The Summer of Math Exposition 2 Script

A tip for making a script:
I found it easier to read the script while recording if everything, even if it is obvious, is written out. For example, I deliberately wrote 'equal to' or 'greater than or equal to' even though I could have written the symbols. And I prefer the numbers to be written as the actual numbers ('8' instead of eight). My tip is to make the script as easy as possible for you to read because this avoids doing many retakes.

Consider two points, S and S ′, randomly placed inside a D-dimensional hyper-rectangular room with walls that are perfect-reflecting (D − 1)-dimensional hyperplane mirrors. How many different light-rays that start from S, reflect N times on one of the walls and N − 1 times on each of the rest, then go to S′? Use D equal to 7 and N equal to 3.

This question is from the open round of the Online Physics Olympiad of 2022. And it has a really elegant and beautiful solution. I will explain the question in 2 dimensions.
We have a rectangular box, and all the sides are mirrors. We pick two random points inside it. Let's take the value of N to be 2. So, our task is to figure out how many light rays reflect 2 times on one wall and 1 time on the other 3 walls before reaching the second point. This is one of the solutions for N equal to 2. And this is one of the solutions for N equal to 3.
You can pause the video if you want to try this out yourself. But before showing the solution, I will give you a hint. So, continue watching if you want to see the hint.

The hint is in the form of a very different and completely unrelated puzzle. But I feel like understanding the trick to this one will hopefully steer you to think about our original question differently. Some of you might have seen this before because it is a pretty famous puzzle.

Consider a 10 cm long stick. There is one ant at its left end that moves to the right. And a random number of ants at random locations on the stick. All the ants move at a speed of 1 centimetre per second. But the directions of all the other ants are random. When two ants bump into each other, they both instantaneously turn around and walk in the other direction. And when one ant gets to the end of the stick, it falls off. How long will it take for all the ants to fall off the stick? And all the ants are point sized. Pause the video and try to figure this out.
The answer is 10 seconds. In the following animation, the ants, or the points, just stop at the end because I really don't want to animate them falling off the stick. And I chose 4, just to demonstrate the question.
The trick here is to realize that when two ants collide, instead of thinking that their velocities get flipped, we can instead think of them as simply passing right through each other. Both of these interpretations have the same effect since the ants are point-sized. So, calculating the time is easy. We just need to keep track of the ant at the left end.
So, the time is just the distance divided by the speed, which is equal to 10 seconds.

Going back to the original question, let's take the simplest case of N=1. Obviously, this only has 4 solutions. The ray reflects from one of the four walls and goes directly to the second point. But can we think about these four solutions in a different way? Pause the video and try to think outside the box, similar to how we did in the ant question.
I didn't say outside the box for no reason because the trick is to literally think outside the box. The idea is to draw mirror images of the original box in all the mirrors and then find the

images of the second point in these four mirrored boxes. And then connect the first point to these mirror images of the second point. Now, the parts of the lines inside the new boxes are exactly the mirror images of the reflected rays in the original box.

But those 4 boxes were just the first reflections of the original box. We can keep doing this process infinitely many times and make an infinite grid of images of the original box.
To understand how this works, let's choose the mirror image of the second point in this box, which is 2 boxes away diagonally. Just like before, we first connect these two points. But now, that line is intersecting the grid four times. If you pause and think about what these intersection points mean, you will soon realize that all these points are just the points of reflection of the ray of light in the original box. And to transform them back to the original intersection points, you just need to keep reflecting them in the walls till you reach the original box.

Now let's see what happens when the value of N is 2. So, we need 1 reflection on 3 walls and 2 on the remaining wall. So, we need a total of 5 reflections. Before, we saw that an intersection with the grid corresponds to a reflection in the original box. So, our task is to first find all the image points, from these infinite image points, that give 5 reflections with the grid. You can pause the video to try to figure this out.
So, these are all the images that give 5 intersections with the grid. But not all of these 20 points are solutions to our question. The problem is that N-1 reflections, or in our case, 4 reflections have to be on different walls. You can pause the video to figure out which points should be eliminated.
I will explain the idea in just the first quadrant because the process is the same in all the quadrants. Let's take this line. Now, if you keep track of which grid walls this line is intersecting and transform the intersection points back to the original box, you can figure out which wall the light ray is intersecting in the original box. In this case, the line intersects the top wall, bottom wall, top wall, bottom wall and finally, the top wall again. Now, if you do this process for the other solutions in this quadrant, you will find out that only 2 images give the required solution.
These are the intersection walls in the original box for the first line.
And these are the intersection walls in the original box for the second line.
After repeating this in all quadrants, we see that these are the final points. Thus, we have 8 solutions. We have 2 boxes along one diagonal direction or one quadrant. And we have 4 diagonal directions. So, we have 8 solutions.

Now we need to figure out what happens for a different value of N. For a 2-dimensional box, we have N-1 reflections on 3 walls and N reflections on 1 wall. But a different way to think about this is that we need N-1 reflections on all the walls and 1 extra reflection on one of the walls. So, what does this geometrically mean in our grid method?
Like I did before, I am going to explain this in the first quadrant. If you observe carefully, the boxes of the two solutions we found for N=2 are just 1 box away from the 2nd box diagonally away from the original box. If you do this for N=3, we will again have just 2 solutions, and they are one box away from the 4th box diagonally away from the original box. And, if you do this for N=5, we again have just 2 solutions, and they are one box away from the 8th box diagonally away from the original box. So, the pattern is that we will always have 2 solutions, but the images get farther away from the original box. And those boxes are one box away from the 2 times $(N-1)^{th}$ box along the diagonal.

So, the value of N does not matter at all as long as N is greater than or equal to 2 because we saw that for N=1 we had 4 solutions for N=1. In two dimensions, we have 8 solutions as long as N is greater than or equal to 2.

Let's see what happens in 3 dimensions. We do the exact same thing as before. And for simplicity, I am taking the value of N equal to 2 because we know that N does not matter if N is greater than or equal to 2. First, make an infinite grid of boxes around the original box. Then go to the boxes that are diagonally 2 away from the origin. And in 2 dimensions, we found 2 solutions along each diagonal direction because we only had 2 boxes available to get one extra reflection. But in 3 dimensions, we have 3 boxes that will give us that one additional reflection.
So, we have 3 boxes along each diagonal direction. And we have a total of 8 diagonal directions. So, we have 24 solutions for N greater than or equal to 2.
Also, for N = 1, the number of solutions is just the number of faces of the cuboid, similar to the answer for the rectangular box. So, the number of solutions is 6 for N equal to 1.

Let’s generalize this. What should we do in a d-dimensional box? We will have d boxes along each diagonal direction. And now we need to calculate the total diagonal directions, which is equal to the number of corners in a d-dimensional hypercube. And we also need the number of faces of this box if N is equal to 1.
I will quickly explain how to calculate the number of corners in a d-dimensional hypercube. The idea is that every time you jump to a higher dimension, you are doubling the number of corners. That is why the total number of corners is 2 to the power d.
For the number of sides, the idea is that there will always be two faces along each dimension. Thus, the number of faces is 2 times d.

In d dimensions and for a given value of N such that N is a positive integer, the total number of light rays is d times 2 to the power of d if N is greater than or equal to 2. And 2 times d if N is equal to 1.

For the actual question, the value of d is 7 and N is 3. So, the final answer is 7 times 2 to the power of 7 which is equal to 896.

Thank you so much for watching. I will leave you with an animation of all the 8 solutions for N equal to 3 in 2 dimensions. You can actually see that the starting direction for all of these lines is always somewhat towards the corners. The codes for these animations are available on my GitHub page in the link in the description. Feel free to ask me questions about this video in the comments. Once again, thank you so much for watching, and I hope you have a great day!