Friday

CH-1 Probability

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1.1 Sample space and events:-

Different types of events:-

- → Mutually exclusive events
- → Exhaustive events
- → Equally-likely events
- → Independent events
- → Dependent events

⇒ Random experiment:-

→ An experiment is random if although it is repeated in the same manner everytime, can result in different outcomes.

→ The set of all possible outcomes is completely determined before carrying it out.

→ Before we carry it out, we cannot predict its outcome.

e.g.:- ~~~~ rolling a dice, tossing a coin, etc.

→ The sample space is the set of all the possible ~~~~ outcomes of a random experiment.

e.g.:- $S = \{H, T\}$, $S = \{1, 2, 3, 4, 5, 6\}$

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⇒ An event is a subset of the sample space (any set of outcomes of the random experiment).

e.g.:- $E_1 = \{H\}$; $E_2 = \{1, 3, 5\}$; $E_3 = \{2, 4, 6\}$

⇒ <u>Examples of random experiment:-</u>

- Tossing a coin: $S = \{H, T\}$
- Rolling a dice: $S = \{1, 2, 3, 4, 5, 6\}$
- Picking an object: $S = \{object\}$
- Tossing two coins simultaneously: 2^n
  $S = \{HH, HT, TH, TT\}$
- Rolling two dice simultaneously: 6^n

$$S = \{11, 12, 13, 14, 15, 16, \\ 21, 22, 23, 24, 25, 26, \\ 31, 32, 33, 34, 35, 36, \\ 41, 42, 43, 44, 45, 46, \\ 51, 52, 53, 54, 55, 56, \\ 61, 62, 63, 64, 65, 66\}$$

e.g.:- Rolling a dice $S = \{1, 2, 3, 4, 5, 6\}$

Let A be the event that the outcome is divisible by 3 $A = \{3, 6\}$; $n(A) = 2$

Let B be the event that the outcome is an even number $B = \{2, 4, 6\}$; $n(B) = 3$

Let C be the event that the outcome is an odd number. $C = \{1, 3, 5\}$; $n(C) = 3$

$\frac{n(A)}{n(S)} = \frac{2}{6} = \frac{1}{3}$; $\frac{n(B)}{n(S)} = \frac{3}{6} = \frac{1}{2}$

$\frac{n(C)}{n(S)} = \frac{3}{6} = \frac{1}{2}$

⇒ For any event E & sample space S;
$n(E) \leq n(S)$

→ $0 \leq \frac{n(A)}{n(S)} \leq 1$ ← this is probability.

Probability :-

→ Probability refers to the extent of the occurence of events. When an event occurs like throwing a ball, picking a card from deck, etc., then their must be some probability associated with that event.

⇒ Probability of an Event :- If there are total p possible outcomes associated with random experiment and q of them are favourable outcomes to the event A, then the probability of event A is denoted by P(A) & is given by: $P(A) = \frac{q}{p}$ $\left[\frac{n(A)}{n(S)}\right]$

→ The probability of non-occurence of event A, i.e.; $P(A') = 1 - P(A)$

NOTE: → If the value of $P(A) = 1$, then event A is called Sure event

→ If the value of $P(A) = 0$, then event A is called Impossible event.

→ Also, $P(A) + P(A') = 1$

⇒ Theorems:-

⇒ Let A, B, C are the events associated with a random experiment, then,

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

- $P(A \cup B) = P(A) + P(B)$, if A & B are mutually exclusive events; & if so, then $P(A \cap B) = 0$

- $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$

- $P(A \cap B') = P(A) - P(A \cap B)$

- $P(A' \cap B) = P(B) - P(A \cap B)$

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Example:- The probability that a person getting an electric contract is 2/5, and probability that it will get plumbing contract is 4/7. If the probability of getting atleast one contract is 2/3, what is the probability of getting both?

→ A: event of getting electric contract
B: event of getting plumbing contract

$P(A) = 2/5$, $P(B) = 4/7$

→ Probability of getting atleast one contract i.e.; it can get either plumbing or electric or both; $P(A \cup B) = \frac{2}{3}$

→ To find probability that he gets both contract;

∴ $P(A \cap B)$;

$\because P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$= \frac{2}{5} + \frac{4}{7} - \frac{2}{3}$

$= \frac{14+20}{35} - \frac{2}{3}$

$= \frac{34}{35} - \frac{2}{3} = \frac{102-70}{105}$

$= \frac{32}{105}$

Types of events:-

⇒ Different types of events:-

⇒ Mutually exclusive events
⇒ Exhaustive events
⇒ Equally-likely events
⇒ Independent events
⇒ Dependent events

⇒ Mutually Exclusive Events:-

⇒ Two or more events associated with a random event are said to be mutually exclusive events if any one of the event occurs, it prevents the occurrence of all other events.

⇒ This means that no two or more ~~discrete~~ events can occur simultaneously at the same time.
i.e.; $E_1 \cap E_2 = \phi$, then E_1 & E_2 are mutually exclusive events.

⇒ Exhaustive Events:-

⇒ Two or more events associated with a random event are said to be exhaustive events if their union is the sample space.

e.g.:- $S = \{1,2,3,4,5,6\}$; $A = \{1,2,3\}$; $B = \{4,5,6\}$;
$\therefore A \cup B = S$ $\therefore P(A) + P(B) = 1$

EXAMPLE: A sample space is given:
sample space $= S = \{1,2,3,4,5,6,7,8,9,10\}$
Let an event $X = \{1,2,3\}$
Event $Y = \{4,5,6\}$
Event $Z = \{7,8,9,10\}$

⇒ Event X, Y, Z are mutually exclusive events
because:- $X \cap Y \cap Z = \phi$.

→ For this, take the union of all events:-
$X \cup Y \cup Z = \{1,2,3\} \cup \{4,5,6\} \cup \{7,8,9,10\}$
$= \{1,2,3,4,5,6,7,8,9,10\}$

→ Events X, Y & Z are exhaustive events, because they form a complete sample space itself.

⇒ Equally likely events:- are events that have the same theoretical probability (or likelihood) of occurring.

Example:- Each numeral on a die is equally likely to occur when the die is tossed.
Sample space of throwing a die: $\{1,2,3,4,5,6\}$

→ Theoretical probability of getting a chosen numeral $= \frac{1}{6}$

$A = \{2,4,6\}$, $B = \{1,3,5\}$
$P(A) = P(B) = \frac{1}{2}$

⇒ ~~The~~ Independent events are those events whose occurrence is not dependent on any other event.

→ for example, if we flip a coin in the air and get the outcome as Head, then again, if we flip the coin but this time we get the outcome as Tail.

e.g.:- Toss two coins simultaneously:

$S = \{HH, HT, TH, TT\}$,

E1: Getting head on first coin
E2: Getting 2 heads

E1 & E2 are not independent

⇒ Dependent events are those events whose occurrence is dependent on any other event.

→ For example, if we roll two dice simultaneously,

A: 3 on first dice
B: sum on two dice is 7

} dependent

GATE-14

e.g.:- Let S be the sample space and two mutually exclusive events A & B be such that $A \cup B = S$. If $P(\cdot)$ denotes the probability of the event. The maximum value of $P(A) \cdot P(B)$ is _____

(A) 0.5 (B) 0.25 ✓ (C) 0.225 (D) 0.125

$\because P(A \cup B) = P(A) + P(B) - P(A \cap B)$
$\therefore 1 = P(A) + P(B)$

Arithmetic mean $\geq$ Geometric mean
$\therefore \frac{P(A)+P(B)}{2} \geq \sqrt{P(A).P(B)}$
$\therefore \left(\frac{1}{2}\right)^2 \geq \left(\sqrt{P(A).P(B)}\right)^2$

$\therefore P(A).P(B) \leq \left(\frac{1}{4}\right) \leq 0.25$

Mean :-

→ Mean is average of a given set of data.

e.g. :- Given : 2, 4, 4, 4, 5, 5, 7, 9
These eight data points have the mean (average) 5.

$Mean = \frac{2+4+4+4+5+5+7+9}{8} = \frac{40}{8} = 5$

⇒ Formula :- $\boxed{\mu = \frac{\sum_{i=1}^{N} x_i}{N}}$

Where, μ is mean & $x_1, x_2, x_3, \ldots, x_i$ are elements.

→ Also note that mean is sometimes denoted by $\bar{x}$.

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Variance:

→ Variance is the sum of squares of differences between all numbers and means.

→ Consider the data 2, 4, 4, 4, 5, 5, 7, 9 with mean 5.
First calculate the deviations of each data point from the mean, and square the result of each:

:- $(2-5)^2 = 9$; $(5-5)^2 = 0$; $(9-5)^2 = 16$
$(4-5)^2 = 1$; $(7-5)^2 = 4$;

$$\text{Variance} = \frac{9+1+1+1+0+0+4+16}{8}$$

$$= \frac{32}{8} = 4$$

⇒ Formula: $\boxed{\sigma^2 = \frac{\sum_{i=1}^{N} (x_i - \mu)^2}{N}}$

where μ is Mean, N is the total number of elements or frequency of distribution.

⇒ For data type with x_i & f_i,
then

$$\mu = \frac{\sum f_i x_i}{\sum f_i}$$

Standard Deviation:-

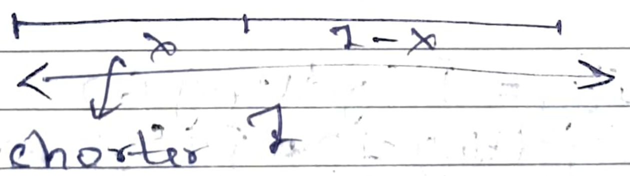
⇒ Standard Deviation is square root of variance. It is a measure of the extent to which data varies from the mean.

→ Consider data 2, 4, 4, 4, 5, 5, 7, 9 with $\mu = 5$ & $\sigma^2 = 4$,

Standard deviation $= \sqrt{\sigma^2} = \sigma = \sqrt{4} = \underline{\underline{2}}$

GATE-14 Suppose you break a stick of unit length at a point chosen uniformly at random. Then the expected length of the shorter stick is ______.

(A.) 0.24 to 0.27 (B.) 0.25 to 0.30
(C.) 0.20 to 0.30 (D.) 0.10 to 0.15

⇒


min ↓ max ↓
$0 \le x \le 0.5$

Average value $= \dfrac{0 + 0.5}{2} = \underline{\underline{0.25}}$

closed is $\underline{0.24 \text{ to } 0.27}$.

⇒ Adding 'x' to each entry in a list add 'x' to the mean of the list.

⇒ Multiplying 'x' to each entry in a list multiplies 'x' to the mean of the list.

⇒ Multiplying 'x' to each entry in a list changes standard deviation by x.

GATE-12 Suppose a fair six-sided die is rolled once. If the value on the die is 1, 2 or 3, the die is rolled up second time. What is the probability that sum of total values turn up is atleast 6?

(A) 10/21 (B) 5/12 (C) 2/3
(D) 1/6

⇒ Probability = P(getting 6)
+ P(getting 1) * P(5 & 6)
+ P(2) · P(4, 5 & 6)
+ P(3) · P(3, 4, 5 & 6)

$$= \frac{1}{6} + \frac{1}{6}\left(\frac{2}{6}\right) + \frac{1}{6}\left(\frac{3}{6}\right) + \frac{1}{6}\left(\frac{4}{6}\right)$$

$$= \frac{1}{6} + \frac{1}{6}\left(\frac{1}{3} + \frac{1}{2} + \frac{2}{3}\right)$$

$$= \frac{1}{6} + \frac{1}{6}\left(\frac{3}{2}\right)$$

$$= \frac{1}{6} + \frac{1}{4} = \frac{3+2}{12} = \frac{5}{12}$$

GATE-10 What is the probability that divisor of 10^{99} is a multiple of 10^{96}?
(A) 1/625 (B) 4/625 (C) 12/625 (D) 16/625

$\Rightarrow n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}$

$18 = 2 \times 3^2$

$10^{96} = (2\times5)^{96} = 2^{96} \times 5^{96}$

$10^{99} = (2\times5)^{99} = 2^{99} \times 5^{99}$

Total no. of divisors $= (k_1+1)(k_2+1)\cdots(k_r+1)$

for 10^{96} no. of divisors $= (96+1)(96+1)$
$= 97 \times 97$

for 10^{99} no of divisor $= (99+1)(99+1)$
$= 100 \times 100$
$= 10000$

$\therefore 10^{96} \times t = 10^{99} \Rightarrow t = 10^3 = 1000$
$\Downarrow$
$2^3 \times 5^3$

$\therefore$ No. of divisors for $1000 = (3+1)(3+1)$
$= 16$

$\therefore$ Probability $= \dfrac{16}{10000} = \dfrac{1}{625}$

GATE-14: Four fair six-sided dice are rolled. The probability that the sum being 22 is $x/1296$. The value of x is ____.
(A) 7 (B) 8 (C) 9 (D) 10

$\Rightarrow$ $6\times6\times6\times6=1296$

6, 6, 6, 4 & 6, 6, 5, 5
These are the two options.

$\because$ 6, 6, 6, 4 $\Rightarrow {}^4P_1 = \underline{4}$
$\because$ 6, 6, 5, 5 $\Rightarrow {}^4P_2 = \frac{4\times3}{2!} = \underline{6}$

$\therefore$ Total $x = 4+6 = \underline{10}$

GATE-20: Let R be the set of all binary relations on the set {1, 2, 3}. Suppose, a relation is chosen from R at random. The probability that the chosen relation is reflexive (round off to 3 decimal places) is ____.

NOTE: This question was Numerical type.
(A) 0.125 (B) 0.25 (C) 0.5 (D) 0.625

$\Rightarrow$ Number of relation $= 2^{n^2} = 2^{3^2} = \underline{2^9}$

Number of reflexive relation $= 2^{n^2-n} = 2^{3^2-3} = \underline{2^6}$

Probability $= \frac{2^6}{2^9} = \frac{1}{8} = \underline{0.125}$

GATE-21: A bag has 'r' red balls & 'b' black balls. All balls are identical except for their colours. In a trial, a ball is randomly drawn from the bag, its colour is noted and the ball is placed back into the bag along with another ball of the same colour. Note that the number of balls in the bag will increase by one, after the trial. A sequence of four such trials is conducted. Which one of the following choices gives the probability of drawing a red ball in the fourth trial?

(A) $\frac{r}{r+b}$ (b) $\frac{r}{r+b+3}$ (c) $\frac{r+3}{r+b+3}$

(D) $\left(\frac{r}{r+b}\right)\left(\frac{r+1}{r+b+1}\right)\left(\frac{r+2}{r+b+2}\right)\left(\frac{r+3}{r+b+3}\right)$

⇒ Consider getting red-balls in all possible trial:

1st — $(r)/(r+b)$

2nd — $(r+1)/(r+b+1)$ * (

3rd

$$\left(\frac{r}{r+b}\right) * \left(\frac{r+1}{r+b+1}\right) + \left(\frac{b}{r+b}\right)\left(\frac{r}{r+b+1}\right)$$

$$= \frac{1}{(r+b)(r+b+1)}(r^2+r+rb)$$

$$= \frac{r(r+1+b)}{(r+b)(r+b+1)}$$

$$= \frac{r}{r+b}$$

⇒ For any number of trials; we will get $\frac{r}{r+b}$

⇒ Coefficient of variation = $\frac{\text{standard deviation}}{\text{mean}} \times 100$

$= \frac{\sigma}{\mu} \times 100$

⇒ Value of standard deviation is zero if value of all entries in input are same.

⇒ If we multiply all values in the input set by '-7', the mean is multiplied by '-7', but the standard deviation is multiplied by 7.

⇒ Standard deviation and variance is a measure that tells how spread out the numbers are. While variance gives you a rough idea of spread, the standard deviation is more concrete, giving you exact distances from the mean.

⇒ Mean, median & mode are the measure of central tendency of data. (either grouped or ungrouped).

GATE-13: Suppose 'p' is the number of cars per minute passing through a certain road junction between 5PM and 6PM, and 'p' has a Poisson distribution with mean 3. What is the probability of observing fewer than 3 cars ~~getting~~ during any given minute in this interval?

(A) $8/(2e^3)$ (B) $9/(2e^3)$ (C) $17/(2e^3)$ (D) $26/(2e^3)$

⇒ A discrete random variable x is said to have a Poisson distribution, with parameter $\lambda > 0$, if it has a probability mass function given by:

$$f(K;\lambda) = P_r(x=K) = \frac{\lambda^K e^{-\lambda}}{K!}$$

where K = number of occurrences ($K = 0, 1, 2, \ldots$)
e = euler's number

→ The positive real number λ is equal to the expected value of x and also to its variance.

$$\lambda = \underline{E(x) = Var(x)}$$

→ Here, $\lambda = 3$, because we have to find probability for less than 3 cars.

$$\therefore P(x<3) = P_r(x=0) + P_r(x=1) + P_r(x=2)$$
$$= f(0,3) + f(1,3) + f(2,3)$$
$$= \frac{3^0 e^{-3}}{0!} + \frac{3^1 e^{-3}}{1!} + \frac{3^2 e^{-3}}{2!}$$

$$= \frac{1}{e^3} + \frac{3}{e^3} + \frac{9}{2e^3}$$

$$= \frac{4}{e^3} + \frac{9}{2e^3}$$

$$= \frac{17}{2e^3}$$

$\Rightarrow$ Mean $= np$

$\Rightarrow$ Varqance $= npq$