1.2.3 Expectation of Bernoulli Distribution

⇒ What is expectation?

→ Roughly, the expectation is the average value of the random variable where each value is weighted according to its probability.

⇒ Expectation of Discrete random variables;

$$E(x) = \sum_i x_i P(x_i)$$

- $E(x)$ is the expectation value of the discrete random variable x
- x is the value of the discrete random variable x
- $P(x)$ is the probability mass function of x.

Example: Let R be the value that comes up with you roll a fair 6-sided die.

⇒ Then the expected value of R is;

x	1	2	3	4	5	6
$P(x)$	1/6	1/6	1/6	1/6	1/6	1/6

→ $E(R) = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$

$= \frac{1}{6}(42) = \frac{7}{2}$

e.g. for tossing a coin $S = \{H, T\}$.
x: getting head
$x = \{0, 1\}$

$\therefore E(X=1) = 0 \times (P(T)) + 1 \times P(H)$
$= P(H) = \frac{1}{2}$

⇒ **Expectation of Bernoulli Distribution:**

→ $P(x=1) = p$ $\quad\therefore P(x) = p^x (1-p)^{1-x}$
$P(x=0) = 1-p$

$\therefore E[x] = P(x=1) \cdot 1 + P(x=0) \cdot 0$
$= p \cdot 1 + (1-p) \cdot 0$
$= p$
$\therefore E[x] = p$

→ Thus, the mean or expected value of a Bernoulli distribution is given by
$E[x] = p$.

⇒ **Variance of Discrete Random variables:**

→ The variance can be defined as the difference of the mean of x^2 and the square of the mean of x.

→ Mathematically, this statement can be written as follows:
$$Var[x] = E[x^2] - (E[x])^2$$

$\therefore E[x] = \sum x_i \cdot P(x_i)$

$\therefore E[x^2] = \sum x_i^2 \cdot P(x_i)$

$E[\phi(x)] = \sum \phi(x_i) \cdot P(x_i)$

e.g. :- for variance of rolling die & getting 6.

[Find the variance & standard deviation when a fair die is rolled]

x	1	2	3	4	5	6
$P(x)$	1/6	1/6	1/6	1/6	1/6	1/6
x^2	1	4	9	16	25	36

$E[x] = \frac{7}{2}$ [from previous question]

$E[x^2] = \frac{1}{6}(1+4+9+16+25+36)$

$= \frac{1}{6}\left(\frac{n(n+1)(2n+1)}{6}\right)$

$= \frac{1}{36}(6)(7)(13)$

$E[x^2] = \frac{91}{6}$

$\therefore$ variance;

$V[x] = \frac{91}{6} - \left(\frac{7}{2}\right)^2$

$V[x] = \frac{91 - 49}{36} = \cancel{\frac{70}{6}}$

$= \frac{35}{12}$

→ Using the properties of $E[x^2]$, we get,

$E[x^2] = \Sigma x^2 . P(X=x)$

$\therefore E[x^2] = 1^2 . p + 0^2 . (1-p) = p$

→ Substituting this value in $Var[x] = E[x^2] - (E[x])^2$

$\therefore Var[x] = p - p^2$

$\therefore Var[x] = p(1-p)$

→ Hence, the variance of a Bernoulli distribution is $Var[x] = p(1-p)$

GATE-2011: If the difference between expectation of the square of a random variable $(E[x^2])$ and the square of the expectation of the random variable $(E[x])^2$ is denoted by R, then?

(A) $R = 0$ (B) $R < 0$ (C) $R \geq 0$ (D) $R > 0$

→ $R = E[x^2] - (E[x])^2 = p(1-p)$

→ The difference between $E[x^2]$ and $(E[x])^2$ is called variance of a random variable.

→ Variance measures how far a set of numbers is spread out. (A variance of zero indicates that all the values are identical.)

→ A non-zero variance is always positive.

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⇒ Properties of variance :-

⇒ $Var(K) = 0$; when K is a constant.

→ $Var[aX+b] = a^2 Var(X)$.