# Essential Singularities and the Great Picard Theorem

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MATD34 Complex Variables II

# Agenda

- Recap the classification Isolated Singularities (C34)
- Linear Fractional Transformations (Mobius)
- Weierstrass-Casorati Theorem
- The Proof
- Properties of Harmonic Functions
- Integral formulas for Harmonic functions
- Harnack-Type Results

Émile Picard (1856-1941)

# Recap: Laurent Series

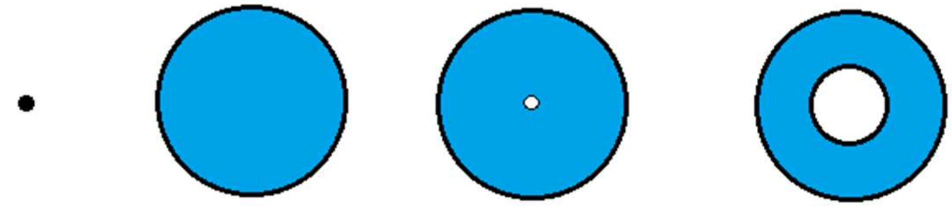
Def: A Laurent series is the infinite sum in terms of variable $z$

$$f(z) = \sum_{-\infty}^{\infty} a_n z^n$$

(Where $a_n \in \mathbb{C}$)

- It is possible that the sum f(z) doesn't converge for any $z \in \mathbb{C}$

Let $S = \{z \in \mathbb{C} \mid f(z) \text{ converges}\} \neq \emptyset$. This subset of $\mathbb{C}$ looks like one of the following 4 possibilities:

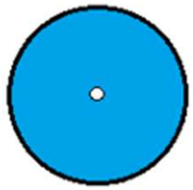


Single Point Open Disk Punctured Disk Annulus

# Recap: Isolated Singularities

Def:
Let $f: \mathbb{C} \to \mathbb{C}$ be any complex valued functions. It has a **singular point** if it is not holomorphic at $z_0$ .

$z_0$ is an **isolated singularity** if f is holomorphic on some punctured disk $B^*(z_0)$



Punctured Disk

The following theorem lets us classify isolated singularities using Laurent series

Theorem: (Laurent's Theorem)

Let f(z) be analytic (i.e. holomorphic) on some annulus $D: r < |z - z_0| < R$. Then there exists a Laurent Series that converges (locally uniformly) on D. We can compute the series

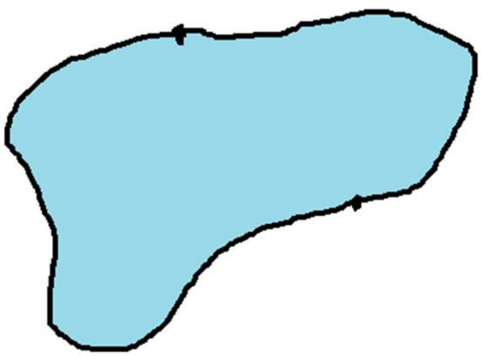
$$f(z) = \sum_{-\infty}^{\infty} a_n z^n \quad \text{forall } z \in D$$

$$\text{where } a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{(z - z_0)^{n+1}} dz$$

Punctured disk is an annulus, thus has a Laurent series that converges to f(z)

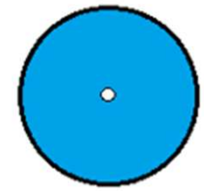
# Recap: Maximum Modulus principle

Theorem:
Non- constant holomorphic functions cannot attain their maximum modulus $|f(z)|$ on a domain



The max can only be attained on the boundary

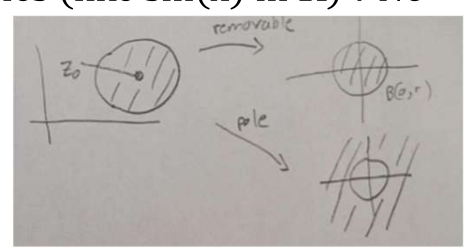
# Recap: Classifying isolated singularities



Theorem:
Let $f(z)$ have an isolated singularity at $z_0$. Let $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ be the Laurent expansion
Let $N = \#$ of negative terms.
Let $\{z_n\}_1^{\infty} \to z_0$ be an arbitrary converging sequence.
Let $B(0, r)$ be an arbitrary disk centered at 0 with radius r.
Let $B^*(z_0)$ be a sufficiently small punctured disk. Then,

| Type | Laurent | Behaviour of Seq. $\{f(z_n)\}_{n=1}^{\infty}$ | Behaviour of Image $f[B^*(z_0)]$ |
|---|---|---|---|
| Removable | $N = 0$ | $\lim_{n\to\infty} f(z_n)$ exists | Bounded by disk $\mathrm{B}(0, \mathrm{r})$ |
| Pole | $0 < N < \infty$ | $\lim_{n\to\infty} f(z_n) = \infty$ | Completely outside $\mathrm{B}(0, \mathrm{r})$ |
| Essential | $N = \infty$ | $\lim_{n\to\infty} f(z_n)\ DNE$ | ? ? ? |

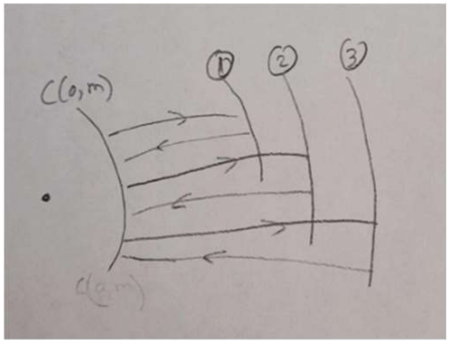
- Can the limit for essential singularity oscillate between two concentric circles (like sin(x) in $\mathbb{R}$) ? No because it becomes bounded



# Recap: Classifying isolated singularities

| **Type** | **Laurent** | **Behaviour of Seq.** $\{f(z_n)\}_{n=1}^{\infty}$ | **Behaviour of Image** $f[B^*(z_0)]$ |
|---|---|---|---|
| Removable | $N = 0$ | $\lim_{n\to\infty} f(z_n)$ exists | Bounded by circle $\mathrm{B}(0, \mathrm{r})$ |
| Pole | $0 < N < \infty$ | $\lim_{n\to\infty} f(z_n) = \infty$ | Completely outside $\mathrm{B}(0, \mathrm{r})$ |
| Essential | $N = \infty$ | $\lim_{n\to\infty} f(z_n)\ DNE$ | ??? |

- Can the limit for essential singularity oscillate between two concentric circles (like $\sin(x)$ in $\mathbb{R}$) ? No because it becomes bounded (Removable singularity)
- Near an essential singularity $f(z)$ runs back and forth from $\infty$ to $C(0, m)$ in order to not be
  1) Bounded
  2) Have $\lim = \infty$



# Mobius Transformations (LFT) and Essential Singularities

Suppose f(z) has an essential singularity at $z_0$

- $\frac{1}{f(z)}$ still has essential singularity at $z_0$
- $\frac{a}{f(z)}$ still has essential singularity at $z_0$
- $\frac{a}{f(z)+d}$ still has essential singularity at $z_0$
- $c + \frac{a}{f(z)+d}$ still has essential singularity at $z_0$

This is intuitively true because the oscillatory behaviour is either inverted or translated, and remains oscillatory. So $\frac{af(z)+b}{cf(z)+d}$ still has essential singularity at $z_0$. [ where $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$ ]

- $f(z+a)$ still has essential singularity, its just moved to $z_0 - a$

Following the same idea, $\mathrm{f}\left(\frac{\mathrm{az+b}}{\mathrm{cz+d}}\right)$ still has an essential singularity, its just moved to $g^{-1}(z_0)$

> Suppose f(z) has an essential singularity at $z_0$. Let $g_1(z), g_2(z)$ be Mobius Transformations, then $g_2 \circ f \circ g_1$ has an essential singularity at $g_1^{-1}(z_0)$

Proof: By contradiction and by using the classification table

We can extend our table

| Type | Laurent | Behaviour of Seq. $\{f(z_n)\}_{n=1}^{\infty}$ | Behaviour of Image $f[B^*(z_0)]$ |
|---|---|---|---|
| Removable | $N = 0$ | $\lim_{n\to\infty} f(z_n)$ exists | Bounded by circle $\mathrm{B}(0,\mathrm{r})$ |
| Pole | $0 < N < \infty$ | $\lim_{n\to\infty} f(z_n) = \infty$ | Completely outside $\mathrm{B}(0,\mathrm{r})$ |
| Essential | $N = \infty$ | $\lim_{n\to\infty} f(z_n)\ DNE$ | ??? |

$\frac{1}{f(z)-w}$ has essential singularity at $z_0$

# Weierstrauss-Casorati (Sokhotski's) Theorem

Theorem:

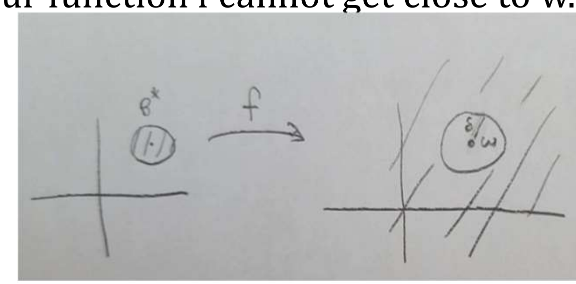
Let $z_0$ be an essential singularity of $f(z)$. Then for any $c \in \mathbb{C}$ there exists a sequence $\{z_n\}_1^\infty \to z_0$ such that $\{f(z_n)\}_1^\infty \to c$

(i.e. $f[B^*(z_0)]$ is dense in $\mathbb{C}$)

Proof: By contradiction,
Suppose $\exists w \in \mathbb{C}$ such that no sequence $\{f(z_n)\}_1^\infty$ converges to $w$. Our function f cannot get close to w. For some fixed $\delta$

$$|f(z) - w| > \delta$$

$$\to \left|\frac{1}{f(z) - w}\right| < \delta$$




This new function $g(z) = \frac{1}{f(z)-w}$ is bounded and g has removable singularity at $z_0$.
But the result from the extend table tells us $z_0$ is an essential singularity of g. Contradiction. Q.E.D.

Theorem:

Let $z_0$ be an essential singularity of $f(z)$. Then for any $c \in \mathbb{C}$ there exists a sequence $\{z_n\}_1^\infty \to z_0$ such that $\{f(z_n)\}_1^\infty \to c$

Ex 1: $e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!\,z^2} + \frac{1}{3!z^3} + \frac{1}{4!z^4} + \ \ldots$

Let $c = 5$

$$\{z_n\} = \left\{\frac{1}{\ln 5}, \frac{1}{\ln(5) + 2\pi i}, \frac{1}{\ln(5) + 4\pi i}, \frac{1}{\ln(5) + 6\pi i}, \ldots\right\}$$

Then

$$\{f(z_n)\} = \{e^{\ln(5)}, e^{\ln(5)+2\pi i}, e^{\ln(5)+4\pi i}, e^{\ln(5)+6\pi i}, \ldots\}$$
$$= \{5, 5, 5, 5, \ldots\}$$

Constant sequence converging to 5. There is an infinite supply of 5's. Does this always happen?

# Entire Injective functions

Ex 2:

## The Great Picard Theorem

Theorem:

$f(z)$ has an essential singularity at $z_0$ iff

in any punctured ball $B^*(z_0)$, $f(z)$ takes on all possible complex values, with at most one exception, infinitely often

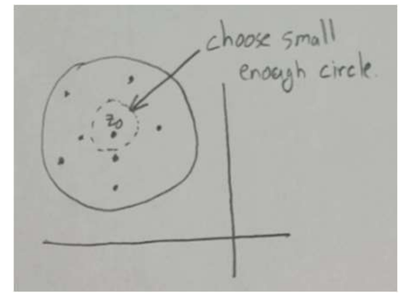
Many proofs of this theorem have been given using (a) the modular function, (b) Schottky and Bloch theorems, (c) a generalization of Schwarz's lemma, (d) Nevannlinna's second fundamental theorem, and (e) probability (see [5] for (a), (b); see [1] for (c); see [6] for(d); and see [2] for (e)). Rickman [10] has obtained an analogue of Picard's theorem for entire quasiregular mappings. He proved

Proof (←)
Clearly $f(z)$ is not bounded on any $B^*(z_0)$
And $f(z)$ is not bounded away from 0 on any $B^*(z_0)$
By process of elimination, since $z_0$ isn't a removable singularity or pole, it must be an essential singularity

**Theorem: The Great Picard Theorem**

$f(z)$ has an essential singularity at $z_0$ iff

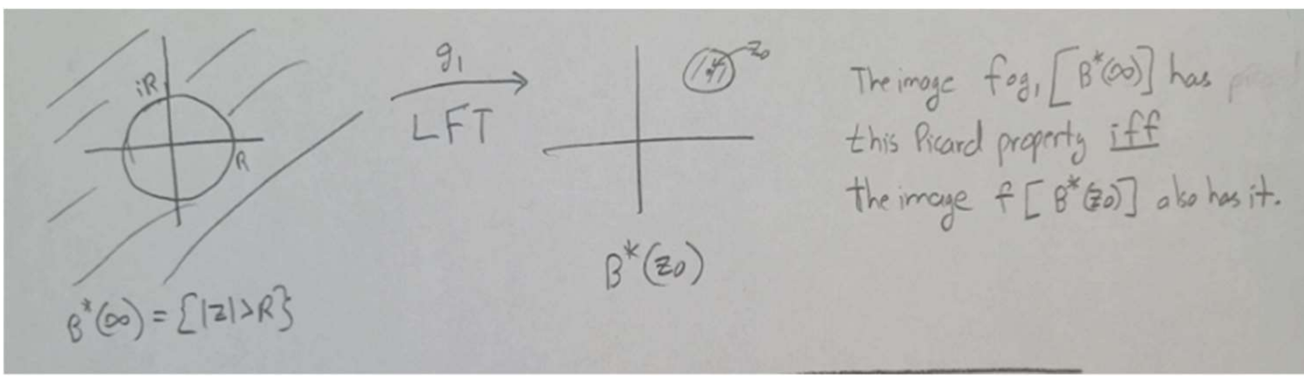
in any punctured ball $B^*(z_0)$, $f(z)$ takes on all possible complex values, with at most one exception, infinitely often




Proof (→) By contradiction
Suppose there are two values $w_1, w_2$ that appear in the image of f finitely many times.
With a sufficiently small punctured disk $B^*(z_0)$ we can omit these two values entirely!
So $f(z) \neq w_1$ and $f(z) \neq w_2$. Consider

$$\frac{f(z) - w_1}{w_2 - w_1}$$

This still has essential singularity at $z_0$ but omits 0, 1 on small enough $B^*(z_0)$. Let $g_2(z) = \frac{z-w_1}{w_2}$




nd $f \circ g_1 = f\left(\frac{1}{z} + z_0\right)$
mages of punctured disks at $\infty$,

Theorem: The Great Picard Theorem

$f(z)$ has an essential singularity at $z_0$ iff

in any punctured ball $B^*(z_0)$, $f(z)$ takes on all possible complex values, with at most one exception, infinitely often

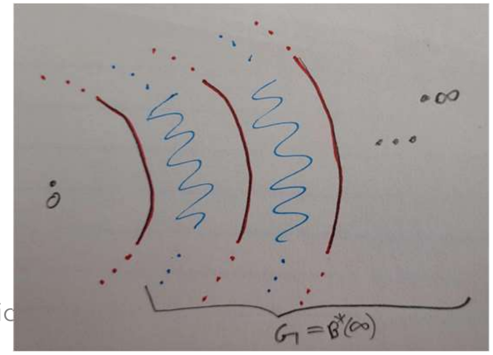
Proof (→)

- $f(z)$ has essential singularity iff $g_2 \circ f \circ g_1(z)$ has essential singularity
- An image $f[B^*(z_0)]$ fails the Picard property iff an image $g_2 \circ f \circ g_1[B^*(\infty)]$ fails the Picard property
- An image $f[B^*(z_0)]$ is bounded iff an image $g_2 \circ f \circ g_1[B^*(\infty)]$ is bounded

This bound is the contradiction we desire, similar to Weierstrass–Casorati theorem.

So WLOG suppose f has essential singularity at $\infty$ and $G = \{z \in \mathbb{C} \mid |z| > R\} = B^*(\infty)$ and $0,1 \notin f[G]$
Then we show $f[G]$ is bounded i.e. removable singularity.
Show $|f(z)|$ is bounded on the boundary and use Maximum Modulus Principle to bound $|f(z)|$ in $G$.

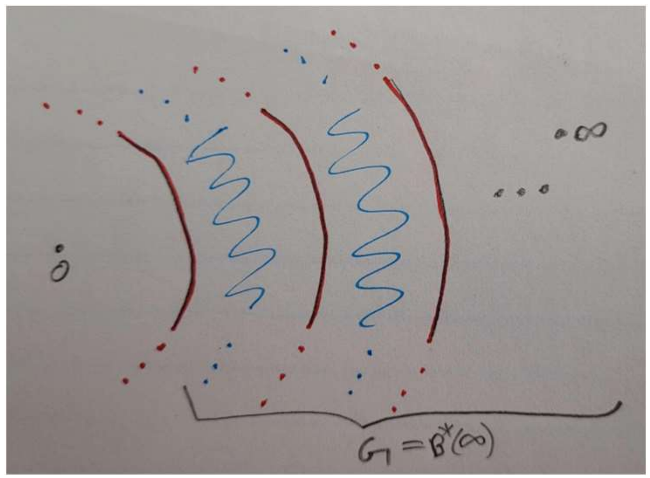
We will show $|f(z)|$ is bounded on each ring



Showing any one of the following is bounded is sufficient:

$$|f(z)|, \qquad |f(z)-1|, \qquad \log|f(z)|, \qquad \log|f(z)-1|$$

- We pick $\log|f(z)|$ and $\log|f(z)-1|$ because they are Harmonic
- Let $u_1(z) = \log|f(z)|, u_2(z) = \log|f(z)-1|$. We want to show the functions $u_1, u_2$ are bounded on the rings



- We pick $\log|f(z)|$ and $\log|f(z)-1|$ because they are Harmonic

Prop 2.

There exists a universal constant B> 1 such that the following is always true:

if real number $\lambda > 0$ and $u_1, u_2$ are two harmonic functions on unit disk $\Delta$ satisfying:

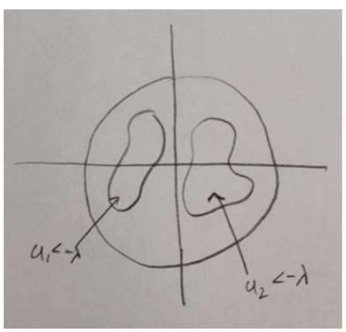
$$\{z \in \Delta \mid u_1(z) < -\lambda\} \text{ and } \{z \in \Delta \mid u_2(z) < -\lambda\} \text{ are disjoint}$$

$$|u_1^+ - u_2^+| < \lambda \quad [\text{def: } f^+ = \max(f, 0)]$$

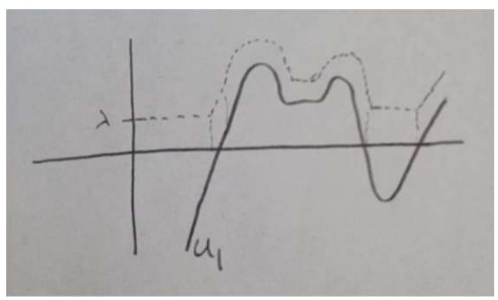
$$|u_j(0)| < \lambda, \qquad \text{for } j = 1, 2$$

Then the both are bounded on $\Delta\left(0, \frac{1}{2}\right)$. Specifically,

$$M\left(\frac{1}{2}, u_j, 0\right) \leq B\lambda. \quad \text{for } j = 1,2$$

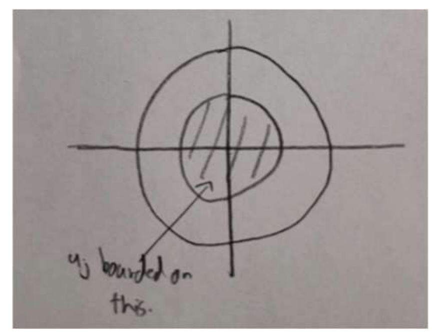



"Sufficiently negative values share the disk but live on disjoint regions"




"$u_2$ must fall below the dotted curve"

Then →




Rest of the proof on chalkboard

# Harmonic functions

Properties of Harmonic functions

- [Average value on circle]

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0 + e^{i\theta})d\theta$$

- [Max/min Modulus Principle]

Non-constant harmonic functions don't attain their max/min on a domain

- Other properties

The zeros of harmonic functions are never isolated

If harmonic function is 0 on some open ball, it is zero everywhere on its domain

Similar to Cauchy's integral formula $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw$

We have the integral formulas for harmonic functions

# Harmonic functions

Similar to Cauchy's integral formula $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}dw$

We have the following integral formulas:

1) Integral formula of harmonic $u(z)$ in terms of itself: [Poisson Integral]

$$u(z) = u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} u\left(\rho e^{i\varphi}\right)\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r\cos(\theta-\varphi)}\,d\varphi$$

2) Integral formula of conjugate $v(z)$ in terms of $u(z)$:

$$v(z) = v(r,\theta) = \beta_0 + \frac{1}{2\pi}\int_0^{2\pi} u\left(\rho e^{i\varphi}\right)\frac{2\rho r\sin(\theta-\phi)}{\rho^2 + r^2 - 2\rho r\cos(\theta-\varphi)}\,d\varphi$$

3) Integral formula of holomorphic $f(z)$ in terms of the real part $u(z)$: [Schwarz's Formula]

$$f(z) = f(r,\theta) = i\beta_0 + \frac{1}{2\pi}\int_0^{2\pi} u\left(\rho e^{i\varphi}\right)\frac{\rho e^{i\varphi} + (z-z_0)}{\rho e^{i\varphi} - (z-z_0)}\,d\varphi$$

Notice that $\frac{\rho e^{i\varphi}+(z-z_0)}{\rho e^{i\varphi}-(z-z_0)} = \frac{\rho^2-r^2}{\rho^2+r^2-2\rho r\cos(\theta-\varphi)} + i\,\frac{2\rho r\sin(\theta-\phi)}{\rho^2+r^2-2\rho r\cos(\theta-\varphi)}$

# Why is this proof called Harnack-Type inequality?

**Harnack's Inequality**
Let u be a **positive harmonic function** in the disk $\Delta(w, R)$, then

$$\frac{1}{3}u(w) \leq u(z) \leq 3u(w), \qquad \forall z \in \overline{\Delta\left(w, \frac{R}{2}\right)}$$

The function must be fairly close to the value at the center *h(w)*

Why?

$$\frac{1}{3} < \frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r\cos(\theta - \varphi)} < 3$$

$$\frac{1}{3}\frac{1}{2\pi}\int_0^{2\pi} u(\rho e^{i\varphi})\, d\varphi \; < \frac{1}{2\pi}\int_0^{2\pi} u(\rho e^{i\varphi})\frac{\rho^2 - r^2}{\rho^2 + r^2 - 2\rho r\cos(\theta - \varphi)}\, d\varphi < \; 3\frac{1}{2\pi}\int_0^{2\pi} u(\rho e^{i\varphi})\, d\varphi$$

Apply Average value and integral formulas to get:

$$\frac{1}{3}u(w) \leq u(z) \leq 3u(w)$$

# Why is this proof called Harnack-Type inequality?

- Lewis's Lemma is a very similar result for bounded harmonic functions
  - Uses Harnack's inequality in the proof (by finding a region in $\Delta$ where $u(z)$ is positive

- Another result that is proved using Harnack's inequality

**Prop 1: Subsequence of Harmonic functions sometimes converges**

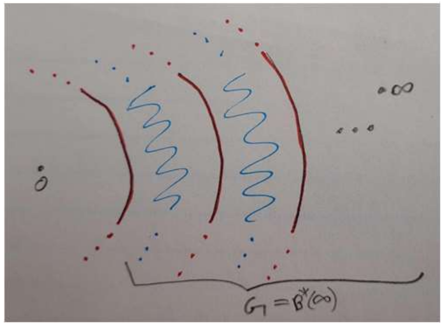
If a sequence of positive harmonic functions on $\Delta$ unit disk $\{u_n\}_{n=1}^{\infty}$ converges at a single point, then

$$\left\{u_{n_k}(z)\right\}_{k=1}^{\infty} \rightarrow u(z)$$

Locally uniformly on $\Delta$

i.e. some subsequence $\left\{u_{n_k}\right\}_{k=1}^{\infty}$ converges uniformly on every compact subset of $\Delta$ to a harmonic function, $u(z)$

- Lewis's Lemma and Proposition 1 are used to prove proposition 2
- Proposition two is applied repeatedly to prove $u(z) = \log|f(z)|$ is bounded on the rings
- The contradiction proves the Great Picard Theorem



Prop 1: Lewis's Lemma

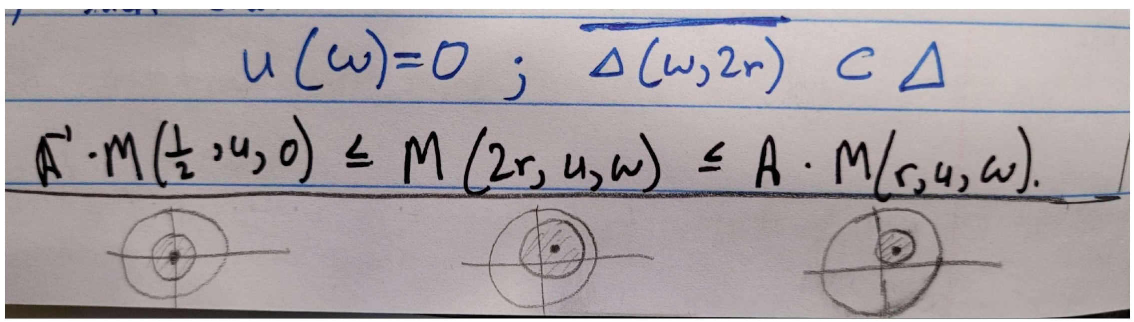
There exists a universal constant A>0 such that the following is always true:

Let $u(z)$ be a **bounded harmonic function** on the unit disk $\Delta$ such that

$$u(0) = 0$$

Then $\exists w \in \Delta$ and an open ball $\overline{B(w, 2r)} \subset \Delta$ such that $u(w) = 0$ and

$$A^{-1} \cdot M\left(\frac{1}{2}, u, 0\right) < M(2r, u, w) < A \cdot M(r, u, w)$$



# References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- E. Stien & R. Shakarchi, *Complex Analysis,* Princeton University Press (2003)
- Course notes from MATC34 and MATD34

# Thank you for listening!