Covariant Density Functional Theory

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- *Motivation*
- *Definitions & Basics of DFT*
- *Covariant form of field equations*
- *QHD - I*
- *Infinite matter EoS*
- *Neutron star EoS & structure*

Nuclear many-body theory

- **One body problem is easier to solve, Eg. Projectile motion, $F = ma$**
- **Two body problem, Eg. satellite orbiting a planet, $F_{12}(x_1,x_2) = m_1a_1$; $F_{12}(x_1,x_2)=m_1a_1$**
- **Three body problem is complicated but solvable**
- **Only approximate solutions can be found for many-body systems**
- **The nuclear force is not known**
- **Average BE of least bound nucleon ~ 8 MeV, Kinetic energy T ~ 40 MeV**
- **Relativistic effect could be important but T << nucleon mass (938 MeV)**

$$v = \sqrt{\frac{2T}{m}} = c\sqrt{\frac{2T}{mc^2}} \approx 0.3\,c$$

- **→ Quantum effects are certainly not negligible**

$$\lambda = \frac{2\pi\hbar}{mv} = \frac{2\pi(\hbar c)}{(mc^2)(v/c)} \approx 4.5 \text{ fm}$$

W. Greiner and J. A. Maruhn. *Nuclear models*. Springer-Verlag, 1997.

The quantum treatment

- **Ideally we should solve the many-nucleon Schödinger equation**

$$H\,|\Psi_n(1,2,.....,A)\rangle = E_n\,|\Psi_n(1,2,.....,A)\rangle$$

$$H = \sum_{i=1}^{A} T_i + \frac{1}{2}\sum_{i,j=1}^{A} V_{ij} + \frac{1}{6}\sum_{i,j,k=1}^{A} V_{ijk} + ...$$

- **→ *microscopic models* ← There is no *a priori* theory for V_{ij}**
- **Various parametrizations are employed which are good for different purposes**
- **Atomic physics: QED is known, only approximations are in methods for solution**
- **A typical microscopic model thus depends on a nucleon-nucleon interaction which necessarily contains parameters fitted to reproduce some experimental data.**
- **Even microscopic approaches are *models*.**
- **Until the nucleon-nucleon interaction can be derived from a more fundamental theory such as quantum chromodynamics, we will have to live with this situation**

W. Greiner and J. A. Maruhn. *Nuclear models*. Springer-Verlag, 1997.

Mean Field Approximation

- **The many-nucleon Schrödinger equation**

$$H\,|\Psi_n(1,2,.....,A)\rangle = E_n\,|\Psi_n(1,2,.....,A)\rangle$$

$$H = \sum_{i=1}^{A} T_i + \frac{1}{2}\sum_{i,j=1}^{A} V_{ij} + \frac{1}{6}\sum_{i,j,k=1}^{A} V_{ijk} + ...$$

0 0

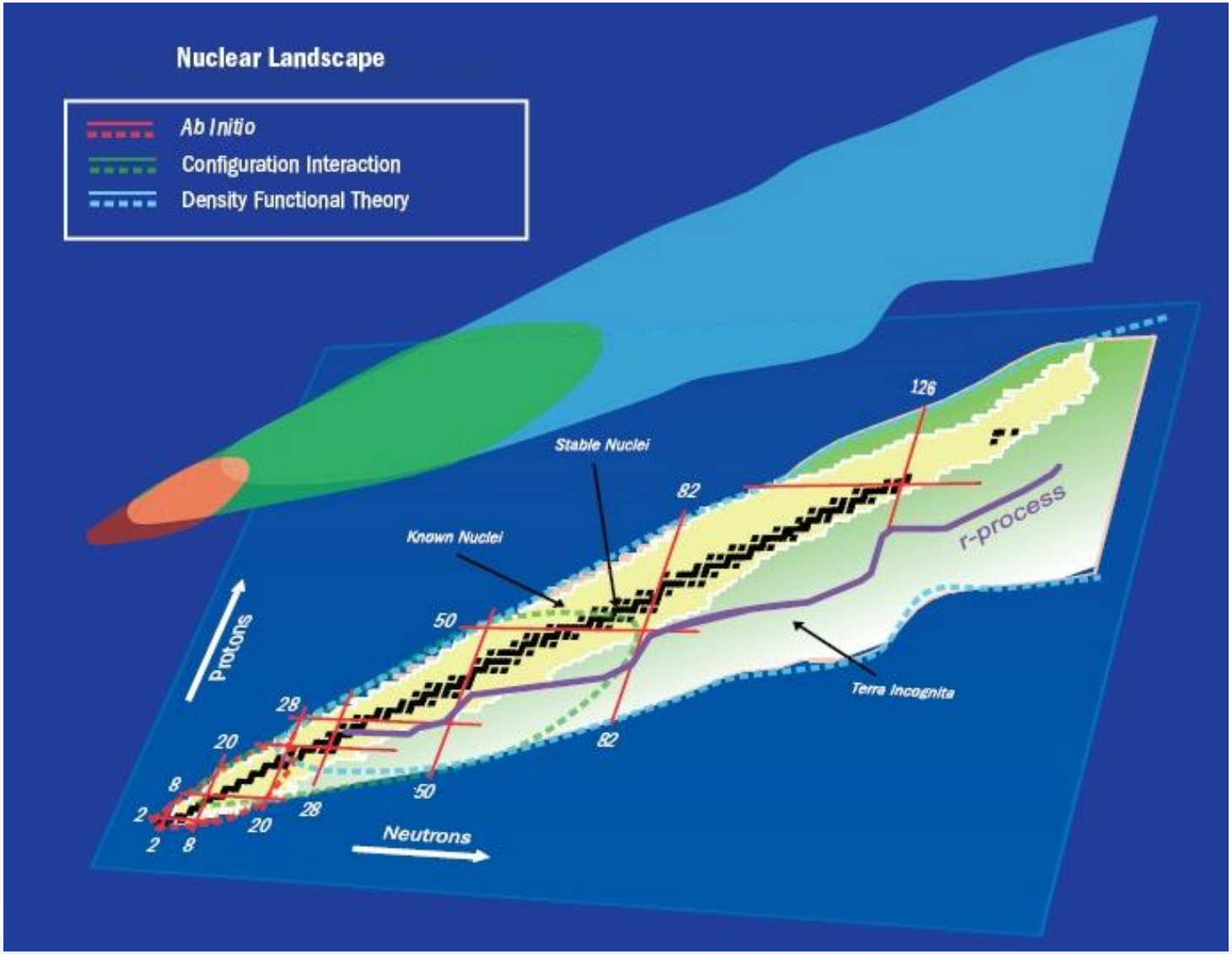
- **Mean Field Approximation in the independent-particle picture**

$$H = \sum_{i=1}^{A}(T_i + U_i) + \left(\frac{1}{2}\sum_{i,j=1}^{A} V_{ij} - U_i\right)$$

0

- **Choice of U_i is crucial**

Present scenario



Motivation: Density dependence

- Compressibility
  - Stiff & Soft matter – Giant resonance
  - BE(rho)
  - Azimuthal anisotropy – RHIC data
- Inhomogeneous distribution
  - Clustering
  - Initial state dependence of RHI collisions
- Asymmetry
  - LDM
  - Neutron skin
  - Neutron stars

Definitions

- Functional: A functional takes a function as input and gives a number as output. An example is: $F[f(x)] = y = \int_{-\infty}^{\infty} f(x)\ dx$
- ab-initio methods can be interpreted as a functional of the wavefunction, with the functional form completely known!

$$E[\Psi] = \frac{\int \Psi^*(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N)\, \hat{H}\, \Psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) d^3\vec{r}_1 d^3\vec{r}_2 \dots d^3\,\vec{r}_N}{\int \Psi^*(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N)\, \Psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) d^3\vec{r}_1 d^3\vec{r}_2 \dots d^3\,\vec{r}_N}$$

- In DFT the key variable is the electron density $n(\vec{r})$, which for a normalized Ψ is given by

$$n(\vec{r}) = N \int \Psi^*(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N)\, \Psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) d^3\vec{r}_1 d^3\vec{r}_2 \dots d^3\,\vec{r}_N$$

- This relation is reversible such that the GS wavefunction

$$\Psi_0(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) = \Psi[n_0]$$

Hohenberg-Kohn theorem 1

The external potential (and hence the total energy), is a unique functional of the electron density.

Corollary: the ground state density uniquely determines the potential and thus all properties of the system, including the many-body wave function.

$$\Psi_0(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) = \Psi[n_0]$$

Eg: The ground state properties of a many-electron system are uniquely determined by an electron density that depends on only 3 spatial coordinates.

Hohenberg-Kohn theorem 2

- **The functional that delivers the ground state energy of the system, gives the lowest energy if and only if the input density is the true ground state density.**

- For any positive integer N and potential $v(\vec{r})$, a density functional $F[n]$ exists such that $E_{(v,N)}[n] = F[n] + \int v(\vec{r})n(\vec{r})d^3r$, obtains its minimal value at the ground-state density of N electrons in the potential $v(\vec{r})$.

- The minimal value of $E_{(v,N)}[n]$ is then the ground state energy of this system.

- ➔ The correct ground state electron density minimizes this energy functional.

Expressions

Ψ is a unique function of n_0: $\Psi_0(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) = \Psi[n_0]$

The GS expectation value: $\mathcal{O}[n_0] = \langle \Psi[n_0] | \hat{\mathcal{O}} | \Psi[n_0] \rangle$

The GS energy: $E_0 = E[n_0] = \langle \Psi[n_0] | \hat{T} + \hat{V} + \hat{U} | \Psi[n_0] \rangle$

The contribution of the external potential $\langle \Psi[n_0] | \hat{V} | \Psi[n_0] \rangle$

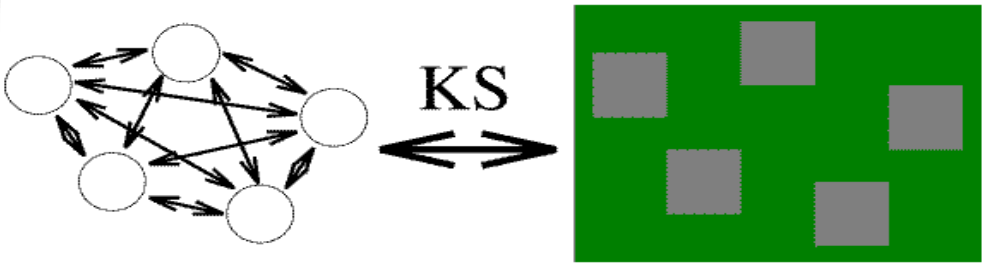
$$V[n_0] = \int V(\vec{r}) n_0(\vec{r}) d^3r$$

Having specified a system, i.e., having specified $\hat{V}$ one then has to minimize the functional

$$E[n] = T[n] + U[n] + \int V(\vec{r}) n(\vec{r}) d^3r$$

with respect to $n(\vec{r})$.

KOHN –SHAM APPROACH



Interacting electrons + real potential

Non-interacting, fictitious particles + effective potential

The system is replaced by a fictitious non-interaction system with same density as the real interacting system.

$$n(\vec{r}) \stackrel{\text{def}}{=} n_s(\vec{r}) = \sum_i^N |\phi_i(\vec{r})|^2$$

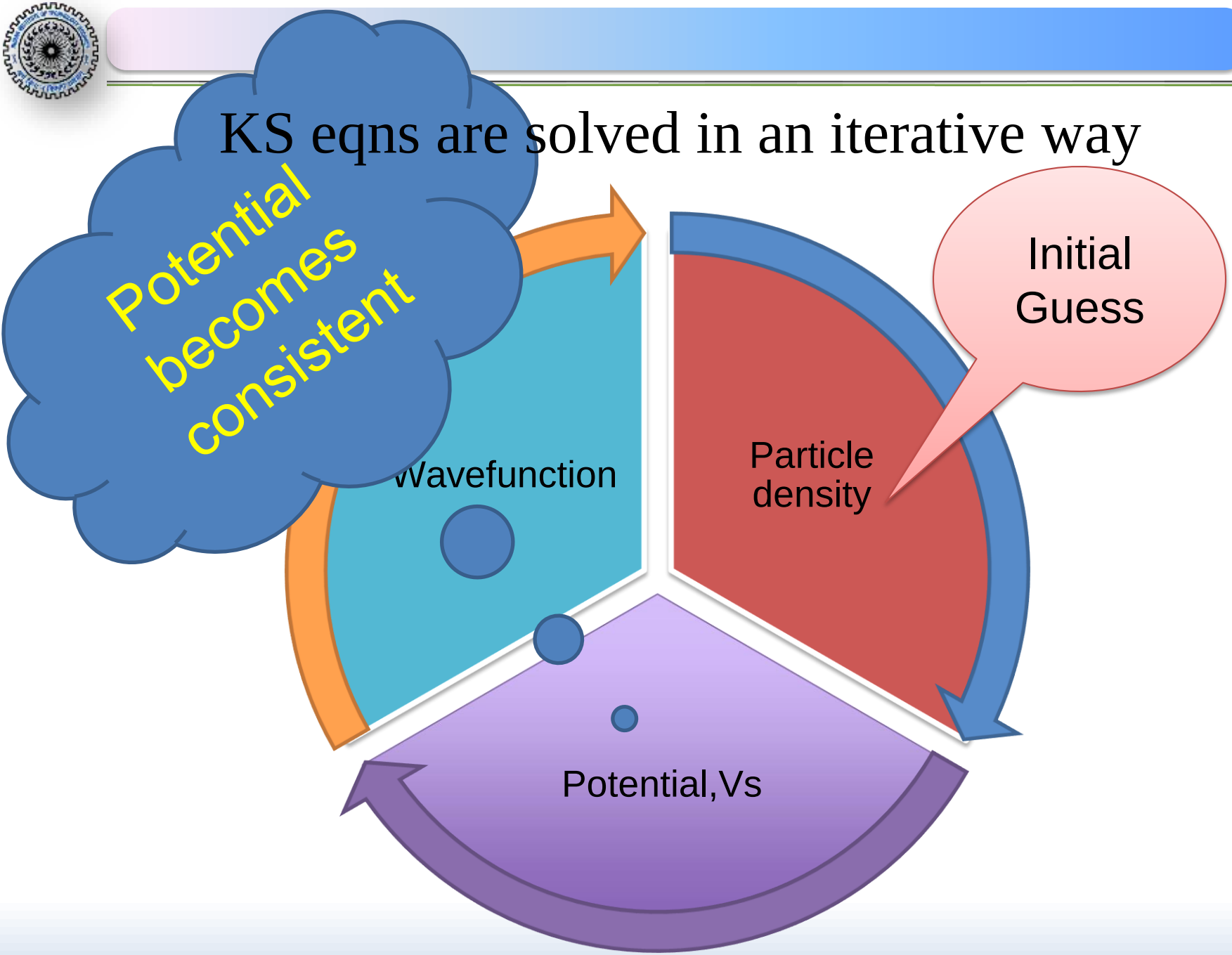
The Hamiltonian doesn't explicitly have an electron-electron interaction term: $H_s = T_s + V_s$

T_s = non-interacting kinetic energy; V_s= effective potential

KS eqns are similar to time independent Schrodinger eqns with single particle wave functions:

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V_s(\vec{r})\right]\phi_i(\vec{r}) = \epsilon_i\,\phi_i(\vec{r})$$

KS eqns are solved in an iterative way



Relativistic fields

- Classical fields: space and time coordinates are treated similarly
- They are treated as parameters specifying the space-time continuum, with which the field variables are defined.
- Relativistic fields that manifests Lorentz covariance: $x'_\mu = a^\nu_\mu x_\nu$.
- Recap:
  - Euler-Lagrange Eqn: $\frac{\partial L}{\partial q_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} = 0$ (6.5)
  - Lagrangian density: $L = \int \mathcal{L}(\hat{x}) d^3x,$ $(\hat{x}: ct, x_1, x_2, x_3)$
  - $\frac{\partial \mathcal{L}}{\partial \psi_k} - \sum_\mu \frac{\partial}{\partial x_\mu}\frac{\partial \mathcal{L}}{\partial \frac{\partial \psi_k}{\partial x_\mu}} = 0$ (6.19)
  - $\frac{\partial \mathcal{L}}{\partial \psi_k} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_k)} = 0$ (6.20)
  - $T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \frac{\partial \psi_k}{\partial x_\nu}}\frac{\partial \psi_k}{\partial x_\mu} - \mathcal{L}\delta_{\mu\nu}$ (6.58)

Covariant form of the KG equation

- Prove that
  - $\mathcal{L}_{KG}(\hat{x}) = \frac{1}{2}\left[\partial^{\mu}\phi(\hat{x})\partial_{\mu}\phi(\hat{x}) - m^2\phi^2(\hat{x})\right]$ satisfies the EL eqn.

(7.14)

  - $H = \int T_{00} d^3x$ (7.17)
- KG Equation
  - $E^2 = c^2p^2 + m^2c^4$
  - $H^2\psi = E^2\psi, \qquad E = i\hbar\frac{\partial\psi}{\partial t} \qquad p = -i\hbar\nabla$
  - $\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{m^2c^2}{\hbar^2}\right]\psi = 0$
  - $\left[\Box^2 - \frac{m^2c^2}{\hbar^2}\right]\psi = 0$
  - $\partial^{\mu} = \frac{\partial}{\partial x_{\mu}}$ ➔ $(\partial^{\mu}\partial_{\mu} + m^2)\psi = 0$

Covariant form of the Dirac equation

$$H = c\,\boldsymbol{\alpha}\cdot\boldsymbol{p} + \boldsymbol{\beta}\,mc^2$$

$$i\hbar\frac{\partial\psi}{\partial t} = (-i\,\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + M\boldsymbol{\beta})\psi$$

$$\boldsymbol{\alpha} = \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}; \qquad \boldsymbol{\beta} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{-1} \end{pmatrix}$$

$$\gamma^0 = \beta; \qquad \gamma = \beta\alpha = (\gamma^1, \gamma^2, \gamma^3)$$

$$\left(i\gamma_\mu\partial^\mu - M\right)\psi = 0$$

$$\mathcal{L}_D = \bar{\psi}\left(i\gamma_\mu\partial^\mu - M\right)\psi$$

- Prove that $\mathcal{L}_D$ satisfies Euler-Lagrange equation

The Lagrangians for various fields

- **Scalar field:** $\mathcal{L}_{KG} = \frac{1}{2}\left[\partial^\mu \phi \partial_\mu \phi - m^2\phi^2\right]$

- **Dirac field:** $\mathcal{L}_D = \bar{\psi}\left(i\gamma_\mu \partial^\mu - M\right)\psi$

- QED Lagrangian (Electromagnetic field)

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - e\bar{\psi}\gamma^\mu \psi A_\mu$$

Electromagnetic field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

- **Scalar-Vector theory:**

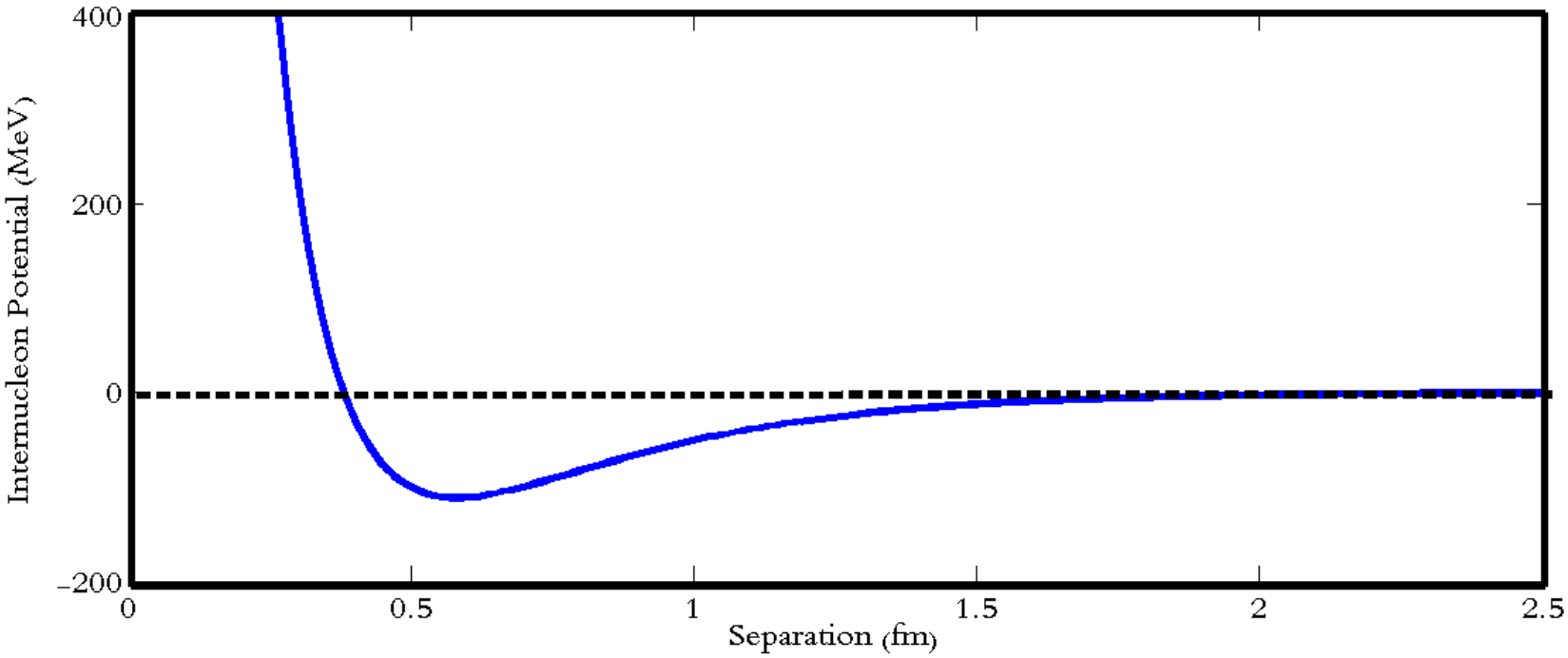
$$L = \bar{\psi}[\gamma_\mu(i\partial^\mu - g_v V^\mu) - (M - g_s\phi)]\psi + \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m_s^2\phi^2) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_v^2 V_\mu V^\mu$$

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$$

QHD

- σ is a neutral scalar meson with spin parity, $J^{\pi} = 0+$ and has mass of about 500 MeV/c^2.
- ω is neutral vector meson with spin parity, $J^{\pi} = 1-$ and has mass of about 782 MeV/c^2.
- Thus, only vector field, V_{μ} (of omega mesons) and a scalar field, ϕ (of sigma mesons) are needed to describe the nuclear interaction.
- The nuclear potential is accounted by the σ & ω mesons, where the hard core repulsive part of the potential is accounted by the ω-mesons and the attractive part is accounted by the σ-mesons.
- The baryons are described by the field, ψ.

QHD-I



$$L = \bar{\psi}[\gamma_\mu(i\partial^\mu - g_v V^\mu) - (M - g_s\phi)]\psi + \frac{1}{2}(\partial_\mu\phi\partial^\mu\phi - m_s^2\phi^2) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m_v^2 V_\mu V^\mu$$

$$F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$$

Equations of motion

$$\frac{\partial}{\partial x^{\mu}}\left[\frac{\partial L}{\partial(\partial q_i/\partial x^{\mu})}\right]-\frac{\partial L}{\partial q_i}=0$$

$$(\partial_{\mu}\partial^{\mu}+m_s^2)\phi=g_s\bar{\psi}\psi$$

$$\partial_{\mu}F^{\mu\nu}+m_v^2V^{\nu}=g_v\bar{\psi}\gamma^{\mu}\psi$$

$$[\gamma^{\mu}(i\partial_{\mu}-g_vV_{\mu})-(M-g_s\phi)]\psi=0$$

$$T_{\mu\nu}=-g_{\mu\nu}L+\frac{\partial q_i}{\partial x^{\nu}}\frac{\partial L}{\partial(\partial q_i/\partial x_{\mu})}$$

Mean field theory

- Field operators ← expectation values ≡ classical fields
  - $\phi \rightarrow \langle \phi \rangle \equiv \phi_0$
  - $V_\mu \rightarrow \langle V_\mu \rangle \equiv \delta_{\mu 0} V_0$

- Now the field equations can be solved for constant ϕ_0 and V_0:
  - $\phi_0 = \frac{g_s}{m_s^2} \langle \bar{\psi} \psi \rangle = \frac{g_s}{m_s^2} \rho_s$
  - $V_0 = \frac{g_v}{m_v^2} \langle \bar{\psi} \gamma^0 \psi \rangle = \frac{g_v}{m_v^2} \rho_B$

- With the classical meson fields the equation for Dirac field becomes linear
  - $\left[i \gamma_\mu \partial^\mu - g_v \gamma^0 V_0 - (M - g_s \phi_0) \right] \psi = 0$

Effective mass

- $\left[i\gamma_\mu \partial^\mu - g_v \gamma^0 V_0 - (M - g_s \phi_0)\right]\psi = 0$

- Stationary state solutions for the free particles
  - $\psi = \psi(\mathbf{k}, \lambda) e^{i\mathbf{k}\cdot\mathbf{x} - i\epsilon(k)t}$
  - $\psi(\mathbf{k}, \lambda)$ → four-component Dirac spinor, λ→ spin index

- $(\boldsymbol{\alpha} \cdot \mathbf{k} + \beta M^*)\psi(\mathbf{k}, \lambda) = [\epsilon(k) - g_v V_0]\psi(\mathbf{k}, \lambda)$

- The effective mass M^* is defined as
  - $M^* = M - g_s \phi_0$

Energy & Pressure

$$\mathscr{E} = T_{00}$$

$$P = \tfrac{1}{3}\langle T_{ii} \rangle$$

$$C_s^2 = g_s^2 \left(\frac{M^2}{m_s^2}\right) = 267.1$$

$$\mathscr{E} = \frac{g_v^2}{2m_v^2}\rho_B^2 + \frac{m_s^2}{2g_s^2}(M - M^*)^2 + \frac{\gamma}{(2\pi)^3}\int_0^{k_F} d^3k (k^2 + M^{*2})^{1/2}$$

$$P = \frac{g_v^2}{2m_v^2}\rho_B^2 - \frac{m_s^2}{2g_s^2}(M - M^*)^2 + \frac{1}{3}\frac{\gamma}{(2\pi)^3}\int_0^{k_F} d^3k \frac{k^2}{(k^2 + M^{*2})^{1/2}}$$

$$\rho_B = \frac{\gamma}{(2\pi)^3}\int_0^{k_F} d^3k = \frac{\gamma}{6\pi^2}k_F^3$$

Calculations

minimizing $\mathscr{E}(M^*)$

$$M^* = M - \frac{g_s^2}{m_s^2}\frac{\gamma}{(2\pi)^3}\int_0^{k_F} d^3k \frac{M^*}{(k^2+M^{*2})^{1/2}}$$

$$M^* = M - \frac{g_s^2}{m_s^2}\frac{\gamma M^*}{4\pi^2}\left[k_F E_F^* - M^{*2}\ln\left(\frac{k_F+E_F^*}{M^*}\right)\right]$$

where $E_F^* = (k_F^2 + M^{*2})^{1/2}$

$$C_s^2 = g_s^2\left(\frac{M^2}{m_s^2}\right) = 267.1 \qquad C_v^2 = g_v^2\left(\frac{M^2}{m_v^2}\right) = 195.9$$

$M = 939$ MeV, $\hbar c = 197.3269631$ MeV fm

http://www.wolframalpha.com integrate x^2/sqrt(x^2+m^2)

$$\int \frac{x^2}{\sqrt{x^2+m^2}}\,dx = \frac{1}{2}x\sqrt{m^2+x^2} - \frac{1}{2}m^2\log\left(\sqrt{m^2+x^2}+x\right) + \text{constant}$$

Solving transcendental equation: Secant method

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}$$

$$x_n = x_{n-1} - f(x_{n-1})\frac{x_{n-1} - x_{n-2}}{f(x_{n-1}) - f(x_{n-2})}$$

```
function func(x)
func=sin(x)
end

external func
x=secmethod(func,3.10)
print *, x
end
```

```
function secmethod(f,x00)
logical flag
x0=x00
f0=f(x0)
x1=x0+x0/100.0d0
f1=f(x1)
flag=.true.
do i=1,100
  x=x1-f1*(x1-x0)/(f1-f0)
  if(abs(x-x1).le.1e-4) then
    flag=.false.
    exit
  end if
  x0=x1
  f0=f1
  x1=x
  f1=f(x)
end do
if (flag) print *, 'No convergence'
secant=x
end
```

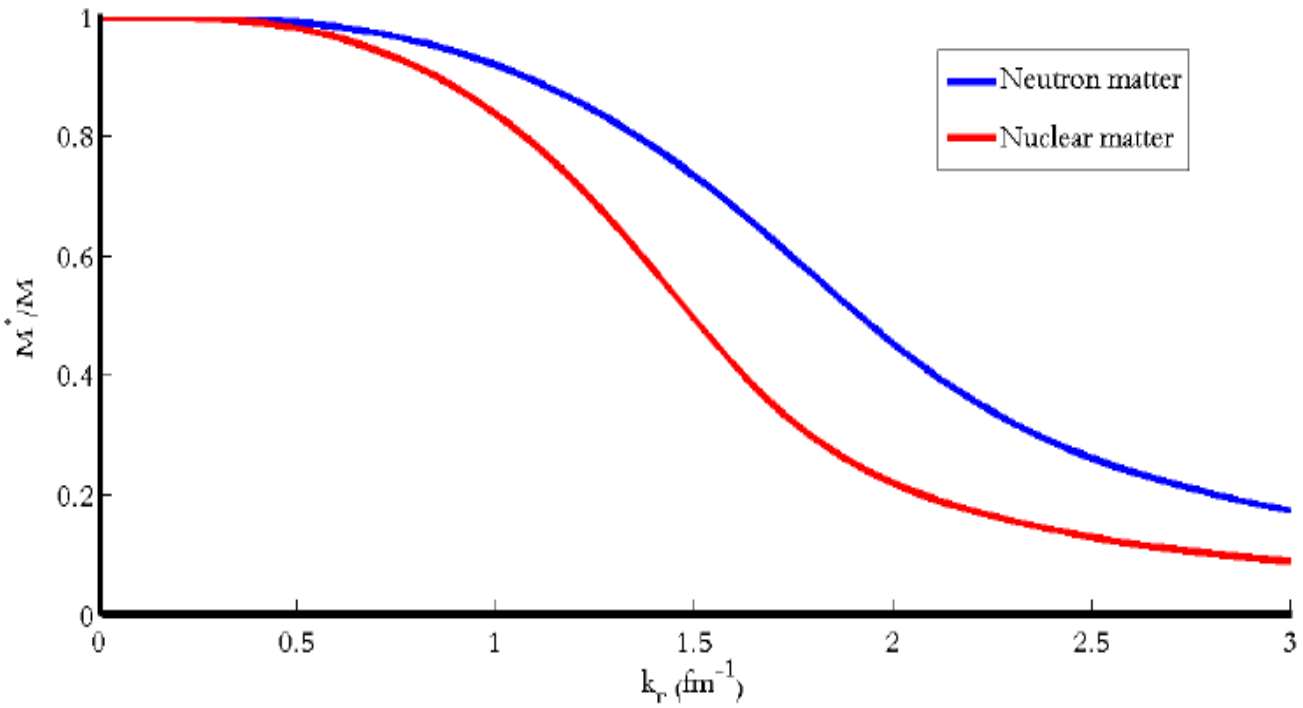
Solving for *M**

```
function rmstar(x)
common /set1/ rkf,gama,am,cs2,sv2,pi
amef=x
efef=sqrt(rkf**2+amef**2)
rmstar=COMPLETE THIS BY YOURSELF
end

external rmstar
common /set1/ rkf,gama,am,cs2,sv2,pi
read *,gama
pi=4.0*atan(1.0)
hc=197.3269804
am=939.0
cs2=267.1
cv2=195.9
amef=939.0
do i=0,50
  akf=float(i)/10.0
  rkf=hc*akf
  amef=secmethod(rmstar,amef)
  print *, akf,amef/939.0
end do
end

include 'secmethod.for'
```

Calculated M^*/M



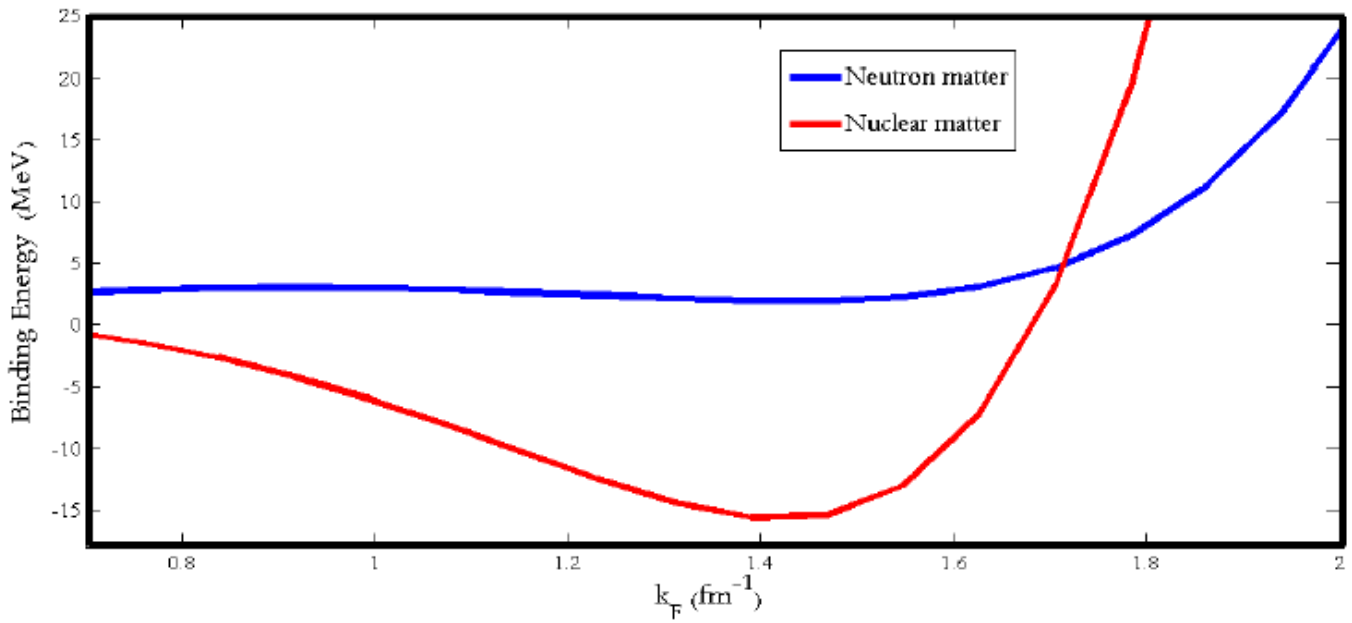
Numerical integration: Simpson's 1/3 rule

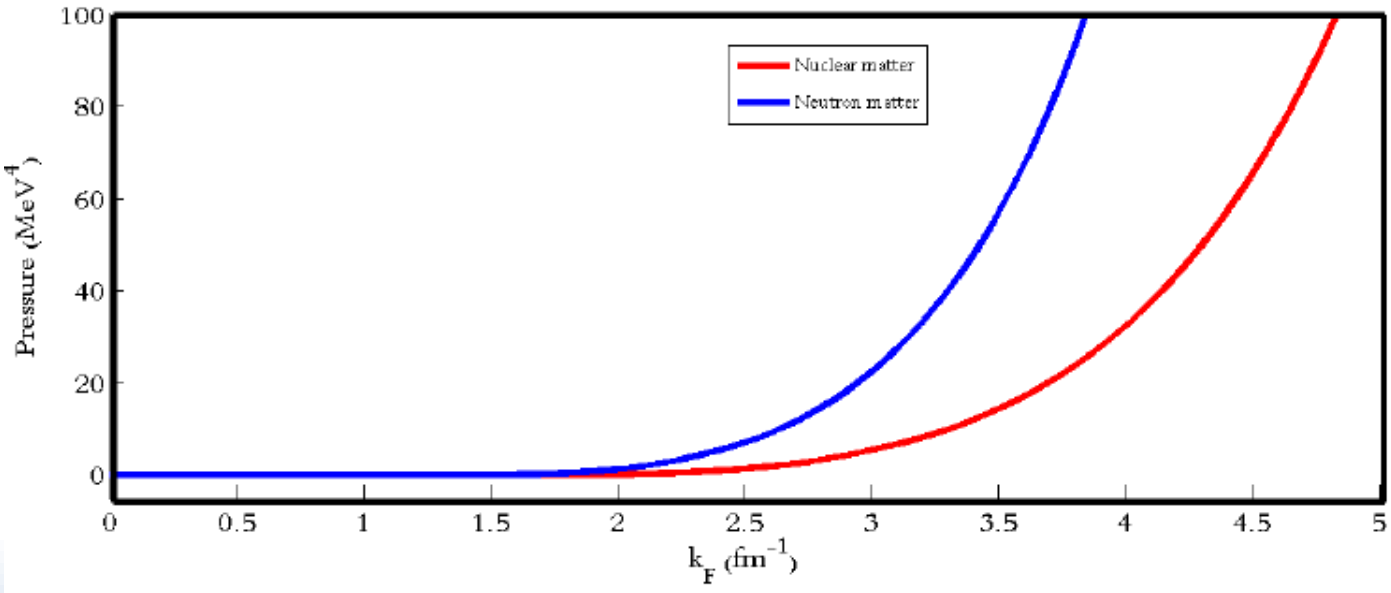
```
function simp(x,y,n)
dimension x(n),y(n)
h=abs(x(n)-x(1))/float(n-1)
sum=0.0
do i=1,n/2
   sum=sum+y(2*i-1)+4.0*y(2*i)+y(2*i+1)
end do
simp=sum*h/3.0
end

function rmstar(x)
common /set1/ rkf,gama,am,cs2,sv2,pi
parameter (nn=101)
dimension xx(nn),yy(nn)
amef=x
do i=1,nn
  xx(i)=dfloat(i-1)*rkf/dfloat(nn-1)
  efef=sqrt(xx(i)**2+amef**2)
  yy(i)=xx(i)**2*amef/efef
end do
rmstar=COMPLETE THIS BY YOURSELF
end
```

Excercise:
Extend this code to calculate E & P

Results of QHD-I





Check the numbers

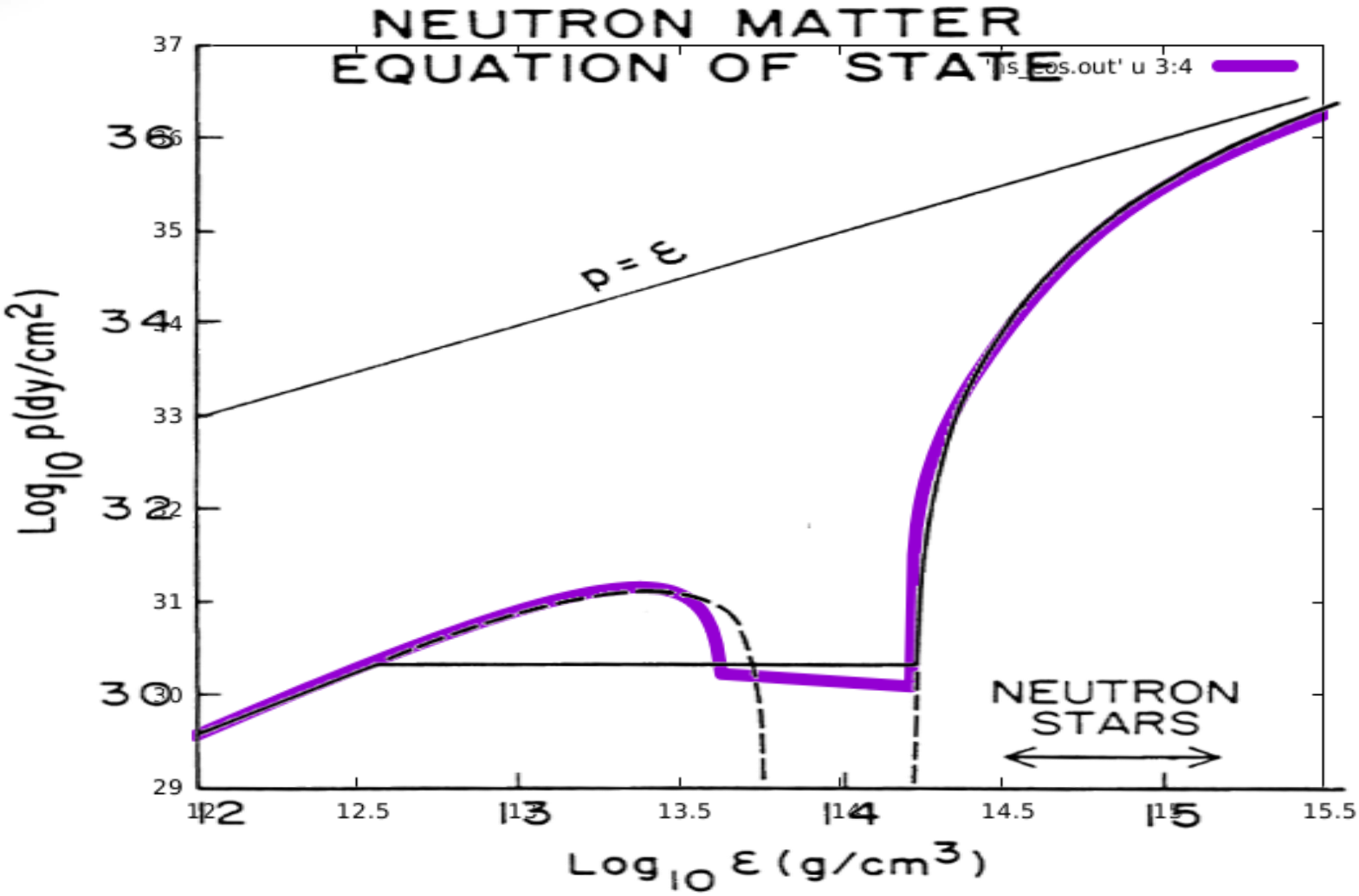


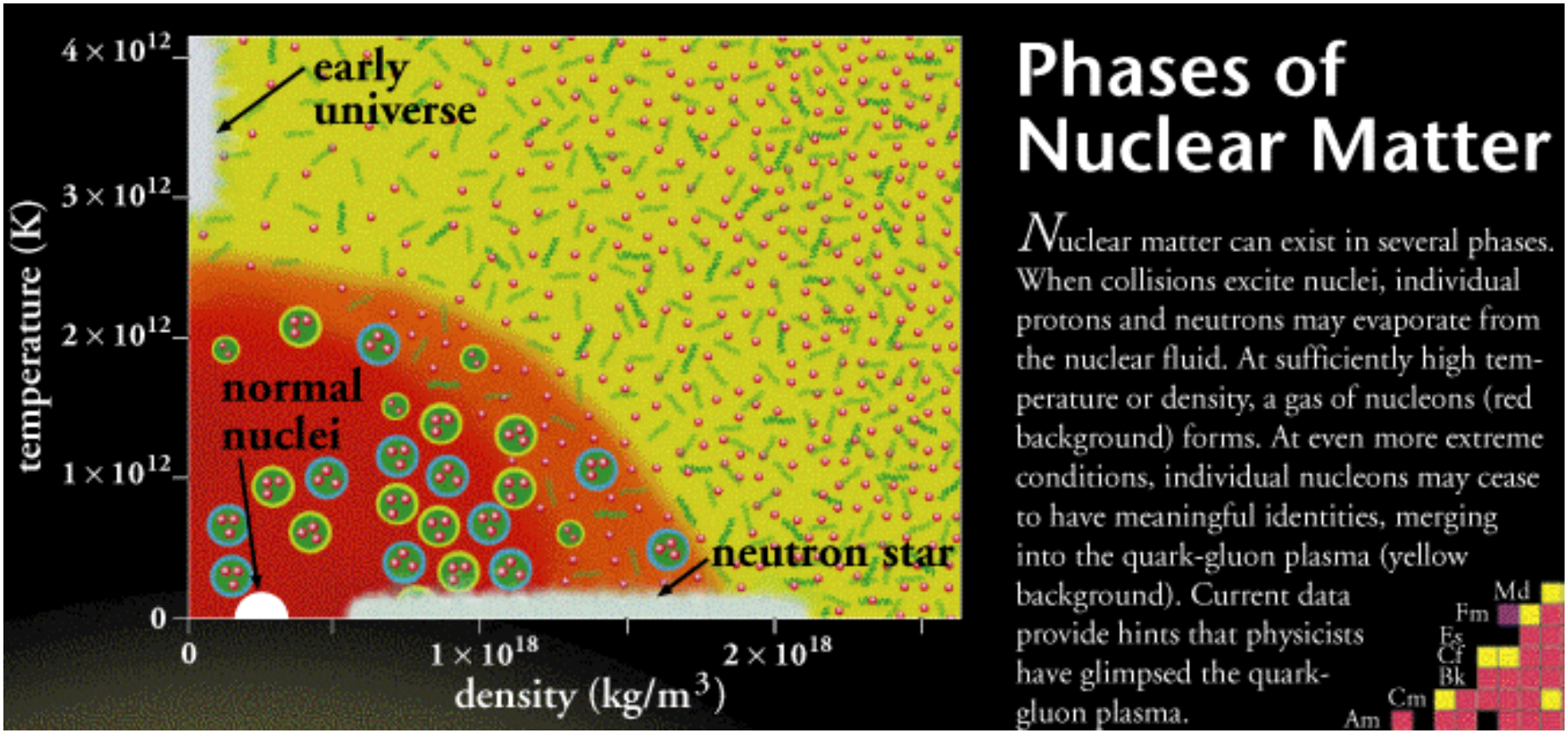
Fig. 14.6 of J.D. Walecka, Theoretical nuclear and subnuclear physics, 2e, World Scientific (2004)

Constants & conversions

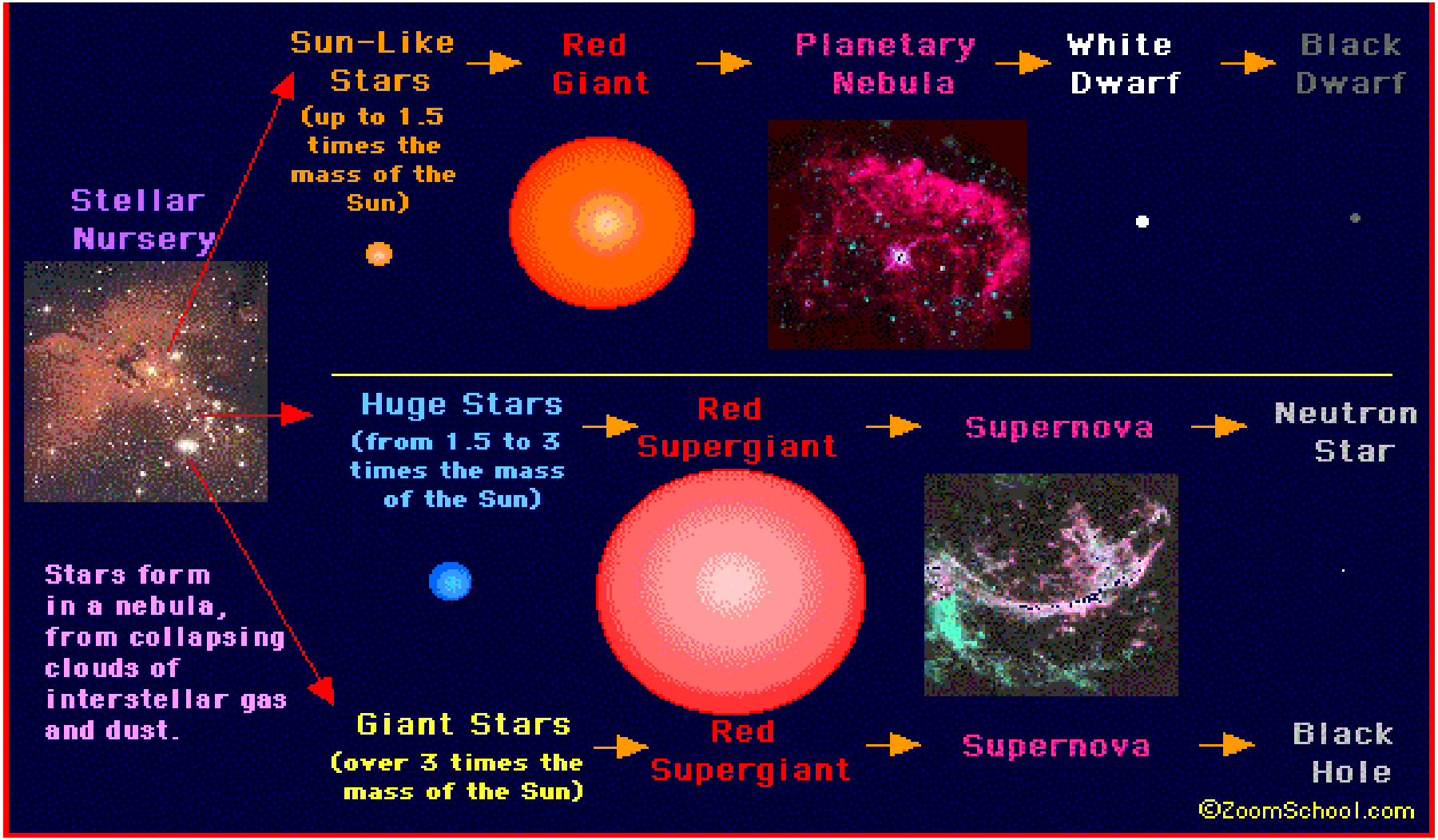
- 1 erg = 1 dyn·cm = 1 g·cm^2/s^2 = 6.241509074×10^4 MeV
- 1 erg = 10^{-7} J
- 1eV=1.602176634 × 10^{-19} J (https://physics.nist.gov/cgi-bin/cuu/Value?evj)
- 1eV=1.602176634 × 10^{-19} × 10^7 erg
- 1MeV=1.602176634 × 10^{-12} × 10^6 erg =1.602176634 × 10^{-6} erg
- 1MeV/fm^3=1.602176634 × 10^{-6} /10$^{-3\times13}$ erg/cm^3
- 1MeV/fm^3=1.602176634 × 10^{33} erg/cm^3=1.602176634 × 10^{33} dyn/cm^2
- 1Mev/c^2=1.782661921 × 10^{-27} g
- 1Mev/fm^3 = 1.782661921 × 10^{12} c^2 g/ cm^3

```
e=e*1.782661921d12/hc**3
p=p*1.602176634d33/hc**3
 write(*,'(2x,4e14.5)') akf,amef/939.0,log10(e),log10(p)
```

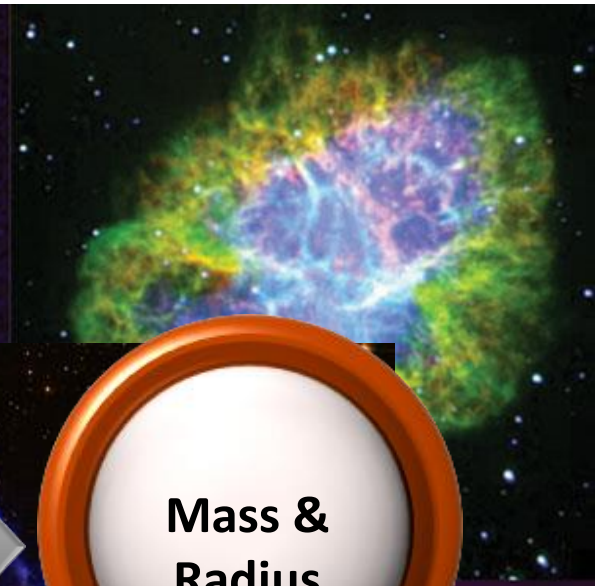
Neutron Star: Another Probe



The Lifecycle of Stars



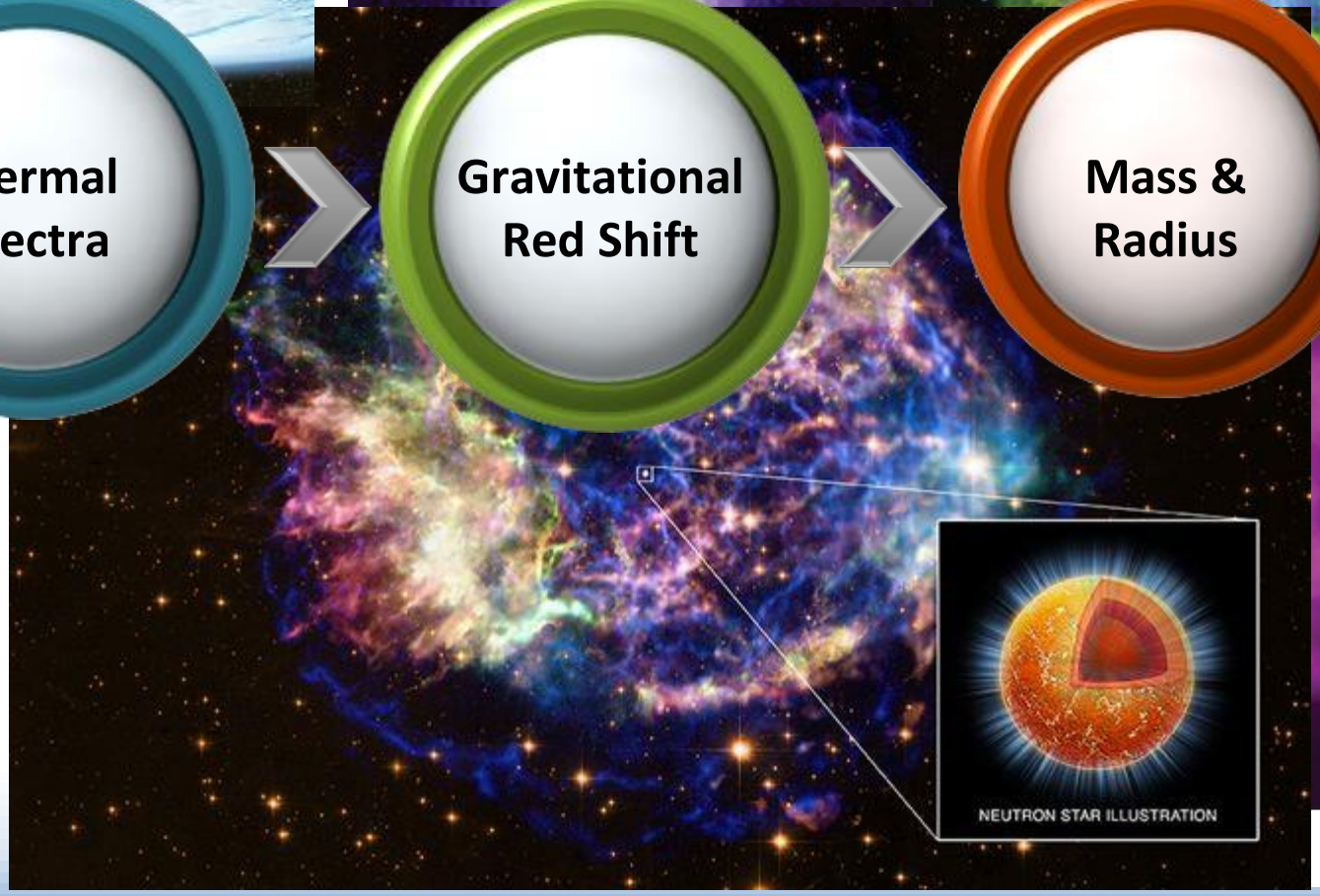
Neutron Star Observables

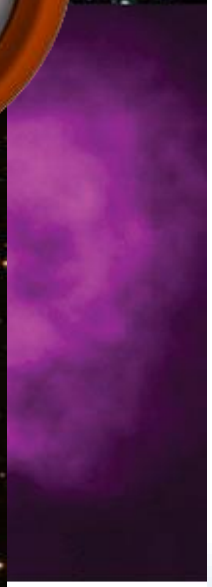


Thermal Spectra

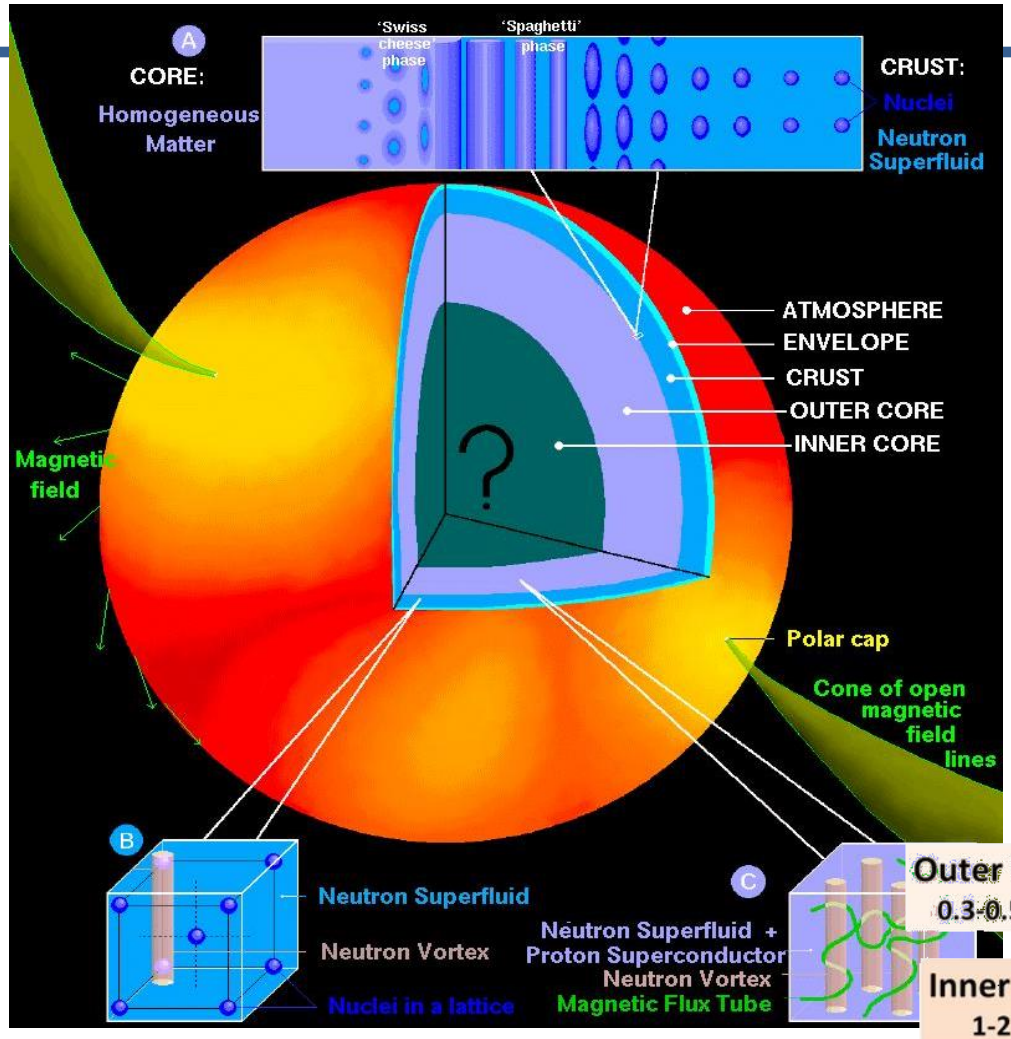
Gravitational Red Shift

Mass & Radius

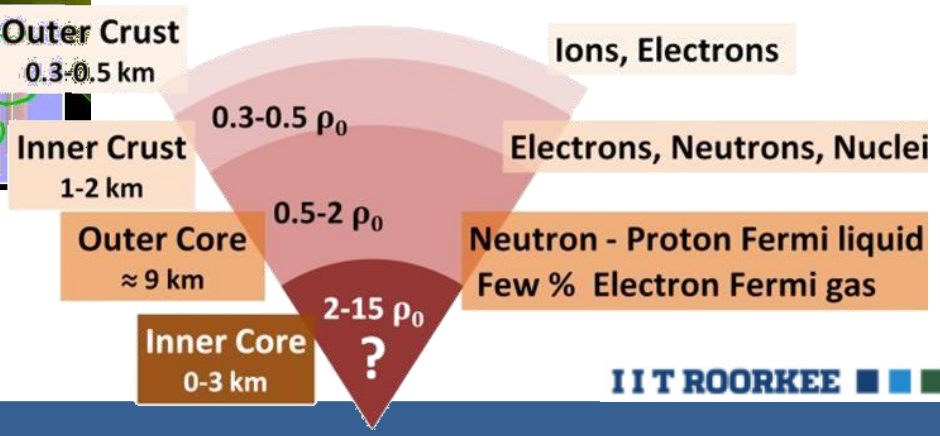




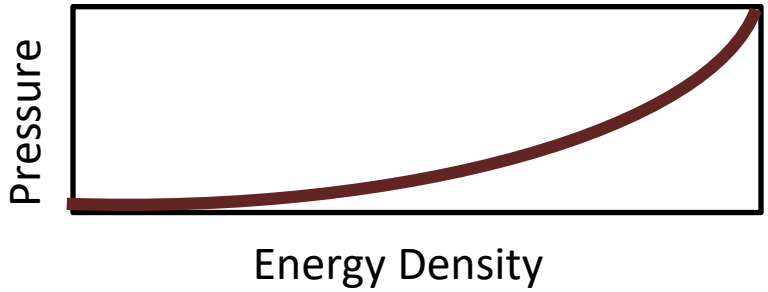
A Neutron Star: Surface and Interior



Lattimer & Prakash ,
Science 304, 536 (2004)



Neutron Star Calculations

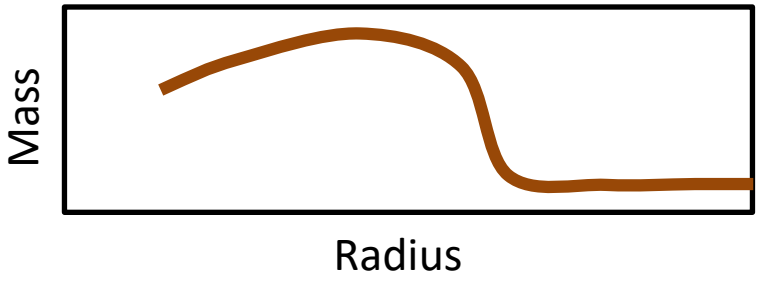


$$\frac{dP}{dr} = -(\mathcal{E} + P)\frac{m + 4\pi r^3 P}{r(r - 2m)}$$

$$\frac{dm}{dr} = 4\pi r^2 \mathcal{E}$$

$m(r)$ iss the total integrated mass out to a radial distance r

R. C. Tolman, Phys. Rev. **55**, 364 (1939),
J. R. Oppenheimer and G.M. Volkoff, Phys. Rev. **55**, 374 (1939).



Solving ODE: Runge-Kutta method

$\frac{dy}{dx} = f(x, y)$ Given (x_0, y_0), the solutions are $(x_1, y_1), (x_2, y_2) \ldots (x_N, y_N)$

$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$$
$$k_3 = hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$$
$$k_4 = hf(x_n + h, y_n + k_3)$$
$$y_{n+1} = y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} + O(h^5)$$

```
      external f
      dimension vx(100),vy(100)
      call rk4(f,0.0,1.0,10.0,100,vx,vy)
      do i=1,100,10
         write(*,'(2x,f4.1,f10.5,e12.4)')
     &         vx(i),vy(i),vy(i)
     &        -(4.0+vx(i)*vx(i))**2/16.0
      end do
      end
```

```
subroutine rk4(f,x0,y0,xn,n,ax,ay)
dimension ax(n),ay(n)
h=(xn-x0)/float(n)
ax(1)=x0
ay(1)=y0
do i=2,n
  x=ax(i-1)
  y=ay(i-1)
  yk1=h*f(x,y)
  yk2=h*f(x+0.5*h,y+0.5*yk1)
  yk3=h*f(x+0.5*h,y+0.5*yk2)
  yk4=h*f(x+h,y+yk3)
  ax(i)=x+h
  ay(i)=y+(yk1+2.0*(yk2+yk3)+yk4)/6.0
end do
end

function f(x,y)
  f=x*sqrt(y)
end
```

$$\frac{dy}{dx} = x\sqrt{y}$$

Wolfram: y'-x*\sqrt(y) = 0

$$y(x) = \frac{1}{16}\left(4c_1 x^2 + 4c_1^2 + x^4\right)$$

II order ODE: Two coupled I order ODEs

$$\frac{d^2y}{dx^2} + q(x)\frac{dy}{dx} = r(x)$$

$$\frac{dy}{dx} = z(x)$$

$$\frac{dz}{dx} = r(x) - q(x)z(x)$$

$$m\frac{d^2x}{dt^2} = F(x)$$

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = F(x)/m$$

$$\frac{dy_i(x)}{dx} = f_i(x, y_1, \ldots, y_N)$$

```
subroutine rk4(x0,y10,y20,xn,n,ax,ay1,ay2,f1,f2)
implicit real*8 (a-h,o-z)
dimension ax(0:n),ay1(0:n),ay2(0:n)

h=(xn-x0)/dfloat(n)
ax(0)=x0
ay1(0)=y10
ay2(0)=y20

do i=1,n
        x=ax(i-1)
        y1=ay1(i-1)
        y2=ay2(i-1)

        y1k1=h*f1(x,y1,y2)
        y2k1=h*f2(x,y1,y2)

        y1k2=h*f1(x+0.5d0*h,y1+0.5d0*y1k1,y2+0.5d0*y2k1)
        y2k2=h*f2(x+0.5d0*h,y1+0.5d0*y1k1,y2+0.5d0*y2k1)

        y1k3=h*f1(x+0.5d0*h,y1+0.5d0*y1k2,y2+0.5d0*y2k2)
        y2k3=h*f2(x+0.5d0*h,y1+0.5d0*y1k2,y2+0.5d0*y2k2)

        y1k4=h*f1(x+h,y1+y1k3,y2+y2k3)
        y2k4=h*f2(x+h,y1+y1k3,y2+y2k3)

        ax(i)=x+h
        ay1(i)=y1+(y1k1+y1k2+y1k2+y1k3+y1k3+y1k4)/6.0d0
        ay2(i)=y2+(y2k1+y2k2+y2k2+y2k3+y2k3+y2k4)/6.0d0
end do
end
```

II order ODE: Excercises

- Projectile motion
- Rutherford scattering
- Solving Schrodinger equation with boundary conditions
  - Particle in an infinite square well potential
  - Particle in a harmonic oscillator potential
  - Particle in finite potentials
- Continuity equation

Solving the TOV equations

```
   real*8 function f1(x,y1,y2)
   implicit real*8 (a-h,o-z)
   common/cons/ams,ecf,c2,gc,g,ak,pi
   f1 = -(gc/(x**2*c2))
&       *(eps(y1) + y1)
&       *(y2 + (4.0d0*pi*x**3*y1)/c2)
&       /(1.0d0 - 2.0d0*gc*y2/(x*c2))
   end

   real*8 function f2(x,y1,y2)
   implicit real*8 (a-h,o-z)
   common/cons/ams,ecf,c2,gc,g,ak,pi
   f2 = 4.0d0*pi*x**2*eps(y1)/c2
   end
```

$$P = K\mathcal{E}^{\gamma} \qquad \mathcal{E} = \left(\frac{P}{K}\right)^{1/\gamma}$$

```
   real*8 function eps(p)
   implicit real*8 (a-h,o-z)
   common/cons/ams,ecf,c2,gc,g,ak,pi
   eps = ecf*(p/(ecf*ak))**(1.0/g)
   end
```

```
implicit real*8 (a-h,o-z)
external f1,f2
dimension ax(0:2000),ap(0:2000),am(0:2000)
common/cons/ams,ecf,c2,gc,g,ak,pi
ak=8.876746957d-4
g = 5.0d0/3.0d0
ams = 1.989d33
ecf = 1.6022d33
c2 = 8.987551787d20
gc = 6.67408d-8
pi = 4.0d0*datan(1.0d0)
np=2000
open(1,file='tov.out',status='unknown')
do j=1,40
  p0 = 10**(dfloat(j)/10.0)*ecf
  call rk4(1.d0,p0,0.d0,20.d5,np,ax,ap,am,f1,f2)
  do i=0,np
    if((ap(i).ne.ap(i)).or.(ap(i).le.0)) exit
  enddo
  write(1,'(2x,2f12.4)')ax(i-1)/1.0d5,am(i-1)/ams
enddo
end
```

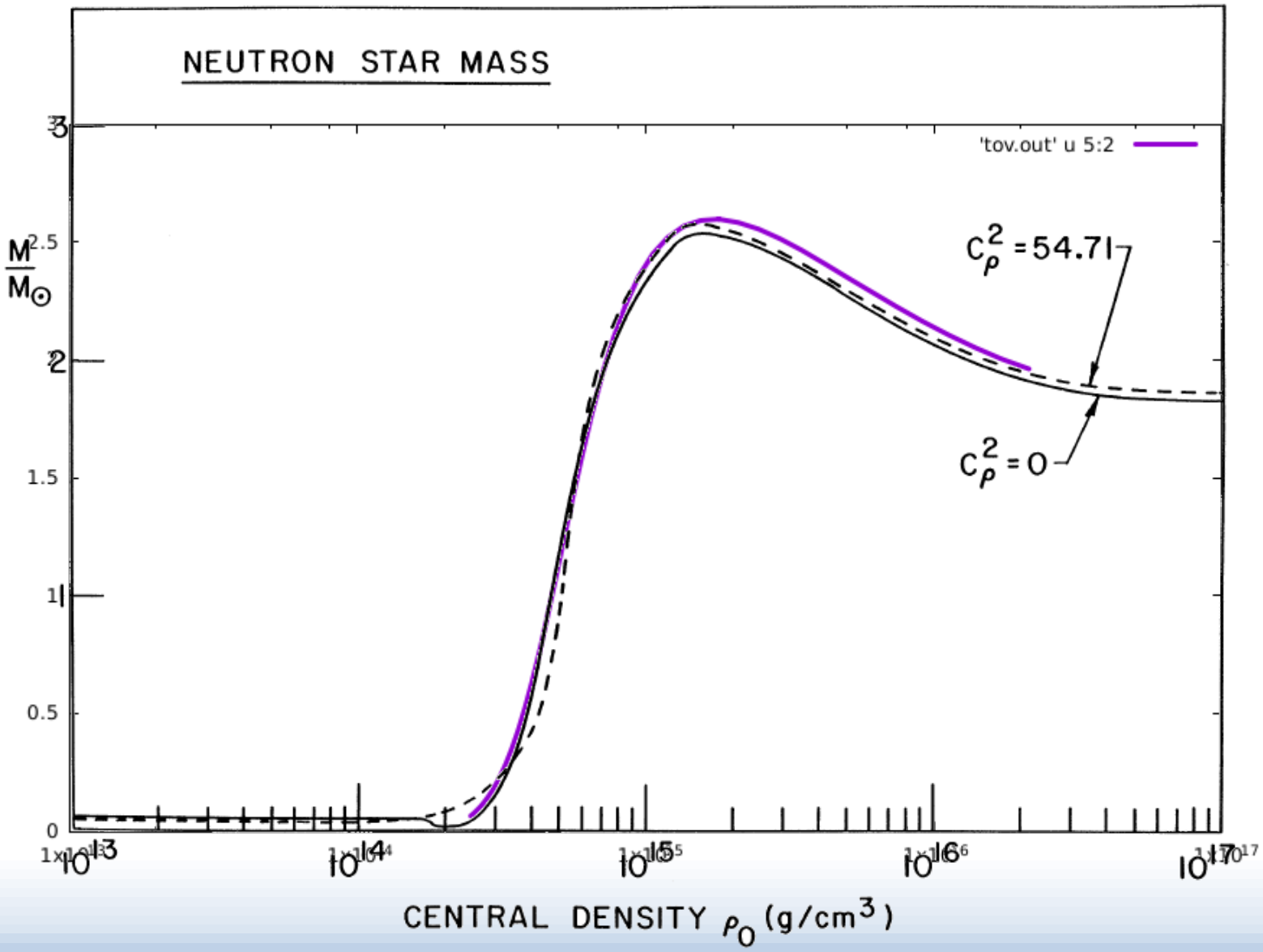
Utilizing a tabulated EoS: Linear interpolation

```
real*8 function lineint(x,xi,yi,ni)
implicit real*8 (a-h,o-z)
dimension xi(ni),yi(ni)
if(x.lt.xi(1))then
  n=1
else
  if(x.gt.xi(ni))then
    n=ni-1
  else
    do i=2,ni
      if(x.le.xi(i))then
        n=i-1
        exit
      endif
    end do
  endif
endif
lineint=(x-xi(n))*(yi(n+1)-yi(n))/(xi(n+1)-xi(n))+yi(n)
end
```

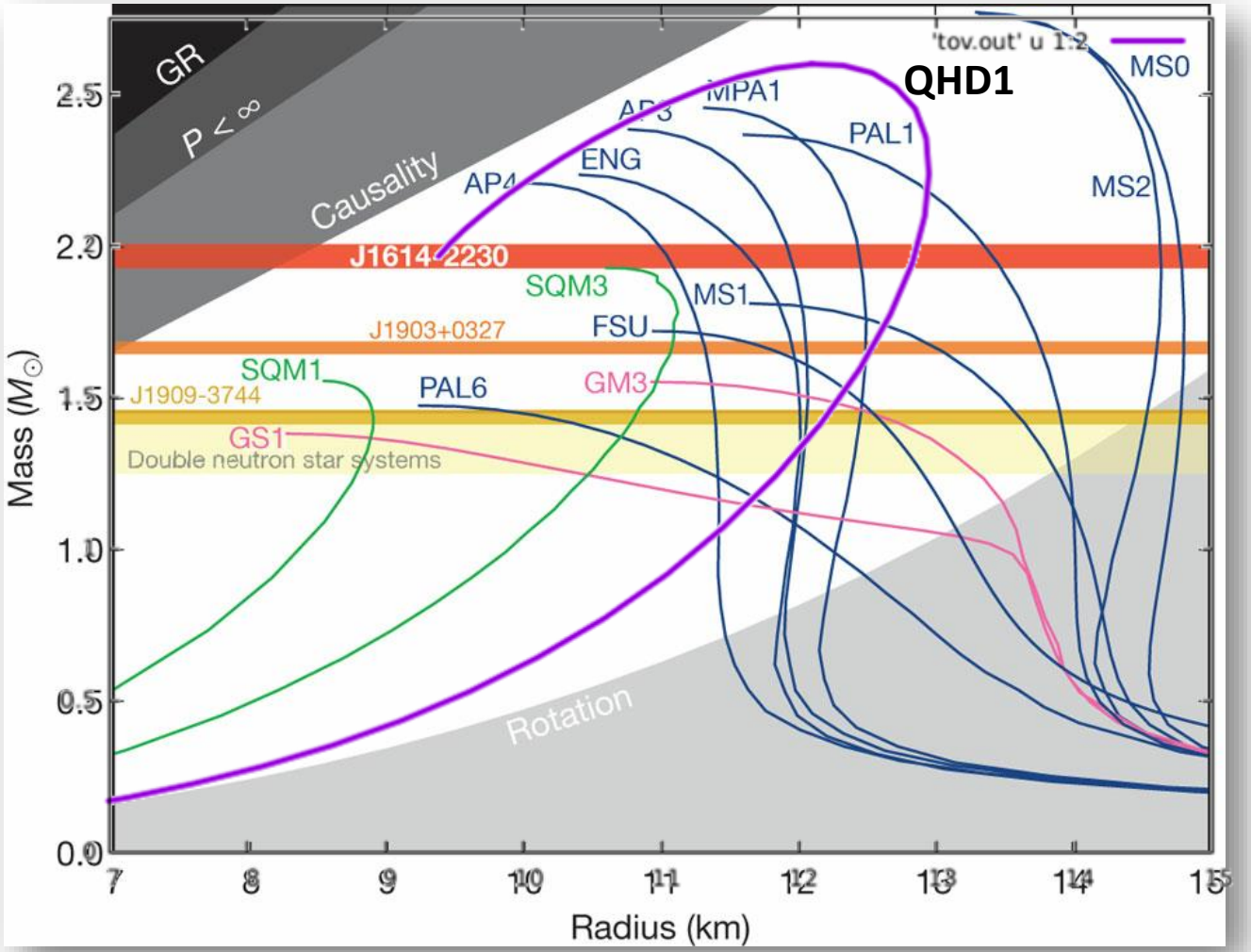
```
real*8 function eps(p)
implicit real*8 (a-h,o-z)
common/cons/ams,ecf,c2,gc,g,ak,pi
common/eos/xi(1000),yi(1000),ni
real*8 lineint
eps = ecf*lineint(p/ecf,xi,yi,ni)
end

open(4,file='eos.dat',status='unknown')
i=1
do while (.true.)
  read(4,*,end=100)yi(i),xi(i)
  print *,i,xi(i),yi(i)
  i=i+1
enddo
100  ni=i-1

e0=eps(p0)
d0=e0/c2
write(*,*) ax(i-1)/1.0d5,am(i-1)/ams,d0
```


NEUTRON STAR MASS
'tov.out' u 5:2
$c_\rho^2 = 54.71$
$c_\rho^2 = 0$
$\frac{M}{M_\odot}$
3
2.5
2
1.5
1
0.5
0
10^{13}
10^{14}
10^{15}
10^{16}
10^{17}
CENTRAL DENSITY ρ_0 (g/cm^3)

The 2 Solar Mass NS



P.B. Demorest, T. Pennucci, S.M. Ransom, M.S.E. Roberts, and J. W. T. Hessels, Nature **467**, 1081 (2010)


Thank You