Problem sheet for the pre-sessional course

27th September 2019

- You should attempt all questions and show all working, as you would do in an exam.
- Solutions must always show how they were obtained, and all non-trivial steps should be justified in a clear way.
- Even if you do not know how to completely solve an exercise, partial solutions are encouraged.
- It is recommended to avoid looking at the solution sheet at least until you have a decent attempt at a solution. Reading the question, thinking "I have no clue how to do this", and immediately checking the solution is not helpful; it is normal to not know immediately how solve an exercise, and thinking about how to do it is a key part of learning.
- The solutions I wrote try to highlight all the important steps that should appear in your solution, even if it is not identical. There may not be a unique way of solving an exercise: if your solution is drastically different from the one provided, make sure it is indeed correct in every part.

1. Compute
$$\int x^n \log x dx.$$

2. Let
$$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \quad t > 0, \quad x \in \mathbb{R}.$$
Compute $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$, and show that $u(x,t)$ is a solution of the heat equation
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}.$$

3. Let $X \sim U(a,b)$, i.e. X is a uniformly distributed continuous random variable with probability density function
$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

Compute the mean, the variance, and the cumulative distribution function of X.

4. Use risk-neutral valuation to find the value of an option on an asset with lognormal distribution the does not pay dividends with payoff at maturity given by $\max\{\, S(T)^\alpha - K, 0 \,\}$, where $\alpha > 0$ is a given constant.

5. Determine the Taylor series for $\sin x$ around 0.

6. Let $g(x) \in \mathcal{C}^\infty$, i.e. it is a infinitely differentiable function.

   (a) Write the quadratic approximation of $e^{g(x)}$ around 0.

   (b) Use the result in (a) to compute the quadratic Taylor approximation around 0 of $e^{(x+1)^2}$.

   (c) Verify the result in (b) by computing the quadratic Taylor approximation around 0 of $e^{(x+1)^2}$ using Taylor approximations of e^x and e^{x^2} instead.

7. Consider the following first order ODE:

$$\begin{aligned} y'(x) &= y(x), \quad x \in [0,1] \\ y(0) &= 1. \end{aligned}$$

   (a) Write a forward finite difference scheme to solve the equation, by discretising the domain using the nodes $x_i = ih$, with $h = \frac{1}{n}$.

   (b) Use the result in (a) to show that $y_i = (1+h)^i$ for $i = 0, \ldots, n$.

8. Assume that the function $V(S, I, t)$ satisfies the following PDE:

$$\frac{\partial V}{\partial t} + S\frac{\partial V}{\partial I} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

   Consider the change of variable

$$V(S, I, t) = SH(R, t), \quad R = \frac{I}{S}.$$

   Show that $H(R, t)$ satisfies the following PDE:

$$\frac{\partial H}{\partial t} + \frac{1}{2}\sigma^2 R^2 \frac{\partial^2 H}{\partial R^2} + (1 - rR)\frac{\partial H}{\partial R} = 0.$$