Design of experimental test setup to study stochastic switching and stochastic resonance in nonlinear systems

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Abstract

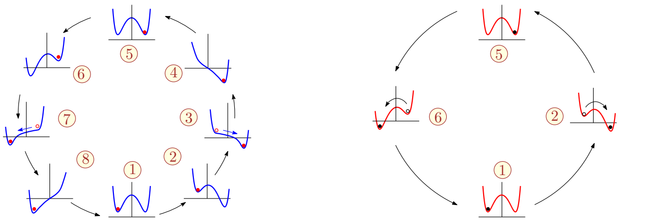
Some non-linear circuits and devices demonstrate enhanced performance in the presence of noise, a counter-intuitive phenomenon called "Stochastic Resonance" (SR). This work reports on the development of an experimental test-setup to characterize such systems, accounting for the effective low-pass behaviour of the system front-end and the limitations of low-cost signal generators. Using our setup, we demonstrate SR in a circuit that mimics the motion of a particle in a double well potential. Our setup can be extended for similar analyses of other bistable systems such as ferroelectric and ferromagnetic devices.

Keywords: Noise, Stochastic switching, Nonlinear systems.

Introduction

Nonlinear systems sometimes exhibit interesting behavior when noise is added to their input. For example, Stochastic Resonance (SR) refers to the improvement of some measurement metric of a nonlinear system due to the addition of noise [1]–[3]. Originally discovered in the context of the theory of ice ages, SR is thought to occur in many biological systems as well. Fig. 1 depicts stochastic resonance in a bistable device/circuit/system. Such systems possess an inherent double well in their energy landscape, such as ferroelectrics, ferromagnets, or circuits like a Schmitt trigger. With a strong periodic forcing function, switching of the system is deterministic and happens via the disappearance of the energy barrier (see states 3,7 in Fig. 1(a)). However, when an optimal amount of white noise is added, the system demonstrates SR – i.e. switching can happen even in the presence of a barrier (see states 2,6 in Fig. 1(b)), enabling weak periodic signals to be detected. Note however that this switching is quasi-periodic, due to the random nature of the noise.

To experimentally characterize the behavior of any nonlinear system to noise, we need to apply a noise signal generated with the desired variance, bandwidth, and power spectral density. In addition, it is important to compensate for any noise filtering that the front-end of the nonlinear system might introduce. The built-in noise function of low-end signal generators (like the Tektronix AFG1022) gives control over the peak-to-peak voltage of the noise (and hence its variance), but not over its bandwidth. Hence, it is possible that the front-end of



(a) Strong forcing function (b) Weak forcing function + Added noise

Fig. 1: (a) Deterministic switching of a double well system, wherein inter-well-transition is accompanied by the disappearance of a barrier. (b) Stochastic switching wherein interwell-transition can happen even in the presence of a barrier when aided by added noise.

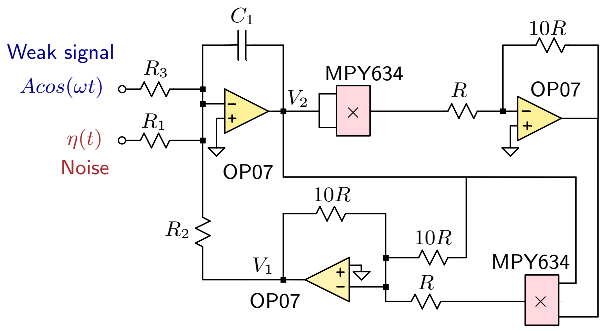


Fig. 2: Circuit model of an overdamped particle performing 1D Brownian motion in a particle in a double well potential, following [6].

the nonlinear system filters out a large portion of the noise, rendering the resultant noise variance too small (even with the largest peak-to-peak voltage setting) to observe SR. In this work, we propose and demonstrate a strategy to overcome these challenges.

To illustrate our approach, we consider the nonlinear circuit [6] in Fig. 2, implemented in hardware in Fig. 3(a). This circuit mimics the dynamics of an over-damped particle performing 1D Brownian motion in a double-well potential. The differential equation for such dynamics is

$$\frac{dx}{dt} = -\frac{dU(x)}{dx} + \eta(t) + Acos(\omega t) \tag{1}$$

$$U(x) = -\frac{a}{2}\left[\frac{x}{c}\right]^2 + \frac{b}{4}\left[\frac{x}{c}\right]^4 \tag{2}$$

where x is the position of the particle, $U(x)$ is the potential function, $\eta(t)$ is zero-mean white Gaussian noise, and $Acos(\omega t)$ is the sinusoidal signal modulating the double-well system. a, b, c are positive constants.

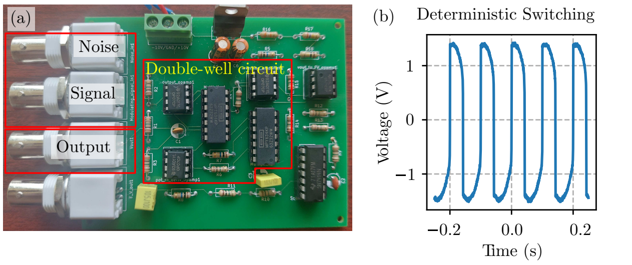


Fig. 3: (a)Hardware implementation of the bistable system depicted in Fig. 2. (b) Deterministic switching measured for a strong input signal without noise, clearly demonstrating the bistable nature of the circuit.

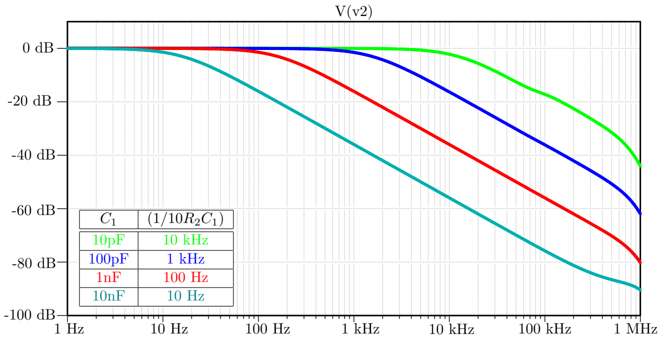


Fig. 4: Frequency response of the circuit model evaluated using AC analysis of LTspice by varying the capacitance C_1 in Fig. 2.

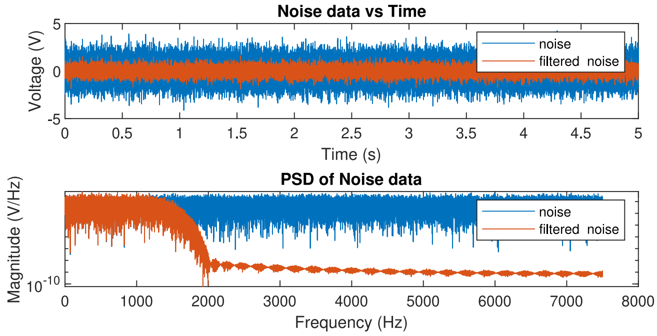


Fig. 5: Simulation results showing that the variance of the noise decreases after low-pass filtering. Hence, this filtering has a large impact on the effectiveness of the noise in causing SR.

The differential equation for the circuit in Fig. 2 is

$$\frac{dV_2}{dt} = -\frac{1}{C_1}\left(\frac{-V_2+V_2^3}{R_2} + \frac{\eta(t)}{R_1} - \frac{A\cos(\omega t)}{R_3}\right) \quad (3)$$

Hence, voltage V_2 is equivalent to position x of the particle in eq. (1). Fig. 3(b) demonstrates the bistable nature of the circuit.

Frequency response analysis of the system

The nonlinear system has a frequency response that determines the properties of the noise to be used to observe expected stochastic behavior. This frequency response of a nonlinear system cannot be evaluated analytically. Instead, it can be estimated by exciting the system with a sinusoid and analyzing the individual harmonics for a range of excitation frequencies [4]. Fig. 4 shows the frequency response of the circuit from Fig. 2 evaluated using LTspice for different values of C_1.

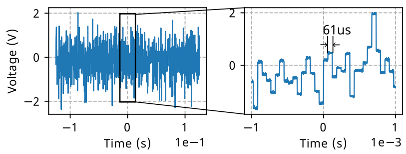
It can be observed that the bandwidth increases with a decreasing value of C_1 or increasing the parameters a and b in eq. (1). This can be interpreted as similar to the result of increasing the system response speed [5]. Alternatively, increasing the derivative term of eq. (1) increases the system response speed and hence, the bandwidth, gives an intuitive understanding of the problem to design the noise bandwidth. The applied noise to the bistable system eq. (1) hence gets low-pass filtered if its bandwidth is higher than the system's bandwidth. Reducing the bandwidth of noise with a flat power spectral density significantly reduces the noise power (and variance), as observed in Fig. 5.

Assuming white noise with variance V_{noise}^2, SR is predicted to occur when the applied subthreshold signal's frequency $\omega = 2 \times r_K$ with $r_K = 1/t_K$ and t_K being the mean first passage time [2]. Since $r_K \propto \exp\left(-\mathtt{constant}/V_{noise}^2\right)$, the unexpected filtering of noise by the front-end of the nonlinear system impacts the observation of SR at the expected frequency and level of noise.

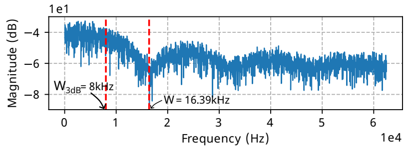
Experimental Noise Generation

White noise generated using low-cost waveform generators gives control over the peak-to-peak voltage of the noise (and hence variance) but not the bandwidth. For example, using the built-in noise function of the arbitrary waveform generator(AWG) AFG1022, the bandwidth is fixed at approximately 37 MHz. Combined with the problem of noise filtering described in the previous system, it becomes very difficult to observe SR using the noise generated by the built-in function.

Instead, we make use of the arbitrary waveform generation capability of the function generator. Uncorrelated noise data can be generated from a normal distribution using MATLAB as a data array and fed to the function generator. The white noise data from MATLAB is reproduced by the function generator as a piece-wise step function as shown in Fig. 6a with a controllable sample hold time Δt. The frequency spectrum is a sinc function, with the 3dB bandwidth W_{3dB} and the bandwidth $W = 1/\Delta t$ of the noise generated in MATLAB, as shown in Fig. 6b. The sinc frequency spectrum can be considered flat over its 3dB bandwidth W_{3dB} which is less than W, and hence the experimental noise can be considered as white noise with this reduced bandwidth. This method enables studying the response of the nonlinear system to different colored noise by designing the MATLAB data for different correlation times as well. Thus the noise data should be generated in MATLAB with a bandwidth determined by numerically solving for 3dB bandwidth of the sinc spectrum to be considered as white noise with the desired bandwidth.



(a) Piece-wise step noise generated by using the arbitrary waveform generation capability of the AFG1022 function generator, using data points generated on a PC running Matlab.



(b) FFT of the noise generated using AFG1022

Fig. 6: Noise generated using arbitrary waveform generation capability of AFG1022 and its FFT.

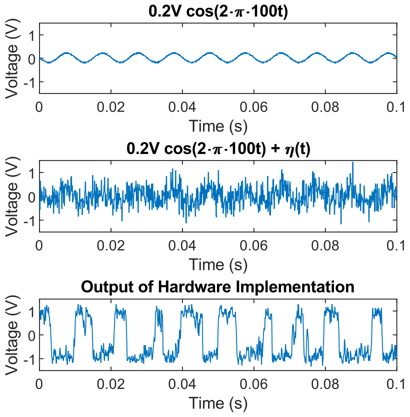


Fig. 7: Stochastic quasi-periodic switching of the output of the system to a 100 Hz subthreshold sinusoidal signal with noise

Stochastic Resonance in Test Circuit

Fig. 7 shows the output of the hardware implementation shows stochastic, quasi-periodic switching between the two stable states for a subthreshold sinusoidal input signal added with noise (whose variance and bandwidth are determined by the previous analyses). The SNR is calculated as the power in the bins containing signal frequency divided by the power in the rest of the bins in the PSD of the output [2]. To account for spectral leakage, the signal power is calculated including a few bins around the signal frequency determined by observation of the PSD. The SNR at the output is maximized for a certain noise variance for this subthreshold signal. For increasing noise variances, the SNR at the output for a subthreshold sinusoidal input to the system is shown in Fig. 8 – the SNR increases before decreasing on further addition of noise which is characteristic of the stochastic resonance phenomenon [6].

Conclusion and Future Work

Analytically when discussing the stochastic switching behavior of nonlinear systems, the noise is assumed to

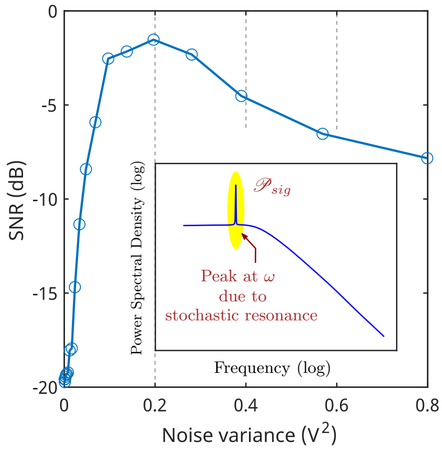


Fig. 8: SNR versus noise variance for 100Hz, 200 mV sinusoid. The increase and subsequent decrease of the SNR is a signature of SR. The inset is a cartoon demonstrating the typical observed PSD of the quasi-periodic output, having a clear peak at the frequency corresponding to the subthreshold input sinusoid.

be uncorrelated, with a flat spectrum. Experimentally, the effective spectrum of noise is not flat due to possible filtering by the front-end of the nonlinear system. To observe stochastic switching behaviors, we propose a simple approach of using the arbitrary waveform generation capability of low-cost function generators to generate the noise. This allows us full control over the variance, bandwidth, and color of the noise. We illustrated the efficacy of this approach by demonstrating stochastic resonance in a nonlinear circuit that mimics a system with a double-well potential. Our proposed test setup should be useful while studying similar behavior in other systems – such as ferroelectrics and ferromagnets - which inherently possess a double well landscape that translates into their bistable characteristics.

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