COM

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Finding COM →

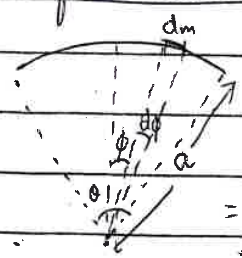
① Find axis of symmetry of body. COM must lie on the axis

② $y_{COM} = \dfrac{\int y\,dm}{M}$ → we have to take small elements whose mass is dm. Take element whose COM is known.

$y = y_{com}$ of element

$\dfrac{dm}{dl} = \lambda$, $\dfrac{dm}{dA} = \sigma$, $\dfrac{dm}{dV} = \rho$

* COM of circular arc →



$\dfrac{dm}{dl} = \lambda = \dfrac{M}{a\theta}$

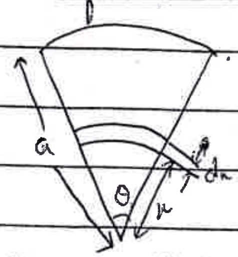
$y_{COM} = \dfrac{\int y\,dm}{M}$

$= \dfrac{\int_{-\theta/2}^{\theta/2} a\cos\phi \, \frac{M}{a\theta} a\,d\phi}{M} = \boxed{\dfrac{2a}{\theta}\sin\left(\dfrac{\theta}{2}\right)}$

$\theta = \pi \Rightarrow y_{com} = \dfrac{2a}{\pi}$

$\theta = 2\pi \Rightarrow y_{com} = 0$

* COM of a sector →



$\sigma = \dfrac{dm}{dA} = \dfrac{M}{a^2\theta/2}$

$\Rightarrow dm = \dfrac{2M(x\cancel{\theta}\,dx)}{a^2\cancel{\theta}}$

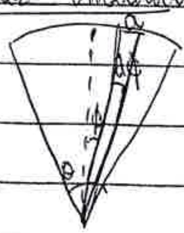
$y_{com} = \dfrac{\int y\,dm}{M} = \dfrac{\int_0^a \frac{2x}{\theta}\sin\left(\frac{\theta}{2}\right)\frac{2\cancel{M}\,x\,dx}{a^2}}{\cancel{M}}$

$= \boxed{\dfrac{4a}{3\theta}\sin\left(\dfrac{\theta}{2}\right)}$

$\theta = \pi \Rightarrow y_{com} = \dfrac{4a}{3\pi}$

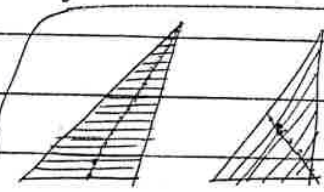
$\theta = 2\pi \Rightarrow y_{com} = 0$

Another method: (using COM of triangle lies on centroid)



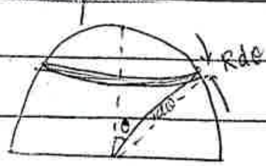
$y_{com} = \dfrac{\int y\,dm}{M} = \dfrac{\int_{-\theta/2}^{\theta/2}\left(\frac{2a}{3}\cos\phi\right)\left(\frac{M\,d\phi}{\theta}\right)}{M}$

$= \dfrac{4a}{3\theta}\sin\left(\dfrac{\theta}{2}\right)$



∴ COM of triangle lies on all 3 medians ⇒ it lies on centroid

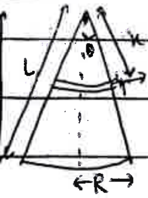
* ~~Hollow~~ Hemisphere



$dm = \dfrac{M}{2\pi R^2}dA = \dfrac{M}{2\pi R^2}2\pi R\sin\theta\,R\,d\theta = M\sin\theta\,d\theta$

$y_{com} = \dfrac{\int y\,dm}{M} = \dfrac{\int_0^{\pi/2} R\cos\theta\,M\sin\theta\,d\theta}{M} = \dfrac{R}{2}\int_0^{\pi/2}\sin 2\theta\,d\theta = \boxed{R/2}$

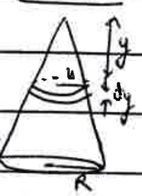
Hollow cone – $dA = 2\pi x \sin\theta \, dx$ $dm = \frac{M}{\pi RL} dA = \frac{M}{\cancel{\pi} RL} 2\cancel{\pi} x \sin\theta \, dx$



$$y_{com} = \frac{\int y\,dm}{M} = \frac{\int_0^L x\cos\theta \; 2x\sin\theta\, dx \frac{M}{RL}}{M} = \frac{2\sin\theta\cos\theta}{RL}\frac{L^3}{3}$$

$$= \frac{2L\cos\theta}{3} = \boxed{\frac{2H}{3}} \text{ (on solving)}$$

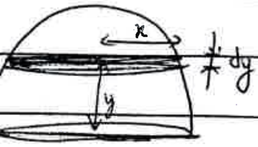
Solid cone – $\rho = \frac{dm}{dV} = \frac{M}{\frac{1}{3}\pi R^2 H} \Rightarrow dm = \frac{3M \cancel{\pi} x^2 dy}{\cancel{\pi} R^2 H}$



$$y_{com} = \frac{\int y\,dm}{M} = \frac{\int y \frac{3\cancel{M} x^2 dy}{R^2 H}}{\cancel{M}} = \frac{3}{H^3}\int_0^h y^3 dy = \boxed{\frac{3H}{4}}$$

$$\frac{x}{R} = \frac{y}{H} \Rightarrow x = \frac{yR}{H}$$

Solid hemisphere → $x^2 + y^2 = R^2$ $dm = \frac{M}{\frac{2}{3}\pi R^3} dV$



$$y_{COM} = \frac{\int y\,dm}{M} = \frac{\int y \frac{M}{\frac{2}{3}\pi R^3}(\pi x^2 dy)}{M} = \frac{M\pi}{\frac{2}{3}\pi R^3 M}\int_0^R y(R^2 - y^2)dy$$

$$= \frac{3}{2R^3}\left[\frac{R^4}{2} - \frac{R^4}{4}\right] = \frac{3}{2R^3}\left[\frac{R^4}{4}\right] = \boxed{\frac{3R}{8}}$$

Centre of mass of composite bodies –

① Find axis of symmetry ② Concentrate mass of each part at its respective COM.
③ Find resultant COM by treating them as point masses.

→ Hollow cone + Hollow hemisphere (σ)

$2\sigma\pi R^2$

$\sigma\pi RL$

$$y_c = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} = \frac{2\sigma\pi R^2 (R/2) + \sigma\pi RL\left(\frac{-L\cos\theta}{3}\right)}{2\pi R^2\sigma + \pi RL\sigma}$$

$$= \frac{R^2 - \frac{L^2\cos\theta}{3}}{2R + L} \text{ [Further simplification]}$$

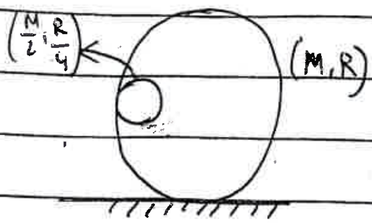
Concept of mass moment →

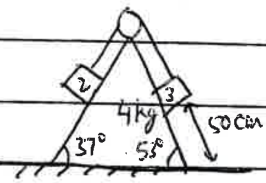
$F_{ext} = Ma_{COM}$ ~~$\Sigma p = mv_{COM}$~~ suppose $V_{COM} = 0$ & $a_{COM} = 0$ [$F_{ext} = 0$]

$s = ut + \frac{1}{2}at^2 \Rightarrow S_{COM} = 0 \Rightarrow S_{COM} = \frac{m_1 s_1 + m_2 s_2 + m_3 s_3 + \dots}{m_1 + m_2 + \dots} = 0$

$m_1 s_1 + \dots = 0$ ms = mass moment

$\Sigma F_{ext} \neq 0$ but it is zero in a particular direction. In that direction, (here x), $m_1 S_{1x} + m_2 S_{2x} + m_3 S_{3x} + \dots = 0$





Find disp. of both spheres/rings Find disp. of wedge.

* $\Sigma F_{ext} = 0$ $[a_{COM} = 0]$ but $V_{COM} \neq 0 \Rightarrow V_{COM}$ = constant

$$\vec{V}_{COM} = \frac{m_1\vec{u}_1 + m_2\vec{u}_2 + \dots}{m_1 + m_2 + \dots} \Rightarrow m_1\vec{u}_1 + m_2\vec{u}_2 + m_3\vec{u}_3 + \dots = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \dots$$

[Law of conservation of momentum]

In absence of net external force, momentum of system is conserved. $\Rightarrow \Sigma p_i = \text{const}$

* COM frame of reference -

$m_1 \longrightarrow u_1$ $m_2 \longrightarrow u_2$

$$v_c = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

$$v_{1c} = v_1 - v_c = \frac{m_2(v_1 - v_2)}{m_1 + m_2}$$

$$v_{2c} = v_2 - v_c = \frac{m_1(v_2 - v_1)}{m_1 + m_2}$$

$$p_{1c} = m_1 v_{1c} = \frac{m_1 m_2 (v_1 - v_2)}{m_1 + m_2} \qquad p_{2c} = \frac{m_2 m_1 (v_2 - v_1)}{m_1 + m_2}$$

$$p_{1c} = -p_{2c} \Rightarrow \boxed{\Sigma p_c = 0}$$

$$K_c = \frac{1}{2} m_1 v_{1c}^2 + \frac{1}{2} m_2 v_{2c}^2 = \boxed{\frac{1}{2}\frac{m_1 m_2}{m_1 + m_2} u_{rel}^2}$$

$$\boxed{\mu = \frac{m_1 m_2}{m_1 + m_2} = \text{reduced mass}}$$

$$K_G = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$= \frac{1}{2} m_1 (v_{1c} + v_c)^2 + \frac{1}{2} m_2 (v_{2c} + v_c)^2$$

$$= \underbrace{\frac{1}{2} m_1 v_{1c}^2 + \frac{1}{2} m_2 v_{2c}^2}_{} + \underbrace{\frac{1}{2}(m_1 + m_2) v_c^2}_{} + \underbrace{(m_1 v_{1c} + m_2 v_{2c}) v_c}_{}$$

$$= \frac{1}{2}\mu u_{rel}^2 + \frac{1}{2} M v_c^2 + 0$$

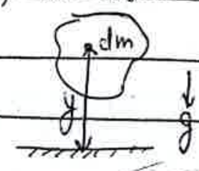
$\therefore$ $K_G = \frac{1}{2}\mu u_{rel}^2 + \underbrace{\frac{1}{2} M v_c^2}_{}$ and $K_c = \frac{1}{2}\mu u_{rel}^2$

$\hookrightarrow$ constant if $F_{ext} = 0$

* If net external force acting on a system is not zero, $\Sigma \vec{F}_{ext} = m_T \vec{a}_c$
* For finding PE of extended body in uniform gravitational field, we can assume mass to be concentrated at COM.

$$U = \int dU = \int dm\, g\, y = g\int y\, dm = g M y_{com} = M g y_{com}$$

$$\left[\because y_{COM} = \frac{\int y\, dm}{M}\right]$$



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Impulse (J) → Impulse = $\int F dt$ = $F\Delta t$ (if F is constant)

~~F is frame-independent, t is frame independent, impulse is frame independent~~

Impulse momentum theorem →

Total impulse of all forces acting on a body is equal to change in its momentum.

$$\Rightarrow \int \vec{F_1} dt + \int \vec{F_2} dt + \int \vec{F_3} dt + \ldots = \Delta \vec{p} = \vec{p_f} - \vec{p_i}$$

Proof → $\vec{F_1} + \vec{F_2} + \vec{F_3} + \ldots = m\vec{a} = m\frac{d\vec{v}}{dt} \Rightarrow \vec{F_1}dt + \vec{F_2}dt + \vec{F_3}dt = m\,d\vec{v} \Rightarrow \int \vec{F_1}dt + \int \vec{F_2}dt + \ldots = m(\vec{v_f} - \vec{v_i})$

* COLLISIONS →

- Types of collisions – $\left[e = \frac{V_B - V_A}{U_A - U_B} = \frac{\text{velocity of separation}}{\text{velocity of approach}} \right.$

1. Elastic collision → No energy loss, $|V_{rel}| = |U_{rel}|$, $e = 1$, $J_{deformation} = J_{reformation}$
2. Partially elastic → Some energy loss, $|V_{rel}| < |U_{rel}|$, $0 < e < 1$, $J_D > J_R$
3. Inelastic collision → Max. energy loss, $V_{rel} = 0$, $e = 0$, $J_R = 0$
   → No reformation [doesn't necessarily mean that they stick to each other]
4. Superelastic collision → Energy is released, $|V_{rel}| > |U_{rel}|$, $e > 1$, $J_R > J_D$

- Energy loss in collision $= \Delta E = \frac{1}{2}\mu u_{rel}^2 - \frac{1}{2}\mu v_{rel}^2 = \frac{1}{2}\mu u_{rel}^2 (1 - e^2)$

From the point of view of impulse, we can categorise forces into 2 categories

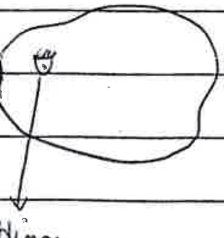
1. Non-impulsive forces → These are the forces for which the value doesn't change all of a sudden. For eg., gravitational force and spring force.
2. Impulsive forces – Are the forces whose values can change suddenly. For eg., tension and normal reaction. Friction can be impulsive if normal

reaction is impulsive and their is tendency of motion relative to surface. For the short duration of impulsive force, the effect of non-impulsive force can be neglected.

ROTATION

(Rig)id body is a body such that distance between any 2 points on the body remains constant. In contrast, a flexible body is one for which the distance between any 2 points may change. In case of a rigid body, the velocity of any point on the body relative to any other point (is) perpendicular to the line joining both of them. So, (w)e can assume that in fixed axis rotation, every point is (m)oving in a circle whose centre is on the axis of rotation. [Any line drawn perpendicular to the axis has the same angular velocity. This is known as **angular velocity** of the rigid body]

(Fo)r rotating a rigid body about the axis of rotation, we need to apply torque on the body. **Torque** is defined as $\vec{r} \times \vec{F}$, here $\vec{r}$ is the position vector of point of application (of) force wrt. to axis of rotation. In case of fixed axis rotation, torque is always along the axis of rotation



Hinge

(A)xis of rotation is $\perp$ to (pl)ane of paper passing through the hinge

$$\text{Torque} = \vec{\tau} = \vec{r} \times \vec{F}$$

$$\Rightarrow \tau = rF\sin\theta$$

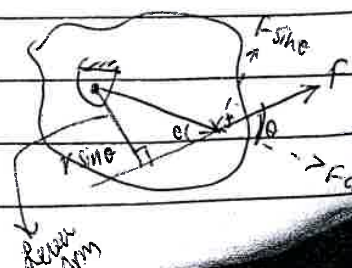
$$\Rightarrow \tau = (r\sin\theta)F = r_{\perp}F$$

$$r_{\perp} = r\sin\theta = \text{lever arm}$$

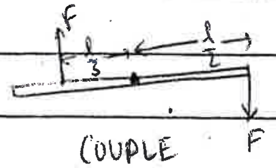
$$\text{Also, } \tau = r(F\sin\theta) = rF_{\perp}$$

$\Rightarrow$

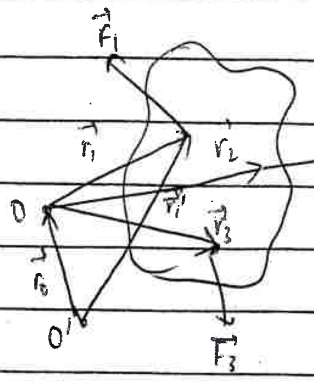
(T)o determine the direction of torque, visualise (th)e sense of rotation of the rigid body (u)nder the influence of that force alone. (Tha)t sense of rotation gives us the direction (of) torque vector (CW or ACW)



Couple - Special situation of torque where net force is zero but net torque is not zero. Often, we apply two equal and opposite forces at two different points.



$F_{net} = 0 \qquad |\tau_{net}| = \frac{5Fl}{6}$



$$\vec{\tau}_O = \vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \vec{r}_3 \times \vec{F}_3 + \dots$$
$$\vec{\tau}_{O'} = \vec{r}_1{}' \times \vec{F}_1 + \vec{r}_2{}' \times \vec{F}_2 + \vec{r}_3{}' \times \vec{F}_3 + \dots$$
$$= (\vec{r}_1 + \vec{r}_0) \times \vec{F}_1 + (\vec{r}_2 + \vec{r}_0) \times \vec{F}_2 + (\vec{r}_3 + \vec{r}_0) \times \vec{F}_3 + \dots$$
$$= \vec{r}_0 \times (\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots) + \vec{\tau}_0$$

0 (in couple, net force = 0)

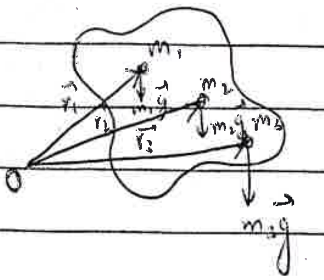
$$\boxed{\vec{\tau}_{O'} = \vec{\tau}_O}$$

Rotational Equilibrium: $\vec{\tau}_{net} = 0$

(generally, question में $\vec{F}_{net} = 0$ भी हो जाता है but it is not necessary)

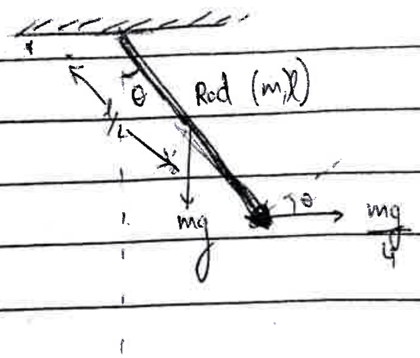
- If $\vec{F}_{net} = 0$, torque about any point in space is same because whenever $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots$ becomes 0, $\vec{\tau}_O = \vec{\tau}_{O'}$

- Torque due to the force in uniform gravitational field can be calculated by assuming the mass at COM.



$$\vec{\tau} = \vec{r}_1 \times m_1\vec{g} + \vec{r}_2 \times m_2\vec{g} + \vec{r}_3 \times m_3\vec{g} + \dots$$
$$= \vec{g}\,(m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + \dots) \times \vec{g}$$
$$= \cancel{\vec{g} \times Mg} \quad M_{sys}\,\vec{r}_{COM} \times \vec{g}$$

Similarly, if we wish to find out torque due to pseudo force ma of a rigid body hinged in a non-inertial frame, pseudo force may be assumed to act at COM.

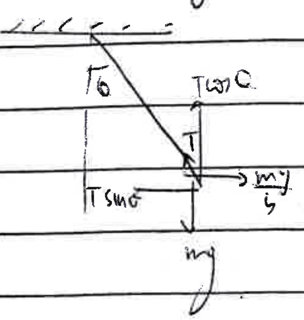


Find θ at eq.

$$mg \sin\theta \frac{l}{2} = \frac{mg}{4} \cos\theta \, l$$

$$\Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{1}{2}\right)}$$ Correct ✓✓✓

If we solve by FBD method (Rope diagram)



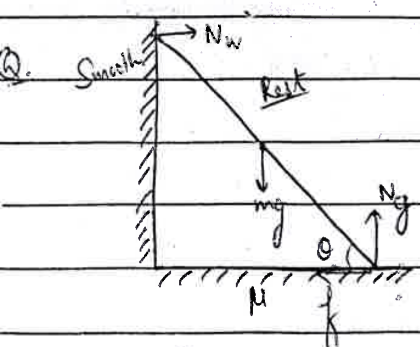
$\theta = \tan^{-1}\left(\frac{1}{4}\right)$. Wrong for rod ✗✗✗

only correct for rope ✓✓

This is because rod has internal shear forces (which are ⊥ to rod) which can't be neglected. Rope has only tension. Rod has tension + shear.

In a rigid body, there are 2 kinds of internal forces at any small cross-section. Forces perpendicular to cross section area are known as tension or compression. Forces parallel to cross-section are known as shear force.

Q.



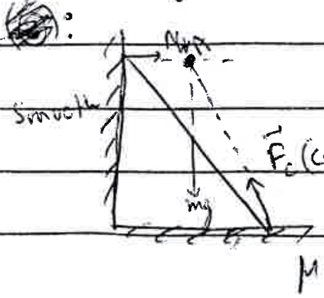
① $f = \frac{mg \cot\theta}{2}$

② $N_g = mg$

③ $N_w = \frac{mg \cot\theta}{2}$

④ $\mu_{min} = \frac{\cot\theta}{2}$

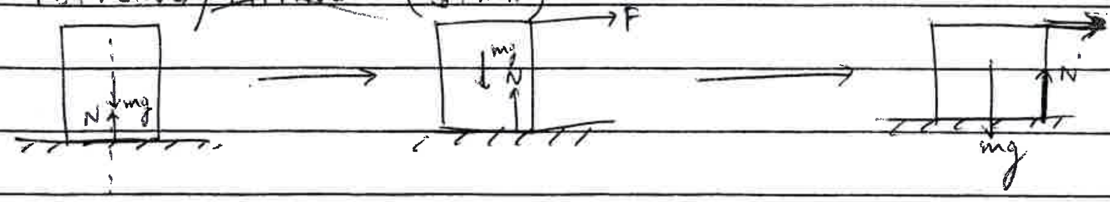
Observation:



All 3 should be concurrent for τ_A to be zero.

• There are 3 forces acting on a body in equilibrium, these 3 forces should be concurrent. (rotational + translational)

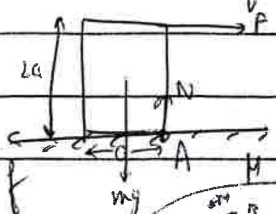
TOPPLING / TIPPING (टिकना)



Verge of toppling

The condition for toppling is that the normal reaction shifts to one of the edges. The body is about to topple about the edge

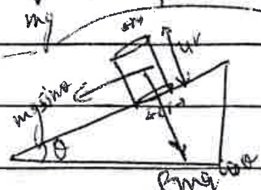
Q. Condition for toppling:



$\tau_A = 2Fa - mg\frac{a}{2} = 0 \Rightarrow \boxed{F = \frac{mg}{4}}$

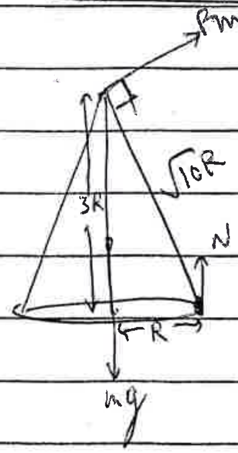
$\mu mg \geq \frac{mg}{4} \Rightarrow \mu \geq \frac{1}{4} \Rightarrow \boxed{\mu_{min} = \frac{1}{4} \text{ for toppling}}$ (not sliding)

Q.



$mg \cancel{\cos}\ \sin\theta(2R) - mg \frac{\cos\theta}{\cancel{\sin\theta}}(R) = 0 \Rightarrow \boxed{\tan\theta = \frac{1}{2}}$

Also, $\mu mg\cos\theta \geq mg\sin\theta \Rightarrow \boxed{\mu_{min} = \tan\theta = \frac{1}{2}}$

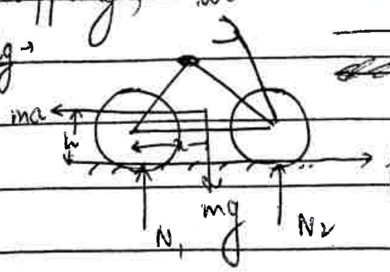


Find min F to topple (not necessarily horizontal).

Ans → $\frac{mg}{\sqrt{10}}$ - F_{min}

F should be perpendicular to slant height of cone

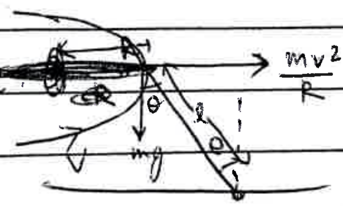
If the body is sliding and also has a tendency for toppling, we should take the torque of pseudo forces as well.



For wheeling ($N_2 = 0$, ~~front~~ wheel lifting off the ground),

$$mah - mgx = 0 \Rightarrow \boxed{a = \frac{gx}{h}}$$

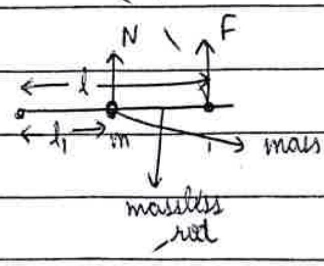
~~Turning~~ ~~of~~ ~~bicycle~~



$$mg\,l\sin\theta = \frac{mv^2}{R}\,l\cos\theta$$

$$\Rightarrow \boxed{\tan\theta = \frac{v^2}{Rg}}$$

MOMENT OF INERTIA - Rotational analog of mass. In the case of rotation, Newton's second law can be written $\Sigma \tau_{ext} = I\alpha$. Here, we find the net ext. torque about the axis of rotation. α is the ang. acc of the rigid body about axis of rotation and I is the MOI about axis of rotation. MOI can be shown to be $\Sigma m_i r_i^2$ where m_i = mass of i^{th} particle and r_i = distance of i^{th} particle from axis of rotation.

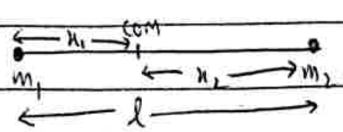


For rod, $Fl = Nl_1$

For mass, $N = m(a_t) = m(l_1\alpha)$ (tangential acc)

$$\tau_{ext} = Fl = Nl_1 = (ml_1\alpha)\,l_1 = ml_1^2\alpha$$

MOI of ~~rod~~ 2 masses about axis through COM



$$I = m_1x_1^2 + m_2x_2^2$$

$$= m_1\frac{m_2^2l^2}{(m_1+m_2)^2} + m_1\frac{m_1^2l^2}{(m_1+m_2)^2}$$

$$= \frac{m_1m_2l^2}{m_1+m_2} = \mu l^2$$

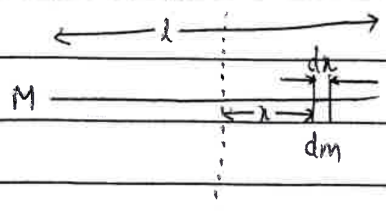
$x_1 = \frac{m_2 l}{m_1+m_2}$, $x_2 = \frac{m_1 l}{m_1+m_2}$

* MOI of continuous mass distribution:

$I = \int r^2 dm$ $\qquad\qquad$ $I = \int dI$

To find the moment of inertia of a continuous mass distribution we should divide entire mass distribution into small elements. The elements should be such that its MOI about the given axis is known. 2 MOIs can be added only if they have the same

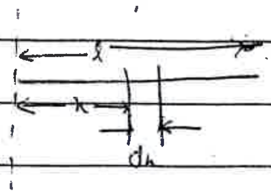
MOI of a thin rod about an ⊥ axis passing through its centre and ⊥ to it.



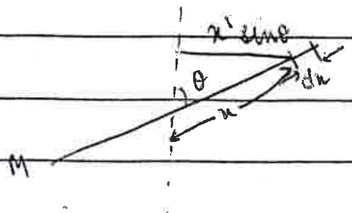
$dm = \lambda\, dx = \frac{M}{l} dx$

$dI = dm\, x^2$

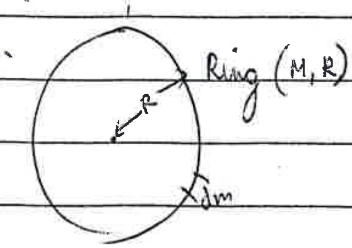
$I = \int dI = \int_{-l/2}^{l/2} \frac{M}{l} dx\, x^2 = \frac{Mx^3}{3l}\Big|_{-l/2}^{l/2} = \frac{Ml^2}{12}$



$I = \int dI = \int_0^l \frac{M}{l} dx\, x^2 = \frac{Mx^3}{3l}\Big|_0^l = \frac{Ml^2}{3}$



$I = \int dI = \int_{-l/2}^{l/2} \frac{M}{l} dx\, (x\sin\theta)^2 = \frac{Mx^3 \sin^2\theta}{3l}\Big|_{-l/2}^{l/2} = \frac{Ml^2}{12}\sin^2\theta$



$I = \int dI = \int R^2 dm = R^2 \int dm = \underline{MR^2}$

when axis is passing

If dimension of rigid body ∥ to a axis of rotation is changed, the expression for MOI remains same.

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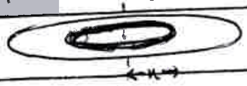
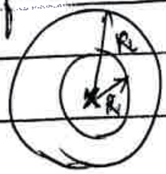
Disc $\quad I = \int dI = \int dm\, x^2 = \int \sigma\, 2\pi x\, dx\, x^2$



$$= 2\pi\sigma \int_0^R x^3\, dx = 2\pi \left(\frac{M}{\pi R^2}\right) \frac{R^4}{4} = \boxed{\frac{MR^2}{2}}$$

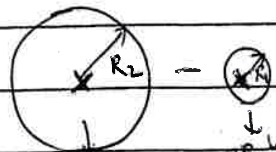
MOI of annular disc



$$I = \int dI = \int dm\, x^2 = \int \sigma\, 2\pi x\, dx\, x^2$$

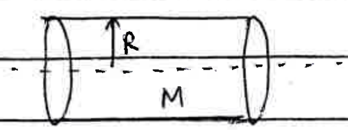
$$= 2\pi\sigma \int_{R_1}^{R_2} x^3\, dx = 2\pi \left(\frac{M}{\pi(R_2^2 - R_1^2)}\right) \frac{(R_2^4 - R_1^4)}{4} = \boxed{\frac{M(R_1^2 + R_2^2)}{2}}$$

Alter :



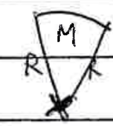
$$I = \frac{\sigma\pi R_2^4}{2} - \frac{\sigma\pi R_1^4}{2} = \frac{\sigma\pi(R_2^4 - R_1^4)}{2} = \frac{M\pi(R_2^4 - R_1^4)}{\pi(R_2^2 - R_1^2)2} = \boxed{\frac{M(R_1^2 + R_2^2)}{2}}$$

Cylinder:



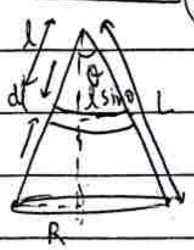
$I = \frac{MR^2}{2}$

Sector:



$I = \frac{MR^2}{2}$

Hollow cone: (Don't compress it into ring)

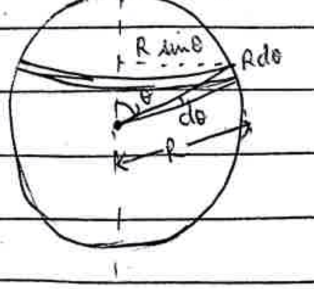


$$dI = dm\,(l \sin\theta)^2 = \sigma\, 2\pi\, l \sin\theta\, dl\, (l\sin\theta)^2$$

$$\Rightarrow I = \int dI = \int \sigma\, 2\pi\, l^3 \sin^3\theta\, dl = 2\pi\sigma \sin^3\theta \frac{L^4}{4}$$

$$= 2\pi \frac{M}{\pi R L} \sin^3\theta \frac{L^4}{4} = \boxed{\frac{1}{2}MR^2}$$

Spherical shell / hollow ~~hemi~~sphere



$$dI = dm(R \sin\theta)^2 = \sigma(R\sin\theta)^3 R\, d\theta\, (2\pi)$$

$$I = \int dI = 2\pi\sigma\, R^4 \int_0^\pi \sin^3\theta\, d\theta$$

$$= 2\pi \left(\frac{M}{4\pi R^2}\right) R^4 \left(\frac{4}{3}\right)$$

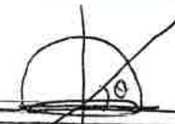
$$= \boxed{\frac{2}{3}MR^2}$$

$\int_0^\pi \sin^3\theta\, d\theta = \frac{4}{3}$

~~Do~~ by $\sin 3\theta$:

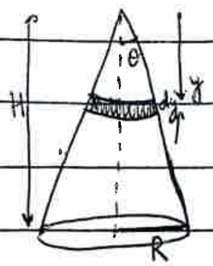
$\frac{3\sin\theta - \sin 3\theta}{4}$

or put ~~tan~~ $\cos\theta = t$



MOI of hemispherical shell about any diameter is $\frac{2}{3}MR^2$.

Solid cone:

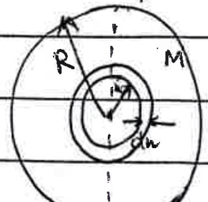


$$dI = \cancel{dm}\ \frac{dm\, r^2}{2} = \frac{\rho \pi r^2 dy\, r^2}{2} = \frac{\rho\pi}{2}(y\tan\theta)^4 dy$$

$$I = \int dI = \frac{\rho\pi}{2}\tan^4\theta \int_0^H y^4 dy = \frac{\rho\pi}{2}\tan^4\theta\, \frac{H^5}{5} = \frac{M\pi}{\frac{1}{3}\pi R^2 H \times 2}\tan^4\theta\,\frac{H^5}{5}$$

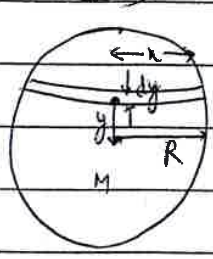
$$= \frac{3}{10}\frac{MH^4\tan^4\theta}{R^2} = \frac{3MR^4}{10R^2} = \boxed{\frac{3}{10}MR^2}$$

Solid sphere:



$$dI = \frac{2}{3}(\rho\, 4\pi x^2 dx)\, x^2$$

$$I = \int dI = \frac{8\pi\rho}{3}\int_0^R x^4 dx = \frac{8\pi}{3}\frac{M}{\frac{4}{3}\pi R^3}\times\frac{R^5}{5} = \boxed{\frac{2MR^2}{5}}$$



$$y^2 + x^2 = R^2 \Rightarrow x^2 = R^2 - y^2$$

$$dI = \cancel{\tfrac{1}{2}dm}\ \frac{1}{2}\pi x^2 dy\, x^2 \rho = \frac{1}{2}\rho\pi(R^2-y^2)^2 dy$$

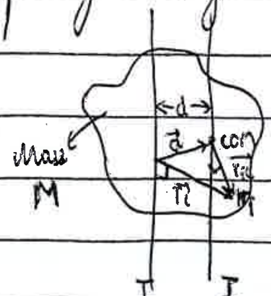
$$I = \int dI = \frac{\rho\pi}{2}\int_{-R}^{R}(R^4 - 2R^2y^2 + y^4)\,dy = \frac{\rho\pi}{2}\left(\frac{16R^5}{15}\right)$$

$$= \frac{M}{\frac{4}{3}\pi R^3}\frac{\pi}{2}\left(\frac{16R^5}{15}\right) = \boxed{\frac{2}{5}MR^2}$$

Radius of gyration: (k) $\quad I = Mk^2$

Eg: $I_{shell} = \frac{2}{3}MR^2 = Mk^2 \Rightarrow \boxed{k = R\sqrt{\frac{2}{3}}}$

Parallel axis theorem: Out of the many different axes parallel to each other, the moment of inertia about the axis passing through COM is minimum.



$$\vec{d} + \vec{r}_{ic} = \vec{r}_i \qquad I_c = \sum m_i r_{ic}^2$$

$$I = \sum m_i r_i^2 = \sum m_i (\vec{d} + \vec{r}_{ic})^2 = \sum m_i d^2 + \sum m_i 2\vec{d}\cdot\vec{r}_{ic} + \sum m_i r_{ic}^2$$

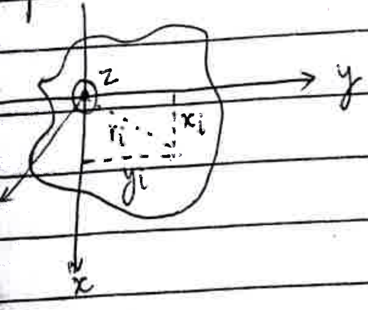
$$= Md^2 + 0 + I_c$$

$$\boxed{I = I_c + Md^2}$$

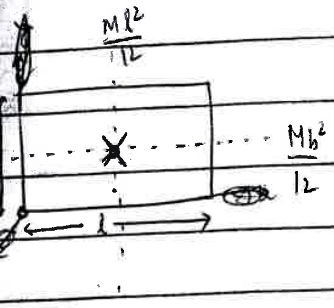
30

$\boxed{I = I_c + Md^2}$ (No assumption, no restriction, always valid)

Perpendicular axis theorem: applicable only for planar bodies

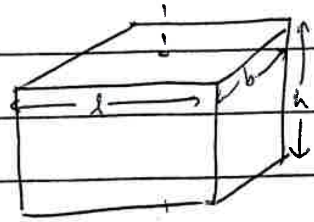


$$I_x = \sum m_i y_i^2$$
$$I_y = \sum m_i x_i^2$$
$$I_z = \sum m_i r_i^2 = \sum m_i (x_i^2 + y_i^2)$$
$$= \sum m_i x_i^2 + \sum m_i y_i^2$$
$$\boxed{I_z = I_x + I_y}$$



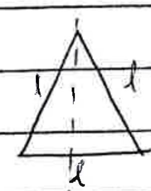
I_z of rectangle $= \dfrac{M(l^2+b^2)}{12}$

MOI of cuboid

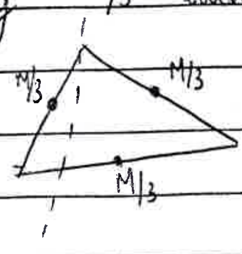


$I = \dfrac{M(l^2+b^2)}{12}$ (extended parallel to axis)

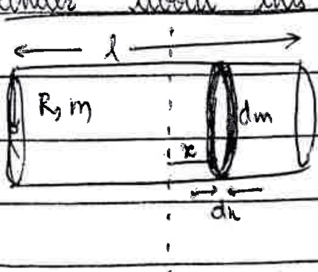
MOI of triangular plate (equilateral) $= \dfrac{ml^2}{24}$ about ~~an~~ median axis.



MOI of any uniform triangular plate can be found be concentrating M/3 each at the mid-points of their sides (any axis).

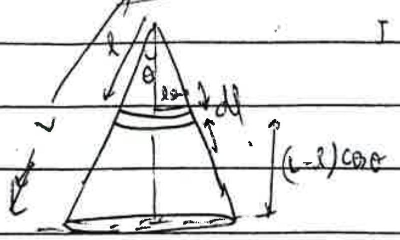


Cylinder about this axis:



$$dm = \frac{m\,dx}{l}$$
$$I = \int dI = \int \frac{dm R^2}{2} + \int dm\, x^2$$
$$= \int_{-l/2}^{l/2} \frac{m\,dx\,R^2}{2l} + \int_{-l/2}^{l/2} \frac{m\,dx}{l} x^2 = \frac{mR^2}{2} + \frac{ml^2}{12}$$

Hollow cone about diameter (axis):

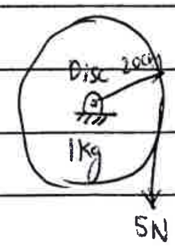


$$I = \int dI = \int dm\,(L-l)^2 \cos^2\theta + \int \frac{1}{2}\, dm\, l^2 \sin^2\theta$$

Fixed axis rotation: $\sum \tau_{ext} = I\alpha$

↳ about axis of rotation.

Eg →

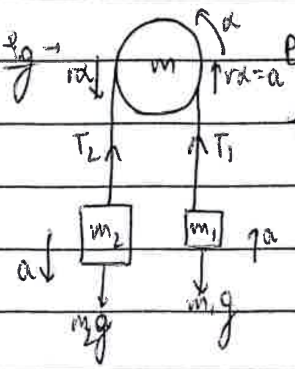


$\tau = 5 \times 0.2 = 1$ N·m $\quad I = \frac{1}{2} \times 1 \times \frac{1}{5} \times \frac{1}{5} = \frac{1}{50}$ kg m²

$$\boxed{\alpha = \frac{1}{1/50} = 50 \text{ rad/s}^2}$$

rallel

Eg →



Pulley is rough, has mass m, string doesn't slip over pulley

$m_2 g - T_2 = m_2 a$

$T_1 - m_1 g = m_1 a$

$(T_2 - T_1)\, r = \frac{1}{2} m r^2 \alpha$

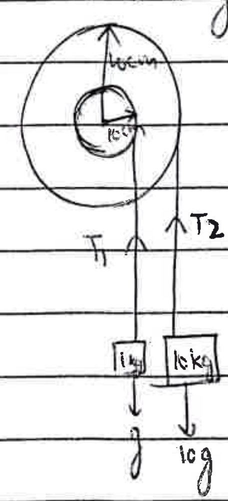
$T_2 - T_1 = \frac{ma}{2}$

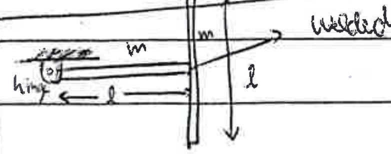
$(m_2 - m_1) g = \left(m_1 + m_2 + \frac{m}{2}\right) a$

$\Rightarrow a = \frac{(m_2 - m_1) g}{\left(m_1 + m_2 + \frac{m}{2}\right)}$

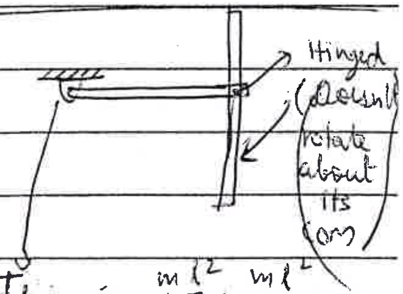
Eg →

$I = 1$ kg m²



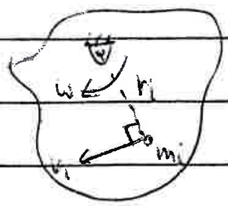


$I_{hinge} = \left(\frac{ml^2}{3}\right) + \left(ml^2 + \frac{ml^2}{12}\right) = \boxed{\frac{17ml^2}{12}}$



$I_{hinge} = \frac{ml^2}{3} + ml^2 = \frac{4ml^2}{3}$

Rotational KE ⇒



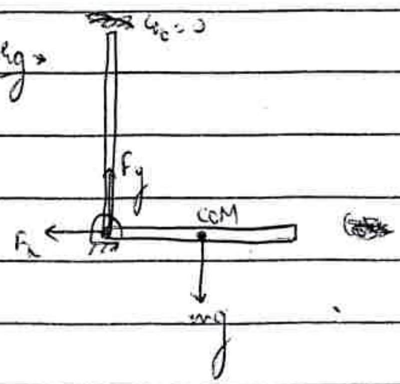
$$k_i = \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_i (\omega r_i)^2 = \frac{1}{2} m_i r_i^2 \omega^2$$

$$K = \sum k_i = \frac{1}{2}\omega^2 \sum m_i r_i^2$$

$$\therefore \boxed{K = \frac{1}{2} I\omega^2}$$

Force exerted by hinge ⇒ The forces exerted by the hinge can't be predicted directly. We need to use Newton's laws to find the force exerted by the hinge. According to newton's law, net force acting on any body = mass × a_{COM}.
In case of pure rotation, COM moves in a circular path. In circular path, COM has two acceleration: tangential and radial. Resolving all the forces around these two directions gives us the forces exerted by the hinge.

Eg ⇒



$$mg\frac{l}{2} = \frac{1}{3}ml^2 \alpha \Rightarrow \alpha = \frac{3g}{2l}$$

$$mg\frac{l}{2} = \frac{1}{2}\frac{1}{3}ml^2\omega^2 \Rightarrow \sqrt{\frac{3g}{l}} = \omega$$

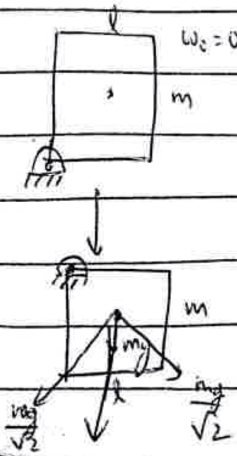
$$F_x = m\omega^2 r_c$$

$$\Rightarrow F_x = m\frac{3g}{l}\frac{l}{2} \quad \boxed{\frac{3mg}{2}}$$

$$F_y - mg = \Rightarrow ma_t = mr_c\alpha$$

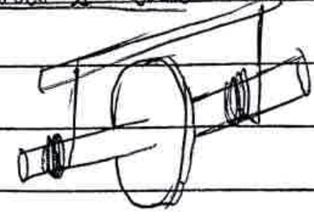
$$\Rightarrow F_y = -m\frac{l}{2}\frac{3g}{2l} + mg = \boxed{\frac{mg}{4}}$$

Eg ⇒



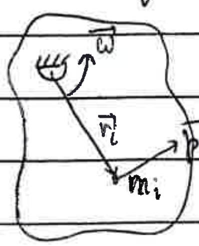
$$\omega = \sqrt{\frac{3g}{l}}, \quad \alpha = \frac{3g}{4l}$$

Maxwell's disc:



→ (Moment of momentum)

Angular momentum: Angular momentum is the rotational analog of momentum. For a point particle, angular momentum is defined about a point as $\vec{L} = \vec{r} \times \vec{p}$. $\vec{r}$ is the position vector of the particle w.r.t. to the point of observation.



$$\vec{p_i} = m_i \vec{v_i} = m_i (\vec{\omega} \times \vec{r_i})$$

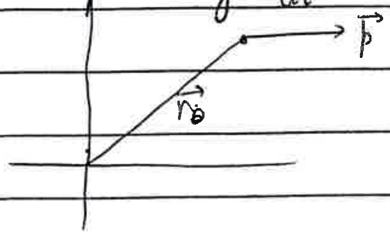
$$\vec{L} = \vec{r} \times \vec{p} \Rightarrow \vec{L_i} = m \vec{r_i} \times (\vec{\omega} \times \vec{r_i}) = m \vec{r_i}^2 \vec{\omega} - m(\vec{r_i} \cdot \vec{\omega}) \vec{r_i}$$

$$\Rightarrow \vec{L_i} = m_i r_i^2 \vec{\omega}$$

$$\Rightarrow \vec{L} = \Sigma \vec{L_i} = \left(\Sigma m_i r_i^2\right) \vec{\omega} = I\vec{\omega}$$

$$\boxed{\vec{L}_{\text{about axis of rotation}} = I \vec{\omega}}$$

For point body, $\frac{d\vec{L}}{dt} = \vec{\tau}$



$$\vec{L} = \vec{r} \times \vec{p} \Rightarrow \frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

$$\Rightarrow \frac{d\vec{L}}{dt} = \vec{v} \times m\vec{v} + \vec{r} \times \vec{F}_{ext}$$

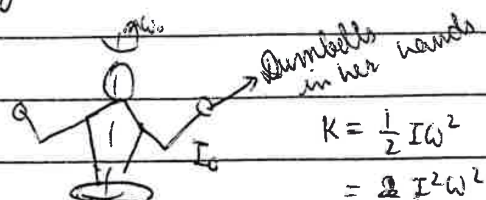
$$\Rightarrow \boxed{\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}_{ext} = \vec{\tau}}$$

For rigid body, $\vec{\tau} = \frac{d\vec{L}}{dt}$

$$\hookrightarrow \vec{L} = I\vec{\omega} \Rightarrow \frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} \Rightarrow \frac{d\vec{L}}{dt} = I\vec{\alpha} \Rightarrow \boxed{\frac{d\vec{L}}{dt} = \vec{\tau}}$$

Law of conservation of angular momentum:

It states that if the net external torque acting on a system is zero, its angular momentum should be conserved.



$\tau_{ext} = 0$

$L = \text{const}$

$\Rightarrow I_0 \omega_0 = I\omega$

$\Rightarrow I\downarrow \ \omega\uparrow$

$$K = \frac{1}{2} I\omega^2 = \frac{I^2\omega^2}{2I} = \frac{L^2}{2I}$$

$I\downarrow, K\uparrow$

As a 'figure skater' pulls her arms inwards, KE of system increases although L remains conserved. As seen from the figure skater's frame of reference the centrifugal force tends to push the dumbells outwards. So she has to work against the centrifugal force and this work increases KE.

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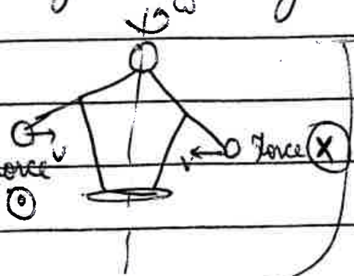
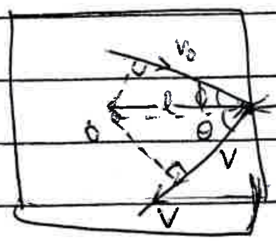
Similarly, from the girls frame of reference, when she moves the dumbells inwards, she experiences a coriolis force $\vec{F}_{cor} = 2m\vec{V}_{rel} \times \vec{\omega}$. The forces on these dumbells are in opposite direction. These constitute a couple which exert a torque to increase the angular velocity. ~~To find a fo~~



To find the point (axis) about which L is conserved, we have to find a point about which the torque is zero. The angular momentum will be conserved about that point.



$L_i = mv_0 l \sin\phi$

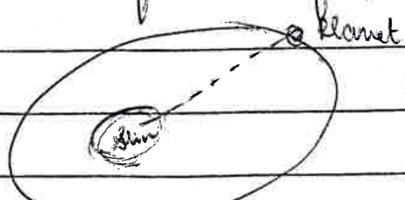
$L_f = mvl \sin\theta$

$\because l\sin\phi = l\sin\theta$ (we know)

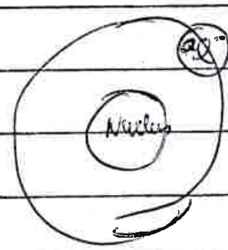
$\Rightarrow \boxed{L_i = L_f}$

Angular momentum about the centre of a 'central force' is always conserved. In case of central force, L of the system about the centre of the force will always remain conserved. eg:

central force $= f(r)\cdot\hat{r}$



$L_{sun} = $ const

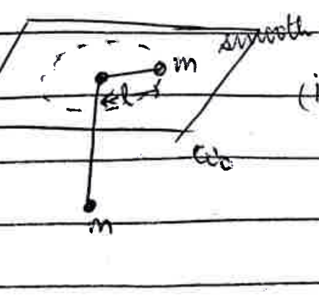


$L = \frac{nh}{2\pi} = mvr$

$=$ constant

Q: (i) Find 'l' ~~to~~ so that particle goes in circular motion.

(ii) If half of the hanging mass drops off, find the max distance of particle of table from hole.



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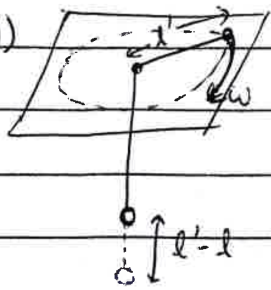
(i) $T = mg$ and $T = m\omega_0^2 \ell$ $\Rightarrow$ $\boxed{\ell = \frac{g}{\omega_0^2}}$

3

(ii) 

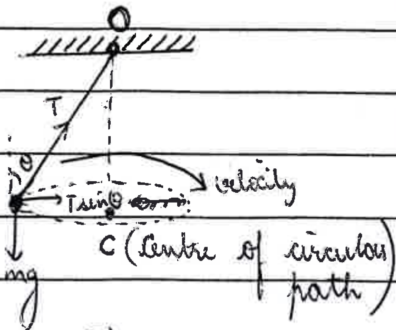
L conservation about centre,

$$m\ell^2\omega_0 = m\ell'^2\omega$$

Energy conservation,

$$\frac{1}{2}m\ell^2\omega_0^2 = \frac{1}{2}m\ell'^2\omega^2 + \frac{mg}{2}(\ell'-\ell)$$

solving these 2, we get $\boxed{\ell' = \ell\left(\frac{1+\sqrt{5}}{2}\right)}$



$\vec{T_0} = \vec{r} \times \vec{mg}$

= horizontal and in a direction opp. to veloc

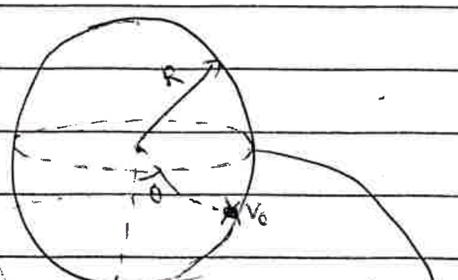
$\Rightarrow \tau_{vertical} = 0$

$\Rightarrow L_v =$ constant

$L = mv\ell[-\cos\theta\,\hat{i} - \sin\theta\,\hat{j}]$

$\vec{\tau_C}$ = constant

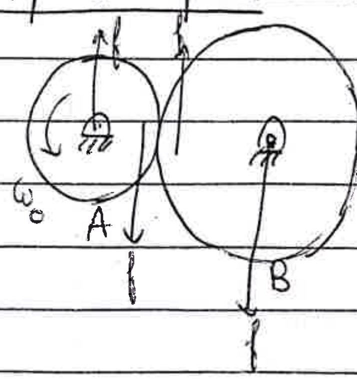
There are some particular situations when L in a particular direction is conserved. So we resolve initial angular momentum and L final in that direction and apply law of cons of L.

Q. 

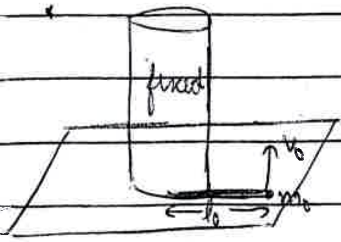
Find v_0 so that it rises only upto ~~diam~~ (horizontal diameter level plan)

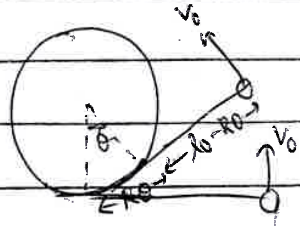
Ans = $v_0 = \sqrt{\frac{2gR}{\cos\theta}}$

Important points:



L_{system} is not conserved





$L_i = mv_0 l_0$

$L_f = mv_0(l_0 - R\theta)$

$$\frac{dL}{dt} = -mv_0 R\frac{d\theta}{dt}$$

$$\omega = \frac{v_0}{l_0 - R\theta}$$

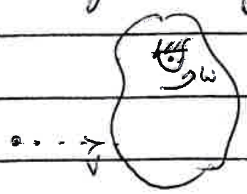
$$T = \frac{dL}{dt} = \frac{-mv_0^2 R}{l_0 - R\theta}$$

KE = constant

L is not conserved (because ground exerts a couple on cylinder to prevent it from rotating due to tension's torque).

Collision of rigid body fixed about an axis with a point particle:

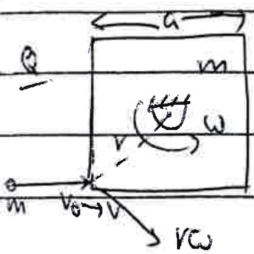
In general, $p \neq$ constant (due to impulse exerted by hinge during collision).



① L_{hinge} = const.

② $e = \dfrac{v_{rel} \text{ in normal direction}}{u_{rel} \text{ in normal direction}}$ (of point of contact)

Q



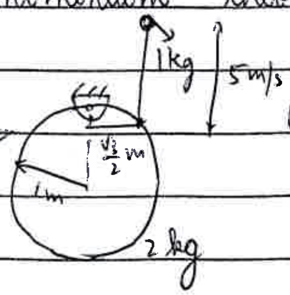
$L_i = L_f \Rightarrow mv_0\frac{a}{2} = mv\frac{a}{2} + \frac{ma^2}{6}\omega$

and $e = 1 \Rightarrow \frac{\omega a}{2} - v = v_0$ $\longrightarrow$ $v = v_0/5$ and $\omega = \frac{12 v_0}{5a}$

$J_{hinge} = p_f - p_i = mv + m\times 0 - mv_0 = \frac{-4mv_0}{5}$

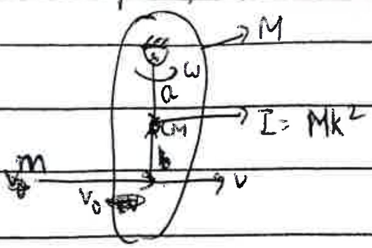
To find impulse exerted by hinge, we should apply impulse-momentum theorem with the momentum of rigid body as MV_{cm}.

Q.



Particle collides and sticks. Find ω after.

Sweet spot / centre of percussion:



Impulse $= 0 \Rightarrow p_i = p_f \Rightarrow mv_0 = mv + M\omega a$

$L_i = L_f \Rightarrow mv_0(a+b) = mv(a+b) + M(k^2+a^2)\omega$

$\Rightarrow m(v_0 - v) = M\omega a$

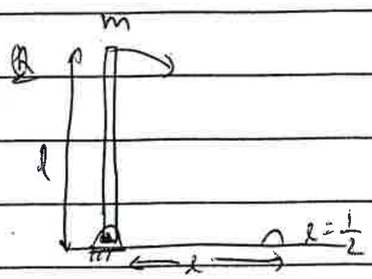
and $m(v_0 - v)(a+b) = M(k^2+a^2)\omega$

Dividing, $a + b = \dfrac{(k^2+a^2)}{a} \Rightarrow \boxed{k^2 = ab}$

Angular impulse:

$\vec{H} = \int \vec{\tau}\cdot dt$ $\qquad \vec{H} = \vec{\tau}\,\Delta t$ (if $\vec{\tau}$ is constant)

$\Sigma \vec{\tau}_{ext} = \dfrac{d\vec{L}}{dt} \Rightarrow \int d\vec{L} = \Sigma \int \vec{\tau}_{ext}\, dt \Rightarrow \Delta \vec{L} = \Sigma \vec{H}$



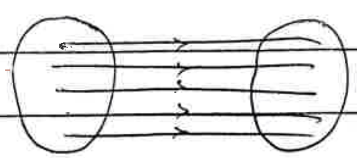
Work done by torque: (when couple is there)

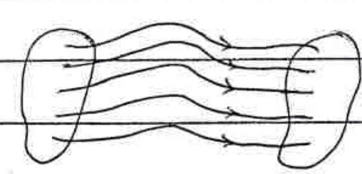
$W = \int \tau\, d\theta$ $\qquad$ if $\tau =$ const., $W = \tau\theta$

Planar motion:

5 kinds of motion for a rigid body:

① Rectilinear translation ② Curvilinear translation

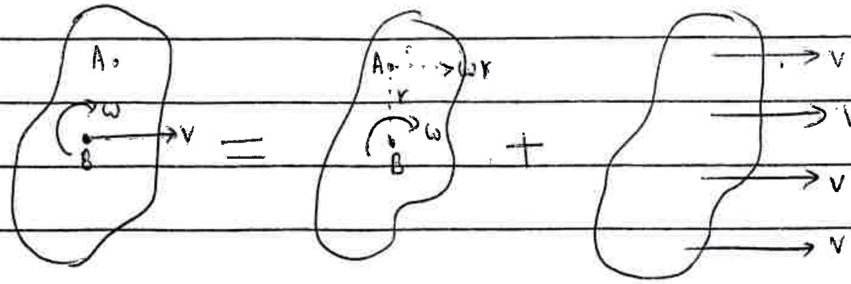




③ Pure rotation: All points are circular paths whose centres lie on same axis.

④ Planar motion: A rigid body is said to undergo planar motion if motion of every point is confined to a plane and all these planes are parallel to each other

(5) Precession: When a rigid body is rotating about an axis which is rotating about another fixed axis. eg → Precession.



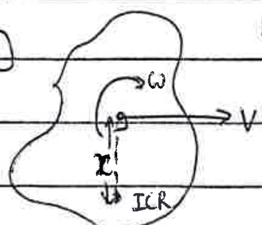
$$\vec{V_{AB}} = \vec{V_A} - \vec{V_B}$$
$$\vec{V_A} = \vec{V_{AB}} + \vec{V_B}$$

$\vec{V_{AB}}$ → Circular wrt. B

$\vec{V_B}$ → translation

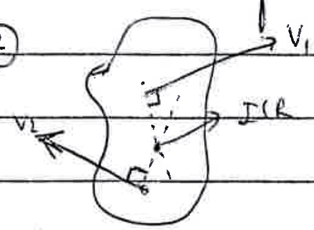
Instantaneous centre of rotation is an imaginary point inside or outside the body whose velocity is zero at that instant. The whole body will seem to rotate purely about that point at that instant. (1) If the velocity of any point on the rigid body is known and the angular velocity is known, draw a ⊥ on the velocity vector. Consider that side of the perpendicular where ω has opposite sense to that of the velocity. The ICR ~~be~~ will lie at the side of the ⊥ at a distance of V/ω.

(1)

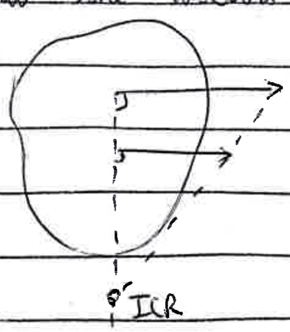


$$V - x\omega = 0$$
$$\Rightarrow \boxed{x = \frac{V}{\omega}}$$

(2)



(3) If the velocity of two points are known and are parallel. Draw the vectors to scale and draw their common perpendicular.



Rolling:

Rolling: Pure rolling motion means point of contact doesn't slip w.r.t. the surface.

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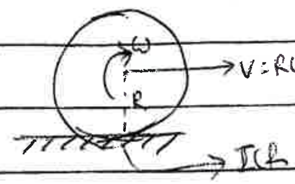
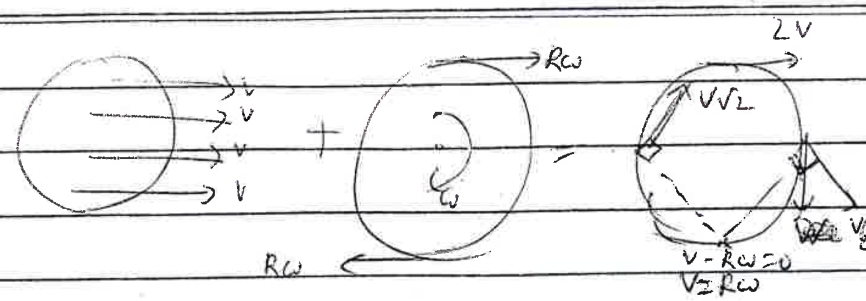
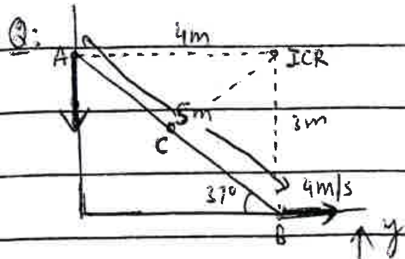


$V_{rel.\ to\ surface} = R\omega$



Q:

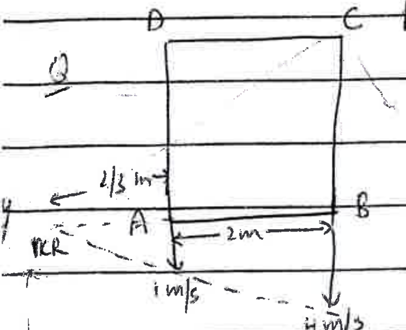


(i) $\vec{\omega} = \frac{v_B}{r_B} = \frac{4}{3}\ rad/s\ \hat{k}$

(ii) $\vec{V_A} = \vec{\omega} \times \vec{r_A} = \frac{16}{3}\ m/s\ (-\hat{j})$

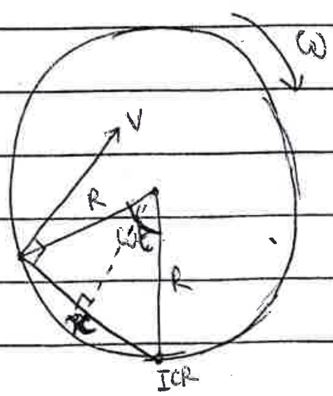
(iii) $\vec{V_C} = 2.5\omega = \frac{10}{3}(\cos 53\,\hat{i} - \sin 53\,\hat{j}) = \left(2\hat{i} - \frac{8}{3}\hat{j}\right) m/s$

Q



① $\vec{\omega} = \frac{1}{2/3}$ or $\frac{4}{8/3} = 1.5\ rad/s\ (-\hat{k})$

② $\vec{v_C} = \frac{10}{3}\omega = 5\left(\frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}\right) = 3\hat{i} - 4\hat{j}$



$V = x\omega = 2R\sin\frac{\omega t}{2}\,\omega = 2R\omega\sin\frac{\omega t}{2}$

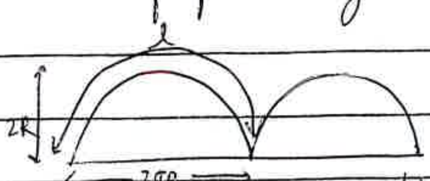
$\vec{V} = 2R\omega\sin\frac{\omega t}{2}\left[\sin\frac{\omega t}{2}\,\hat{i} + \cos\frac{\omega t}{2}\,\hat{j}\right]$

$\vec{v} = R\omega\left[(1-\cos\omega t)\hat{i} + \sin\omega t\,\hat{j}\right]$

$\frac{dx}{dt} = R\omega(1-\cos\omega t) \Rightarrow x = R(\omega t - \sin\omega t)$

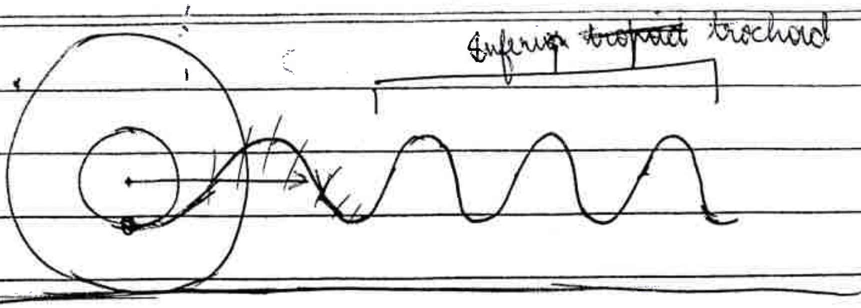
$\frac{dy}{dt} = R\omega\sin\omega t \Rightarrow y = R(1-\cos\omega t)$

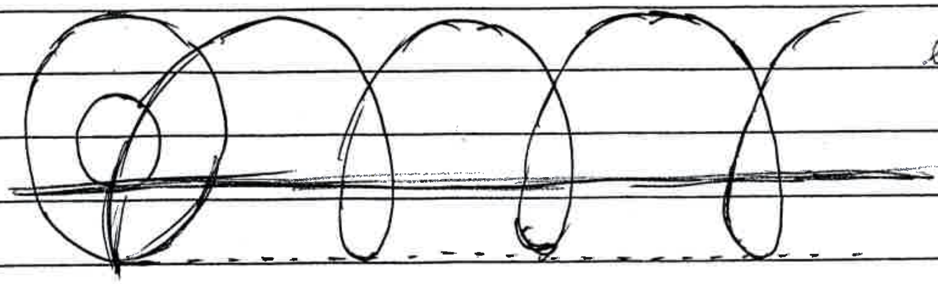
Path of point: Cycloidal path

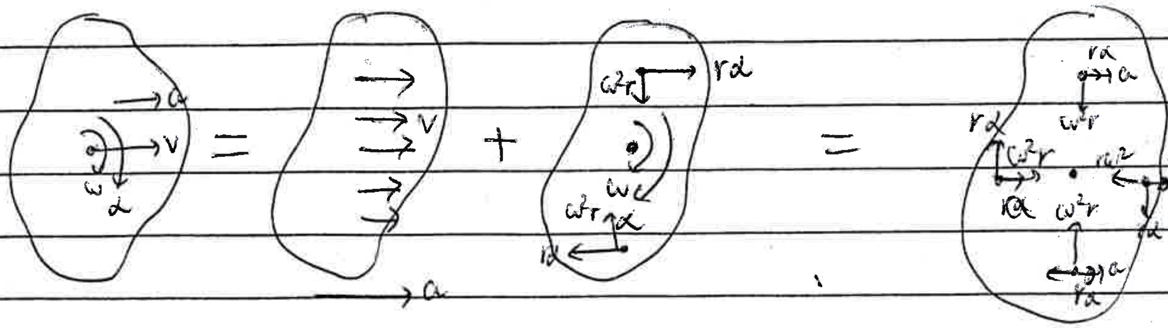


$l = \int_0^{2\pi/\omega} v\,dt = \int_0^{2\pi/\omega} 2R\omega\sin\frac{\omega t}{2}\,dt = 4R\left(-\cos\frac{\omega t}{2}\right)\Big|_0^{2\pi/\omega} = \underline{\underline{8R}}$

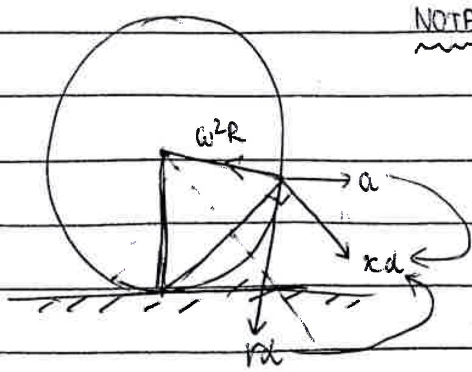
~~Inferior trochoid~~ trochoid







To find the acceleration of any point on the rolling body, find the distance of that point from ICR. If the distance is x, one part of acceleration is $x\alpha$ which is in direction ⊥ to x. Another part of acc is $\omega^2 R$ which is towards the centre.



NOTE: These are not radial and tangential accelerations. "Radial/Normal" and "Tangential" are defined w.r.t to the path of the particle, which is cycloidal and not circular here.

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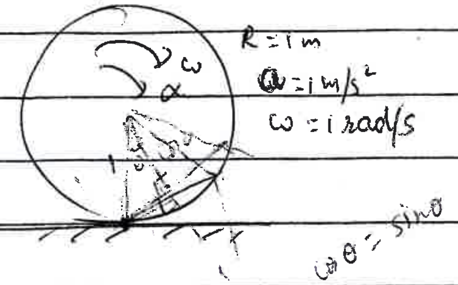
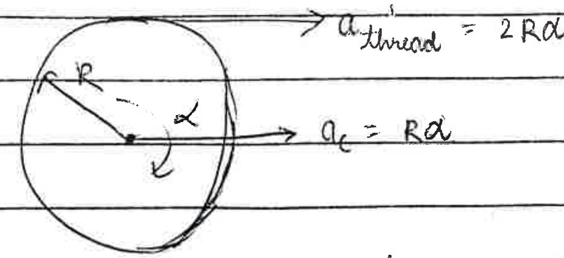
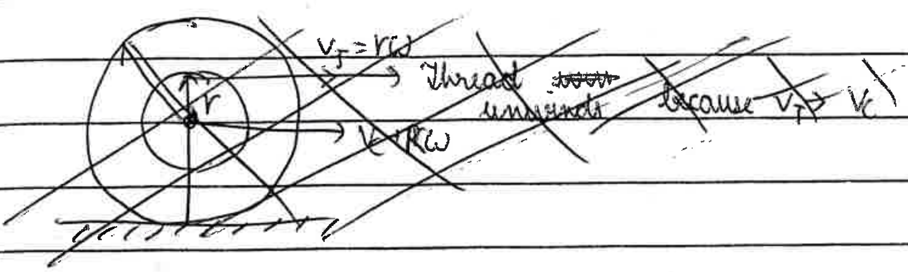
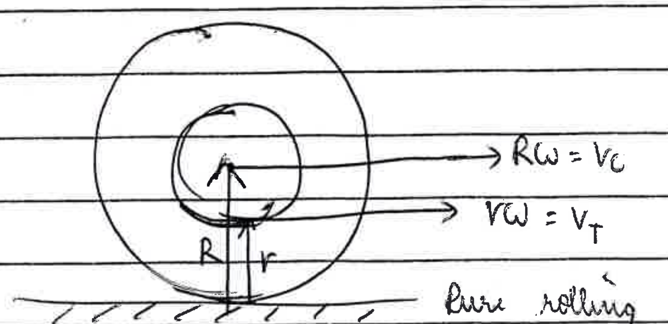
Q. Find point where acc is 0.



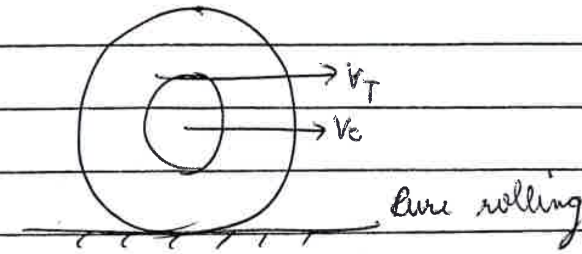




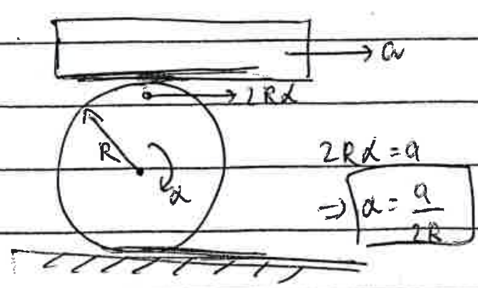


$V_C > V_T \Rightarrow$ Thread winds

(In real, there is impure rolling)

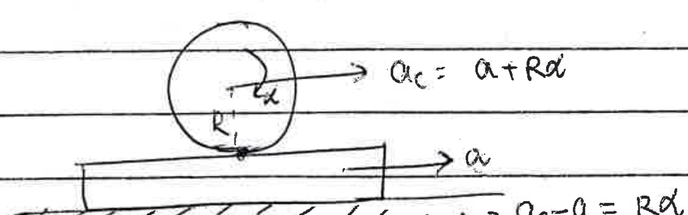


$V_T > V_C \Rightarrow$ Thread unwinds



$2R\alpha = a$

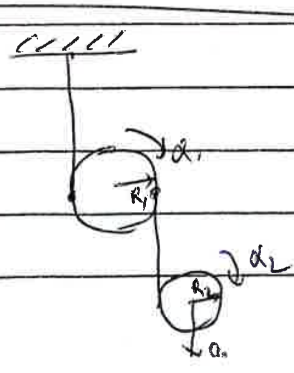
$\Rightarrow \boxed{\alpha = \frac{a}{2R}}$



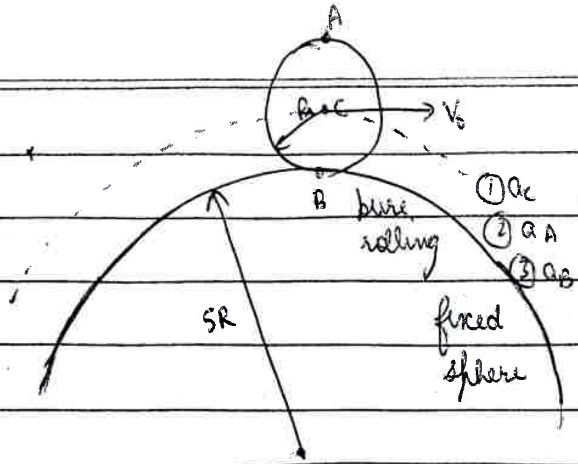
$a_{rel} = a_C - a = R\alpha$

$\Rightarrow a_C = a + R\alpha$

Q. Find relation among $R_1, \alpha_1, a, \alpha_2, R_2$ & a_2



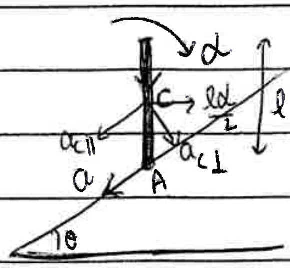
Ans: $\boxed{2\alpha_1 R_1 + R_2\alpha_2 = a_2}$



① $a_C = \frac{v_0^2}{6R} \downarrow$ (circular path with $r = 6R$)

② $a_A = a_{AC} + a_C = \frac{v_0^2}{R} + \frac{v_0^2}{6R} = \frac{7v_0^2}{6R} \downarrow$

③ $a_B = a_{BC} + a_C = \frac{v_0^2}{R}\uparrow + \frac{v_0^2}{6R}\downarrow = \frac{5v_0^2}{6R} \uparrow$

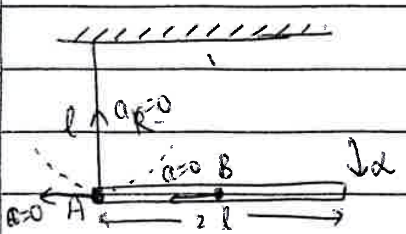


Find relation between acc. of centre and angular acceleration.

Relative to point A, C has acc $\frac{l\alpha}{2}$

$a_{C\parallel}$ (rel A) $= \frac{l\alpha}{2}\cos\theta \Rightarrow \boxed{a_{C\parallel} = a - \frac{l\alpha\cos\theta}{2}}$

$a_{C\perp}$ (rel A) $= \frac{l\alpha}{2}\sin\theta \Rightarrow \boxed{a_{C\perp} = \frac{l\alpha\sin\theta}{2}}$



Find relation between a_A and a_B in terms of α.

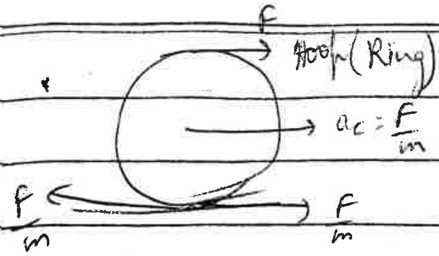
$a_A = 0$

$a_B = l\alpha \downarrow$

Dynamics of Rolling:

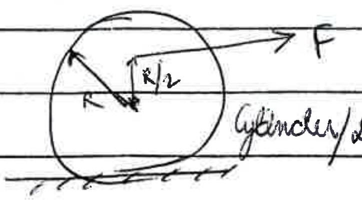
Step-1: Draw the FBD of the rolling bodies concerned.

To find the direction of friction, assume the surface to be frictionless. With this assumption, find the direction of motion of point of contact relative to the surface. If the point of contact moves in forward direction relative to surface, friction is backward and vice-versa.



Also $FR = MR^2\alpha \Rightarrow R\alpha = \frac{F}{M}$

No ~~accelera~~ slipping of point of contact.
$\Rightarrow$ No friction



Cylinder/Disc $\Rightarrow$ No friction.

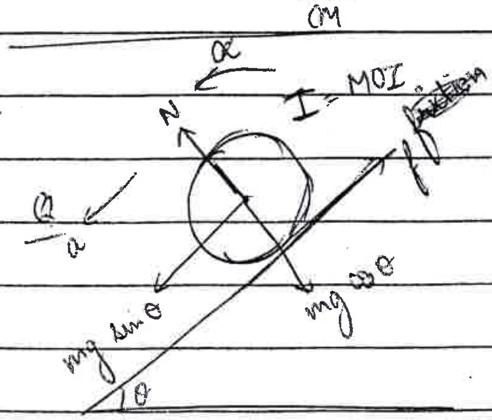
Step-2 : Find the relation between acceleration of the centre (COM) and the angular acceleration.

Step-3 : $\sum \vec{F}_{ext} = M\vec{a}_{CM}$

~~Step-4: $\sum \vec{\tau}_{ext}$ about~~

Step-4 : $\sum \vec{\tau}_{ext\,\text{about CM}} = I_{CM}\,\alpha$

Pure rolling, uniform body.
$\sum \vec{\tau}_{ICR} = I_{ICR}\,\alpha$



$a = R\alpha$

$mg\sin\theta - f = ma$

$fR = I\alpha = \frac{Ia}{R} \Rightarrow f = \frac{Ia}{R^2}$

$\Rightarrow mg\sin\theta = ma + \frac{Ia}{R^2}$

$\Rightarrow \boxed{a = \frac{g\sin\theta}{1 + I/MR^2}}$

$\Rightarrow \boxed{f = \frac{Ia}{R^2} = \frac{mg\sin\theta}{1 + \frac{MR^2}{I}}} \le \mu_s N = \mu_s mg\cos\theta$

$\Rightarrow \boxed{\mu_s \ge \frac{\tan\theta}{1 + \frac{MR^2}{I}}}$

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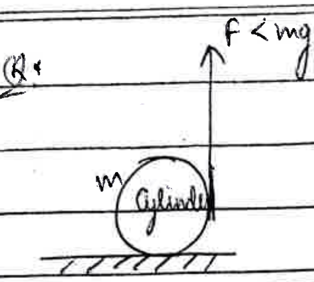
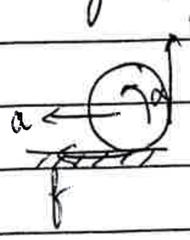
Q.



$$\left.\begin{array}{l} ① \ a \\ ② \ f \\ ③ \ \mu_{min} \end{array}\right\} \text{for pure rolling of cylinder}$$

Using CM,



$$FR - fR = \frac{1}{2}mR^2\alpha$$

$$a = R\alpha$$

$$f = ma$$

$$\Rightarrow \boxed{a = \frac{2F}{3m}, \ f = \frac{2F}{3}}$$

$$f \le \mu N$$

$$\Rightarrow \boxed{\mu_{min} > \frac{2F}{3(mg - F)}}$$

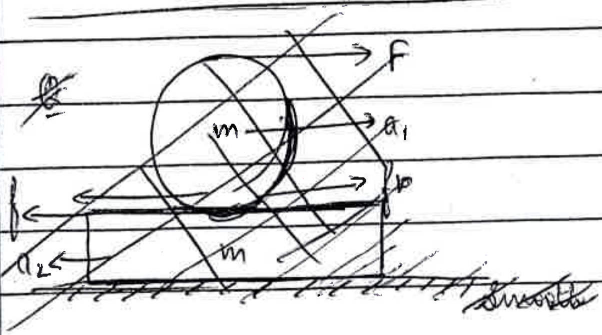
Using ICR,

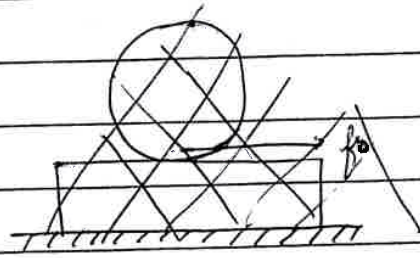
$$FR = \frac{3}{2}mR^2\alpha$$

$$\Rightarrow \boxed{a = \frac{2F}{3m}}$$

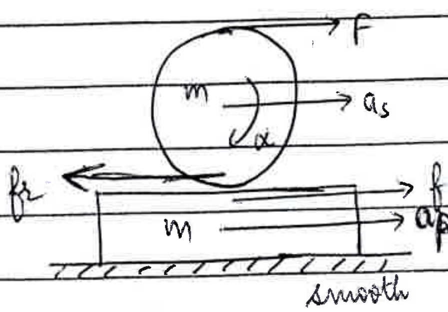
$$\cancel{\Rightarrow f = \frac{2F}{3}}$$

~~Q~~





Q



$$F - f = ma_s$$

$$f = ma_p$$

$$a_s - a_p = R\alpha$$

$$(F + f)R = \frac{2}{5}mR^2\alpha$$

$$\Rightarrow \frac{F - f}{m} - \frac{f}{m} = \frac{5}{2}\frac{(F + f)}{m} \Rightarrow F - 2f = \frac{5}{2}(F + f) \Rightarrow \boxed{f = -\frac{F}{3}}$$

$$a_s = \frac{4F}{3m}, \quad a_p = \frac{-F}{3m}$$

* Friction is not necessary for pure rolling, a body can roll without slipping even on a smooth surface, provided that $\underline{a = R\alpha}$.

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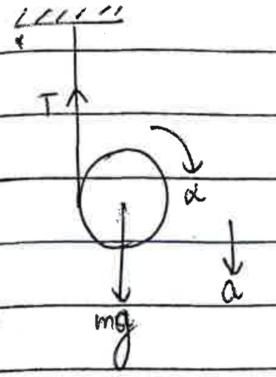
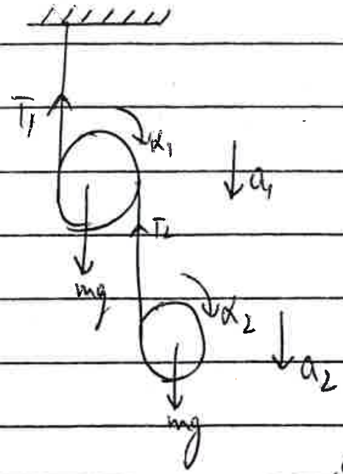


$$mg - T = ma$$
$$TR = \frac{1}{2}MR^2\alpha$$
$$a = \alpha R$$
$$\Rightarrow \boxed{a = \frac{2g}{3}}$$
$$\boxed{T = \frac{mg}{3}}$$

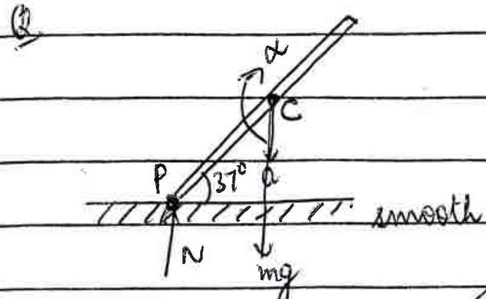


$$mg + T_2 - T_1 = ma_1$$
$$mg - T_2 = ma_2$$
$$\cancel{a_2 = a_1} \quad a_1 = R\alpha_1$$
$$2a_1 = a_2 - R\alpha_2$$
$$T_2 R = \frac{1}{2}MR^2\alpha_2$$
$$(T_1 + T_2)R = \frac{1}{2}MR^2\alpha_1$$

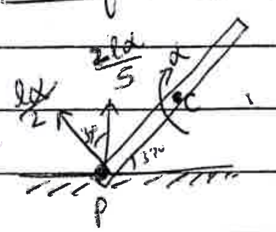
Ans $\Rightarrow a_1 = \frac{2g}{3}\downarrow \quad a_2 = g\downarrow$

T_2 will get −ve, string slack, $a_2 = g$ (free fall)

Q.



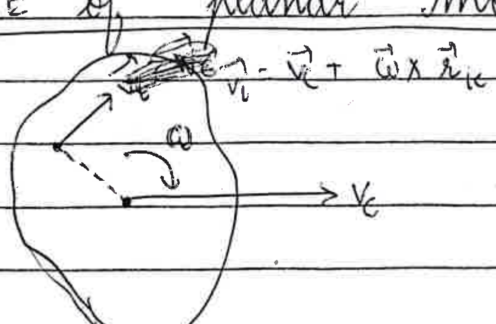
COM frame:



This $\frac{2l\alpha}{5}$ acc. in vertical direction (wrt com)

$$\frac{2l\alpha}{5} = a_{rel\,y} = a_{P_y} - a_{C_y}$$
$$\Rightarrow \boxed{a_{C_y} = \frac{2l\alpha}{5}\downarrow}$$

KE of planar motion:

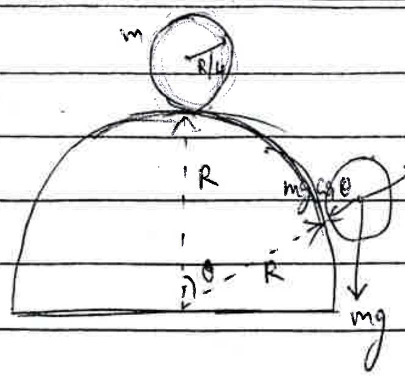


$$\vec{v_i} = \vec{v_c} + \vec{\omega}\times\vec{r_{ic}}$$
$$K_i = \frac{1}{2}m_i v_i^2$$
$$= \frac{1}{2}m_i v_c^2 + \frac{1}{2}m_i\omega^2 r_{ic}^2 + \frac{1}{2}m_i\, 2v_c\omega r_{ic}$$
$$K = \Sigma K_i = \frac{1}{2}(\Sigma m_i)v_c^2 + \frac{1}{2}(\Sigma m_i r_{ic}^2)\omega^2 + \frac{1}{2}2v_c\omega(\Sigma m_i r_i)$$
$$\boxed{K = \frac{1}{2}M v_c^2 + \frac{1}{2}I_c\omega^2}$$

→ Rotational KE

→ Translational KE

Q.



Find θ when hoop looses contact.

$N = 0$

$\Rightarrow mg\cos\theta = \dfrac{mv^2}{\frac{5}{4}R}$

And $\Rightarrow mg\,\dfrac{5R}{4}(1-\cos\theta) = \dfrac{1}{2}mv^2 + \dfrac{1}{2}I\omega^2$

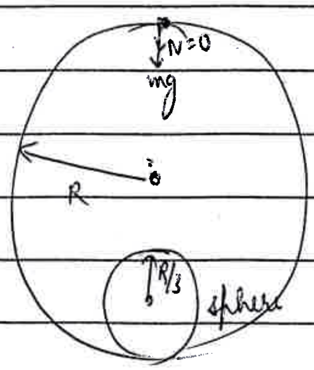
On solving, $\Rightarrow$ we get $\cos\theta = \frac{1}{2} \Rightarrow \boxed{\theta = 60^\circ}$

$\omega = \dfrac{V}{R/4} = \dfrac{4V}{R}$

$\omega_c = V/\tfrac{5R}{4} = \dfrac{4V}{5R} = \dfrac{\omega}{5}$ $\Rightarrow$ Total $\theta = \theta + \theta_c = 360^\circ$ (Same direction ω)

$\Rightarrow \theta_c = \dfrac{\theta}{5} \Rightarrow \theta = 5\theta_c = 300^\circ$

Q.



Pipe

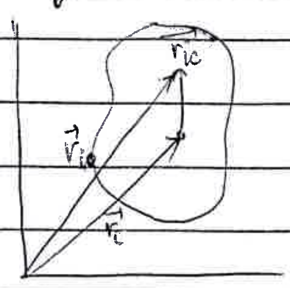
$\dfrac{mv^2}{2R/3} = mg$

$-mg\left(\dfrac{4R}{3}\right) = \dfrac{7}{10}(mv^2 - mv_0^2)$

$\Rightarrow v_0 = \sqrt{\dfrac{18gR}{7}}$

$\left[\because \dfrac{1}{2}mv^2 + \dfrac{1}{2}I\omega^2_{sphere} = \dfrac{7}{10}mv^2\right]$

Angular momentum in planar motion:



$\vec{L}_i = \vec{r}_i \times m_i\vec{v}_i$

$\vec{v}_i = \vec{v}_c + \vec{\omega}\times\vec{r}_{ic}$

~~$\Rightarrow \vec{v}_i = \vec{v}_c +$~~

$\vec{L}_i = \vec{r}_i \times m_i\vec{v}_i$

$\vec{L}_i = \vec{r}_i \times (m_i\vec{v}_c + m_i\vec{\omega}\times\vec{r}_{ic})$

$\vec{L}_i = (\vec{r}_c + \vec{r}_{ic}) \times (m_i\vec{v}_c + m_i(\vec{\omega}\times\vec{r}_{ic}))$

$\Sigma\vec{L}_i = \Sigma\,\vec{r}_c \times m_i\vec{v}_c + \Sigma\, m_i\vec{r}_c \times (\vec{\omega}\times\vec{r}_{ic}) + \Sigma\, m_i\vec{r}_{ic}\times\vec{v}_c + \Sigma\, m_i\vec{r}_{ic}\times(\vec{\omega}\times\vec{r}_{ic})$

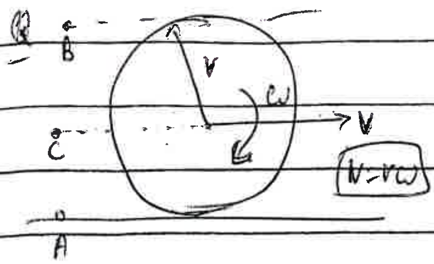
$= M\vec{r}_c\times\vec{v}_c + \vec{r}_c\times(\vec{\omega}\times\Sigma m_i\vec{r}_{ic}) + \Sigma\, m_i(r_{ic}^2\vec{\omega} - \vec{\omega}\cdot\vec{r}_{ic})\vec{r}_{ic})$

$\Rightarrow \boxed{\vec{L} = \vec{r}_c \times M\vec{v}_c + I_c\vec{\omega}}$

Spin angular momentum

Orbital angular momentum

Q.

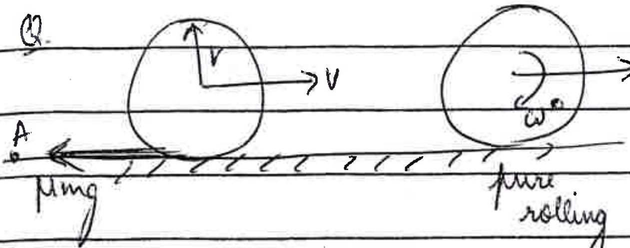


① $L_A = \frac{1}{2}mr^2\omega + mvr = \frac{3}{2}mr^2\omega$

② $L_B = -\frac{1}{2}mr^2\omega$ (considering ω as +ve)

③ $L_C = \frac{1}{2}mr^2\omega$

Q.



$\tau_A = 0$, L_A = const.

$mvr = mv'r + \frac{2}{5}mr^2\omega$

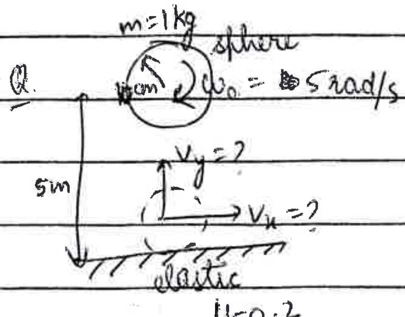
$\Rightarrow mvr = \frac{7}{5}mr^2\omega$

$\Rightarrow \boxed{\omega = \frac{5v}{7R}}$

Time taken, ⇒ $-\mu mg = ma \Rightarrow a = -\mu g \Rightarrow v' = v - \mu g t \Rightarrow t = \frac{v'-v}{\mu g}$

$\Rightarrow \boxed{t = \frac{2v}{7\mu g}}$

Q.



Let us assume that the body slips all the time dt during collision

$\Rightarrow \int N\,dt = mv_y - (-mu) = 1(10+10) = 20\ Ns$

$\mu\int N\,dt = mv_x \Rightarrow 0.2 \times 20 = 1\,v_x \Rightarrow v_x = 4\ m/s$

$R\cdot\mu\int N\,dt = \cancel{I(\omega-\omega_0)}\ I\omega - (-I\omega_0)$

$\Rightarrow \frac{1}{10}\times\frac{1}{5}\times 20 = \frac{2}{5}\times 1\times\frac{1}{10}\times\frac{1}{10}(\omega+5) \Rightarrow \boxed{\omega = 95\ rad/s}$ CCW

So, our assumption is wrong.

The sphere will start pure rolling before completion of collision

Irodov: 1.51 to 1.54 , 1.252 to 1.262
HCV: 37, 59, 60, 61, 62, 63, 64, 70 to 86.

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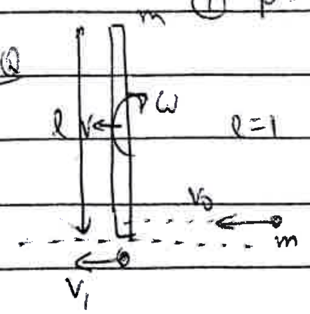
[illegible] collision between a free rigid body and a point particle:

① p = const. ② L = const. (about any point) ③ Equation for e

Q



$e = 1$

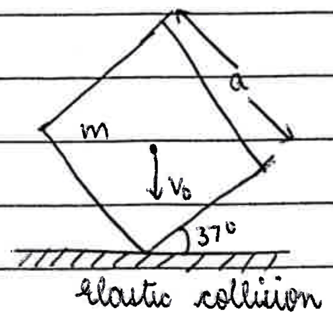
$mv_0 = mv + mv_1$

$$e = 1 = \frac{\left(v + \frac{l\omega}{2}\right) - v_1}{v_0 - 0}$$

$$L_i = L_f = 0 = mv\frac{l}{2} - \frac{ml^2\omega}{12}$$

(we have taken L about lowest point and not COM)

Q



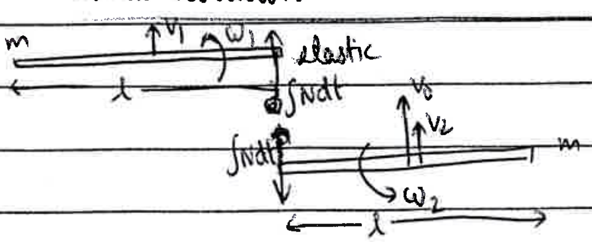
Elastic collision

Find final velocity and ω.

Ans: $v = \frac{47v_0}{53}$, $\omega = \frac{60v_0}{53a}$

~~[illegible]~~ Write 3 equations for v_1

Q.1



$\int N\,dt = mv_1$

$\int N\,dt = m(v_2 - v_0)$

$\frac{L}{2}\int N\,dt = \frac{mL^2\omega_1}{12}$

$\frac{L}{2}\int N\,dt = \frac{mL^2}{12}\omega_2$

$\omega_1 = \omega_2$

$$e = 1 = \frac{\left(v_1 + \frac{\omega L}{2}\right) - \left(v_2 - \frac{\omega L}{2}\right)}{v_0 - 0}$$

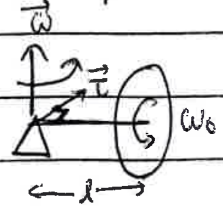
$\Rightarrow v_1 - v_2 + \omega L = v_0$

$$\Rightarrow \frac{\int N\,dt}{m} - \left(v_0 - \frac{\int N\,dt}{m}\right) + 6\frac{\int N\,dt}{m} = v_0$$

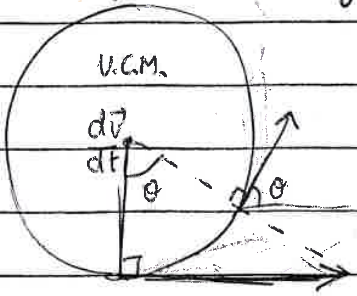
$$\Rightarrow \int N\,dt = \frac{mv_0}{4} \Rightarrow \boxed{v_1 = \frac{v_0}{4}}, \boxed{v_2 = \frac{3v_0}{4}}, \boxed{\omega_1 = \omega_2 = \omega = \frac{3v_0}{2L}}$$

Precession:

Precession refers to the situation where a rigid body is rotating about an axis which is itself rotating about another stationery axis. The simplest example of this is a gyroscope which consists of a spinning wheel. When this spinning wheel connected to an axle is suspended from a support as shown, the wheel doesn't fall down but rotates about the ~~support~~ support point.



In this situation, the $\vec{\tau}$ of gravity about the support point is always perpendicular to the angular momentum vector. The situation is analogous to the case of UCM where ~~acc. is~~ $\vec{a} \perp \vec{v}$. So, the acceleration rotates the velocity vector with an angular velocity of $\vec{\omega}$



$$\vec{a} = \vec{\alpha} \times \vec{r} + \vec{\omega} \times \vec{v}$$

$$\boxed{\frac{d\vec{v}}{dt} = \vec{\omega} \times \vec{v}} \qquad (\alpha = 0)$$

Similarly,

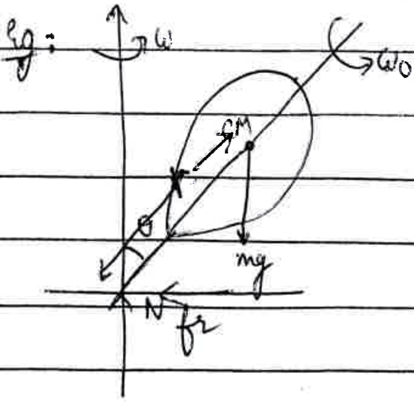
$$\frac{d\vec{L}}{dt} = \vec{\omega} \times \vec{L}$$

where $\vec{L}$ rotates with angular velocity $\vec{\omega}$.

For the gyroscope of example,

$$mgl = \underset{\sin 90^\circ}{\omega I \omega_0} \Rightarrow \boxed{\omega = \frac{mgl}{I\omega_0}}$$

eg:



$\vec{\tau} = \vec{\omega} \times \vec{L}$

$\Rightarrow mgr \sin\theta = \omega I \omega_0 \sin\theta$

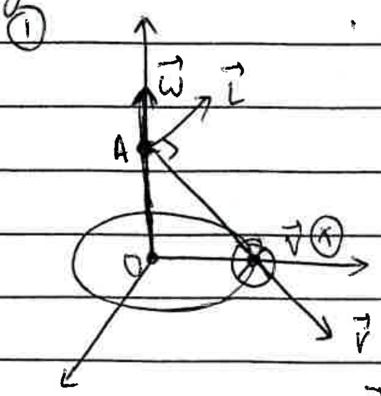
$\Rightarrow \boxed{\omega = \dfrac{mgr}{I\omega_0}}$

$\boxed{N = mg}$ $\quad f = m\omega^2 r \sin\theta$

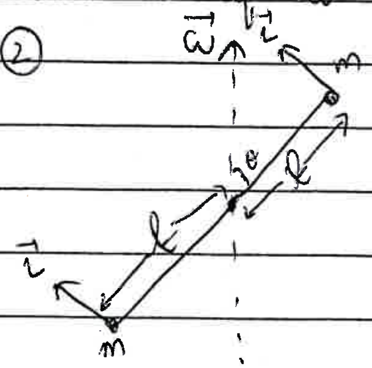
$= \dfrac{m^3 g^2 r^3 \sin\theta}{I^2 \omega_0^2}$

If we wish to calculate angular momentum about a point, in general $\vec{L}$ is not in the direction of $\vec{\omega}$ velocity

eg: ①

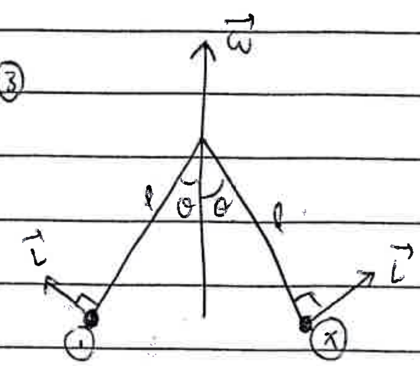


②



$\vec{L}$ & $\vec{\omega}$ are not in same direction

③



If the line perpendicular to the axis of rotation is not an axis of symmetry, $\vec{L}$ & $\vec{\omega}$ are not in same direction.

- Rigid body is rotating about a pivot. (Pivot is taken to be origin)

In general,

$$\begin{bmatrix} L_x \\ L_y \\ L_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

$I_{xx} = \int r_x^2 dm \rightarrow$ moment of inertia

$I_{xy} = \int x^2 dm \rightarrow$ products of inertia

In case we can find a line passing through the pivot which can form an axis of symmetry, that line is known as principal axis. We can choose any 2 mutually perpendicular axis passing through the pivot and perpendicular to this principal axis, they will also be the principle axis. In such a case, product of inertia will be zero and L can be given as:

$$\vec{L} = I_{xx}\vec{\omega_x} + I_{yy}\vec{\omega_y} + I_{zz}\vec{\omega_z}$$

$$I_{xy} = I_{yz} = I_{xz} = I_{yx} = I_{zy} = I_{zx} = 0$$

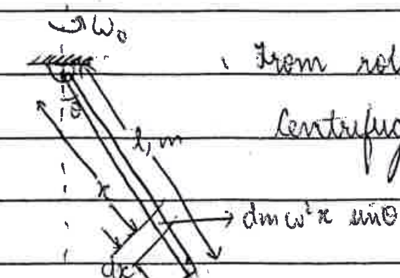
In case, the pivot is also moving (for general planar motion):

$$\vec{L} = \vec{r} \times M\vec{v_c} + I_{xx}\vec{\omega_x} + I_{yy}\vec{\omega_y} + I_{zz}\vec{\omega_z}$$

Pivoted: $K = \frac{1}{2}I_{xx}\omega_x^2 + \frac{1}{2}I_{yy}\omega_y^2 + \frac{1}{2}I_{zz}\omega_z^2$

If not pivoted: $K = \frac{1}{2}Mv_c^2 + \frac{1}{2}I_{xx}\omega_x^2 + \frac{1}{2}I_{yy}\omega_y^2 + \frac{1}{2}I_{zz}\omega_z^2$

Eg:

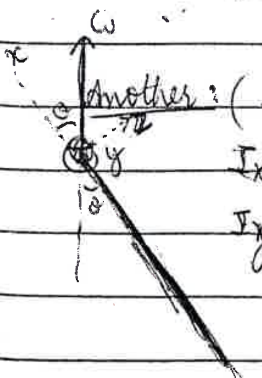


From rotating frame of reference,

Centrifugal $\tau = \int dm\, \omega^2 x \sin\theta\; x\cos\theta$

$$= \frac{m}{l}\,\omega^2 \frac{l^3}{3}\sin\theta\cos\theta = \frac{m\omega^2 l^2 \sin\theta\cos\theta}{3}$$

Also, $\frac{mgl\sin\theta}{2} = \frac{m\omega^2 l^2 \sin\theta\cos\theta}{3} \Rightarrow \boxed{\cos\theta = \frac{3g}{2\omega^2 l}}$



Another: (from ground frame of reference).

$I_{xx} = 0$ $\qquad \omega_x = \omega\cos\theta$

$I_{yy} = I_{zz} = \frac{1}{3}ml^2$ $\qquad \omega_z = \omega\sin\theta$

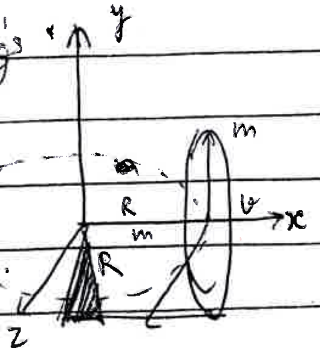
$\omega_y = 0$

$$\Rightarrow L = \frac{1}{3}ml^2\omega\sin\theta$$

Also, here $\vec{\tau} \perp \vec{L}$ always

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$\Rightarrow \vec{\tau} = \vec{\omega} \times \vec{L} \Rightarrow \frac{mgl}{2}\sin\theta = \omega\left(\frac{1}{3}ml^2\omega\sin\theta\right)\cos\theta \Rightarrow \boxed{\frac{3g}{2l\omega^2} = \cos\theta}$$



Find ① $\vec{\omega}$ ② $\vec{\alpha}$ ③ $\vec{L}$ (about 0,0) ④ ~~Com~~ K

① $\vec{\omega} = \frac{V}{R}\hat{i} - \frac{V}{R}\hat{j}$ → due to clockwise rotation of com about y

from disc

② $\vec{\alpha}$ ~~$= \vec{\omega} \times \vec{\omega}$~~ $\frac{d\vec{A}}{dt} = \vec{\omega} \times \vec{A}$

$$\Rightarrow \vec{\alpha} = \left(\frac{-V}{R}\hat{j}\right)\left(\frac{V}{R}\hat{i} - \frac{V}{R}\hat{j}\right)$$

$$\Rightarrow \boxed{\vec{\alpha} = \frac{V^2}{R^2}\hat{k}}$$

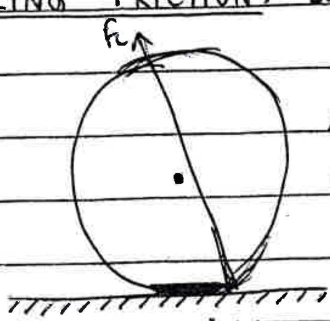
③ $\vec{L} = I_{xx}\vec{\omega_x} + I_{yy}\vec{\omega_y} = \frac{1}{2}MR^2\left(\frac{V}{R}\right)\hat{i} + \frac{19}{12}mR^2\left(\frac{-V}{R}\right)\hat{j}$

$$= \frac{MVR}{2}\hat{i} - \frac{19\,MVR}{12}\hat{j}$$

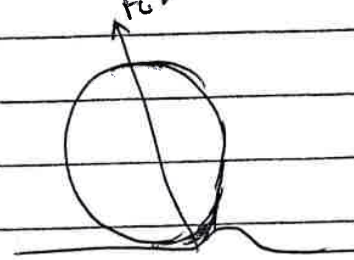
④ $K = \frac{1}{2}I_{xx}\omega_x^2 + \frac{1}{2}I_{yy}\omega_y^2$

$$= \frac{1}{2} \times \frac{1}{2}MR^2 \times \frac{V^2}{R^2} + \frac{1}{2} \times \frac{19}{12}MR^2\left(\frac{V^2}{R^2}\right) = \frac{25mV^2}{24}$$

ROLLING FRICTION: occurs because body and surface are not perfectly rigid



Surface rigid
Body elastic



Surface ~~rig~~ not rigid
Body rigid

Energy is dissipated due to collision with surface

$\tau_c = I\alpha$

$a = R\alpha$

$a = F_c$

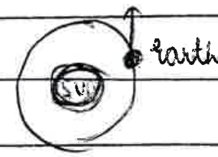
S.H.M.

Oscillatory: To & fro about a position

Periodic: Motion repeats after regular time intervals

$x = A \sin(t^2)$

Oscillatory but not periodic



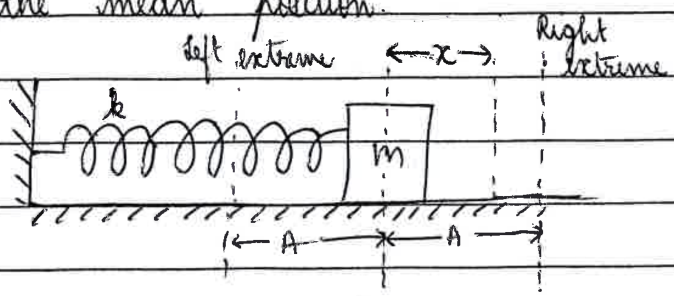
Periodic but not oscillatory

*** Simple harmonic motion:**

Motion is said to be if:

(i) it is periodic (ii) it is oscillatory

(iii) the net force is any position is proportional to the displacement from the mean position and directed towards the mean position.



$$ma = F = -kx$$

$$\Rightarrow a = \frac{-k}{m}x$$

Let $\frac{k}{m} = \omega^2 \Rightarrow \boxed{a = \omega^2 x}$

* **Amplitude** is the maximum displacement of the body from mean position

$$a = -\omega^2 x \Rightarrow \frac{v\,dv}{dx} = -\omega^2 x \Rightarrow \int v\,dv = -\omega^2 \int x\,dx$$

$$\Rightarrow \frac{v^2}{2} = \frac{-\omega^2 x^2}{2} + C \qquad \Rightarrow \frac{v^2}{2} = \frac{\omega^2 A^2}{2} - \frac{\omega^2 x^2}{2}$$

At $x = A$, $v = 0$,

$$0 = \frac{-\omega^2 A^2}{2} + C$$

$$\Rightarrow C = \frac{\omega^2 A^2}{2}$$

$$\Rightarrow \boxed{v = \omega\sqrt{A^2 - x^2}}$$

$v = \omega\sqrt{A^2 - x^2}$

$\Rightarrow \frac{dx}{dt} = \omega\sqrt{A^2-x^2}$

$\Rightarrow \int \frac{dx}{\sqrt{A^2-x^2}} = \int \omega \, dt$

$\Rightarrow$ Put $x = A \sin\theta$

$\Rightarrow dx = A\cos\theta \, d\theta$

$\int \frac{A\cos\theta \, d\theta}{A\cos\theta} = \int \omega \, dt$

$\Rightarrow \theta = \omega t + \phi$

$\Rightarrow \sin^{-1}\left(\frac{x}{A}\right) = \omega t + \phi$

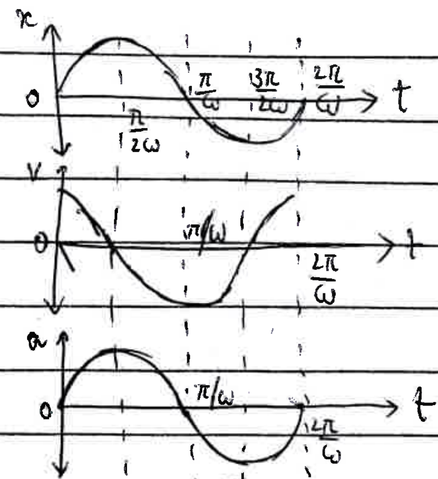
$\Rightarrow x = A\sin(\omega t + \phi)$.

Let us say $\phi = 0$

$x = A \sin \omega t$

$v = A\omega \cos \omega t$

$a = -\omega^2 x = -\omega^2 A \sin \omega t$



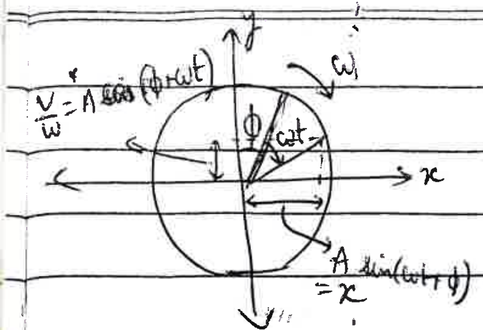
$\boxed{\frac{2\pi}{\omega} = T = \text{Time period}}$ (x, v, a repeat)

$\boxed{T = 2\pi\sqrt{\frac{m}{K}}}$ $\left(\because \omega = \sqrt{\frac{K}{m}}\right)$

• Time period is the min. time after which velocity, position and acceleration repeat their value.

ω = Angular frequency
f = frequency = $\frac{1}{T} = \frac{\omega}{2\pi}$
$\omega t + \phi$ = phase of SHM
ϕ = initial phase / initial epoch / phase constant

* Phase diagram is an imaginary circle in x-y plane centred at origin. Here, we consider the radius of the circle ~~is~~ equal to the amplitude of the particle executing SHM. The radius is assumed to rotate with a constant angular velocity ω in the clockwise direction. At $t=0$, the angle made by the radius with y-axis is ϕ.

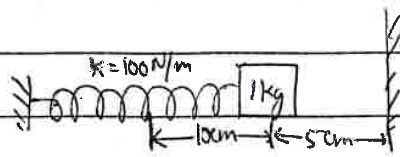


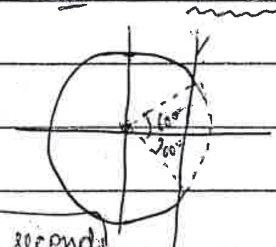
In this model, the displacement from mean position is given by the projection on the x-axis. The projection on y-axis gives us V/ω.

Q. A particle is oscillating with $A = 2$cm, $T = 4$ seconds. At $t=0$, it is at $x=-1$ cm, moving towards left extreme. When will it first reach mean position? Ans: $5/3$ seconds.

Q. Particle, $A = 2$cm, $T = 6$ cm, $x = -1$ cm at $t=0$ (moving towards left extreme)
Find ① x ② v ③ a ④ Distance ⑤ v_{avg} at $t=4$
Ans: ① $+2$cm ② 0 ③ $\frac{-2\pi^2}{9}$ cm/s^2 ④ 5cm ⑤ 0.75 cm/s

Q Two particles are executing SHM about same mean position and same amplitude and same time period. The first particle is at mean moving towards left extreme. 2nd particle is 1cm right of mean moving towards right extreme. $A = 2$cm, $T = 4$s. Find the time at will they will collide. Ans: $11/6$ seconds

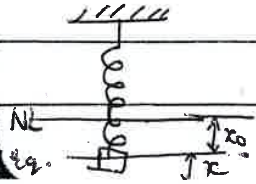
eg. 

⇒ Phasor diagram: 

$\omega = 10$ rad/s

Time period for oscillation (not SHM) $= \frac{4\pi/3}{10} = \boxed{\frac{2\pi}{15}\text{ seconds}}$

* If the block is made to oscillate vertically as shown, it executes SHM with the same ω. But the mean position is not the natural length but the position where $kx_0 = mg$.

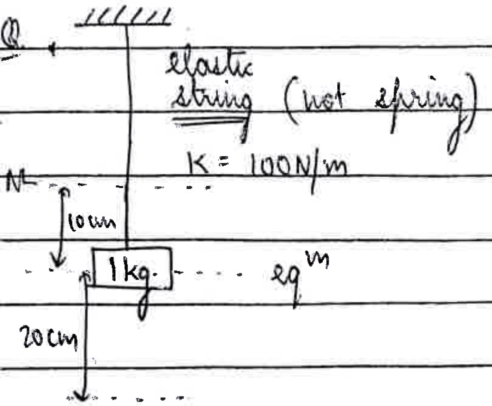


Mean ⇒ $mg = Kx_0$

$mg - K(x+x_0) = ma \Rightarrow -Kx = ma \Rightarrow a = -\omega^2 x$

$\Rightarrow \boxed{\omega = \sqrt{\frac{K}{m}}\text{, same}}$

Q. Find T of oscillation.



Ans: SHM ~~for~~ till NL

Time $= \dfrac{4\pi/3}{10} = \dfrac{2\pi}{15}$ seconds

For free-fall (after that),

Time $= \dfrac{2u}{g} = \dfrac{2\sqrt{3}}{10} = \dfrac{\sqrt{3}}{5}$ seconds

$$T_{oscillation} = \boxed{\left(\frac{2\pi}{15} + \frac{\sqrt{3}}{5}\right) \text{sec}}$$

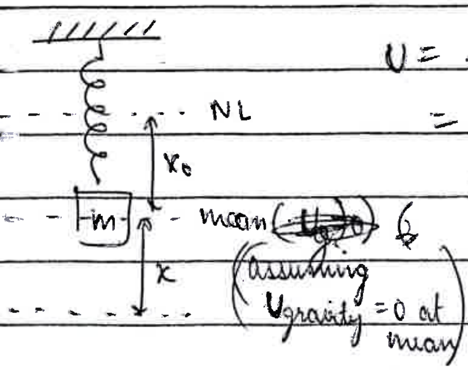
Energy in SHM:

$U = \frac{1}{2}kx^2$ $\qquad K = \frac{1}{2}mv^2 = \frac{1}{2}m\,\omega^2(A^2 - x^2)$ $\qquad \because v = \omega\sqrt{A^2 - x^2}$

$= \frac{1}{2}k(A^2 - x^2)$

$E = U + K = \frac{1}{2}kA^2 \longrightarrow$ Energy of SHM

$\hookrightarrow$ Total mechanical energy.



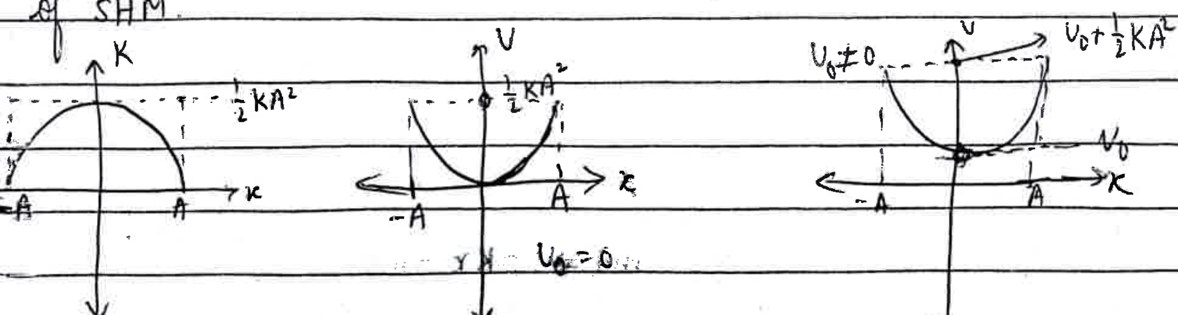
$U = \frac{1}{2}k(x_0 + x)^2 - mgx$

$= \frac{1}{2}k(x_0^2 + x^2 + \cancel{2x_0x}) - \cancel{mgx}$ $\qquad (\because mg = kx_0)$

$= \frac{1}{2}kx_0^2 + \frac{1}{2}kx^2 = \boxed{\frac{1}{2}kx^2 + U_0}$

$\boxed{E = \frac{1}{2}kA^2 + U_0}$ $\longrightarrow$ Energy of mean position

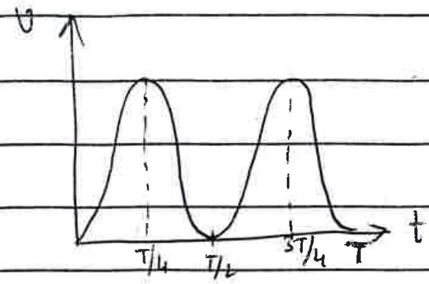
If the potential energy of spring in the mean position is not zero, the total mechanical energy is not equal to the energy of SHM.



$U = \frac{1}{2}kx^2 \Rightarrow U = \frac{1}{2}kA^2\sin^2\omega t$

$K = \frac{1}{2}k(A^2 - x^2) \Rightarrow K = \frac{1}{2}kA^2\cos^2\omega t$

$(x = A\sin\omega t)$

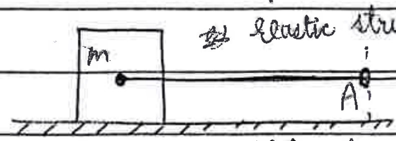


frequency of ~~osc~~ oscillation of energy $= 2f$.

$$U_{rms} = \sqrt{\frac{\int_0^T U^2 dt}{T}} = \sqrt{\frac{A^2\omega^2 \int_0^T \cos^2\omega t\, dt}{T}} = \sqrt{\frac{A^2\omega^2}{T}\int_0^T \left(\frac{1+\cos 2\omega t}{2}\right)dt}$$

$$= \sqrt{\frac{A^2\omega^2}{2T}\left(\frac{2\pi}{\omega}\right)} = \boxed{\frac{A\omega}{\sqrt{2}}}$$

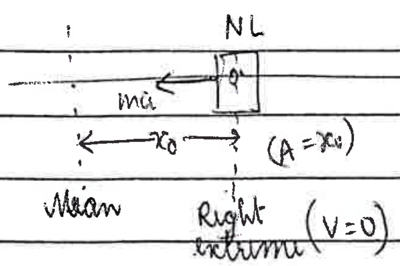
Q.



Find position of block as function of 't'.

Ans: $\frac{1}{2}at^2 - \frac{ma}{K}(1-\cos\omega t)$ where $\omega = \sqrt{\frac{K}{m}}$

(go to the frame of point A).



Displacement (w.r.t. A)

$= -A + A\sin(\omega t + 90°)$

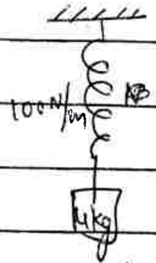
$= -x_0 + x_0\cos\omega t$

$= -\frac{ma}{K}(1-\cos\omega t)$

$ma = Kx_0 \quad \omega = \sqrt{\frac{K}{m}}$

$\Rightarrow \boxed{\text{Position} = \frac{1}{2}at^2 - \frac{ma}{K}\left(1-\cos\sqrt{\frac{K}{m}}\,t\right)}$

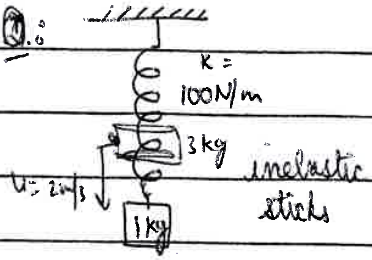
Q.



The block is executing vertical SHM with $A = 10$ cm. When it is at the lowest extreme, $3/4^{th}$ of the mass falls off without exerting any impulse on the remaining mass. Find the extension in spring as a f^n of time.

Ans: Extension $= (10 + 40 \cos 10t)$ cm ($t = 0$ at lower extreme)

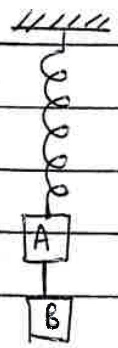
Q.:

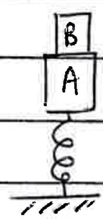


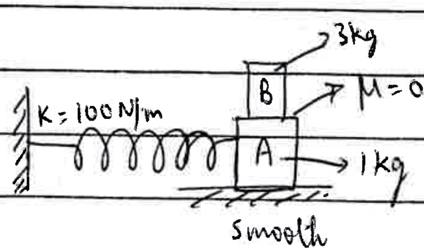
Find ① A ② ϕ ③ ω.

Ans: ① $30\sqrt{2}$ cm ② $45° = -\frac{\pi}{4}$ ③ 5 rad/s

* Connected SHM:







$f \le \mu_s N = 6N$

$f = 3(\omega^2)x$

$\Rightarrow 3\omega^2 x < 6$

$\Rightarrow x \le \frac{2}{\omega^2}$ $\omega = \sqrt{\frac{100}{3+1}} = 5$

$\Rightarrow \boxed{x \le 8 \text{ cm}}$

Spring should not go beyond N.L.

$a = -\omega^2 x$

$N - m_b g = -m_b \omega^2 x$

$N \ge 0 \Rightarrow \boxed{x \le \frac{g}{\omega^2}}$

$\omega = \sqrt{\frac{k}{m_A + m_B}}$

$\Rightarrow A \le \frac{g(m_A + m_B)}{K}$

(x_{max})

$\Rightarrow$ Spring should not go beyond N.L.

Date: / / Page no:

Dynamics of SHM →

① Force method →

Step (i) : Find the eq. position of the oscillating body.

(ii) : Displace the body from eq. position, by a small distance x in the +ve direction

(iii) Find the F_{net} on body in this position.

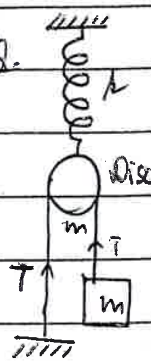
(iv) If F_{net} is directed towards eq. position and proportional to the displacement x from the eq. position, then the motion is SHM.

$$F_{net} = -m\omega^2 x \qquad \omega = 2\pi/T$$

Q. A particle is constrained to move along x-axis such that $U(x) = x^2 - 2x + 1$, $m = 2$ kg and initially it is at origin, rest. Find $x(t)$.

Ans: $U(x) = x^2 - 2x + 1 \Rightarrow F(x) = \frac{-dU(x)}{dx} = -2x + 2$

$\Rightarrow$ eq. position $= x = 1$ $(F_{net} = 0)$ $\Rightarrow x(t) = 1 - \cos \omega t$ $(\phi = -90°)$.

Q. 

① T ② A_{max}

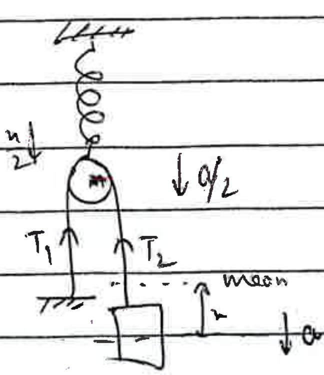
Sol: $2T + mg = Kx_0$

$\Rightarrow 3mg = Kx_0$

$\Rightarrow \boxed{x_0 = \frac{3mg}{K}}$

$A_p \leq x_0 = \frac{3mg}{K}$

$A_b = 2A_p \leq \boxed{\frac{6mg}{K} = A_{max}}$



$mg - T_2 = ma$

$T_2 + T_1 + mg - \cancel{Kx_0} - K\left(x_0 + \frac{x}{2}\right) = \frac{ma}{2}$

$T_2R - T_1R = \frac{1}{2}mR^2\alpha$

$\Rightarrow T_2 - T_1 = \frac{1}{2}mR\alpha = \frac{ma}{4}$

$a = 2R\alpha$

② Energy method.

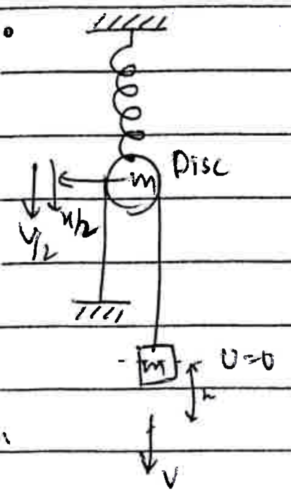
(i) Find the mean position.

(ii) Displace the body from the mean position by a distance 'x' and find the P.E. and KE of the system in this position.

(iii) $E = U + K$

$$\Rightarrow \frac{dE}{dt} = 0 \quad \left(\text{or} \quad \frac{dE}{dx} = 0 \right)$$

$\Rightarrow \hookrightarrow$ Equations of motion



$$U = -mgx - mg\left(\frac{x}{2}\right) + \frac{1}{2}k\left(x_0 + \frac{x}{2}\right)^2$$

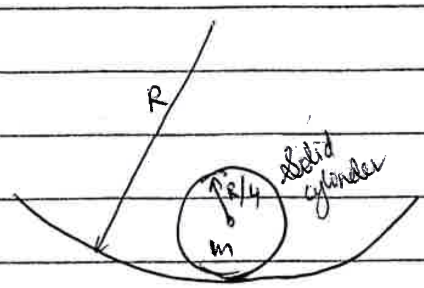
$$K = \frac{1}{2}mv^2 + \frac{1}{2}m\left(\frac{v}{2}\right)^2 + \frac{1}{2}\left(\frac{1}{2}mR^2\right)\left(\frac{v}{2R}\right)^2 = \frac{11}{16}mv^2$$

$$0 = \frac{dE}{dt} = -\frac{3mg}{2}v + \frac{1}{2}k\left(x_0 + \frac{x}{2}\right)2\,\frac{v}{2} + \frac{11}{16}m\,2va$$

$$\Rightarrow 0 = \cancel{\left(\frac{-3mg}{2} + kx_0\right)}\frac{v}{2} + \frac{kxv}{4} + \frac{11}{8}mva$$

$$\Rightarrow a = -\frac{2kx}{11m} \Rightarrow \omega = \sqrt{\frac{2k}{11m}} \Rightarrow T = 2\pi\sqrt{\frac{11m}{2k}}$$

Q



Sufficient friction to cause pure rolling.

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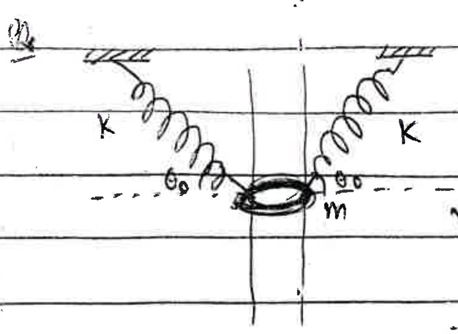
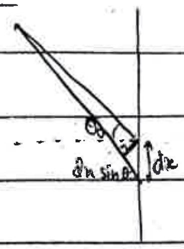
Q. Find period of oscillation for small displacement of the ring.





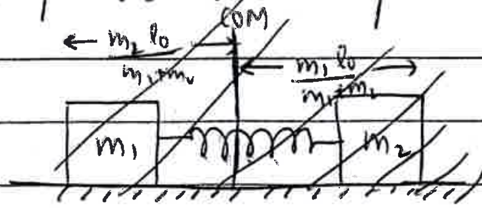
$$2k(dx \sin\theta_0)\sin\theta_0 = ma$$

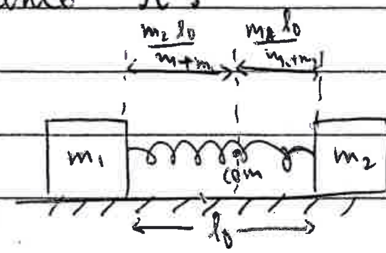
$$\Rightarrow a = \frac{2k\sin^2\theta_0}{m}x$$

$$\Rightarrow \boxed{\omega = \sqrt{\frac{2k}{m}}\sin\theta_0 \Rightarrow T = \frac{2\pi}{\sin\theta_0}\sqrt{\frac{m}{2k}}}$$

* 2- block system :

Q. Two blocks are connected by spring. They are pulled out and released. They oscillate from COM frame of reference. Both the blocks have same time period but different amplitudes and phase difference π.

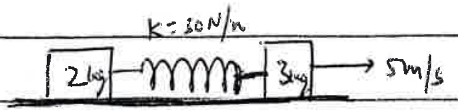




$$k l_0 = k_1\frac{m_2 l_0}{m_1+m_2} = k_2\frac{m_1 l_0}{m_1+m_2}$$

$$\omega = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{k}{\frac{m_1 m_1}{m_1+m_2}}} = \sqrt{\frac{k}{\mu}} \qquad \omega = \sqrt{\frac{k_2}{m_2}} = \sqrt{\frac{k_2}{\frac{m_1 m_2}{m_1+m_2}}} = \sqrt{\frac{k}{\mu}}$$

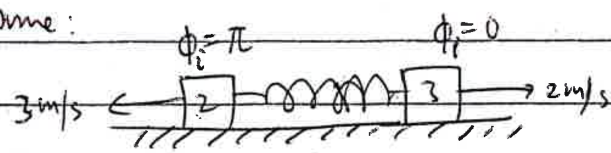
Q.



CM frame : $\omega = \sqrt{\frac{k}{\mu}} = \sqrt{\frac{30}{6/5}} = 5$ rad/s

$v_{CM} = \frac{3 \times 5}{3+2} = 3$ m/s

→ CM frame :



$x_2(t) = 3t - \frac{3}{5}\sin 5t$

$x_3(t) = 3t + \frac{2}{5}\sin 5t$

Spring at NL

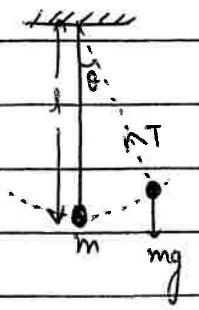
$\Rightarrow$ force = 0

$\Rightarrow$ Mean position

Angular SHM :

when a body undergoes pure rotation such that it oscillates to and fro about a mean position and the torque in any position is proportional to the angular displacement from the mean position and is directed towards the mean position, then the motion is said to be angular SHM.

$$\tau = mgl \sin\theta = ml^2\alpha$$



$$\Rightarrow \alpha = \frac{g \sin\theta}{l} = -\frac{g\theta}{l} \quad (\sin\theta \approx \theta \text{ for small } \theta)$$

$$\Rightarrow \omega = \sqrt{\frac{g}{l}} \Rightarrow \boxed{T = 2\pi\sqrt{\frac{l}{g}}}$$

$$\underbrace{\frac{d^2x}{dt^2} = -\omega^2 x}\ \cancel{\theta} \text{ and } \underbrace{\frac{d^2\theta}{dt^2} = -\omega^2\theta}$$

Final solⁿ will be analogous

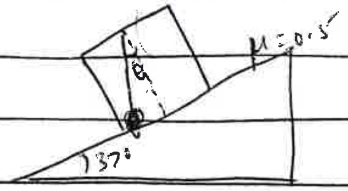
$x = A \sin(\omega t + \phi)$
$v = A\omega \cos(\omega t + \phi)$
$a = -A\omega^2 \sin(\omega t + \phi)$
$K = \frac{1}{2} kA^2 \cos^2(\omega t + \phi)$
$U = \frac{1}{2} kA^2 \sin^2(\omega t + \theta)$
$= \frac{1}{2} kx^2$

$\theta = \theta_0 \sin(\omega t + \phi)$
$\Omega = \theta_0 \omega \cos(\omega t + \phi)$
$\alpha = -\theta_0 \omega^2 \sin(\omega t + \phi)$
$K = \frac{1}{2} ml^2 \theta_0^2 \omega^2 \cos^2(\omega t + \phi)$
$= mgl \frac{\theta_0^2}{2} \cos^2(\omega t + \phi)$
$U = mgl \frac{\theta^2}{2} \quad \left(\because \Delta h = l(1 - \cos\theta) = \frac{l\theta^2}{2}\right)$
$= mgl \frac{\theta_0^2}{2} \sin^2(\omega t + \phi)$

$E = K + U = \frac{mgl\theta_0^2}{2}$ = energy of SHM

If in accelerating frame,

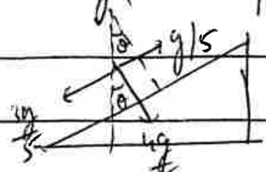
$$T = 2\pi\sqrt{\frac{l}{g_{eff}}} \quad \text{where } \vec{g_{eff}} = \vec{g} - \vec{a}$$

Q 

Find (i) θ_{mean}

(ii) T for small oscillations

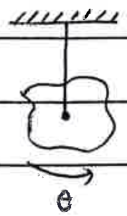
$a = g(\sin\theta - \mu\cos\theta) = g/5$



(i) $\theta_{mean} = \tan^{-1}\left(\frac{1}{2}\right)$

(ii) $T_{oscillation} = 2\pi\sqrt{\frac{l}{g_{eff}}} = 2\pi\sqrt{\frac{5l}{\sqrt{20}g}}$

* Torsional Pendulum:

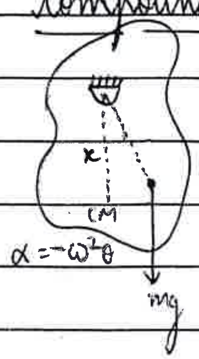


$\tau = -c\theta$ (c = torsional constant)

$I\alpha = -c\theta \Rightarrow \alpha = \frac{-c}{I}\theta \Rightarrow \boxed{\omega = \sqrt{\frac{c}{I}}}$

$\int du = -\int dW = \int c\theta\, d\theta = \frac{1}{2}c\theta^2$

• Compound / Physical Pendulum:



I = MI about axis of rotation

$\tau = mgx\sin\theta = I\alpha$

$\Rightarrow \alpha = \frac{mgx\sin\theta}{I}$

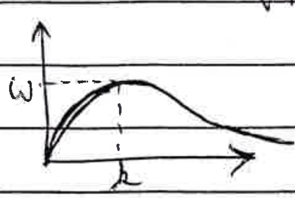
$\Rightarrow \alpha = \left(\frac{-mgx}{I}\right)\theta \quad (\sin\theta \approx \theta)$.

$\Rightarrow \omega = \sqrt{\frac{mgx}{I}}$ Now, $I = I_c + mx^2 = mk^2 + mx^2$

$\Rightarrow \omega = \sqrt{\frac{gx}{k^2+x^2}} = \sqrt{\frac{g}{\frac{k^2}{x}+x}}$

$x = 0 \rightarrow \omega = 0$

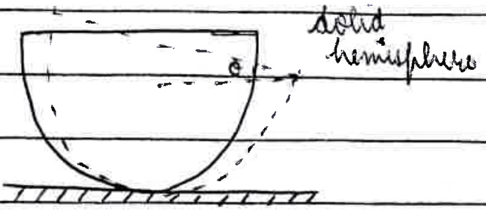
$x = \infty \rightarrow \omega = 0$



ω is max at $x = k$ $\left(\frac{k^2}{x} + x = \text{min at } x = k\right)$

$\omega_{max} = \sqrt{\frac{g}{2k}}$

* Rocking pendulum :



Superposition of SHM :

When a particle is subjected to two simultaneous SHMs, the resultant displacement is the sum of displacements of each of the SHMs.

Case-1: Both SHMs are in same direction & same angular frequency ω

$x_1 = A_1 \sin \omega t \longrightarrow (\phi = 0$ for simplicity$)$

$x_2 = A_2 \sin(\omega t + \phi)$

$x = \cancel{A_1} \; x_1 + x_2 = A_1 \sin \omega t + A_2 \sin(\omega t + \phi)$

$= A_1 \sin \omega t + A_2 \sin \omega t \cos \phi + A_2 \cos \omega t \sin \phi$

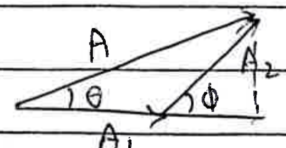
$= (A_1 + A_2 \cos \phi) \sin \omega t + (A_2 \sin \phi) \cos \omega t$

$\Rightarrow A^2 = A_1^2 + A_2^2 \cos^2 \phi + 2 A_1 A_2 \cos \phi + A_2^2 \sin^2 \phi$

$\Rightarrow \boxed{A^2 = A_1^2 + A_2^2 + 2 A_1 A_2 \cos \phi}$

$x = A \cos \theta \sin \omega t + A \sin \theta \cos \omega t$

$= A \sin(\omega t + \theta)$



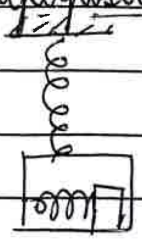
When a particle is subjected to 2 SHMs in the same direction, the resulting SHM has an amplitude and phase which can be obtained by vectorially adding the 2 amplitudes such the angle between these two vectors is equal to the

• phase difference between these two SHMs.

If a body is subjected to two simultaneous SHMs in the same direction but they do not have the same angular frequency, the resultant motion is not simple harmonic.

$x = A_1 \sin \omega_1 t + A_2 \sin(\omega_2 t + \phi)$

Superposition of two mutually $\perp$ SHMs:



Case-1: $\omega_y = \omega_x$

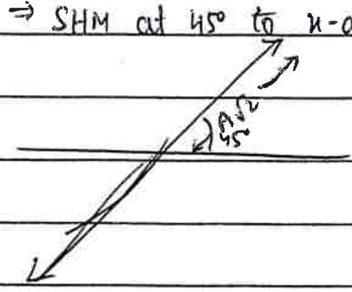
(a) $A_x = A_y$

$\phi = 0$

$x = A \sin \omega t$

$y = A \sin \omega t$

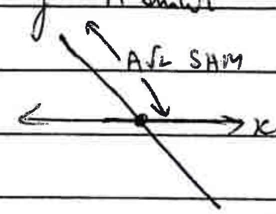
$\Rightarrow$ SHM at 45° to x-axis



(b) $\phi = \pi$

$x = A \sin \omega t$

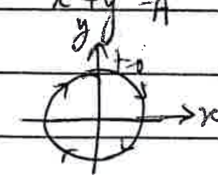
$y = -A \sin \omega t$



(c) $x = A \sin \omega t$

$y = A \cos \omega t$

$x^2 + y^2 = A^2$



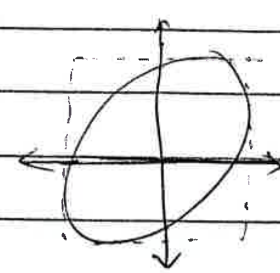
(d) $x = A \sin \omega t$

$y = A \sin(\omega t + \pi/3)$ → or any other angle.

$\Rightarrow y = \frac{A}{2} \sin \omega t + \frac{\sqrt{3}A}{2} \cos \omega t$

$= \frac{x}{2} + \frac{\sqrt{3}}{2} \sqrt{A^2 - x^2}$

$\Rightarrow \left(y - \frac{x}{2}\right)^2 = \frac{3}{4}(A^2 - x^2) \Rightarrow$ Ellipse

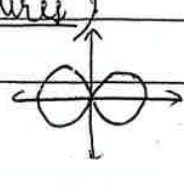


Case-2: $\omega_x \neq \omega_y$

(Lissajous' figures)

$T_x = 2T_y$

$A_x = A_y$



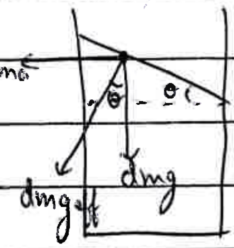
$x = A \sin \omega t$

$y = A \sin(2\omega t)$

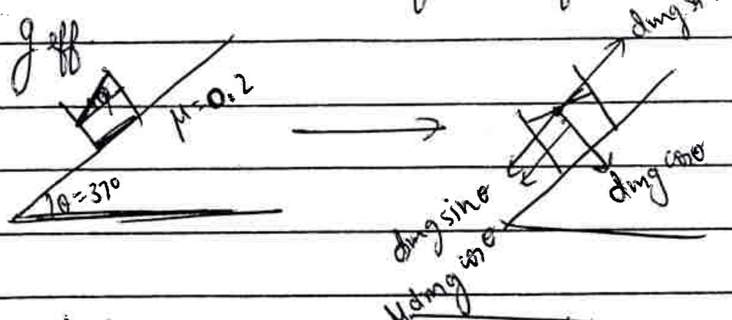
FLUID MECHANICS:

Date: / / Page no:

Fluid has property to flow. In other words, it can't withstand shear force in hydrostatic condition. The free surface of liquid is perpendicular to g_{eff}.

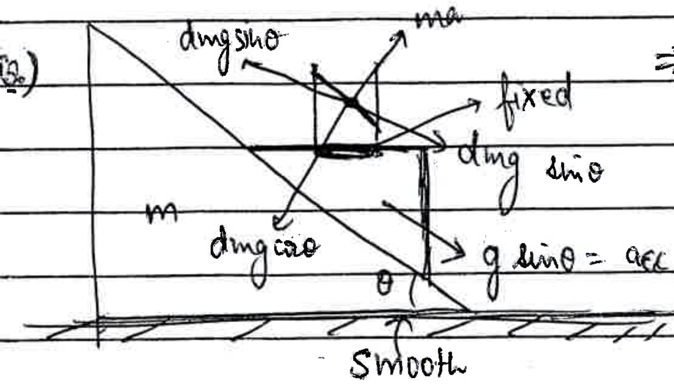


$\tan\theta = \frac{a}{g}$



$\Rightarrow \tan\phi = \mu \Rightarrow \boxed{\phi = \tan^{-1}(\mu)}$

(2.)

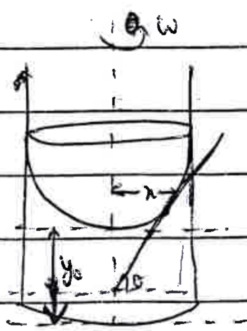


$\rightarrow g\sin\theta =$ acc in this direction

$\Rightarrow$ Fluid surface will be parallel to the ~~lower~~ inclined surface of the lower wedge.

(because only mg sin θ will act in this direction wrt. ground frame)

(3.)



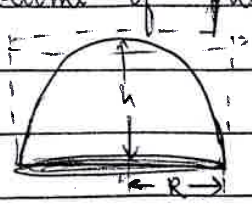
$\tan\theta = \frac{a}{g} \Rightarrow \tan\theta = \frac{\omega^2 x}{g} = \frac{dy}{dx}$

$$\Rightarrow \int_{y_0}^{y} dy = \frac{\omega^2}{g}\int_0^x x\,dx$$

$\Rightarrow$ ~~$y = y_0 +$~~ $\boxed{y - y_0 = \frac{\omega^2 x^2}{2g}}$ $\Rightarrow$ ~~The~~ Surface is paraboloid.

Volume of paraboloid $= \frac{1}{2}\pi R^2 h$

$= \frac{1}{2} \times$ vol. of cylinder corresponding to it

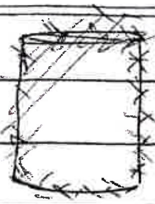


Q. A cylindrical tank has radius 20 cm and height 40 cm filled with water upto 30 cm;

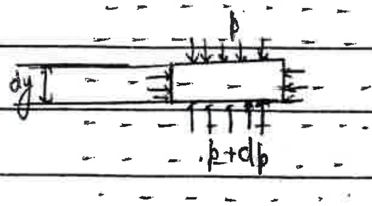
(i) Max. ω(rotation) so that water doesn't overflow. $\rightarrow$ 10 rad/s

(ii) ~~ω~~ ω at which we begin to see the base. $\rightarrow 10\sqrt{2}$ rad/s

(iii) How much water flowed till ω of (ii) part? $\rightarrow 4\pi$ litres



* Hydrostatic pressure:



$$A\cdot p + (dm)g = (p+dp)A$$

$$\Rightarrow (dm)g = p\,A\cdot dp$$

$$\Rightarrow \rho A\,dy\,g = \cancel{p}\,A\cdot dp$$

$$\Rightarrow \boxed{\frac{dp}{dy} = \rho g} \rightarrow \text{Working formula}$$

→ if acc. is there, g would become g_{eff} $\Rightarrow \frac{dp}{dy} = \rho g_{eff}$

(i) ρ & g constant

$$\Rightarrow \int_{p_0}^{p} dp = \rho g \int_0^h dy$$

$$\Rightarrow \boxed{p = p_0 + \rho g h}$$

↳ We can use it only when ρ, g constant.

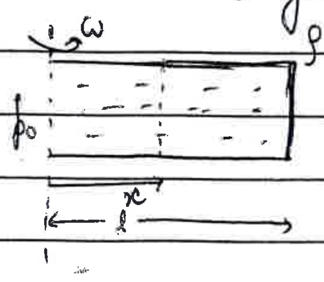
(ii) ρ is variable, g is constant.

$$\frac{dp}{dy} = -\rho g = -\frac{pMg}{RT} \quad (-ve)\,(\text{going upwards})$$

$$\Rightarrow \int_{p_0}^{p} \frac{dp}{p} = -\frac{Mg}{RT}\int_0^h dy$$

$$\Rightarrow \boxed{p = p_0\, e^{-\frac{Mgh}{RT}}} \quad \text{Barometric formula}$$

(iii) ρ constant, g variable (acceleration)



$$\frac{dp}{dx} = \rho\omega^2 x \Rightarrow \int_{p_0}^{p} dp = \rho\omega^2 \int_0^x x\,dx$$

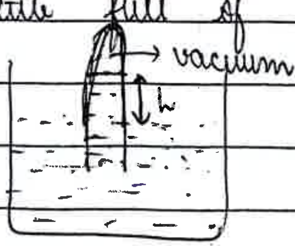
$$\Rightarrow p = p_0 + \frac{\rho\omega^2 x^2}{2}$$

$p = p_0 + \rho g y$ → gauge pressure → manometer

p ↓ absolute pressure

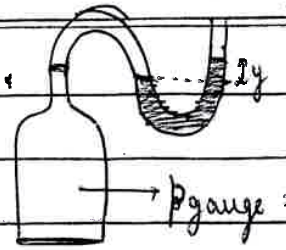
p_0 ↓ atmospheric pressure → barometer

atmospheric pressure is measured with the help of barometer. We take a tube full of mercury and invert it in a tub full of mercury.

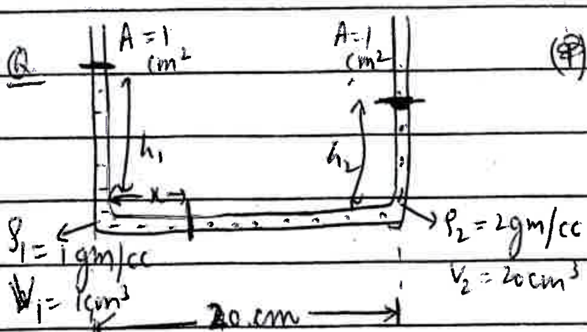


$$hA\rho g = p_0 A$$

$$\Rightarrow \boxed{p_0 = h\rho g}$$

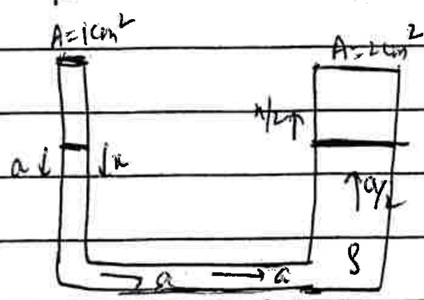


$\rightarrow p_{gauge} = \rho g y$

Q. 

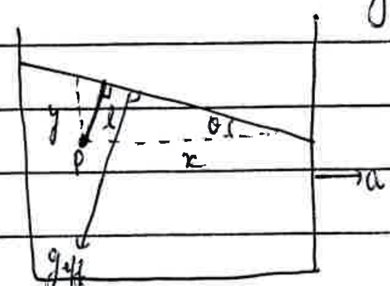
Find x, h_1, h_2

$x = \frac{20}{3}$ cm, $h_1 = \frac{40}{3}$ cm, $h_2 = \frac{20}{3}$ cm

Q. 

Find time period of oscillation.
(It will not be SHM).
(Two half SHMs are there).

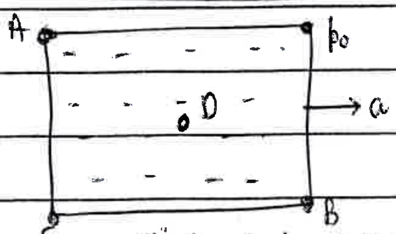
Pressure in accelerating fluid:



$p_P = p_0 + \rho g_{eff} l$

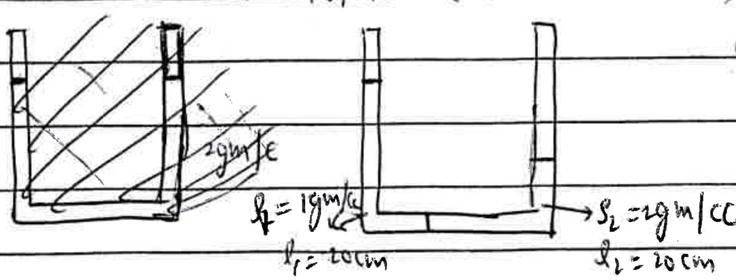
$l = \perp$ distance from free surface

$\tan\theta = \frac{a}{g} = \frac{y}{x}$, $\sin\theta = \frac{l}{x} = \frac{a}{g_{eff}}$

Q 

$$\Rightarrow \boxed{\begin{aligned} P &= p_0 + \rho g_{eff} l \\ &= p_0 + \rho g y \\ &= p_0 + \rho a x \end{aligned}}$$

Find P at A, B, C, D. (Questions like this)

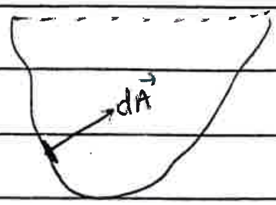
Q. 

(i) ~~How~~ What should be the minimum horizontal acc so that the two levels ~~are~~ are the same?

(ii) If the u-tube is stopped suddenly in this position, what is the acceleration of the fluid?

- Force due to pressure:

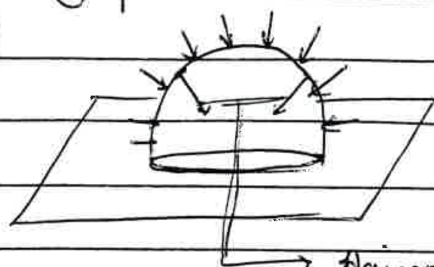
$p = \frac{dF}{dA}$



$d\vec{F} = p\, d\vec{A}$

$\Rightarrow \vec{F} = \int p \cdot d\vec{A}$

(I) $p =$ const.



$\vec{F} = \int p \cdot d\vec{A}$

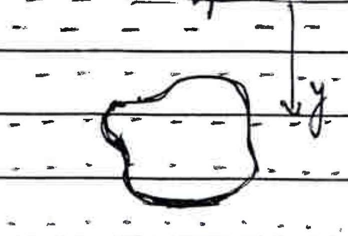
$\vec{F} = p_0 \int d\vec{A}$

$= p_0 \pi R^2$

$\left[\int d\vec{A} = \pi R^2 \text{ and } \int dA = 2\pi R^2\right]$

Horizontal component cancelled.

- Force on $\boxed{\text{plane}}$ surface when pressure is variable:



$dF = (p_0 + \rho g y)\, dA$

$\Rightarrow F = \int p_0\, dA + \rho g \int y\, dA$

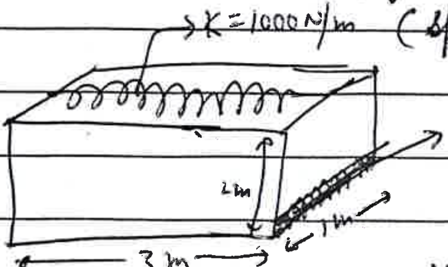
$\Rightarrow \boxed{F = (p_0 + \rho g y_{CM})\, A}$ Use only for force, Not for torque.

$y_{CM} = \frac{\int y\, dm}{M} \Rightarrow \int y\, dm = M y_{CM}$

Assuming uniform mass distribution (surface),

$\int y\, \sigma\, dA = \sigma A\, y_{CM} \Rightarrow \boxed{\int y\, dA = y_{CM}}$

Q.



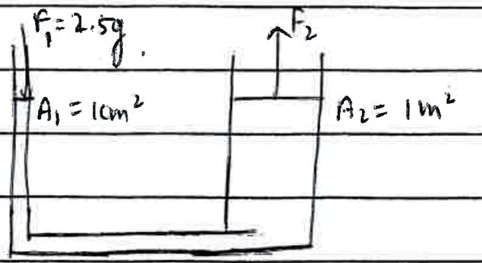
Find max. height upto which water can be filled (no water leak).

Ans: $\boxed{H = \sqrt[3]{0.6} \text{ metres}}$

Q. (Curved surface).

Pascal's law :

Pascal's law states that if pressure is increased at any point of the fluid, the same change is transmitted to entire fluid ~~fo~~ without diminision.



Δp is same throughout the fluid

$$\Rightarrow \frac{F_1}{A_1} = \frac{F_2}{A_2}$$

$$\Rightarrow \boxed{F_2 = \frac{F_1 A_2}{A_1}}$$

Force is increased, work or energy is not.

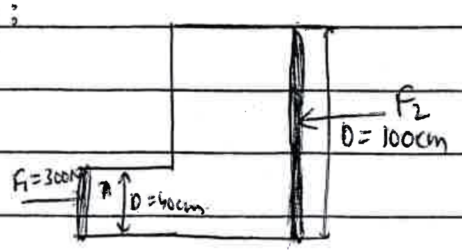
$x_1 A_1 = x_2 A_2$ (incompressible fluid).

$W_1 = F_1 x_1 = \Delta p \, A_1 x_1$

$W_2 = F_2 x_2 = \Delta p \, A_2 x_2$

$\Rightarrow \boxed{W_1 = W_2}$

eg :



Find F_2.

Archimedes Principle :

Archimedes principle states that whenever a body is dipped in a liquid, the upthrust on the ~~liquid~~ body is equal to the weight of liquid displaced. This upthrust is known as buoyant force. It acts due to tendency of any system to minimize its potential energy.

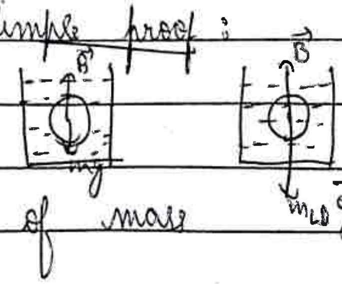
$$\vec{B} = -\rho_L V_{LD} \vec{g}_{eff}$$

ρ_L = density of liquid

V_{LD} = volume of liquid displaced

$\vec{g}_{eff} = \vec{g} - \vec{a}$

Simple proof :



$\Rightarrow \vec{B} = m_{LD} \vec{g}$

This buoyant force acts at the centre of mass of the displaced liquid.

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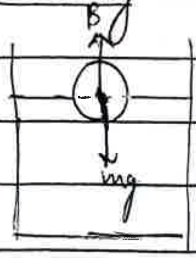
(i) Floating:



$mg = \rho_b V_b g$

$B = \rho_L V_{LD} g$

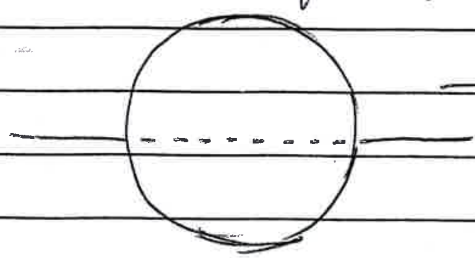
$\because mg = B$

$\Rightarrow \rho_b V_b = \rho_L V_{LD}$

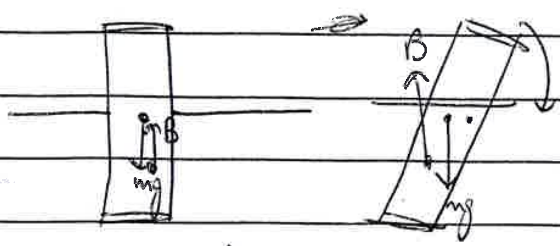
$\Rightarrow \dfrac{\rho_b}{\rho_L} = \dfrac{V_{LD}}{V_b} \le 1$

~~Centre~~ Centre of buoyancy:

Buoyant force can be assumed to act at the centre of mass ~~gravity~~ of the displaced fluid, this point is known as centre of buoyancy.

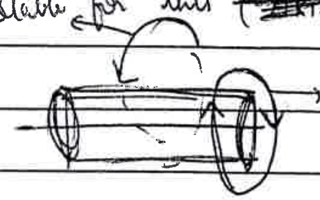


→ Stable equilibrium for vertical oscillation
Neutral eq. for rotational oscillation

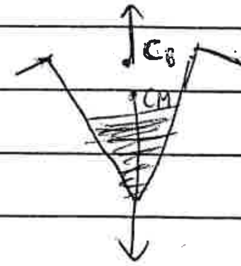


Stable eq. for vertical oscillation
Unstable for rotational

Stable for this (~~...~~ vertical



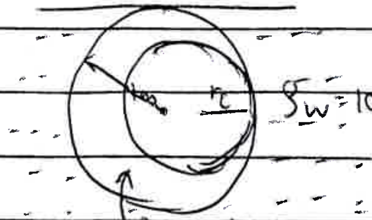
Stable for vertical
Neutral for this



Ship deck
(Stable eq).

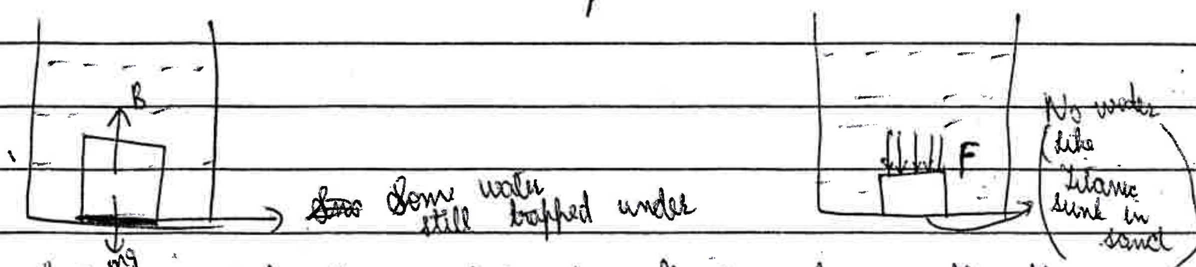
For stable equilibrium, the centre of buoyancy should be located such that the torque of buoyant force about COM in the displaced position should bring it back to the equilibrium position.

Q.



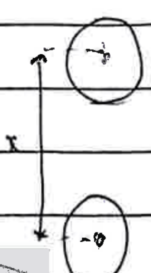
Find (i) r_c for trans. eq
(ii) α

(•) Buoyant force & force exerted by liquid are not identical concepts. For example, in the situation shown, the buoyant force exerted by liquid 1 is non-zero but the force exerted by liquid 1 is zero. But ~~the~~ net buoyant force will be equal to the net force exerted by the liquid. However, this also has an exception.



If the body is not surrounded by fluids from all the sides, we can't apply the concept of ~~bo~~ buoyant force. We have to apply the concept of force exerted by the liquid

(•) Buoyant force appears due to force exerted by gravity (g_{eff}) on the liquid. So, ~~the~~ work done by buoyant force is equal to work done by gravity on the liquid.

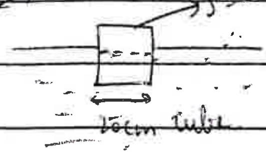


$W_B = -\rho_L g V x$

$W_{\text{gravity on liquid}} = -\rho_L g V x$

$\Rightarrow W_B = W_{g\ on\ liq}$ (Works even when body is not fully submerged)

• Q: Find min. $W_{ext.}$ to lift cube out of water.



Sol: $W_{ext} + W_B + W_g = \Delta KE = 0$

W_B: W_{g2} — $4 \times 10 \times 0.05$; W_g: $-4 \times 10 \times 0.1$

$\Rightarrow \boxed{W_{ext} = 2J}$

OR by integration

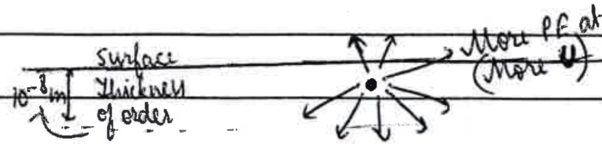
$F - 4g + (1000 \times 0.04(\ )(0.1 - x)\,10) = 0$

$\Rightarrow F = 400x$

$\Rightarrow W_{ext} = \int F_{ext} \cdot dx = 400\int_0^{0.1} x\,dx = \boxed{2J}$

SURFACE TENSION:

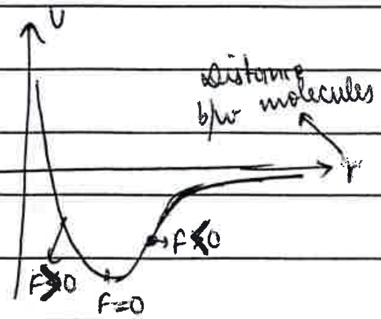
The free surface of a liquid behaves like a stretched membrane, this phenomenon is known as surface tension.



$U \propto$ No. of molecules on the surface $\propto$ Area

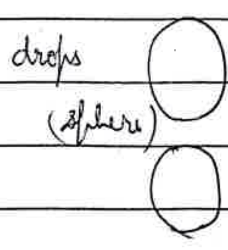
$\Rightarrow U = \sigma A$

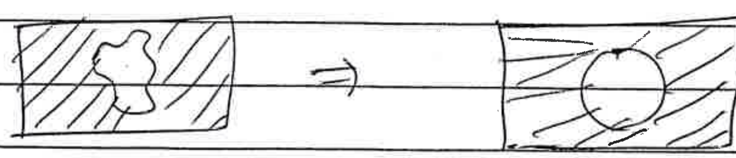
σ = surface tension



• For a given volume, area of sphere is minimum. For a given perimeter, area of circle is maximum.

So, drops acquire spherical shape ($A\downarrow \Rightarrow U\downarrow$)





For min. area of soap film, area of loop = max. $\Rightarrow$ (loop must be circular)

Q. A drop of radius 1 cm is dropped from height h. ($\sigma_{water} = 0.075\ N/m$) What is h_{min} so that after collision it may split into 1000 identical droplets? (Consider final & initial droplets as sphere)

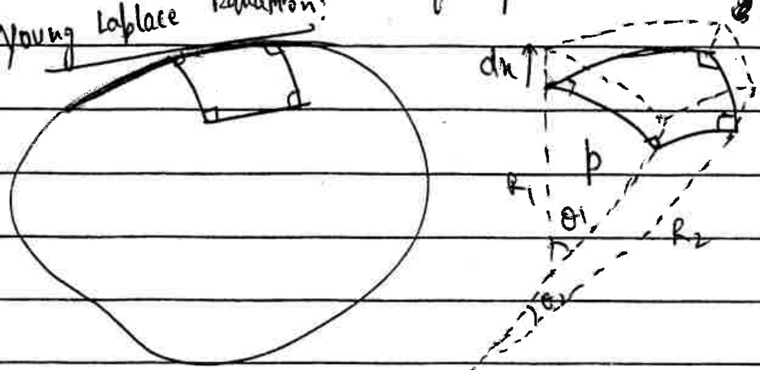
Sol. When it gets ~~con~~ split into more droplets, surface area ↑ ⇒ U ↑
This ~~increase~~ potential energy comes from G.P.E.

$\rho \cdot \frac{4}{3}\pi R^3 = \rho \cdot \frac{4}{3}\pi r^3 \cdot 1000 \Rightarrow \boxed{r = \frac{R}{10}}$

$mgh + \sigma \cdot 4\pi R^2 = 1000 \cdot \pi \cdot 4\pi r^2 \Rightarrow h = (0.075 \times 27\pi)\,cm$
$= 2.025\,cm.$

• Excess pressure:
Free surface of liquid behaves likes a stretched membrane, so the pressure inside the liquid drop is more than the pressure outside. This difference of pressure is known as excess pressure.

Young Laplace Equation:



θ_1, θ_2 = small

• Principle of virtual work: It states that if a system is in equilibrium and we displace it slightly from the equilibrium, then net work done by all the forces in that displacement is zero.

Now, ~~in the~~ when we slightly displace the surface,

$dW_p = pR_1\theta_1 R_2\theta_2 \, dx$ $\qquad (R_1\theta_1)(R_2\theta_2) = \text{Area}$

$dW_{p_0} = -p_0 R_1\theta_1 R_2\theta_2 \, dx$

$dW_{\text{surface tension}} = -dU$
$= -\sigma\left((R_1+dx)\theta_1 (R_2+dx)\theta_2 - R_1\theta_1 R_2\theta_2\right)$

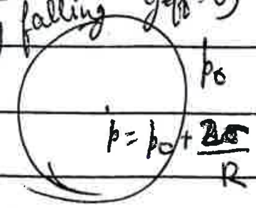
Now, $dW_p + dW_{p_0} + dW_\sigma = 0$

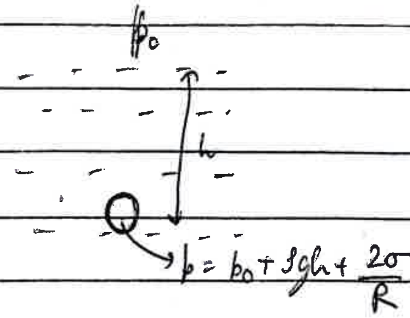
$\Rightarrow pR_1\theta_1 R_2\theta_2 \, dx - p_0 R_1\theta_1 R_2\theta_2 \, dx - \sigma(R_1+R_2)\theta_1\theta_2 \, dx$

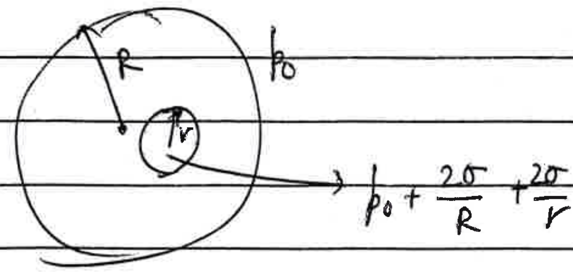
$\Rightarrow \boxed{p = p_0 + \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}$

- The pressure is more on the side where the centre of curvature of ~~bubb~~ surface lies.

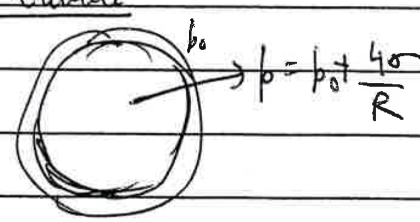
Liquid drop:
(Freely falling $g_{eff} = 0$)

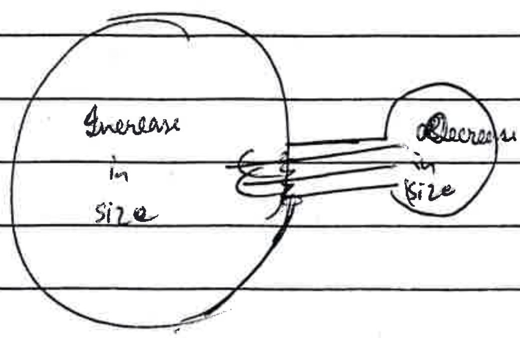


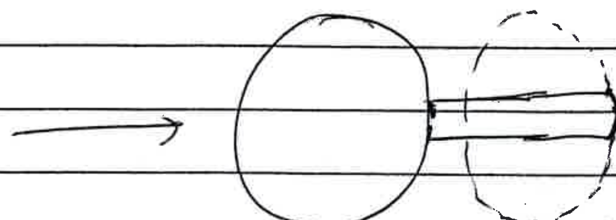




- **Soap bubble**

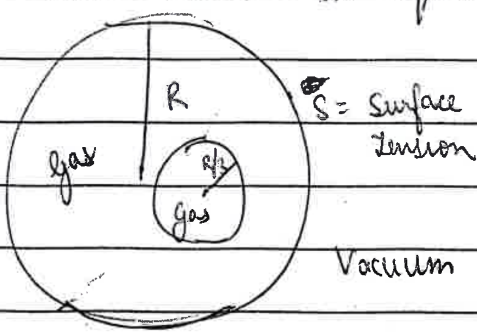






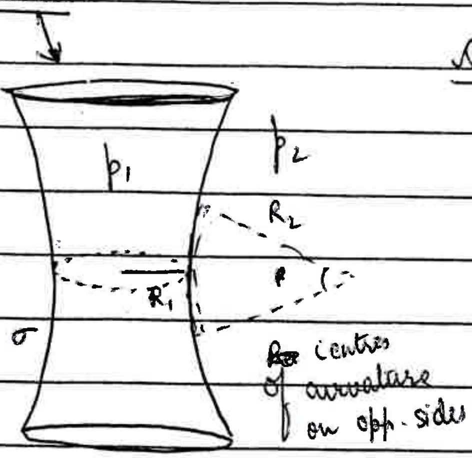
Till they become of the same size

Q. Find radius in equilibrium condition if inner bubble burst

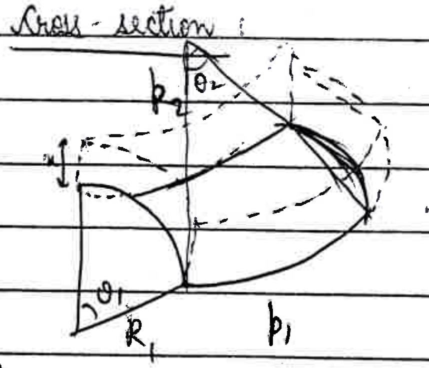


- Don't apply conservation of energy. Apply conservation of mass of gas.

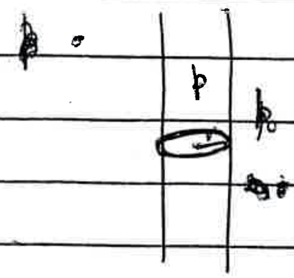
Spandrel:



Cross-section :

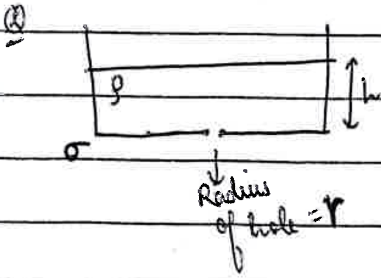


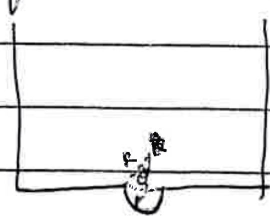
$$p_1 R_1\theta_1 R_2\theta_2 x - p_2 R_1\theta_1 R_2\theta_2 x$$
$$= \sigma(R_1+x)\theta_1(R_2-x)\theta_2 - \sigma(R_1\theta_1 R_2\theta_2) \Rightarrow \boxed{p_1 - p_2 = \sigma\left(\frac{1}{R_1} - \frac{1}{R_2}\right)}$$



$$p' = p_0 + \frac{\sigma}{r}$$
$$r_1 = r$$
$$r_2 = \infty$$

Q. Find max height so that liq. doesn't flow from hole.

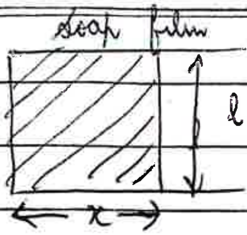




$$\rho g h = \frac{2\sigma \sin\theta}{r}$$
$$\Rightarrow h = \frac{2\sigma \sin\theta}{\rho g r}$$

$r = R\sin\theta \Rightarrow \boxed{h_{max} = \frac{2\sigma}{\rho g r}}$

Surface tension can also be defined as force per unit length of the boundary. Since the liquid surface acts like a stretched membrane, it pulls the boundary towards itself. This force per unit length is surface tension.



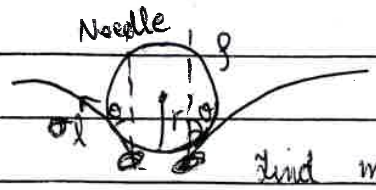
$U = 2\sigma l x$ → Two surfaces of soap film

$2F = -\frac{dU}{dx}$

$\Rightarrow F = \sigma l$

$\Rightarrow \boxed{\sigma = \frac{F}{l}}$

Q.



Find max "r" for floating.

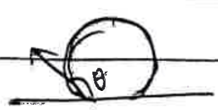
$2\sigma l \cos\theta = \rho(\pi r^2 l) g$

$\Rightarrow r = \sqrt{\frac{2\sigma\cos\theta}{\rho\pi g}}$

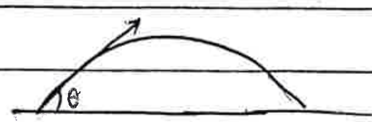
$\Rightarrow r_{max} = \sqrt{\frac{2\sigma}{\rho\pi g}}$

- **Contact angle:**

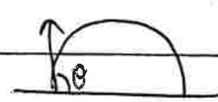
When a liquid drop is kept on a solid surface, it may spread on the solid surface or may not spread at all. This depends on the nature of the surface, the impurities present on the surface and the properties of the liquid. The angle between free surface of the liquid and liquid-solid boundary is known as contact angle. The concept of contact angle is valid only if the liquid hasn't reached the edge of the surface.



Water on lotus leaf



Water on glass



Water on gold plate

θ = Contact angle

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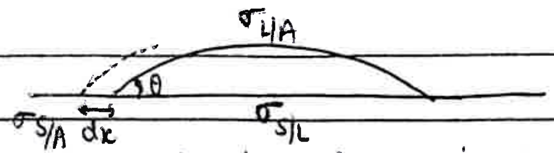


By principle of virtual work,

$$\sigma_{ls}\, l\,dx - \sigma_{sa}\, l\,dx + \sigma_{la}\, l\,dx \cos\theta = 0$$

l = dimension inside the plane of paper

$$\Rightarrow \boxed{\cos\theta = \frac{\sigma_{sa}-\sigma_{ls}}{\sigma_{la}}}$$

- If the contact angle is acute, the liquid is said to wet the surface. If the contact angle is 0°, the liquid wets the surface completely. For example, water on clean glass.
- If the contact angle is obtuse, the liquid doesn't wet the surface.
- $-1 \le \cos\theta \le 1$

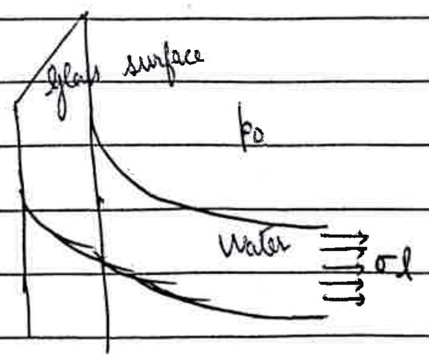
If $\frac{\sigma_{sa}-\sigma_{ls}}{\sigma_{la}} > 1 \Rightarrow \sigma_{sa} > \sigma_{ls} + \sigma_{la} \longrightarrow$ Contact angle is zero

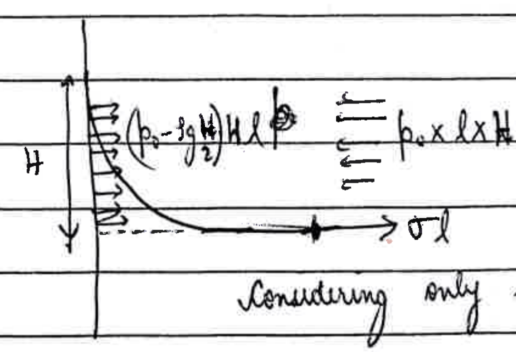
We can imagine σ_{sa} pulling the surface ~~without~~ ~~no~~ ~~influence~~ easily and σ_{ls} & σ_{la} not able to do anything.

$\sigma_{sa} < \sigma_{ls}+\sigma_{la}$ Not observed

- If force of cohesion > force of adhesion, contact angle is obtuse.
  If force of adhesion > force of cohesion, contact angle is acute

Shape of liquid near a solid surface:





Considering only horizontal forces:

$$\sigma \cancel{l} = \frac{\rho g H^2 \cancel{l}}{2} \Rightarrow \boxed{H = \sqrt{\frac{2\sigma}{\rho g}}}$$

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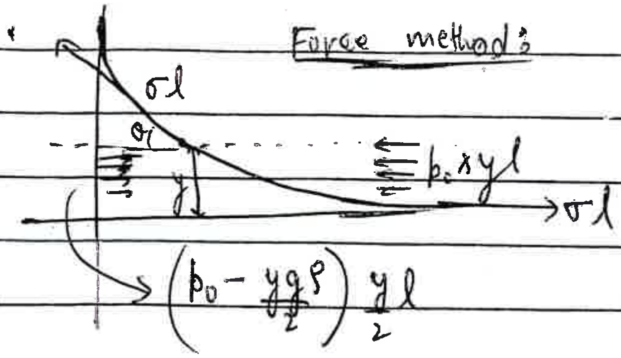
Force method:



$$\sigma l \cos\theta + p_0 yl = \sigma x + \left(p_0 - \frac{yg\rho}{2}\right)\frac{yl}{2}$$

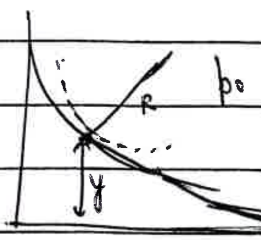
$$\Rightarrow \cos\theta = \frac{2\sigma - y^2\rho g}{2\sigma}$$

$\Rightarrow$ $\tan\theta$ from here

$\Rightarrow$ and $\frac{dy}{dx} = \tan(\pi-\theta) = -\tan\theta$

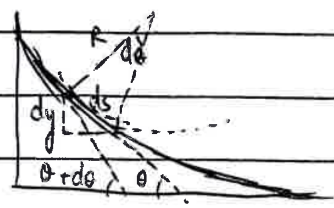
Integrate to get $y = f(x)$.

Pressure Method:



$$p_0 - \frac{\sigma}{R} = p_0 - \rho y g$$

$$\Rightarrow \boxed{\frac{\sigma}{\rho y g} = R}$$



$\frac{dy}{ds} = \sin\theta$ $\qquad$ $\rho\, ds = R\, d\theta$

$$\Rightarrow R = \frac{dy}{\sin\theta\, d\theta} = \frac{\sigma}{\rho y g}$$

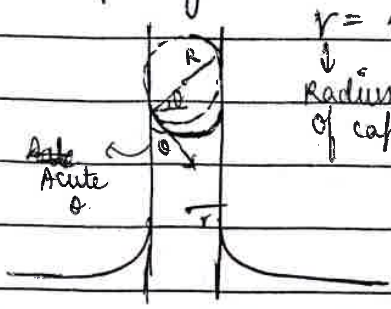
$$\Rightarrow \int_0^y y\,dy = \frac{\sigma}{\rho g}\int_0^\theta \sin\theta\, d\theta$$

If $\theta = 90°$, $y = H$, $H = \sqrt{\frac{2\sigma}{\rho g}}$

$$\Rightarrow \frac{y^2}{2} = \frac{\sigma}{\rho g}(1-\cos\theta)$$

$$\Rightarrow y = \sqrt{\frac{\sigma}{\rho g}} \times 2\sin\frac{\theta}{2}$$

- **Capillary ~~Tube~~ Rise:**

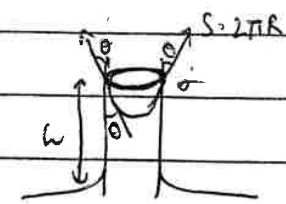


$r = R\cos\theta$
↓
Radius of capillary

$$p_0 - \frac{2\sigma}{R} = p_0 - \rho g h$$

$$\Rightarrow h\rho g = \frac{2\sigma\cos\theta}{r}$$

$$\Rightarrow \boxed{h = \frac{2\sigma\cos\theta}{\rho g r}}$$



$$S \cdot 2\pi R\cos\theta = mg = \rho g(\pi R^2 h)$$

$$\Rightarrow \boxed{h = \frac{2S\cos\theta}{\rho g R}}$$

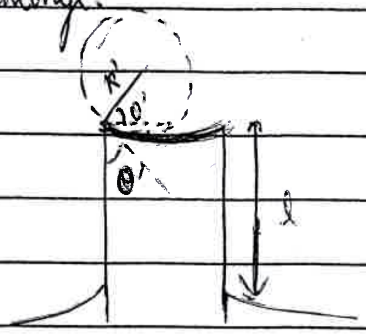
When $\theta = 0°$ and r is not negligible than h,

$$S \times 2\pi r = \left(\pi r^2 h + \frac{2}{3}\pi r^3\right)\rho g$$

$$\Rightarrow h = \frac{2S}{r\rho g} + \frac{2r}{3}$$

- If the contact angle is acute, $h = +ve \Rightarrow$ liquid rises in the capillary.
- If the contact angle is obtuse, liquid gets depressed in the capillary. ($\hookrightarrow h = -ve$)

If the capillary tube is of insufficient length, i.e., $l < \frac{2S\cos\theta}{R\rho g}$, the liquid will rise to the top and the contact angle will change.

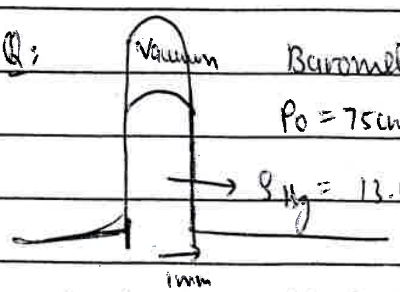


$$S\cdot 2\pi r\cos\theta' = l\times\pi r^2\rho g$$

$$\Rightarrow \frac{r}{\cos\theta'} = \frac{2S}{l\rho g}$$

$$\boxed{\frac{r}{\cos\theta'} = R' = \frac{2S}{l\rho g}}$$

Q:



Barometer

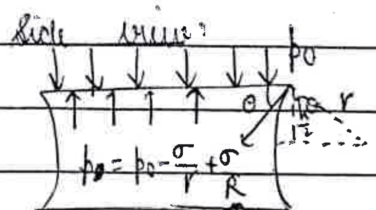
$P_0 = 75\text{ cm Hg}$

$\rho_{Hg} = 13.6\text{ g/cc}$

Contact angle $= 143^\circ$

Find height obtained by barometer.

Force of attraction between two discs wetted by liquid:



$r = \frac{h}{2}\sec\theta$

$p \approx p_0 - \frac{\sigma}{r}$

$= p_0 - \frac{2\sigma\cos\theta}{h}$

$F_{attraction} = (p_0 - p)\pi R^2 + \sigma\, 2\pi R\sin\theta$

$= 2\pi\sigma R\left(\frac{R}{h}\cos\theta + \sin\theta\right)$

$\boxed{R \gg h}$

$$\Rightarrow \boxed{F \approx \frac{2\pi\sigma R^2\cos\theta}{h}}$$

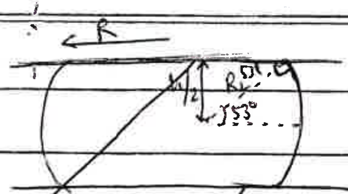
Q

$m = 10\,gm$

$\rho_{Hg} = 13.6\ g/cc$

$V = 10\pi\ cm^3$

$\sigma = 0.5\ N/m \quad \theta = 143°$



$\frac{h}{2R_2} = \sin 53°$

$\Rightarrow R_2 = \frac{5h}{8}$

$\pi R^2 h = V$

$\Rightarrow h = \frac{V}{\pi R^2}$

$p = p_0 + \sigma\left(\frac{1}{R} + \frac{1}{R_2}\right)$

$= p_0 + \sigma\left(\frac{1}{R} + \frac{8}{5h}\right)$

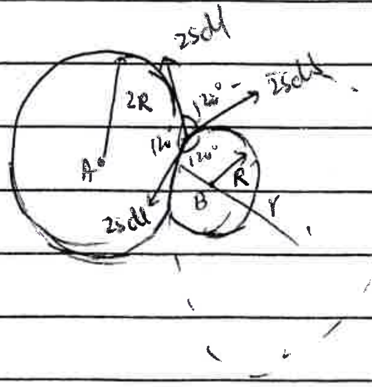
$= p_0 + \sigma\left(\frac{1}{R} + \frac{8\pi R^2}{5V}\right)$

Now, $p_0 \pi R^2 + mg + \sigma \cdot 2\pi R \cos 53° = p\pi R^2$

$\Rightarrow$ equation ---- $\Rightarrow$ we can't solve by known techniques

$R_2 \times R_1, h \times R \times$

Q:

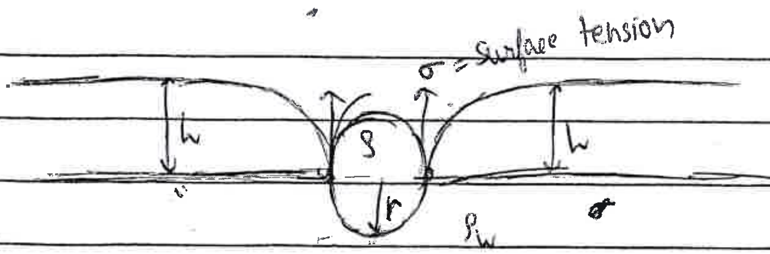


(i) $r = ?$ $\qquad r = 2R$

(ii) (AB) length = ?

$AB = \left[(2R)^2 + R^2 - 2\cdot(2R)\,R\cos 60\right]^{1/2}$

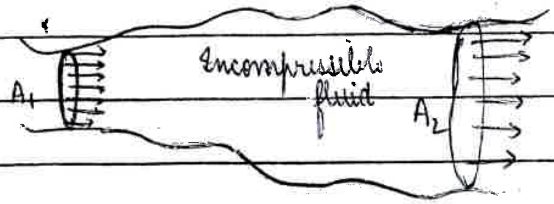
$= \underline{\sqrt{3}R}$



(consider buoyancy)

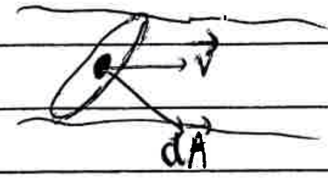
$$h = \frac{\rho g \pi R^2 - \rho_w g \frac{\pi R^2}{2} - 2\sigma}{2\rho_w g r}$$

FLUID DYNAMICS :



velocity flux

$$\frac{dV}{dt} = \int \vec{V}\cdot d\vec{A}$$



$$\left(\frac{dm}{dt}\right)_1 = \left(\frac{dm}{dt}\right)_2$$

$$\Rightarrow \rho\left(\frac{dV}{dt}\right)_1 = \rho\left(\frac{dV}{dt}\right)_2$$

$$\Rightarrow A_1\left(\frac{dl}{dt}\right)_1 = A_2\left(\frac{dl}{dt}\right)_2$$

$$\Rightarrow \boxed{A_1V_1 = A_2V_2}$$ (equation of continuity)

~~So~~, we ~~have~~ should take dot product to take only perpendicular projection of area.

$$\oint \vec{V}\cdot d\vec{A} = 0$$

Here, v is velocity w.r.t pipe / container.

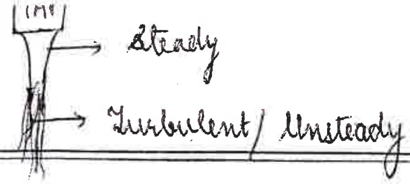
~~Rain~~

Q Rain is falling at 2 m/s @ 37° to the vertical. We have a vertical container which gets filled in 20 minutes. Now, the container is moved with a velocity of $2\hat{i}+4\hat{j}$. Find the time in which container will get filled.

Bernoulli's theorem:

- Assumptions: ① Incompressible fluid
  ② Non-viscous fluid
  ③ Steady / laminar / streamlined
  ④ Irrotational

- Flow is said to be viscous when there is fluid friction. In non-viscous flow, we can apply law of conservation of energy.



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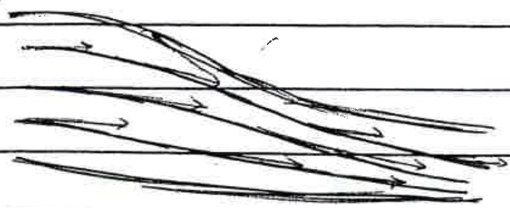
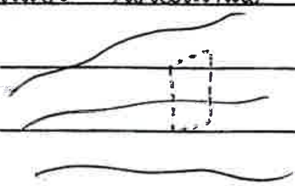
- Flow is said to be steady when velocity at any point of the fluid flow doesn't change with time.

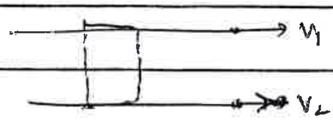


- Streamlines are imaginary lines whose tangents give us the direction of fluid flow at that point of fluid flow.

- ~~Flow~~ is said to be irrotational if fluid ~~particles~~ (molecules) don't rotate about their COM. This means that the fluid particles don't have rotational KE.

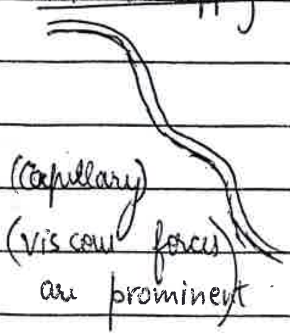


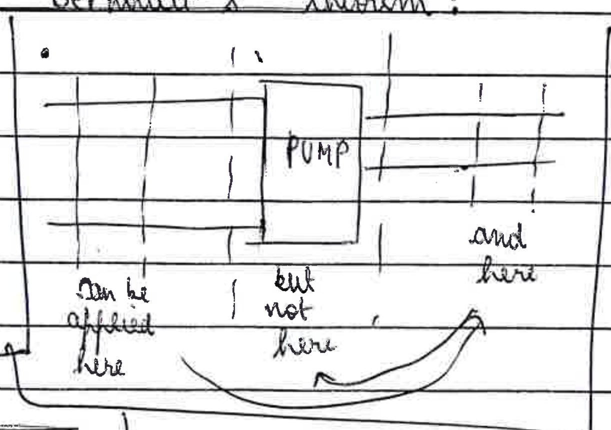
$\oint \vec{v} \cdot \vec{dl} = 0$ for irrotational flow.

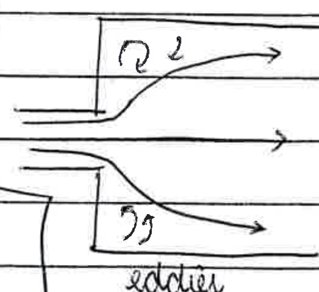


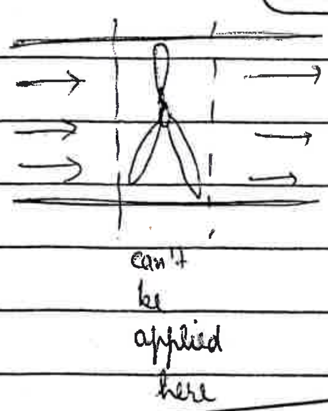
$\oint \vec{v} \cdot \vec{dl} = v_1 l - v_2 l \neq 0$ (rotational flow)

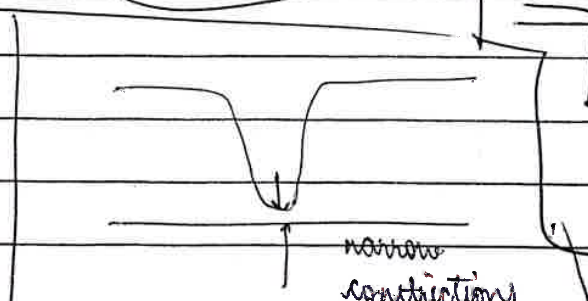
- Not to apply Bernoulli's theorem:

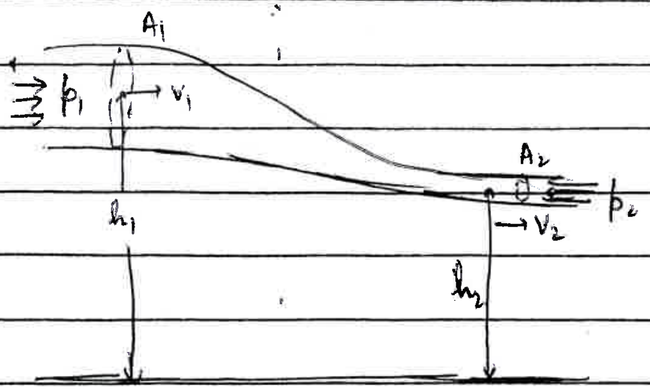












$$\frac{dm}{dt} = \rho\frac{dV}{dt}$$

$$= \rho A_1 V_1 = \rho A_2 V_2$$

$$\left(\frac{dW}{dt}\right)_p = \cancel{\rho_1 A_1 V_1}\; p_1 A_1 V_1 - p_2 A_2 V_2 = (p_1 - p_2) A_1 V_1$$

$$\left(\frac{dW}{dt}\right)_g = \left(\frac{dm}{dt}\right) g(h_1 - h_2) = \rho A_1 V_1 g(h_1 - h_2)$$

$$\frac{dK}{dt} = \frac{1}{2}\frac{dm}{dt}(v_2^2 - v_1^2) = \frac{1}{2}\rho A_1 V_1 (v_2^2 - v_1^2)$$

$$\left(\frac{dW}{dt}\right)_p + \left(\frac{dW}{dt}\right)_g = \frac{dK}{dt} \Rightarrow (p_1 - p_2) A_1 V_1 + \rho A_1 V_1 g(h_1 - h_2) = \frac{1}{2}\rho A_1 V_1 (V_2^2 - V_1^2)$$

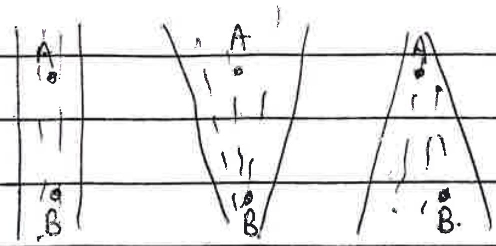
$$\Rightarrow \boxed{\frac{p_1}{\rho g} + \frac{V_1^2}{2g} + h_1 = \frac{p_2}{\rho g} + \frac{V_2^2}{2g} + h_2}$$

Bernoulli's theorem states that the sum total of pressure head, velocity head and gravitational head is constant.

$\frac{p}{\rho g}$ = pressure head $\qquad \frac{V^2}{2g}$ = velocity head

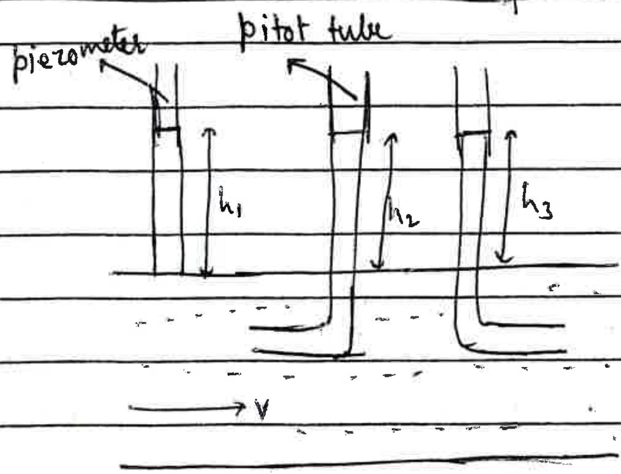
h = gravitational head.

To apply Bernoulli's theorem, we need to consider 2 different points in a fluid flow. Both the points should be preferably on the same streamline. This means that the same fluid is flowing at both these points. ρ is the density of that fluid which is flowing at that point. If the fluid is flowing in a particular direction, the pressure in that direction can't be governed by $p = p_0 + \rho gh$. But in a direction perpendicular to the flow, we can apply $p = p_0 + \rho gh$.



$p_B = p_A + \rho g h$ | p_A & p_B can't be compared | $p_B > p_A$

Q



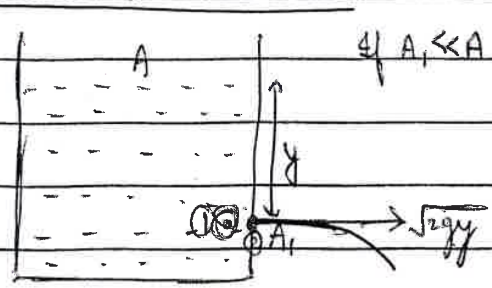
Ans: $h_3 = h_1 < h_2$

$h_1 = h_3$ (both conditions are equivalent)

Also, $\rho g h_1 + \underbrace{\frac{1}{2}\rho v^2}_{\text{Dynamic pressure}} = \rho g h_2$

$\Rightarrow h_2 = h_1 + \frac{v^2}{2g} > h_1$

Torricelli's theorem:



If $A_1 \ll A$, $\frac{p_0 + \rho g y}{\rho g} + \frac{0^2}{2g} + 0 = \frac{p_0}{\rho g} + \frac{v^2}{2g} + 0$

$\Rightarrow v^2 = 2gy \Rightarrow \boxed{v = \sqrt{2gy}}$

$A_1 v = A\, v_{surface}$

$\frac{p_0 + \rho g y}{\rho g} + \frac{v_s^2}{2g} + 0 = \frac{p_0}{\rho g} + \frac{v^2}{2g}$

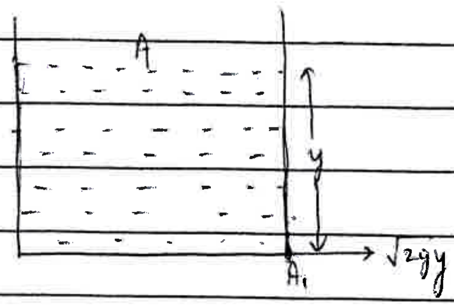
$\Rightarrow 2gy = (v^2 - v_s^2) \Rightarrow 2gy = v^2\left(1 - \frac{A_1^2}{A^2}\right)$

$\Rightarrow \boxed{v = \sqrt{\frac{2gy}{1 - A_1^2/A^2}}}$

$$F_{thrust} = v_{rel}\frac{dm}{dt} = v \times \frac{dm}{dV} \times \frac{dV}{dt} = v\rho A_1 v = \rho A_1 v^2$$
$$= 2\rho A_1 g y$$

Q. Water tank has an area of 1 m². Water is filled upto a height of 1 m. A hole is made on the side wall close to the bottom. Area of hole = 1 cm². Find ① initial velocity of efflux. ② Time taken to empty the tank. ③ momentum imparted to the water till then.

Ans:



$$A\frac{dy}{dt} = -\sqrt{2gy}\,A_1$$

$$\Rightarrow \int_1^0 \frac{dy}{\sqrt{y}} = \int_0^t \sqrt{2g} \times 10^{-4}\,dt$$

$$\Rightarrow \boxed{t = \frac{10^4}{\sqrt{5}}\text{ seconds}}$$

Another method:

$$v_s = 10^{-4}\sqrt{2gy}$$

$$\Rightarrow a_s = v_s\frac{dv_s}{dy}$$

$$= 10^{-4}\sqrt{2gy} \times \frac{10^{-4}}{2}\sqrt{\frac{2g}{y}}$$

$$= g \times 10^{-8} = 10^{-7}\text{ m/s}^2 \text{ (constant)}$$

$$t = \sqrt{\frac{2\times 1}{10^{-7}}} = \sqrt{\frac{2s}{a}}$$
$$= \frac{10^4}{\sqrt{5}}\text{ sec}$$

$$\int dp = \int dm \times \sqrt{2gy}$$

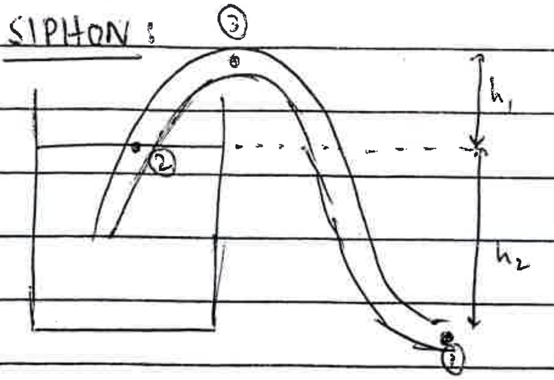
$$\Rightarrow p = \int \frac{dm}{A\,dy} \times dy\sqrt{2gy}$$

$$= 10^3\int \sqrt{20}\sqrt{y}\,dy$$

$$= 10^3\sqrt{20}\,\frac{2}{3}$$

$$= \boxed{\frac{2\sqrt{20}}{3} \times 10^3\text{ kg m/s}}$$

* SIPHON :



① & ③ :

$$\frac{p_0}{\rho g}+\frac{v^2}{2g}+0=\frac{p_3}{\rho g}+\frac{v^2}{2g}+h_1+h_2$$

$$\Rightarrow \boxed{p_3=p_0-(h_1+h_2)\rho g}$$

~~Also~~, ① & ② : $\boxed{p_2=p_0-\rho g h_2}$

2 outside & 2 inside :

$$\frac{p_0}{\rho g}+\frac{0^2}{2g}+0=\frac{p_2}{\rho g}+\frac{v^2}{2g}+0 \Rightarrow \boxed{p_2=p_0-\frac{\rho v^2}{2}}$$

$V=\sqrt{2gh_2}$

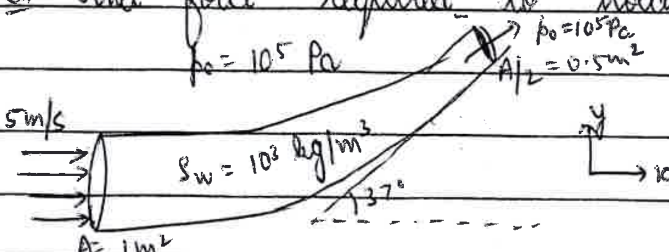
Since −ve pressure doesn't exist,

$p_2>0 \Rightarrow p_0-\rho g h_2>0 \Rightarrow \boxed{h_2<\frac{p_0}{\rho g}}$

$p_3>0 \Rightarrow \boxed{h_1+h_2<\frac{p_0}{\rho g}}$

Condition for siphon

Q. Find force required to hold the pipe

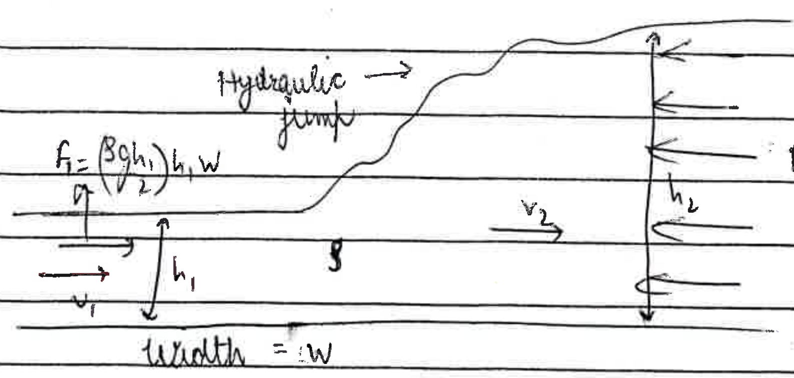


Ans: ~~$(15000\hat{i}+30000\hat{j})$ N~~

$3.75\times10^4\,\hat{i}+\vec{F}=1.5\times10^4\,\hat{i}+3\times10^4\,\hat{j}$

$\Rightarrow \vec{F}=-2.25\times10^4\,\hat{i}+3\times10^4\,\hat{j}$

Don't forget the pressure (excess over p_0) at the left end, that is also acting at the (pipe + water inside pipe) system.



$$\boxed{v_2=\frac{h_1 v_1}{h_2}}$$

Now, $\left(\frac{dw}{dt}\right)_p+\left(\frac{dw}{dt}\right)_g-\frac{dE}{dt}=\frac{dK}{dt}$

$$\left(\frac{dw}{dt}\right)_P = F_1v_1 - F_2v_2 = \frac{\rho g w}{2}\left(h_1^2v_1 - h_2^2v_2\right)$$

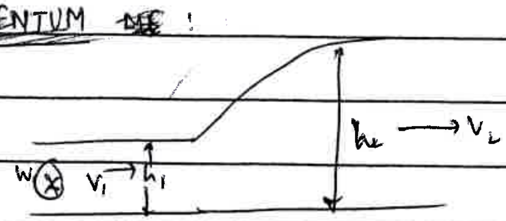
$$\left(\frac{dw}{dt}\right)_g = \frac{dm}{dt}\,g\left(\frac{h_1-h_2}{2}\right) \qquad \frac{dK}{dt} = \frac{1}{2}\frac{dm}{dt}\left(v_2^2 - v_1^2\right)$$

$$\frac{dE}{dt} = \frac{\rho_w g v_1 h_1}{2}\left(h_1-h_2\right)\left(2 - \frac{v_1}{gh_2^2}(h_1+h_2)\right)$$

$$\Delta \text{ head} = \frac{\frac{dE}{dt}}{\frac{dm}{dt}g} = \left(h_2-h_1\right)\left(\frac{v_1}{2gh_2^2}(h_1+h_2)-1\right)$$

can also be done by difference of heads on both sides.

MOMENTUM ~~EE~~ :



$$F_{left} = wh_1 \times \frac{\rho g h_1}{2} \qquad F_{right} = wh_2 \times \frac{\rho g h_2}{2}$$

$$\frac{\rho g w}{2}\left(h_1^2 - h_2^2\right) = \frac{dm}{dt}\left(v_f - v_i\right)$$

$$\Rightarrow \frac{\rho g w}{2}\left(\cancel{h_1-h_2}\right)(h_1+h_2) = \rho w h_1 v_1 v_1\left(\frac{\cancel{h_1-h_2}}{h_2}\right)$$

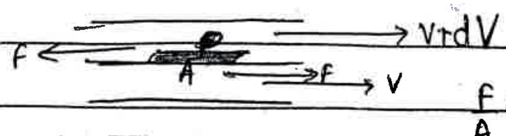
$$\Rightarrow h_2^2 + h_1h_2 = \frac{2h_1v_1^2}{g} \Rightarrow \boxed{h_2 = \frac{-h_1 + \sqrt{h_1^2 + \frac{8h_1v_1^2}{g}}}{2}}$$

$h_2 > h_1$
$\Rightarrow \boxed{v_1 > \sqrt{gh_1}}$ for hydraulic jump

VISCOSITY: (fluid friction)

No slip condition: If a fluid is in contact with a solid surface, the fluid doesn't move relative to the surface.
$\Rightarrow$ Fluid flows in layers. Different layers move relative to each other.

① Liquids:

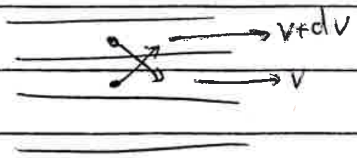


$F = F_{\parallel}$ to surface = Shear force

$$\frac{F}{A} \propto \frac{dV}{dy}$$

Shear force is due to attractive force between liquid molecules.

(2) Gases: Intermolecular forces are negligible.



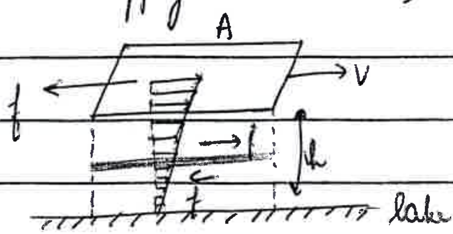
There is interlayer mixing. So there is a friction force between different layers.

$$\boxed{\frac{df}{dA} = -\eta \frac{dv}{dy}}$$ η = coefficient of viscosity (depends on temp.)

In case of liquids, η decreases with ↑ in temp.
In case of gases, η increases with ↑ in temp.

$\boxed{\frac{df}{dA} = \eta \frac{dv}{dy}}$ ⇒ Newton's law of viscosity ← (Not exactly true)
Newtonian fluids follow this law. ← (No such fluid exists)

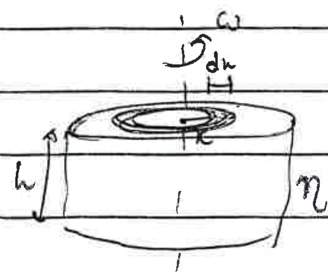
To apply this law, we have to find velocity profile $\frac{dv}{dy}$.



A for all layers is same
⇒ f is same for all layers.
⇒ $\frac{f}{A} = -\eta \frac{dv}{dy}$ ⇒ $\frac{dv}{dy} = \text{const} = \frac{v}{h}$

⇒ $\boxed{f = \frac{\eta A v}{h}}$

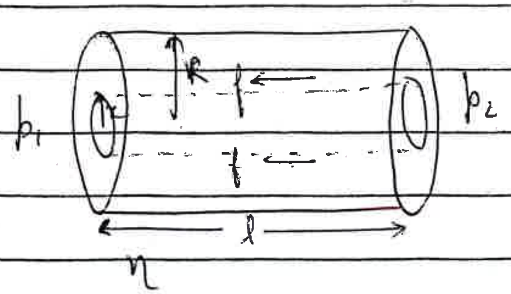
Q:



Find τ required.

$$\tau = \int d\tau = \int_0^R \frac{\eta(2\pi x\, dx)(x\omega)\, x}{h} = \boxed{\frac{\pi\eta\omega R^4}{2h}}$$

Q



$$\frac{f}{A} = \frac{(p_1-p_2)\pi x^2}{2\pi x l} = -\eta \frac{dv}{dx}$$

$$\Rightarrow \int_0^R \frac{(p_1-p_2)\, x\, dx}{2l\eta} = \int_{v_0}^{0} -dv$$

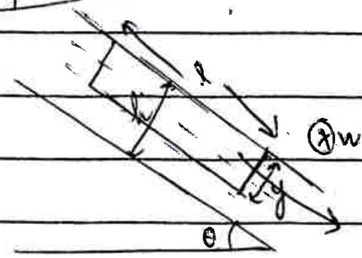
$$\Rightarrow v_0 = \frac{(p_1-p_2)\cancel{x} R^2}{4l\eta}$$

(1) $v = f(x)$
(2) v_0 at centre $= \frac{(p_1-p_2)R^2}{4\eta l}$
(3) $\frac{dV}{dt}$

$$v = f(x) = v_0\left(1 - \frac{x^2}{R^2}\right)$$

$$\frac{dV}{dt} = \int 2\pi x\, dx\; v = \int_0^R v_0\left(1 - \frac{x^2}{R^2}\right) 2\pi x\, dx = \cancel{v_0 \frac{2\pi R^2}{2} - v_0 \frac{2\pi R^4}{4R^2}} = v_0 \frac{\pi R^2}{2}$$

$$\Rightarrow \boxed{\frac{dV}{dt} = \frac{(p_1 - p_2)\pi R^4}{8 l \eta}}$$ Poiseuille's equation

Q. 

$$\frac{F}{A} = \frac{\rho l y w g \sin\theta}{wl} = -\eta \frac{dv}{dy}$$

$$\Rightarrow \int_0^v dv = -\int_h^y \frac{\rho g \sin\theta}{\eta} y\, dy$$

$$\Rightarrow \boxed{v = \frac{\rho g \sin\theta}{\eta}\left[h^2 - y^2\right]} = v_{surface}\left[1 - \frac{y^2}{h^2}\right]$$

$$\Rightarrow \boxed{v_{surface} = \frac{\rho g h^2 \sin\theta}{\eta}}$$

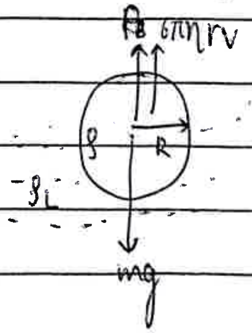
$$\frac{dV}{dt} = \int v\, w\, dy$$

$$= \frac{\rho g \sin\theta\, w}{\eta}\int_0^h (h^2 - y^2)\, dy$$

$$= \frac{2\rho g \sin\theta\, w h^2}{3\eta}$$

- According to Stokes' law, the viscous force by a fluid on a spherical ~~ball~~ body moving through it is given by

$$\vec{F} = -6\pi\eta r\vec{v}$$



$$mg - B - 6\pi\eta r v = ma$$

$$\Rightarrow \rho \frac{4}{3}\pi R^3 g - \rho_L \frac{4}{3}\pi R^3 g - 6\pi\eta R v = \rho \frac{4}{3}\pi R^3 a$$

$$\Rightarrow a = g\left(1 - \frac{\rho_L}{\rho}\right) - \frac{9\eta v}{2\rho R^2}$$

$$\Rightarrow \vec{v}_{terminal} = \frac{2}{9}\frac{R^2 \vec{g}(\rho - \rho_L)}{\eta}$$

~~The~~ Terminal velocity is the velocity attained by the body in the steady state, i.e., when the acceleration becomes zero.

$$v = v_T\left(1 - e^{\frac{-g\left(1-\frac{\rho_L}{\rho}\right)t}{v_T}}\right)$$

Q. Assuming the buoyant force as negligible, if a ball is dropped from a large height. It strikes the ground. $e = 0.5$ (Coeff. of restitution). Find acc. just after striking.

Ans. Acc. $= \frac{3g}{2}\downarrow$ use: $mg = 6\pi\eta r v$ (before ~~fallen~~ collision)

Reynold's number is a dimensionless number whose value is

$$Re = \text{Reynold's number} = \frac{\rho v d}{\eta}$$

(e is not in subscript)

Reynold's number characterises the fluid flow. If ~~th~~ it is less than a critical value, the flow is said to be streamlined.

ρ = density of fluid. η = coefficient of viscosity

d = characteristic length

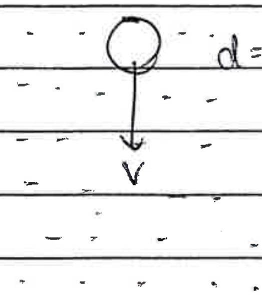
v = velocity of fluid flow

Pipe flowing full

d = Diameter of pipe

Re < 1000	Re > 3000
streamlined	turbulent

1000 < Re < 3000 (transition)



d = diameter of ball

Kinematic viscosity $= \frac{\eta}{\rho} = \nu$ (nu)

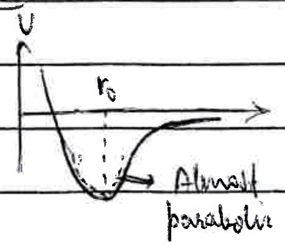
SI unit : Stokes

Elasticity:

A body is said to be **elastic** if it gets deformed on application of an appropriate force but regains its shape or size on removal of ~~the~~ ext. force.

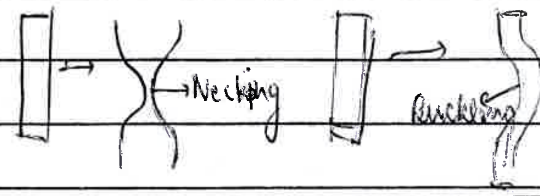
A body is said to be **plastic** if it gets deformed on application of force but doesn't regain its original shape and size on removal of force

Reason:



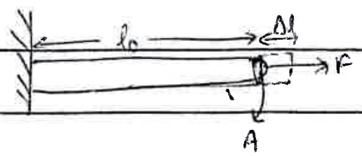
For small x,
$F \propto x$.

Expands. Compress:

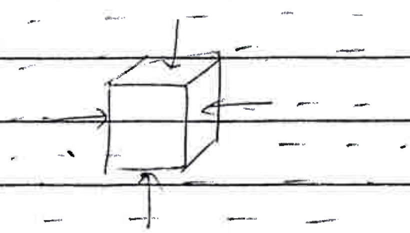


Stress: **Strain**

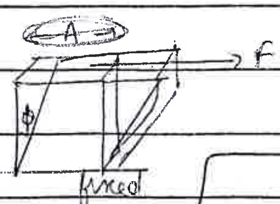
$\text{Stress} = \dfrac{\text{Force}}{\text{Area}}$



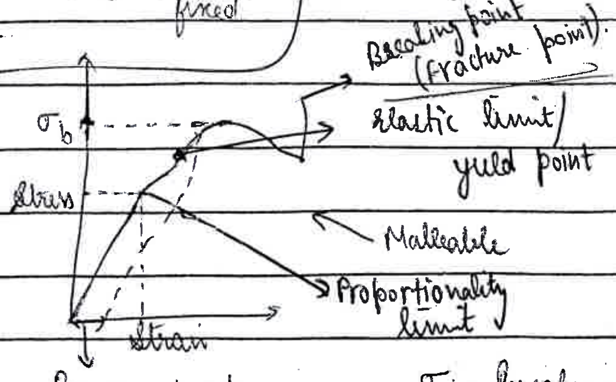
F = longitudinal force
F/A = longitudinal stress
$\Delta l / l_0$ = longitudinal strain

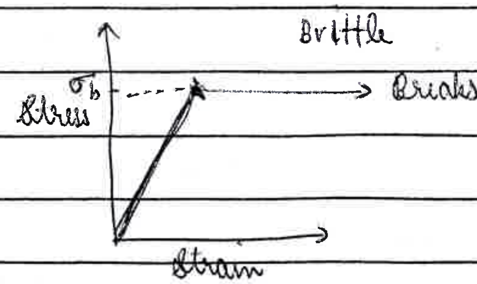


$F/A = p$ = volumetric stress
$\Delta V / V_0$ = volumetric strain

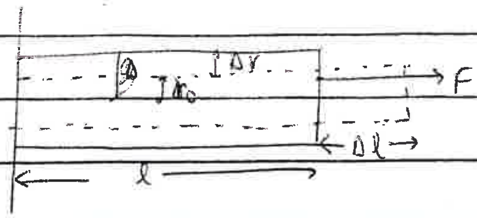


ϕ = shear strain
F/A = shear stress





σ_b = breaking ~~point~~ stress



$\Delta l / l_0 =$ Longitudinal strain

$\Delta r / r_0 =$ transverse strain

$$\frac{-\Delta r / r_0}{\Delta l / l_0} = \sigma \; : \; \text{Poisson's ratio}$$

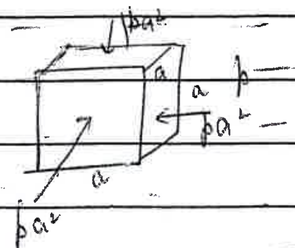
$$-1 < \sigma < 1/2 \quad (\text{Theoretical})$$

Longitudinal ~~strain~~ stress $\propto$ longitudinal strain

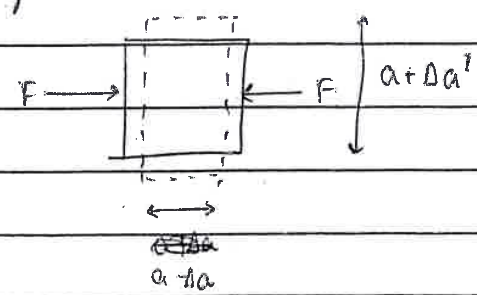
$$\frac{F}{A} = Y\frac{\Delta l}{l_0} \qquad Y = \text{young's modulus of elasticity}$$

$$p = -B\Delta V / V_0 \qquad B = \text{Bulk modulus of elasticity}$$

$$F/A = \eta\phi \qquad \eta = \text{Modulus of rigidity / shear modulus}$$



In single dimension,



$a' = a - \Delta a + 2\Delta a'$

$V = a^3$

$\ln V = 3 \ln a$

$$\frac{\Delta V}{V} = 3\,\frac{a' - a}{a}$$

$$\frac{\Delta V}{V} = \frac{3}{a}\left(\frac{-pa}{Y} + \frac{2p\sigma a}{Y}\right)$$

$$p = \frac{\Delta a}{a}\ldots \quad p = \frac{F}{a^2} = Y\frac{\Delta a}{a}$$

Also, $\frac{\Delta a'}{a} = \sigma\frac{\Delta a}{a}$

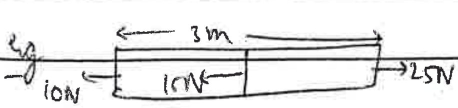
$$p = B\frac{\Delta V}{V}$$

$$\boxed{B = \frac{Y}{3(1-2\sigma)}}$$

$\because B > 0$

$\Rightarrow 1 - 2\sigma > 0 \Rightarrow \sigma < \frac{1}{2}$

eg



$Y = 10^{11}\ N/m^2$

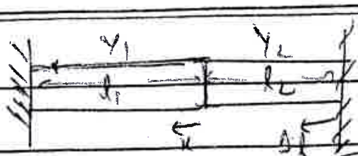
$A = 1\,cm^2$

Find $\Delta l = 5.25\ \mu m$

① stress at x from lower end $= \rho g x$

② Δr at x from lower end $= -\frac{\sigma \rho g h}{Y}$

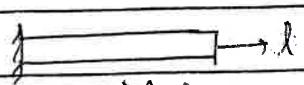
③ $\Delta l = \frac{\rho g l^2}{2Y}$



① $\Delta x = \dfrac{Y_2/L_2 \quad \Delta l}{Y_1/L_1 + Y_2/L_2} = \dfrac{Y_2 L_1 \Delta l}{Y_1 l_2 + Y_2 l_1}$

② T (tension in rod)

$F/A = Y \dfrac{\Delta l}{l} \Rightarrow \boxed{F = \dfrac{Yl}{A} \Delta l} \Rightarrow \boxed{F = k \Delta l}$ where $\boxed{k = \dfrac{YA}{l}}$

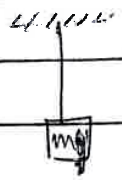


$U = \frac{1}{2} k x^2 \qquad U = \frac{1}{2} \frac{YA}{l} \Delta l^2$

$\dfrac{U}{Al} =$ Energy density $= \dfrac{1}{2} Y \left(\dfrac{\Delta l}{l}\right)^2 = \dfrac{1}{2} Y\ \text{strain}^2$

$= \dfrac{1}{2}$ stress strain

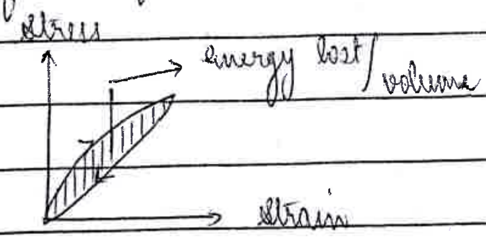
$= \dfrac{1}{2} \dfrac{\text{stress}^2}{Y}$



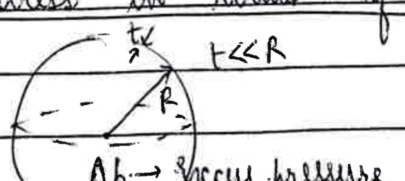
$U = \frac{1}{2} mg \Delta l \qquad W_g = mg \Delta l$

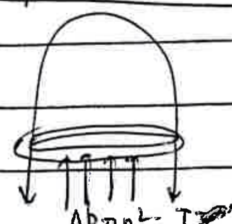
Remaining energy is lost in form of heat in rubbing of internal layers, resulting in increase in temperature of wire.

When the block is ~~released~~ released from the N.L position of wire the block doesn't oscillate for very long but settles at the eq. position very quickly. The remaining energy gets dissipated in the form of heat. This loss is known as hysteresis loss.



Stress in walls of a spherical container:





$\text{Stress} = \dfrac{T}{A} = \dfrac{\pi R^2 \Delta p}{2\pi R t}$

$= \boxed{\dfrac{\Delta p R}{2t}}$

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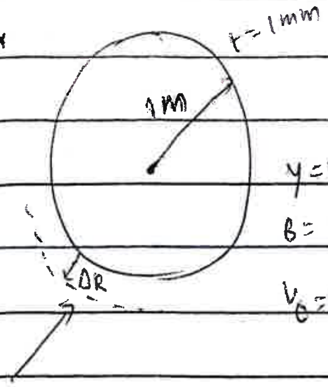


Find (i) eq. radius of shell
(ii) Pressure in liquid
(iii) Stress in walls of tank

$Y = 10^{11}\ N/m^2$

$B = 10^9\ N/m^2$

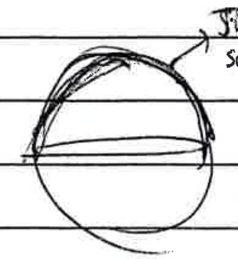
$V_0 = \left(\frac{4}{3}\pi + 4\pi \times 10^{-6}\right) m^3$ (Neglect P_{atm}, gravity)

Sol: $\cancel{\Delta V_0}$ $\Delta V_{liquid} = 4\pi\,(\Delta R - 10^{-6})$

$p = -B\frac{\Delta V}{V_0} = 3\times10^{9}\,(10^{-6} - \Delta R)$

$\frac{T}{A} = \frac{\Delta p R}{2t} = \frac{3\times10^9\,(10^{-6} - \Delta R)}{2\times10^{-3}} = \frac{T}{A} = Y\frac{\Delta R}{R} = 10^{11}\times \Delta R$

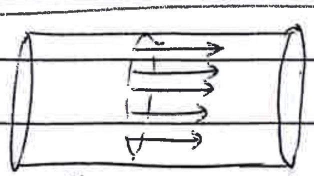
$\Rightarrow \boxed{\Delta R = \frac{15}{16}\,\mu m}$



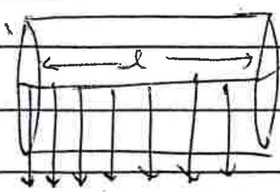
$\pi R \to \pi(R + \Delta R)$

$\Rightarrow \frac{\Delta l}{l} = \frac{\pi \Delta R}{\pi R} = \frac{\Delta R}{R}$

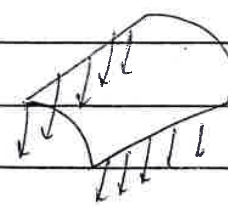
Stress in walls of a cylinder container:



Transverse stress



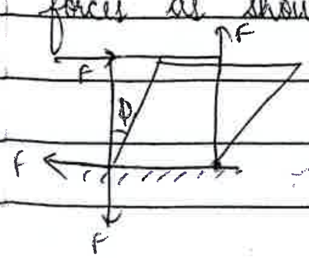
Longitudinal stress

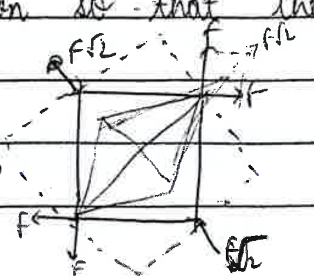


$\frac{T}{A} = \frac{\Delta p\cdot \pi R^2}{(2\pi R t)} = \boxed{\frac{\Delta p R}{2t}}$

$\frac{T}{A} = \frac{\Delta p\cdot 2Rl}{2lt} = \boxed{\frac{\Delta p R}{t}}$

A body is said to be in pure shear if it is subjected to shear forces as shown so that there is no internal torque.





$\frac{F\sqrt{2}}{\sqrt{2}a^2} = Y\frac{\Delta l}{a\sqrt{2}}$

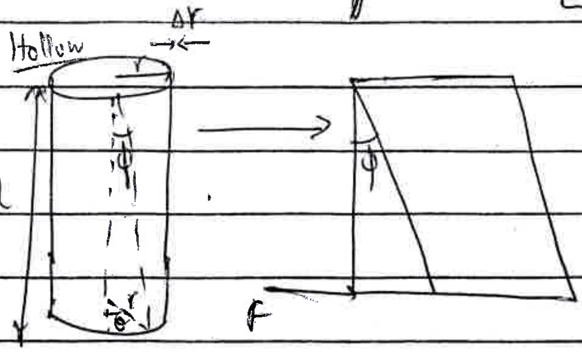
$\Rightarrow \frac{F\sqrt{2}}{aY} = \Delta l$

$\Delta l = \Delta l + \Delta l'$

$= \frac{F\sqrt{2}}{aY}(1+\sigma)$

Also, $\alpha\phi \cos 45° =$, $\Delta l = \frac{Fl}{aY}(1+\sigma) \Rightarrow \boxed{\eta = \frac{Y}{2(1+\sigma)}}$

Eg:



$F = A\eta\phi$

$= 2\pi r \Delta r\, \eta\phi$

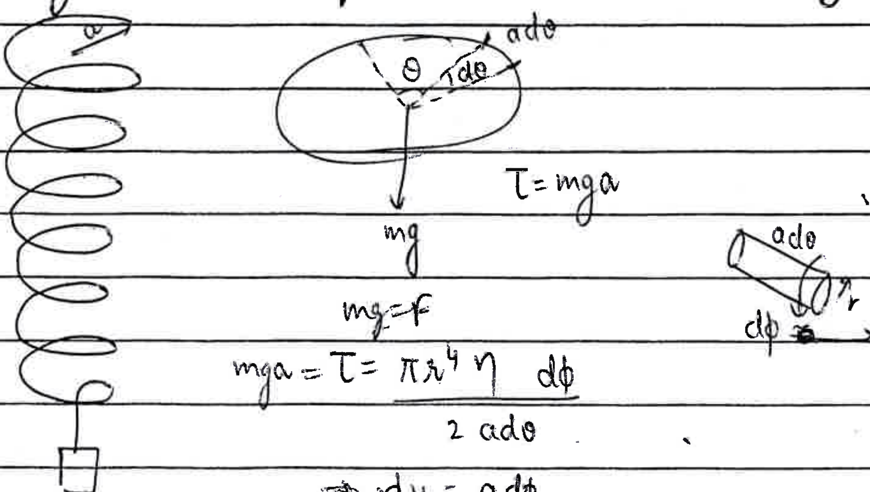
$l\phi = r\theta$

$\Rightarrow \tau = Fr$

$= \frac{2\pi r^3 \Delta r\, \eta\theta}{l}$

Solid cylinder: $d\tau = \frac{2\pi r^3 dr\, \eta\theta}{l}$ $\quad \tau = \frac{\pi r^4 \eta\theta}{2\, l}$

Spring constant of axially loaded spring:



$\tau = mga$

$mg = F$

$dφ \rightarrow$ angle of twist

$mga = \tau = \frac{\pi r^4 \eta}{2} \frac{d\phi}{a\,d\theta}$

$\Rightarrow dy = a\,d\phi$

$2mg\, a^3 \int_0^{2\pi n} d\theta = \pi \eta r^4 \int_0^y dy$

$\Rightarrow y = \frac{2mga^3 \cdot 2\pi n}{\pi \eta r^4}$

$mg = \frac{\eta r^4}{4na^3} y$

$\boxed{K = \frac{\eta r^4}{4na^3}}$

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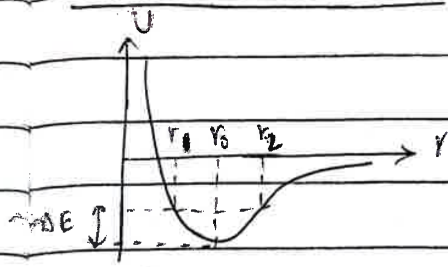
THERMAL EXPANSION:

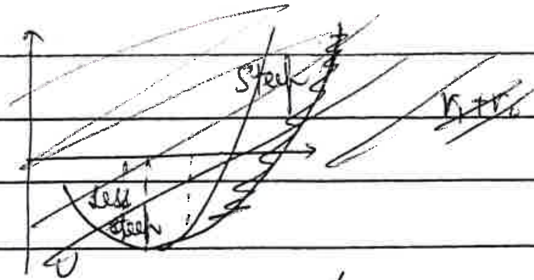


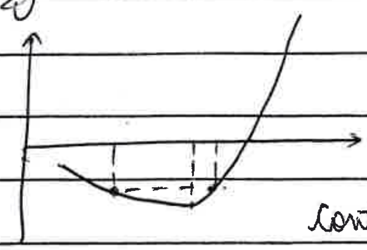
At T = 0K, particles at r_0.
As T ↑, particles start oscillating between r_1 & r_2.

$r_{avg} = \frac{r_1 + r_2}{2} > r_0$

(Right less steep than left part)

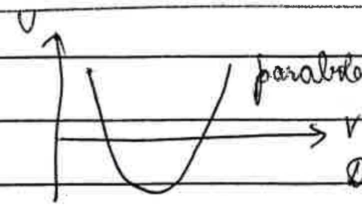
$\boxed{\text{expands on heating}}$





$\frac{r_1+r_2}{2} < r_0$

contracts on heating (eg. rubber)



$\frac{r_1+r_2}{2} = r_0$

Doesn't expand/contract on heating

$dL \propto L$ (more bonds, more ~~or~~ total expansion)

$dL \propto \Delta T$ (~~experimental~~ experimental) (not actually true)

$dL = L\alpha\Delta T$

α = coefficient of linear expansion $\sim 10^{-5}/°C$

$L' = L + dL$

$= L(1 + \alpha\Delta T)$

$\int_{L_0}^{L} \frac{dL}{L} = \alpha \int_{T}^{T+\Delta T} dT \Rightarrow L = L_0 e^{\alpha\Delta T} = L_0\left(1 + \frac{\alpha\Delta T}{1!} + \frac{\alpha^2\Delta T^2}{2!} + \dots\right)$

$T_0 \longrightarrow T_0 + \Delta T$ $\quad L_1 = L_0(1+\alpha\Delta T)$

$\longleftarrow$ $\quad L_2 = L_1(1-\alpha\Delta T)$

$\longrightarrow$ $\quad L_2 = L_0(1-\alpha^2\Delta T^2)$

$\longleftarrow$ $\quad$????

$\longrightarrow$

Reheat / Recool — How length decreased?

$\Rightarrow$ Use $e^{\alpha\Delta T}$

Error due to approximation

eg:

$T = 2$ sec

$g = \pi^2$

$\alpha = 4 \times 10^{-5}/°C$

If T ↑ by 10°C, how much time will clock gain/lose in 10^5 seconds?

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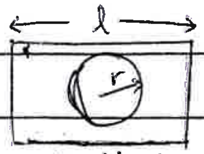
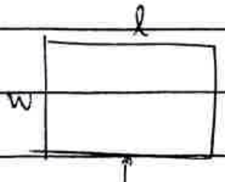


Here, the expansion is 'photographic' every dimension increases in the same proportion

r, l all increase



$$A = lw = l_0 w_0 (1+\alpha \Delta T)^2$$

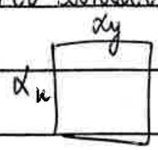
$$\Rightarrow A = A_0 (1 + 2\alpha \Delta T)$$

$$A = A_0 (1+\beta \Delta T) \qquad \beta = \text{coefficient of superficial expansion}$$

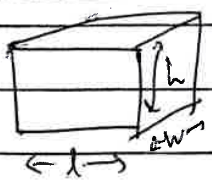
Isotropic solid

$$\beta = 2\alpha$$

For anisotropic solid (properties not same in all directions)



$$\beta = \alpha_x + \alpha_y$$



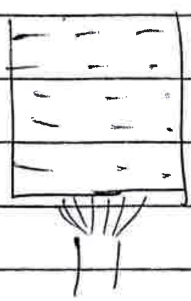
$$V = lhw = l_0 h_0 w_0 (1+\alpha \Delta T)^3$$

$$= V_0 (1+3\alpha \Delta T) = V_0 (1+\gamma \Delta T)$$

$$\gamma = 3\alpha$$

γ = coefficient of volume expansion
or coefficient of cubical expansion

In case of liquids, only coefficient of volume expansion is valid as they don't have a fixed surface (~~area~~ SA)



$$\Delta V = V_{liq} - V_{container}$$

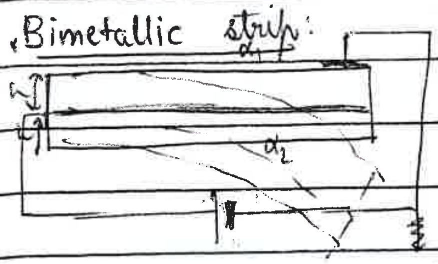
$$= V_0 (1+\gamma_L \Delta T) - V_0 (1+3\alpha \Delta T)$$

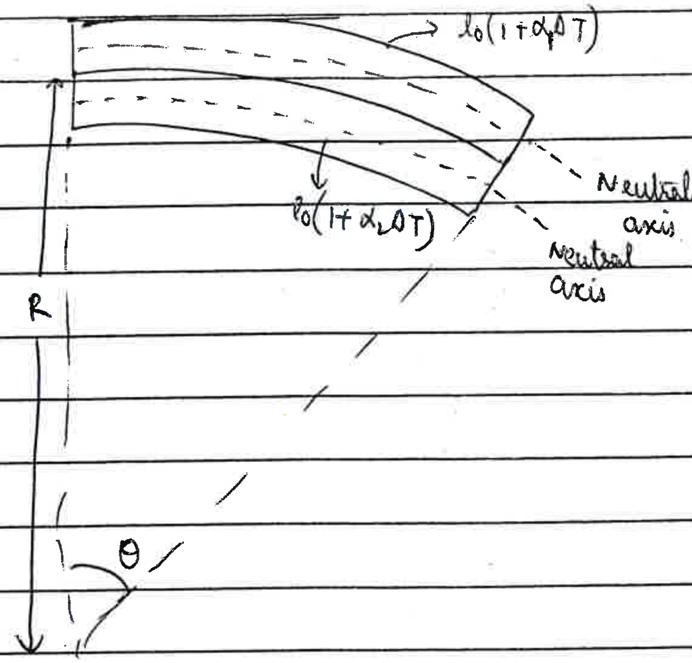
$$= V_0 \Delta T (\gamma_L - 3\alpha) \Delta T$$

$$\therefore \gamma_{apparent} = \frac{\Delta V}{V_0 \Delta T} = \gamma_L - 3\alpha$$

$$\rho_L = \frac{m}{V_0 (1+\gamma \Delta T)} \Rightarrow \boxed{\rho_L = \frac{\rho_0}{1+\gamma \Delta T}}$$

Bimetallic strip:





$$l_0(1+\alpha_1\Delta T) = \left(R+\frac{h}{2}\right)\theta$$

$$l_0(1+\alpha_2\Delta T) = \left(R-\frac{h}{2}\right)\theta$$

$$\theta = \frac{l_0(\alpha_1-\alpha_2)\Delta T}{h}$$

$$R = \frac{h}{2}\left(\frac{2+(\alpha_1+\alpha_2)\Delta T}{(\alpha_1-\alpha_2)\Delta T}\right)$$

Q Hydrometer is a device used to measure density of a liquid. It is known that hydrometer reads 10.1 mm at 20°C. What will be the temp. at which hydrometer will read 10.2 mm? ($\gamma_L = 10^{-3}/°C$) Ans: $\left(20+\frac{1000}{101}\right)$°C.

- For measurement of temp., we use some property of substance which linearly depends on temp.

$T = ax+b$

↳ a property linearly varying with temp, using which temp. is inferred.

① Mercury thermometer: $\Delta V = V_0\gamma\Delta T$

② Platinum-resistance thermometer: $\Delta R = \alpha_R R\Delta T$

③ Constant volume gas thermometer: $\Delta P = \left(nR/V\right)\Delta T$

④ Constant pressure gas thermometer: $\Delta V = \left(nR/P\right)\Delta T$

• **Calorimetry:**

• Heat is the energy in transit. It goes from a body at higher temp. to a body at lower temp.

• Zeroth law of thermodynamics: states that if A and B are in thermal equilibrium with each other and B and C are also in thermal equilibrium with each other, then A and C will also be in thermal eq^m with each other. Thermal equilibrium means if no (net) heat flows between two bodies.

• Reservoir of heat: Reservoir is a body from which any amount of heat can be extracted or rejected without any change in its temperature.
- → Source: Reservoir at high temperature
- → Sink: Reservoir at low temperature

Adiabatic valve is a valve which is a perfect heat insulator. It does not allow heat to flow through it.

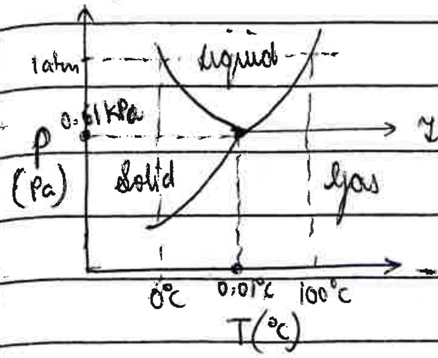
Diathermic valve is a valve which doesn't have temp. difference across it. Allows heat flow freely.

→ Mass which changes phase

H (Heat)
- → phase change → $H = mL$ $\quad$ L = latent heat
- → temperature change → $H = ms\Delta T$ $\quad$ S = specific heat capacity
- → phase change as well as temp. change

If the phase of ~~a~~ body is changing without change in temp., for example, a solid is melting, that temp. is known as melting point.

If the liquid is boiling, then the liquid evaporates without change in temp. That temp is known as boiling point.

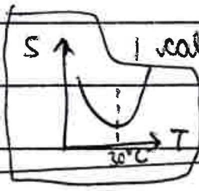


Remember: $L_f = 80\ cal/gm$

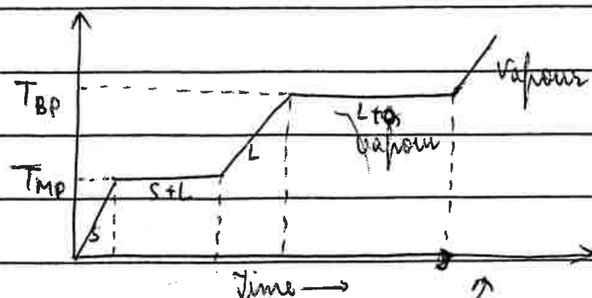
$L_v = 536\ cal/gm \approx 540\ cal/gm$

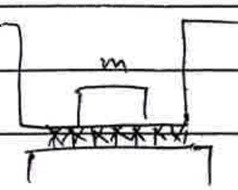
$s_{ice} = 0.5\ cal/gm°c$

$s_w = 1\ cal/gm°c$



1 cal = heat required to change 1 gm of water from 14.5°C to 15.5°C.





(Continuous stirring to prevent local heating)

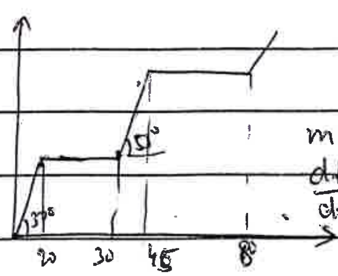
$\frac{dH}{dt}$ = constant for graph

$T < T_{mp}$: $\frac{dH}{dt} = m s_s \frac{dT}{dt}$

$T = T_{mp}$: $\frac{dH}{dt} = \frac{dm}{dt} L_f$

$T_{bp} > T > T_{mp}$: $\frac{dH}{dt} = m s_L \frac{dT}{dt}$

Q



$m = 10\ gm$

$\frac{dH}{dt} = 1W$

(a) $s_s = \frac{400}{3}\ J/kg°C$

(b) $s_L = \cancel{\frac{300}{4}\ J/kg°C} = 75\ J/kg°C$

(c) $L_f = 1000\ J/kg$

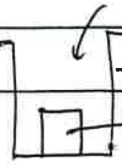
(d) $L_{vapourisation} = 3500\ J/kg$

Heat capacity, $C = ms$ $\qquad \Delta H = ms\Delta T = C\Delta T$

• Water equivalent is equivalent mass of water which has the same heat capacity as that of the given body.

$m_w s_w = ms \Rightarrow m_w = \frac{ms}{s_w} = \frac{C}{1}$ $\quad (s_w = 1\ cal/gm°c)$

Q.



5 gm steam (100°C)

→ $m_w = 10\ gm$ (container)

→ 40 gm ice at −20°C

Find T_{final}

Ans : 0°C → 37.5 gm water

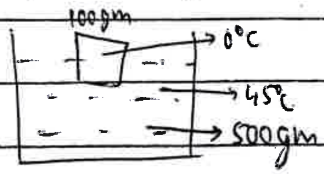
→ 7.5 gm ice

If s is not constant, $dH = ms\,dT$

$\Delta H = m\int s\,dT$ Date: / / Page no:

Q. 50 gm ice at $-10°C$ is kept in calorimeter of water eq. 20 gm. We add 10 gm of steam at $100°C$. Find T_{final} and composition of the mixture. Ans: $24.375°C$. 100% water

Q:



$S_w = 4200\ J/kg°C$

$S_b = 4200\left(1+\frac{\theta}{10}\right)$

Ans: Final temp $= 30°C$.

~~do~~ Supercooling & superheating:

Water can be raised to temp. above the boiling point (100°C) without a change in phase. This happens when water is very pure, so that there are no centres of nucleation. If some impurity is added, the highly energetic molecules get centres of nucleation and they immediately go into vapour state.

Similarly, supercooled water is water at temp. less than melting point. This is very pure water which doesn't have centres of nucleation. If we suddenly add some impurity to it, a part of water converts into ice. The latent heat thus released will raise the temp. of water and ice to melting point.

Q ~~mass~~ Ice + supercooled water at $-21°C$ and total mass is 0.6 kg. ~~latent heat of fusion for ice is~~

$l_f = 3.3\times10^5\ J/kg$ $S_W = 4200\ J/kg°C$ $S_{ice} = 2100\ J/kg°C$.

Temp. rises at uniform rate of $0.1°C/sec$, on heating by heater. Find the (i) P_{heater} $=168W$ (ii) Time t (from starting) when ice starts melting $=210$ sec (iii) t', ice melts completely ≈ 995 sec

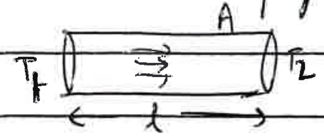
Heat capacity of water & ice are equal.

Q A block of ice at 0°C is dropped from height h. Find h (min) so that it melts completely on collision with ground.

HEAT TRANSFER :

① Conduction ② Convection ③ Radiation

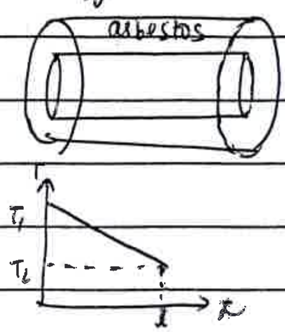
• Conduction is a of heat transfer where heat flows from higher temp to lower temp. through a material medium without the physical transportation of particles of the medium.



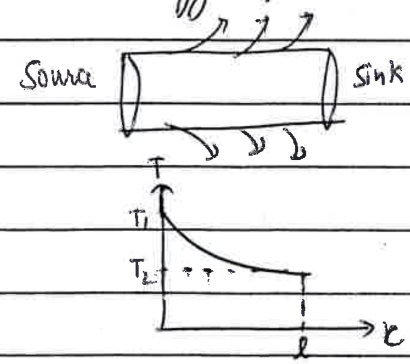
$\frac{dH}{dt} \propto A$ $\frac{dH}{dt} \propto T_1 - T_2$ $\frac{dH}{dt} \propto \frac{1}{L}$

$\boxed{\frac{dH}{dt} = \frac{kA(T_1 - T_2)}{l}}$: Fourier's law of heat conduction
k = coefficient of thermal conductivity

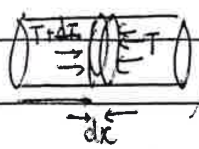
① well lagged / well insulated



② unlagged / uninsulated



Fourier's law in differential form: $\boxed{\frac{dH}{dt} = -kA\frac{dT}{dx}}$

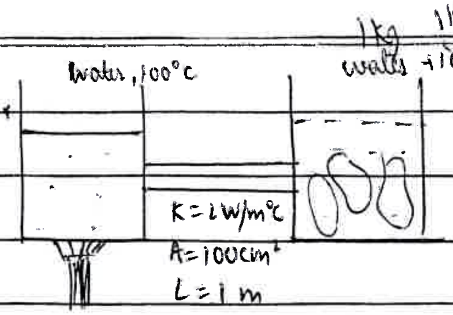


$\frac{dT}{dx}$ = temp. gradient.

• $k_{metals} >> k_{non\text{-}metals}$.

• Heat conduction is metals is by free electrons. There is no net transfer of electrons, otherwise current would be flowing...

Steady state: When the rate of heat flow through any cross-section is same, we say that the conductor is in a steady state. Fourier's law is applicable only for steady state.

Q 

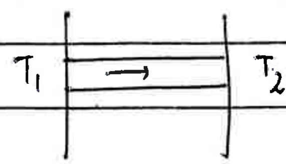
① Find the rate at which ice is melting

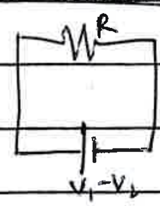
② After ice is completely melted, find

(i) initial rate of rise of temp.

(ii) ~~rate~~ rate of rise of temp at $T = 10°C$.

<u>Electrical & thermal conductance similarity</u>



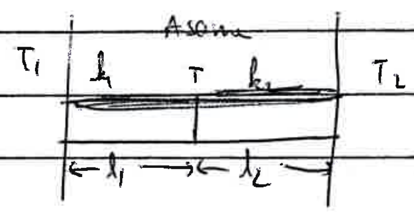


$$i = \frac{V_1 - V_2}{R} \qquad R = \frac{\rho l}{A} = \frac{l}{\sigma A}$$

$$\frac{dH}{dt} = \frac{kA}{l}(T_1 - T_2)$$

$$\Rightarrow \boxed{i_{thermal} = \frac{T_1 - T_2}{l/kA}} \;\; \equiv \;\; \boxed{i = \frac{V_1 - V_2}{l/\sigma A}}$$

Electrical	Thermal
1) $i = dq/dt$	$i_{Th} = \frac{dH}{dt}$
2) $R = l/\sigma A$	$R_{Th} = l/KA$
3) $V_1 - V_2$	$T_1 - T_2$
4) series resistors; $R = R_1 + R_2$; i same	rods in series; $R = R_1 + R_2$; i same
5) V_1, V_2 parallel resistors; $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$; $V_1 - V_2$ same	T_1, T_2 parallel rods; $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$; $T_1 - T_2$ same



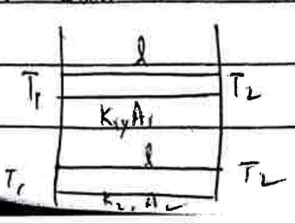
① $$T = \frac{\frac{k_1 T_1}{l_1} + \frac{k_2 T_2}{l_2}}{k_1/l_1 + k_2/l_2}$$

② $k_{eq} = ?$ $\qquad R_{eq} = R_1 + R_2$

$$\frac{l_1 + l_2}{k_{eq} A} = \frac{l_1}{K_1 A} + \frac{l_2}{K_2 A}$$

$$\Rightarrow k_{eq} = (l_1 + l_2) \Big/ \left(\frac{l_1}{k_1} + \frac{l_2}{k_2}\right)$$

<u>Equivalent thermal resistance</u> is the resistance of a rod which can replace the system of rods between the same set of reservoirs so that the rate of heat flow remain the same.

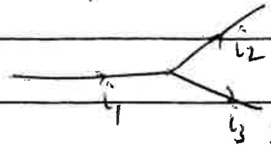


① $$R_{eq} = \frac{l}{K_1 A_1 + K_2 A_2} \left(= \frac{l}{(A_1 + A_2) k_{eq}} \right)$$

② $$k_{eq} = \frac{k_1 A_1 + k_2 A_2}{A_1 + A_2}$$

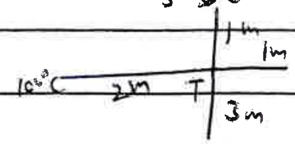
Kirchoff's law.

Kirchoff's current law states that total current entering the junction is equal to the current leaving the junction.



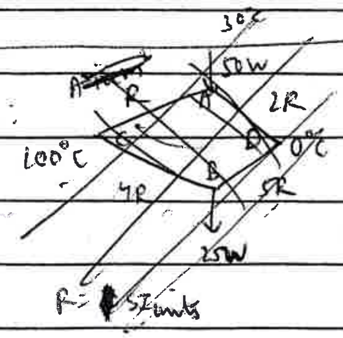
$i_1 = i_2 + i_3$

Eg:

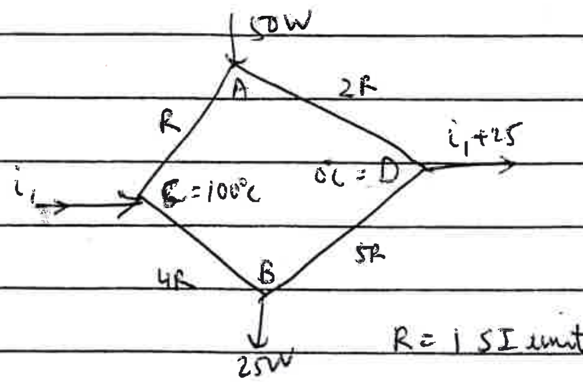


$T = \frac{780}{13}$ °C

Eg:



① T_A
② T_B
③ $\left(\frac{dH}{dt}\right)_C$
④ $\left(\frac{dH}{dt}\right)_D$

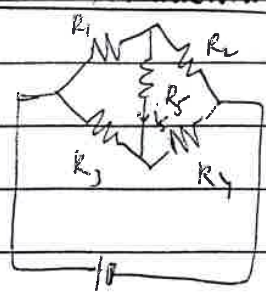


$$\frac{T_A - 100}{1} + \frac{T_A - 0}{2} = 50$$

$$\frac{100 - T_B}{4} + \frac{0 - T_B}{5} = 25$$

Balanced Wheatstone Bridge: (Balanced if $\frac{R_1}{R_3} = \frac{R_2}{R_4}$)

$\Rightarrow i_5 = 0$



Coefficient of thermal conductivity or area of cross section is variable:

$$\frac{dH}{dt} = kA\frac{dT}{dx}$$

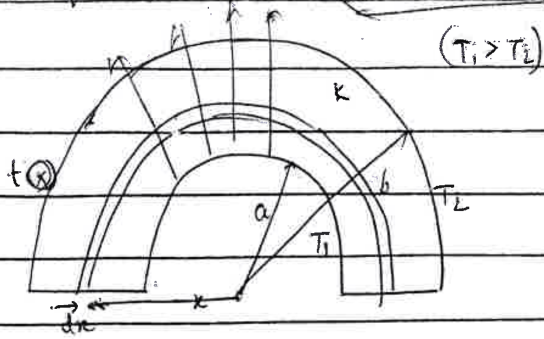
For finding out thermal resistance, we should divide it in small elements. The element should be such that the coeff. of thermal conductivity (K) and the area of cross-section (A) is the same at every point.

If the elements are in series, $R_{eq} = \int dR$

If the elements are in parallel, $\frac{1}{R_{eq}} = \int \frac{1}{dR}$

- If correct element is taken, integration will be easy, otherwise it will ~~off~~ get impossible
- Heat flow should be (perpendicular to area of cross section and along the length) of element

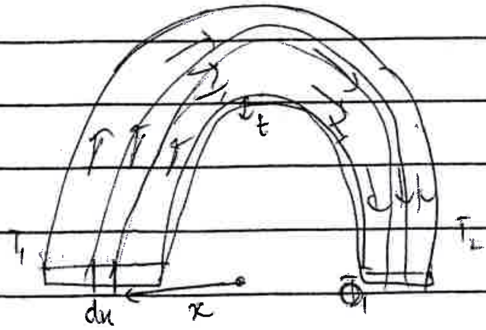
eg:



$(T_1 > T_2)$

$$\int dR = \int_a^b \frac{dx}{K\pi x t}$$

$$R = \int_a^b dR = \frac{1}{K\pi t} \ln\left(\frac{b}{a}\right)$$

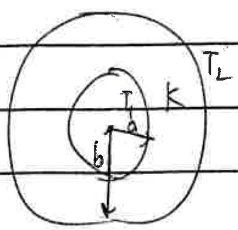


$$dR = \frac{\pi x}{k t\, dx}$$

$$\Rightarrow \frac{1}{R_{eq}} = \int \frac{1}{dR} = \frac{kt}{\pi} \int_a^b \frac{dx}{x}$$

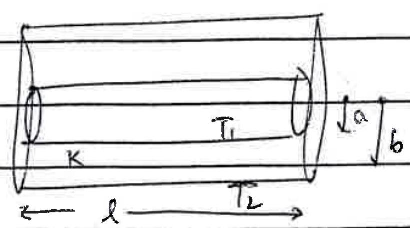
$$\boxed{R_{eq} = \frac{\pi}{kt \ln\left(\frac{b}{a}\right)}}$$

Q



① $R_{eq} = \frac{1}{4\pi K}\left(\frac{1}{a} - \frac{1}{b}\right)$

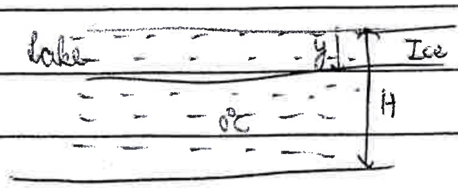
② $\left(\frac{T_1 + T_2}{2}\right)$ temp position $= \frac{2ab}{a+b}$ from centre



① $R_{eq} = \frac{1}{2\pi l k} \ln\left(\frac{b}{a}\right)$

② $\left(\sqrt{T_1 T_2}\right)$ position $= a\left(\frac{b}{a}\right)^{\sqrt{T_1}/(\sqrt{T_1}+\sqrt{T_2})}$

Q



$\rho_w = \rho_{ice} = \rho_0$

k_{ice} Latent heat of fusion $= L_f$

Thickness of ice as function of time

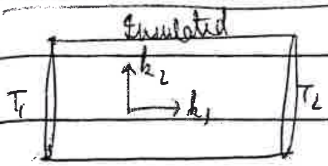
$$y = \sqrt{\frac{2Kt\theta}{\rho_0 L_f}}$$

$$\frac{dH}{dt} = \frac{kA\theta}{y} = \rho_0 A \frac{dy}{dt}$$

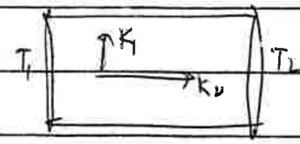
If $\rho_{water} \neq \rho_{ice}$,

$$y = \sqrt{\frac{2kt\theta}{\rho_{ice} L_f}}$$

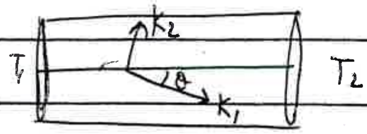
Heat conduction through anisotropic substances (different value of k in different direction)

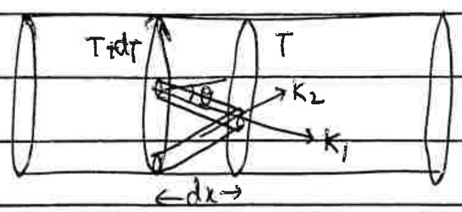


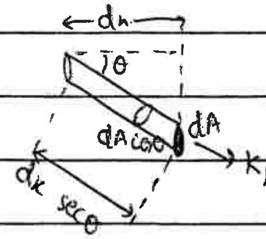
$$\frac{dH_1}{dt} = \frac{k_1 A}{l}(T_1 - T_2) ,$$



$$\frac{dH_2}{dt} = \frac{k_2 A}{l}(T_1 - T_2)$$







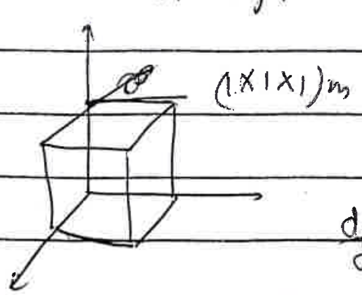
$$\frac{dH_1}{dt} = \frac{k_1 \, dA \cos\theta \, dT}{dx \sec\theta} = k_1 \, dA \cos^2\theta \frac{dT}{dx}$$

$$\frac{dH_2}{dt} = k_2 \, dA \sin^2\theta \frac{dT}{dx}$$

$$\Rightarrow \frac{dH}{dt} = (k_1 \cos^2\theta + k_2 \sin^2\theta) A \frac{dT}{dx} = (k_1 \cos^2\theta + k_2 \sin^2\theta) \frac{A}{l}(T_1 - T_2)$$

Q. When heat flows along +x-direction for $\Delta T = 100°C$, $\frac{dH}{dt} = 2W$.
When heat flows along +y, $\Delta T = 100°C$, $dH/dt = 3W$
When heat flows along +z, $\Delta T = 100°C$, $dH/dt = 4W$

(i) $k_x, k_y, k_z = 0.02, 0.03, 0.04$.



(2) When a cylinder is carved out along body diagonal, find $\left(\frac{dH}{dt}\right)_{cylinder}$

$A_{cyl} = 1 \, cm^2$.

$$\frac{dH}{dt} = \frac{kA\Delta T}{l} = \frac{(0.02 + 0.03 + 0.04)\left(\frac{1}{\sqrt{3}}\right)^2 \times (1 \times 10^{-4}) \, 100}{\sqrt{3}}$$

$$= \sqrt{3} \times 10^{-4} \, W$$

$$(K = K_1 \cos^2\theta_1 + K_2 \cos^2\theta_2 + K_3 \cos^2\theta_3) \quad \left(\cos\theta = \frac{1}{\sqrt{3}}\right)$$

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Convection :

1. Natural convection : Buoyant force, coriolis force
2. Forced convection

- Convection is a process of heat transfer where heat flows from one point to another by physical transportation of the particles of the medium. There are two kinds of convection, natural and forced. Natural convection occurs due to natural forces like buoyant force or coriolis force. Forced convection is when the fluid is forced from one point to another by an external agency like AC, fan, cooler, etc.

T_∞

$$\frac{dH}{dt} = hA(T - T_\infty)$$; for natural convection

h ↓ Coefficient of convective heat transfer (Not a constant)

(Depends heavily on atmospheric conditions, velocity, turbulence, etc.)

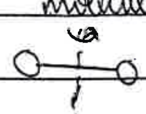
A

T

RADIATION :

- Radiation is the process of heat transfer by the mean of EM waves. The human body absorbs appreciably and radiates appreciably EM waves in the infrared region. So, thermal radiations are many a times also called as ~~infraa~~ infrared radiations. These radiations travel in space with the speed of light. So, ~~the~~ radiation is the fastest mode of heat transfer.

- IR radiations are generally produced due to change in rotational kinetic energy levels of molecules. Other atomic & molecular transitions are high energy.



Free e^- don't absorb radiation. So metals 'shine' (don't absorb) but non-metals absorb it.

For free e^-,

$$\boxed{\frac{hc}{\lambda} = \frac{mv^2}{2}}, \quad \boxed{\frac{h}{\lambda} = mv} \Rightarrow \boxed{\cancel{v = 2c}}$$

momentum conservation not applicable for bound electrons

If a body has large no. of free e^- as in a metal, it is a bad radiator of heat because if the body is heated, the free e^- move with high velocity but they are not accelerated. Accelerated charges radiate EM waves. But if we have a non-metal, the molecules can rotate with higher angular velocity on being heated. These charges radiate EM waves. The process of absorption is reverse of emission. If there are large no of free e^-, they don't absorb radiations incident on them but bound electrons can absorb radiations easily. So, in general, metals are bad radiators and bad absorbers of heat but non-metals are good absorber and good radiator of heat.

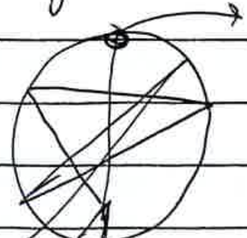
- **Prevost's theory of heat exchange:**

When any body at any temperature above absolute zero continuously radiates heat energy and also absorbs heat energy.

- The rate of heat emitted depends on the area of body, the nature of the surface and its temperature.
- The rate at which heat will be absorbed depends on its own surface area, temp. of the surroundings and the nature of its surface

$\rightarrow$ better radiator, better absorber] more surface area.

- **Blackbody:**
- A blackbody is a body which absorbs all electromagnetic radiations falling on it.



Fabry perot black body

- Any isothermal enclosure acts like a blackbody.

- **Greybody** is a body which absorbs a certain fixed percentage of all the electromagnetic radiations incident on it.
- **Whitebody** is a body which reflects all the radiation incident on it. This means it absorbs no radiation incident on it.

- **Emissive power (E)**: Emissive power of any body is the energy radiated per unit area per unit time. (W/m^2)

- **Absorptive power / Absorptivity (a)**: It is the energy absorbed per unit area per unit time divided by energy incident per unit area per unit time.

$$\underset{\text{whitebody}}{0} \le a \le \underset{\text{blackbody}}{1}$$

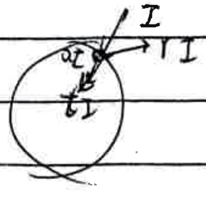
Emissivity (e): Ratio of emissive power of a body and the emissive power of a black body at the same temperature.

$$e = \frac{E}{E_{blackbody}} \qquad e_{blackbody} = 1$$

Reflectivity / albedo (r): Ratio of energy reflected per unit area per unit time and energy incident per unit area per unit time.

$$r = \frac{\frac{dE_{reflected}}{A\,dt}}{\frac{dE_{incident}}{A\,dt}}$$

Transmissivity (t): $\dfrac{\frac{dE_{transmitted}}{A\,dt}}{\frac{dE_{incident}}{A\,dt}}$



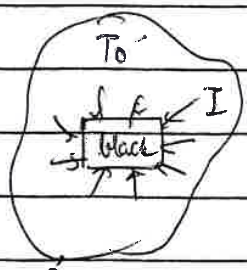
$I = aI + rI + tI \Rightarrow a + r + t = 1$

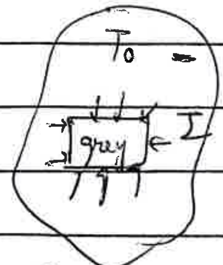
$a + r = 1$ ($t \approx 0$; dealing with opaque bodies in this topic)

Kirchoff's law:

Kirchoff's law states that ratio of emissive power and absorptive power of a body is constant and is equal to emissive power of a blackbody at the same temperature.

$$\frac{E}{a} = E_{black}$$





$$\Rightarrow \boxed{\frac{E}{E_{black}} = a = e}$$

$E_{black} = I$

= energy incident/ area/ time

$E = aI$

Stefan's Boltzmann law:

$E_{black} \propto T^4$

$\Rightarrow E_{black} = \sigma T^4$

σ = Stefan Boltzmann constant $= 5.67 \times 10^{-8}\ W m^{-2} K^{-4}$

$= \frac{17}{3} \times 10^{-8}\ W/m^2K^4$

energy radiated by a black body:

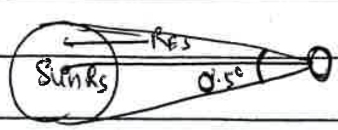
$$\frac{dH}{dt} = \sigma A T^4$$

$$E_{grey} = a\, E_{black} \Rightarrow \left(\frac{dH}{dt}\right)_{radiated} = \sigma a A T^4 \ (= \sigma e A T^4)$$

estimation of sun's temperature:

S = solar constant = 1500 W/m² $= \left(\frac{dH}{dt}\right)$ on earth surface

$$\frac{E \times 4\pi R_s^2}{4\pi R_{ES}^2} = 1500$$



$$\Rightarrow E = \sigma T^4 = 1500\left(\frac{R_{ES}}{R_S}\right)^2 = 1500 \times \left(\frac{720}{\pi}\right)^2$$

$$\Rightarrow \boxed{T \approx 6000\ K}$$

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Intensity $= S$

$\left(\frac{dH}{dt}\right)_{incident} = S\pi R^2$ (Area Projection)

$\left(\frac{dH}{dt}\right)_{absorbed} = a S\pi R^2$

In steady state,

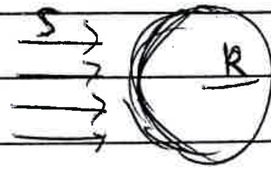
$\left(\frac{dH}{dt}\right)_{absorbed} = \left(\frac{dH}{dt}\right)_{radiated}$

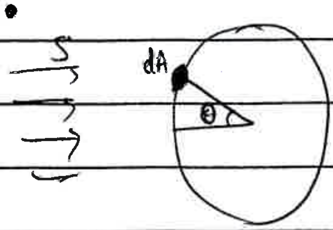
$\Rightarrow \alpha S \cancel{\pi R^2} = \sigma e 4 \cancel{\pi R^2}\, T^4$

$\Rightarrow \sqrt[4]{\frac{S}{4\sigma}} = T \approx \underline{282\,K}$

$T =$ Temp. (earth)

$R =$ Radius of earth





$\left(\frac{dH}{dt}\right)_{abs} = \left(\frac{dH}{dt}\right)_{radiated}$

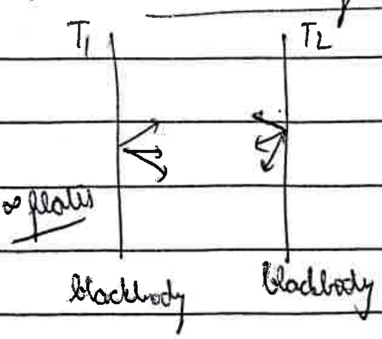
$\alpha S \cancel{dA} \cos\theta = \sigma e \cancel{dA}\, T^4$

$\Rightarrow \sqrt[4]{\frac{S\cos\theta}{\sigma}} = T$

For the earth, summer in Northern Hemisphere coincides with winter in the Southern hemisphere. At this point, we are farthest away from sun but the sun rays are incident almost normally at the Tropic of Cancer. So, the temperature is maximum.

Similarly, at our peak winter, there is summer in the Southern Hemisphere when the sun rays are incident normally at the Tropic of Capricorn

~~Heat~~ Heat exchange between two ~~large~~ infinite planes:

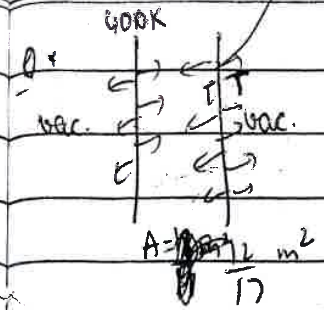


$\left(\frac{dH}{dt}\right)$ emitted by 1 $= \sigma A T_1^4$

$\left(\frac{dH}{dt}\right)$ absorbed by 1 $= \sigma A T_2^4$

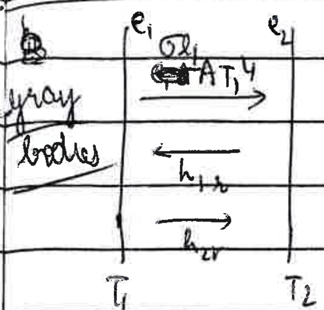
$\left(\frac{dH}{dt}\right)_{net} = \sigma A (T_1^4 - T_2^4)$

perfectly conducting to maintain same temp. T on both sides of plate.



① What is the power of heater required to maintain temp of left plate? 1536 K.

② Find T of second plate ~~1536~~ ~~1536 K~~.



h_1 absorbed $= e_1 e_2 A \sigma T_1^4$

$h_{1r} = \sigma e_1 (1-e_1) A T_1^4$

$h_{2r} = \sigma e_1 (1-e_1)(1-e_2) A T_1^4$

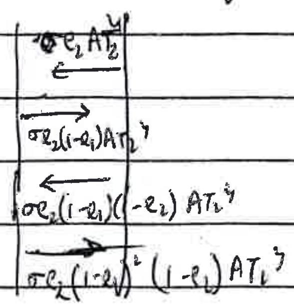
h_2 absorbed $= \sigma e_1 e_2 (1-e_1)(1-e_2) A T_1^4$

$$\left(\frac{dH}{dt}\right)_{\text{absorbed by }2} = \sigma e_1 e_2 A T_1^4 \left[1 + (1-e_1)(1-e_2) + (1-e_1)^2(1-e_2)^2 + \cdots\right]$$

$$= \frac{\sigma e_1 e_2 A T_1^4}{1-(1-e_1)(1-e_2)} = \frac{\sigma A T_1^4}{\frac{1}{e_1} + \frac{1}{e_2} - 1}$$

This is the one which was originally radiated by 1 and reflected multiple times.

Now, we will find the heat which was radiated by 2 and again absorbed by 2 after multiple reflections.



$h_1 = \sigma e_2^2 (1-e_1) A T_2^4$

$h_2 = \sigma e_2^2 (1-e_1)^2 (1-e_2) A T_2^4$

$$\frac{dH}{dt} = e_2^2 (1-e_1) A \sigma T_2^4 \left(1 + (1-e_1)(1-e_2) + \cdots\right)$$

$$= \frac{e_2^2 (1-e_1) A \sigma T_2^4}{1-(1-e_1)(1-e_2)}$$

$$= \frac{e_2^2 (1-e_1) A \sigma T_2^4}{e_1 + e_2 - e_1 e_2}$$

$$\left(\frac{dH}{dt}\right)_{\text{absorbed by B total}} = \frac{A \sigma e_2}{e_1 + e_2 - e_1 e_2}\left[e_1 T_1^4 + e_2 T_2^4 - e_1 e_2 T_2^4\right]$$

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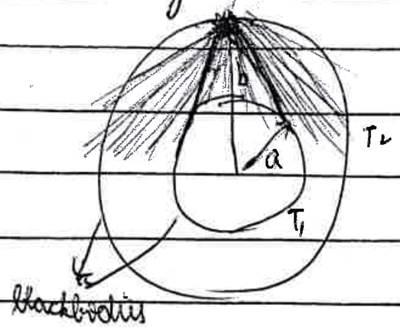
Heat exchange between spherical bodies :



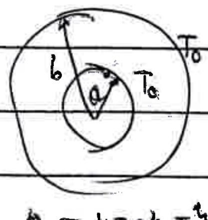
$$\left(\frac{dH}{dt}\right)_{\text{radiated by } 1} = \sigma\, 4\pi a^2 T_1^4$$

$$\left(\frac{dH}{dt}\right)_{\text{radiated by } 2} = \sigma\, 4\pi b^2 T_2^4$$

$$\left(\frac{dH}{dt}\right)_{\text{received by } 1} = f\, \sigma\, 4\pi b^2 T_2^4$$

$f \rightarrow$ view factor / shape factor

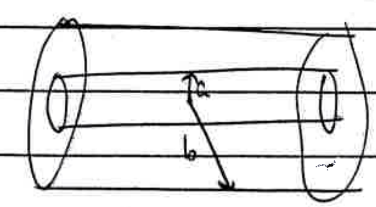
$$= \sigma\, 4\pi a^2 T_2^4$$



$$\left(\frac{dH}{dt}\right)_{net} = \sigma\, 4\pi a^2 \left(T_1^4 - T_2^4\right)$$

$$\sigma\, 4\pi a^2 T_0^4 = f\,\sigma\, 4\pi b^2 T_0^4$$

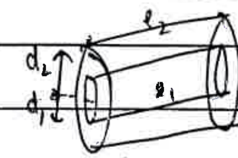
$$\Rightarrow f = \frac{a^2}{b^2}$$



$$\frac{dH}{dt} = 2\pi a l\, \sigma \left(T_1^4 - T_2^4\right)$$

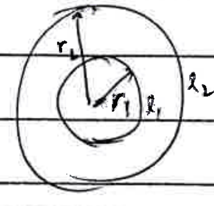
For concentric ~~circles~~ cylinders

$$ff = \frac{1}{\frac{1}{e_1} + \frac{d_1}{d_2}\left(\frac{1}{e_2} - 1\right)}$$



For concentric spheres:

$$ff = \frac{1}{1/e_1 + \frac{r_1^2}{r_2^2}\left(\frac{1}{e_2} - 1\right)}$$

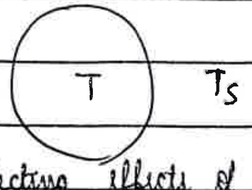


$$\frac{dH}{dt} = \cancel{ff \times \sigma A T}$$

$$ff \times \sigma A_{inside}\left(T_1^4 - T_2^4\right)$$

Newton's law of cooling:

$$\frac{dT}{dt} \propto \left(T - T_{surr}\right)$$



$$\frac{dH}{dt} = \sigma e A \left(T^4 - T_S^4\right)$$

$$= \sigma e A \left((T_S + \Delta T)^4 - T_S^4\right)$$

$$= \sigma e A T_S^4 \left(\left(1 + \frac{\Delta T}{T_S}\right)^4 - 1\right)$$

$$= \sigma e A T_S^4 \left(1 + \frac{4\Delta T}{T_S} - 1\right) = 4\sigma e A T_S^3 \Delta T$$

(Neglecting effects of conduction, convection, because radiation is the major mode of heat transfer here.)

$\frac{dH}{dt} = 4\sigma e A T_s^3 \Delta T$

$\Rightarrow ms\frac{dT}{dt} = 4\sigma e A T_s^3 (T-T_s)$

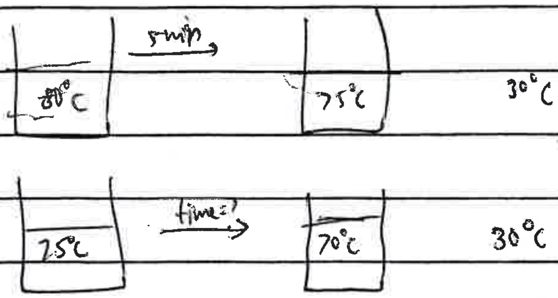
$\Rightarrow \frac{dT}{dt} = \frac{4\sigma e A T_s^3}{ms}(T-T_s)$

$\Rightarrow \boxed{\frac{dT}{dt} = -C(T-T_s)}$ Newton's law of cooling

If $T<T_s$

$\boxed{\frac{dT}{dt} = C(T_s-T)}$ Newton's law of heating

Q:

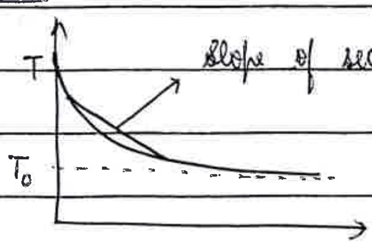


Exact method

$\int_{80}^{75}\frac{dT}{T-T_0} = -C\int_0^5 dt \Rightarrow \frac{1}{5}\ln\frac{\cancel{50}^{10}}{\cancel{45}_{9}} = c = \frac{1}{5}\left(\frac{1}{9}\right) = \frac{1}{45}$

$\int_{75}^{70}\frac{dT}{T-T_0} = -\frac{1}{45}\int_0^t dt \Rightarrow 45\ln\frac{9}{8} = \text{time} = 45\times\frac{1}{8} = 5.625$ min

Approx. method



Slope of secant ≈ slope of tangent at mid point

$\frac{T_1-T_2}{\Delta T} = \left(\frac{dT}{dt}\right)_{\text{mid-point}} = C\left(\frac{T_1+T_2}{2}-T_s\right)$

$\Rightarrow \frac{5}{5} = C(77.5-30) \Rightarrow C = \frac{1}{47.5}$

$\frac{5}{\Delta T} = 42.5\,C \Rightarrow \Delta T = \frac{5\times 47.5}{42.5} =$

Q The water from the geyser is at a $T = 80°C$ and has a volume of 10L. You wait for 10 mins and then mix it with 10L water at 20°C, the resultant temp is 45°C. If you had mixed the 20°C water in the beginning, how long will you have to wait for the temp to become 45°C? Assume that emissivity and area remains the same. Use exact method / Not approximate. Ans: 20 mins ✓ / 110/9 mins X

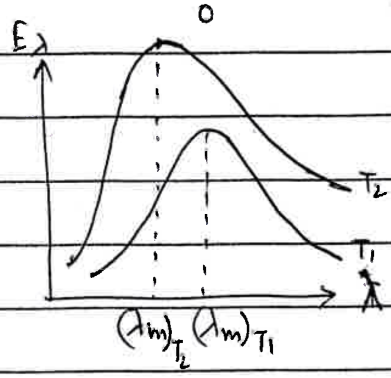
Q Heater rated at ~~400 W~~ 420 W. Room temp. is 20°C. Neglect heat capacity of heater & coil. 1 L water at 20°C. It is seen that after sometime, temp becomes steady at 90°C. Find the temp of water as a function of time. Ans: $T = 90 - 70e^{-t/700}$

Spectral emissive power:

$E_\lambda = \frac{dE}{d\lambda} =$ Spectral emissive power

Intensity radiated by the black body in wavelength range λ to $\lambda + d\lambda$ is dE.

$$E = \int dE$$
$$= \int_0^\infty E_\lambda \, d\lambda = \sigma T^4$$



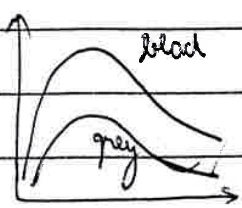
$T_2 > T_1$

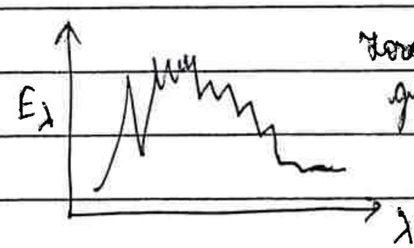
Area under graph $= \sigma T^4$

⟶ Spectral energy distribution curve

Wien's displacement law: $\lambda_m \cdot T = b$

$b = 2.88 \times 10^{-3}$ mK





For earth, graph is like this