SSY281
Assignment 5
MPC Stability

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Stability

1 Question 1

For the given system which is in the form $x^+ = Ax$, the corresponding Lyapunov function can be derived using Lyapunov's equation as shown in Equation 1

$$\begin{aligned} V(x^+) = (x^+)^T S(x^+) = (Ax)^T S(Ax) = x^T A^T SAx \\ = x^T Sx - x^T Qx \\ \implies S = A^T SA + Q \end{aligned} \tag{1}$$

By plugging in the given values of A and taking Q which is the cost function as an identity matrix, in the Equation 1, the S matrix in the Lyapunov function $V = x^T Sx$ is found to be as shown in Equation 2

$$S = \begin{bmatrix} 1.289 & 0.592 \\ 0.592 & 2.631 \end{bmatrix} \tag{2}$$

2 Question 2

The Lyapunov function in its common quadratic form appears as $V = x^T Sx$ with S being a positive definite matrix. On these lines, the given function $V(u, x_0)$ reflects the solution of the optimization criterion of an Infinite horizon LQ control whose optimal cost to go function is given as shown in Equation 3

$$V^0_\infty(x(0)) = x^T(0)Px(0) \tag{3}$$

This optimal cost to go function is the obtained by minimizing the given optimization criterion of the infinite horizon LQ control where P is the solution of the algebraic Riccati equation which is calculated by using dare command in MATLAB and it turns out to be a positive definite matrix as shown below, which proves that the obtained solution on minimizing the given optimization criterion is a Lyapunov function, by definition.

$$P = \begin{bmatrix} 13.65 & 7.99 \\ 7.99 & 39.43 \end{bmatrix} \tag{4}$$

3 Question 3a

Using the DP_57 function which performs dynamic programming of the given RH controller, the shortest length of prediction horizon N was found to be 5, above which the closed loop poles of the system tends to fall outside the unit circle of stability region.

4 Question 3b

With $N = 1$, changing values of Q seemed to have no effect on the closed loop poles hence the stability of the system remained unaffected.

5 Question 3c

At N = 1, the state gets close enough to origin and the constraints will no loger be active making the constrained LQ identical to unconstrained LQ which allows us to rely on the properties of infinite horizon LQ controller by choosing the terminal cost as the value function for the unconstrained LQ problem as shown below

$$V_f(x) = V_{\infty}^{uc}(x) = x^T P x \tag{5}$$

where P is the solution of algebraic Riccati equation which is obtained by using the dare command in MATLAB, hence the logic behind the formulation of function Pf_57.

6 Question 3d

Alternate value of R for which system is stable at $N = 1$ is 0.1.

7 Question 4

The feedback gain of the unconstrained LQ controller K is as shown in Equation 8

$$K = -(B^T P_f B + R)^{-1} B^T P_f A \tag{6}$$

On plugging in the values of the respective values of the matrices in Equation 8 and following the parameter conditions as given in the question, the value of the controller gain is found to be 0. Therefore, the closed loop poles which are the eigen values of the closed loop system matrix $A + BK$ takes the form as shown below in 7

$$\lambda_i(A + BK) = \lambda_i(A + 0) = \lambda_i(A) \tag{7}$$

Since the open loop system matrix A is said to be unstable, and following the situation shown in 7, the established RHC cannot control the system, regardless of the values of Q and R.

8 Question 5a

Using Equation 8 and the condition for the closed loop system to be stable i.e.

$$\lambda_i(A + BK) < 1 \tag{8}$$

at N = 1, using the values of the given matrices, the characteristic equation of the closed loop system $A + BK$ is derived by taking the determinant of the matrix shown in Equation 9

$$\begin{bmatrix} 1-\lambda_1 & 1 \\ \frac{R}{R+1} & \frac{5R}{R+1} - \lambda_2 \end{bmatrix} = \mathbf{0} \tag{9}$$

Since R should be positive, from the following relation obtained by one of the roots of the characteristic equation derived from 9, the range of R required for the system to be stable is found to be $0 < R < 0.2$ as shown below,

$$\begin{aligned} \frac{5R}{R+1} + \frac{R}{R+1} &< 1 \\ R &< \frac{1}{5} \end{aligned} \tag{10}$$

9 Question 5b

Following the similar approach, at N = 2, but with a change in the formulation of controller gain K(0) as,

$$K(0) = -(R + B^T P(1)B)^{-1} B^T P(1)A \tag{11}$$

where P(1) is given as

$$P(1) = Q + A^T P_f A - A^T P_f B(R + B^T P_f B)^{-1} B^T P_f A \tag{12}$$

again by substituting the values of the appearing matrices in the equation, the values of P1 and K0 were found as shown in Equation 13

$$\begin{aligned} P1 &= \begin{bmatrix} 3 - \frac{1}{R+1} & \frac{5}{R+1} - 4 \\ \frac{5}{R+1} - 4 & 27 - \frac{25}{R+1} \end{bmatrix} \\ K0 &= \begin{bmatrix} \frac{31\,R+1}{R^2+28\,R+2} & -\frac{131\,R+11}{R^2+28\,R+2} \end{bmatrix} \end{aligned} \tag{13}$$

Using these values and evaluating the eigen values of the matrix as shown in Equation 8, the following relation was established

$$\begin{bmatrix} \frac{37\,R+\sqrt{12\,R^4-252\,R^3+661\,R^2+26\,R+1}+6\,R^2+1}{2\,(R^2+28\,R+2)} \\ \frac{37\,R-\sqrt{12\,R^4-252\,R^3+661\,R^2+26\,R+1}+6\,R^2+1}{2\,(R^2+28\,R+2)} \end{bmatrix} < \mathbf{1} \tag{14}$$

On solving this equation and accounting for the real roots and complex roots, interestingly the solution converges according to the condition at two ranges of R. For real roots/strictly real valued closed loop poles, the range of R was found to be

$$0 < R < 0.3819 \tag{15}$$

For the complex roots within the unit circle, the range of R was found to be

$$2.618 < R < 4.481 \tag{16}$$

10 Question 6

The stability of the system can be proved with an approach of investigating the global assymptotic stability and the decaying property of the framed Lyapunov functions in this question. With $V(x) = x^T Px$, the lyapunov function of the evolved state would be $V(x^+) = (A + BK)^T x^T P(A + BK)x$. Therefore in order to investigate the decaying property of the lyapunov function i.e if $V(x^+) - V(x) \leq 0$, the values of the involved matrices and variables were substituted and analysed as follows

$$\begin{aligned} V(x^+) - V(x) &= (A + BK)^T x^T P(A + BK)x - x^T Px \\ &= x^T (A^T PA + B^T K^T PBK - A^T PBK - K^T B^T PA - P)x \end{aligned} \tag{17}$$

From the algebraic riccati equation solution, the solutions P and K can be rewritten as

$$\begin{aligned} P &= Q + A^T PA - A^T PBK \\ -A^T PBK &= -Q + P - A^T PA \\ K &= -(B^T PB + R)^{-1} B^T PA \\ RK &= B^T P(A - BK) \end{aligned} \tag{18}$$

From Equation, 17 and 18, following equations hold,

$$\begin{aligned} V(x^+) - V(x) &= x^T(-Q - K^T B^T P(A - BK))x \\ &= x^T(-Q - K^T RK)x < 0 \end{aligned} \tag{19}$$

The resulting terms on the RHS of Equation 19 involve Q and R which are positive definite associated with negative signs, which implies that the decaying property of the formulated Lyapunov functions holds, which is true only when all the closed loop poles of the system are within the unit circle of stability region hence, proving the stability of the system.