Fundamental Theory of Phase Field Method

The phase field method (PFM) is based upon Griffith's theory of brittle fracture, where total energy functional (elastic strain energy + fracture dissipation energy) is minimised. The variational formulation of PFM shows the Γ-convergence and makes its implementation straightforward. The linearisation of this variational formulation results in two equations: one for the displacement variable and the other for the phase field variable. The value of the phase field variable ranges from 0 to 1 and describes the failure: 0 refers to intact material, and 1 presents the crack. The crack can only grow or keep its state during the loading process, which means it cannot be healed; this is included in the formulation by making the phase field variable irreversible. The spectral decomposition of strain allows to model the crack growth only when the material is subjected to positive strain energy density. The mathematical formulation of PFM and the spectral decomposition is described in the following section.

1. Mathematical Formulation

A solid body $\Omega \subset \mathbb{R}^n$ is the reference geometry, which has an external boundary $\partial\Omega \subset \mathbb{R}^{n-1}$. The body force $\boldsymbol{f}$, along with prescribed displacement $\vec{\boldsymbol{u}}$ and traction $\vec{\boldsymbol{t}}$ are applied on the external boundary $\partial\Omega_u$ and $\partial\Omega_t$ respectively. For any admissible deformation, the total energy of the system can be written as,

$$\vartheta(\boldsymbol{u},\phi) = \overbrace{\int_\Omega \underbrace{\omega(\phi)}_{\substack{\text{Degradation}\\ \text{function}}}\ \psi_0\left(\varepsilon(\boldsymbol{u})\right) dV}^{\text{Stored Energy}} + \overbrace{\int_\Omega \frac{G_c}{c_w l_0}\left(\underbrace{\gamma(\phi)}_{\text{Phase field evolution function}} + l_0{}^2 \nabla\phi.\nabla\phi\right) dV}^{\text{Dissipation Energy}} - \overbrace{\int_{\partial\Omega_T} F_{ext}.\boldsymbol{u}\, dS}^{\substack{\text{Work of}\\ \text{external}\\ \text{forces}}}$$

where, $\boldsymbol{u}$ and ϕ are the displacement and phase field variables, respectively. $\omega(\phi)$ is the degradation function, $\psi_0\left(\varepsilon(\boldsymbol{u})\right)$ is the elastic strain energy, G_c is the fracture energy, l_0 is the characteristic length, $\gamma(\phi)$ is the phase field evolution function and c_w is the scaling factor. The phase field evolution function is defined as,

$$\gamma(\phi) = \begin{cases} 2\phi - \phi^2 & \text{Higher order model} \\ \phi^2 & \text{AT2 model} \\ \phi & \text{AT1 model} \end{cases}$$

The scaling factor c_w is,

$$c_w = \begin{cases} \pi & \text{Higher order model} \\ 2 & \text{AT2 model} \\ 8/3 & \text{AT1 model} \end{cases}$$

The generalised energy degradation function is described as,

$$\omega(\phi) = \frac{(1-\phi)^2}{(1-\phi)^2 + Q(\phi)}$$

For AT1 and AT2 models, the energy degradation function is defined as,

$$\omega(\phi) = (1-\phi)^2, \qquad \textit{where} \quad Q(\phi) = 1-(1-\phi)^2$$

For the higher order model, the energy degradation function is defined as,

$$\omega(\phi) = \frac{(1-\phi)^2}{(1-\phi)^2 + Q(\phi)} \qquad \textit{where} \quad Q(\phi) = a_1\phi + a_1a_2\phi^2 + a_1a_2a_3\phi^3$$

where, a_1, a_2, a_3 are the numerical constants that can control the softening behaviour. The constant a_1 depends on the material properties as given below,

$$a_1 = \frac{2EG_c}{f_t^2}\frac{2}{\pi l_0}$$

where, E is the Young's modulus of the material and f_t is the failure strength. a_2 and a_3 are numerical constants that depend on the ultimate crack opening and initial slope of the traction-separation characteristic curve. The values of constants a_2 and a_3 for general softening behaviours are tabulated in Table 1.

Table 1: Values of numerical constants a_2 and a_3 for general softening behaviours.

Softening type	a_2	a_3
Liner	-0.5	0
Exponential	0.1748	0
Hyperbolic	0.5379	0
Cornelissen	1.3868	0.6567

2. Spectral Decomposition

The elastic energy is divided into two parts: the active part and the passive part. The active part is degraded with the evolution of the phase field, whereas the passive part remains unaffected as follows,

$$\omega(\phi)\psi_0(\boldsymbol{\varepsilon}(\boldsymbol{u})) = \omega(\phi)\psi_0^+(\boldsymbol{\varepsilon}(\boldsymbol{u})) + \psi_0^-(\boldsymbol{\varepsilon}(\boldsymbol{u}))$$

The spectral decomposition of the strain tensor is implemented to divide the elastic energy as shown below,

$$\psi_0^{\pm}\left(\boldsymbol{\varepsilon}\left(\boldsymbol{u}\right)\right) = \frac{1}{2}\lambda\left\langle tr\left(\boldsymbol{\varepsilon}\left(\boldsymbol{u}\right)\right)\right\rangle_{\pm}^{2} + \mu tr\left[\left(\boldsymbol{\varepsilon}^{\pm}\left(\boldsymbol{u}\right)\right)^{2}\right]$$

where, λ and μ are the Lamé constants, $\langle\bullet\rangle$ is the Macaulay bracket, $tr\left(\bullet\right)$ is the trace of the tensor and $\boldsymbol{\varepsilon}^{\pm}\left(\boldsymbol{u}\right) = \sum_{i=i}^{3}\left\langle\varepsilon_i\right\rangle_{\pm}\boldsymbol{n}_a \otimes \boldsymbol{n}_a$ being ε_i and $\boldsymbol{n}_a$ are the eigenvalues and corresponding eigenvectors of the strain tensor.