1) Design a PDA ✓(Problem)

2) PDA to CFG (Proofs) (or) (Algorithm)

3) CFG to PDA (Proofs) (or) (Algorithm)

4) Pumping lemma for CFL ✓(Problem)

5) Closure properties of CFL (Theory)

6) Turing machines ✓(Proof & Problem)

7) Decidability & Undecidability.

8) Post correspondance & Halting problem

Now, we can prove that if $w \in N(A)$ then $w \in L(G)$. As $w \in N(A)$, we have $(q, w, S) \overset{*}{\vdash} (q, \Lambda, \Lambda)$. By taking $u = w$, $v = \Lambda$, $\alpha = \Lambda$ and applying (7.25), we get $S \overset{*}{\Rightarrow} w\Lambda = w$, i.e. $w \in L(G)$. Thus,

$$L(G) = N(A)$$

Theorem 7.4 If $A = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ is a pda, then there exists a context-free grammar G such that $L(G) = N(A)$.

Proof We first give the construction of G and then prove that $N(A) = L(G)$.

Step 1 (Construction of G). We define $G = (V_N, \Sigma, P, S)$, where

$$V_N = \{S\} \cup \{[q, Z, q'] \mid q, q' \in Q, Z \in \Gamma\}$$

i.e. any element of V_N is either the new symbol S acting as the start symbol for G or an ordered triple whose first and third elements are states and the second element is a pushdown symbol.

The productions in P are induced by moves of pda as follows:

R_1: S-productions are given by $S \to [q_0, Z_0, q]$ for every q in Q.

R_2: Each move erasing a pushdown symbol given by $(q', \Lambda) \in \delta(q, a, Z)$ induces the production $[q, Z, q'] \to a$.

R_3: Each move not erasing a pushdown symbol given by $(q_1, Z_1Z_2 \ldots Z_m) \in \delta(q, a, Z)$ induces many productions of the form

$$[q, Z, q'] \to a[q_1, Z_1, q_2][q_2, Z_2, q_3] \ldots [q_m, Z_m, q']$$

where each of the states $q', q_2, \ldots, q_m$ can be any state in Q. Each move yields many productions because of R_3. We apply this construction to an example before proving that $L(G) = N(A)$.

EXAMPLE 7.8

Construct a context-free grammar G which accepts $N(A)$, where

$$A = (\{q_0, q_1\}, \{a, b\}, \{Z_0, Z\}, \delta, q_0, Z_0, \emptyset)$$

and δ is given by

$$\delta(q_0, b, Z_0) = \{(q_0, ZZ_0)\}$$
$$\delta(q_0, \Lambda, Z_0) = \{(q_0, \Lambda)\}$$
$$\delta(q_0, b, Z) = \{(q_0, ZZ)\}$$
$$\delta(q_0, a, Z) = \{(q_1, Z)\}$$
$$\delta(q_1, b, Z) = \{(q_1, \Lambda)\}$$
$$\delta(q_1, a, Z_0) = \{(q_0, Z_0)\}$$

Solution

Let

$$G = (V_N, \{a, b\}, P, S)$$

where V_N consists of S, $[q_0, Z_0, q_0]$, $[q_0, Z_0, q_1]$, $[q_0, Z, q_0]$, $[q_0, Z, q_1]$, $[q_1, Z_0, q_0]$, $[q_1, Z_0, q_1]$, $[q_1, Z, q_0]$, $[q_1, Z, q_1]$.

The productions are

$$P_1: S \to [q_0, Z_0, q_0]$$

$$P_2: S \to [q_0, Z_0, q_1]$$

$\delta(q_0, b, Z_0) = \{(q_0, ZZ_0)\}$ yields

$$P_3: [q_0, Z_0, q_0] \to b[q_0, Z, q_0][q_0, Z_0, q_0]$$

$$P_4: [q_0, Z_0, q_0] \to b[q_0, Z, q_1][q_1, Z_0, q_0]$$

$$P_5: [q_0, Z_0, q_1] \to b[q_0, Z, q_0][q_0, Z_0, q_1]$$

$$P_6: [q_0, Z_0, q_1] \to b[q_0, Z, q_1][q_1, Z_0, q_1]$$

$\delta(q_0, \Lambda, Z_0) = \{(q_0, \Lambda)\}$ gives

$$P_7: [q_0, Z_0, q_0] \to \Lambda$$

$\delta(q_0, b, Z) = \{(q_0, ZZ)\}$ gives

$$P_8: [q_0, Z, q_0] \to b[q_0, Z, q_0][q_0, Z, q_0]$$

$$P_9: [q_0, Z, q_0] \to b[q_0, Z, q_1][q_1, Z, q_0]$$

$$P_{10}: [q_0, Z, q_1] \to b[q_0, Z, q_0][q_0, Z, q_1]$$

$$P_{11}: [q_0, Z, q_1] \to b[q_0, Z, q_1][q_1, Z, q_1]$$

$\delta(q_0, a, Z) = \{(q_1, Z)\}$ yields

$$P_{12}: [q_0, Z, q_0] \to a[q_1, Z, q_0]$$

$$P_{13}: [q_0, Z, q_1] \to a[q_1, Z, q_1]$$

$\delta(q_1, b, Z) = \{(q_1, \Lambda)\}$ gives

$$P_{14}: [q_1, Z, q_1] \to b$$

$\delta(q_1, a, Z_0) = \{(q_0, Z_0)\}$ gives

$$P_{15}: [q_1, Z_0, q_0] \to a[q_0, Z_0, q_0]$$

$$P_{16}: [q_1, Z_0, q_1] \to a[q_0, Z_0, q_1]$$

P_1–P_{16} give the productions in P.

Using the techniques given in Chapter 6, we can reduce the number of variables and productions.

Step 2 Proof of the construction, i.e. $N(A) = L(G)$.

Before proving that $N(A) = L(G)$, we note that a variable $[q, Z, q']$ indicates that for the pda the current state is q and the topmost symbol in PDS is Z. In the course of a derivation, a state q' is chosen in such a way that the PDS is emptied ultimately. This corresponds to applying R_2. (Note that the production given by R_2 replaces a variable by a terminal.)