Problem sheet for the pre-sessional course

Solutions

27th September 2019

Question 1. *Compute*

$$\int x^n \log x dx.$$

If $n = -1$, then $x^{-1} \log x = (\log x)' \log x$, hence

$$\int \frac{\log x}{x} dx = \frac{1}{2} \log^2 x + C.$$

If $n \neq -1$, then we can integrate by parts. Set $f'(x) = x^n$ and $g(x) = \log x$, so that $f(x) = \frac{x^{n+1}}{n+1}$ and $g'(x) = \frac{1}{x}$. Then

$$\begin{aligned} \int x^n \log x dx &= \frac{x^{n+1}}{n+1} \log x - \int \frac{x^{n+1}}{n+1} \cdot \frac{1}{x} dx \\ &= \frac{x^{n+1}}{n+1} \log x - \frac{x^{n+1}}{(n+1)^2} + C. \end{aligned}$$

Question 2. *Let*

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \quad t > 0, \quad x \in \mathbb{R}.$$

Compute $\frac{\partial u}{\partial t}$ *and* $\frac{\partial^2 u}{\partial x^2}$*, and show that* $u(x,t)$ *is a solution of the heat equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

We can compute directly using the product rule.

$$\begin{aligned} \frac{\partial u}{\partial t} &= \left(\frac{\partial}{\partial t} \frac{1}{\sqrt{4\pi t}} \right) e^{-\frac{x^2}{4t}} + \left(\frac{1}{\sqrt{4\pi t}} \right) \frac{\partial}{\partial t} e^{-\frac{x^2}{4t}} \\ &= -\frac{1}{2t\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} + \frac{1}{\sqrt{4\pi t}} \frac{x^2}{4t^2} e^{-\frac{x^2}{4t}}. \end{aligned}$$

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= -\frac{x}{2t}\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}. \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \\
&= \left(\frac{\partial}{\partial x}\left(-\frac{x}{2t}\right)\right)\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}} + \left(-\frac{x}{2t}\right)\left(\frac{\partial}{\partial x}\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}\right) \\
&= -\frac{1}{2t\sqrt{4\pi t}}e^{-\frac{x^2}{4t}} + \frac{x^2}{4t^2}\frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}.
\end{aligned}
$$

Question 3. *Let $X \sim U(a,b)$, i.e. X is a uniformly distributed continuous random variable with probability density function*

$$
f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \textit{otherwise.} \end{cases}
$$

Compute the mean, the variance, and the cumulative distribution function of X.

By using the definitions, we obtain

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{+\infty} x f(x)dx = \int_a^b \frac{x}{b-a}dx = \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{a+b}{2}. \\
E[X^2] &= \int_{-\infty}^{+\infty} x^2 f(x)dx = \int_a^b \frac{x^2}{b-a}dx = \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b = \frac{a^2+ab+b^2}{3}. \\
\operatorname{Var} X &= E[X^2] - E[X]^2 = \frac{a^2+ab+b^2}{3} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{12}. \\
F_X(x) &= P(X \le x) = \int_{-\infty}^{x} f(t)dt.
\end{aligned}
$$

If $x < a$, then

$$
\int_{-\infty}^{x} f(t)dt = 0.
$$

If $a \le x \le b$, then

$$
\int_{-\infty}^{x} f(t)dt = \int_{\infty}^{a} f(t)dt + \int_{a}^{x} f(t)dt = \int_a^x \frac{1}{b-a}dt = \left[\frac{t}{b-a}\right]_a^x = \frac{x-a}{b-a}.
$$

If $x > b$, then

$$
\int_{-\infty}^{x} f(t)dt = \int_{-\infty}^{a} f(t)dt + \int_a^b f(t)dt + \int_b^x f(t)dt = \int_a^b f(t)dt = \frac{b-a}{b-a} = 1.
$$

Therefore,

$$
F_X(x) = \begin{cases} 0 & x < a, \\ \frac{x-a}{b-a} & a \le x \le b, \\ 1 & x > b. \end{cases}
$$

Question 4. *Use risk-neutral valuation to find the value of an option on an asset with lognormal distribution the does not pay dividends with payoff at maturity given by* $\max\{\, S(T)^\alpha - K, 0\,\}$, *where* $\alpha > 0$ *is a given constant.*

As usual, we assume

$$S(T) = S(0)\exp\bigg(\Big(r - \frac{\sigma^2}{2}\Big)T + \sigma\sqrt{T}Z\bigg),$$

where Z is the standard normal random variable.

Using risk-neutral valuation, the value of the option is

$$V(0) = e^{-rT}E_{RN}[V(T)] = e^{-rT}E_{RN}[\max\{\, S(T)^\alpha - K, 0\,\}],$$

so we need to compute

$$E[h(Z)] = \int_{-\infty}^{+\infty} h(x)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx,$$

where $h(Z) = \max\{\, S(T)^\alpha - K, 0\,\}$.

First, we observe that h is non-zero when $S(T)^\alpha > K$, or $S(T) > k^{\frac{1}{\alpha}}$, so by rearranging the terms this is equivalent to

$$Z > \frac{1}{\sigma\sqrt{T}}\bigg(\log\frac{K^{\frac{1}{\alpha}}}{S(0)} - \Big(r - \frac{\sigma^2}{2}\Big)T\bigg) =: -a.$$

We can now use the explicit expression for $S(T)^\alpha$ to write

$$V(0) = e^{-rT}\int_{-a}^{+\infty}\bigg(S(0)^\alpha \exp\bigg(\alpha\Big(r - \frac{\sigma^2}{2}\Big)T + \alpha\sigma\sqrt{T}x\bigg) - K\bigg)\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx.$$

We can split the integral into its two summands by linearity. One of the two terms is well known:

$$-e^{-rT}\int_{-a}^{+\infty} K\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx = -Ke^{-rT}P(Z \geq -a) = -Ke^{-rT}N(a).$$

The other term is the following:

$$\frac{e^{-rT}S(0)^\alpha}{\sqrt{2\pi}}\int_{-a}^{+\infty}\exp\bigg(\alpha rT - \frac{\alpha\sigma^2}{2}T + \alpha\sigma\sqrt{T}x - \frac{x^2}{2}\bigg)\,dx. \tag{1}$$

In order to compute it, we can reduce it to the integral of the probability density function of the standard normal variable. First, we can complete the square to isolate the x-terms:

$$\frac{-x^2}{2} + \alpha\sigma\sqrt{T}x = -\frac{(x - \alpha\sigma\sqrt{T})^2}{2} + \frac{\alpha^2\sigma^2T}{2},$$

hence

$$\alpha rT - \frac{\alpha\sigma^2}{2}T + \alpha\sigma\sqrt{T}x - \frac{x^2}{2} = -\frac{(x-\alpha\sigma\sqrt{T})^2}{2} + \alpha rT - \frac{\alpha\sigma^2 T}{2} + \frac{\alpha^2\sigma^2 T}{2}.$$

We can then rewrite (1) as

$$S(0)^\alpha \exp\bigg((\alpha-1)\Big(r+\frac{\alpha\sigma^2}{2}\Big)T\bigg)\int_{-a}^{+\infty}\frac{1}{\sqrt{2\pi}}\exp\bigg(-\frac{(x-\alpha\sigma\sqrt{T})^2}{2}\bigg)dx,$$

and use the change of variable $y = x - \alpha\sigma\sqrt{T}$ to get

$$S(0)^\alpha \exp\bigg((\alpha-1)\Big(r+\frac{\alpha\sigma^2}{2}\Big)T\bigg)\int_{-a-\alpha\sigma\sqrt{T}}^{+\infty}\frac{1}{\sqrt{2\pi}}\exp\Big(-\frac{y^2}{2}\Big)dy.$$

Now the integral is known and we can rewrite it as

$$S(0)^\alpha \exp\bigg((\alpha-1)\Big(r+\frac{\alpha\sigma^2}{2}\Big)T\bigg)N(a+\alpha\sigma\sqrt{T}).$$

Therefore,

$$V(0) = S(0)^\alpha \exp\bigg((\alpha-1)\Big(r+\frac{\alpha\sigma^2}{2}\Big)T\bigg)N(a+\alpha\sigma\sqrt{T}) - Ke^{-rT}N(a),$$

where

$$a = \frac{1}{\sigma\sqrt{T}}\bigg(\log\frac{S(0)}{K^{\frac{1}{\alpha}}} + \Big(r-\frac{\sigma^2}{2}\Big)T\bigg).$$

Question 5. *Determine the Taylor series for* $\sin x$ *around 0.*

Recall that

$$f(x) = f(a) + f'(a)(x-a) + \frac{f^{(2)}(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots.$$

Since $a = 0$ and

$$\begin{aligned} f(x) &= \sin x, \\ f'(x) &= \cos x, \\ f^{(2)}(x) &= -\sin x, \\ f^{(3)}(x) &= -\cos x, \\ f^{(4)}(x) &= \sin x = f(x), \end{aligned}$$

we have that

$$\begin{aligned} f^{(4k)}(0) &= 0, \\ f^{(4k+1)}(0) &= 1, \\ f^{(4k+2)}(0) &= 0, \\ f^{(4k+3)}(0) &= -1. \end{aligned}$$

Therefore, we can write

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots,$$

or

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

Question 6. *Let $g(x) \in \mathcal{C}^\infty$, i.e. it is a infinitely differentiable function.*

(a) Write the quadratic approximation of $e^{g(x)}$ around 0.

(b) Use the result in (a) to compute the quadratic Taylor approximation around 0 of $e^{(x+1)^2}$.

(c) Verify the result in (b) by computing the quadratic Taylor approximation around 0 of $e^{(x+1)^2}$ using Taylor approximations of e^x and e^{x^2} instead.

(a) In order to write a quadratic approximation, we need terms up to the second derivative. We compute

$$\begin{aligned} f(x) &= e^{g(x)}, \\ f'(x) &= g'(x)e^{g(x)}, \\ f''(x) &= g''(x)e^{g(x)} + g'(x)\Big(g'(x)e^{g(x)}\Big) = \Big(g''(x) + (g'(x))^2\Big)e^{g(x)}. \end{aligned}$$

Then, the quadratic approximation around 0 is given by

$$\begin{aligned} f(x) &= f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + O(x^3) \\ &= e^{g(0)} + g'(0)e^{g(0)}x + \frac{g''(0) + (g'(0))^2}{2}e^{g(0)}x^2 + O(x^3). \end{aligned}$$

(b) For $g(x) = (x+1)^2$, we have $g'(x) = 2(x+1)$ and $g''(x) = 2$. Therefore, by substituting in the previous expression of $f(x)$ we obtain

$$e^{(x+1)^2} = e + 2ex + 3ex^2 + O(x^3).$$

(c) First, expand $e^{(x+1)^2} = e \cdot e^{2x} \cdot e^{x^2}$. Recall the Taylor formula for e^x:

$$e^x = 1 + x + \frac{x^2}{2} + O(x^3),$$

so we can compute

$$\begin{aligned} e^{2x} &= 1 + (2x) + \frac{(2x)^2}{2} + O((2x)^3) = 1 + 2x + 2x^2 + O(x^3), \\ e^{x^2} &= 1 + (x^2) + \frac{(x^2)^2}{2} + O((x^2)^3) = 1 + x^2 + O(x^4). \end{aligned}$$

We can now substitute in the two expressions to compute

$$\begin{aligned} e^{(x+1)^2} &= e \cdot e^{2x} \cdot e^{x^2} \\ &= e(1 + 2x + 2x^2 + O(x^3))(1 + x^2 + O(x^4)) \\ &= e(1 + x^2 + 2x + 2x^2 + O(x^3)) \\ &= e + 2ex + 3ex^2 + O(x^3). \end{aligned}$$

Note that here we have $x^m O(x^n) = O(x^{m+n})$ and $O(x^n)O(x^m) = O(x^{m+n})$, so we ignore any term of higher order when computing the product.

Question 7. *Consider the following first order ODE:*

$$\begin{aligned} y'(x) &= y(x), \quad x \in [0, 1] \\ y(0) &= 1. \end{aligned}$$

(a) Write a forward finite difference scheme to solve the equation, by discretising the domain using the nodes $x_i = ih$, with $h = \frac{1}{n}$.

(b) Use the result in (a) to show that $y_i = (1 + h)^i$ for $i = 0, \ldots, n$.

(a) Recall that the first order forward approximation of the first derivative is given by

$$y'(x) = \frac{y(x + h) - y(x)}{h} + O(h),$$

and since our ODE is $y'(x) = y(x)$, we have

$$y(x) = \frac{y(x + h) - y(x)}{h} + O(h).$$

By using $x = x_i$ we have $x + h = x_{i+1}$, and by using the approximation $y_i \approx y(x_i)$, we get

$$y_i = \frac{y_{i+1} - y_i}{h},$$

hence by rearranging the terms we obtain the forward scheme

$$y_{i+1} = hy_i + y_i = (1 + h)y_i.$$

(b) Since $y(0) = 1$ then we have $y_0 = 1$, and using the scheme it is straightforward to observe that $y_1 = 1 + h, y_2 = (1 + h)y_1 = (1 + h)^2, \ldots, y_i = (1 + h)^i$.

Question 8. *Assume that the function $V(S, I, t)$ satisfies the following PDE:*

$$\frac{\partial V}{\partial t} + S\frac{\partial V}{\partial I} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Consider the change of variable

$$V(S, I, t) = SH(R, t), \quad R = \frac{I}{S}.$$

Show that $H(R,t)$ satisfies the following PDE:

$$\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + (1 - rR)\frac{\partial H}{\partial R} = 0.$$

In order to do the change of variable, we have to replace all terms in V, including all derivatives.

We start by computing some derivatives that will be needed later:

$$\begin{aligned}
\frac{\partial R}{\partial I} &= \frac{1}{S}, \\
\frac{\partial R}{\partial S} &= -\frac{I}{S^2}, \\
\frac{\partial H}{\partial I} &= \frac{\partial H}{\partial R}\frac{\partial R}{\partial I} + \frac{\partial H}{\partial t}\frac{\partial t}{\partial I} = \frac{1}{S}\frac{\partial H}{\partial R}, \\
\frac{\partial H}{\partial S} &= \frac{\partial H}{\partial R}\frac{\partial R}{\partial S} + \frac{\partial H}{\partial t}\frac{\partial t}{\partial S} = -\frac{I}{S^2}\frac{\partial H}{\partial R}.
\end{aligned}$$

We can now compute the derivatives of V:

$$\begin{aligned}
V &= SH \\
\frac{\partial V}{\partial t} &= S\frac{\partial H}{\partial t} \\
\frac{\partial V}{\partial I} &= S\frac{\partial H}{\partial I} = \frac{\partial H}{\partial R} \\
\frac{\partial V}{\partial S} &= H + S\frac{\partial H}{\partial S} = H - \frac{I}{S}\frac{\partial H}{\partial R} = H - R\frac{\partial H}{\partial R} \\
\frac{\partial^2 V}{\partial S^2} &= \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \frac{\partial H}{\partial S} - \frac{\partial R}{\partial S}\frac{\partial H}{\partial R} - R\frac{\partial}{\partial S}\frac{\partial H}{\partial R}
\end{aligned}$$

Observe that $\frac{\partial H}{\partial S} = \frac{\partial H}{\partial R}\frac{\partial R}{\partial S}$, so the first two terms cancel out. Then,

$$\frac{\partial}{\partial S}\frac{\partial H}{\partial R} = \frac{\partial^2 H}{\partial R^2}\frac{\partial R}{\partial S} + \frac{\partial^2 H}{\partial R \partial t}\frac{\partial t}{\partial S} = \frac{\partial^2 H}{\partial R^2}\frac{\partial R}{\partial S} = -\frac{I}{S^2}\frac{\partial^2 H}{\partial R^2}.$$

Therefore,

$$\frac{\partial^2 V}{\partial S^2} = -R\frac{\partial}{\partial S}\frac{\partial H}{\partial R} = \frac{RI}{S^2}\frac{\partial^2 H}{\partial R^2} = \frac{R^2}{S}\frac{\partial^2 H}{\partial R^2}.$$

We can now substitute these expressions in the original equation:

$$\begin{aligned}
0 &= \frac{\partial V}{\partial t} + S\frac{\partial V}{\partial I} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV \\
&= S\frac{\partial H}{\partial t} + S\frac{\partial H}{\partial R} + \frac{\sigma^2 S^2}{2}\frac{R^2}{S}\frac{\partial^2 H}{\partial R^2} + rS\left(H - R\frac{\partial H}{\partial R}\right) - rSH \\
&= S\frac{\partial H}{\partial t} + S\frac{\sigma^2 R^2}{2}\frac{\partial^2 H}{\partial R^2} + S(1 - rR)\frac{\partial H}{\partial R},
\end{aligned}$$

and after dividing by S we obtain the required PDE.