5.4 Generalized PnC:-

Permutations with repetitions:-

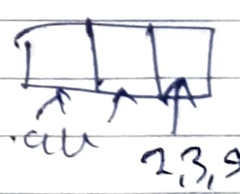
⇒ Counting permutations when repetition of elements can be easily done using the product rule.

Example:- Number of strings of length r is 26^r, since for every character there are 26 possibilities.

→ Thus, the no. of r-permutations of a set of n objects with repetition is n^r

Question:- How many 3-digit odd numbers can be formed by using the digits 1, 2, 3, 4 and 5?
(a.) if repetition is allowed
(b.) if repetition is not allowed.

⇒ (a.) if repetition is allowed:-



∴ No. of such odd numbers can be formed are $= 3 \times 5 \times 5 = 75$

(b.) if no repetition allowed.

∴ No. of such odd numbers can be formed are $= 3 \times 4 \times 3 = 36$

Question-2 The number of 4-digit numbers having their digits in non-decreasing order (from left to right) constructed by using the digits belonging to the set {1, 2, 3} is ______ {Gate-2015}

(a) 12 (b) 13 (c) 14 (d) 15

=> 4-digit numbers: | | | | |

-> Digits allowed: {1, 2, 3}

Non-decreasing means either increasing or same.

-> Consider 1 as first digit | 1 | 1 | 1 | 1 |

2 3

2

| 1 | 1 | 1 | 2 |

2
3

3

| 1 | 2 | 2 | 2 |

2 3
3 3

3

| 1 | 1 | 3 | 3 |

1

| 1 | 3 | 3 | 3 |

1

-> Consider 2 as first digit | 2 | 2 | 2 | 2 |

2

3

| 2 | 2 | 3 | 3 |

1

| 2 | 3 | 3 | 3 |

1

→ Consider 3 as first digit: $\boxed{3|3|3|2}$

1

∴ Total ways = 1 + 1 + 1 + 2 + 1 + 3 + 3 + 2
= 3 + 3 + 6 + 2
= 14 + 1 = 15

Combinations with repetitions:-

→ Counting the no. of ~~repetitions~~ combinations with repetition is a bit more complicated than counting permutations.

→ Consider a set of n types of objects.

→ There are $C(r+n-1, r) = C(r+n-1, n-1)$ r-combinations from a set with n elements when repetition is allowed.

EXAMPLE-1:- In how many ways can 4 drinks can be chosen out of 6 possible types of drinks? There are no restriction on the number of drinks of a type that can be chosen and drinks of the same type are indistinguishable.

⇒ Drinks to be chosen: r = 4
Types of drinks: n = 6

∴ No. of combinations = $C(4+6-1, 4)$ $= C(9,4) = \frac{9 \times 8 \times 7 \times 6}{24} = 126$

EXAMPLE-2:- How many solutions does the equation $x_1+x_2+x_3=11$ have, where x_1, x_2, x_3 are non-negative integers?

$\Rightarrow$ $x_1+x_2+x_3=11$

$\rightarrow$ It means, every variables can be having values betn [0, 11]. Total such values are having range: $r=11$

$\rightarrow$ No. of variables: $n=3$

$\therefore$ No. of such combinations possible are:
$C(11+3-1, 11) = C(13, 11)$
$= \frac{13 \times 12^{6}}{2} = \underline{78}$

EXAMPLE-3: Consider the same equation as e.g. 2 but with the additional constraint that $x_3 >= 1$.

$\Rightarrow$ Given $x_3 >= 1$
$x_1+x_2+x_3=11$

$\rightarrow$ Let's say x_3 is atleast 1; substituting $x_3=1$, we get
$x_1+x_2=10$

No. variables: $n=3$, $r=10$

$C(10+3-1, 2) = C(12, 2) = \frac{12 \times 11}{2} = \underline{66}$

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EXAMPLE-4:- Consider the same question as e.g. 2 but with the additional constraint that $x_1 < 2$.

⇒ for $x_1 < 2$; possible values are $x_1 = 0, 1$.

$x_1 + x_2 + x_3 = 11$

→ Suppose there isn't such condition, so,
no. of ways $= C(3+11-1, 3-1)$
$= C(13,2) = \frac{13 \times 12}{2} = \underline{78}$

→ Consider $x_1 >= 2$, we get the following equation,
$2 + x_2 + x_3 = 11$
$\therefore x_2 + x_3 = 9$
$\therefore$ Possible ways $= C(9+2-1, 2-1)$
$= C(10, 1) = \underline{10}$

→ Suppose $x_1 = 0$; $x_2 + x_3 = 11$

$\therefore$ No. of ways $= C(11+2-1, 2-1)$
$= C(12, 1) = \underline{12}$

→ For $x_1 = 1$; we get: $x_2 + x_3 = 10$.

$\therefore$ No. of ways $= C(10+2-1, 2-1)$
$= C(11, 1) = \underline{11}$

$\therefore$ Total possible ways $= 11 + 12 = \underline{23}$

EXAMPLE-5:- Consider the same question as in e.g. 2, but with an inequality, i.e., $x_1+x_2+x_3 <= 11$.

$\Rightarrow$ There are two ways to solve this problem.

1.) Tedious way:

$x_1+x_2+x_3=0 \Rightarrow C(2,2)=1$
$x_1+x_2+x_3=1 \Rightarrow C(3,2)=3$
$x_1+x_2+x_3=2 \Rightarrow C(4,2)=6$
$x_1+x_2+x_3=3 \Rightarrow C(5,2)=10$
$x_1+x_2+x_3=4 \Rightarrow C(6,2)=15$
$x_1 \quad \dots \quad =5 \Rightarrow C(7,2)=21$
$\quad \dots \quad =6 \Rightarrow C(8,2)=28$
$\quad \dots \quad =7 \Rightarrow C(9,2)=36$
$\quad \dots \quad =8 \Rightarrow C(10,2)=45$
$\quad \dots \quad =9 \Rightarrow C(11,2)=55$
$\quad \dots \quad =10 \Rightarrow C(12,2)=66$
$\quad \dots \quad =11 \Rightarrow C(13,2)=78$

Total would be = 364 ways

2.) The above inequality equation is same as finding the no. of solutions for below equations:- $x_1+x_2+x_3+x_4=11$.

$\Rightarrow r=11, n=4$

$$C(11+4-1, 4-1) = C(14,3)$$
$$= \frac{14\times13\times12^2}{6}$$
$$= 364$$

→ Combinatorial problems can be rephrased in several ways, the most common of which is in terms of distributing balls into boxes.

→ The balls and boxes can be either distinguishable or indistinguishable and the distribution can take place either with or without exclusion.

→ With Exclusion - In case of exclusion, distribution is the same as counting r-permutations, as there are 'n' choices for the first ball, (n-1) for the second and so on.

→ Without exclusion - When the distribution is without exclusion, i.e., there is no restriction on the minimum no. of balls a box has to have, the no. of ways are: n^m.

↳ This is because every ball has choices.

→ Fixed no. of balls :- If the distribution is such that each box should only have a fixed no. of balls then the number of ways are:-

$$\frac{m!}{m_1!\, m_2! \cdots m_k!}$$

where m_k is ~~the~~ the number of balls to put in the k^{th} box.

EXAMPLE-1: In how many ways can 10 prizes be distributed among 5 people without exclusion?

⇒ 1 prize can go to 5 people
2 prize ... 5 people
|
|
10^{th} prize can go to 5 people

∴ Total ways $= 5 \times 5 \times \ldots \times 5$ 10 times
$= 5^{10}$

EXAMPLE-2: How many ways are there to distribute hands of 5 cards to each of 4 players from a standard deck of 52 cards?

⇒ As once cards distributed cannot be taken back;

No. of ways $= {}^{52}C_5 \times {}^{47}C_5 \times {}^{42}C_5 \times {}^{37}C_5$

$= \frac{52!}{\cancel{47!}\,5!} \times \frac{\cancel{47!}}{\cancel{42!}\,5!} \times \frac{\cancel{42!}}{\cancel{37!}\,5!} \times \frac{\cancel{37!}}{32!\,5!}$

$= \frac{52!}{5!\,5!\,5!\,5!\,32!}$

⇒ We can also solve it as:
there will 5 groups: 5, 5, 5, 5 and 32
not distinct ones

$\therefore$ No. of ways $= \dfrac{52!}{5!\,5!\,5!\,32!}$

⇒ Indistinguishable and distinguishable balls.

⇒ Counting the number of ways of placing indistinguishable balls into distinguishable boxes with exclusion is the same as counting r-combinations without repetition of elements.

→ But if the distribution is without exclusion then the problem is the same as counting the number of r-combinations where elements can be repeated.

Question:- In how many ways 5 identical balls can be placed in 3 different boxes?

⇒ No. of balls: 5
No. of different boxes: 3

$\therefore x_1 + x_2 + x_3 = 5$

⇒ Using the previous concept equation, we get

$C(5+3-1, 3-1) = C(7,2)$

$= \dfrac{7 \times 6}{2} = 21$

GATE-2003:- m identical balls are to be placed in n distinct bags. You are given that $m \geq kn$ where, k is a natural number ≥ 1. In how many ways can the balls be placed in the bags if each bag must contain at least 'k' balls?

(a.) $\binom{m-k}{n-1}$ (b.) $\binom{m-kn+n-1}{n-1}$

(c.) $\binom{m-1}{n-k}$ (d.) $\binom{m-kn+n+k-2}{n-k}$

$b_1 + b_2 + b_3 + \dots + b_n = m \geq kn$

and $b_i \geq k$ for $\forall i \in [1, n]$

$r = m$
$n = n$

$k + k + \dots + k$ n times + remaining = m

$\therefore$ remaining = $m - kn$
still
n-bags are there

$\therefore$ It would be distributed as $\binom{m-kn+n-1}{n-1}$

⇒ **Distinguishable and Indistinguishable boxes:-**

→ There is no simple closed formula for counting the number of ways of distributing distinguishable balls into indistinguishable boxes, but there is a complex one involving Stirling number of the second kind.

→ The Stirling number is denoted by $S(m, j)$ where 'm' is the number of balls and 'j' is the number of non-empty boxes.

$$S(m, j) = \frac{1}{j!} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} (j-i)^m$$

⇒ So, the number of ways are:

$$\sum_{j=1}^{n} S(m, j)$$

m = no. of distinguishable things
j / n = no. of indistinguishable boxes

EXAMPLE: Find number of ways of distributing 5 distinguishable balls into 3 indistinguishable boxes, if no box is empty.

$\Rightarrow$ $m = 5$
$j = 3$

$$\therefore S(m,j) = \frac{1}{j!} \sum_{i=0}^{j-1} (-1)^i (j-i)^m \binom{j}{i}$$

$$\therefore S(5,3) = \frac{1}{3!} \sum_{i=0}^{2} (-1)^i (3-i)^5 \binom{3}{i}$$

$$= \frac{1}{6}\left[(3-0)^5\binom{3}{0} - (3-1)^5\binom{3}{1} + (3-2)^5\binom{3}{2}\right]$$

$$= \frac{1}{6}\left[243 - 96 + 3\right]$$

$$= \frac{1}{6}\left[243 - 93\right]$$

$$= \frac{1}{\cancel{6}}\left[\overset{25}{\cancel{150}}\right] = \underline{\underline{25}}$$

$\Rightarrow$ Indistinguishable balls and Indistinguishable boxes:-

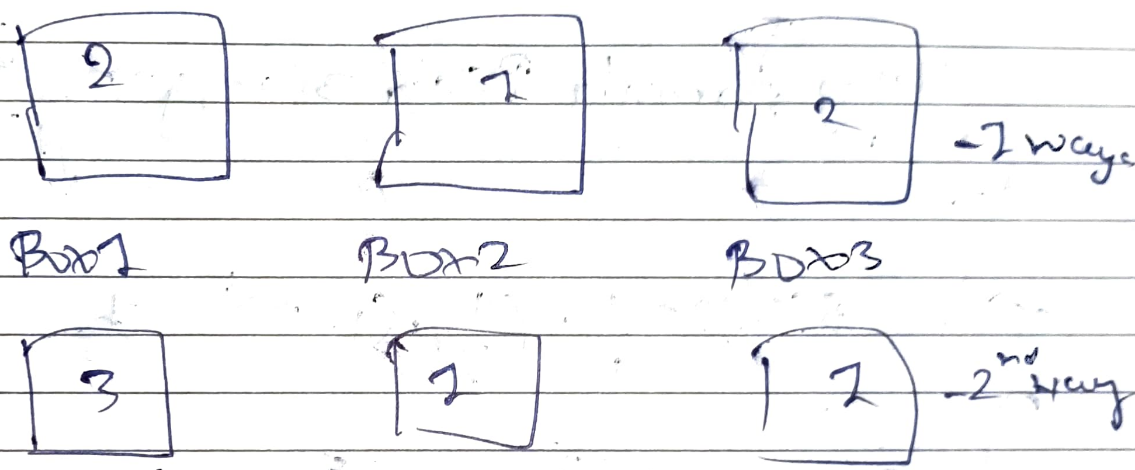
$\rightarrow$ Counting the number of ways of 'm' distinguishable balls into 'n' indistinguishable objects is analogous to finding the $\sum_{k=1}^{n} P(m,k)$,

where $P(m,k)$ is number of partitions of a positive integer.

→ If no box is empty then number of ways $P(n, r)$

EXAMPLE:- No. of ways for 5 balls to be placed in 3 boxes, if no box is empty is $P(5,3)$.

⇒



Total no. of ways = $1+1=2$

→ Sometimes we get some greater 'n', it would be difficult for us to find out the partitions and count the no. of partitions. So, we use the following formula:

$$P(n,r) = \sum_{k=1}^{r} P(n-r, k)$$

Example:- No. of ways for 9 balls to be placed in 3 boxes, if no box is empty.

→ n = no. of balls = 9
r = no. of boxes = 3

$\therefore P(9,3) = \sum_{k=1}^{3} P(9-3, k)$

$= P(6,1) + P(6,2) + P(6,3)$

$= \cancel{6+30+120}$

Using formula $S(m,j) = \frac{1}{j!} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} (j-i)^m$

$\cancel{S(6,1) = \frac{1}{1} \cdot \binom{1}{0} \cdot (1-0)^6 = 1}$

$\cancel{S(6,2) = \frac{1}{2}\left[\binom{2}{0}(2-0)^6 - \binom{2}{1}(2-1)^6\right]}$

$\cancel{= \frac{1}{2}(32-2) = 15}$

P(6,1) : 1 way to put balls into boxes

P(6,2) : 3+3
4+2
5+1 } 3 way

P(6,3) : 3+2+1
4+1+1
2+2+2] 3 ways

$\therefore$ Total ways $= 1+3+3 = \underline{7}$

EXAMPLE:- How many ways are there to put 4 indistinguishable balls into 3 indistinguishable boxes?

⇒ $n = 4$ balls
$r = 3$ boxes

$P(4,1) + P(4,2) + P(4,3)$

$P(4,1) \to 1$

$P(4,2) \to 2+2,\ 3+1 \to 2$

$P(4,3) \to 1+1+2 \to 1$ way

Total ways $= 1+2+1 = \underline{4 \text{ ways}}$

GATE-2000:- The minimum number of cards to be dealt from an arbitrarily shuffled deck of 52 cards to guarantee that three cards are from some same suit is:-

[A.] 3 [B.] 8 [C.] 9 [D.] 12

⇒ C − 2
D − 2
H − 2
S − 2

$+1 = 3$ card of same suit
$= 8+1 = \underline{9}$

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GATE-1998:- In a room containing 28 people, 18 people speak English, 15 people who speak Hindi, and 22 people who speak Kannada. 9 persons speak both English and Hindi, 11 persons speak both Hindi and Kannada whereas 13 persons speak both Kannada and English. How many speak all 3 languages?

1. 9
2. 8
3. 7
4. 6

⇒ $E = 18$ $\quad E \cup H \cup K = 28$

$H = 15$

$K = 22$

$E \cap H = 9$

$H \cap K = 11$

$K \cap E = 13$

$\therefore E \cup H \cup K = E + H + K - E \cap H - H \cap K - K \cap E + K \cap E \cap H$

$\therefore 28 = 18 + 15 + 22 - 9 - 11 - 13 + K \cap E \cap H$

$\therefore 28 = 9 + 4 + 9 + K \cap E \cap H$

$\therefore K \cap E \cap H = 28 - 22$

$= 6$

GATE-2001 :- How many 4-digit even numbers have all 4-digits distinct?

⇒ [A.] 2240 [B.] 2296 [C.] 2520 [d.] 4536

⇒ 1st case: 0 in last

~~0, 2, 4, 6, 8~~

$9 \times 9 \times 8 \times 7$

$= 63 \times 40$

$= 2520$

$\boxed{}\boxed{}\boxed{}\boxed{0}$

7 8 9

$= 56 \times 9$

$= 504$

2nd case: 0 is not last

8 8 7 4 $= 28 \times 64$

$= 1792$

∴ Total ways $= 1792 + 504$

$= 2296$

Gate-2001 :- In how many ways can we distribute 5 distinct balls: B1, B2, B3 ... B5 in 5 distinct cells C1, C2 ..., C5 such that Ball Bi is not in cell Ci, ∀ i = 1, 2 ..., 5 and each cell contains exactly one ball?

[A.] 44 [B.] 96 [C.] 120 [D.] 3125

⇒ 1 2 3 4 5

1 2 3 4 5

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⇒ We want every one of the 5 balls to be in wrong box. This is nothing but the number of dearrangements of a set of 5 elements = D_5.

⇒ i.e., we need to compute D_5;

$$D_n = \sum_{r=2}^{n} (-1)^r \frac{n!}{r!}$$

$$D_5 = \sum_{r=2}^{5} (-1)^r \cdot \frac{n!}{r!}$$

$$= \frac{5!}{2!} - \frac{5!}{3!} + \frac{5!}{4!} - \frac{5!}{5!}$$

$$= \frac{\cancel{120}^{60}}{\cancel{2}} - \frac{\cancel{120}^{20}}{\cancel{6}} + \frac{\cancel{120}^{5}}{\cancel{24}} - 1$$

$$= 60 - 20 + 5 - 1$$

$$= 44$$

GATE-2005:- What is the minimum no. of ordered pairs of non-negative numbers that should be chosen to ensure that there are two pairs (a, b) and (c, d) in the chosen set such that "$a \equiv c \bmod 3$" and "$b \equiv d \bmod 5$".

(A) 4 (C) 16
(B) 6 (D) 24

⇒ Given: $a \equiv c \bmod 3 = \{0,1,2\}$
$b \equiv d \bmod 5 = \{0,1,2,3,4\}$

Pairs (a,b) and (c,d)

⇒ such ordered pairs possible for (a,b)
$a = 0,1,2$ and $b = 0,1,2,3,4$

$\underbrace{(0,0),(0,1)\ldots\ldots}_{3 \times 5 = 15 \text{ pairs}}, (2,4)$

⇒ For (c,d): It can be any; like $(24,7)$
$+1$

∴ Minimum no. of ordered pair required are: $15+1 = \underline{16}$.

GATE-2005: Let $n = p^2q$, where p and q are distinct prime numbers. How many numbers m satisfy $1 \le m \le n$ and $\gcd(m,n) = 1$? Note that $\gcd(m,n)$ is the greatest common divisor of m and n.

⇒ $n = p^2q$ p and q distinct prime numbers

⇒ To find no. of numbers for m such that $1 \le m \le n$ and $\gcd(m,n) = 1$

→ Factor of $n = p^2 q$ are p, q and pq

→ Number that are divisible by $p = pq$

METHOD-I :- The number of numbers from 1 to n, which are relatively prime to n, i.e., $\gcd(m, n) = 1$, is given by Euler totient function $\phi(n)$.

→ If 'n' is broken down into its prime factors as:-

$n = p_1^{n_1} \cdot p_2^{n_2} \dots$

where p_1, p_2 etc are distinct prime numbers,

then,

$$\phi(n) = \phi(p_1^{n_1}) \cdot \phi(p_2^{n_2})$$

→ Here, $n = p^2 q$,

so, $\phi(n) = \phi(p^2) \times \phi(q)$

→ Now, using this property,

$\phi(p^k) = p^k - p^{k-1}$

$\therefore \phi(p^2) = p^2 - p$

→ Similarly $\phi(q) = q - 1$

$$\begin{aligned}\therefore \phi(n) &= \phi(p^2) \cdot \phi(q) \\ &= (p^2 - p) \cdot (q - 1) \\ &= p(p-1) \cdot (q-1)\end{aligned}$$

→ The euler's totient function (ϕ) gives the count of positive integers upto a given number 'n', that are coprime with n.

→ It can also be calculated using formula :-

$$\phi(n) = n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$

METHOD-2 :- We are given $n = p^2 q$, where p and q are distinct prime numbers, we want to find the number of integers 'm' such that $1 \le m \le n$ and $\gcd(m, n) = 1$.

→ To find the number of integers 'm' ~~that~~ such that $\gcd(m, n) = 1$, we first need to determine the numbers that are divisible by either 'p' or 'q', and then subtract them from the total.

1. Numbers divisible by p in the range $1 \le m \le n$: $p, 2p, 3p, \ldots \left\lfloor \frac{n}{p} \right\rfloor p$. There are $\left\lfloor \frac{n}{p} \right\rfloor$ such numbers.

2. Numbers divisible by q in the range $1 \le m \le n$: $q, 2q, 3q, \ldots \left\lfloor \frac{n}{q} \right\rfloor . q$. There are $\left\lfloor \frac{n}{q} \right\rfloor$ such numbers.

3. Numbers divisible by both p and q in the range $1 \le m \le n$: $pq, 2pq, 3pq, \ldots, \left\lfloor \frac{n}{pq} \right\rfloor \cdot pq$. There are $\left\lfloor \frac{n}{pq} \right\rfloor$ such numbers.

→ Therefore, the total number of integers m such that $1 \le m \le n$ and $\gcd(m, n) = 1$ is:

$$n - \left(\left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{q} \right\rfloor - \left\lfloor \frac{n}{pq} \right\rfloor \right)$$

$$= p^2q - \left(\left\lfloor \frac{p^{\cancel{2}}q}{\cancel{p}} \right\rfloor + \left\lfloor \frac{p^2\cancel{q}}{\cancel{q}} \right\rfloor - \left\lfloor \frac{p^{\cancel{2}}\cancel{q}}{\cancel{pq}} \right\rfloor \right)$$

$$= p^2q - (pq + p^2 - p)$$

$$= p^2q + p - pq - p^2$$

$$= p \cdot (pq + 1 - q - p)$$

$$= p(pq - q - p + 1)$$

$$= p \cdot (q(p-1) - 1(p-1))$$

$$= p(q-1) \cdot (p-1)$$

(A.) $p \cdot (q-1)$
(B.) pq
(C.) $(p^2-1) \cdot (q-1)$
(D.) $p(p-1)(q-1)$

GATE-2008 - The exponent of 11 in the prime factorization of 300! is

(A) 27
(B) 28
(C) 29
(D) 30

$\Rightarrow$ $300! = \left\lfloor \frac{300}{11} \right\rfloor = \lfloor 27.27 \rfloor = 27$

$\Rightarrow$ These 27 contains numbers which can further be divisible by 11 $= \left\lfloor \frac{27}{11} \right\rfloor$

$= 2$

$\therefore$ Total exponent on 11 is $27 + 2 = 29$