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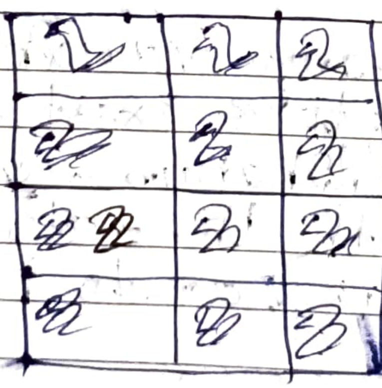
5.2 The Pigeonhole Principle:

→ Suppose that a flock of 13 pigeons flies into a set of 12 pigeonholes to roost.

→ Because there are 13 pigeons but only 12 pigeonholes, at least one of these 12 pigeonholes must have at least two pigeons in it.

⇒ A general principle called the Pigeonhole principle, which states that if there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons ~~on~~ in it.



EXAMPLE-1:- How many students must be in a class to guarantee that at least two students receive the same score on final exam, if the exam is graded from 0 to 100 points?

⇒ Score scale: $\underbrace{0, 1, 2, \ldots, 100}$

101 + 1

= 102 students required to receive same score

⇒ Here '102' is minimum no. of students required.

⇒ Theorem:

I.) If "A" is the average number of pigeons per hole, where A is not an integer then

- At least one pigeon hole contains $\lceil A \rceil$ (smallest integer greater than or equal to A) pigeons

- Remaining pigeon holes contain at most $\lfloor A \rfloor$ (largest integer less than or equal to A) pigeons

II.) We can say as, if n+1 objects are put into 'n' boxes, then atleast one box contains two or more objects.

⇒ The abstract formulation of the principle:

- Let X and Y be finite sets and let $f: A \to B$ be a function.

- If X has more elements than Y, then f is not one-to-one.

- If X and Y have the same number of elements and f is onto, then f is one-to-one.

- If X and Y have the same number of elements and f is one-to-one, then f is onto.

EXAMPLE:- If $(kn+1)$ pigeons are kept in 'n' pigeon holes where 'K' is a positive integer, what is the average no. of pigeons per pigeon hole?

$\Rightarrow$ Avg. pigeons for 'n' holes $= \frac{kn+1}{n}$

$$= k + \left(\frac{1}{n}\right)$$

$\therefore \rightarrow$ Atleast $k+1$ pigeons in one hole and remaining holes have atmost 'K' pigeons.

EXAMPLE:- A bag contains 10 red marbles, 10 white marbles, and 10 blue marbles. What is the minimum no. of marbles you have to choose randomly from the bag to ensure that we get 4 marbles of same color?

⇒ Red marbles: 10
White marbles: 10
Blue marbles: 10

→ We need 4 marbles of same color

For that pick: $\left.\begin{array}{l} R-3 \\ W-3 \\ B-3 \end{array}\right\}$ +1 of any color will make 4 marbles of same color.

Another way:-

- No. of colors (pigeonholes) $n=3$
- No. of marbles of same color required $K+1=4$
  $\therefore K=3$

⇒ Simply putting these values into formula $Kn+1$
$= 3(3)+1$
$= \underline{10}$ marbles are minimum required

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EXAMPLE:- How many cards must be selected from a standard deck of 52 cards to guarantee that atleast three cards of same suit chosen?

=> No. of suits = 4 ∴ n = 4

No. of cards chosen to be of same suit = 3

∴ $K+1 = 3$

∴ $K = 2$

∴ Minimum no. of cards to be chosen

$= Kn+1$

$= 2(4)+1$

$= 8+1$

$= 9$

EXAMPLE:- How many cards must be selected from a standard deck of 52 cards to guarantee that atleast three hearts are selected?

=> Types of cards $n=4$ [Hearts, square, spade, club]

Atleast no. of cards required $= K+1 = 3$

∴ $K = 2$

∴ Minimum no. of cards required to select cards $= Kn+1$

$= (2)(4)+1 = 9$

→ worst case pickup of cards:-

$$\left.\begin{array}{l} H = 2 \\ S = 13 \\ C = 13 \\ D = 13 \end{array}\right\} + 1$$ will guarantee atleast 3 hearts

⇒ Pigeon hole principle → strong form:-

→ Theorem:- Let $q_1, q_2, \dots, q_n$ be 'n' positive integers.

→ If $q_1 + q_2 + \dots + q_n - n + 1$ objects are put into 'n' boxes, then either the 1st box contains at least q_1 objects, or the 2nd box contains at least q_2 objects, ..., the n^{th} box contains at least q_n objects.

EXAMPLE:- In a computer science department, a student club can be formed with either 10 members from first year or 8 members from second year or 6 from third year or 4 from final year. What is the minimum no. of students we have to choose randomly from department to ensure that a student that a students club is formed?

→ students required from first year = 10
second year = 8
third year = 6

∴ fourth year = 4

∴ To form a student club; applying pigeon-hole strong form directly as there is not given any minimum requirement of students

= 10 + 8 + 6 + 4 − 4 + 1
= 24 + 4 − 4 + 1
= 25 students

EXAMPLE :- A box contains 6 red, 8 green, 10 blue, 12 yellow and 15 white balls. What is the min. no of balls we have to choose randomly from the box to ensure that we get 9 balls of same color?

⇒ Red - 6 ; Green - 8 ; Blue - 10 ; Yellow - 12, white - 15.

We require atleast 9 - balls of same color.

→ In this case, we will directly add red, green after applying pigeon-hole to remaining ones.

⇒ For blue, yellow and white balls;

= ~~10 + 12 + 15 − 3 + 1 = 37 − 2 = 35 balls~~
= 6 + 8 + 8 + 1 = 25 balls

∴ Now, minimum no. of balls required

$$= 25 + 8 + 6$$

$$= \underline{39 \text{ balls}}$$

Question: There are 38 different time periods during which classes at a university can be scheduled. If there are 677 different classes, what is the minimum number of different rooms that will be needed?

⇒ Time periods: 38

Different classes: 677

$n = 38$

$k = 677$

∴ No. of different classes required $= \left\lceil \frac{677}{38} \right\rceil$

$$= \underline{18}$$