1.29. Expectation of Exponential Distribution:

→ The PDF is as follows:

$$f_x(x) = \begin{cases} \lambda \cdot e^{-\lambda x} ; & x \in [0, \infty) \\ 0 ; & \text{otherwise} \end{cases}$$

→ CDF = $P(x \leq t) = F(t) = 1 - e^{-\lambda t}$

⇒ Expected value:

→ To find out the expected value, we simply multiply the probability distribution function with x and integrate over all possible values (support).

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$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

$$= \int_{0}^{\infty} x \cdot \lambda \cdot e^{-\lambda x} dx \quad [-\infty \text{ to } 0 \text{ is } 0]$$

$$= \lambda \int_{0}^{\infty} x \cdot e^{-\lambda x} dx \quad \left[\int uv \cdot dx = u \int v \cdot dx - \int \left(\frac{du}{dx}\right) \cdot \int v \cdot dx \cdot dx\right]$$

$$= \lambda \left[\left[x \int_{0}^{\infty} e^{-\lambda x} \cdot dx\right] - \left[\int \frac{d \cdot x}{dx} \left(\int e^{-\lambda x} \cdot dx\right) \cdot dx\right]_{0}^{\infty} \right]$$

$$= \lambda \left[\left[x \cdot \frac{e^{-\lambda x}}{-\lambda}\right]_{0}^{\infty} - \left[\int \frac{e^{-\lambda x}}{-\lambda} dx\right]_{0}^{\infty} \right]$$

$$= \left[\frac{-x}{e^{\lambda x}}\right]_{0}^{\infty} + \left[\int e^{-\lambda x} \cdot dx\right]_{0}^{\infty}$$

$$= \left[\frac{-x}{e^{\lambda x}}\right]_{0}^{\infty} + \left[\frac{e^{-\lambda x}}{-\lambda}\right]_{0}^{\infty}$$

$$= [0 - 0] - \frac{1}{\lambda}[0 - 1]$$

$$= \frac{1}{\lambda}$$

$$\therefore E[X] = \frac{1}{\lambda}$$

⇒ Variance and Standard deviation:-

⇒ The variance ~~and standard~~ of the exponential distribution is given by-

$$Var[X] = E[X^2] - E[X]^2$$

$$= \int_{-\infty}^{\infty} x^2 . f_X(x) . dx - \frac{1}{\lambda^2}$$

$$= \lambda \int_{0}^{\infty} x^2 . e^{-\lambda x} . dx - \frac{1}{\lambda^2}$$

$$= \lambda \left[\left[x^2 . \int e^{-\lambda x} dx \right]_0^\infty - \left[\int \frac{d}{dx} x^2 \left(\int e^{-\lambda x} . dx \right) . dx \right]_0^\infty \right] - \frac{1}{\lambda^2}$$

$$= \lambda . \left[\left[\frac{x^2 . e^{-\lambda x}}{-\lambda} \right]_0^\infty - \left[\int 2x . \frac{e^{-\lambda x}}{-\lambda} . dx \right]_0^\infty \right] - \frac{1}{\lambda^2}$$

$$= \left[\frac{-x^2}{e^{\lambda x}} \right]_0^\infty + \frac{2}{\lambda} \left[\int x . e^{-\lambda x} . dx \right]_0^\infty - \frac{1}{\lambda^2}$$

$$= [0 - 0] + \frac{2}{\lambda} * \frac{1}{\lambda} - \frac{1}{\lambda^2}$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

$$\boxed{\therefore Var[X] = \frac{1}{\lambda^2}}$$

→ The standard deviation of the distribution:-

$$\sigma = \sqrt{Var[x]} = \sqrt{\frac{1}{\lambda^2}} = \frac{1}{\lambda}$$

$$\therefore \sigma = \frac{1}{\lambda}$$

EXAMPLE:- Let x denote the time between detections of a particle with a Geiger counter and assume that x has ~~an~~ an exponential distribution with $E[x] = 1.4$ minutes. What is the probability that we detect a particle within 30 seconds of starting the counter?

⇒ $E[x] = \frac{1}{\lambda} = 1.4$ minutes

$$\therefore \lambda = \frac{1}{1.4}$$

→ To find a particle detection probability within 30 seconds,

$\therefore t = 30$ sec $= 0.5$ minute

$$\therefore P(x \le 0.5) = 1 - e^{-\lambda t}$$

$$= 1 - e^{-\left(\frac{1}{1.4}\right)(0.5)}$$

$$= 1 - 0.6997$$

$$P(x \le 0.5) = 0.3003$$

GATE-2021 :- The lifetime of a component of a certain type is a random variable whose probability density function is exponentially distributed with parameter 2. For a randomly picked component of this type, the probability that its lifetime exceeds the expected lifetime (rounded to 2 decimal places) is ______.

→ Expected lifetime = $E[x]$.

Parameter is only one ; and its $\lambda = 2$

$E[x] = \frac{1}{\lambda} = \frac{1}{2} = 0.5$

$P(x \geq 0.5) = 1 - P(x \leq 0.5)$

$= 1 - [1 - e^{-0.5 \times \lambda}]$

$= e^{-0.5 \times 2} = e^{-1}$

$= 0.3678$

$\simeq 0.37$

(A.) 0.37 (B.) 0.25 (C.) 0.50 (D.) 0.18