Unit-5

Solving Ordinary Differential Equations:

An equation which uses differential calculus to express relationship between variables is known as differential equation.
Differential equations are of two types:
→ Ordinary differential equations (ODE)
→ Partial differential equations (PDE).

Ordinary differential equations:

A differential equation with single independent variable (i.e. Quantity with respect to which the dependent variable is differentiated) is called ordinary differential equation.

For Example:

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + y = \sin x.$$

Partial differential equations:

A differential equation with more than one independent variable is called partial differential equation.

For Example:

$$3\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x^2 + y^2$$

where u is the dependent variable (i.e. Quantity being differentiated) and x & y are independent variables.

Note: Order = highest derivative & degree = power of highest derivative.

For example $x^3\frac{d^3y}{dx^3} + x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = e^x$ has order 3 & degree 1.

& $\left(\frac{dy}{dx} + 1\right)^2 + x^2\frac{dy}{dx} = \sin x$ has order 1 & degree 2.

General vs. Particular Solution of Differential Equations:-

Relationship between dependent and independent variables that satisfies differential equation is called solution of the differential equations.

For example. $y = 3x^2 + x$ is the solution of $y' = 6x + 1$.

⇒ A solution of the differential equation that contains arbitrary constants such that it can be modified to represent any condition is called general solution.

For example $y = 3x^2 + x + c$ is general solution of $y' = 6x + 1$.

& A particular solution is defined as a solution that satisfies the differential equation and some initial or boundry conditions.

Initial vs. Boundry Values Problems:-

⇒ If all the conditions are specified at the same value of the independent variable, then the problem is called initial-value problem.

For example : Solve the equation $y' = x^2 + y^2$, given $y(0) = 1$.

⇒ If the conditions are known at different values of the independent variable usually at boundry points of a system, is called boundry-value problem.

For example: Solve the equation $y' = y$, given $y(0) + y(1) = 2$.

A) Solving Initial Value Problems:-

1) Taylors Series Method:-

Taylors Series expansion of function $y(x)$ about a point $x = x_0$ is given by the relation:

$$y(x) = y(x_0) + (x - x_0) y'(x_0) + (x - x_0)^2 \frac{y''(x_0)}{2!} + \dots + (x - x_0)^n \frac{y^n(x_0)}{n!}$$

Note $f = f(x, y) = \frac{dy}{dx}$

f_x = partial derivative of $f(x,y)$ with respect to x

f_y = partial derivative of $f(x,y)$ with respect to y.

⇒ Here, $y' = f(x,y)$, $y'' = f_x + f_y \cdot f$ & $y''' = f_x f_x + 2 f f_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2$

Example 1. Solve the differential equation $y' = 3x^2$ such that $y=1$ at $x=1$. Find y for $x=2$ by using first four terms.

Solution:

From Taylor's Series Method, we have,

$$y' = 3x^2$$

$$y'' = f_x + f_y f = 6x$$

$$y''' = f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_x f_y + f f_y^2 = 6$$

Now, Values of derivatives can be calculated at $x=1$ as below:

$$y' = 3$$

$$y'' = 6$$

$$y''' = 6.$$

Substituting above values in the Taylor's series.

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + (x-x_0)^2 \frac{y''(x_0)}{2!} + \dots + (x-x_0)^n \frac{y^n(x_0)}{n!}$$

we get,

$$y(x) = 1 + (x-1)\times 3 + (x-1)^2 \times \frac{6}{2} + (x-1)^3 \times \frac{6}{6}$$

$$= 1 + 3(x-1) + 3(x-1)^2 + (x-1)^3$$

This is the solution of given differential equation.

Now put $x=2$ in above equation, we get,

$$y(2) = 1+3+3+1 = 8.$$

2) Picard's Method:

Let we are given the differential equation $y' = \frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$.

Now we can write the given differential equation as;

$$dy' = f(x,y)\cdot dx.$$

Integrating both sides we get,

$$\int_{y_0}^{y} dy' = \int_{x_0}^{x} f(x,y)\cdot dx$$

or, $$y - y_0 = \int_{x_0}^{x} f(x,y)\cdot dx$$

$$\Rightarrow \boxed{y = y_0 + \int_{x_0}^{x} f(x,y)\cdot dx}$$ This is called integral equation.

So first we will write integral equation of given differential equation then we will apply successive approximations on that equation as follows;

First Approximation:

while approximating change will be only at this marked position

$$y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx.$$

Second Approximation:

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx.$$

In General n^{th} Approximation: $$y^{(n)} = y_0 + \int_{x_0}^{x} f(x, y^{(n-1)})\, dx.$$

$\Rightarrow$ We repeat the process till two values of y becomes same or reaches desired accuracy.

Example: Solve the equation $y' = 1 + xy$ by using Picard's method with the initial condition $y(0) = 1$. Find the value of $y(0.2)$ correct up to 3 decimal places.

Solution

Given, $y' = 1 + xy$, $y_0 = 1$ & $x_0 = 0$

First approximation:

We have: $y = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$

n $n-1$

$$\Rightarrow y^{(1)} = y_0 + \int_{x_0}^{x} f(x, y^{(0)})\, dx.$$

from question

$$= 1 + \int_0^x 1 + xy^{(0)} \cdot dx$$

since $y(0) = 1$ so, $\int_0^x 1 + x \cdot dx$

$$= 1 + x + \frac{x^2}{2}$$

At $x = 0.2$

$$y(x) = 1 + x + \frac{x^2}{2}$$

$$= 1 + 0.2 + 0.02$$

$$= 1.22$$

Second Approximation:

$$y^{(2)} = y_0 + \int_{x_0}^{x} f(x, y^{(1)})\, dx.$$

या निर मात्रै previous y को value राखेर change गर्ने.

$$\Rightarrow y^{(2)} = 1 + \int_0^x 1 + xy^{(1)}\, dx.$$

$$= 1 + \int_0^x 1 + x\left\{1 + x + \frac{x^2}{2}\right\} dx$$

$$= 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2}\right) dx$$

$$= 1 + \int_0^x 1\,dx + \int_0^x x\,dx + \int_0^x x^2\,dx + \int_0^x \frac{x^3}{2}\,dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

At $x=0.2$

$$y(x)=1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}$$

$$=1+0.2+0.02+0.00266+0.0002$$

$$=1.22286.$$

Third Approximation:

$$y^{(3)}=1+\int_{x_0}^{x} f(x,y^{(2)}).dx.$$

$$\Rightarrow y^{(3)}=1+\int_{0}^{x} f(x,y^{(2)})\,dx.$$

$$=1+\int_0^x 1+xy^{(2)}\,dx.$$

$$=1+\int_0^x 1+x\left\{1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}\right\}dx$$

$$=1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}+\frac{x^5}{15}+\frac{x^6}{48}.$$

At $x=0.2$

$$y(x)=1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{8}+\frac{x^5}{15}+\frac{x^6}{48}$$

$$=1+0.2+0.02+0.00266+0.0002+0.0000213+0.00000133$$

$$=1.222883.$$

Since comparing value of $y(x)$ at $y^{(2)}$ and $y^{(3)}$ the values are correct upto 3 decimal places. Hence, the solution is 1.222.

3) Euler's Method:

For any given differential equation $\frac{dy}{dx}=f(x,y)$ with initial condition $y(x_0)=y_0$ the General equation for Euler's method is;

$$\boxed{y(x_{i+1})=y(x_i)+hf(x_i,y_i).}$$

where, $n=0,1,2,3..,$
$h=$ step size

⇒ We should continue steps by adding step size each time until it reaches to given approximate value.

Example: Approximate the solution of the initial-value problem $y' = x^2 + y^2$, $y(0) = 1$ by using Euler method with step size of 0.2. Approximate the value of $y(0.6)$.

Solution:

Given, $f(x,y) = x^2 + y^2$ & step size $= 0.2$ (h)

We know that,

$$y(x_{i+1}) = y(x_i) + hf(x_i, y_i)$$

Step-1

$y(0) = 1$, $x_0 = 0$, $y_0 = 1$

initial step size

Here

$$y(x_1) = y(0.2)$$
$$= y(x_0) + 0.2 \times f(x_0, y_0)$$
$$= y(0) + 0.2 \times 1$$
$$= 1 + 0.2 \times 1$$
$$= 1.2$$

Step-2

$x_1 = 0.2$, $y_1 = 1.2$

0.2 लै (i.e, h) add गर्दै जाने until ~~upto~~ it reaches to approximate value given.

Now,

$$y(x_2) = y(0.4)$$
$$= y(x_1) + 0.2 \times f(x_1, y_1)$$
$$= y(0.2) + 0.2 \times f(0.2, 1.2)$$
$$= 1.2 + 0.2 \times (0.2^2 + 1.2)^2$$
$$= 1.496$$

Since $f(x,y) = x^2 + y^2$ given

Step-3

$x_2 = 0.4$, $y_2 = 1.496$

reached approximate value so we stop after this step

Now,

$$y(x_3) = y(0.6)$$
$$= y(0.4) + 0.2 \times f(x_2, y_2)$$
$$= 1.496 + 0.2 \times (0.4^2 + 1.496^2)$$
$$= 1.976$$

Thus, $y(0.6) = 1.976$ **Ans.**

4) Heun's Method:

Since the error rate of Euler's method was high so to minimize error rate Euler's method was modified. So, Heun's method is the modified Euler's method.

Let given differential equation is $\frac{dy}{dx} = f(x,y)$ with initial condition $y(x_0) = y_0$. Then in this method first we will find slopes m_1 and m_2 as;

$$m_1 = f(x_i, y_i)$$

& $m_2 = f(x_1, y(x_0) + hf(x_0, y_0))$

Then finally we use Heun's formula as;

$$\boxed{y(x_{i+1}) = y(x_i) + \frac{h}{2}(m_1 + m_2)}$$

⇒ We will continue steps by adding step size each time until it reaches to given approximate value as we did in Euler's Method.

Example:- Approximate the solution of the initial-value problem $y' = x^2 + y$, $y(0) = 1$ by using Heun's method with step size of 0.05. Approximate the value of $y(0.2)$.

Solution:

Here, $f(x,y) = x^2 + y$

step size $(h) = 0.05$

Iteration 1

$x_0 = 0$, $y_0 = 1$

Now,

$m_1 = f(x_0, y_0) = 0^2 + 1^2 = 1$

$m_2 = f(x_1, y(x_0) + hf(x_0, y_0))$

similar to hm_1 ; $f(x_0, y_0)$

Step size added to x_0 ∴ $x_1 = x_0 + h$

$= f(0.05, 1 + 0.05 \times 1)$

$= f(0.05, 1.05)$

$= 1.0525$

$y(x_1) = y(0.05)$

$= y(x_0) + \frac{h}{2}(m_1 + m_2)$

Since $y(x_0) = y(0) = 1$

$= 1 + \frac{0.05}{2}(1 + 1.0525)$

$= 1.0513$

Iteration 2

$x_1 = 0.05 \quad , \quad y_1 = 1.0513$

$m_1 = f(x_1, y_1) = 1.054$

$m_2 = f(x_2, y(x_1) + hf(x_1, y_1))$

h added to 0.05 in each steps

$= f(0.1, 1.0513 + 0.05 \times 1.054)$

$= f(0.1, 1.104)$

$= 1.114$

$y(x_2) = y(0.1)$

$= y(x_1) + \frac{h}{2}(m_1 + m_2)$

$= 1.0513 + \frac{0.05}{2}(1.054 + 1.114)$

$= 1.105$

Iteration 3

$x_2 = 0.1 \quad , \quad y_2 = 1.105.$

$m_1 = f(x_2, y_2) = 1.115$

$m_2 = f(x_3, y(x_2) + hf(x_2, y_2))$

$= f(0.15, 1.105 + 0.05 \times 1.115)$

$= f(0.15, 1.104)$

$= 1.61$

$y(x_3) = y(0.15)$

$= y(x_2) + \frac{h}{2}(m_1 + m_2)$

$= 1.105 + \frac{0.05}{2}(1.115 + 1.161)$

$= 1.162$

Iteration 4

$x_3 = 0.14 \quad , \quad y_3 = 1.162$

$m_1 = f(x_3, y_3) = 1.184$

$m_2 = f(x_4, y(x_3) + hf(x_3, y_3)) = f(0.2, 1.162 + 0.05 \times 1.184)$

$= f(0.02, 1.22)$

$= 1.221$

We reached given approximate value so we will stop after this iteration

$y(x_4) = y(0.2)$

$= y(x_3) + \frac{h}{2}(m_1 + m_2)$

$= 1.162 + \frac{0.05}{2}(1.184 + 1.221) \quad = 1.122$

Thus, $y(0.2) = 1.122$ Ans.

5) Fourth Order Runge-Kutta Method (RK-4th Order Method):

Further refining Heun's Method is Fourth Order Runge-Kutta Method. This refinement in Heun's method improves the order of approximation from h^2 to h^4.

Given the initial-value problem:

$$\frac{dx}{dy} = f(x,y),\ y(x_0) = y_0.$$

For a fixed constant value of h; $y(x_n+h)$ can be approximated by:

$$\boxed{y(x_n+h) = y_{n+1} = y_n + \frac{1}{6}h\left(m_1 + 2m_2 + 2m_3 + m_4\right)}$$

where,

$$m_1 = f(x_i, y_i)$$

$$m_2 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}hm_1\right)$$

$$m_3 = f\left(x_i + \frac{1}{2}h,\ y_i + \frac{1}{2}hm_2\right)$$

$$m_4 = f\left(x_i + h,\ y_i + hm_3\right)$$

Example: Use the Fourth oder Runge-Kutta method with step size 0.2 to estimate $y(0.4)$ if $y' = x^2 + y^2$, $y(0) = 0$.

Solution:

Here,

$f(x,y) = x^2 + y^2$

step size $(h) = 0.2$

Now, Iteration 1

$x_0 = 0$, $y_0 = 0$

$$m_1 = f(x_0, y_0) = f(0,0) = 0$$

$$m_2 = f\left(x_0 + \frac{1}{2}h,\ y_0 + \frac{m_1 h}{2}\right) = f\left(0 + \frac{0.2}{2},\ 0 + \frac{0 \times 0.2}{2}\right) = f(0.1, 0) = 0.01$$

$$m_3 = f\left(x_0 + \frac{0.2}{2},\ y_0 + \frac{m_2 \times 0.2}{2}\right) = f(0.1, 0.001) = 0.01$$

$$m_4 = f(x_0 + 0.2,\ y_0 + m_3 h) = f(0.2, 0.002) = 0.04$$

$$y(x_1) = y(0.2)$$
$$= y(x_0) + \frac{1}{6} \times h \, (m_1 + 2m_2 + 2m_3 + m_4)$$
$$= 0 + \frac{1}{6} \times 0.2 \, (0 + 2 \times 0.01 + 2 \times 0.01 + 0.04)$$
$$= 0.00267$$

Iteration 2:

$$x_1 = 0.2 \, , \; y_1 = 0.00267$$
$$m_1 = f(x_1, y_1) = f(0.2, 0.00267) = 0.04$$
$$m_2 = f\left(x_1 + \frac{0.2}{2}, \, y_1 + \frac{m_1 \times 0.2}{2}\right) = f(0.3, 0.00667) = 0.09004$$
$$m_3 = f\left(x_1 + \frac{0.2}{2}, \, y_1 + \frac{m_2 \times 0.2}{2}\right) = f(0.3, 0.0117) = 0.090136$$
$$m_4 = f(x_1 + 0.2, \, y_1 + m_3 h) = f(0.4, 0.0207) = 0.1604$$

reached to given approx. value so we step after this step

$$y(x_2) = y(0.4)$$
$$= y(x_1) + \frac{1}{6} h \, (m_1 + 2m_2 + 2m_3 + m_4)$$
$$= 0.00267 + \frac{1}{6} \times 0.2 \, (0.04 + 2 \times 0.09004 + 2 \times 0.090136 + 0.1604)$$
$$= 0.02135$$

Hence $y(0.4) = 0.02135$

B) Solving System Of Ordinary Differential Equations:-

Example 1: Solve the following two simultaneous first order differential equations with step size 0.25.

$$\frac{dy}{dx} = z = f_1(x, y, z) \quad , \quad y(0) = 1$$

$$\frac{dz}{dx} = e^{-x} - 2z - y = f_2(x, y, z) \, , \quad z(0) = 2$$

दुई वटा differential equation दिएको बेला यसरि solve गर्ने

Use Euler method to find $y(0.75)$.

Solution:-