

## Unit-1

### A) Errors and Mathematical Review

(lesser important section)  
1 question for 5 marks  
may be asked from A section.

#### ① Error and Types of errors in numerical computing:-

Error → Error is defined as the measure of the estimated difference between the observed or calculated value of a quantity and its true value.

In any numerical analysis errors will arise during the calculation. Errors may come in various forms and sizes among them some are avoidable and some are not. To establish good numerical computing system we need to minimize the errors as per our requirement.

#### Types of Errors:

i) True Error → True error is denoted by  $E_t$ , is the difference between the true value and the approximate value.

i.e, True Error = True value (OR Exact value) - Approximate value

Eg. The derivative formula for function  $f(x)$  at a particular value of  $x$  can be approximately calculated by,

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

ii) Relative Error → Relative Error is denoted by  $E_r$  and is defined as the ratio between the true error and the true value.

i.e, Relative true Error =  $\frac{\text{True Error}}{\text{True Value}}$

E.g. true value = 9.514

true error = -0.75061 then,

$$\begin{aligned} \text{Relative error (E}_r\text{)} &= \frac{\text{true error}}{\text{true value}} \\ &= \frac{-0.75061}{9.514} \end{aligned}$$

$$= -0.078895$$

Relative error can also be represented in percentage, for e.g.

$$E_r = -0.078895 \times 100\%$$

$$= -7.8895\%$$

Absolute errors may also need to be calculated in such case as,

$$|E_r| = |-0.078895|$$

$$= 0.078895 \text{ OR } 7.8895\%$$

iii) Approximate error  $\rightarrow$  True errors can be calculated only if true values are known but mostly we don't know the true values. In such cases we want to find approximate value. When we are solving a problem numerically, we may only have access to approximate value. It is denoted by  $E_a$ , and is defined as the difference between the present approximation and previous approximation.

i.e., Approximate Error ( $E_a$ ) = Present approx. - previous approx.

E.g. The deviation of function  $f(x)$  at a particular value of  $x$  can be calculated approximately by,  $f'(x) = \frac{f(x+h) - f(x)}{h}$

iv) Relative Approximate Error ( $E_{ra}$ )  $\rightarrow$  Relative approximate error is denoted by  $E_{ra}$  and is defined as the ratio between the approximate error and the present approximation.

i.e.,  $E_{ra} = \frac{\text{Approximate error}}{\text{Present approximation}}$ .

E.g. from previous example;

$$\text{approximate error} = -0.38474$$

$$\text{present approximation} = 9.87799$$

Then, the relative approximate error is calculated as;

$$E_{ra} = \frac{-0.38474}{9.87799}$$

$$= -0.03894$$

Similarly as before this can also be represented in percentage as  $E_{ra} = -0.03894 \times 100\%$

$$= -3.894\%$$

& Absolute relative approximate error may also need to be calculated as;  $|E_{ra}| = |-0.03894|$   
 $= 0.03894 \text{ OR } 3.894\%$

Q1. For  $f(x) = 7e^{0.5x}$  and  $h = 0.3$ . Find the approximate value of  $f'(x)$ , if  $x = 2$ , the true value of  $f'(x)$  and the true error.

Solution: Approximate value of derivative is calculated by formula,

$$f'(x) = \frac{f(x+h) - f(x)}{h}$$

$$\text{Now, } x = 2 \text{ & } h = 0.3$$

$$\begin{aligned}
 f'(x) &\approx \frac{f(2+0.3) - f(2)}{0.3} \\
 &\approx \frac{f(2.3) - f(2)}{0.3} \\
 &\approx \frac{7e^{0.5 \times 2.3} - 7e^{0.5 \times 2}}{0.3} \\
 &\approx 10.256
 \end{aligned}$$

$$\therefore f'(2) \approx 10.256$$

Now,

Exact value of  $f'(x)$  at  $x=2$  can be calculated by using our knowledge of differential calculus;

$$f(x) = 7e^{0.5x}$$

$$\text{Then, } f'(x) = 7 \times 0.5 \times e^{0.5x}$$

$$\text{So, } f'(2) = 7 \times 0.5 \times e^{0.5 \times 2}$$

$$\begin{aligned}
 \therefore \text{true error} &= \text{true value} - \text{approximate value} \\
 &= 9.514 - 10.256
 \end{aligned}$$

$$E_t = -0.75061.$$

Q2. For  $f(x) = 7e^{0.5x}$  and  $x=2$ . Find the following;

a)  $f'(2)$  using  $h=0.3$

b)  $f'(2)$  using  $h=0.15$

c) Approximate error for the value of  $f'(2)$ .

Solution.

The approximate exp<sup>n</sup> for the derivative of function is;

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}$$

a) for  $x=2$  and  $h=0.3$

$$\begin{aligned}
 f'(2) &\approx \frac{f(2+0.3) - f(2)}{0.3} \\
 &= \frac{7e^{0.5 \times 2.3} - 7e^{0.5 \times 2}}{0.3} \\
 &= 10.285
 \end{aligned}$$

b) Similarly  $f'(2)$  at  $x=2$  and  $h=0.15$  is 9.8799.

c) The approximate error ( $E_a$ ) is = Present approximation - Previous approximation

$$= 9.8799 - 10.265$$

$$\begin{aligned}
 |E_a| &= |-0.38474| \\
 &= 0.38474
 \end{aligned}$$

## ~~laser imp.~~ • \* Numerical Computing, Process of Numerical Computing & Characteristics:

Numerical Computing → Numerical computing is an approach for solving complex mathematical problems using only simple arithmetic operations.

- The approach involves formulation of mathematical models of physical situations that can be solved with arithmetic operations.
- It requires development, analysis and use of algorithms.
- Numerical Computing Methods deals with the following topics:
  - \* finding the roots of equations.
  - \* Solving system of linear algebraic equations.
  - \* Interpolation and regression analysis.
  - \* Numerical Integration.
  - \* Numerical Differentiation etc.

### Process of Numerical Computing:

- Numerical Computing involves formulation of mathematical models of physical problems that can be solved by using basic mathematical operations.
- The problem of ~~in~~ numerical computing can be roughly divided into the following four phases.
  - a) Formulation of mathematical model.
  - b) Construction of approximate numerical method.
  - c) Implementation of method to obtain a solution.
  - d) Validation of the solution.

### Characteristics of Numerical Computing:

- 1) Accuracy → The results we obtain must be sufficiently accurate to serve the purpose for which mathematical model was build.
- 2) Efficiency → A method that requires less computing time and less programming effort and yet achieves the desired accuracy is always preferred.

iii) Rate of convergence → Many numerical methods are based on the idea of an iterative process. This process involves generation of a sequence of approximations with the hope that the process will converge to the required solution. The certain method converge faster than others. Some methods may not converge at all. It is therefore important to test for convergence before a method is used.

iv) Numerical stability → In some cases errors tend to grow exponentially with disastrous computational result. A computing process that shows such exponential growth is said to be numerically unstable. We must choose methods that are not only fast but also stable.

### Sources of errors:

Errors in solving an engineering or science problem can arise due to several factors. First the errors may be in the modeling technique (i.e, A mathematical model). Second errors may arise from mistakes in programs themselves or in measurement of physical quantities. But in application of numerical methods, the main two errors to be focussed are;

#### 1) Round off error:

- A computer can only represent a number approximately.
- Round off errors occur when a fixed number of digits are used to represent exact numbers.
- For example, a number like  $\frac{1}{3}$  may be represented as 0.33333 on PC. Then the round off error in this case is  $\frac{1}{3} - 0.33333 = 0.0000033$ .
- There are other numbers like  $\pi$  and  $\sqrt{2}$  that cannot be represented exactly.

#### 2) Truncation error:

- Truncation error is defined as the error caused by truncating mathematical procedure.
- For example, the Maclaurin series for  $e^x$  is given as;

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

→ This series has infinite number of terms but when using this series to calculate  $e^x$ , only a finite number of terms can be used. For example, if one uses three terms to calculate  $e^x$ , then,

$$e^x \approx 1 + x + \frac{x^2}{2!}$$

Now the truncation error for such an approximation is,

$$\text{Truncation error} = e^x - \left( 1 + x + \frac{x^2}{2!} \right),$$

$$= \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

### Propagation of Errors:

- If a calculation is made with numbers that are not exact, then the calculation itself will have an error.
- Propagation of error is the effect of variables errors on the uncertainty of a function based on them.
- When the variables are the values of experimental limitations which propagate to the combination of variables in the function.

Eg. Find the bounds for the propagation error in adding two numbers, if one is calculating  $X+Y$  where,  $X = 1.5 \pm 0.05$

Soln

By looking at the numbers, the maximum possible value of  $X$  and  $Y$  are  $X = 1.55$  and  $Y = 3.44$ .

$$\therefore Y = 3.4 \pm 0.04$$

The minimum possible values of  $X$  and  $Y$  are  $X = 1.45$  and  $Y = 3.66$ .  
Hence,

$$X+Y = 1.45 + 3.36$$

= 4.81 is the max value of  $X+Y$

Hence,

$$4.81 \leq X+Y \leq 4.99$$

## Review of Taylor theorem:

Taylor's theorem is given by,

$$f(x+h) = f(x) + f'(x)h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \dots$$

provided all derivatives of  $f(x)$  exist and are continuous between  $x$  and  $x+h$ .

This can also be written as:  $f(h+x) = f(h) + f'(h)x + \frac{f''(h)}{2!}x^2 + \frac{f'''(h)}{3!}x^3 + \dots$

Putting  $h=0$  we get MacLaurian series as;

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

Example: Find the value of  $e^{0.25}$  using the first five terms of MacLaurian series.

Solution: The first five terms of MacLaurian series for  $e^x$  can be calculated as.

$$f(x) = e^x, \Rightarrow f(0) = 1$$

$$f'(x) = e^x, \Rightarrow f'(0) = 1$$

$$f''(x) = e^x, \Rightarrow f''(0) = 1$$

$$f'''(x) = e^x, \Rightarrow f'''(0) = 1$$

$$f''''(x) = e^x, \Rightarrow f''''(0) = 1$$

Thus, from MacLaurian series,

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f''''(0)}{4!}x^4 + \dots$$

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$$

$$e^{0.25} \approx 1 + 0.25 + \frac{0.25^2}{2!} + \frac{0.25^3}{3!} + \frac{0.25^4}{4!}$$

$$e^{0.25} \approx 1.2840$$

The exact value of  $e^{0.25}$  up to 5 significant digits is 1.2840.

Note: If  $f(x) = \sin x \Rightarrow f(0) = 0$

then,  $f'(x)$  will be  $\cos x \Rightarrow f'(0) = 1$

$\therefore f''(x)$  will be  $-\sin x \Rightarrow f''(0) = 0$

$f'''(x)$  will be  $-\cos x \Rightarrow f'''(0) = -1$

## ④ Mean Value Theorem (Review)

Suppose  $f$  is a continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Draw the secant line joining the point  $(a, f(a))$  to point  $(b, f(b))$ . Clearly there's a point  $c$  where  $a < c < b$  such that the tangent line to the graph of  $f$  at  $x=c$  is parallel to the secant line. It has same slope. The slope of the secant line is  $\frac{f(b) - f(a)}{b-a}$  and that of the tangent line is  $f'(c)$ .

$$\text{So, } f'(c) = \frac{f(b) - f(a)}{b-a}$$

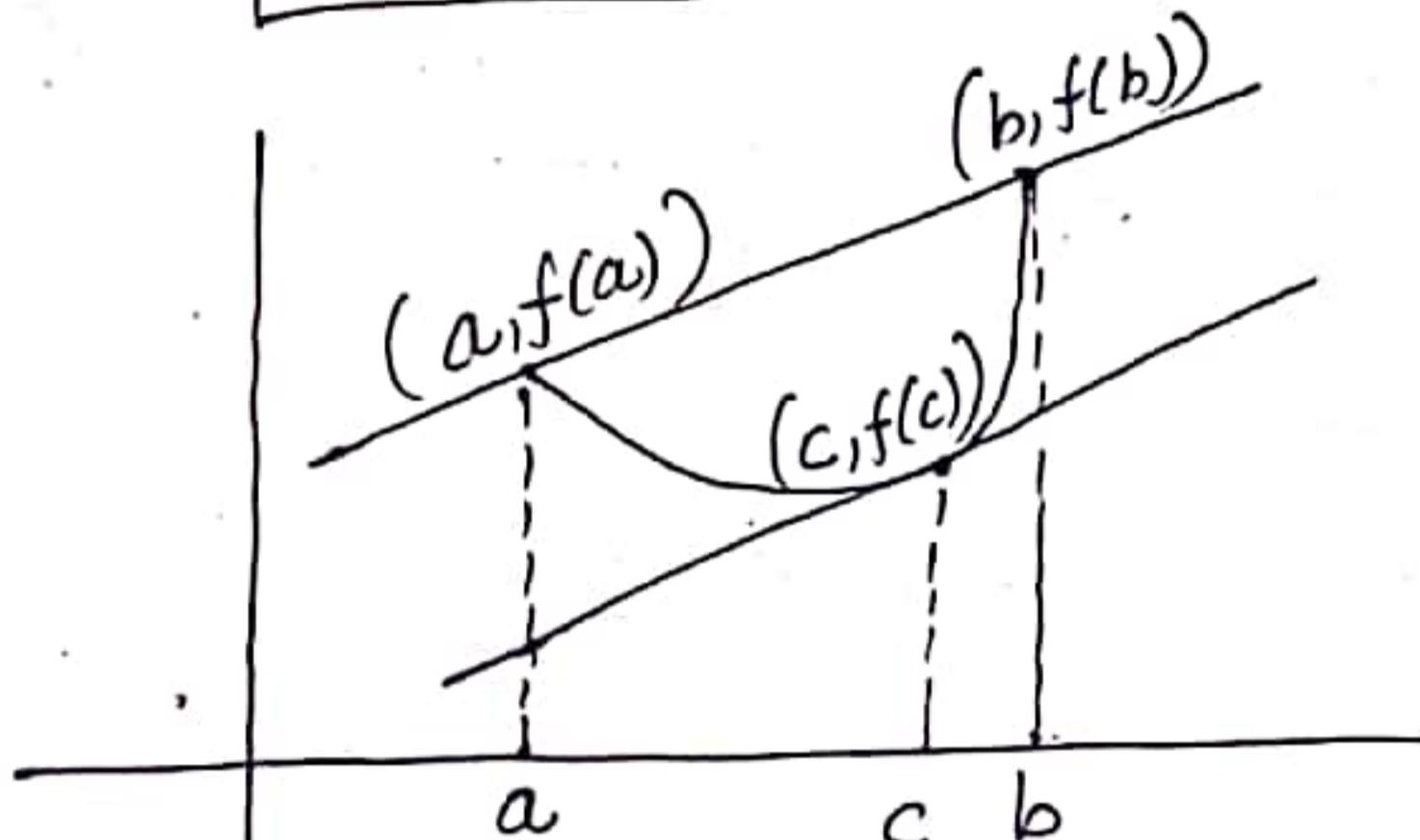


fig. Graphical Interpretation of Mean Value Theorem.

Example: Consider  $f(x) = x^3 + 3x^2$  on the interval  $-5 \leq x \leq 1$ . Since  $f$  is a polynomial,  $f$  is continuous on  $-5 \leq x \leq 1$  and differentiable on  $-5 < x < 1$ .

$$\frac{f(1) - f(-5)}{1 - (-5)} = \frac{4 + 50}{6} = 9.$$

So, we should be able to find a number  $c$  between  $-5$  and  $1$  such that  $f'(c) = 9$ .

$$f'(x) = 3x^2 + 6x, \text{ so, } f'(c) = 3c^2 + 6c.$$

Now, we set  $f'(c)$  equal to  $9$  and solve for  $c$ , we get.

$$\begin{aligned} 3c^2 + 6c &= 9 \\ \Rightarrow c^2 + 2c &= 3 \\ \Rightarrow c^2 + 2c - 3 &= 0 \\ \Rightarrow c &= -3 \text{ or } c = 1. \end{aligned}$$

$c = 1$  is not in the interval  $-5 < x < 1$ , but  $c = -3$  is. Therefore  $c = -3$  is a number satisfying the conclusion of MVT.

## Algorithm Analysis:

Complexity analysis of an algorithm is very hard if we try to analyze exact. We know that complexity of an algorithm is the mathematical function of the size of the input. So, if we analyze the algorithm in terms of bound (upper and lower) then it would be easier. For this purpose we need the concept of following asymptotic notations.

1) Big Oh ( $O$ ) notation → When we have only asymptotic upper bound then we use  $O$  notation. A function  $f(x) = O(g(x))$  [we read it as  $f(x)$  is big oh of  $g(x)$ ] iff there exists two constants  $c$  and  $k$  such that for all  $x \geq k$ ,  $f(x) \leq c * g(x)$ . This relation says that  $g(x)$  is upper bound of  $f(x)$ .

2) Big Omega ( $\Omega$ ) notation → Big omega gives asymptotic lower bound. A function  $f(x) = \Omega(g(x))$  [read as  $f(x)$  is big omega of  $g(x)$ ] iff there exists two positive constants  $c$  and  $k$  such that for all  $x \geq k$ ,  $c * g(x) \leq f(x)$ . This relation says that  $g(x)$  is lower bound of  $f(x)$ .

3) Big Theta ( $\Theta$ ) notation → When we need asymptotically tight bound then we use Big- $\Theta$ . A function  $f(x) = \Theta(g(x))$  [read as  $f(x)$  is big theta of  $g(x)$ ] iff there exists three positive constants  $c_1, c_2$  and  $k$  such that for all  $x \geq k$ ,  $c_1 * g(x) \leq f(x) \leq c_2 * g(x)$ . This relation shows that  $f(x)$  is order of  $g(x)$ .

## B). Solving Non-Linear Equations: (This onwards important)

Any function of one variable which does not graph as a straight line in two dimensions or any function of two variables which does not graph as a plane in three dimensions can be said to be non-linear.

### Types of non-linear equation:

i) Algebraic equation → An eqn of type  $y = f(x)$  is said to be algebraic if it can be expressed in the form;

$$f_n x^n + f_{n-1} x^{n-1} + f_{n-2} x^{n-2} + \dots + f_1 x^1 + f_0 = 0$$

E.g.

$$3x + 5y - 21 = 0 \text{ (linear)}$$

$$2x + 5xy - 25 = 0 \text{ (non-linear)}$$

$$x^3 + xy - 3y^3 = 0 \text{ (non-linear)}$$

ii) Polynomial equations → Polynomial equations are a simple class of algebraic equations that are represented as follows:-

$$a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 = 0.$$

e.g.  $5x^5 + x^3 + 2x^2 + 5 = 0$ .

iii) Transcendental equation → A non-algebraic equation is called transcendental equation. These include, trigonometric, exponential and logarithmic function.

Eg.  $2 \sin x - x = 0$

$$x - e^{1/x} = 0$$

$$\log x^2 - 1 = 0 \text{ etc.}$$

### Methods of Solving non-linear equations:-

There are number of ways to find roots of non-linear equations. They include.

Not imp i) Direct analytical method.

ii) Graphical method

iii) Trial and Error Method

Imp iv) Iterative methods [Bisection method, Newton Raphson method, Secant method & Fixed-point method].

## ④ Direct Analytical Method:

In this method equations are solved by using derived formulas. We just need to apply the value of input parameters in the equation.

E.g. Solution of algebraic equation  $ax^2 + bx + c = 0$  is given by the formula,  $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

## ⑤ Graphical Method:

This is the simplest method to determine the root of an equation  $f(x) = 0$ . The procedure is quite straight forward as;

→ Plot the function  $f(x)$

→ Observe where it crosses the  $x$ -axis, this point represent the value for which  $f(x) = 0$ .

## ⑥ Trial & Error Method:

→ In this method we make a series of guesses for  $x$  & then evaluate  $f(x)$  at that  $x$ . If it is close to zero, it is one of the approximate root of given non-linear equation. Otherwise we make another guess for  $x$  and repeat the same process again, until the  $x$  for which  $f(x) = 0$  is found.

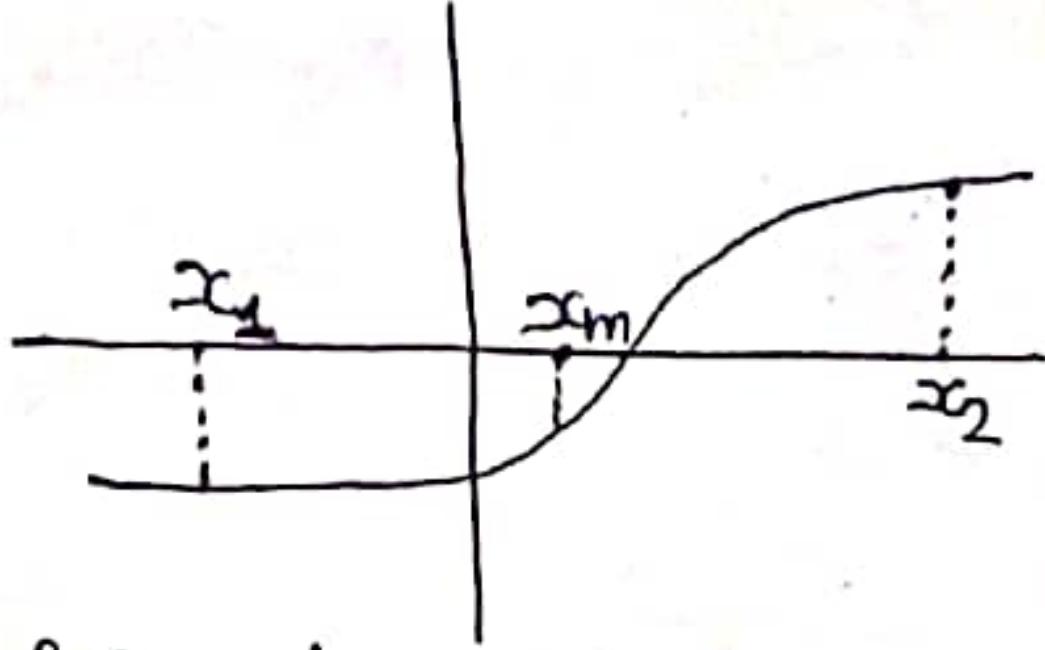
## ⑦ Iterative Methods:

### 1) Bisection Method:

This method is also known as half-interval method. It is based on the fact that if  $f(x)$  is real and continuous in the interval  $a < x < b$  and  $f(a)$  and  $f(b)$  are of opposite signs, i.e.,  $f(a) \cdot f(b) < 0$ . Then there is at least one real root in the interval between  $a$  &  $b$ .

- Let  $x_1 = a$  &  $x_2 = b$ . Let us define another point  $x_m$  to be the mid-point between  $a$  &  $b$ . i.e.,  $x_m = \frac{x_1 + x_2}{2}$
- Now there exist the following conditions:-
- \* If  $f(x_m) = 0$ , we have a root at  $x_m$ .
  - \* If  $f(x_1) \cdot f(x_m) < 0$ , then there is a root between  $x_1$  &  $x_m$ .
  - \* If  $f(x_2) \cdot f(x_m) < 0$ , then there is a root between  $x_2$  &  $x_m$ .

By testing the sign of the function at mid-point, we decide which part of the interval contains the root. This can be illustrated in the figure as follows:-



### Advantages of Bisection Method:

- 1) The bisection method is always convergent since the method brackets the roots.
- 2) As iterations are conducted, the interval gets halved. So one can guarantee the decrease in error in the solution of the equation.

### Drawbacks of Bisection Method:

1) The convergence of bisection method is slow as it is simply based on halving the interval.

2) If one of the initial guesses is closer to the root, it will take larger number of iterations to reach the root.

Q. Using the bisection method solve  $x^2 - 4\cos x = 0$ , correct up to 2 significant figures.

Solution:-

$$\text{Assume initial guess as } x_l = 1 \text{ and } x_u = 2. \quad \text{let } f(x) = x^2 - 4\cos x$$

#### Iteration 1

$$x_l = 1$$

$$f(x_l) = -1.161$$

$$x_m = 1.5$$

since,  $f(x_l) \times f(x_m) < 0$

$$\text{so, } x_u = x_m = 1.5$$

$$\text{Error} = 0.500$$

$$x_u = 2$$

$$f(x_u) = 5.664$$

$$f(x_m) = 1.97$$

$$\left( \because x_m = \frac{x_l + x_u}{2} \right)$$

where  $x_l$  and  $x_u$  are initial guess.

#### Iteration 2

$$x_l = 1$$

$$f(x_l) = -1.161$$

$$x_m = 1.25$$

$$x_u = 1.5$$

$$f(x_u) = 1.97$$

$$f(x_m) = 0.301$$

Since,  $f(x_l) \times f(x_m) < 0$

$$\text{so, } x_u = x_m = 1.25$$

$$\text{Error} = 0.333$$

### Iteration 3

$$\begin{array}{ll} x_l = 1 & x_u = 1.25 \\ f(x_l) = -1.161 & f(x_u) = 0.301 \\ x_m = 1.125 & f(x_m) = -0.459 \end{array}$$

Since,  $f(x_u) \times f(x_m) < 0$

$$x_l = x_m = 1.125$$

$$\text{Error} = 0.200$$

Continuing the process in table below;

Iteration	$x_l$	$x_u$	$x_m$	$f(x_l)$	$f(x_u)$	$f(x_m)$	Error
1	1	2	1.5	-1.161	5.665	1.967	0.500
2	1	1.5	1.25	-1.161	1.967	0.301	0.333
3	1	1.25	1.125	-1.161	0.301	-0.459	0.200
4	1.125	1.25	1.188	-0.459	0.301	-0.086	0.100
5	1.188	1.25	1.219	-0.086	0.301	0.106	0.050
6	1.188	1.219	1.203	-0.086	0.106	0.010	0.025
7	1.188	1.203	1.195	-0.086	0.010	-0.038	0.013
8	1.195	1.203	1.199	-0.038	0.010	-0.014	0.006

Thus, root = 1.199

### Convergence of Bisection Method

In bisection method interval is halved in every iteration.

After  $n$ th iteration, size of iteration is reduced to;

$$\Delta_n = \frac{(x_u - x_l)}{2^n}$$

Now we can say that maximum error after  $n$ th iteration is  $E_n = \pm \Delta_n$ .

$$\Rightarrow |E_n| = \frac{(x_u - x_l)}{2^n}$$

Similarly, after  $(n+1)$ th iteration maximum error is given by,

$$E_{n+1} = \frac{(x_u - x_l)}{2^{n+1}}$$

$$E_{n+1} = \frac{(x_u - x_l)}{2^n \times 2}$$

$$E_{n+1} = \frac{E_n}{2}$$

This shows that error is halved after each iteration therefore we can say that bisection method converges linearly.

## 2) Newton Raphson Method:

Newton-Raphson method is based on the principle that if the initial guess of the root of  $f(x)=0$  is at  $x_i$ , and if one draws the tangent to the curve at  $f(x_i)$ , the point  $x_{i+1}$  where the tangent crosses the  $x$ -axis is an improved estimate of the root.

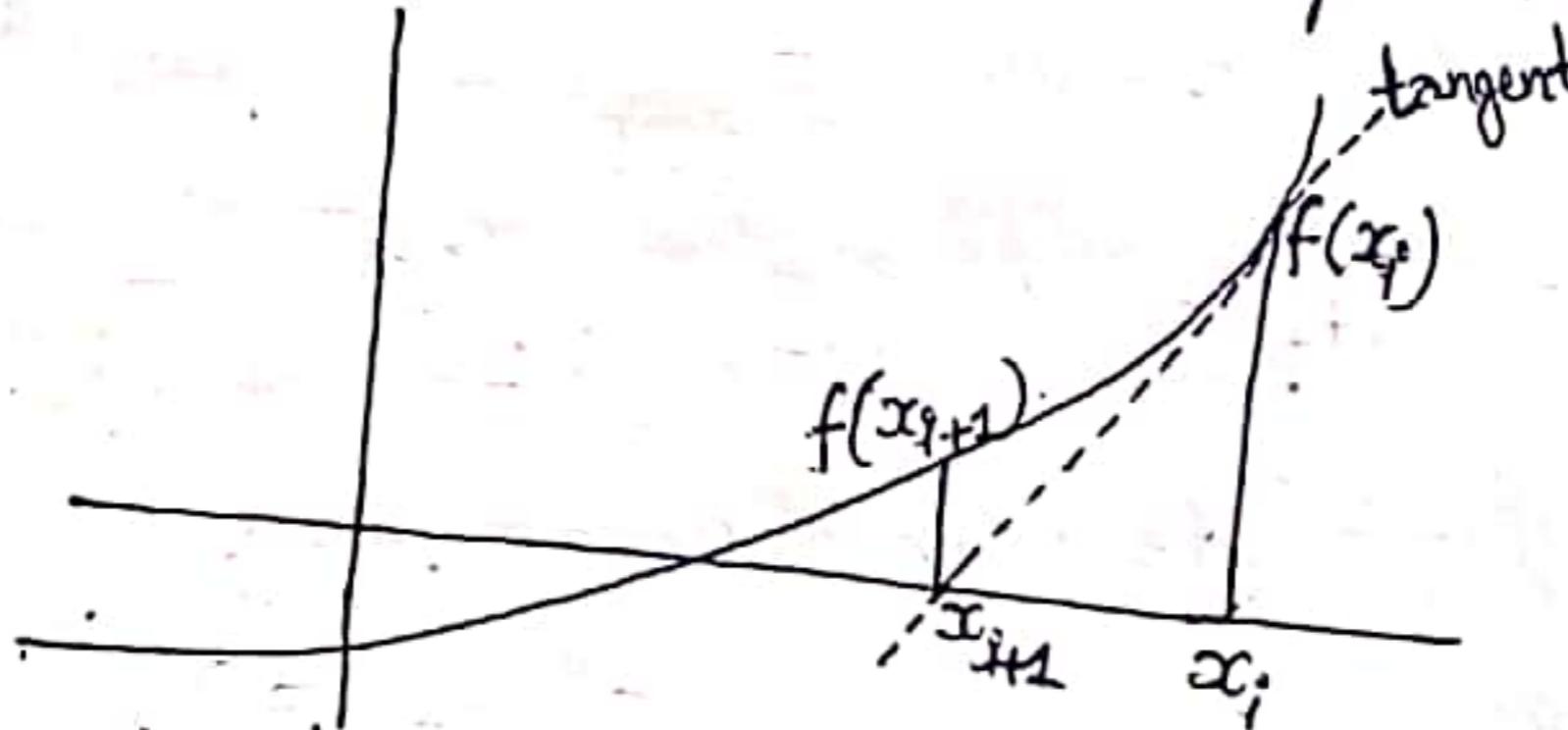


fig. Geometrical interpretation of Newton-Raphson Method.

Using the definition of the slope of a function, at  $x=x_i$ .

$$f'(x_i) = \frac{f(x_i) - f(x_{i+1})}{x_i - x_{i+1}}$$

If  $f(x)$  have root at  $x_{i+1}$ , then  $f(x_{i+1})=0$ . Putting this in above equation, we get

$$f'(x_i) = \frac{f(x_i)}{x_i - x_{i+1}}$$

This gives,

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

This equation is called the Newton-Raphson formula for solving non-linear equations of the form  $f(x)=0$ . So starting with an initial guess  $x_i$ , we find the next guess  $x_{i+1}$ , by using the above equation. We can repeat this process until the root within a desirable tolerance is found.

### Deriving Newton Raphson formula from Taylors Series

Let  $h$  is an small increment in  $x_n$  to get next estimate of root

$$\text{i.e., } x_{n+1} = x_n + h \quad \text{--- (P)}$$

$$\text{or, } h = x_{n+1} - x_n \quad \text{--- (Q)}$$

We know that Taylors series can be written as;

$$f(x+h) = f(x) + \frac{f'(x)h}{1!} + \frac{f''(x)h^2}{2!} + \frac{f'''(x)h^3}{3!} + \dots$$

Since  $h$  is very small, we can neglect the terms containing second and higher order terms. Then, above equation can be written as:

$$f(x+h) = f(x) + \frac{f'(x)h}{1!}$$

$$\text{or, } f(x_n+h) = f(x_n) + \frac{f'(x_n)h}{1!}$$

value of  $h$   
from ①

from ①  
 $x_{n+1} = x_n + h$

$$\text{or, } f(x_{n+1}) = f(x_n) + \frac{f'(x_n)(x_{n+1} - x_n)}{1!} \quad \text{--- (ii)}$$

If  $x_{n+1}$  is root of the given polynomial then,

$$f(x_{n+1}) = 0 \quad \text{from (ii)}$$

$$\Rightarrow f(x_n) + \frac{f'(x_n)(x_{n+1} - x_n)}{1!} = 0$$

Solving above equation to get the value of  $x_{n+1}$  we get,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

It will be better to understand if we write algorithm before solving question of each method which are after few pages.

This equation is called Newton Raphson formula.

Q. Solve the equation  $x^2 + 4x - 9 = 0$  by using Newton-Raphson method. The solution correct upto 4 significant digit.

Soln

$$f(x) = x^2 + 4x - 9 = 0.$$

$$f'(x) = 2x + 4$$

Let us assume initial guess is 4.

Iteration 1.

$$x_0 = 4$$

$$f(x) = 23 \quad \& \quad f'(x) = 12$$

$$x_1 = 4 - \frac{23}{12} = 2.0833 \quad (\text{Using formula})$$

$$\text{Error} = \left| \frac{(x_1 - x_0)}{x_1} \right|$$

$$= \left| \frac{2.08 - 4}{2.08} \right|$$

$$= 0.920$$

## Iteration 2

$$x_0 = 2.0833$$

$$f(x) = 3.678$$

$$f'(x) = 8.166$$

$$x_1 = 1.633$$

$$\text{Error} = 0.275,$$

Continuing above process in table to get solution as;

Iteration	$x_i$	$f(x)$	$f'(x)$	Error
1.	4	23	12	
2	2.0833	3.678	8.166	0.920
3	1.633	0.202	7.267	0.275
4	1.606	0.001	7.211	0.017
5	1.606	0.000	7.211	0.000

Thus, root = 1.606.

## Drawbacks of Newton-Raphson Method.

1) Division by zero → Consider a function  $f(x) = 1-x^2$ . If it has a maximum at  $x=0$  and solutions of  $f(x)=0$  at  $x=\pm 1$ . If we start iterating from the  $x_0=0$  (where the derivative is zero),  $x_1$  will be undefined,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0 - \frac{1}{0}$$

$$= \text{undefined.}$$

→ The same issue occurs if, instead of starting point, any iteration point is stationary.

→ Even if the derivative is small but not zero, the next iteration will be a far worse approximation.

2) Oscillations near local maximum and minimum → For some functions, some starting points may enter an infinite cycle, preventing convergence. Let  $f(x) = x^3 - 2x + 2$  and take 0 as starting point. In some cases the sequence of iteration will alternate between certain points without converging to a root.

## Convergence of Newton-Raphson Method:

Suppose  $x_r$  is root of  $f(x)=0$  and  $x_n$  is estimated root of  $f(x)=0$  such that  $|x_r - x_n| = \delta \ll 1$ . Then by Taylor's Series expansion:

$$f(x_r) = f(x_n + \delta) = f(x_n) + f'(x_n)(x_r - x_n) + R_1 \dots \textcircled{P}$$

where,  $R_1$  is Lagrange form of the Taylor series expansion remainder and is given by the equation,

$$R_1 = \frac{1}{2!} f''(\xi)(x_r - x_n)^2$$

where,  $\xi$  is in between  $x_n$  and  $x_r$ . Since  $x_r$  is the root, eqn \textcircled{P}, becomes;

$$0 = f(x_n) + f'(x_n)(x_r - x_n) + \frac{1}{2!} f''(\xi)(x_r - x_n)^2 \text{ --- } \textcircled{Q}$$

Dividing equation \textcircled{Q} by  $f'(x_n)$  and rearranging gives,

$$\frac{f(x_n)}{f'(x_n)} + (x_r - x_n) = -\frac{f''(\xi)}{2f'(x_n)} (x_r - x_n)^2 \text{ --- } \textcircled{R}$$

We know that according to Newton Raphson formula  $x_{n+1}$  is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Thus eqn \textcircled{R} can be written as;

$$x_r - x_{n+1} = -\frac{f''(\xi)}{2f'(x_n)} (x_r - x_n)^2 \text{ --- } \textcircled{S}$$

Let  $E_n = x_r - x_n$  and  $E_{n+1} = x_r - x_{n+1}$ , Eqn \textcircled{S} can be written as;

$$E_{n+1} = -\frac{f''(\xi)}{2f'(x_n)} \cdot E_n^2$$

Taking absolute value on both sides gives;

$$|E_{n+1}| = \frac{|f''(\xi)|}{2|f'(x_n)|} E_n^2 \text{ --- } \textcircled{T}$$

Eqn \textcircled{T} shows that Newton-Raphson method have quadratic rate of convergence.

### 3) Secant Method:

This method uses two initial estimates but does not require to bracket the root. Geometrical representation of secant method is as shown in the figure below:-

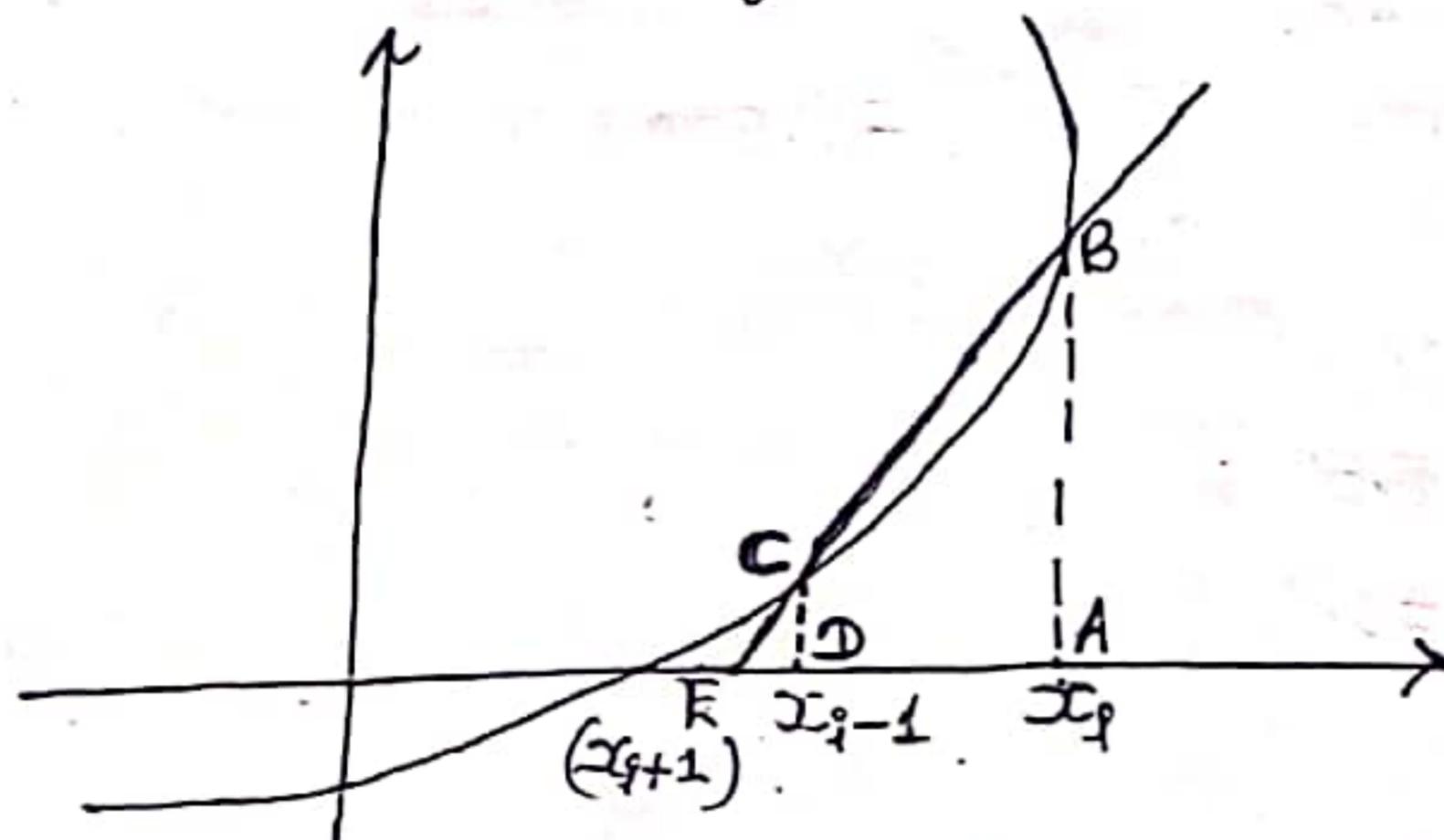


fig. Geometrical representation of the secant method

Let us consider two initial guesses  $x_i$  and  $x_{i-1}$  and draw a straight line between  $f(x_i)$  and  $f(x_{i-1})$  passing through the x-axis at  $x_{i+1}$ . Now,  $ABF$  and  $DCE$  are similar triangles and therefore following relation holds;

$$\frac{AB}{AF} = \frac{DC}{DE}$$

$$\text{or, } \frac{f(x_i)}{x_i - x_{i+1}} = \frac{f(x_{i-1})}{x_{i-1} - x_{i+1}}$$

Rearranging the terms we get,

$$x_{i+1} = \frac{f(x_{i-1})x_i - f(x_i)x_{i-1}}{f(x_{i-1}) - f(x_i)}$$

Adding and subtracting  $f(x_i)x_i$  from numerator, we get.

$$x_{i+1} = x_i - \frac{f(x_i)(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

This equation is called secant formula.

Q. Solve the equation  $2x^2 + 4x - 10 = 0$  with error precision 0.001

Sol'n

let us assume that initial guess  $x_1 = 2$  &  $x_2 = 6$ .

If specified error precision  $E = 0.04$ .

### Iteration 1:

$$\begin{array}{ll} x_1 = 2 & f(x_1) = 6 \\ x_2 = 6 & f(x_2) = 86 \end{array}$$

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)}$$

$$x_3 = 1.7 \leftarrow \text{Error} = \frac{x_3 - x_2}{x_2} = \frac{1.7 - 6}{6} = 2.529$$

### Iteration 2:

$$x_1 = 6 \quad f(x_1) = 86$$

$$x_2 = 1.7 \quad f(1.7) = 2.58$$

$$x_3 = 1.568 \quad \text{Error} = 0.828$$

### Iteration 3:

$$x_1 = 1.7 \quad f(x_1) = 2.58$$

$$x_2 = 1.568 \quad f(x_2) = 1.189$$

$$x_3 = 1.4551 \quad \text{Error} = 0.0775.$$

Continuing above solution in table 28;

Iteration	$x_1$	$x_2$	$x_3$	$f(x_1)$	$f(x_2)$	Error.
4	1.568	1.4551	1.4496	1.189	0.055	0.003

Thus the root is 1.4496.

### Convergence of Secant Method:

The secant formula is given by;

$$x_{q+1} = x_q - \frac{f(x_q)(x_q - x_{q-1})}{f(x_q) - f(x_{q-1})} \quad \textcircled{P}$$

Let  $x_r$  be the actual root of  $f(x)$  and  $e_q$  be the error estimate in  $q$ th iteration then,

$$x_{q+1} = e_{q+1} + x_r$$

$$x_q = e_q + x_r$$

$$x_{q-1} = e_{q-1} + x_r.$$

Substituting these values in  $\textcircled{P}$ , we get,

$$e_{q+1} = e_q - \frac{f(x_q)(e_q - e_{q-1})}{f(x_q) - f(x_{q-1})}$$

$$\text{or, } e_{q+1} = \frac{e_{q-1} \cdot f(x_q) - e_q \cdot f(x_{q-1})}{f(x_q) - f(x_{q-1})} \quad \textcircled{P}$$

According to Mean Value Theorem, there exists at least one point say  $x = R_i$  in the interval  $x_j$  and  $x_r$  such that,

$$f'(R_i) = \frac{f(x_r) - f(x_j)}{x_r - x_j} \quad \text{--- (III)}$$

Since  $f(x_r) = 0$  and  $e_g = x_r - x_j$ , therefore eqn (III) becomes.

$$f'(R_i) = \frac{f(x_j)}{e_g}$$

$$\text{or, } f(x_j) = e_g \cdot f'(R_i)$$

$$\text{Similarly, } f(x_{j-1}) = e_{j-1} \cdot f'(R_{j-1})$$

Substituting these in the numerator of eqn (II) we get,

$$e_{j+1} = e_j \cdot e_{j-1} \cdot \frac{f(R_i) - f'(R_{j-1})}{f(x_j) - f(x_{j-1})}$$

i.e, we can say,

$$e_{j+1} \propto e_j \cdot e_{j-1} \quad \text{--- (IV)}$$

We know that, the order of convergence of an iteration process is  $p$  iff

$$e_j \propto e_{j-1}^p$$

$$\text{or, } e_{j+1} \propto e_j^p$$

Substituting these values in eqn (IV).

$$e_j^p \propto e_{j-1}^p \cdot e_{j-1}$$

$$\text{or, } e_j^p \propto e_{j-1}^{p+1}$$

$$\text{or, } e_j \propto e_{j-1}^{(p+1)/p}$$

i.e  $p^2 - p - 1 = 0$  which has the solution,

$$p = \frac{1 \pm \sqrt{5}}{2}$$

Since  $p$  is always positive we have  $p = 1.618$

It follows that the order of convergence of the secant method is 1.618 & convergence is reflected to as superlinear convergence.

#### A) Fixed-Point Iteration Method:

In fixed-point iteration method we rearrange the function  $f(x)=0$  such that  $x$  is on the left-hand side of the equation. This can be expressed as below:

$$\text{as } f(x)=0 \quad \text{(P)}$$

written as,

$$x=g(x) \quad \text{(P)}$$

Eqn (P) is called fixed point equation. This can be achieved by algebraic manipulation or by simply adding  $x$  to both sides of the original equation, example;

$$x^2+3x-1=0 \Rightarrow x=\frac{1-x^3}{3}$$

$$5m(x)=0 \Rightarrow x=5m(x)+x.$$

Since eqn (P) and eqn (P) are similar, root of equation (P) is also root of equation (P). Root of eqn (P) is given by the point of intersection of straight line  $y=x$  and the curve  $x=g(x)$ . This point of intersection is called fixed point of  $g(x)$ . Point  $p$  is a fixed point of function  $f(x)$  if and only if  $f(p)=p$ . For example, if function  $f$  is defined on the real numbers by  $f(x)=x^2-3x+4$ , then 2 is a fixed point of  $f$ , because  $f(2)=2$ .

Eqn (P) provides a convenient way of predicting the next value of  $x$  on the basis of current value of  $x$ . If  $x_0$  is the initial guess to the root, next approximated value of root can be calculated as  $x_1=g(x_0)$ .

Similarly, further approximation can be given as;

$$x_2=g(x_1)$$

After generalizing this, we get  $\boxed{x_{n+1}=g(x_n)} \quad \text{(P)}$

This point is called fixed point formula and gives rise to the sequence  $x_0, x_1, x_2, \dots$  which is hoped to converge to a point  $x$  (fixed point).

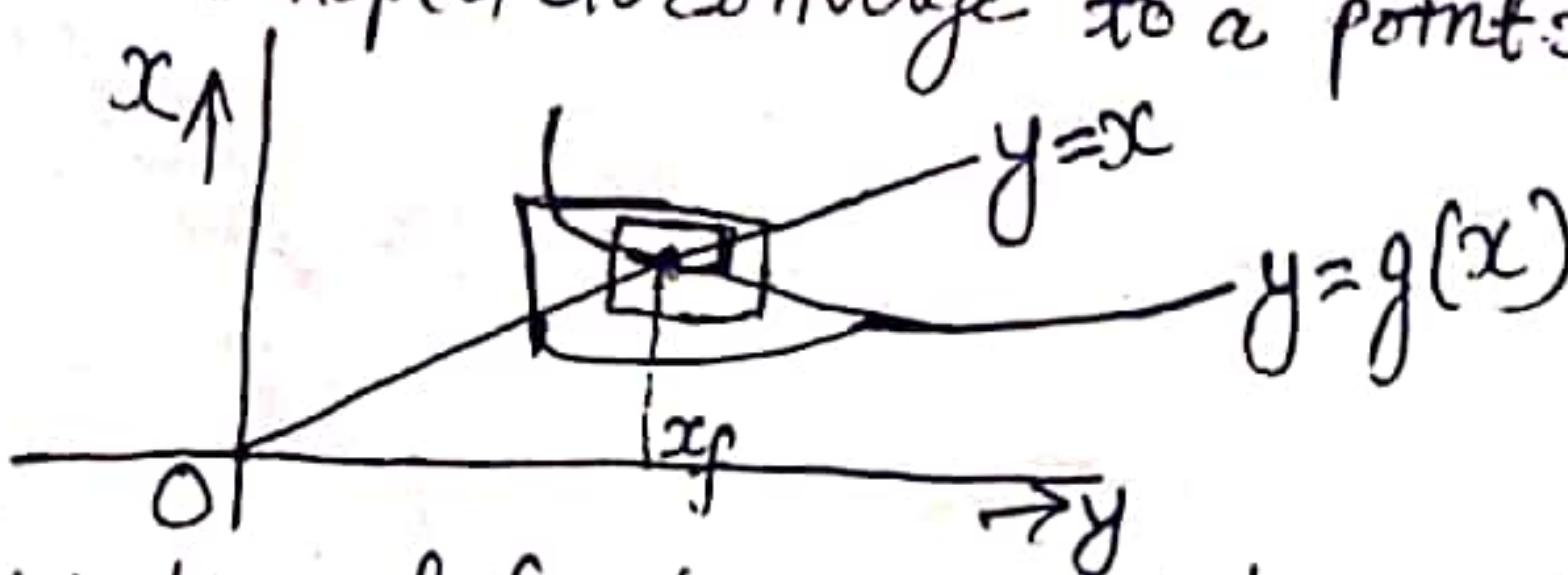


fig. Working of fixed point method

Q. Solve the equation  $x^2 - 6x + 8 = 0$  by assuming error precision 0.01.

Solution:

Above equation can be written as  $x = x^2 - 5x + 8$ .

Assume that initial value of guess is 1.

Iteration 1

$$x_0 = 1$$

$$x_1 = 4$$

$$\text{Error} = \frac{4-1}{4} = 0.75$$

$$x = x^2 - 5x + 8$$

$$= 1^2 - 5 \times 1 + 8$$

Iteration 2

$$x_0 = 4$$

$$x_1 = 4$$

$$\text{Error} = 0.0000$$

Since, Absolute value of error is less than specified limit.

$$\therefore \text{Root} = 4.$$

Q. Solve the equation  $1 + 1/2 \sin(x) - x = 0$  by assuming error precision 0.001.

Solution:

Above equation can be written as  $x = 1 + 1/2 \sin(x)$ .

Assume that initial ~~gu~~ value of guess is 0.

Iteration 1:

$$x_0 = 0$$

$$x_1 = 1$$

$$\text{Error} = 1$$

Iteration 2:

$$x_0 = 1$$

$$x_1 = 1.42$$

$$\text{Error} = 0.296$$

Iteration 3:

$$x_0 = 1.42$$

$$x_1 = 1.494$$

$$\text{Error} = 0.0492$$

Iteration 4:

$$x_0 = 1.494$$

$$x_1 = 1.4985$$

$$\text{Error} = 0.0027$$

Iteration 5:

$$x_0 = 1.4985$$

$$x_1 = 1.4986$$

$$\text{Error} = 0.000103$$

Since, Absolute value of error is less than specified limit. Root = 1.4986.

## Convergence of Fixed-point Method:

Convergence of this iteration process depends on the nature of  $g(x)$ . We can theoretically prove this as follows:-

→ The iteration formula is,  $x_{n+1} = g(x_n) \quad \text{--- (P)}$

Let  $x_f$  be the root of given equation then,  
 $x_f = g(x_f) \quad \text{--- (P')}$

Subtracting above two equations we get,

$$x_f - x_{n+1} = g(x_f) - g(x_n) \quad \text{--- (P'')}$$

According to MVT, there is at least one point say  $x=R$  in the interval between  $x_f$  &  $x_n$  such that,

$$g'(R) = \frac{g(x_f) - g(x_n)}{x_f - x_n}$$

$$\text{or, } g(x_f) - g(x_n) = g'(R)(x_f - x_n)$$

Substituting this equation in (P'').

$$x_f - x_{n+1} = g'(R)(x_f - x_n) \quad \text{--- (Q)}$$

→ If  $e_g$  represents the error in the  $g$ th iteration, then eqn (Q) becomes,

$$e_{n+1} = g'(R)e_n \quad \text{--- (R)}$$

→ This gives that the error decreases with each iteration only if  $g'(R) < 1$ .

→ Hence we say that fixed point method converges only under the condition  $g'(R) < 1$ . i.e, linearly convergent.

## ④ Algorithms for each Iterative Methods:

### 1) Bisection Method: (Algorithm)

- Decide initial values for  $x_1$  and  $x_2$  and stopping criteria  $E$ .
- Compute  $f_1 = f(x_1)$  &  $f_2 = f(x_2)$
- If  $f_1 * f_2 > 0$ , then  $x_1$  &  $x_2$  do not bracket the root and go to stop.
- Compute  $x_0 = (x_1 + x_2)/2$  and compute  $f_0 = f(x_0)$
- If  $f_1 * f_2 < 0$  then, set  $x_2 = x_0$ 
  - else, Set  $f(x_1) = f(x_0)$
  - Set  $x_1 = x_0$
  - set  $f_1 = f_0$
- If absolute value of  $(x_2 - x_1)/x_2$  is less than error  $E$ , then
  - root =  $(x_1 + x_2)/2$
  - write the value of root
  - goto Stop.
- else, goto step 4.
- Stop.

### 2) Newton-Raphson Method: (Algorithm)

- Assign an initial guess to  $x_0$ , say  $x_0$  of precision  $E$ .
- Evaluate  $f(x_0)$  &  $f'(x_0)$ .
- Find the improved estimate of  $x_0$  as;
- $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$
- Check for the accuracy of latest estimate.  
"Compare the relative approximate error with predefined value  $E$ ."  
"If  $|(x_1 - x_0)/x_1| < E$  stop, otherwise continue"
- Replace  $x_0$  by  $x_1$  & repeat step 3 & 4.

Error

### 3) Secant Method : (Algorithm):

- Input two initial guesses (say  $x_0$  &  $x_1$ ) & precision  $E$ .
- Evaluate  $f(x_0)$  &  $f(x_1)$ .
- Estimate the new value of the root as;

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)}$$

- Find the absolute relative approximate error  $[E_a]$  as;

$$E_a = \left| \frac{x_2 - x_1}{x_2} \right|$$

- Compare the approximate error ( $E_a$ ) with prespecified relative error ( $E$ ).

If  $E_a > E$ ,

$$x_0 = x_1$$

$$f(x_0) = f(x_1)$$

$$x_1 = x_2$$

else, goto step 3.  
goto Stop.

→ Stop

### 4) Fixed-Point Iteration Method : (Algorithm)

- Input initial guess (say  $x_0$ ) & error estimate (say  $E$ ).
- Convert  $f(x) = 0$  to the form  $x = g(x)$ .
- Now estimate the value of the root  $x_1$  as;  $x_1 = g(x_0)$
- Find the absolute relative approximate error ( $E_a$ ) as;

$$|E_a| = \left| \frac{x_1 - x_0}{x_1} \right|$$

- Compare absolute relative approximate error,  $E_a$  with the pre-specified relative error  $E$ .

If  $|E_a| > E$ ,

$$x_0 = x_1$$

else, goto step 3.

→ Stop.

Note: According to these algorithms practice C program for each method.

(Lesser imp. topics onwards this)  
don't waste much time on this

## Q. Synthetic Division:

- It is a method of performing long divisions of polynomial with less writing and fewer calculations.
- Let  $P(x)$  be a polynomial of degree  $n$ . If we divide  $P(x)$  by  $(x - c)$  we get another polynomial  $q(x)$ , which is quotient of degree  $n-1$ .

→ Assume,

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$$

And,

$$q(x) = b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \dots + b_0$$

So,

$$P(x) = q(x) \cdot (x - a)$$

$$\text{or, } a_n x^n + a_{n-1} x^{n-1} + \dots + a_0 = (b_{n-1} x^{n-1} + b_{n-2} x^{n-2} + \dots + b_0)(x - c)$$

→ Comparing the coefficients on both sides of the equation,

$$b_{n-1} = a_n$$

$$b_{n-2} = a_{n-1} + a_n b_{n-1}$$

:

$$b_0 = a_1 + a_n b_1$$

Thus,  $b_n = 0$  &  $b_{s-1} = a_s + a_n b_s$ , where  $s = 1 \dots n$ .

Q. Solve the following problem using synthetic division method.

$$(x^4 - 3x^3 + 5) \div (x - 4)$$

Sol'n

Given,  $(x^4 - 3x^3 + 5) \div (x - 4)$

Set the synthetic division,

$$\begin{array}{c} x-4 \\ \hline x^4 + 0x^3 + 0x^2 - 3x + 5 \end{array}$$

Then it becomes,

$$\begin{array}{r} 4 | 1 & 0 & 0 & -3 & 5 \\ \underline{+} 4 | 4 & 16 & 64 & 244 \\ \hline & 1 & 4 & 16 & 61 & 249 \end{array}$$

coefficients multiplied by constant term 4 at each time to bottom term.

Remainder = 249

## ④. Remainder Theorem:

- Remainder theorem states that if the polynomial  $f(x)$  is divided by  $(x-c)$ , then the remainder is  $f(c)$ .
- This means that we can apply synthetic division of the last two numbers is the remainder. This remainder is the functional value of  $f(x)$  at  $x=c$ .
- Thus the remainder theorem is useful for evaluating polynomial at a given value of  $x$ .

Q. Solve using remainder theorem  $f(x) = 3x^3 + x^2 + x - 5$  to find  $f(-2)$ .

Soln

We have to use synthetic division as described above. What the difference is what the final answer is going to be. This time we are looking for functional value, so our answer will not be a quotient, but only the remainder,

$$\text{Given, } f(x) = 3x^3 + x^2 + x - 5.$$

$$\begin{array}{r} -2 \\ \hline 3 & 1 & 1 & -5 \\ & \downarrow & -6 & 10 & -22 \\ \hline & 3 & -5 & 11 & -27 \end{array}$$

$$\text{Remainder} = -27$$

$$\text{So, } f(-2) = -27.$$

## ⑤. Finding Multiple roots by using Newton-Raphson method:

All real roots of polynomials can be found by repeatedly applying Newton's Method and synthetic division. Actually synthetic division is used to obtain polynomial of degree  $n-1$  from polynomial of degree  $n$ . Root of polynomial can be found by using Newton's formula given below and initial guess;

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Suppose  $x_r$  is one of the root of polynomial of degree  $n$ . Now deflate the polynomial by dividing it by  $x - x_r$  to get another polynomial of degree  $n-1$  and use  $x_r$  as initial guess for finding next root. Continue this process until polynomial of degree one is achieved as quotient. This polynomial will be of the form  $a_1x + a_0 = 0$ . Now final root can be calculated as;  $x_r = -\frac{a_0}{a_1}$ .

### Algorithm:

→ Input the degree & coefficient of polynomial.

→ Input initial guess  $x_0$  & limit E.  
while  $n > 1$ ,

→ Find the root using Newton Raphson method as;

$$x_r = x_0 - \frac{f(x_0)}{f'(x_0)}$$

→ Divide the polynomial by  $x - x_r$  to get the polynomial of degree  $n-1$ .

→ Set  $x_0 = x_r$

end while. →  $n = n-1$

$$1^{\text{st}} \text{ root} = -\frac{a_0}{a_1}$$

→ Stop.

Example: Find roots of equation  $(x^4 - 2x^3 - 13x^2 + 38x - 24) = 0$  using newton raphson method. fourth synthetic division.

Solution: Assume, Initial guess  $(x_0) = 0.5$ , Error Limit ( $E$ ) = 0.01

Step 1: Use Newton Raphson method to get root, thus, 4<sup>th</sup> root of the equation = 1.

Step 2: Now use synthetic division deflate the polynomial by  $(x-1)$ , which gives the polynomial:

$$x^3 - x^2 - 14x + 24 = 0$$

Again use Newton Raphson method, with,

Initial guess  $(x_0) = 1$ , Error limit ( $E$ ) = 0.01

Third root of the equation = 2

Step 3: Again, Use synthetic division to deflate the polynomial by  $(x-2)$ , which gives the polynomial:  $x^2 + x - 12 = 0$ .

From Newton Raphson method,

Initial guess  $(x_0) = 2$ , Error Limit ( $E$ ) = 0.01

Second root of the equation = 3.

Step 4: Again, Use synthetic division to deflate the polynomial by  $(x-3)$ , which gives the polynomial  $x+4=0$ .

Now, first root of the polynomial is calculated using the formula given below:

$$\text{root} = -\frac{a_0}{a_1}$$

Thus, First Root = -4

## Horner's Method for Polynomial Evaluation:

→ It is a way to optimize the task of evaluating a polynomial evaluation. The method splits the polynomial into its individual terms & solve them individually. This method separates the lowest degree term in the polynomial. Thus it represent any polynomial in the form:  $K * x + c$ .

where,  $K$  is the group of all terms with degree higher than one &  $c$  is the term with degree zero.

→ Consider a general polynomial equation,

$$P(x) = \sum_{i=0}^n a_i x^i \\ = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$$

where,  $a_0, a_1, a_2, \dots, a_n$  are the real numbers & we wish to evaluate the polynomial at point say  $x_0$ .

→ To accomplish this, we need to define the given polynomial in terms of nested multiplication.

$$P(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$$

$$\text{or, } P(x) = a_0 + x(a_1 + a_2 x + \dots + a_n x^{n-1})$$

$$\text{or, } P(x) = a_0 + x(a_1 + x(a_2 + \dots + a_n x^{n-2}))$$

$$\vdots \\ P(x) = a_0 + x(a_1 + x(a_2 + x(a_3 + x(a_4 + \dots + a_{n-1} + x(a_n))))$$

→ Now we can define a new sequence of constants as follows:-

$$b_n = a_n$$

$$b_{n-1} = a_{n-1} + b_n x_0$$

Then  $b_0$  is the value of  $p(x_0)$ .

### Algorithm:

- Enter the degree of the polynomial.
- Enter the value at which the polynomial to be evaluated,  $x$ .
- Initially set  $b_n = a_n$ .
- While  $n > 0$ 
  - $b_{n-1} = a_{n-1} + b_n * x$
- End While.
- Display the value of  $b_0$ , which is the value of polynomial at  $x$ .
- Stop.

Q. Evaluate the polynomial  $p(x) = 3x^3 - 4x^2 + 5x - 6$  at  $x=2$ .

#### Solution:

We know that,

$$a_3 = 3, a_2 = -4, a_1 = 5, a_0 = -6$$

Now new sequence of constants can be determined by using recursive formula as below:

$$b_3 = a_3 = 3$$

$$\begin{aligned}b_2 &= a_2 + b_3 * x \\&= -4 + 3 * 2 \\&= 2\end{aligned}$$

$$\begin{aligned}b_1 &= a_1 + b_2 * x \\&= 5 + 2 * 2 \\&= 9\end{aligned}$$

$$\begin{aligned}b_0 &= a_0 + b_1 * x \\&= -6 + 9 * 2 \\&= 12\end{aligned}$$

Thus  $p(2) = 12$ .