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1.3.5 Bayes's Theorem

→ The formula provides the relationship ~~betw~~ between $P(A|B)$ and $P(B|A)$.

⇒ Bayes's formula :- $P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$

$$\therefore P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

$$= \frac{\frac{P(A \cap B)}{P(A)} \cdot P(A)}{P(B)} = \frac{P(A \cap B)}{P(B)}$$

$$\therefore P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

e.g. :- To find $P(\text{King}|\text{face})$.

$$P(\text{King}|\text{face}) = \frac{P(\text{face}|\text{King}) \cdot P(\text{King})}{P(\text{face})}$$

$$= \frac{1 \cdot (4/52)}{(12/52)} = \frac{1}{3}$$

EXAMPLE :- A company buys 70% of its computers from company X and 30% from company Y. Company X produces 1 faulty computer per 20 computers. A computer is found faulty, what is the probability that it was bought from company X?

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NOTE:- Bayes theorem is the method in which the calculated probabilities are revised with values of new probabilities.

$\Rightarrow$ Given: $P(X) = 0.7$; $P(Y) = 0.3$
$P(F|X) = 0.2$; $P(F|Y) = 0.05$

To find $P(X|F) = \dfrac{P(F|X) \cdot P(F)}{P(X)}$

$P(F) = P(F|X) \cdot P(X) + P(F|Y) \cdot P(Y)$
$= (0.2)(0.7) + (0.05)(0.3)$
$= 0.14 + 0.015$
$= 0.155$

$\therefore P(X|F) = \dfrac{P(F|X) \cdot P(X)}{P(F)}$

$= \dfrac{P(F|X) \cdot P(X)}{P(F)}$

$= \dfrac{0.2 \times 0.7}{0.155}$

$P(X|F) = 0.9032$

EXAMPLE:- Toss a coin 5 times. Let H_1 = "first toss is heads" and let H_A = "all 5 tosses are heads". Then $P(H_1|H_A) = 1$, but $P(H_A|H_1) = 1/16$.

$$\Rightarrow P(HA|H_1) = \frac{P(H_1|HA) \cdot P(HA)}{P(H_1)}$$

$$P(HA) = \frac{1}{2^5} = \frac{1}{32}$$

$$P(H_1) = \frac{1}{2}$$

$$= \frac{1 \times 1/32}{1/2}$$

$$\therefore P(HA|H_1) = 1/16$$

GATE-2015 :- Box P has 2 red balls & 3 blue balls and box Q has 3 red balls and 1 blue ball. A ball is selected as follows:
1. Select a box
2. Choose a ball from the selected box such that each ball in the box is equally likely to be chosen.

The probabilities of selecting boxes P and Q are 1/3 & 2/3 respectively. Given that a ball selected in the above process is a red ball, the probability that it came from the box P is

(A) 4/19 (B) 5/19 (C) 2/9 (D) 19/30

⇒ $P(P|R)$ is to be found out.
Given ; $P(P) = 1/3$

$$P(P|R) = \frac{P(R|P) \cdot P(P)}{P(R)}$$

$$\therefore P(R) = P(R|P) \cdot P(P) + P(R|Q) \cdot P(Q)$$
$$= (2/5) \cdot (1/3) + (3/4) \cdot (2/3)$$
$$= \frac{2}{15} + \frac{1}{2}$$

$$= \frac{19}{30}$$

$$P(P|R) = \frac{\left(\frac{2}{5}\right) \cdot \left(\frac{1}{3}\right)}{9/20}$$

$$= \frac{2/15}{19/30}$$
$$= \frac{4}{19}$$