Unit-4

Solutions of Linear Algebraic Equations:

⊛. System of linear Equations:

A general set of m linear equations and n unknowns is:

$$a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n=b_1$$
$$a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n=b_2$$
$$\vdots$$
$$a_{m1}x_1+a_{m2}x_2+\dots+a_{mn}x_n=b_m$$

This can be written in matrix form as;

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

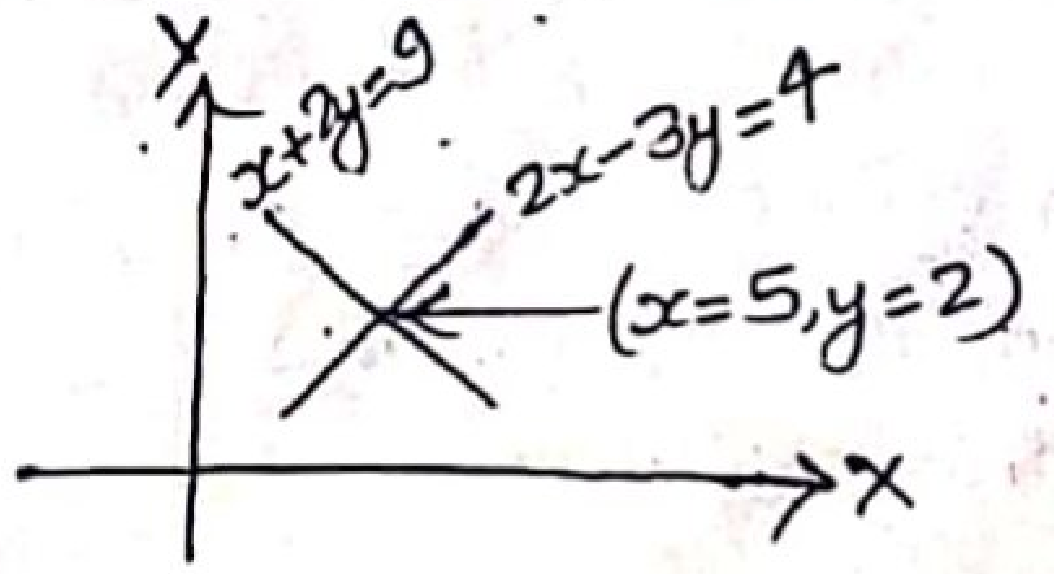
Which is of the form $AX=B$
where A is coefficient matrix, B is called right hand side matrix & X is called solution vector.

⊛. Existence of solution of system of linear Equations:-

There are four conditions in solution of system of linear equations:-
i) Unique solution ii) No solution iii) ILL-condition
iv) No-unique solution.

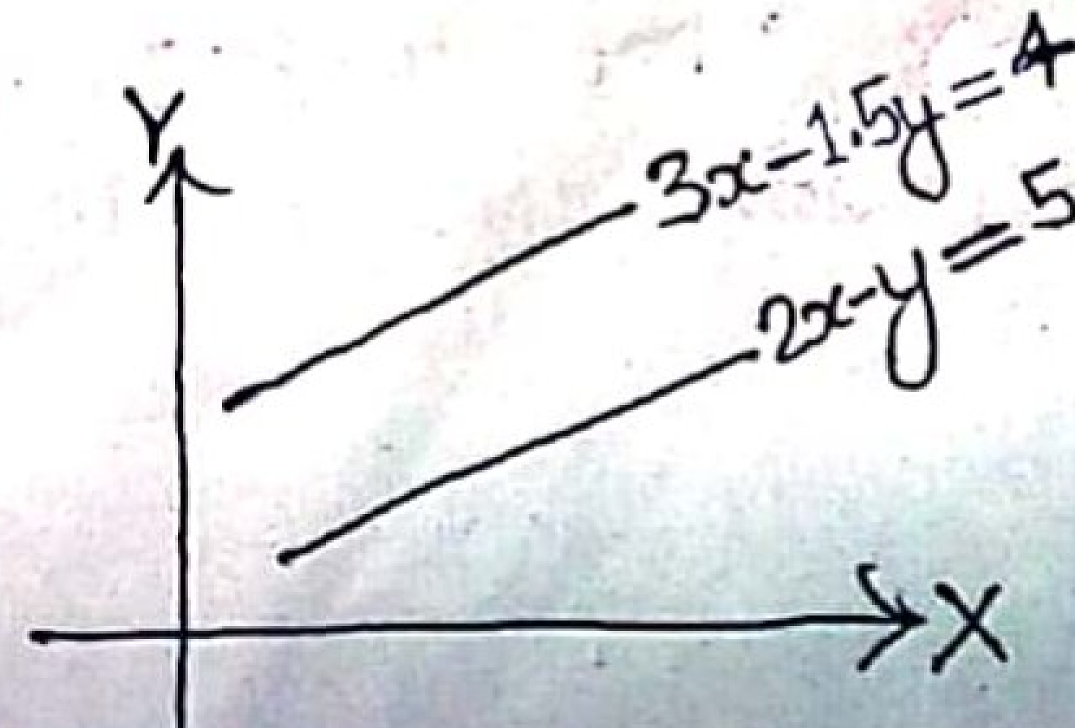
i) Unique solution:

Since the equation of lines $x+2y=9$ and $2x-3y=4$ have no other solution than $(x=5, y=2)$. So the solution is said to be unique.



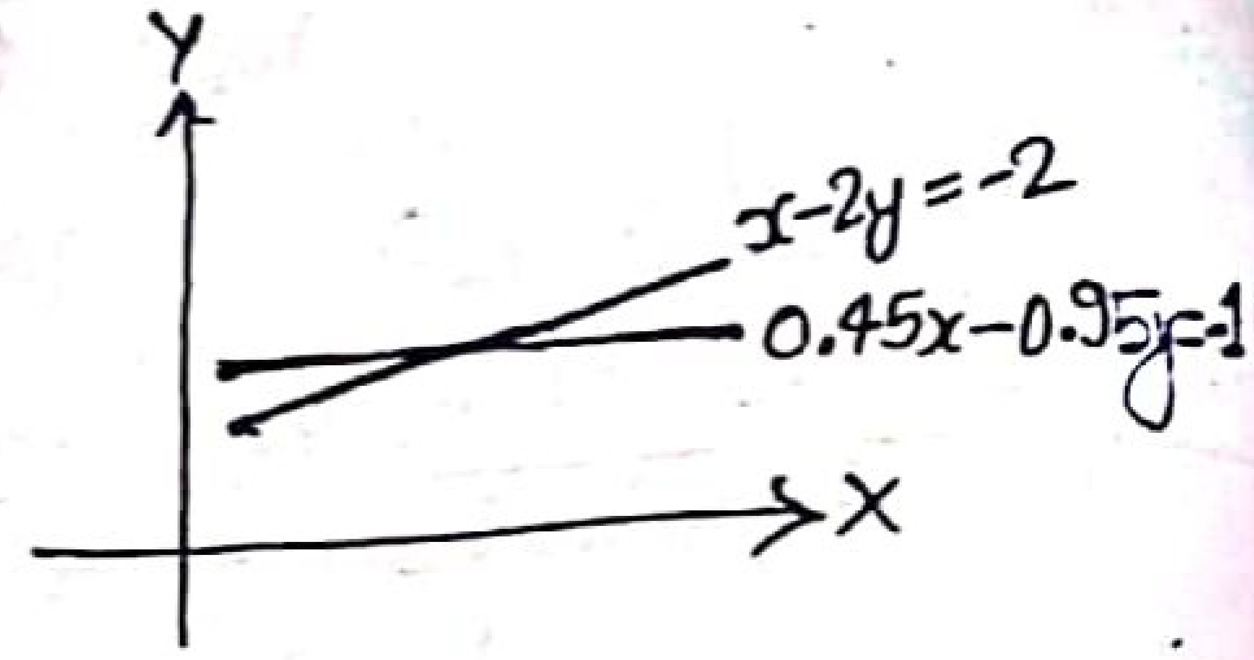
ii) No-solution:

The two lines $3x-1.5y=4$ and $2x-y=5$ are parallel to eachother so they never intersect. Thus there is no-solution.



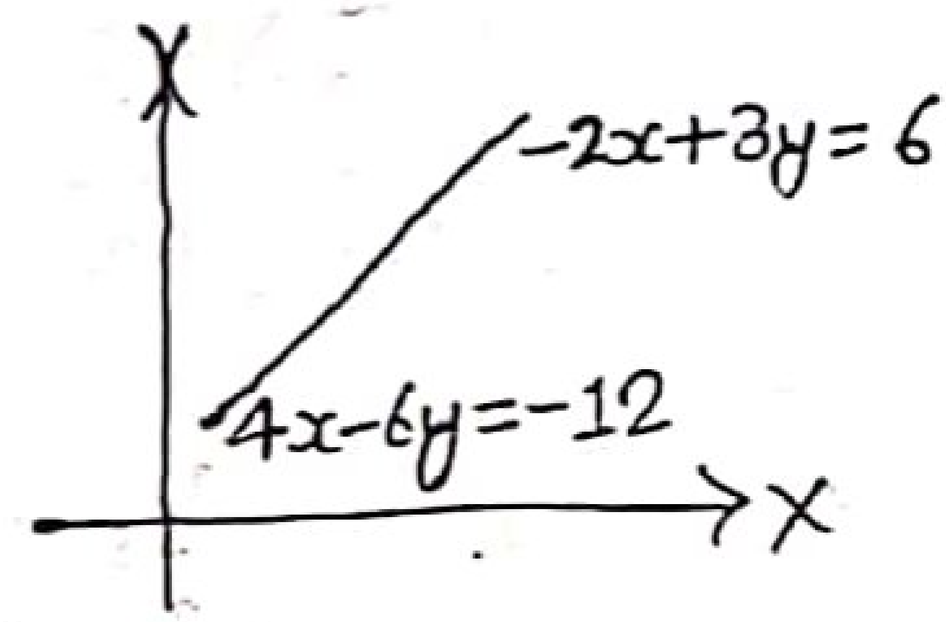
iii) Ill-condition:

It has a solution but it is very difficult to identify the exact point of line intersect. This is called Ill-condition and such systems are ill-conditioned systems.



iv) No-unique solution:

If one system is scalar multiple of other then both system form a single line. In this case one line overlaps another forming many intersection points. Hence in this case there are many solutions.



Solving system of linear equations:-

Solution techniques of system of linear equations can be fundamentally divided into two groups;

A) Elimination Approach (Direct method).

B) Iterative Approach.

A) Direct Methods For Solving System of Linear Equations:-

a. Gauss Elimination Method:-

1) Naive (or Basic) Gauss Elimination Method:-

Gauss Elimination method consists of following two steps:

i) Forward elimination of unknowns:- In this step, the unknowns are eliminated from each equation starting with the first equation.

ii) Back Substitution:- In this step, starting from the last equation, value of each of the unknowns is found.

Example: Use Naive Gauss elimination to solve

$$x_1 - 3x_2 + x_3 = 4$$
$$2x_1 - 8x_2 + 8x_3 = -2$$
$$-6x_1 + 3x_2 - 15x_3 = 9$$

Solution

Given equations are;

$$x_1 - 3x_2 + x_3 = 4$$
$$2x_1 - 8x_2 + 8x_3 = -2$$
$$-6x_1 + 3x_2 - 15x_3 = 9$$

Representing the above equations in matrix form, we get

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & -8 & 8 \\ -6 & 3 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 9 \end{bmatrix}$$

i) **Forward Elimination of Unknowns** i.e, making matrix as upper triangular matrix i.e, all elements below diagonal elements are zero.

Now; Performing $R_2 = R_2 - 2R_1$, the resulting matrix is:

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & -2 & 6 \\ -6 & 3 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 9 \end{bmatrix}$$

Performing $R_3 = R_3 + 6R_1$, the resulting matrix is;

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & -2 & 6 \\ 0 & -15 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ 33 \end{bmatrix}$$

Performing $R_3 = R_3 - 15/2\, R_2$, the resulting matrix is.

$$\begin{bmatrix} 1 & -3 & 1 \\ 0 & -2 & 6 \\ 0 & 0 & 54 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -10 \\ -108 \end{bmatrix}$$

ii) **Back Substitution:-**

Now we can solve the above equations by back substitution starting from the third row,

$$54x_3 = -108$$
$$\Rightarrow x_3 = -2$$

Substituting value of x_3 in the second row,

$$-2x_2 + 6x_3 = -10$$
$$\text{or, } -2x_2 - 12 = -10$$
$$\Rightarrow x_2 = -1$$

Substituting the value of x_2 and x_3 in first row,

$$x_1 - 3x_2 + x_3 = 4$$
$$\Rightarrow x_1 - 3\times(-1) + (-2) = 4$$
$$\Rightarrow x_1 = 4 - 1$$
$$\Rightarrow x_1 = 3$$

Hence the solution is $[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$

Algorithm:

1. Start
2. Read Dimension of System of equations, say n
3. Read coefficients of matrix row-wise.
4. Read RHS vector.
5. Perform forward elimination as below;

For gauss elimination with partial pivoting which comes after this method has also same algorithm. We just add following three lines and go to else part excluding if part.
→ Find largest of a[p][k] for p=k,k+1,...n.
→ Swap row k and row p in coefficient matrix
→ Swap row k and row p in RHS vector.

```
For k=1 to n-1
    pivot = a[k][k]
    if (pivot <0.000001)
        Display the message "Method failed".
    else
        for i=k+1 to n
            term = a[i][k]/pivot.
Multiply row k of coefficient matrix by "term" and subtract it from row i.
Multiply row k of B matrix by "term" and subtract it from row i.
        End for
    End for
```

6. Perform back substitution as below;

```
x[n] = b[n] /a[n][n]
for i=n-1 to 1
    sum=0
    for j=i+1 to n
        sum = sum+a[i][j]*x[j].
    End for
    x[i] = (b[i]-sum)/a[i][i]
End for
```

7. Display solution vector and terminate.

Drawbacks of Naive Gauss Elimination Method:

i) Division by zero → This method may suffer from division by zero problems at the beginning of the each step of forward elimination. This occurs when pivot element is zero.

ii) Round-off Error → Round-off Error can be a serious problem when there are large number of equations as errors propagate. Also, if there is subtraction of floating point numbers from each other, it may create large errors.

2) Gauss Elimination with Partial Pivoting:

This method avoids division by zero problem and round-off errors which was problem for naive gauss elimination method.

→ In this method, if there are n equations, then there are $n-1$ forward elimination steps, similar to Naive Gauss elimination.

→ At the beginning of the k^{th} step of forward elimination, one finds the maximum of

$$|a_{kk}|, |a_{k+1,k}|, \ldots, |a_{nk}|.$$

i.e, maximum of $a_{11}, a_{21}, \& a_{31}$ for 1st column and so on.

→ Then if the maximum of these values is $|a_{pk}|$ in p^{th} row, $k \le p \le n$, then switch rows p and k.

i.e interchange and replace 1st row by maximum having row

→ The other steps of forward elimination are same as the Naive Gauss elimination method.

→ The back substitution steps stay exactly the same as the Naive Gauss elimination method.

Example: Solve following system of linear equations by using gauss elimination with partial pivoting.

$$2x_1 + x_2 + x_3 = 5$$
$$4x_1 - 6x_2 = -2$$
$$-2x_1 + 7x_2 + 2x_3 = 9$$

Solution:-

Given equations are;

$$2x_1 + x_2 + x_3 = 5$$
$$4x_1 - 6x_2 = -2$$
$$-2x_1 + 7x_2 + 2x_3 = 9.$$

Representing the above eqations in matrix form, we get

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

i) Forward Elimination of Unknowns:

Since, largest absolute value among a_{11}, a_{12} and a_{13} is 4. So, switch row 1 and row 2. i.e, Performing $R1 \leftrightarrow R2$, the resulting matrix is;

$$\begin{bmatrix} 4 & -6 & 0 \\ 2 & 1 & 1 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 9 \end{bmatrix}$$

Perform $R_2 = R_2 - \frac{1}{2} R_1$, the resulting matrix is;

$$\begin{bmatrix} 4 & -6 & 0 \\ 0 & 4 & 1 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 9 \end{bmatrix}$$

Perform $R_3 = R_3 + \frac{1}{2} R_1$, the resulting matrix is;

$$\begin{bmatrix} 4 & -6 & 0 \\ 0 & 4 & 1 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 8 \end{bmatrix}$$

elimination गर्सकेको आर्को column मा कुन्हा maximum वाला apply गर्ने

Since, largest absolute value among a_{22}, a_{23} is 4, we do not need to switch rows.

So, directly perform $R_3 = R_3 - R_2$, the resulting matrix is;

$$\begin{bmatrix} 4 & -6 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \\ 2 \end{bmatrix}$$

ii) Back Substitution:

We can now solve the above equations by back substitution starting from the third row;

$$x_3 = 2$$

Substituting the value of x_3 in second row:

$$4x_2 + x_3 = 6$$

$$4x_2 = 4$$

$$\Rightarrow x_2 = 1.$$

Substituting value of x_2 and x_3 in first equation.

$$4x_1 - 6x_2 = -2$$
$$\Rightarrow 4x_1 = 4$$
$$\Rightarrow x_1 = 1.$$

Hence the solution is;
$$[X] = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Note:- Algorithm is same as in Naive Gauss Elimination method but just a small change which is discussed already in same algorithm.

(b) Gauss–Jordan Method: [V.Imp]

This method is the variation of Gauss elimination method. The differences between Gauss Elimination method and Gauss Jordan method are described below:

- → All rows are normalized by dividing them by their pivot element and when an unknown is eliminated from an equation, it is also eliminated from all other equations. Hence the elimination step results in an identity matrix rather than a triangular matrix.
- → Back substitution is not required. We can obtain solution directly from identity matrix obtained from elimination step.

General matrix form of Gauss-Jordan Method is as follows;

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & b_1 \\ 0 & 1 & 0 & b_2 \\ 0 & 0 & 1 & b_3 \end{array}\right]$$

where, b_1, b_2, b_3 are constants and remaning are coefficients of given equations.

Example: Solve the following system of linear equations by Gauss-Jordan Method.

$$2x_1 - x_2 + 4x_3 = 15$$
$$2x_1 + 3x_2 - 2x_3 = 1$$
$$3x_1 + 2x_2 - 4x_3 = -4$$

Solution:

The augmented matrix is:

$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & 15 \\ 2 & 3 & -2 & 1 \\ 3 & 2 & -4 & -4 \end{array}\right]$$

Performing $R_1 = R_1/2$