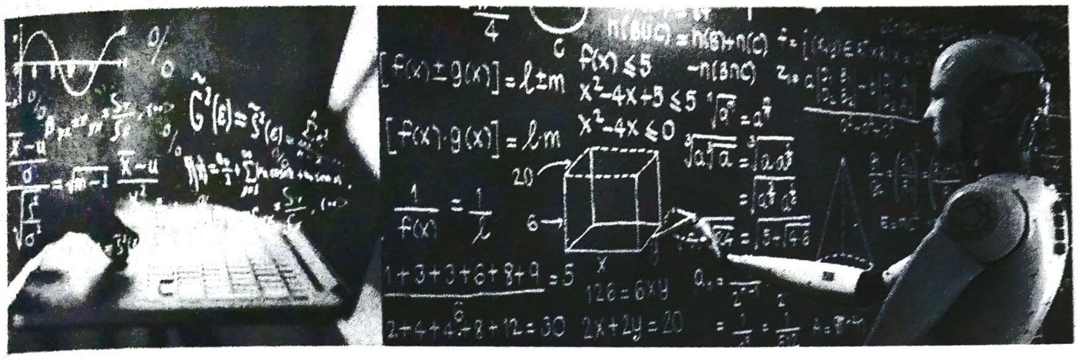


CHAPTER

2

SETS

2.1 INTRODUCTION

A set is a well defined collection of distinct objects. The objects in a set are called elements or members of the set. Here 'well-defined' means that the criteria for deciding if an object belongs to a set or not. Objects, elements and members of a set are synonymous terms.

The following are examples of sets.

(i) All counting numbers.

(ii) All vowels in English alphabets.

(iii) All prime numbers less than 50.

(iv) All universities in India.

(v) All rivers in India.

The following are examples which are not sets.

(i) All small towns in Odisha.

(ii) All big shops in Cuttack.

(iii) All good students in Ravenshaw College.

(iv) All big rivers in Odisha.

The great German Mathematician George Cantor (1845–1918) was the first man to denote and use the concept of set in Mathematics. The concept of set is fundamental and is used in almost all branches of modern Mathematics. It is very much useful because its language is precise and accurate.

Notations : The sets are usually denoted by capital letters A, B, C, ... etc and its elements are usually denoted by small letters a, b, c, ... etc.

The symbol '$\in$' stands for "belongs to".

The symbol '$\notin$' stands for "does not belong to".

2.2 REPRESENTATION OF SETS

Sets are usually represented in two forms :

(i) Roaster or Tabular or Extension form.

(ii) Set builder or set selector or rule form or intension form.

Roaster Form

In this form of writing set, the elements are separated by commas and are enclosed within curly brackets.

For example :

Set of vowels in English alphabets

$$= \{a, e, i, o, u\}$$

Set builder Form

In this form, a set is defined by stating the property which characterises all the elements of the set.

For example :

Set of vowels in English alphabets

$$= \{x : x \text{ is a vowel in English alphabets}\}$$

A set of standard sets with their usual notations is given below :

N = Set of all natural numbers = $\{1, 2, 3, \ldots\ldots\}$

Z = Set of all integers = $\{0, \pm 1, \pm 2, \pm 3, \ldots\ldots\}$

Z^+ = Set of all positive integers = $\{1, 2, 3, \ldots\ldots\}$

Z^- = Set of all negative integers = $\{-1, -2, -3, \ldots\ldots\}$

Q = Set of all rational numbers = $\left\{\frac{p}{q} : p, q \in Z \text{ and } q \neq 0\right\}$

Q^1 = Set of all irrational numbers.

R = Set of all real numbers.

C = Set of all complex numbers.

$= \{z : x + iy, x, y \in R, i = \sqrt{-1}\}$

Types of Set

(i) Singleton Set : A set having one element is called a singleton set.
e.g. $A = \{1\}$

(ii) Finite Set : A set having finite number of elements is called a finite set.
e.g. $B = \{1, 2, 3, 4, 5\}$

(iii) Infinite Set : A set having infinite number of elements is called an infinite set.
e.g. $N = \{1, 2, 3, 4, 5, \ldots\ldots\}$

(iv) Empty Set : A set having no element is called an empty set. It is denoted by ϕ.
e.g. $\phi = \{x : x \neq x\}$

EXAMPLE 2.2.1. Give an example of set which has exactly 10 elements and express it through a defining property.

SOLUTION

$$A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$$
$$= \{x : x \text{ is a digit in decimal number system}\}$$

EXAMPLE 2.2.2. Write the following sets in the form of lists :

(i) $\{x : x \text{ is a prime number and } 1 \le x \le 20\}$

(ii) $\{x : x^2 - 5x + 6 = 0\}$

(iii) $\{x : x = 1 \text{ or } x = 2 \text{ or } x = 3\}$

(iv) $\{x \in Q : x(x^2 - 2)(2x + 3) = 0\}$

(v) $\{x \in N : x < -4 \text{ and } x > 4\}$

SOLUTION

(i) $\{2, 3, 5, 7, 11, 13, 17, 19\}$

(ii) $\{2, 3\}$

(iii) $\{1, 2, 3\}$

(iv) $\{0, -\frac{3}{2}\}$

(v) $\{5, 6, 7, 8, \dots \dots\}$

EXAMPLE 2.2.3. Write the following sets in the intension form :

(i) $\{a\}$ (ii) ϕ (iii) $\{1, 2\}$ (iv) $\{1, 2, 3, 4, 5\}$

SOLUTION

(i) $\{x : x - a = 0\}$

(ii) $\{x : x \neq x\}$

(iii) $\{x : x^2 - 3x + 2 = 0\}$

(iv) $\{x : x \in N \text{ and } 1 \le x \le 5\}$

Definition (Cardinality of a Set) : The number of elements in a set is called cardinality of the set. Cardinality of a set A is denoted by $|A|$ or $0(A)$.

For example :

If $\quad V = \{a, e, i, o, u\}$ then $|V| = 5$

Definition (Equivalent Sets) : The sets are said to be equivalent if they have same number of elements.

For example :

Let $\quad A = \{1, 2, 3, 4, 5\}$ and $V = \{a, e, i, o, u\}$

Then $\quad |A| = 5 = |V|$

So, A and V are equivalent.

Definition (Equal Sets) : Two sets are said to be equal if they have the same elements. It is denoted by $A = B$.

For example :

Let $\quad A = \{1, 2\}$ and $B = \{x : x^2 - 3x + 2 = 0\}$

Then $\quad B = \{1, 2\}$

Hence $\quad A = B$

Note : All the equal sets are equivalent. But equivalent sets may not be equal.

Definition (Subset) : A set A is said to be a subset of a set B if every element of A is also an element of B. It is denoted by $A \subseteq B$.

For example :

Let $A = \{1, 2\}$ and $B = \{1, 2, 3\}$

Then A is a subset of B as each element of A is an element of B.

Theorem 2.2.1. For any Set A

(i) $A \subseteq A$ and $\phi \subseteq A$

Prof. (i) If $a \in A$, then $a \in A$.

So, $A \subseteq A$

(ii) To prove that $\phi \subseteq A$.

Assume that $\phi \subseteq A$ is false.

Then there must exist some x in ϕ such that $x \notin A$.

There is no x in ϕ.

This is a contradiction.

Hence our assumption is wrong.

Therefore $\phi \subseteq A$.

Definition (Proper Subset) : Set A is said to be a proper subset of B if A is a subset of B and $A \neq B$. It is denoted by $A \subset B$.

e.g. $\{a, b\} \subset \{a, b, c\}$

Note : Empty set ϕ has no proper subset.

Definition (Power Set) : The power set of a set A is denoted by P(A) and is defined by P(A) is the set of all subsets of A.

e.g. if $A = \{1, 2, 3\}$, then

$$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}.$$

Note : If $|A| = n$, then $|P(A)| = 2^n$.

Definition (Universal Set) : If all the sets under consideration are subsets of U, then U is an universal set.

EXAMPLE 2.2.4. Find the power sets of each of the following sets :

(i) ϕ (ii) $\{\phi\}$ (iii) $\{\phi, \{\phi\}\}$

SOLUTION

(i) Let $A = \phi$

Then $P(A) = \{\phi\}$

(ii) Let $A = \{\phi\}$

Then $P(A) = \{\phi, \{\phi\}\}$

(iii) Let $A = \{\phi, \{\phi\}\}$

Then $P(A) = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$

EXAMPLE 2.2.5. Find the number of elements of :

(i) $P(P(\phi))$ (ii) $P(P(P(\phi))$ (iii) $P(P(P(P(\phi))))$

SOLUTION

(i)
$$|\phi| = 0$$
$$|P(\phi)| = 2^0 = 1$$
$$|P(P(\phi))| = 2^1 = 2$$

(ii)
$$|P(P(P(\phi)))| = 2^2 = 4$$

(iii)
$$|P(P(P(P(\phi))))| = 2^4 = 16$$

EXAMPLE 2.2.6. Let A be a set and suppose $x \in A$. Is $x \subseteq A$ also possible ? Explain.

SOLUTION

Let A be a set and $x \in A$.

$x \subseteq A$ is possible.

For example set $\quad x = \{1\}$, and $A = \{1, \{1\}\}$

Then $\quad x \subseteq A$.

EXAMPLE 2.2.7. If P(A) has 256 elements, how many elements are there in A.

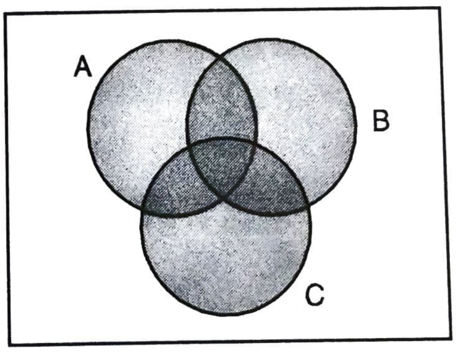
SOLUTION

Let $\quad |A| = n \Rightarrow |P(A)| = 2^n \Rightarrow 256 = 2^n \Rightarrow 2^8 = 2^n$

Hence $\quad n = 8$ *i.e.* $|A| = 8$

Venn-diagrams : The relationship between sets can be better understood with the help of diagrams. These diagrams are called Venn-diagrams named after the famous English Logician John Venn (1834–1883). In Venn-diagrams, the universal set is usually denoted by a rectangle while its subsets are denoted by small circles.

Venn-diagram containing E as universal set and A, B, C as its subsets is given below :



2.3 OPERATIONS ON SETS

In this section, we discuss ways in which two or more sets can be combined in order to form a new set. There are five set operations. They are :

1. Union
2. Intersection
3. Difference
4. Symmetric difference
5. Complementation

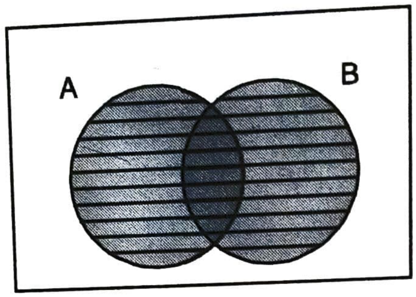
1. Union of two Sets

The union of two sets A and B is a set whose elements belong to either A or B or both. The union of A and B is denoted by $A \cup B$.

Symbolically, $A \cup B = \{x : x \in A \vee x \in B\}$

For example if $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$, then $A \cup B = \{1, 2, 3, 4, 5\}$

Venn-diagram of $A \cup B$



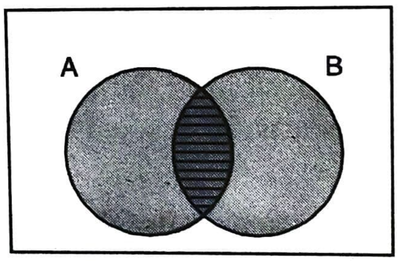
Note : (i) $A \cup A = A$ (ii) $A \cup \phi = A$ (iii) $A \cup E = E$ (iv) $\phi \cup \phi = \phi$

2. Intersection of two Sets

The intersection of two sets A and B is the set containing common elements, which are in both A and B. Symbolically, $A \cap B = \{x : x \in A \wedge x \in B\}$

For example if $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$, then $A \cap B = \{2, 3\}$

Venn-diagram of $A \cap B$



Note : (i) $A \cap A = A$ (ii) $A \cap \phi = \phi$ (iii) $A \cap E = A$ (iv) $\phi \cap \phi = \phi$

Note : A and B are said to be disjoint if $A \cap B = \phi$.

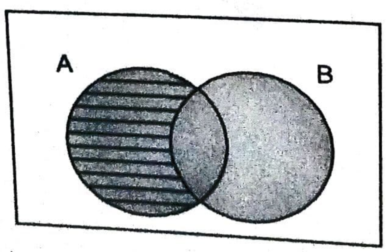
3. Difference of two Sets

The difference of two sets A and B (say A difference B) is the set of all elements of A which are not in the set B.

Symbolically, $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$

For example if $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$, then $A \setminus B = \{1\}$

Venn-diagram of $A \setminus B$



Note : (i) $A \setminus A = \phi$ (ii) $A \setminus \phi = A$ (iii) $A \setminus B = A \setminus (A \cap B)$ (iv) $\phi \setminus A = \phi$

4. Symmetric difference of two Sets

The symmetric difference of two sets A and B is denoted by $A \oplus B$ and is defined as

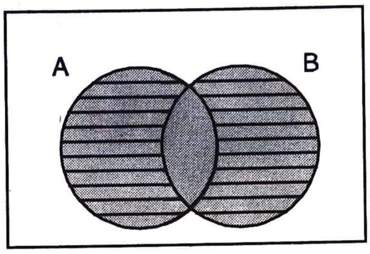
$$A \oplus B = \{x : x \in (A \setminus B) \vee x \in (B \setminus A)\}$$

For example if $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$

then
$$\begin{aligned} A \oplus B &= (A \setminus B) \cup (B \setminus A) \\ &= \{1\} \cup \{4, 5\} \\ &= \{1, 4, 5\} \end{aligned}$$

Note : The notation $A \Delta B$ is used by some authors to denote symmetric difference.

Venn-diagram of $A \oplus B$



Note :

(i) $A \oplus B = (A \setminus B) \cup (B \setminus A)$

(ii) $A \oplus B = (A \cup B) \setminus (A \cap B)$

(iii) $A \oplus A = \phi$

(iv) $A \oplus \phi = A$

(v) $A \oplus B = B \oplus A$

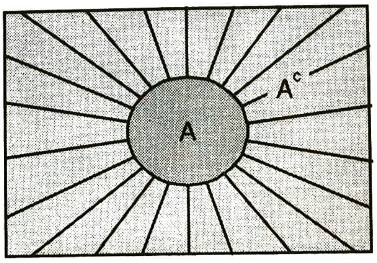
(vi) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

5. Complement of a Set

Let A be a subset of a universal set E. Then the complement of A is denoted by A^c or A^1 and is defined by A^c is the set of all elements which are in E but not in A.

For example if $E = \{1, 2, 3, 4, 5\}$ and $A = \{1, 3, 5\}$ then $A^c = \{2, 4\}$

Venn-diagram of A^c



Note : (i) $E^c = \phi$ (ii) $\phi^c = E$ (iii) $A \cup A^c = E$ (iv) $A \cap A^c = \phi$

(v) $(A^c)^c = A$ (vi) $(A \cup B)^c = A^c \cap B^c$ (vii) $(A \cap B)^c = A^c \cup B^c$

Intervals : Let a and b be real numbers with $a < b$.

Then

$[a, b] = \{x \in R : a \leq x \leq b\}$ Closed interval

$$(a, b) = \{x \in R : a < x < b\}$$ Open interval

$$(a, b] = \{x \in R : a < x \leq b\}$$ Semi-open interval

$$[a, b) = \{x \in R : a \leq x \leq b\}$$ Semi-open interval

EXAMPLE 2.3.1. Find A^c (with respect to E = R in each of the following cases.)

(i) $A = (1, \infty) \cup (-\infty, -2]$

(ii) $A = (-3, \infty) \cap (-\infty, 4]$

(iii) $A = \{x \in R : x^2 \leq 1\}$

SOLUTION

(i) $A^c = (-2, 1]$

(ii) $A^c = (-\infty, -3] \cup (4, \infty)$

(iii) $A^c = (-\infty, -1) \cup (1, \infty)$

SET IDENTITIES

The most important set of identities are given in the following table. It can be observed that there exist similarity between set identities and the logical equivalences discussed in section 1.5.

TABLE–1

Set Identities

Identity	Name
$A \cup \phi = A$	Identity Laws
$A \cap U = A$	
$A \cup U = \cup U$	Domination Laws
$A \cap \phi = \phi$	
$A \cup A = A$	Idempotent Laws
$A \cap A = A$	
$(A^c)^c = A$	Double Complementation Law
$A \cup B = B \cup A$	Commutative Laws
$A \cap B = B \cap A$	
$A \cup (B \cup C) = (A \cup B) \cup C$	Associative Laws
$A \cap (B \cap C) = (A \cap B) \cap C$	
$A \cup (B \cap C) = (A \cup B) \cap (A \cap C)$	Distributive Laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	
$(A \cup B)^c = A^c \cap B^c$	De Morgan's Laws
$(A \cap B)^c = A^c \cup B^c$	
$A \cup (A \cap B) = A$	Absorption Laws
$A \cap (A \cup B) = A$	
$A \cup A^c = U$	Complement Laws
$A \cap A^c = \phi$	

Note : $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$

EXAMPLE 2.3.2. If $A \cup B = A \cap B$, then prove that $A = B$.

SOLUTION

Given that $A \cup B = A \cap B$

To prove that $A = B$

Let $x \in A$

$\Rightarrow$ $x \in A \cup B$ $(\because A \subseteq A \cup B)$

$\Rightarrow$ $x \in A \cap B$ $(\because A \cup B = A \cap B)$

$\Rightarrow$ $x \in B$ $(\because A \cap B \subseteq B)$

So, $A \subseteq B$ .... (1)

Let $x \in B$

$\Rightarrow$ $x \in A \cup B$ $(\because B \subseteq A \cup B)$

$\Rightarrow$ $x \in A \cap B$ $(\because A \cup B = A \cap B)$

$\Rightarrow$ $x \in A$ $(\because A \cap B \subseteq A)$

So, $B \subseteq A$ .... (2)

From (1) and (2), we get

$$A = B$$

EXAMPLE 2.3.3. Prove the following :

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

SOLUTION

(i) Let $x \in A \cup (B \cap C)$

$\Leftrightarrow$ $x \in A$ or $x \in (B \cap C)$

$\Leftrightarrow$ $x \in A$ or $(x \in B$ and $x \in C)$

$\Leftrightarrow$ $(x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$(\because p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r))$

$\Leftrightarrow$ $x \in (A \cup B)$ and $x \in (A \cup C)$

$\Leftrightarrow$ $x \in (A \cup B) \cap (A \cup C)$

So, $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ .... (1)

and $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$ .... (2)

From (1) and (2), we get

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

(iii) Left to the reader.

EXAMPLE 2.3.4. Prove the following :

(i) $(A \cup B)^c = A^c \cap B^c$

(ii) $(A \cap B)^c = A^c \cup B^c$

SOLUTION

(i) Let $x \in (A \cup B)^c$

$\Leftrightarrow$ $x \notin (A \cup B)$

$$x \notin A \text{ and } x \notin B \qquad (\because \sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q)$$

$$\Leftrightarrow \quad x \in A^c \text{ and } x \in B^c$$

$$\Leftrightarrow \quad x \in A^c \cap B^c$$

$$\Leftrightarrow \quad (A \cup B)^c \subseteq A^c \cap B^c \qquad \ldots. (1)$$

So, $\quad A^c \cap B^c \subseteq (A \cup B)^c \qquad \ldots. (2)$

and

From (1) and (2), we get

$$(A \cup B)^c = A^c \cap B^c$$

(iii) Left to the reader.

EXAMPLE 2.3.5. Prove that

$(A \setminus B) \setminus C = A \setminus (B \cup C)$ for any sets A, B and C by using set identities.

SOLUTION

We know that $\quad X \setminus Y = X \cap Y^c$

$$\begin{aligned} \text{LHS} &= (A \setminus B) \setminus C \\ &= (A \setminus B) \cap C^c && (\because X \setminus Y = X \cap Y^c) \\ &= (A \cap B)^c \cap C^c && (\because X \setminus Y = X \cap Y^c) \\ &= A \cap (B^c \cap C^c) && \text{(By associative Law)} \\ &= A \cap (B \cup C)^c && \text{(By De Morgan's Law)} \\ &= A \setminus (B \cup C) && (\because X \setminus Y = X \cap Y^c) \\ &= \text{RHS} \end{aligned}$$

EXAMPLE 2.3.6. Let A, B and C be subsets of some universal set U.

(a) Prove that $A \cap B \subseteq C$ and $A^c \cap B \subseteq C \rightarrow B \subseteq C$.

(b) Given that $A \cap B = A \cap C$ and $A^c \cap B = A^c \cap C$, does it follow that $B = C$? Justify your answer.

SOLUTION

(a) $\quad A \cap B \subseteq C$ and $A^c \cap B \subseteq C$

$\Rightarrow \quad (A \cap B) \cup (A^c \cap B) \subseteq C \cup C$

$\Rightarrow \quad (A \cup A^c) \cap B \subseteq C \cup C \qquad$ (Distributive Law)

$\Rightarrow \quad (A \cup A^c) \cap B \subseteq C \qquad$ (Idempotent Law)

$\Rightarrow \quad U \cap B \subseteq C \qquad$ (Complement Law)

$\Rightarrow \quad B \subseteq C \qquad$ (Identity Law)

(b) Given that $\quad A \cap B = A \cap C$ and $A^c \cap B = A^c \cap C$

To verify that $\quad B = C$

$\quad A \cap B = A \cap C$ and $A^c \cap B = A^c \cap C$

$\Rightarrow \quad A \cap B \subseteq C$ and $A^c \cap B \subseteq C$

$\Rightarrow \quad B \subseteq C \qquad \ldots (1)$

(By Example 2.3.6 (a))

Again

$$A \cap C = A \cap B \text{ and } A^c \cap C = A^c \cap B$$

$\Rightarrow$ $$A \cap C \subseteq B \text{ and } A^c \cap C \subseteq B$$

$\Rightarrow$ $$C \subseteq B \quad \ldots (2)$$

(By Example 2.3.6 (a))

From (1) and (2), we get

$$B = C$$

EXAMPLE 2.3.7 . Prove that

$$A \oplus B = \phi \Leftrightarrow A = B$$

SOLUTION

$$A \oplus B = \phi$$

$\Rightarrow$ $$(A \setminus B) \cup (B \setminus A) = \phi$$

$\Rightarrow$ $$A \setminus B = \phi \text{ and } B \setminus A = \phi$$

$\Rightarrow$ $$A \subseteq B \text{ and } B \subseteq A$$

$\Rightarrow$ $$A = B$$

EXAMPLE 2.3.8. Prove that

$$A \oplus B = B \oplus A$$

SOLUTION

$$\begin{aligned} A \oplus B &= (A \setminus B) \cup (B \setminus A) \\ &= (B \setminus A) \cup (A \setminus B) \quad \text{(Commutative Law)} \\ &= B \oplus A \end{aligned}$$

EXAMPLE 2.3.9. Prove that

$P(A \cap B) = P(A) \cap P(B)$ where $P(A)$ is the power set of A.

SOLUTION

Let X be an arbitrary set belonging to $P(A \cap B)$

$\therefore$ $$X \in P(A \cap B)$$

$\Leftrightarrow$ $$X \subseteq A \cap B$$

$\Leftrightarrow$ $$X \subseteq A \text{ and } X \subseteq B$$

$\Leftrightarrow$ $$X \in P(A) \text{ and } X \in P(B)$$

$\Leftrightarrow$ $$X \in P(A) \cap P(B)$$

Hence $$P(A \cap B) \subseteq P(A) \cap P(B)$$

and $$P(A) \cap P(B) \subseteq P(A \cap B)$$

Therefore $$P(A \cap B) = P(A) \cap P(B)$$

EXAMPLE 2.3.10. For sets A and B, prove that

$$A \cup B = B \text{ if and only if } A \subseteq B.$$

SOLUTION

($\Rightarrow$) Given that $A \cup B = B$

To prove that $A \subseteq B$

Let $x \in A$

$\Rightarrow x \in A \cup B$ $\quad (\because A \subseteq A \cup B)$

$\Rightarrow x \in B$ $\quad (\because A \cup B = B)$

So, $A \subseteq B$

($\Rightarrow$) Given that $A \subseteq B$

To prove that $A \cup B = B$

$A \subseteq B$

$\Rightarrow$ $A \cup B \subseteq B \cup B$

$\Rightarrow$ $A \cup B \subseteq B$ .... (1)

But $B \subseteq A \cup B$ .... (2)

From (1) and (2), we get $A \cup B = B$

EXAMPLE 2.3.11. Use the first law of De Morgan to prove the second : $(A \cap B)^c = A^c \cup B^c$.

SOLUTION

$(A \cap B)^c$

$= (X^c \cap Y^c)^c$ $\quad$ Taking $A = X^c$ and $B = Y^c$

$= ((X \cup Y)^c)^c$ $\quad$ (By First law of De Morgan)

$= (X \cup Y)$ $\quad$ (By Double complementation Law)

$= A^c \cup B^c$ $\quad$ $(\because A^c = (X^c)^c = X$ and $B^c = (Y^c)^c = Y)$

EXAMPLE 2.3.12. Let A, B and C be subsets of some universal set $\cup$. Use set theoretic identities to prove that $A\backslash(B\backslash C) = (A \backslash B) \cup (A \backslash C^c)$.

SOLUTION

$A \backslash (B \backslash C)$

$= A \backslash (B \cap C^c)$ $\quad (\because X \backslash Y = X \cap Y^C)$

$= A \cap (B \cap C^c)^c$ $\quad (\because X \backslash Y \backslash X \cap Y^c)$

$= A \cap (B^c \cup (C^c)^c)$ $\quad$ (By De Morgan's Law)

$= A \cap (B^c \cup C)$ $\quad$ (By Double complementation Law)

$= (A \cap B^c) \cup (A \cap C)$ $\quad$ (By Distributive Law)

$= (A \cap B^c) \cup (A \cap (C^c)^c)$ $\quad$ (By Double complementation Law)

$= (A \backslash B) \cup (A \backslash C^c)$ $\quad (\because X \backslash Y = X \cap Y^c)$

Example 2.3.13. Let A, B, C and D be subsets of a universal set $\cup$. Use set theoretic identities to simplify the expression $[(A \cup B)^c \cap (A^c \cup C)^c]^c \setminus D^c$

SOLUTION

$$
\begin{aligned}
& [(A \cup B)^c \cap (A^c \cup C)^c]^c \setminus D^c \\
&= [(A^c \cap B^c) \cap ((A^c)^c \cap C^c)]^c \setminus D^c && \text{(By De Morgan's Law)} \\
&= [(A^c \cap B^c) \cap (A \cap C^c)]^c \setminus D^c && \text{(By Double Complementation Law)} \\
&= [(B^c \cap A^c) \cap (A \cap C^c)]^c \setminus D^c && \text{(By Commutative Law)} \\
&= [B^c \cap \{A^c \cap (A \cap C^c)\}]^c \setminus D^c && \text{(By Associative Law)} \\
&= [B^c \cap \{(A^c \cap A) \cap C^c\}]^c \setminus D^c && \text{(By Associative Law)} \\
&= [B^c \cap (\phi \cap C^c)]^c \setminus D^c && \text{(By Complement Law)} \\
&= [B^c \cap \phi]^c \setminus D^c && \text{(By Domination Law)} \\
&= \phi^c \setminus D^c && \text{(By Domination Law)} \\
&= \cup \setminus D^c && (\phi^c = \cup) \\
&= \cup \cap (D^c)^c && (X \setminus Y = X \cap Y^c) \\
&= \cup \cap D && \text{(By Double Complementation Law)} \\
&= D && \text{(By Identity Law)}
\end{aligned}
$$

2.4 APPLICATIONS OF SETS

We have already learnt about sets, type of sets, subsets, different operations and cardinal number of set. Now we shall discuss some important results based on number of elements in a set, which are highly useful. It is clear that if A and B are disjoint sets, then $|A \cup B| = |A| + |B|$. Basing on this concept, we can derive the following important results :

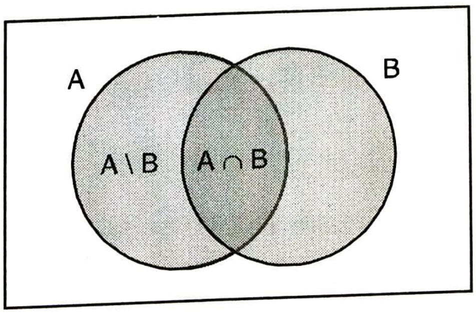
(i) $|A \setminus B| = |A| - |A \cap B|$

(ii) $|A \cup B| = |A| + |B| - |A \cap B|$

(iii) $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$

Proof :

(i)



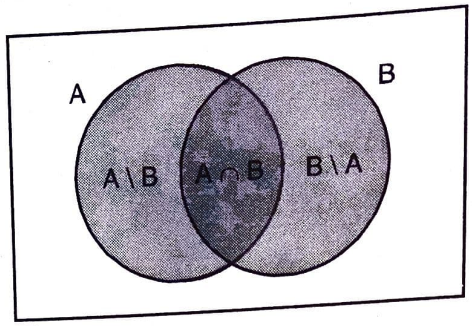
Clearly, $A = (A \setminus B) \cup (A \cap B)$

$\Rightarrow$ $|A| = |(A \setminus B) \cup (A \cap B)|$

$\Rightarrow$ $|A| = |(A \setminus B)| + |(A \cap B)|$ ($\because$ $(A \setminus B)$ and $(A \cap B)$ are disjoint)

$\Rightarrow$ $|A \setminus B| = |A| - |A \cap B|$

(ii)



Clearly, $A \cup B = A \cup (B \setminus A)$

$\Rightarrow$ $|A \cup B| = |A \cup (B \setminus A)|$

$\Rightarrow$ $|A \cup B| = |A| + |B \setminus A|$ .... (1)

($\because$ A and $B \setminus A$ are disjoint sets)

Also, $B = (B \setminus A) \cup (A \cap B)$

$\Rightarrow$ $|B| = |(B \setminus A) \cup (A \cap B)|$

$\Rightarrow$ $|B| = |(B \setminus A)| + |(A \cap B)|$ ($\because$ $(B \setminus A$ and $(A \cap B)$ are disjoint sets)

$\Rightarrow$ $|B \setminus A| = |B| - |A \cap B|$ .... (2)

From (1) and (2), we can get

$$|A \cup B| = |A| + |B| - |A \cap B|$$

(iii)

$$\begin{aligned}
\text{LHS} &= |A \cup B \cup C| \\
&= |A \cup D| && (\text{Taking } B \cup C = D) \\
&= |A| + |D| - |A \cap D| \\
&= |A| + |B \cup C| - |A \cap (B \cup C)| && (\text{Putting } D = B \cup D) \\
&= |A| + |B \cup C| + |(A \cap B) \cup (A \cap C)| \\
&= |A| + |B| + |C| - |B \cap C| - \{|A \cap B| + |A \cap C| - |A \cap B \cap A \cap C|\} \\
&= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C| \\
&= \text{RHS}
\end{aligned}$$

EXAMPLE 2.4.1. In a mathematics seminar attended by 100 delegates 75 can take Tea and 40 can take coffee. How many can take Tea only ? How many can take coffee only ? How many can take both Tea and Coffee?

SOLUTION

Let T denotes the set of delegates who can take Tea.

Let C denotes the set of delegates who can take coffee.

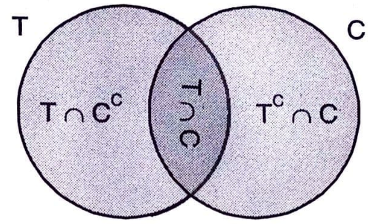
Thus, we have $|T \cup C| = 100, |T| = 75$ and $|C| = 40$

We know that $|T \cup C| = |T| + |C| - |T \cap C|$

$\Rightarrow$ $100 = 75 + 40 - |T \cap C|$

$\Rightarrow$ $|T \cap C| = 75 + 40 - 100 = 15$

So, 15 delegates can take both tea and coffee.



To find the number of delegates who can take tea only.

$$|T \cap C^c| = |T| - |T \cap C| = 75 - 15 = 60$$

So, 60 delegates can take tea only.

To find the number of delegates who can take coffee only.

$$|T^c \cap C| = |C| - |T \cap C| = 40 - 15 = 25$$

So, 25 delegates can take coffee only.

EXAMPLE 2.4.2. In a survey of 70 readers, it is found that 30 read 'The Samaj', 31 read 'The Samaya', 29 read 'The Dharitri', 12 read both Samaja and Samaya, 14 read both Samaya and Dharitri, 9 read both Dharitri and Samaj and 5 read all the three newspapers. Find the number of people who read atleast one of the three newspapers and also find the number of readers who read only 'The Samaj'. Find the number of readers who do not read any one of the three newspapers.

SOLUTION

Let A, B and C be the set of readers reading 'The Samaj', 'The Samaya' and 'The Dharitri' respectively. According to the question :

$$|A| = 30, |B| = 31, |C| = 29$$

$$|A \cap B| = 12, |B \cap C| = 14, |C \cap A| = 9, |A \cap B \cap C| = 5, |E| = 70$$

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |B \cap C| - |C \cap A| + |A \cap B \cap C|$$
$$= 30 + 31 + 29 - 12 - 14 - 9 + 5 = 60$$

So, the number of people who read atleast one of the three newspapers = 60.

Number of readers reading only 'the Samaj'.

$$= |A| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$
$$= 30 - 12 - 9 + 5 = 14$$

Number of readers who do not read any one of the three newspapers

$$= |E| - |A \cup B \cup C| = 70 - 60 = 10$$

2.5 CARTESIAN PRODUCT OF TWO SETS

Let A and B be two sets. The Cartesian product of A and B is denoted by A × B and is given by

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}$$

EXAMPLE 2.5.1. Find A × B and B × A if A = {1, 2} and B = {2, 4, 6}.

SOLUTION

$$A = \{1, 2\} \text{ and } B = \{2, 4, 6\}$$

$$A \times B = \{1, 2\} \times \{2, 4, 6\} = \{(1, 2), (1, 4), (1,6), (2, 2), (2, 4), (2, 6)\}$$

$$\begin{aligned} B \times A &= \{2, 4, 6\} \times \{1, 2\} \\ &= \{(2, 1), (2, 2), (4, 1), (4, 2), (6, 1), (6, 2)\} \end{aligned}$$

EXAMPLE 2.5.2. Prove the following :

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

SOLUTION

(i) Let $(x, y) \in A \times (B \cup C)$

$\Leftrightarrow$ $x \in A,\ y \in (B \cup C)$

$\Leftrightarrow$ $x \in A,\ (y \in B \text{ or } y \in C)$

$\Leftrightarrow$ $(x \in A, y \in B)$ or $(x \in A, y \in C)$

$\Leftrightarrow$ $(x, y) \in (A \times B)$ or $(x, y) \in (A \times C)$

$\Leftrightarrow$ $(x, y) \in (A \times B) \cup (A \times C)$

So, $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

and $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$

Hence $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(ii) Let $(x, y) \in A \times (B \cap C)$

$\Leftrightarrow$ $x \in A,\ y \in (B \cap C)$

$\Leftrightarrow$ $x \in A,\ (y \in B \text{ and } y \in C)$

$\Leftrightarrow$ $(x \in A,\ y \in B)$ and $(x \in A, y \in C)$

$\Leftrightarrow$ $(x, y) \in (A \times B)$ and $(x, y) \in (A \times C)$

$\Leftrightarrow$ $(x, y) \in (A \times B) \cap (A \times C)$

So, $A \times (B \cap C) \subseteq (A \times B) \cap (A \times C)$

and $(A \times B) \cap (A \times C) \subseteq A \times (B \cap C)$

Hence $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Note : (1) $|A \times B| = |A||B|$

(2) $(x, y) = (a, b) \Leftrightarrow x = a, y = b$

EXAMPLE 2.5.3. If $|A| = m$, $|B| = n$, then find $|P(A) \times P(B)|$

SOLUTION

$$|P(A) \times P(B)| = |P(A)||P(B)| = 2^m . 2^n = 2^{m+n}$$

EXAMPLE 2.5.4. Find x and y if $(2x + y, 5) = (3, 3x + 2y)$

SOLUTION

$$(2x + y, 5) = (3, 3x + 2y)$$

$\Rightarrow$ $2x + y = 3$ .... (1)

$3x + 2y = 5$ .... (2)

Equ. (1) × 2 $\quad 4x + 2y = 6$

Equ. (2) × 1

$$3x + 2y = 5$$
$$\underline{- \quad - \qquad -}$$
$$x = 1$$
$$2x + y = 3$$
$$2 + y = 3$$
$$\Rightarrow \quad y = 1$$
$$\therefore \quad x = 1, y = 1$$

EXAMPLE 2.5.5. If A and B are non-empty sets, then show that $A \times B = B \times A$ if and only if $A = B$.

SOLUTION

($\Rightarrow$:) Given that $\quad A \times B = B \times A$

To prove that $\quad A = B$

Let $\quad x \in A$

$\Rightarrow \quad (x, b) \in A \times B$

$\Rightarrow \quad (x, b) \in B \times A \qquad (\because A \times B = B \times A)$

$\Rightarrow \quad x \in B$

So, $\quad A \subseteq B \qquad$ .... (1)

Let $y \in B$

$\Rightarrow \quad (a, y) \in A \times B$

$\Rightarrow \quad (a, y) \in B \times A \qquad (\because A \times B = B \times A)$

$\Rightarrow \quad y \in A$

So, $\quad B \subseteq A \qquad$ .... (2)

From (1) and (2), we get

$$A = B$$

(: $\Leftarrow$) Given that $\quad A = B$

To prove that $\quad A \times B = B \times A$

$$\text{LHS} = A \times B$$
$$= A \times A \qquad (\because B = A)$$
$$\text{RHS} = B \times A$$
$$= A \times A \qquad (\because B = A)$$
$$\text{LHS} = \text{RHS}$$

EXERCISE –2

1. Prove the following :

   (i) $A \setminus B = A \cap B^c = B^c - A^c$

   (ii) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

   (iii) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

   (iv) $A \setminus \bigcup_{i=1}^{n} B_i = \bigcap_{i=1}^{n} (A \setminus B_i)$

(v) $A \setminus \bigcap_{i=1}^{n} B_i = \bigcup_{i=1}^{n} (A \setminus B_i)$

(vi) $A \subseteq B \Rightarrow B' \subseteq A'$

(vii) $A \cup B = E \Leftrightarrow A' \subseteq B$

(viii) $A \oplus (B \oplus C) = (A \oplus B) \oplus C$

(ix) $A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

(x) $A \cup B = U$ and $A \cap B = \phi \Rightarrow B = A'$

2. Prove that $P(A) \subseteq P(B)$ if and only if $A \subseteq B$.
3. If $\{x : p_1(x)\} = \{x : p_2(x)\}$, then show that for each x, $p_1(x)$ and $p_2(x)$ have the same truth values.
4. If A, B and C are three sets, then prove that

   $A \times (B - C) = (A \times B) - (A \times C)$
5. If A, B and C are three sets, then prove that

   $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
6. List all the subsets of the set $\{a, b, c, d\}$ that contain
   (i) four elements;
   (ii) three elements;
   (iii) two elements;
   (iv) one element;
   (v) no element.
7. (i) How many elements are in the power set of power set of the empty set ?

   (ii) Suppose A is a set containing one element. How many elements are in P(P(A)) ?
8. For sets A and B, prove that $A \cap B = A$ if and only if $A \subseteq B$.
9. For $n \in Z$, let $A_n = \{a \in Z \mid a \leq n\}$.

   Find each of the following sets

   (i) $A_3 \cup A_{-3}$ (ii) $A_3 \cap A_{-3}$

   (ii) $A_3 \cap (A_{-3})^c$ (iv) $\bigcap_{i=0}^{4} A_i$
10. Let $n \geq 1$ be a natural number. How many elements are in the set $\{(a, b) \in N \times N \mid a \leq b \leq n\}$?
11. Let A and B be sets. Find a necessary and sufficient condition for $A \oplus B = A$.
12. Let A and B be sets. Find a necessary and sufficient condition for $A \cup B = A \cap B$.
13. Let a and b be real numbers with $a < b$. Find $(a, b)^c$, $[a, b]^c$, $(a, \infty)^c$ and $(-\infty, b]^c$.
14. Prove that $(A \cap (B \setminus C))^C \cap A = (A \setminus B) \cup (A \cap C)$ for any sets A, B and C by using set identities.

ANSWERS

6. (i) $\{a, b, c, d\}$

   (ii) $\{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}$

   (iii) $\{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}$

(iv) $\{a\}, \{b\}, \{c\}, \{d\}$

(v) ϕ

7. (i) 2, (ii) 4

9. (i) A_3 (ii) A_{-3} (iii) $\{-3, -2, -1, 0, 1, 2, 3\}$ (iv) A_0

10. $\dfrac{n(n+1)}{2}$

11. $B = \phi$

12. $A = B$

13. $(a, b)^c = (-\infty, a] \cup [b, \infty)$

$[a, b]^c = (-\infty, a] \cup [b, \infty)$

$(a, \infty)^c = (-\infty, a]$

$(-\infty, b]^c = (b, \infty)$

□□□