## Abstract

We study the heat equation as second order parabolic partial differential equation. We use Fourier transform technique(one of the many techniques) to derive the general solution of the heat equation. We estimate these solutions of heat equations in $L^\infty$ using Maximum Principle. We prove maximum principle using Divergence theorem. We show the uniqueness. We give a statement about smoothness. We calculate the priori energy estimates in $L^\infty$ by using integration by parts. Then, We generalize the heat equation as general second order partial differential equations. We construct the weak formulations of such second order partial differential equations using test function from infinitely differentiable space and derive priori estimate of weak solution. Before, we show the existence and uniqueness of classical solution using maximum principle, here we use priori energy estimate and Galerkin approximation technique (other technique can be Lax Milgram Methods)to show the existence and uniqueness of the weak solutions. To prove existence, we try to construct Galerkin approximation to certain extend followed by citations of literature for explicit proof. Then, we prove uniqueness and moved to regularity. Using energy estimate and other inequalities (mainly related to Sobolev space), we talk about regularities. We see theorems(without proof) on how lower order derivatives can estimate higher order derivative. We say, with further construction, using regularities and Sobolev embedding theorem, we can see weak solutions are infact smooth, classical solution solutions. We say this regularity estimation mirros with similiar estimate for elliptical case.(stated in the Evans book of Partial Differential Equations [1]) For further study, we can reflect, we can derive analytical estimate from structural algebraic assumption.
The main project question is to derive maximum principle using divergence theorem for which I have done some calculations in chapter 3 (supported by Appendix part). Another, question is "Does the $u$ from priori estimate suffice maximum principle? Is $u$ regular?" This thesis is written trying to answer these questions.

# Contents

# 1 Introduction

We study the main theory about classical and weak solutions of heat equation from [1] and [2] followed by some results and description [3] and [4] presented in easier way. We have used [5] and [6] to do some calculations in Maximum principle part. We have used [7] to understand more about Glarekin method and [7] as a certain tool while proving uniqueness. We have used [8] to get insight on Sobolev embeddings. Other, [9] , [10] , [11] and [12] has been used to understand more on the same topics priori estimates, existence, uniqueness and regularity of solutions of second order parabolic PDEs.
Heat equations are being studied from time of Fourier and even before. While studying, both theoretical and applied motivations were realized. In applied part, we realized, to study Black Scholes equation (in sense of evolution) ,heat equation is useful and methods like Galerkin approximation's connection with Finite elements method in a way. In theoretical part, we realized regularities of solutions of heat equations can be studied or have connection to group theory and calculus of variation. But, we don't go in details about them here.

# 2 Heat Equation

The heat equation

$$\partial_t = \Delta u + f \tag{1}$$

is a parabolic partial differential equation. It is given with suitable initial condition and boundary conditions to make well equipped problem with a unique solution. For example, Let's take Drichlet's boundary conditions on a bounded open set $\Omega \in \mathbb{R}^n$ for $u : \Omega \times [0, \infty) \to \mathbb{R}$ (initial boundary value problem)

$$\partial_t u = \Delta u + f(x,t) \qquad \text{for} \qquad x \in \Omega \quad and \quad t > 0 \tag{2}$$

$$u(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega \quad and \quad t > 0$$

$$u(x,0) = g(x) \qquad \text{for} \quad x \in \Omega \quad \text{Initial} \quad \text{Condition}$$

Physically, the evolution in time of temperature $u(x,t)$ of a body in $\Omega$
boundary temperature = 0 (fixed)
initial temperature = g $\quad g : \Omega \to \mathbb{R}$
heat source=f/volume $\quad f : \Omega \times (0, \Omega) \to \mathbb{R}$

# 3 Maximum principle

We can estimate solutions of heat equations above(2) in $(L^\infty)$ using **Maximum principle.**

If $f \leq 0, \quad T > 0$;

$$\Gamma = \Omega \times \{t = 0\} \cup \partial\Omega \times [0, T)$$

$$\mathbf{Max}_{\Omega\times[0,T]}\mathbf{u} = \mathbf{Max}_{\Gamma}\mathbf{u}$$

Proof (Using divergence theorem):
Let $(u - m)_+$: $=$ max{0, u-m}
Multiply heat equation above(1) with test function $(u - m)_+$ and integrate on both sides under domain $\Omega$

$$\int_\Omega \partial_t u(u-m)_+\, dx = \int_\Omega \Delta u(u-m)_+\, dx + \int_\Omega f(u)(u-m)_+\, dx$$

Given in the condition; $f \leq 0$
So,

$$\int_\Omega f(u)(u-m)_+ \leq 0$$

Let, assume the contradiction,
Let, 'I' be the set where $(u - m) > 0$

$$\Rightarrow \int_I \partial_t(u-m)(u-m)\, dx \leq \int_I \Delta(u-m)(u-m)\, dx$$

- $$LHS^1 = \int_I \partial_t(u-m)(u-m)\, dx = \int_I \frac{d}{dt}(u-m)(u-m)\, dx = \frac{1}{2}\frac{d}{dt}\int_I (u-m)^2\, dx$$

- RHS (Using divergence theorem and integration by parts)[2]

  $$= \int_I \Delta(u-m)(u-m)\, dx = \int_{\partial I} n.\nabla(u-m)(u-m)\, ds - \int_I |\nabla(u-m)|^2\, dx$$

  or ( see equation(21.7) of [6] )

  $$\Rightarrow \frac{1}{2}\frac{d}{dt}\int_I (u-m)^2\, dx \leq \int_{\partial I} n.\nabla(u-m)(u-m)\, ds - \int_I |\nabla|^2\, dx$$

We look at the each terms of RHS:

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[1]

We know this differentiation and integration interchange,

$$\int_a^b \frac{d}{dt}\Big[\frac{du}{dx}\Big]\, dx = \frac{d}{dt}\int_a^b \Big[\frac{du}{dx}\Big]\, dx$$

[2]The calculation of use of divergence theorem and integration by parts to get RHS is given in the appendix secction as use of divergence theorem or we can directly use Green's first identity from equation(21.7) of [6]

- $$\int_{\partial I} n.\nabla(u-m)(u-m)\,dx = 0 \quad \text{because} \quad \text{at}, \partial\Omega \quad u = 0$$

  or (u-m) isnot isn't integrable by the boundary $\partial\Omega$

- $$-\int_I |\nabla(u-m)|^2\,dx \leq 0 \quad \text{because} \quad |\nabla(u-m)|^2 = +ve$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\int_I (u-m)^2\,dx \leq -\int_I |\nabla(u-m)|^2\,dx \leq 0$$

$$\Rightarrow \frac{d}{dt}\int_I (u-m)_+^2 \leq 0$$

$$\text{at} t = 0, \quad u \leq m, \quad so, (u-m)_+ = 0 \quad QED$$

which is contradiction to above assumption $(u-m) > 0$
Argument, note 'u' is smooth function that attains maximum at $x \in \Omega$ and $0 < t \leq T$. Then,we have two options:
$\rightarrow \partial_t u = 0 \quad \text{if} \quad 0 < t < T$
or $\rightarrow \partial_t u \leq 0 \quad \text{if} \quad t = T \quad \text{and} \quad \Delta u \leq 0$

$$\partial_t - \Delta u \geq 0$$

which is impossible if

$$f < 0$$

.
So,'u' is maximum on $\partial\Omega \times [0, T]$ where $u = 0$ or at $t = 0$.

**Uniqueness** (This proof is given in the book by Evans)
$\exists$ at most one solution $u \in C(\bar{\Omega}) \cap C^2(\Omega)$ to the problem,

$$\partial_t u - \Delta u = f \qquad \text{in} \quad \Omega$$

$$u = g \quad \text{on} \quad \Gamma$$

Proof: Let, $u =$ difference between two solutions to this problem. Then, 'u' solves same problem.
But, if $f = 0$ and $g = 0$, 'u' achieves min and max at $\Gamma$. But, there $u = 0 \Rightarrow \quad u = 0$ everywhere. QED

**Question?**:Does $L^\infty(0, T; L^2) \bigcap L^2(0, T); L^2$ suffice Maximum principle argument? Yes, Because of the following theorem. **Smoothness** (The proof of this theorem is given in page 59 of [1])
Lets, take $U_T$. Suppose $u \in C_1^2(U_T)$ solves the heat equation in $U_T$.Then, $u \in C^\infty(U_T)$. To understand more, See Theorem(3.4) of [3] or page 39 section(2.6) of [10]

# 4 Solving the heat equation using fourier transform

[3]

We suppose the following initial condition,

$$u_t - \Delta u = 0, \qquad -\infty < x < -\infty, \qquad t > 0; \tag{3}$$
$$u(x,0) = g(x), \qquad -\infty < x < \infty$$
$$u(x,t) \to 0, u_x(x,t) \to 0; \quad \text{as} \quad |x| \to \infty$$

Let $\tilde{u}(x,t) = \mathbb{F}\{u(x,t)\}$ We transform both sides of (3) to obtain; $\frac{\partial \tilde{u}}{\partial t} = -w^2\tilde{u}$ (Transform of derivatives)
It is a first order ODE(ordinary differential equation). It has a general solution

$$\tilde{u}(w,t) = A(w)e^{-w^2 t} \tag{4}$$

We transform initial conditon to obtain $\tilde{u}(w,0) = \tilde{g}(w)$.We combine with (3) to form $A = \tilde{g}$. Therefore,$\tilde{u}(x,t) = \tilde{g}(w)e^{-w^2 t}$
We compute the following to recover $u(x,t)$;
$\mathbb{F}_{-1}\{g(w)\widetilde{e^{-w^2 t}}\} = \mathbb{F}^{-1}\{\tilde{g}(w)\tilde{f}(w,t)\}$
We apply the convolution theorem [4]

$$\tilde{F}^{-1}\{\tilde{g}(w)\} = g(x)f(x,t) = \tilde{F}^{-1}\{e^{-w^2 t}\} = \frac{e^{\frac{-x^2}{4t}}}{\sqrt{2t}}$$

This follows from Fourier transform of Gaussian function.

Thus,by convolution theorem,

$$u(x,t) = \frac{1}{\sqrt{2\pi}}(f*g)(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-p,t)g(p)\,dp$$
$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(p)\frac{exp\frac{-(x-p)^2}{4t}}{\sqrt{2t}}\,dp$$
$$= \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} g(p)exp\frac{-(x-p)^2}{4t}\,dp$$

So, this is the nature of general solution of the heat equation.

[3] Fourier transform of f=f(x);

$$F\{f(x)\} = \tilde{f}(w) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-iwx}\,dx$$

We can find the inverse.

[4] The convolution of two functions f and g is given by;

$$(f*g) = \int_{-\infty}^{\infty} f(p)g(x-p), dp = \int_{-\infty}^{\infty} f(x-p)g(p),$$

# 5 Energy estimates

We can integrate the heat equation in ($L^2$) by integration of the equation. We multiply the heat equation (1) by u, integrate over $\Omega$ , apply the divergence theorem, use the boundary condition that u = 0 on $\partial\Omega$:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\,dx + \int_\Omega |Du|^2\,dx = \int_\Omega fu\,dx$$

Integrate the equation wrt time and use initial condition:

$$\frac{1}{2}\int_\Omega u^2(x,t)\,dx + \int_0^t\int_\Omega |Du|^2\,dx\,ds = \int_0^t\int_\Omega fu\,dx\,ds + \frac{1}{2}\int_\Omega g^2\,dx \tag{5}$$

For $0 \le t \le T$, use Cauchy inequality with $\epsilon$:

$$\int_0^t\int \Omega fu\,dx\Omega ds \le \Big(\int_0^t\int_\Omega f^2\,dx\,ds\Big)^{\frac{1}{2}}$$
$$\le \frac{1}{4\epsilon}\int_0^T\int_\Omega f^2\,dx\,ds + \int_0^T\int_\Omega u^2\,dx\,ds$$
$$\le \frac{1}{4}\int_0^T\int_\Omega f^2,\,ds + \epsilon T\max_{0\le t\le T}\int_\Omega u^2\,dx$$

Taking supremum of (3) over $t \in [0,T]$ and using this inequality with $\epsilon T = \frac{1}{4}$ :

$$\frac{1}{4}\max_{[0,T]}\int_\Omega u^2(x,t)\,dx + \int_0^T\int_\Omega |Du|^2\,dx\,dt \le T\int_0^T\int_\Omega f^2\,dx\,dt + \frac{1}{2}\int_\Omega g^2\,dx$$

This gives a priori energy estimate:

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H_0^1)} \le C\Big(\|f\|_{L^2(0,T;L^2)} + \|g\|_{L^2}\Big) \tag{6}$$

where $C = C(T)$ is constant. We use this energy estimate to construct weak solutions.[5]

# 6 Generalization of second-order parabolic PDEs

Theorems, definations and generalization of heat equation are based on [1] and [2] mostly.But, the sign convention used mostly in every literature or books has been used.

We replace Laplacian $-\Delta$ by uniformly elliptic operator L on $\Omega \times (0,T)$. L in divergence form is given as follows:

$$L = -\sum_{i,j=1}^{n}\partial_i(a^{ij}\partial_j u) + \sum_{j=1}^{n} b^j\partial_j u + cu \tag{7}$$

[5]We can estimate $u$ and $Du$ if $f = 0$ and initial data $g$, so we can say there is parabolic smoothing in the heat equation.

where $a^{ij}(x,t), b^i(x,t), c(x,t)$ are coefficient functions
$a_{ij} = a^{ji}$
Assume $\exists\, \theta > 0$ s.t.

$\forall (x,t) \in \Omega \times (0,T)$ and $\xi \in R^n$

Corresponding parabolic PDE:

$$u_t + \sum j = 1^n b^j \partial_j u + cu = \sum_{i,j=1}^{n} \partial_i (a^{ij} \partial_j u + f \tag{8}$$

Drichlet boundary cnditions:

u$_t + Lu = f(9)$

$$u(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega \quad \text{and} \quad t > 0,$$

$$u(x,0) = g(x) \quad \text{for} \quad x \in \overline{\Omega}$$

As same estimate as heat equation holds, so we use $L^2$-energy estimate to prove esixtence of weak solutions of above .

# 7 Weak formulation

Theorems, definations and construction of weak formulation are based on [1] and [2] mostly.But, the sign convention used mostly in every literature or books has been used. Let, $\Omega$ , the coefficient of L and u be smooth. We multiply (9) by the a test function $v \in C_C^\infty(\Omega Uniform Ellipticity) \sum_{ij=1}^n a^{ij}(x,t)\xi_i\xi_j \geq \theta|\xi|^2$
(10)

$\forall (x,t) \in \Omega \times (0,T)$ and $\xi \in R^n$

Corresponding parabolic PDE:

$$u_t + \sum j = 1^n b^j \partial_j u + cu = \sum_{i,j=1}^{n} \partial_i (a^{ij} \partial_j u + f \tag{11}$$

Drichlet boundary cnditions:

u$_t + Lu = f(12)$

$$u(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega \quad \text{and} \quad t > 0,$$

$$u(x,0) = g(x) \quad \text{for} \quad x \in \overline{\Omega}$$

As same estimate as heat equation holds, so we use $L^2$-energy estimate to prove esixtence of weak solutions of above .

# 8 Weak formulation

Theorems, definations and construction of weak formulation are based on [1] and [2] mostly.But, the sign convention used mostly in every literature or books has been used. Let, $\Omega$ , the coefficient of L and u be smooth. We multiply (9) by the a test function $v \in C_C^\infty(\Omega$, integrate the results over $\Omega$ and apply divergence theorem:

$$(u_t, v)_{L^2} + a(u(t), v; t) = (f(t), v)_{L^2} \text{for} 0 \leq t \leq T \tag{13}$$

$(.,.)_{L^2}$ denotes the $L^2$-inner product

$(u, v)_{L^2} = \int_\Omega u(x)v(x)\, dx,$

$a$ is the **bilinear form** associated with L

$$a(u, v; t) = \sum_{i,j=1}^{n} \int_\Omega a^{ij}(x, t)\partial_i u(x)\partial_j u(x)\, dx + \sum_{j=1}^{n} \int_\Omega b^j(x, t)\partial_j u(x)v(x)\, dx + \int_\Omega c(x, t)u(x)v(t)\, dx \tag{14}$$

In (11) change of vector valued viewpoint and write u(x) =u(.,t).
**Assumption(1)**: The set $\Omega \subset R^n$ is bounded and open, $T > 0$, and:

1. the coefficients of a in (12) satisfy $a^{ij}, b^j, c \in L^\infty(\Omega \times (0, T))$;

2. $a^{ij} = a^{ji}$ for $1 \leq i, jn$ and the uniform ellipticity condition (8) holds for some constant $\theta > 0$

3. $f \in L^2(0, T; H^{-1}(\Omega)$ and $g \in L^2(\Omega)$

we let f to take values in $H^{-1}(\Omega) = H_0^1(\Omega)$' and duality pairing, $H^{-1}(\Omega)$and$H_0^1(\Omega)$' by

$$\langle ., . \rangle : H^{-1}(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$$

coefficients of a are uniformly bounded in time so it follows from theorem (2) page 14 of this thesis that:

$$a : H_0^1(\Omega) \times H_0^1(\Omega) \times (0, T) \to \mathbb{R}$$

$\exists\ C > 0$ and
$gamma \in \mathbb{R}$ s.t. for every $u, v \in H_0^1(\Omega)$ **Coercivity**, see page 14 theorem(2) of this thesis

$$C\|U\|_{H_0^1} \leq a(u,u;t) + \gamma\|u\|_{L^2}^2 \tag{15}$$

$$|a(u,v;t)| \leq C\|U\|_{H_0^1}\|v\|_{H0^1} \tag{16}$$

**Weak Solution Defination(1)**: A function $u : [0,t] \to H_0^1(\Omega)$ is a weak solution of (10) if:

1. $$u \in L^2(0,T; H_0^1(\Omega) \text{and} u_t \in L^2(0,T; H^{-1}(\Omega));$$

2. $$For \quad every$$

   v $\in H_0^1$,
   $\langle u_t, v\rangle + a(u(t), v; t) = \langle f(t), v\rangle$ (17)

   for t pointwise a.e. in [0,T] where a is defined in (12).

3. u(0) = g

PDE in (15) is in weak sense and the BC $u = 0$ on $\partial\Omega$ by $u(t) \in H_0^1(\Omega)$.

- Time derivative in (15) is a distributional time derivative ($u_t = w$ if

$$\int_0^T \phi(t)u(t)\,dt = -\int_0^T \phi_,(t)w(t)\,dt \tag{18}$$

  for every $\phi : (0,T) \leftarrow \mathbb{R}$ with $\phi \in C_C^\infty(0,T)$
  It is **generalization of the weak derivative** of $\mathbb{R}$-valued function. The integral in (16) are vector valued Lebesgue integrals defined analogous way to Lebesgue integral of an integrable $\mathbb{R}$-valued function.

- it isn't trivial that $u(0) = g$ (initial condition) make sense in defnation[1] . Any explicit continuity on u isn't required and $u \in L^2(0,T; H^{-1}(\Omega))$ is defined only upto pointwise value everywhere equivalence in t $\in [0,T]$. So, we say $u \in L^2(0,T; H_0^1(\Omega))$ and $u_t \in L^2(0,T; H^{-1}(\Omega))$ implies that
  $u \in C([0,T]; L^2(\Omega))$. So, u with its continuous [6]

  representative, initial condition makes sense.

[6]page 202, proposition (6.34) of [2]

**Existence Theorem**: Let the conditions in assumption (1) are satisfied. Then for every $f \in L^2(0,T;H^{-1}(\Omega))$ and $g \in H_0^1(\Omega)$, $\exists$ a unique solution

$$u \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$$

of (10) in the sense of the defination (2) with $u_t \in L^2(0,T;H^{-1}(\Omega))$ Constant C depending only on $\Omega$, T and the coefficients of L, s.t.

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;H_0^1)} + \|u_t\|_{L^2(0,T;H^{-1})} \leq C(\|f\|_{L^2}(0,T;H^{-1}) + \|g\|_{L^2})$$

# 9 The Galerkin Method

We note most of the ideas about Glarekin contruction, Existence and Uniqueness of weak solution are from [2] and [1]. **Approximation construction** Galerkin approximation allows us to approximately solve a large or infinite dimensional problem by searching an approximate solution in a smaller finite dimensional space of our choosing. They are closely related to Variational formulation of PDE(eg, time dependent elliptic) Idea:For existence

- We approximate $u : (0,T] \rightarrow H_0^1(\Omega)$ by functions $u_N : [0,T] \leftarrow E_N$. (It takes values in finite-dmensional subspace $E_N H_0^1(\Omega)$ of dimension N.)
- We project PDE onto $E_N$. (We need $u_N$ that satisfies the PDE upto a residual which is orthogonal to $E_N$)
- We then get a system of ODE for $u_N$ which has a solution
- All $u_N$ satisfies energy estimate of the same form as the priori estimates.
- These estimates are uniform in N
- So,pass to $Limit \rightarrow \infty$ and obtain solution of the PDE.

Construction(for our analysis):(Choice of $E_N$ is suitable for existence proof)

- Let,
$$E_N = \langle w_1, w_2, \ldots, w_n \rangle \tag{19}$$
be linear space spanned by first N vectors.

- Orthonormal basis of $E_N = w_k : k \in \mathbb{N}$of$L^2(\Omega)$
Assume Orthonormal basis of $H_0^1 = w_k : k \in \mathbb{N}$of$L^2(\Omega)$
Let $w_k(x)$ be eigenfunction of Drichlet laplacian $\Omega$;

-$\Delta w_k = \lambda_k w_k \quad w_k \in H_0^1(\Omega) \quad k \in \mathbb{N}$

(20)

- Explicitly, [1]

$$\int_{\Omega} w_j w_k dx = \left\{ \begin{array}{lll} 1, & \text{if} & \text{j=k} \\ 0, & \text{if} & \text{j} \neq k \end{array} \right\}$$

$$\int_{\Omega} Dw_j.Dw_k\, dx = \left\{ \begin{array}{ll} \lambda_j & \text{if} \\ j = k0 & \text{if} \\ j \leq k & \end{array} \right\}$$

- We expand, $u \in H_0^1(\Omega)$ and the series is convergent in $H_0^1(\Omega)$ iff [7]

$$\sum_{k \in \mathbb{N}} \lambda_k |c^k|^2 < \infty$$

- $P_N : L^2(\Omega) \to E_N \subset L^2(\Omega)$ the orthogonal projection onto $E_N$

$$P_N\Big(\sum_{k \in \mathbb{N}} c^k w_k\Big) = \sum_{k=1}^{N} c^k w_k$$

(21)

- Similarly,
  $P_N : H_0^1 \to E_N \subset H_0^1$ (restricting)
  $P_N : H{-1} \to E_N \subset H^{-1}$ (extending)

  So, $P_N$ is defined on $H_0^1(\Omega)$ by (6.17) and on $H^{-1}$ by
  $\langle P_N u, v\rangle = \langle u, P_N v\rangle$for all$v \in H_0^{1(\Omega)}$

[1]Existence theory for solutions of elliptic PDEs,(the Dirchlet Laplacian on a bounded open set is a self-adjoint operator with compact resolvent, so normalized set of eigenfunctions have the required properties)

[7]Similar to expansion $u \in L^2(\Omega)$ is $L^2$-convergent series as $u(x) = \sum_{k \in \mathbb{N}} c^k w_k(x)$ where $c^k = (u, w^k)_{L_2}$ and $u \in L^2(\Omega)$ iff, $\sum_{k \in \mathbb{N}} |c^k|^2 < \infty$

## 9.1 Existence of weak solutions

We first prove the existence of the weak solutions, then uniqueness and briefly talk about their regularity. For existence proof; we do follow these three steps: (1) Construct approximate solutions (2) Derive energy estimate for approximate solution (3) Converge approximate solutions to a solution.

- Let,
  $E_N$ = N-dimensional subspace of $H_0^1(\Omega)$ (17)(18) and,
  $P_N$ = orthogonal projection onto $E_N$.

  Definition (Approximate solution)(as 6): A function $u_N : [0,T] \to E_N$ is an approximate solution of (10) if:

  1. $u_N \in L^2(0,T : E_N)$ and $u_{Nt} \in L^2(0,T : E_N)$
  2. for every $v \in E_N$,
  $$(u_{Nt}(t), v)_{L^2} + a(u_N(t), v; t) =< f(t), v > \tag{22}$$
  pointwise a.e. in $t \in (0,T)$;
  3. $u_N(0) = P_{Ng}$.

- Since $u_N \in H^1(0,T; E_N)$, so $u_N \in C(0,T; E_N)$;(Sobolev embedding theorem [8] for function of single variable t.) , so the initial condition (3) is realized.

- For condition (2) $u_N$ should satisfy the weak formulation (15) of the PDE (test functions v are restricted to $E_N$), i.e.
  $$u_{Nt} + P_N L u_N = P_N f$$
  for $t \in (0,T)$ pointwise a.e., $\Rightarrow$u$_N$ takes values in $E_N$ and satisfies the projection of the PDE onto $E_N$.

- Now, we rewrite their definition explicitly ( IVP for an ODE). We expand,

- $$u_N(t) = \sum_{k=1} {}_N c_{Nk}(t)(w)_k \tag{23}$$
  where $c_N^k : [0,T] \to \mathbb{R}$ (continuous)

- It suffice to impose (20) for $v = w_1, ......, w_N$ due to linearity. Thus, (21) is an approximate solution iff;
  $$c_N^k \in L^2(0,T), c_{Nt}^k \in L^2(0,T) \text{ for } 1 \leq k \leq N,$$
  and $c_N^1, ....., c_N^N$ satisfies the system of ODEs
  $$c_{Nt}^j + \sum_{k=1}^{N} a^{jk} c_N^k = f^j, c_N^j(0) = g^j \text{ for } 1 \leq j \leq N \tag{24}$$
  where,
  $$a^{jk}(t) = a(w_j, w_k; t), f^j(t) =< f(t), w_j >, g^j = (g, w_j)_{L^2}$$

[8] We have defined Sobolev embedding theorem on chapter 10 of this thesis using [8]

- Equation (22) in vector form for $\vec{c}:[0,T]\leftarrow[]R^N$ is

$$\vec{c}_{Nt} + A(t)\vec{c}_N = \vec{f}(t), \vec{c}_N(0) = \vec{g} \tag{25}$$

  where

$$\vec{c}_N = \{c_N^1, ....c_N^N\}^T, \vec{f}_N = \{f_N^1, ....f_N^N\}^T, \vec{g}_N = \{g_N^1, ....g_N^N\}^T$$

- and $A : [0,T] \rightarrow R^{N\times N}$ is a matrix-valued function of $t$ with coefficients $(a^{jk})_{j,k=1,N}$

- **Proposition(1)** . For every $N \in N$, $\exists$ a unique approximate solution $u_N : [0,T] \rightarrow E_N$ of (10).

$$c_{Nt}^j + \sum_{k=1}^{N} a^{jk}c_N^k = f^j, c_N^j(0) = g^j \text{ for } 1 \leq j \leq N$$

Proof: Proof for this proposition can be found on proposition(6.5) page 184 of (ch6.pdf) of [2] or chapter 3 of [3].It has been said it follows from standard ODE theory. (also, contraction mapping theorem)

# Step 2:Energy estimates for approximate solutions

.
Similar to priori estimate for the heat equation, we get energy estimate for aprroximate solutions. We take the test function $v = u_N$ in the Galerkin equations.

**Proposition (2)**: There exists a constant $C$, depending only on $T, W$, and the coefficient functions $a^{ij}, b^j, c$, such that for every $N \in N$ the approximate solution $u_N$ constructed in Proposition(1) satisfies:

$$||u_N||_{L^\infty(0,T;L^2)} + ||u_N||_{L^2(0,T;H_0^1)} + ||u_{Nt}||_{L^2(0,T;H^{-1})} \leq C\left(||f||_{L^2(0,T;H_0^1)} + ||g||_{L^2}\right)$$

Proof: Proof for this proposition can be found on proposition(6.6) page 185 of (ch6.pdf) of [2] or chapter 3 of [3]. Different inequalities like Cauchy inequality, Cauchy-Schwartz inequality and projection inequality has been used.

# Step 3:Convergence of approximate solutions.

We use compactness argument to show that a subsequence of approximate solutions converges to a weak solution. **Weak convergence**,

- – Dual space of $L^2(0,T;H_0^1(\Omega)) = L^2(0,T;H^{-1}(\Omega))$.

–

– Action of $f \in L^2(0,T;H^{-1}(\Omega))$ on $u \in L^2(0,T;H_0^1(\Omega))$;

$$<< f,u >>= \int_0^T < f,u > dt$$

, where $<< .,. >>$ is the duality pairing between $L^2(0,T;H^{-1})$ and $L^2(0,T;H_0^1)$ and $< .,. >$ is the duality pairing between $H^{-1}$ and $H_0^1$.

- $u_N \to u \quad \text{in} \quad L^2(0,T;H_0^1(\Omega)) \Rightarrow$

$$\int_0^T < f(t), u_N(t) > dt \to \int_0^T < f(t), u(t) > dt \quad \forall f \in L^2(0,T;H^{-1}(\Omega))$$

- $f_N \to f$ in $L^2(0,T;H^{-1}(\Omega)) \Rightarrow$

$$\int_0^T < f_N(t), u(t) > dt \to \int_0^T < f(t), u(t) > dt \quad \forall u \in L^2(0,T;H_0^1(\Omega))$$

- $u_N \to u$ in $L^2(0,T;H_0^1(\Omega))$ and $f_N \to f$ strongly in $L^2(0,T;H^{-1}(\Omega))$, or conversely, then $< f_N, u_N > \to < f, u >$.

**Proposition (3)**. A subsequence of approximate solutions converges weakly in $L^2(0,T;H^{-1}(\Omega))$ to a weak solution:

$$u \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$$

of 6.8 with $u_t \in L^2(0,T;H^{-1}(\Omega))$. Further, there is a constant $C$ s.t.

$$||u||_{L^\infty(0,T;L^2)} + ||u||_{L^2(0,T;H_0^1)} + ||u_t||_{L^2(0,T;H^{-1})} \le C \left( ||f||_{L^2(0,T;H^{-1})} + ||g||_{L^2} \right)$$

Proof: Proof for this proposition can be found on proposition(6.7) page 186 of (ch6.pdf) of [2] or chapter 3 of [3].It has been said it follows from Banach Alaoglu-Theorem (see Theorem(1.19) of [3]).

**Note: It has been said Existence of weak solution can be proved using Lax-Milgram theorem(page 28 of [10] )**

## 9.2 Uniqueness of weak solutions

We assume same data f,g , if $u_1, u_2$ are two solutions. then linearity gives $u = u_1 - u_2$ is a solution with zero data $f = 0$, $g = 0$. Showing the only weak solution with zero data is $u = 0$ suffice to prove uniqueness argument.
$u(t) \in H_0^1(\Omega)$,
We take $v = (t)$ as a test function in (15) with $f = 0$,

$$\angle u_t, u\rangle + a\langle u, u; t\rangle = 0$$

it holds pointwise a.e. in [0,T] in the sense of weak derivatives. We use ((6.46)(see page 208 of ch6A.pdf of [2]) ) and the coercivity estimate (13) and find that there are constants $\beta > 0$ and $-\infty < \gamma < \infty$ s.t:

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \beta\|u\|_{H_0^1}^2 \leq \gamma\|u\|_{L^2}^2$$

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2}^2 + \leq \gamma\|u\|_{L^2}^2, \qquad u(0) = 0$$

and since $\|u(0)\|_{L^2} = 0$, **Gronwall's inequality** implies that $\|u(t)\|_{L^2} = 0$ for all $t \geq 0$, so $u = 0$. Similarly, we get continuous dependence of weak solutions on the data. It $u_i$ is the weak solutions with data $f_i, g_i$ for $i = 1, 2$, then there is a constant C independent of the data s.t;

$$\|u_1 - u_2\|_{L^\infty}(0, T; L^2) + \|u_1 - u_2\|_{L^2}(0, T; H_0^1) \leq C(\|f_1 - f_2\|_{L^2}(0, T; H^{-1})) + \|g_1 - g_2\|_{L^2}$$

(From wikipedia defination and [7])(Gronwall's inequalityin differential form) It bounds a function (which satisfy a certain differentialor integral inequality) by a solution of corresponding differential or integral equation. Let I denote an interval of the real line of the form $[a, \infty)$ or $[a, b]$ or $[a, b)$ with $a < b$. Let $\beta$ and u be real-valued continuous functions defined on I. If u is differentiable in the interior $I^o$ of I (the interval I without the end points a and possibly b) and satisfies the differential inequality.

$$u' \leq \beta(t)u(t), \quad t \in I^o,$$

then u is bounded by the solution of the corresponding differential equation $v'(t) = \beta(t)v(t)$:

$$u(t) \leq u(a)exp\Big(\int_\alpha^t \beta(s)\, ds\Big)$$

for all $t \in I$ Remark: There is no assumption on the signs of the functions $\beta$ and u

**Theorem(2)Coercivity(Bilinear Form)**:

$$a : H_0^1(\Omega) \times H_0^1 \times (0, T) \to \mathbb{R}$$

then, $\exists$ constants $C > 0$ and $\gamma \in \mathbb{R}$s.t. for every u,v $\in H_0^1$

$$C\|u\|_{H_0^1}^2 \leq a(u, u; t) + \gamma\|u\|_{L^2}^2$$

$$|a(u, v; t)| \leq C\|u\|_{H_0^1}\|v\|_{H_0^1}$$

# 10 Regularity

Based on section (6.3) , section (7.1.3) from [1] and a thesis by [3], we talk about the regularity. Our final aim is to prove our weak solution 'u' to the initial-boundary value problem is smooth, provided coefficients of partial differential equations, boundary are smooth.

Consider, initial value problem,

$$u_t - \triangle u = f \text{ in } R^n \times (0,T] u = g \text{ on } R^n \times t = 0, \tag{26}$$

For heuristic purposes, we assume 'u' is smooth and 'u' goes to zero as $|x| \rightarrow and \quad we \quad compute \quad following \quad for \infty \; 0 \leq t \leq T$,

$$\int_{R^n} f^2 dx = \int_{R^n} (u_t - \triangle u)^2 dx = \int_{R^n} u_t^2 - 2\triangle u u_t + (\triangle u)^2 dx = \int_{R^n} u_t^2 + 2Du.Du_t + (\triangle u)^2 dx \tag{27}$$

$\Rightarrow 2Du.Du_t = \frac{d}{dt}(|Du|^2)$

$$\int_0^t \int_{R^n} 2Du.Du_t dxds = \int_{R^n} |Du^2|dx|_{s=0}^{s=t}$$

As we can see from (6.3) of [1];

$$\int_{R^n} (\triangle u)^2 dx = \int_{R^n} |Du^2|^2 dx$$

Using two equalities above equation and integrating in time;

$$\begin{aligned} \sup_{0 \leq t \leq T} int_{R^n} |Du|^2 dx + \int_0^t \int_{R^n} u_t^2 + |D^2 u|^2 dxdt \\ \leq C\left( \int_0^t \int_{R^n} f^2 dxdt + \int_{R^n} |Dg|^2 dx \right) \end{aligned}$$

We see, using $L^2$-norm of 'f', we can estimate $L^2$-norm of second derivative. $L^2$-norm of 'f' on $\mathbb{R}^n \times (0,T)$ and $L^2$-norm of $D_g$ on $\mathbb{R}^n$ estimates, $L^2$-norm of $u_t$ and $D^2 u$ within $\mathbb{R}^n \times (0,T)$.

We now differentiate the partial differential equations(initial value condition) above w.r.t $t$, set $\tilde{u} := u_t$.

$$\begin{aligned} \tilde{u}_t - \triangle \tilde{u} = \tilde{f} \text{ in } R^n \times (0,T] \\ \tilde{u} = \tilde{g} \text{ on } R^n \times t = 0 \end{aligned}$$

for $\tilde{f} := f_t, \tilde{g} := u_t(.,0) = f(.,0) + \triangle g$.
We multiply by $\tilde{u}$, integrate by parts and use Gronwall's inequality;

$$\sup_{0 \leq t \leq T} \int_{R^n} |u_t|^2 dx + \int_0^T \int_{R^n} |Du_t|^2 dxdt \leq C\left( \int_0^t \int_{R^n} f^2 dxdt + \int_{R^n} |Dg|^2 + f(.,0)^2 dx \right) \tag{28}$$

From Theorem 2(iii) of (5.9.2) from [1], we have,

$$\max_{0 \leq t \leq T} ||f(.,t)||_{L^2(R^n)} \leq C(||f||_{L^2(R^n \times (0,T))} + ||f_t||_{L^2(R^n \times (0,T))})$$

We write $-\triangle u = f - u$ and according to (6.3) of [1],we get,

$$\int_{R^n} |D^2 u|^2 dx \leq C \int_{R^n} f^2 + u_t^2 dx$$

We combine above equations, we get,

$$\sup_{0\leq t\leq T}\int_{R^n}|u_t|^2+|D^2u|^2dx+\int_0^T\int_{R^n}|Du_t|^2dxdt$$
$$\leq C\left(\int_0^T\int_{R^n}f_t^2+f^2dxdt+\int_{R^n}|D^2g|^2dx\right)$$

for constant $C$.

We can apply same method, use first derivatives of f to estimate $L_2$-norm of the third derivatives.Further, we can estimate $L^2$-norm of $(m+2)^{nd}$ derivatives by $L^2$-norm of $m^{th}$ derivatives of f, for $m=0,1,...$. For, $m=\infty$ where u belongs to $H^m$ for all $m=1,...$ and hence to $C^\infty$. Above wasn't real proof but we assumed u smooth for calculation. We have presented below some calculations which are technically difficult but very powerful. These calculations comes from ellipticity. We derive analytical estimates from structural algebraic assumption.Coupling this procedure (and Galerkin method) with Sobolev embedding theorem, we can construct weak solutions are smooth,classical solutions provided data follows compatible relation. We present those higher estimates below:

**Proof of Regularity for our boundary condition can be found in Theorem(3.4) of [3] or page 39 section(2.6) of [10]**

(Theorem 5 of (7.1.3) of [1])(Imrpoved regularity), (i) Assume

$$g\in H_0^1(U), f\in L^2(0,T;L^2(U))$$

Also assume $u\in L^2(0,T,H_0^1(U))$, with $u'\in L^2(0,T;H^{-1}(U))$ is the weak solution of:

$$\begin{aligned}u_t+Lu&=f \text{ in } U_T\\ u&=0 \text{ on } \partial U\times[0,T]\\ u&=g \text{ on } U\times t=0\end{aligned}$$

Then,

$$u\in L^2(0,t;H^2(U))\cap L^\infty(0,T;H_0^1(U)), u'\in L^2(0,T;L^2(U)),$$

and we have the estimate

$$\text{ess sup }_{0\leq t\leq T}||u(t)||_{H_0^1(U)}+||u||_{L^2(0,T;H^2(U))}+||u'||_{L^2(0,T;L^2(U))}$$
$$\leq C\left(||f||_{L^2(0,T;H^2(U))}+||g||_{H_0^1(U)}\right)$$

the constant $C$ depending only on $U,T$ and the coefficients of $L$. (ii) If, in addition,

$$g\in H^2(U), f'\in L^2(0,T;H^2(U))$$

then

$$u\in L^\infty(0,T;H^2(U)), u'\in L^\infty(0,T;L^2(U))\cap L^2(0,T;H_0^1(U)), u''\in L^2(0,T;H^{-1}(U))$$

with the estimate

$$\text{ess sup}_{0\leq t\leq T}(||u(t)||_{H^2(U)} + ||u'(t)||_{L^2(U)} + ||u'||_{L^2(0,T;H^1_0(U))} + ||u''||_{L^2(0,T;H^{-1}(U))} \leq C(||f||_{H^1(0,T,L^2(U))} + ||g||_{H^2(U)}))$$

(Theorem 6 of (7.1.3) of [1]) (Higher regularity) Assume

$$g \in H^{2m+1}(U), \frac{d^k f}{dt^k} \in L^2(0,T;H^{2m-2k}(U))(k=0,....,m)$$

Suppose the following $m^{th}$ -order compatibility conditions hold:

$$g_0 := g \in H^1_0(U), g_1 := f(0) - Lg_0 \in H^1_0(U),$$
$$...., g_m := \frac{d^{m-1}f}{dt^{m-1}}(0) - Lg_{m-1} \in H^1_0(U),$$

Then,

$$\frac{d^k u}{dt^k} \in L^2(0,T;H^{2m+2-2k}(U))(k=0,....,m+1);$$

and we have,

$$\sum_{k=0}^{m+1}\left|\left|\frac{d^k u}{dt^k}\right|\right|_{L^2(0,T;H^{2m+2-2k}(U))}$$
$$\leq C\left(\sum_{k=0}^{m}\left|\left|\frac{d^k f}{dt^k}\right|\right|_{L^2(0,T;H^{2m+2-2k}(U))} + ||g||_{H^{2m+1}(U)}\right)$$

the constant $C$ depending only on $m, U, T$ and the coefficients of $L$.

(Theorem 7 of (7.1.3) of [1])(Infinite differentiability). Assume

$$g \in C^\infty(\tilde{U}), f \in C^\infty(\tilde{U_T})$$

and the $m^{th}$ order compatibility conditions hold for m = 0,1,... Then the parabolic initial/boundary-value problem (IBVP of Galerkin approximation's above) has a unique solution:

$$u \in C^\infty(\tilde{U_T})$$

# 11 Sobolev Embedding

(We find the following shortic defination based on [8])

**(Gagliardo-Nirenberg-Sobolev inequality)(GNS)**: Assume $1 \leq p < n$. $\exists$ a constant C, depending only on p and n, such that

$$\|u\|_{L^{p*}}(\mathbb{R}^n) \leq C\|Du\|_{L^p(\mathbb{R}^N)}$$

$\forall u \in C^1_c(\mathbb{R}^n)$. Sobolev embedding theorem
follows directly from GNS. **we note GNS is very useful tool for analysis in Sobolev space**

(Sobolev embedding ): Let $1 \leq p \leq \infty, k \in \mathbb{Z}_+$ and $\mathbb{Q}$
be a boundary Lipschitz domain in $\mathbb{R}^n$.

- Case 1: $kp > n$

$$W^{k,p}(\Omega) \hookrightarrow C(\Omega)$$

- Case 2: $kp = 1$

$$W^{k,p}(\Omega) \hookrightarrow L(\Omega), \quad \forall q \in [1, \infty)$$

$$\text{Also,} \quad W^{n,1} \hookrightarrow C(\tilde{\Omega})$$

- Case 3: $kp < n$

$$W^{k,p}(\Omega) \hookrightarrow L^q(\Omega), \qquad \text{with} \quad \frac{1}{q} = \frac{1}{p} - \frac{k}{p}$$

# 12 Appendix

## 12.1 Calculation for Chapter 3 Maximum Principle of this thesis : Applying Divergence Theorem

Let $I \subset \mathbb{R}^n$ be an open bounded with $\partial I$ being $C^1$. Suppose $w \in C^1(\bar{I})$. Then,Divergence Theorem:

$$\int_I \nabla w \, dx = \int_{\partial I} w.k^i \, ds \quad (i = 1, ..., n) \tag{30}$$

where $k^i = n$ denotes outward pointing unit vector field to the region $I$.
Let, $w = (u - m) \times v$.

$$\text{LHS} = \int_I \nabla w \, dx = \int_I \nabla(u - m)v \, dx$$

Using product rule of integration(Integration by parts)

$$\int_I \nabla(u - m)v \, dx = \int_I \nabla(u - m)v - \int_I (u - m)\nabla v \, dx$$

Now we use the divergence theorem from above:

$$\int_I \nabla(u - m)v \, dx = \int_{\partial I} (u - m)v.n \, ds - \int_I (u - m)\nabla.v \, dx$$

Let, $v = \nabla(u-m)$

$$\int_I \nabla(u-m)\nabla(u-m) = \int_{\partial I} (u-m)\nabla(u-m).n\, ds - \int_I (u-m)\nabla.\nabla(u-m)$$

we know, $\nabla.\nabla = \Delta$

$$\int_I \Delta(u-m) = \int_{\partial I} \nabla(u-m)(u-m).n\, ds - \int_I |\nabla(u-m)|^2\, dx \tag{31}$$

Or We can directly use Green's first identity(which is also given in ( see equation(21.7) of [6] )

# References


[1] Lawrence C Evans. *Partial differential equations*, volume 19. American Mathematical Society, second edition, 2010.

[2] John K. Hunter. Notes on partial differential equations. University of California at Davis, 2014. URL https://www.math.ucdavis.edu/~hunter/pdes/. lecture notes.

[3] Said Mohammed Said Hamrouni Ahmed. Faedo-galerkin method for heat equation. Research India Publication, 2017. URL https://www.ripublication.com/gjpam17/gjpamv13n4_06.pdf.

[4] Mathew A. Johnson. Energy methods in parabolic pde theory. University of Kansas. lecture notes.

[5] Yongheng Zhang. Proof of gauss green theorem. Math Stack Exchange.

[6] Birne Binegar. Lecture 21: Green's identities. Oklahoma State University. URL https://math.okstate.edu/people/binegar/5233-S96/5233-l21.pdf. lecture note.

[7] Gronwall's inequality. Wikipedia. https://en.wikipedia.org/wiki/Gr

[8] Long Chen. Sobolev space and elliptic equations. University of California Irvine. lecture notes.

[9] Francisco Pinto Berkemeier. A priori regularity of parabolic partial differential equations. King Abdullah University of Science and Technology, 2018. Masters Thesis.

[10] Marcel Schaub. Advanced partial differential equations. Ludwig Maximilian University of Munich, 2015. URL https://www.mathematik.uni-muenchen.de/~schaub/PDE2.pdf.

[11] Nicholas Michalowski Stephen Pankavich. A short proof of increased parabolic regularity. Electronic Journal of Differential Equations, 2015.

[12] Dr Chris Tisdell. Fourier transforms: Heat equation. youtube.