SSY281 Model Predictive Control

Assignment 5

MPC Stability

Due February 26 at 23:59

Systems & Control

Department of Electrical Engineering

Chalmers University of Technology

February 2020

Instructions

This assignment is **individual** and must be solved according to the following rules and instructions:

- Written report:
  - For those questions where a text (motivations, explanations, observations from simulations) rather than a numerical value is asked, the answers should be concisely written in the report.
  - Figures included in the report should have legends, and axes should be labeled.
  - The report should be uploaded *before the deadline* in Canvas.
  - Name the report as A5_X.pdf, where X is your *group* number.
- Code:
  - Your code should be written in the Matlab template provided with this assignment following the instructions therein.
  - Name the Matlab script as A5_X.m and Pf_XX.m where X is your *group* number.
  - Strictly follow the instructions in the Matlab template.
- Grading:
  - This assignment if worth **15 points** in total.

1 Stability

Question 1 (Points: 1). *Consider the discrete-time system*

$$x(k+1) = \begin{bmatrix} 0.5 & 1 \\ -0.1 & 0.2 \end{bmatrix} x(k), \tag{1}$$

where $x(k) \in \mathbb{R}^2$ *and* $k \geq 0$. *Find a quadratic Lyapunov function, i.e.,* $V(x(k)) = x(k)^\top S x(k)$, *for the system to prove its stability.*

Question 2 (Points: 1). *Consider the discrete-time system*

$$x(k+1) = Ax(k) + Bu(k), \tag{2}$$

where

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 5 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; k \geq 0.$$

Show that $V(u, x_0)$ *in* (3) *with the parameters in* (4) *is a Lyapunov function for the system* (2) *when the controller is obtained by minimizing* $V(u, x_0)$.

$$V(u, x_0) = \sum_{i=0}^{\infty} \big(x(i)^T Q x(i) + u(i)^T R u(i) \big), \tag{3}$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; R = 1. \tag{4}$$

Question 3 (Points: 3). *Consider system* (2), *the cost function* (5), *and given parameters in* (4), (6).

$$V_N(x) = x(N)^T P_f x(N) + \sum_{i=0}^{N-1} \big(x(i)^T Q x(i) + u(i)^T R u(i) \big), \tag{5}$$

$$P_f = Q. \tag{6}$$

(a) *Find the shortest* N *such that the RH controller designed with the cost* (5) *and the system and weighting matrices in* (4) *stabilizes the system.*

(b) *What is the effect of* Q *on the stability when* $N = 1$?

(c) *Find a* P_f *for which the system is stable with* $N = 1$. *Fill in the function* `Pf_X.m` *which takes* A, B, Q, *and* R *as inputs and returns* P_f *for which the system is stable with* $N = 1$. *Concisely motivate your answer in the report.*

(d) *Consider given parameters in* (4) *and* (6)*; find another* R *for which the system is stable with* $N = 1$.

Question 4 (**Points: 3**). *Consider the following parameters for system* (2)

$$A = \begin{bmatrix} \bar{A}_{(n-1)\times n} \\ 0_{1\times n} \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \ P_f = diag(\begin{bmatrix} p_1 & p_2 & \dots & p_n \end{bmatrix}), \ N = 1,$$

where $\bar{A}$ *is an arbitrary matrix,* P_f *is a positive definite diagonal matrix,* (A, B) *are controllable, and* A *is unstable. Show that a RH controller cannot stabilize the system with the cost function* (5) *regardless of* Q *and* R.

Question 5 (**Points: 4**). *Consider system* (2)*, the cost function* (5)*, and given parameters in* (4) *and* (6).

(a) *Find the range of* R *for which the system is stable with* $N = 1$.

(b) *Find the range of* R *for which the system is stable with* $N = 2$.

Question 6 (**Points: 3**). *Show that system* (2) *is stable when controlled with an infinite-time LQ controller. Note that* R *and* Q *are positive definite.* ***Hint****: Use* $V(x) = x^T P x$ *as the Lyapunov function where matrix* P *is the solution to the Riccati equation.*