# Homework 1

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## 1 Linear Transformation

### 1.1 $\mathbb{E}[\boldsymbol{y}] = \mathbb{E}[A\boldsymbol{x} + b] = A\mathbb{E}[\boldsymbol{x}] + \boldsymbol{b}$

Since $\boldsymbol{y}$ is a continuous function, the expected value can be calculated as follows:

$$\begin{aligned}
\mathbb{E}[\boldsymbol{y}] &= \int f(\boldsymbol{y}) \cdot \boldsymbol{y} d\boldsymbol{y} \\
&= \int f(A\boldsymbol{x} + \boldsymbol{b})(A\boldsymbol{x} + \boldsymbol{b}) d\boldsymbol{x}
\end{aligned}$$

Note: since $\boldsymbol{x}$ is the only random variable, the probability density function, $f(A\boldsymbol{x} + \boldsymbol{b}) = f(\boldsymbol{x})$ and $\mathbb{E}[x] = \int f(\boldsymbol{x}) \cdot \boldsymbol{x} d\boldsymbol{x}$

$$\begin{aligned}
&= \int f(\boldsymbol{x}) \cdot (A\boldsymbol{x} + \boldsymbol{b}) d\boldsymbol{x} \\
&= A \int f(\boldsymbol{x}) \cdot \boldsymbol{x} d\boldsymbol{x} + \boldsymbol{b} \int f(\boldsymbol{x}) d\boldsymbol{x} \\
&= A\mathbb{E}[\boldsymbol{x}] + \boldsymbol{b}
\end{aligned}$$

### 1.2 $\textbf{cov}[\boldsymbol{y}] = A\boldsymbol{\Sigma}A^T$

$$\begin{aligned}
\text{cov}[\boldsymbol{y}] &= \mathbb{E}[(\boldsymbol{y} - \mathbb{E}[\boldsymbol{y}])(\boldsymbol{y} - \mathbb{E}[\boldsymbol{y}])^T] \\
&= \mathbb{E}[(A\boldsymbol{x} + \boldsymbol{b} - \mathbb{E}[A\boldsymbol{x} + \boldsymbol{b}])(A\boldsymbol{x} + \boldsymbol{b} - \mathbb{E}[A\boldsymbol{x} + \boldsymbol{b}])^T] \\
&= \mathbb{E}[A(\boldsymbol{x} - \mathbb{E}[\boldsymbol{x}])(\boldsymbol{x} - \mathbb{E}[\boldsymbol{x}])A^T] \\
&= A\mathbb{E}[(\boldsymbol{x} - \mathbb{E}[\boldsymbol{x}])(\boldsymbol{x} - \mathbb{E}[\boldsymbol{x}])]A^T \\
&= A\boldsymbol{\Sigma}A^T
\end{aligned}$$

## 2 Linear Regression - Least Squares Solution

$\mathcal{D} = \{(0, 1), (2, 3), (3, 6), (4, 8)\}$

### 2.1 Cramer's Rule

to find the least squares solution to the dataset, we can apply the cramer's rule as follows

$$m = \frac{n \sum_{i=1}^{n} x_i y_i - (\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} y_i)}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}$$

$$b = \frac{(\sum_{i=1}^{n} x_i^2)(\sum_{i=1}^{n} y_i) - (\sum_{i=1}^{n} x_i)(\sum_{i=1}^{n} x_i y_i)}{n \sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2}$$

$$\sum_{i=1}^{n} x_i = 0 + 2 + 3 + 4 = 9$$
$$\sum_{i=1}^{n} y_i = 1 + 3 + 6 + 8 = 18$$
$$\sum_{i=1}^{n} x_i^2 = 0 + 4 + 9 + 16 = 29$$
$$\sum_{i=1}^{n} x_i y_i = 0 + 6 + 18 + 32 = 56$$

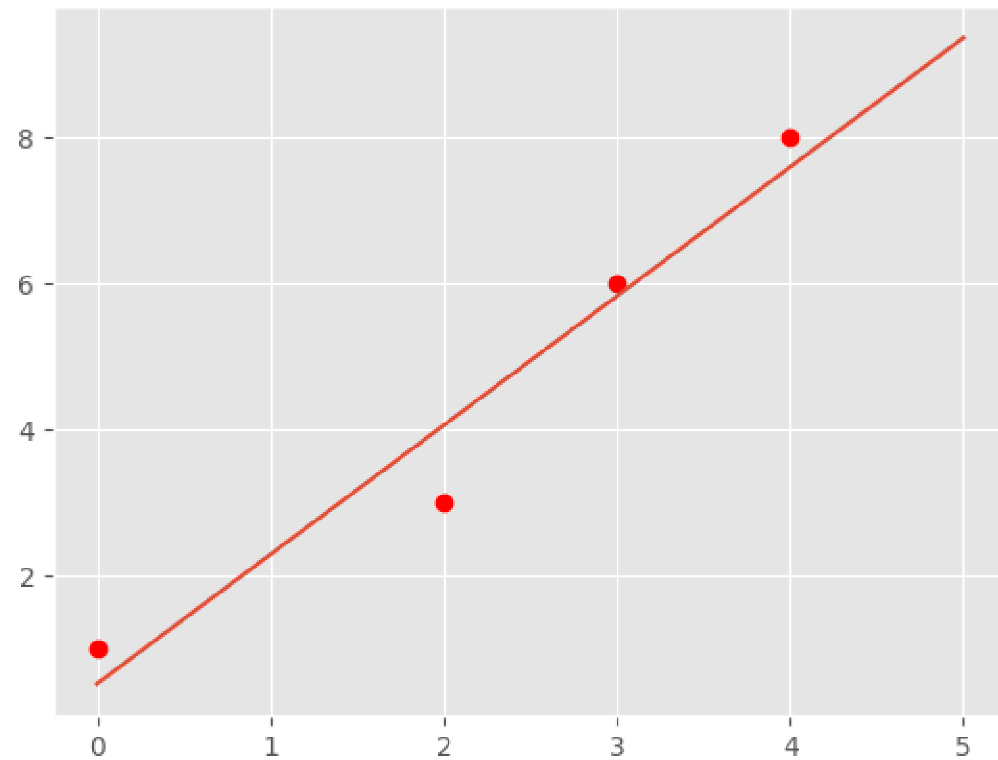
$$m = 62/35 \text{ and } b = 18/35$$

## 2.2 Normal Equations

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}$$

$$\begin{aligned} \boldsymbol{\theta}* &= (X^T X)^{-1} X^T \boldsymbol{y} \\ &= \left( \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 6 \\ 8 \end{bmatrix} \\ &= \frac{1}{35} \begin{bmatrix} 18 \\ 62 \end{bmatrix} \end{aligned}$$

### 2.3 Graph: least square solution



### 2.4 Graph: 100pts with Gaussian Noise