Properties of Weierstrass $\wp$ function and Projective Curves

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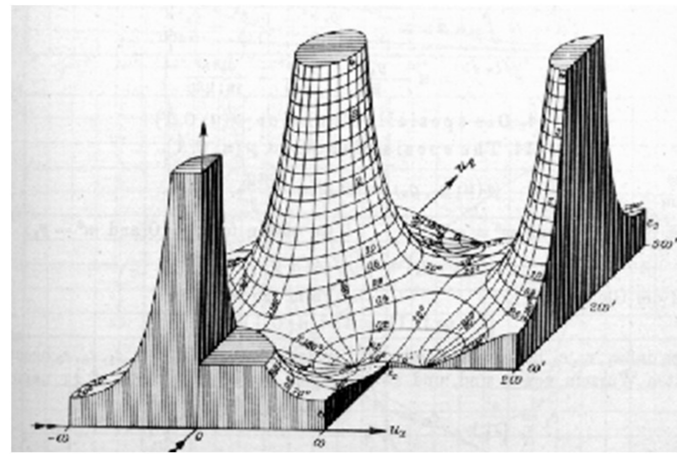
MATD92 Mathematics Project – Riemann Surfaces

Agenda

- Preliminary Results
- Elliptic function and lattices
- Weierstrass function
- Properties of Elliptic function
- Projective Curves as Riemann Surfaces
- Holomorphic function $u\colon \mathbb{C}/\Lambda \to C_\Lambda$



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Preliminary Results

Prop 1: Weierstrass M-test

Let $\{f_{n,m} : W \to \mathbb{C}\}_{\mathbb{Z}\times\mathbb{Z}}$ be holomorphic functions on open set W
Suppose there exists $M_{n,m}$ for all n, m s.t.

1) $|f_{n,m}(z)| < M_{n,m} \quad \forall z \in W$
2) $\sum_{\mathbb{Z}\times\mathbb{Z}} M_{n,m}$ converges

Then the following sum converged uniformly to some holomophic function $f(z)$ on W

$$f(z) = \sum_{\mathbb{Z}\times\mathbb{Z}} f_{n,m}(z)$$

The derivative is obtained by summing the term wise derivatives

$$f'(z) = \sum_{\mathbb{Z}\times\mathbb{Z}} f_{n,m}'(z)$$

Preliminary Results

- Zeros and poles of meromorphic functions (single valued, complex functions) are isolated
- Valency(order) of a zero: the first nth non-zero derivative at the zero

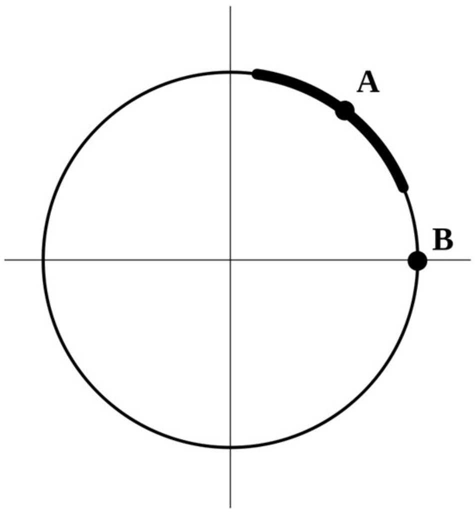
$$f(z) = a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \cdots$$
$$= (z - z_0)^n g(z)$$

- Valency(order) of a pole: the smallest exponent of the laurent series centered at pole

$$f(z) = a_{-n}(z - z_0)^{-n} + a_{-n+1}(z - z_0)^{-n+1} + \cdots$$
$$= (z - z_0)^{-n} g(z)$$

- Implicit function theorem:

Let polynomial $P(x, y) \in \mathbb{C}[x, y]$ define the curve $C = \{(x, y) \mid P(x, y) = 0\}$. Let $(a, b) \in C$ such that $\partial P/\partial y \neq 0$. Then there exists a holomorphic function $f: U(a) \to V(b)$ open neighbourhoods such that if $x \in U(a)$ and $y \in V(b)$ then $y = f(x) \leftrightarrow (x, y) \in C$



Elliptic Function and Lattices

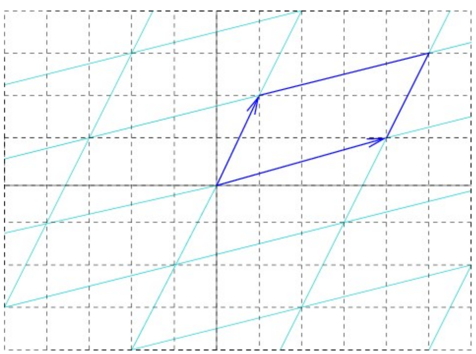
Def: Doubly Periodic Functions

A function that satisfies $f(z+\omega_1) = f(z+\omega_2) = f(z)\ \forall z \in \mathbb{C}$ where $\omega_1, \omega_2 \in \mathbb{C}$ are linearly independent over $\mathbb{R}$ is called *doubly periodic.* The ω_1, ω_2 are periods of $f(z)$

Examples of periodic functions:

$$\sin(x)\,,\cos(x)\,, e^x$$

The period of e^x is $2i\pi$. This is the smallest positive period, no other period is smaller in magnitude. The set of all periods are integer multiples of $2i\pi$



Def: Lattice is the set $\Lambda = \{\, n\omega_1 + m\omega_2 |\ (n, m) \in \mathbb{Z}\,\}$

Def: Elliptic Function $f(z)$ is a meromorphic and doubly periodic functions

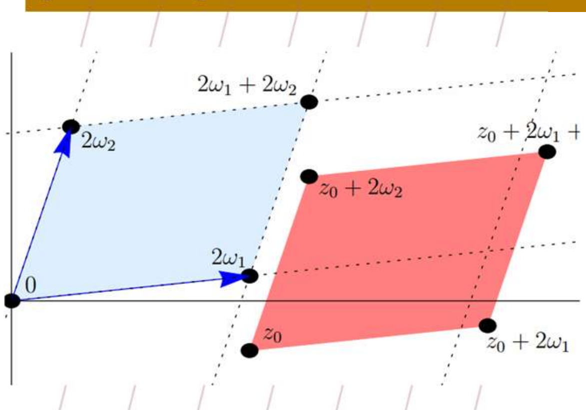
Non − constant Elliptic function $f(z)$ → $\exists$ lattice Λ s.t. $\forall z\ f(z+\omega) = f(z) \leftrightarrow \omega \in \Lambda$ → What generates the lattice?

Elliptic Function and Lattices

Def. Fundamental Period Parallelogram

Two linearly independent periods ω_1, ω_2 of $f(z)$ form a parallelogram. It is the ***Fundamental Period Parallelogram*** if no other period lies in it (boundaries included, vertices excepted) and ω_1, ω_2 are called the ***fundamental periods.***

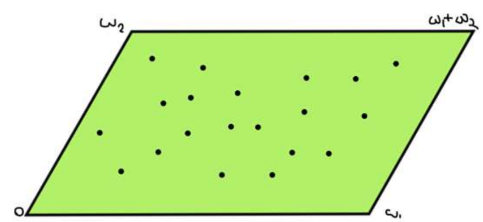
Easy to see the lattice generated by ω_1, ω_2 contains all the periods of $f(z)$ since otherwise the parallelogram will not be fundamental



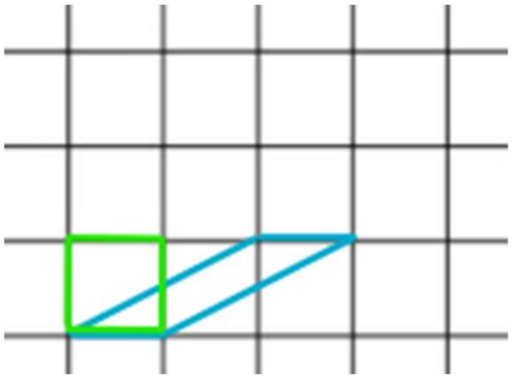
We can translate the fundamental period parallelogram by adding z_0 to get a cell, denoted $P(z_0)$

Why should there exists a fundamental Period Parallelogram?

[pf: Conv. Subsequence + isolated zeros/poles]



Choice of Fundamental Periods



Notice we have 2 pairs of fundamental periods for the lattice $\Lambda(1, i)$

$(\omega_1, \omega_2) = (1, i)$ defines the lattice
$(\omega_1', \omega_2') = (1, 2 + i)$ also defines the lattice

Instead of a unique fundamental period, there is a unique equivalence class defining the lattice. The following are equivalent

(ω_1, ω_2) and (ω_1', ω_2') define the same lattice Λ

$\leftrightarrow$ (ω_1, ω_2) and (ω_1', ω_2') produce a fund. period parallelograms in Λ

$$\leftrightarrow \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = A \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \text{ where } A \in SL(2, \mathbb{Z})$$

Def. Equivalence Relation on $\mathbb{C}$:

$$z_0 \sim z_1 \text{ iff } (z_0 - z_1) \in \Lambda$$

This means $z_0 \sim z_1 \rightarrow f(z_0) = f(z_1)$

Example: Weierstrass $\wp$ function

Def. Given a lattice $\Lambda(\omega_1, \omega_2)$ the associated Weierstrass function is

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda - \{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

Its derivative is

$$\wp'(z) = \sum_{\omega \in \Lambda} \frac{-2}{(z-\omega)^3}$$

By Weierstrauss M-test [Kirwan 5.10], these functions are meromorphic on $\mathbb{C}$. The lattice points are the only poles

Prop. $\wp$ is even and doubly periodic (i.e. periodic on the lattice Λ)

- $\wp(-z) = \frac{1}{(-z)^2} + \sum_{\omega \in \Lambda - \{0\}} \frac{1}{(-z-\omega)^2} - \frac{1}{\omega^2} = \frac{1}{(z)^2} + \sum_{(-\omega) \in \Lambda - \{0\}} \frac{1}{\left(z-(-\omega)\right)^2} - \frac{1}{(-\omega)^2} = \wp(z)$ $\therefore$ it is even
- $\wp'(z)$ is clearly doubly periodic

So $\wp'(z+\omega) = \wp'(z) \rightarrow integrate\ wrt\ z \ \rightarrow \wp(z+w) = \wp(z) + C(\omega)$

plug in $z = -\omega$ $\quad \wp\left(\frac{\omega}{2}\right) = \wp\left(-\frac{\omega}{2}\right) + C(w) \Rightarrow C(\omega) = 0$ $\therefore$ it is doubly periodic and an elliptic curve

Def.

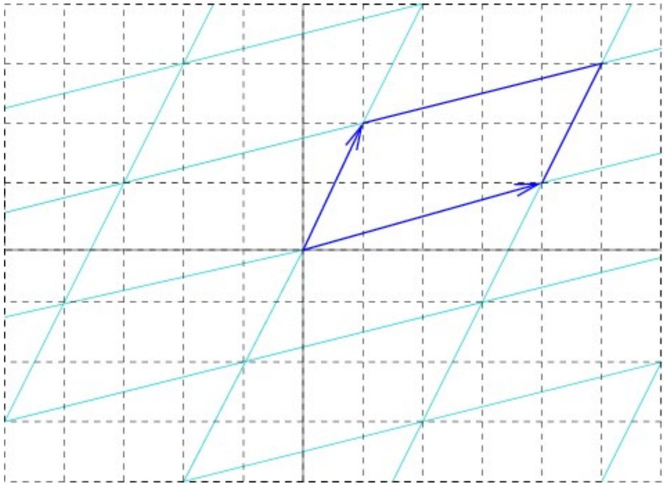
Irreducible set of zeros of an elliptic function are all the zeros that lie strictly inside a cell $P(z_0)$.

This cell should have no zeros or poles on its boundary

Similarly define *Irreducible set of poles.*

- A irreducible set of zeros are congruent to all zeros in $\mathbb{C}$
- No two zeros in this set are congruent to each other
- Any two congruent zeros have the same order
- Same is true for irreducible set of poles

- All the poles of Weierstrass $\wp$ functions have the same order



Laurent Series @$z = 0$ and order of poles

In an open ball at the origin:

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda - \{0\}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

The order of pole at 0 is 2 since the Laurent series looks like

$$\wp(z) = \frac{1}{z^2} + \frac{0}{z} + 0 + a_2 z^2 + a_4 z^4 + \ \dots$$

By considering some combinations of $[\wp']^2, \ \wp, \ \wp^3, 1$ we can create a doubly periodic holomorphic

$$\wp(z) = \qquad \frac{1}{z^2} \qquad + a_2 z^2 + a_4 z^4 + \ \dots$$

$$\wp^3(z) = \quad \frac{1}{z^6} + \ \frac{3a_2}{z^2} + \ 3a_4 \ + \_z^2 \ + \cdots$$

$$[\wp'(z)]^2 \ = \frac{4}{z^6} - \ \frac{8a_2}{z^2} - \ 16a_4 \ + \_z^2 \ \ + \cdots$$

The combination is $[\wp'(z)]^2 \ - 4\wp^3(z) + 20a_2\wp(z) + 28a_4 = \_z^2 + \_z^4 + \dots$

Laurent Series @$z = 0$ and order of poles

The combination is $[\wp'(z)]^2 - 4\wp^3(z) + 20a_2\wp(z) + 28a_4 = \_z^2 + \_z^4 + \ldots$

Written a bit concisely, the function $[\wp'(z)]^2 - 4\wp^3(z) + g_2\wp(z) + g_3$ is a doubly periodic holomorphic functions where

$$g_2 = 20a_2 = 60 \sum_{\omega \in \Lambda - \{0\}} \frac{1}{\omega^4}$$

$$g_3 = 28a_4 = 140 \sum_{\omega \in \Lambda - \{0\}} \frac{1}{\omega^6}$$

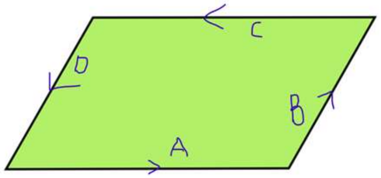
It follows by a property [see next slide] that the holomorphic function identically equals to 0. This $\wp(z)$ solves the differential equation

$$\left(\frac{dy}{dz}\right)^2 = 4y^3 - g_2 y - g_3$$

Properties of Elliptic function

1) Sum of Residues over an irreducible set of poles of an elliptic function is 0

Idea [Residue Thm] $\sum Res(f; z_i) = \frac{1}{2i\pi} \oint_{\partial_{cell}} f(z) = 0$



- Example: Residue of the only pole of $\wp(z)$ is 0.

2) A holomorphic elliptic function is constant

Idea [continuous function bounded on compact parallelogram + periodic + Liouville's Thm]

3) Number of roots of $f(z) = w_0$ in a cell is equal to the number of poles of $f(z)$ in a cell (counting multiplicity). This is independent of w_0

We call the number of poles counting multiplicity the *order* of elliptic function $f(z)$

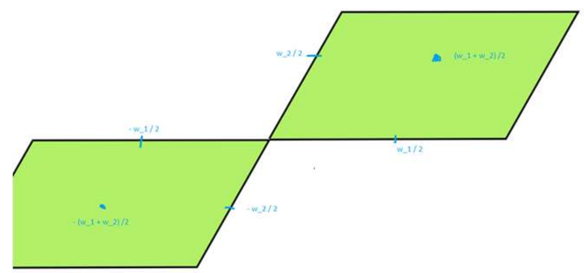
Idea [Argument principle + property 1]

Let $g(z) = \frac{f'(z)}{f(z)-w_0}$. It is elliptic function so $\frac{1}{2i\pi}\oint_{\partial_{cell}} g(z) = 0$. By argument principle, $\frac{1}{2i\pi}\oint_{\partial_{cell}} g(z)$ = # zeros - # poles = 0

- Elliptic curves are surjective
- Order of $\wp$ is 2. It has single pole of order 2.
- Means $\wp(z)$ is surjective "twice" i.e. $\wp(z) = w_0$ has two distinct zeros in irred. set $\{z_0, -z_0\}$
- Actually, sometimes $-z_0 \sim z_0$. In this case the single zero has multiplicity 2

This happens at exactly 3 points $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$ (called half-periods)

Notice multiplicity is 2 -> first derivative is 0 (also called stationary points)



4) The smallest order of an elliptic function is 2

Weierstrass $\wp(z)$ function, which has order 2, has the smallest possible order

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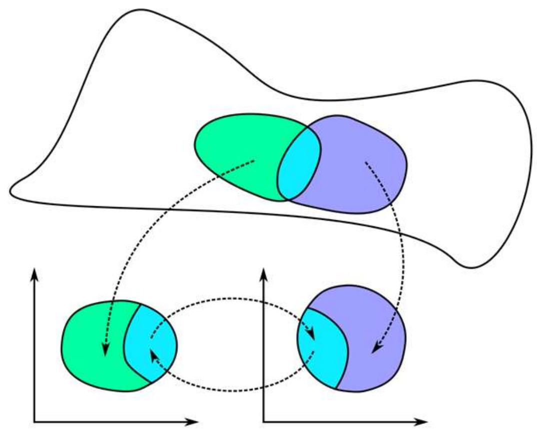


Riemann surfaces have
holomorphic transition maps

Some More Preliminary Results

- Closed subsets of compact space is compact
- Compact subset of Hausdorff is closed
- Projective space $\boldsymbol{P_n}$ is compact and Hausdorff
- Projective Curves (subsets of $\boldsymbol{P_n}$) are closed, compact, Hausdorff
- Continuous bijection between a compact and Hausdorff space is a homeomorphism

[W.A. Sutherland, *Introduction to metric and topological spaces*]
[F. Kirwan, *Complex Algebraic Curves*]

Complex Algebraic Curves as Riemann Surfaces

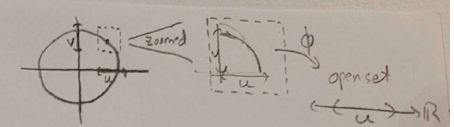
Prop [Kirwan 5.27]

If C is a complex algebraic curve in $\mathbb{C}^2$ defined by polynomial $P(x, y)$ then $C - Sing(C)$ has a holomorphic atlas (aka is a Riemann surface)

Take any point $(a,b) \in C - Sing(C)$

Either $\frac{\partial P}{\partial x} \neq 0$ or $\frac{\partial P}{\partial y} \neq 0$.

e.x.



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By Implicit Function Theorem

$\exists f : U(a) \to V(b)$ between open neigh. of a, b

s.t. $y = f(x)$ iff $(x,y) \in \underbrace{C \cap (U \times V)}_{\text{this is open}}$

So chart $\Phi_\alpha : C \cap (U \times V) \to U$

$(x,y) \mapsto x$ is a homeomorphism

$\Phi_\alpha^{-1} : x \mapsto (x, f_\alpha(x))$

Similarly $\Psi_\beta : C \cap (U \times V) \to V$

$(x,y) \mapsto y$

$\Psi_\beta^{-1} : y \mapsto (g_\beta(y), y)$

Transition maps

$t_{1,2} = \Phi_2 \circ \Phi_1^{-1}$ in this case $x \xmapsto{\Phi_1^{-1}} (x, f_1(x)) \xmapsto{\Phi_2} x$ identity map

$t_{1,2} = \Psi_2 \circ \Psi_1^{-1}$ in this case $y \xmapsto{\Psi_1^{-1}} (g_1(y), y) \xmapsto{\Psi_2} y$ identity map.

$t_{1,2} = \Phi_2 \circ \Psi_1^{-1}$ in thi case $y \xmapsto{\Psi_1^{-1}} (g_1(y), y) \xmapsto{\Phi_2} g_1(y)$ holo. by implicit. thm.

$t_{1,2} = \Psi_2 \circ \Phi_1^{-1}$ in this case $x \xmapsto{\Phi_1^{-1}} (x, f_1(x)) \xmapsto{\Psi_2} f_1(x)$ " " " "

Complex Algebraic Curves as Riemann Surfaces

Prop [Kirwan 5.27]

If C is a complex algebraic curve in $\mathbb{C}^2$ defined by polynomial $P(x, y)$ then $C - Sing(C)$ has a holomorphic atlas (aka is a Riemann surface)

Any point $(a, b) \in C - Sing(C)$

[Implicit Function theorem]

$\exists f: U(a) \to V(b)$

s.t. $\quad y = f(x)$ iff $(x, y) \in C$

So the chart $\quad \phi_\alpha : C \cap (U \times V$

$(x, y) \mapsto x$

Inverse chart $\phi_\alpha^{-1} : x \mapsto (x, f_\alpha$

Take any point $(a,b) \in C - Sing(C)$

Either $\frac{\partial P}{\partial x} \neq 0$ or $\frac{\partial P}{\partial y} \neq 0$.

. . .

By Implicit Function Theorem

$\exists f: U(a) \to V(b)$ between open neigh. of a,b

s.t. $y = f(x)$ iff $(x,y) \in \underbrace{C \cap (U \times V)}_{\text{this is open}}$

Similarly $\psi_\beta : C \cap (U \times V) \to V$

$(x,y) \mapsto y$

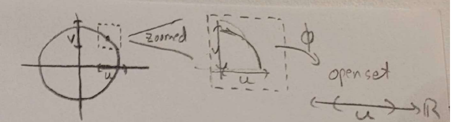
$\psi_\beta^{-1} : y \mapsto (g_\beta(y), y)$

So chart $\phi_\alpha : C \cap (U \times V) \to U$

$(x,y) \mapsto x$ is a homeomorphism

$\phi_\alpha^{-1} : x \mapsto (x, f_\alpha(x))$

e.x.



<u>Transition maps</u>

$t_{1,2} = \phi_2 \circ \phi_1^{-1}$ in this case $x \xmapsto{\phi_1^{-1}} (x, f_1(x)) \xmapsto{\phi_2} x$ identity map

$t_{1,2} = \psi_2 \circ \psi_1^{-1}$ in this case $y \xmapsto{\psi_1^{-1}} (g_1(y), y) \xmapsto{\psi_2} y$ identity map.

$t_{1,2} = \phi_2 \circ \psi_1^{-1}$ in thi case $y \xmapsto{\psi_1^{-1}} (g_1(y), y) \xmapsto{\phi_2} g_1(y)$ holo. by implicit. thm.

Projective Curves as Riemann Surfaces

Prop [Kirwan 5.28]

If C is a projective curve in $\boldsymbol{P_2}$ defined by polynomial $P(x, y, z)$ then $C \ - Sing(C)$ has a holomorphic atlas (aka is a Riemann surface)

- Using the fact P is homogenous function and P_2 projective space, we can reduce to 2 variables and use Implicit Function Theorem

Atlases look like $\phi : U \ \rightarrow \mathbb{C}$

$$[\mathrm{x}, \mathrm{y}, \mathrm{z}] \mapsto \frac{\mathrm{x}}{\mathrm{z}}, \frac{\mathrm{z}}{\mathrm{x}}, \frac{\mathrm{y}}{\mathrm{z}}, \frac{\mathrm{z}}{\mathrm{y}}, \frac{\mathrm{x}}{\mathrm{y}}, \frac{\mathrm{y}}{\mathrm{x}}$$

Transition maps look like

$$w \mapsto w, \frac{1}{w}, g(w), \frac{1}{g(w)}, \frac{w}{g(w)}, \frac{g(w)}{w}$$

Projective Curves as Riemann Surfaces

Prop [Kirwan 5.19, 5.20]

The projective curve C_Λ in $\boldsymbol{P_2}$ defined by the polynomial

$$Q_\Lambda(x, y, z) = y^2 z - 4x^3 + g_2 x z^2 + g_3 z^3$$

is non-singular. Therefore it is also a Riemann Surface.

Recall the half periods $\frac{\omega_1}{2}, \frac{\omega_2}{2}, \frac{\omega_1+\omega_2}{2}$. They make $f' = 0$. Thus differential eq. is 3rd degree cubic with distinct roots. $y(z) = \wp(z)$ and $\wp' = 0$

$$g_2 = 20a_2 = 60 \sum_{\omega \in \Lambda - \{0\}} \frac{1}{\omega^4}$$

$$g_3 = 28a_4 = 140 \sum_{\omega \in \Lambda - \{0\}} \frac{1}{\omega^6}$$

$$\left(\frac{dy}{dz}\right)^2 = 4y^3 - g_2 y - g_3$$

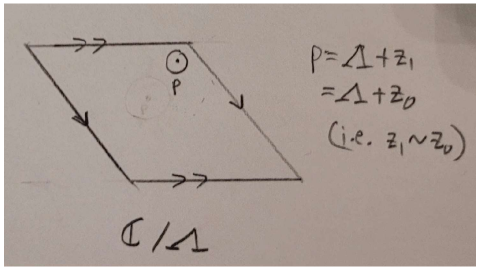
$$= 4(y - e_1)(y - e_2)(y - e^3)$$

$$= 4[(y - e_2)(y - e_3) + (y - e_1)(y - e_3) + (y - e_1)(y - e_2)]$$

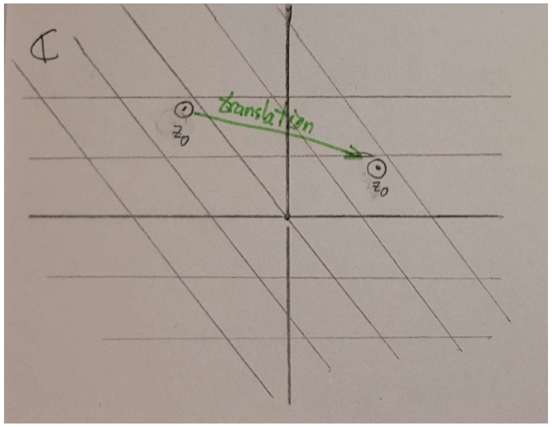
Complex Torus as Riemann Surfaces

Prop [Kirwan 5.42]

Let $\omega_1, \omega_2 \in \mathbb{C}$ be linearly independent over $\mathbb{R}$. Then the set $\mathbb{C}/\Lambda(\omega_1, \omega_2)$ with quotient topology has a holomorphic atlas (aka is a Riemann surface)



- In this case, the quotient map $\Pi : \mathbb{C} \to \mathbb{C}/\Lambda$ is open and continuous (but not injective!)
- Take a point $\Lambda + z_1 \in \mathbb{C}/\Lambda$. We could use a different z_i
- Consider an open ball $U(z_1)$ small enough to make Π injective. It becomes a homeomorphism between open set in $\mathbb{C}$ to neighbourhood of $\Lambda + z_1$. Π^{-1} is the chart.
- Recall we could've chosen any z_i. Thus multiple homeomorphisms exist for same neighbourhood of $\Lambda + z_1$. However, the open set in $\mathbb{C}$ are plane translations. Thus transition maps are holomorphic



Holomorphic maps between Riemann surfaces

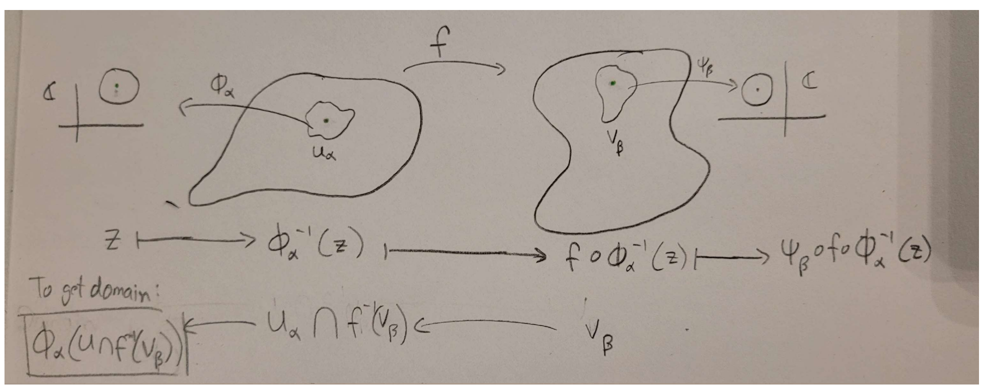
Def [Kirwan 5.33]

Let S and T be surfaces with holomorphic atlases Φ and Ψ. A continuous map $f : S \to T$ is called holomorphic with respect to Φ and Ψ if the map

$$\psi_\beta \circ f \circ \phi_\alpha^{-1}$$

is holomorphic

- Equivalently, it can be defined pointwise by asking if there exists 2 charts the cover the point and its image under f such that $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is holomorphic



Holomorphic map $u\colon\ \mathbb{C}/\Lambda \to C_\Lambda$

Def [Kirwan 5.22, 5.43]

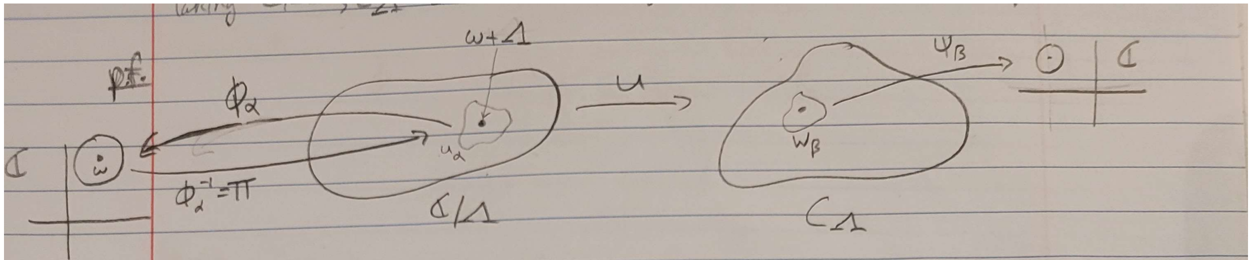
The map $u\colon\ \mathbb{C}/\Lambda \to C_\Lambda$ defined by

$$\Lambda + z \mapsto \begin{cases} [\wp(\mathrm{z}), \wp'(\mathrm{z}), 1] & \text{if } z \notin \Lambda \\ [0, 1, 0] & \text{if } z \in \Lambda \end{cases}$$

is a homeomorphism (topological). Treating $\mathbb{C}/\Lambda$ and C_Λ as Riemann surfaces, u is also holomorphic.

Insert Image for proof

pf.



I can pick one chart on each Riemman surface and show that u is pointwise holo everywhere

Case 1

$\boxed{\omega \notin \Lambda}$ The open ball $U(\omega)$ will contain no lattice points.

Let $z \in U(\omega)$ $\quad z \xmapsto{\pi} \Lambda + z \xmapsto{u} [\wp(z), \wp'(z), 1] \xmapsto{\psi_\beta} \frac{\wp(z)}{1}$ $\quad$ [form: $\frac{x}{z}$]

Since $\wp(z)$ is holo away from the lattice points it is holo on $U(\omega)$

Case 2

$\boxed{\omega \in \Lambda}$ The open ball $U(\omega)$ will contain no stationary points i.e $f'(z) \neq 0$.

Let $z \in U(\omega)$ $\quad z \xmapsto{\pi} \Lambda + z \xmapsto{u} \begin{cases} [0,1,0] \\ [\wp(z), \wp'(z), 1] \end{cases} \xmapsto{\psi_\beta} \begin{cases} 0 \\ \frac{\wp(z)}{\wp'(z)} \end{cases}$ $\quad$ [form: $\frac{x}{y}$]

Since both $\wp(z)$, $\wp'(z)$ have poles at origin, we inspect order of poles

$$\wp(z) = \frac{g(z)}{z^2} \quad ; \quad \wp'(z) = \frac{h(z)}{z^3} \qquad \text{when } g,h \text{ holo. near } 0.$$

Therefore u is a holo. map.

Summary

- Elliptic functions are doubly periodic, meromorphic functions
- Non-singular projective curves are compact Riemann surfaces
- Used the Weierstass elliptic function $\wp$ to prove conformal equivalence between a complex torus and a projective curve defined on a given lattice

Sources

- F. Kirwan, *Complex Algebraic Curves*
- A.F. Beardon, *A Primer on Riemann Surfaces*
- G. Pastras, *Four Lectures on Weierstrass Elliptic Function and Applications in Classical and Quantum Mechanics*

Thank You for Listening
Any Questions?