SSY281 Model Predictive Control

Assignment 1

Basic Control

Deadline: January 29, 23:59

Systems & Control

Department of Electrical Engineering

Chalmers University of Technology

January 2020

Instructions

This assignment is **individual** and must be solved according to the following rules and instructions:

- Written report:
  - For those questions where a text (motivations, explanations, observations from simulations) rather than a numerical value is asked, the answers should be concisely written in the report.
  - Figures included in the report should have legends, and axes should be labelled.
  - The report should be uploaded *before the deadline* in Canvas.
  - Name the report as A1_X.pdf, where X is your *group* number (see Canvas page).
- Code:
  - Your code should be written in the Matlab template provided with this assignment following the instructions therein.
  - Name the Matlab script as A1_X.m, where X is your *group* number.
- Grading:
  - This assignment if worth **10 points** in total. Each question is worth 1 point.

Objectives

The purpose of this assignment is to refresh your knowledge about basic control concepts that are needed in the course.

1 Discrete state-space model

A pendulum with length l and point mass m, subject to gravity force and controlled by a motor at the pivot point, giving an external torque τ can be described by the following differential equation:

$$ml^2\ddot{\theta} = \tau + mgl\sin\theta, \tag{1}$$

where θ is the angle relative to the vertical direction. By defining the state variables $x_1 = \theta$ and $x_2 = \dot{\theta}$, and using the small angle approximation $\sin\theta \approx \theta$, an approximate linear model in the state-space form can be rewritten as

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t), \\
\dot{x}_2(t) &= g/l \cdot x_1(t) + 1/ml^2 \cdot \tau(t) = \alpha x_1(t) + \beta u(t), \\
y(t) &= x_1(t),
\end{aligned} \tag{2}$$

where $u(t) = \tau(t)$.

Question 1. *Using the largest possible sampling interval $h = 0.1$ and $\alpha = 0.5$, $\beta = 1$ in (2), find the matrices A, B, C in the the following discrete-time model.*

$$\begin{aligned}
x(k+1) &= Ax(k) + Bu(k), \\
y(k) &= Cx(k).
\end{aligned} \tag{3}$$

Hint: the commands `sys=ss(Ac,Bc,Cc,Dc)` and `sysd=c2d(sys,h)` in MATLAB can be used.

Question 2. *In case that the continuous system (2) has an input delay of $0.5h$ second, calculate A_a, B_a, C_a in the following model using MATLAB.*

$$\begin{aligned}
\zeta(k+1) &= A_a\zeta(k) + B_a u(k) \\
y(k) &= C_a\zeta(k)
\end{aligned}, \quad \zeta(k) = \begin{bmatrix} x(k) \\ u(k-1) \end{bmatrix} \tag{4}$$

Hint: The commands `expm` and `int` in MATLAB can be used.

2 Controllability and Observability

Question 3. *Check the controllability and the observability of systems* (2), (3)*, and* (4)*.*

Hint: The commands `ctrb`, `obsv`, and `rank` in MATLAB to check can be used.

Question 4. *Find a non-zero matrix C for the system* (2)*, i.e. redefine $y(t)$, such that the system is not observable.*

Answer the following questions:

1. *Can you conclude that the same output (C matrix) makes system* (3) *unobservable?*

2. *Can you conclude that the same output (C matrix) makes system* (4) *unobservable?*

Briefly motivate your answers in the report.

Question 5. *Calculate $x(k)$ of system* (3) *given $y(k)$, $y(k+1)$, and $u(k)$. Try to calculate the state when C is defined as in the previous question and explain your outcome.*

3 Feedback Design

In this section, you use the *pole placement* technique to design l*inear controllers and observers* for the pendulum.

Hint: The commands `place` and `step` in MATLAB can be used for pole-placement and plotting the step response, respectively.

Question 6. *Consider the discrete-time model* (3) *of the pendulum in closed-loop with the state-feedback control law $u(k) = -Kx(k)$.*

Find the controller gain K such that the poles of the closed-loop system in discrete time matches the complex poles $\lambda_{1,2} = -4 \pm 6i$, given for the continuous-time system (2)*.*

Plot the step responses of the closed loop discrete-time systems (3) *and* (4)*. Note to use the same feedback policy in both cases, i.e., for the latter, append one zero to the K vector of the former! Explain the difference you observe.*

For the delayed system (4)*, find a new state-feedback gain to recover the closed-loop behavior of system* (3)*. Plot a the two previous and the new step responses in one figure (on top of each other not in different subplots!), compare the results and explain your observations in the report.*

4 Set-point tracking and disturbance rejection

In this section, we try to regulate the pendulum angle at a desired position and reject an input disturbance. For the questions in this section, simulate your system using a simple `for` loop and without `step` function.

Question 7. *For system* (4)*, find the steady-state and input* (x_s, u_s)*, when* $y_s = \frac{\pi}{6}$*. Use these steady-state input and output and the feedback gain that you calculated in the last part of the previous question to regulate the output to its desired value when the initial condition is zero. Plot the output and explain your observations in the report.*

Question 8. *Consider the following discrete model where* $d(k)$ *is a constant disturbance.*

$$\begin{aligned} \zeta(k+1) &= A_a\zeta(k) + B_a u(k) + B_d d(k) \\ y(k) &= C_a \zeta(k) \end{aligned}, \quad \zeta(k) = \begin{bmatrix} x(k) \\ u(k-1) \end{bmatrix}, \quad B_d = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{5}$$

Augment the state vector with the disturbance and define the new augmented system as

$$\begin{aligned} \xi(k+1) &= A_e\xi(k) + B_e u(k) \\ y(k) &= C_e \xi(k) \end{aligned}, \quad \xi(k) = \begin{bmatrix} \zeta(k) \\ d(k) \end{bmatrix}. \tag{6}$$

Find the matrices A_e*,* B_e*, and* C_e *and check the stabilizability and the detectability of the augmented system.*

Question 9. *Consider the discrete-time system* (6)*. Design a linear controller to place the three poles of the closed-loop system as in the last part of **Question 6** and the fourth pole at one (provide feedback gain* K*). Could you place the last pole somewhere else? Why?*

Consider system

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k), \\ y(k) &= Cx(k), \end{aligned} \tag{7}$$

is detectable; then, a linear observer can be designed to estimate the system states, based on its measured output and input.

The observer includes a "copy" of the system dynamics and a difference, between the estimated and measured outputs, to adjust the estimated state as follows

$$\begin{aligned} \hat{x}(k+1) &= A\hat{x}(k) + Bu(k) + L(y(k) - \hat{y}(k)), \\ \hat{y}(k) &= C\hat{x}(k), \end{aligned}, \tag{8}$$

where L is a constant matrix which is called the observer gain. By defining the estimation error as $e(k) = x(k) - \hat{x}(k)$, one can calculate the error dynamics as

$$e(k+1) = (A - LC)e(k). \tag{9}$$

Consequently, by designing L such that $e(k)$ is exponentially stable, then the error converges to zero and the state estimation converges to the real state. The observer gain L can be calculated with the `place` command in MATLAB, by using $A^\top$ and $C^\top$ instead of A and B as input arguments, while the output is $L^\top$.

Question 10. *Design a simple linear observer for the augmented system* (6) *with poles placed at* $\{0.1, 0.2, 0.3, 0.4\}$.