SSY281
Assignment 1
Basic Control

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Introduction

This assignment involves the basic knowledge and application of Linear Control Systems like state space modelling of the given system in different equivalent forms such as continuous time and discrete time model. This also emphasises the method of extending a given state space model by adding an additional state. Furthermore, the concepts of controllability and observability are extended with an effort to understand their importance in the feedback controller design and linear observer design along with Set-point tracking and disturbance rejection.

Discrete State Space Model

1 Question 1

The given set of equation that define the dynamics of the pendulum are as shown in the Equation 1.

$$ml^2\ddot{\theta} = \tau + mgl\sin\theta \tag{1}$$

Where θ is the angle relative to the vertical direction and τ is the external torque input controlling the pendulum at the pivot point. On taking the state variables as θ and $\dot{\theta}$, the state space form of Equation 1 becomes, see Equation 2

$$\begin{aligned} \dot{x_1}(t) &= x_2(t) \\ \dot{x_2}(t) &= g/l \cdot x_1(t) + 1/ml^2 \cdot \tau(t) \\ y(t) &= x_1(t) \end{aligned} \tag{2}$$

As mentioned in the question, the terms g/l and $1/ml^2$ are assigned to variable α and β with values of 0.5 and 1 respectively. By taking the sampling time $h = 0.1$ seconds and using c2d command in MATLAB, a discrete time equivalent system of system 2 is designed as shown in Equation 3

$$
\begin{aligned}
x(k+1) &= Ax(k) + Bu(k) \\
y(k) &= Cx(k)
\end{aligned}
\tag{3}
$$

Where,

$$
A = \begin{bmatrix} 1.0025 & 0.1001 \\ 0.05 & 1.0025 \end{bmatrix} \quad B = \begin{bmatrix} 0.005 \\ 0.1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{4}
$$

2 Question 2

By inserting the delayed input as an additional state in the state vector of the system 3, the new system matrix A_a, input matrix B_a and output matrix C_a are calculated using the convolution solution of system 3 as shown in 5 and splitting its integral function about the time delay constant τ

$$
\begin{aligned}
x(k+1) &= e^{Ac(h-\tau)}x(kh+\tau) + \int_0^{h-\tau} e^{A_c s} B_c ds \cdot u(k) \\
&= e^{A_c h}x(kh) + e^{A_c(h-\tau)} \int_0^{\tau} e^{A_c s} B_c ds \cdot u(k-1) + \int_0^{h-\tau} e^{A_c s} B_c ds \cdot u(k) \\
&= Ax(k) + B1u(k-1) + B2u(k)
\end{aligned}
\tag{5}
$$

where,

$$
\begin{aligned}
&A = e^{A_c h}; \quad B1 = e^{A_c(h-\tau)} \int_0^{\tau} e^{A_c s} B_c ds; \quad B2 = \int_0^{h-\tau} e^{A_c s} B_c ds; \\
&A_a = \begin{bmatrix} A & & B1 \\ \\ 0 & 0 & 0 \end{bmatrix} \qquad B_a = \begin{bmatrix} B2 \\ \\ 0 \end{bmatrix} \qquad C_a = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
\zeta(k+1) &= A_a \zeta(k) + B_a u(k) \\
y(k) &= C_a x(k)
\end{aligned}
\tag{7}
$$

$$
A_a = \begin{bmatrix} 1.0025 & 0.1001 & 0.0038 \\ 0.05 & 1.0025 & 0.0501 \\ 0 & 0 & 0 \end{bmatrix} \quad B_a = \begin{bmatrix} 0.0013 \\ 0.005 \\ 0.1 \end{bmatrix} \quad C_a = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{8}
$$

Controllability and Observability

3 Question 3

The cotrollability and observability matrix of system 2, 3 and 7 are as shown in 9, 10 and 11 respectively.

$$S_{system_2} = \begin{bmatrix} 0 & 1.0 \\ 1.0 & 0 \end{bmatrix} \quad O_{system_2} = \begin{bmatrix} 1.0 & 0 \\ 0 & 1.0 \end{bmatrix} \tag{9}$$

$$S_{system_3} = \begin{bmatrix} 0.005 & 0.0150 \\ 0.10 & 0.1005 \end{bmatrix} \quad O_{system_3} = \begin{bmatrix} 1.0 & 0 \\ 1.002 & 0.100 \end{bmatrix} \tag{10}$$

$$S_{system_7} = \begin{bmatrix} 0.0012 & 0.01 & 0.02 \\ 0.05 & 0.100 & 0.101 \\ 1.0 & 0 & 0 \end{bmatrix} \quad O_{system_7} = \begin{bmatrix} 1.0 & 0 & 0 \\ 1.002 & 0.1 & 0.0037 \\ 1.01 & 0.2 & 0.0087 \end{bmatrix} \tag{11}$$

The rank of all the matrices shown in the Equations 9, 10 and 11 are found to be full rank, hence System 2, 3 and 7 are both controllable and observable.

4 Question 4

In order to make the system 2 as non observable, the observability matrix of the system must have a determinant equal to 0. To satisfy this condition, let C_x be the output matrix of the system 2 as shown in 12 and A_1 is the system matrix of the system 2 as shown in in 12. The observability matrix of this system O(A_1,C_x) now becomes, see Equation 13

$$C_x = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \quad A_1 = \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix} \tag{12}$$

$$O_{system_1} = \begin{bmatrix} C_x \\ C_x A_1 \end{bmatrix} \tag{13}$$

On equating the determinant of the matrix in 13 to 0, we get the relation between two elements of C_x matrix as shown in 14

$$x_1 = \pm\sqrt{0.5} \cdot x_2 \tag{14}$$

Originally the C matrix for system 2 has x_1 as 1, which implies from 14 that x_2 must be $\sqrt{2}$, hence the new non-zero C matrix for which the system 2 becomes unobservable is as shown in 15

$$C_{unobservable} = \begin{bmatrix} 1 & \sqrt{2} \end{bmatrix} \tag{15}$$

5 Question 4.1

The system 2 and 3 represent the same system with the same set of states but in continuous time and discrete time model respectively. Hence the output matrix C which makes the system 2 unobservable can make the system 3 unobservable as well.

6 Question 4.2

System 7 is an extended state model of the system 2. Hence the same output matrix cannot be used for 7.

7 Question 5

Let $y(k)$ = a, $y(k+1)$ = b, $u(k)$ = c, in 3, where a, b and c are the known values as mentioned in the question. On unwinding the equation from the state space model of system 3 by substituting the output matrix C as the matrix calculated in 15, we get these set of equations as shown in 16. On solving these equations the states $x_1(k)$ and $x_2(k)$ tend to cancel each other out. This occurrence can be credited for the use of the C matrix causing unobservability of the states.

$$\begin{aligned} b &= x_1(k+1) + \sqrt{2}x_2(k+1) \\ a &= x_1(k) + \sqrt{2}x_2(k) \\ x_1(k+1) &= x_1(k) + 0.1x_2(k) + 0.005 \cdot c \\ x_2(k+1) &= 0.05x_1(k) + x_2(k) + 0.1c \end{aligned} \tag{16}$$

Feedback Design

8 Question 6

The controller gain K for the closed loop discrete system 3 to have its poles equivalent to the poles of continuous system 2 at $-4 \pm 6i$ is as shown in 17 which makes the poles of the system 3 to be $0.5532 \pm 0.378i$

$$K_3 = \begin{bmatrix} 34.77 & 7.24 \end{bmatrix} \tag{17}$$

The step response for the system 3 and 7 with controller K_3 is as shown in the figure 1 and 2 respectively.

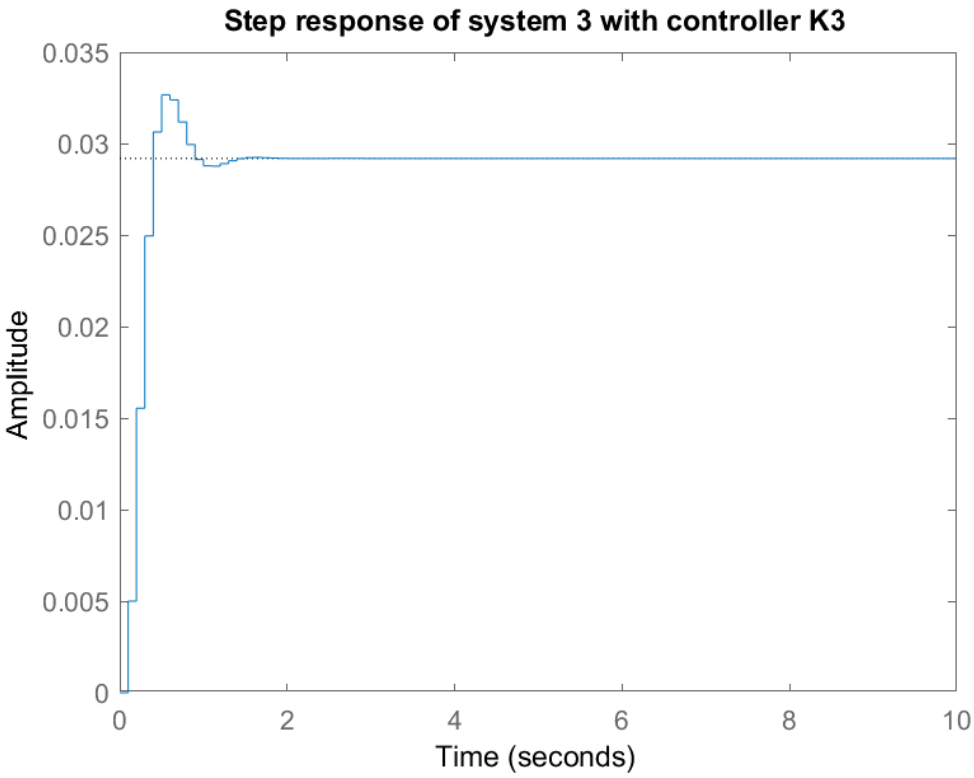
Step response of system 3 with controller K3
Amplitude
Time (seconds)
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
0
2
4
6
8
10

Figure 1

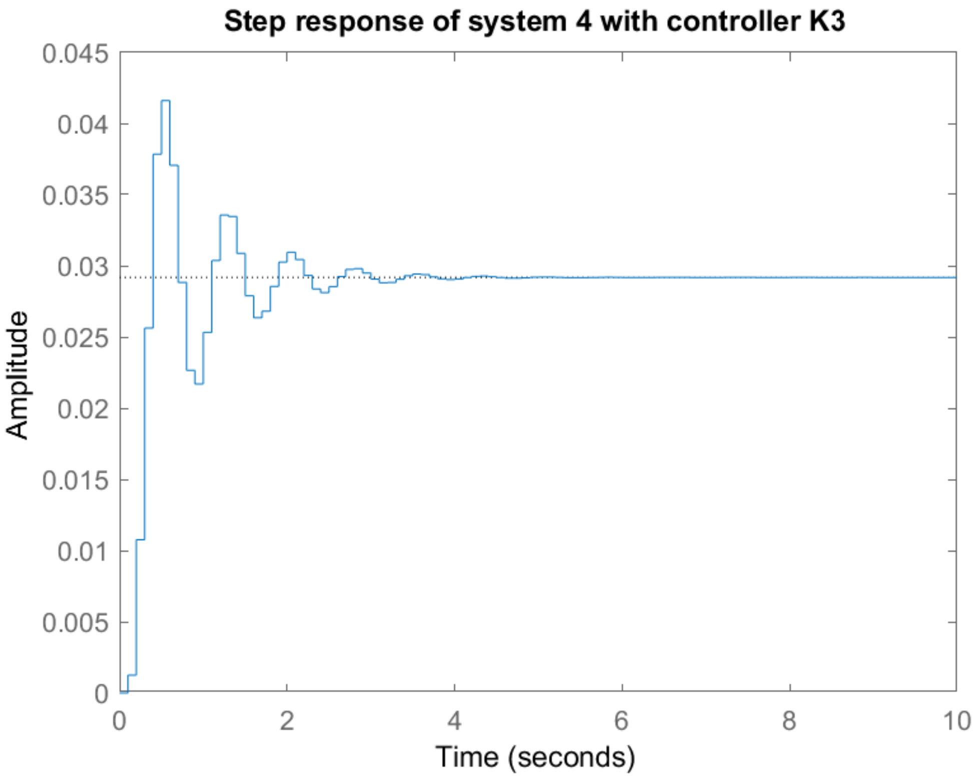

Figure 2

The step response in system 7 seems to take longer time to stabilize its response than the system 3 because of the presence of delay in the input signal in 7.

In order to make the system 7 recover the properties of the system 3, a new controller is designed as shown in the equation 18. On comparing the responses of the system 7 with the previous calculated controller K_3 and new controller K_7 as shown in Figure 3, one can observe that the system 7 stabilizes its response deviation much faster than the one with the previous calculated controller K_3. This is due to the fact that the the new controller K_7 can account for the effects of the delayed input state as seen in 18 which is nit controlled by the previously calculated controller as seen in 17

$$K_7 = \begin{bmatrix} 34.9736 & 8.9834 & 0.4055 \end{bmatrix} \tag{18}$$

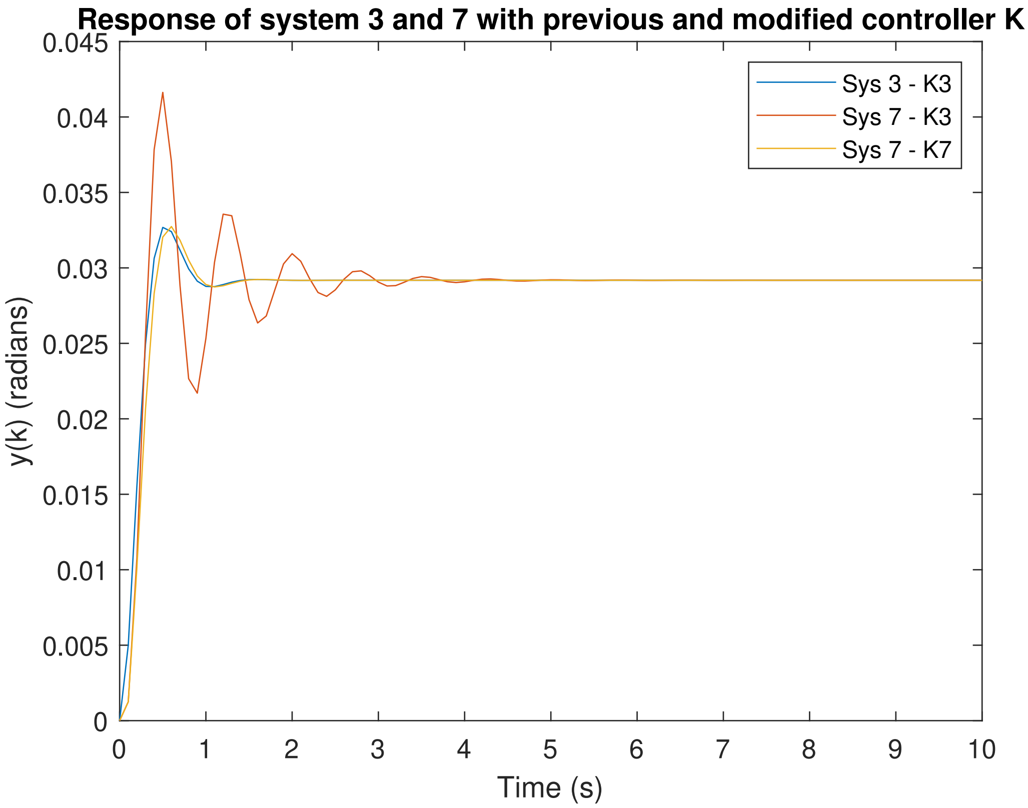

Figure 3

Set-point tracking and disturbance rejection

9 Question 7

The input required to maintain the steady response of the system 7 at $y_s = \frac{\pi}{6}$ is 17.94 which gives output plot made using for loop in MATLAb as shown in the Figure 4

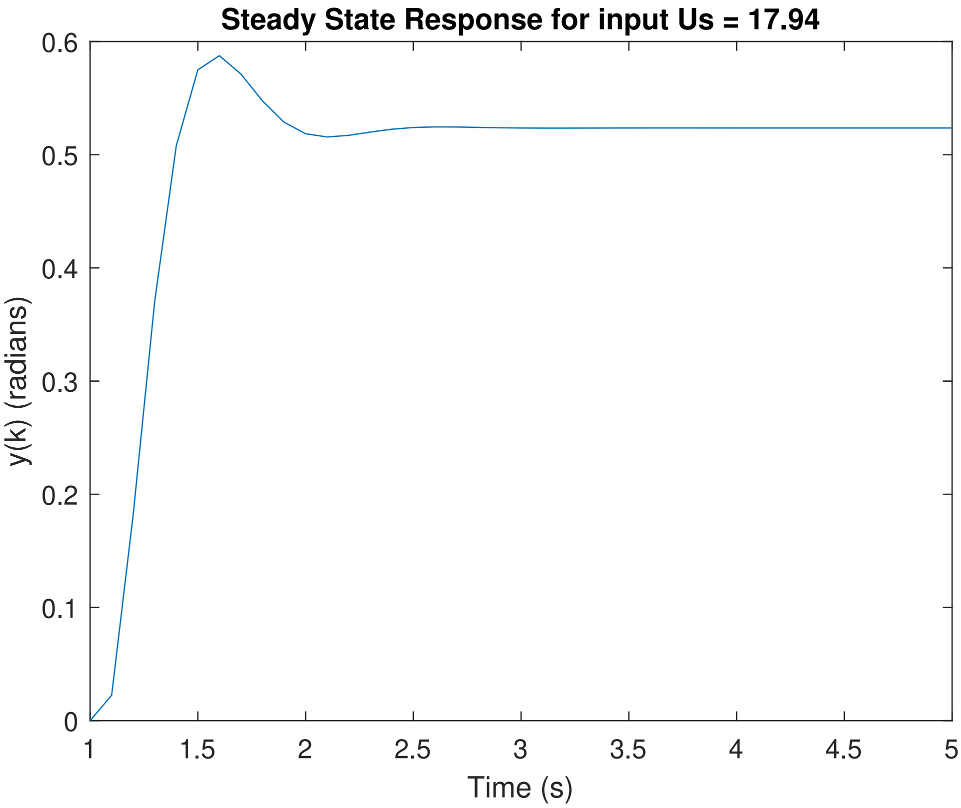

Figure 4: Steady state response maintained at $y_s = \frac{\pi}{6}$

10 Question 8

On introducing the disturbance signal as a state vector to the system 7, the model gets extended as shown in 19

$$\begin{aligned} \xi(k+1) &= A_e \xi(k) + B_e u(k) \\ y(k) &= C_e \xi(k) \end{aligned} \qquad \xi(k) = \begin{bmatrix} \zeta(k) \\ d(k) \end{bmatrix} \tag{19}$$

where,

$$A_e = \begin{bmatrix} 1.002 & 0.1 & 0.004 & 0 \\ 0.05 & 1.002 & 0.05 & 1.0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0 \end{bmatrix} \quad B_e = \begin{bmatrix} 0.0012 \\ 0.05 \\ 1.0 \\ 0 \end{bmatrix} \quad C_e = \begin{bmatrix} 1.0 & 0 & 0 & 0 \end{bmatrix} \tag{20}$$

The controllability matrix and the observability matrix of the system 19 is as shown in the Equation 21. One can observe that the controllability matrix here is not a full rank matrix but the observabilty matrix is. Hence, the system is not stabilizable, but detectable.

$$S_{19} = \begin{bmatrix} 0.001 & 0.01 & 0.02 & 0.03 \\ 0.05 & 0.10 & 0.101 & 0.102 \\ 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad O_{19} = \begin{bmatrix} 1.0 & 0 & 0 & 0 \\ 1.002 & 0.100 & 0.003 & 0 \\ 1.01 & 0.2 & 0.008 & 0.1 \\ 1.022 & 0.302 & 0.013 & 0.3 \end{bmatrix} \tag{21}$$

11 Question 9

The controller gain for the system 19 with its last pole at 1 is as shown in 22

$$K_{19} = \begin{bmatrix} 34.9736 & 8.9834 & 0.4055 & -26.8717 \end{bmatrix} \tag{22}$$

A system's controllability matrix must be a full rank matrix in order to place its pole arbitarily. Since system 19 is not controllable, the last pole of this system cannot be placed arbitarily somewhere else.

12 Question 10

The linear observer for the system 19 is as shown in Equation 23

$$L = \begin{bmatrix} 2.0050 & 13.6608 & -1.9198 & 3.0215 \end{bmatrix}^T \tag{23}$$