5.3 Binomial Theorem:

→ Binomial theorem is used to solve binomial expressions in a simple way.

→ It gives an expression to calculate the expansion of $(a+b)^n$ for any positive integer n.

→ The Binomial theorem is stated as:-

$$(a+b)^n = {}^nC_0 a^n + {}^nC_1 a^{n-1} \cdot b^1 + {}^nC_2 a^{n-2} \cdot b^2 + \cdots + {}^nC_{n-1} a^1 \cdot b^{n-1} + {}^nC_n \cdot a^0 \cdot b^n$$

$$= \binom{n}{0} \cdot a^n \cdot b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} \cdot a^{n-2} \cdot b^2 + \cdots$$

$$+ \binom{n}{n-1} \cdot a^1 \cdot b^{n-1} + \binom{n}{n} \cdot a^0 \cdot b^n$$

↓ Binomial coefficient

⇒ Binomial coefficients:

→ The r- combinations from a set of n elements, denoted by $\binom{n}{r}$.

→ This number is also called a binomial coefficient since it occurs as a coefficient in the expansion of powers of binomial expressions.

→ The binomial theorem gives a power of a binomial expression as a sum of terms involving binomial coefficients.

EXAMPLE :- What is the coefficient of $x^{12} \cdot y^{13}$ in the expansion of $(2x-3y)^{25}$?

$\Rightarrow ((2x)+(-3y))^{25}$

$\Rightarrow$ $n-1=12$

$\therefore n=13$

$\therefore \binom{25}{13} \cdot (2)^{12} \cdot (-3)^{13} x^{12} \cdot y^{13}$ } by using binomial theorem.

Properties of coefficients :-

$\Rightarrow$ Symmetry rule: $\binom{n}{r} = \binom{n}{n-r}$

$\Rightarrow$ Sum over K: $\sum_{K=0}^{n} \binom{n}{K} = 2^n$

→ Consider this, $(x+y)^n = \sum_{K=0}^{n} \binom{n}{K} x^{n-K} \cdot y^K$

if $x=1$ & $y=1$

$\Rightarrow (2)^n = \sum_{K=0}^{n} \binom{n}{K}$

→ If $x=1$ and $y=2$

$$\therefore (3)^n = \sum_{K=0}^{n} \binom{n}{K} \cdot 2^K \cdot (1^{n-K}=1)$$

→ If $x=1$ and $y=-1$

$$\therefore (0)^n = \sum_{K=0}^{n} \binom{n}{K} (1)^{n-K} (-1)^K$$

$$\therefore \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \binom{n}{3} + \dots + \binom{n}{n}(-1)^n = 0$$

Properties of Binomial theorem:

1.) There are $(n+1)$ terms in an expansion of a binomial expression with index n, i.e. more than the power (index) of the binomial theorem ~~expression~~ expression.

∴ e.g.:- The no. of terms presente in the expansion of $(a+b)^n$ is equal to $(n+1)$.

2.) The expansion of $(a+b)^n$ has first term is equal to a^n while the last one is equal to b^n.

3.) In any term the sum of the indices (exponents) of 'a' and 'b' is equal to 'n' (i.e., the power of the binomial).

4.) From the beginning of the expansion of $(a+x)^n$, the powers of 'a' decrease from 'n' upto '0', and the powers of 'x' increase from '0' upto 'n'.

5.) The binomial coefficients in the expansion are arranged in an array, which is called Pascal's triangle. This pattern developed is summed up by the binomial theorem formula.

6.) When $a=b$ or $a+b=n$ only, then ${}^nC_a = {}^nC_b$. The coefficient of each term equidistant from the beginning and the end are equal.

→ Such coefficients are called as the binomial coefficients and ${}^nC_r = {}^nC_{r-1}$, where r is $0, 1, 2, \ldots, n$.

7.) Sum over k, $\sum_{k=0}^{n} {}^nC_k = 2^n$

EXAMPLE:- How many ways we can select 6 students in a class of 10?

$\Rightarrow$ $^{10}C_6 = \cancel{\frac{10\times 9\times 8\times 7}{4\times 3\times 2}}$

$= {}^{10}C_4 = \frac{10\times 9\times 8\times 7}{4\times 3\times 2}$

$= 210$ ways

$\rightarrow$ Let's consider below:

1 2 3 4 5 6 7 8 9 10

a b a b a b a b ... a b

a = probability of getting chosen = 0.5
b = probability of not getting chosen = 0.5

$\therefore = (a+b) * (a+b) * \dots * (a+b) = \left(\frac{1}{2}\right)$

$= (a+b)^{10}$

$= \binom{10}{0} a^{0} b^{10} + \binom{10}{1} . a^{9} . b^{1} \dots + \binom{10}{10} . a^{10} b^{0}$

$\rightarrow$ We want $\binom{10}{6} a^{6} . b^{4}$

$\uparrow$
no. of ways $= 210$ ways

EXAMPLE: How many ways we can select at least one student in a class of 6?

$\Rightarrow$ $(a+b)^6 = \binom{6}{0} a^6 b^0 + \binom{6}{1} a^5 b^1 + \dots + \binom{6}{6} a^0 b^6$

coefficient for none student getting selected

$= \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} + \binom{6}{6}$

$= 6 + \left(\frac{6\times5}{2}\right) + \left(\frac{6\times5\times4}{6}\right) + \left(\frac{6\times5}{2}\right) + 6 + 1$

$= 12 + 30 + 20 + 1$

$= 51 + 12$

$= \underline{63}$

GATE-CS-2003: n couples are invited to a party with the condition that every husband should be accompanied by his wife. However, a wife need not be accompanied by her husband. The number of different gatherings possible at the party is:-

[A] $\binom{2n}{n} * 2^n$ [B] 3^n [C] $\frac{(2n)!}{2^n}$ [D] $\binom{2n}{n}$

⇒ Let x be husband and y be wife

⇒ Possible solutions are $(x,y)=(1,1)$ and $(x,y)=(0,1)$. As husband should be accompained by her wife but wife can also come alone.

∴ ⇒ As there are n-couple, total people will be $2n$.

Husbands	1	2	3	4 — — —	n
wives	1	2	3	4 - - -	n

⇒ There are three options for every couple.
1.) Nobody goes to gathering
2.) wife alone goes
3.) Both goes.

⇒ So, for every couple there are 3 options

⇒ Hence; it will be $3\times3\times\ldots$ n times
$= 3^n$ ways

⇒ Binomial theorem for negative or fractional index is:-

+ $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}\cdot x^2 + \frac{n(n-1)(n-2)}{3!}x^3$
$+ \cdots$ upto ∞ where $|x|<1$.

→ General term of $(1+x)^n$ where n is negative $= \frac{n.(n-1).(n-2)\cdots(n-k+1)}{k!}\cdot x^k$

GATE-CS-2016: The coefficient of x^{12} in $(x^3+x^4+x^5+x^6+\cdots)^3$ is.

⇒ $= (x^3+x^4+x^5+x^6+\cdots)^3$

$= (x^3)^3 (\underbrace{1+x+x^2+x^3+\cdots}_{\text{This is expansion of } \frac{1}{1-x}})^3$

$= x^9 \cdot \left(\frac{1}{1-x}\right)^3$

$= x^9 \cdot (1-x)^{-3}$

Here the fourth term will have x^3, hence we will find that

$= x^9 \cdot \frac{(-3)(-3-1)(-3-2)}{3!}(-x)^3$

$= x^9 \cdot \frac{(+\cancel{3})(+\cancel{4}^2)(+5)}{\cancel{6}} + x^3$

$= 10x^{12}$

Coefficient of x^{12}: 10

⇒ **Another method:-**

$= x^9 (1 + x + x^2 + \dots)^3$

$= (1 + x + x^2 + \dots)^3$

⇒ Using binomial expansion,

$(1 + (x + x^2 + x^3))^3$

$= \binom{3}{0} + \binom{3}{1} \cdot (x + x^2 + x^3)$
$+ \binom{3}{2} (x + x^2 + x^3)^2$
$+ \binom{3}{3} (x + x^2 + x^3)^3$

Coefficients of $x^3 = \binom{3}{1} \cdot x^3$
$+ \binom{3}{2} (x^2 + x^4 + x^6 + \underline{2x^3} + \dots)$

$+ \binom{3}{3} \cdot x^3$

$= \left(3 + \frac{3 \times 2}{2!} + 1\right) x^3$

$= \underline{10x^3}$

∴ Multiplying x^3 with $x^9 = x^{12}$

∴ Coefficient $= \underline{10}$