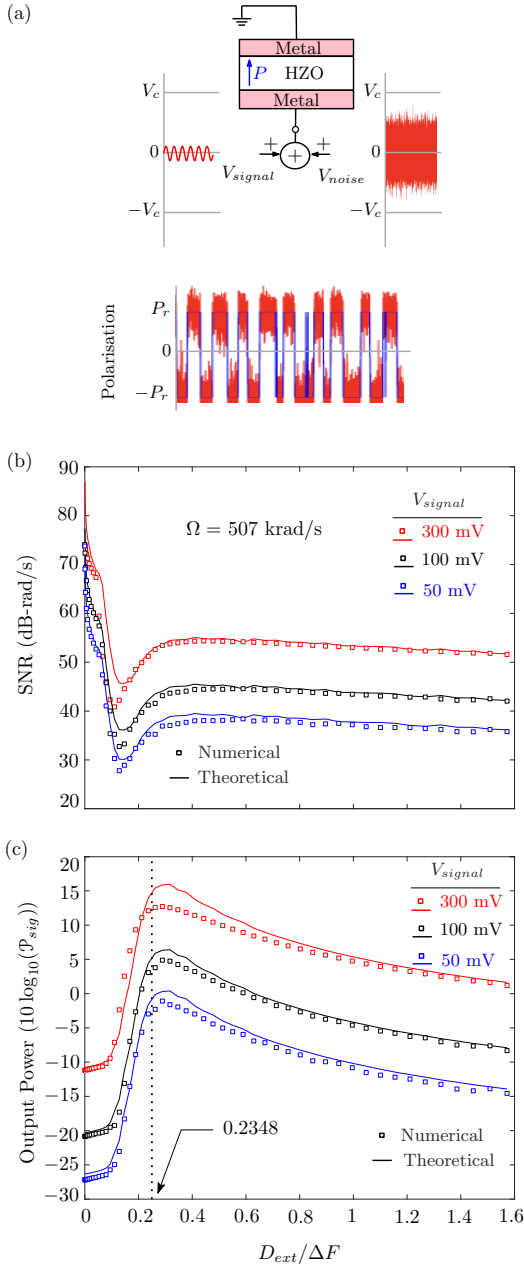
(a)

(b)

(c)

Fig. 4. (a) Schematic showing quasi-periodic switching of a ferroelectric due to a weak sinusoidal input (amplitude below V_c) and noise. (b,c) SNR and Output Power as a function of normalised noise intensity ($D_{ext}/\Delta F$). The dotted line shows the approximate noise intensity value that yields maximum output power, determined from Kramers' theory ($\Omega/2\pi \approx r_K/2$).

with an increase in noise, but then rises with a further increase in noise. The match between the theoretical and numerical results supports the use of LRT to understand SR in HZO based ferroelectric capacitors.

SR is very closely linked to Kramers' escape problem. The output power $\mathcal{P}_{sig}(D_{ext})$ is maximum [51] when the frequency of the weak periodic signal $\Omega/2\pi \approx r_K/2$. Noting the relationship (eq. 9) between r_K and D_{ext}, the optimum noise strength that should be added depends on the frequency of the weka periodic signal that has to be detected. At this *resonant* condition, the polarisation switching in the ferroelectric would be synchronised (with some phase shift, (eq. (17)) with the input signal, thereby giving the desired output with maximal power. For our choice of $\Omega = 0.01 \times \Omega_0$, we predict the optimum noise strength $D_{ext}/\Delta F = 0.2348$, which is consistent with our results presented in Figure 4(c).

IV. CONCLUSION

We presented a comprehensive analysis of Kramers' escape problem in the context of a ferroelectric ($\text{Hf}_{0.5}\text{Zr}_{0.5}\text{O}_2$ or HZO), under the assumption of homogeneous switching. Subsequently, we proposed two engineering applications using white noise: True random number generation (TRNG) and Stochastic resonance (SR). The former can be set by appropriately tuned voltage pulses in the presence of an optimal amount of noise. Our simulation results match the theory, implying that it may be possible to experimentally realise a TRNG with a ferroelectric capacitor. The second application exploits the phenomenon of SR to realise a weak periodic signal detector using the same capacitor. The simulation results suggest that SR should occur and can be explained well using linear response theory. Additionally, we validate that an optimum noise strength, related to Kramers rate r_K, can be chosen to to maximize the output power.

value at $\omega = \Omega$. The noise term in the numerator of eq. (23) is negligible.

Figure 4 (b,c) presents our numerical and theoretical results for SR in a HZO based ferroelectric capacitor. The horizontal axis is depicted in terms of the dimensionless quantity $D_{ext}/\Delta F$. Following Ref. [51], $\Omega_0 \equiv 2\alpha/\rho$ defines a frequency scale in eq. (4). We choose $\Omega = 0.01 \times \Omega_0 \equiv 507$ krad/s (80.6 kHz) in our simulations. We use an ensemble of 400 copies of the PSDs to determine $\chi(\omega; D_{ext})$ required to estimate the theoretical and numerical SNR curves. Note the characteristic trend of SR in Figure 4(b) - the SNR first drops

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