Unit-II Time Response Analysis

Time Response Analysis

→ Time Response of the S/m is the o/p of the closed loop system as a function of time [c(t)]

→ The response c(t) can be obtained from the transfer function & the i/p to the S/m.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + H(s)G(s)} = M(s)$$

$$C(s) = R(s)\,M(s).$$

$$c(t) = L^{-1}\left[R(s)\,M(s)\right]$$

→ The time response of a control S/m consists of two parts.

(i) transient response

(ii) steady state response.

→ Transient response: is the response of the s/m when the i/p changes from one state to another.

→ steady state response: is the response as time (t) approaches infinity.

→ standard test signals :-

Name of the signal (i/p)	Time domain equation of signal $r(t)$	$L[r(t)]$ (ie) $R(s)$
step	A	A/s
unit step	1	$1/s$
Ramp	At	A/s^2
unit Ramp	t	$1/s^2$
Parabolic	$At^2/2$	A/s^3
unit parabolic	$t^2/2$	$1/s^3$
Impulse	$\delta(t)$	1

→ Impulse Response ⇒ with i/p as impulse signal

$$R(s) = 1$$

$$M(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{C(s)}{R(s)}$$

$$\therefore C(s) = R(s)\left[\frac{G(s)}{1+G(s)H(s)}\right]$$

$$C(s) = \frac{G(s)}{1+G(s)H(s)}$$

$$c(t) = L^{-1}\left[\frac{G(s)}{1+G(s)H(s)}\right]$$

∴ Impulse response is the inverse LT of transfer function.

Order of a s/m :-

→ The i/p & o/p relationship of a ctrl s/m can be expressed by n^{th} order differential equation,

$$a_0 \frac{d^n}{dt^n} p(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} p(t) + \dots + a_n p(t) = b_0 \frac{d^m}{dt^m} q(t)$$

$$+ b_1 \frac{d^{m-1}}{dt^{m-1}} q(t) + \dots + b_m q(t).$$

$p(t) \rightarrow$ o/p / Response
$q(t) \rightarrow$ I/p / Excitation.

→ Also, order can be determined from the transfer function of the s/m.

$$T(s) = \frac{P(s)}{Q(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}.$$

→ $P(s) \Rightarrow$ Numerator polynomial.
$Q(s) \Rightarrow$ Denominator polynomial.

→ The order of the s/m is given by the maximum power of 's' in the denominator polynomial [Q(s)]

$$\therefore Q(s) = a_0 s^n + a_1 s^{n-1} + \dots + a_n.$$

$n \rightarrow$ order of the s/m.

$n = 0 \Rightarrow$ Zero order s/m
$n = 1 \Rightarrow 1^{st}$ order s/m
$n = 2 \Rightarrow 2^{nd}$ order s/m.

→ Type of the s/m : The numerator and denominator polynomial can be expressed in the factor form as shown in,

$$T(s) = \frac{P(s)}{Q(s)} = \frac{(s+z_1)(s+z_2)\ldots\ldots(s+z_m)}{(s+P_1)(s+P_2)\ldots\ldots(s+P_n)}.$$

→ here 'n' is the no. of poles. Therefore order of the s/m is given by the number of poles of the tfr function.

→ The No. of poles at the origin gives the type of the s/m.

Eg: $Q(s) = s^2(s+1)(s+2)$

no. of poles at origin ⇒ 2.

∴ It is type 2 s/m.

If no pole present at the origin, then it is type-0 s/m.

Recall : Partial fraction expansion.

Case 1 : Function with separate/distinct poles.

$$T(s) = \frac{K}{s(s+P_1)(s+P_2)}.$$

$$\frac{K}{s(s+P_1)(s+P_2)} = \frac{A}{s} + \frac{B}{s+P_1} + \frac{C}{s+P_2}.$$

$$A = T(s) \times s \Big|_{s=0}$$

$$B = T(s) \times (s+P_1) \Big|_{s=-P_1}$$

$$C = T(s) \times (s+P_2) \Big|_{s=-P_2}$$

Case 2: Tfr function with Multiple poles.

$$T(s) = \frac{K}{s(s+P_1)(s+P_2)^2}$$

$$\frac{K}{s(s+P_1)(s+P_2)^2} = \frac{A}{s} + \frac{B}{s+P_1} + \frac{C}{(s+P_2)} + \frac{D}{(s+P_2)^2}.$$

$$A = T(s) \times s \Big|_{s=0}$$

$$B = T(s) \times (s+P_1) \Big|_{s=-P_1}$$

$$C = \frac{d}{ds}\left[T(s) \times (s+P_2)^2\right]\Big|_{s=-P_2}.$$

$$D = T(s) \times (s+P_2)^2 \Big|_{s=-P_2}$$

Case 3: Tfr function with complex conjugate poles.

$$T(s) = \frac{K}{(s+P_1)(s^2+bs+c)}$$

$$\frac{K}{(s+P_1)(s^2+bs+c)} = \frac{A}{(s+P_1)} + \frac{Bs+C}{(s^2+bs+c)} \quad \text{—(1)}$$

$$A = T(s) \times (s+P_1) \Big|_{s=-P_1}$$

To solve for B & C, cross multiply the above equation (1), sub the value of (A) and the equate the like power of s.

eg: $1 = (s^2+s+1) + Bs^2 + 2Bs + Cs + 2C$

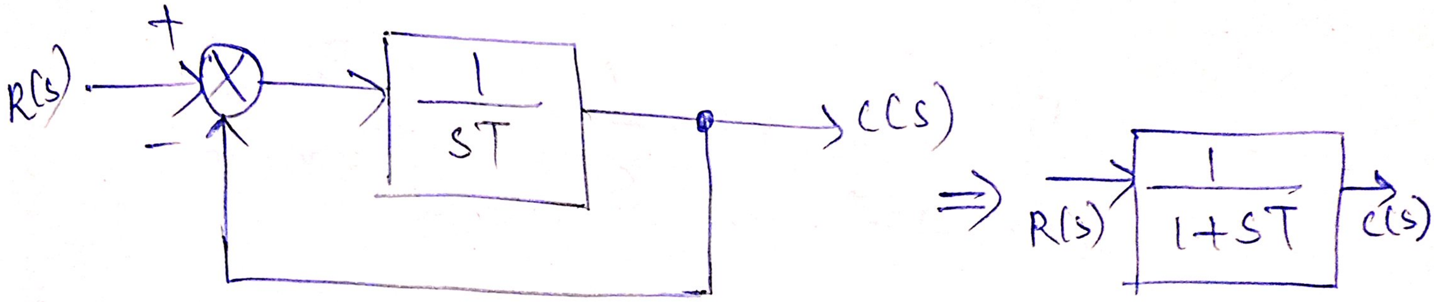
$1 = (1+B)s^2 + (1+2B+C)s + (1+2C)$

co-eff of s^2 $\Rightarrow$ $1+B=0$

co-eff of s $\Rightarrow$ $1+2B+C=0$

~~co-eff~~ constant $\Rightarrow$ $1+2C=1$.

Response of First-order s/m for unit-step i/p



$$\rightarrow \frac{C(s)}{R(s)} = \frac{1}{1+ST}.$$

→ unit step i/p, $r(t) = 1 \Rightarrow R(s) = 1/s$.

$$\rightarrow C(s) = R(s)\left[\frac{1}{1+ST}\right]$$

$$C(s) = \frac{1}{s}\left[\frac{1}{1+ST}\right]$$

$$C(s) = \frac{1}{s(1+ST)}$$

$$\rightarrow \frac{1}{s(1+sT)} = \frac{1}{sT\left(s+\frac{1}{T}\right)} = \frac{(1/T)}{s(s+1/T)}$$

$$\rightarrow \frac{1/T}{s(s+1/T)} = \frac{A}{s} + \frac{B}{(s+1/T)}$$

$$\rightarrow A = \frac{(1/T)}{\cancel{s}(s+1/T)} \times \cancel{s}\Bigg|_{s=0} = \frac{(1/T)}{(1/T)} = 1$$

$$\rightarrow B = \frac{(1/T)}{s\cancel{(s+1/T)}} \times \cancel{(s+1/T)}\Bigg|_{s=-1/T} = \frac{(1/T)}{(-1/T)} = -1$$

$$\rightarrow C(s) = \frac{1}{s(1+sT)} = \frac{A}{s} + \frac{B}{(s+1/T)}$$

$$= \frac{1}{s} + \frac{-1}{s+1/T}$$

$$\rightarrow c(t) = L^{-1}\left[\frac{1}{s} + \frac{(-1)}{s+1/T}\right]$$

$$\boxed{c(t) = \left(1 - e^{-t/T}\right)} \qquad \boxed{LT\left(e^{-at}\right) = \frac{1}{s+a}}$$

→ unit step response.

→ For step Response, (Amplitude $\neq 1$)

$$\boxed{c(t) = A\left(1 - e^{-t/T}\right)}, \quad T \rightarrow \text{time constant}$$

$$\rightarrow \boxed{c(t) = \begin{cases} (1 - e^{-t/T}), & \text{unit step i/p} \\ A(1 - e^{-t/T}), & \text{step i/p.} \end{cases}}$$

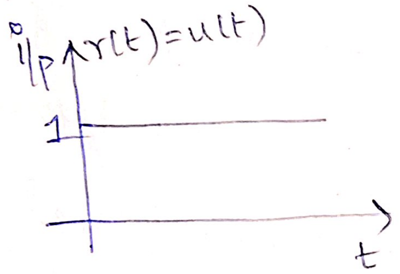
$\rightarrow$ $t = 0$, $c(t) = 1 - 1 = 0$

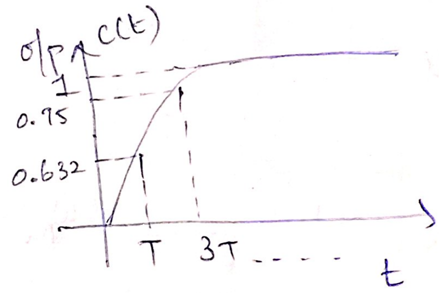
$t = 1T$, $c(t) = 1 - e^{-1} = 0.632$

$t = 2T$, $c(t) = 1 - e^{-2} = 0.865$

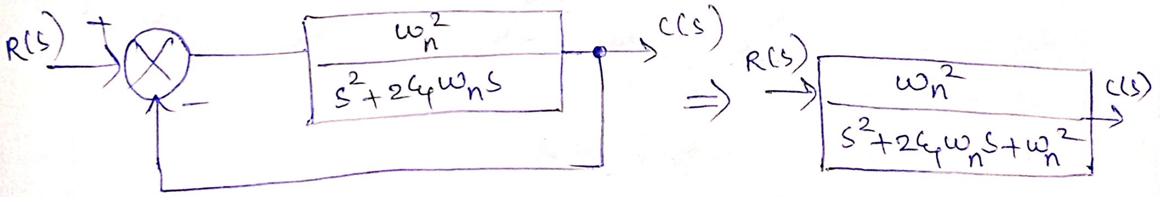
$t = 3T$, $c(t) = 1 - e^{-3} = 0.95$

$t = \infty$, $c(t) = 1 - e^{-\infty} = 1$





Second order s/m



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

$\omega_n \rightarrow$ undamped natural frequency (rad/sec)

$\xi \rightarrow$ Damping Ratio. (ξ)

$\rightarrow$ If $\xi = 0 \Rightarrow$ undamped

$0 < \xi < 1 \Rightarrow$ under damped s/m

$\xi = 1 \Rightarrow$ critically damped s/m

$\xi > 1 \Rightarrow$ Overdamped s/m.

→ characteristic equation of IInd - order s/m

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0.$$

Roots $\boxed{s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2 - 1}}$.

ξ	s_1, s_2	Roots & s/m
$\xi = 0$	$\pm j\omega_n$	Roots are purely imaginary. s/m → undamped.
$\xi = 1$	$-\omega_n$	Roots are Real & Equal. s/m → critically damped.
$\xi > 1$	$-\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$	Roots are real & unequal s/m → overdamped.
$0 < \xi < 1$	$-\xi\omega_n \pm \omega_n\sqrt{(-1)(1-\xi^2)}$ $-\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$ $-\xi\omega_n \pm j\omega_d$ where $\boxed{\omega_d = \omega_n\sqrt{1-\xi^2}}$ ↳ damped frequency of oscillation	Roots → complex conjugates s/m → underdamped

#1) Response of undamped II-order s/m for unit step i/p [$\zeta = 0$]

$$\rightarrow \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

$\rightarrow$ undamped s/m $\Rightarrow \zeta = 0$.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}.$$

$$C(s) = R(s)\left[\frac{\omega_n^2}{s^2+\omega_n^2}\right]$$

$\rightarrow$ i/p $\Rightarrow$ step i/p, $\therefore r(t) = u(t)$ & $R(s) = 1/s$.

$$C(s) = \left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2+\omega_n^2}\right)$$

$$\frac{\omega_n^2}{s(s^2+\omega_n^2)} = \frac{A}{s} + \frac{Bs+C}{(s^2+\omega_n^2)}.$$

$$\rightarrow A = \frac{\omega_n^2}{\cancel{s}(s^2+\omega_n^2)} \times \cancel{s}\Bigg|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$$\rightarrow \omega_n^2 = A(s^2+\omega_n^2) + Bs^2 + Cs.$$

$$\omega_n^2 = s^2 + \omega_n^2 + Bs^2 + Cs.$$

$$\cancel{\omega_n^2} = s^2(1+B) + \cancel{\omega_n^2} + Cs$$

$1 + B = 0 \Rightarrow B = -1$

$Cs = 0 \Rightarrow C = 0.$

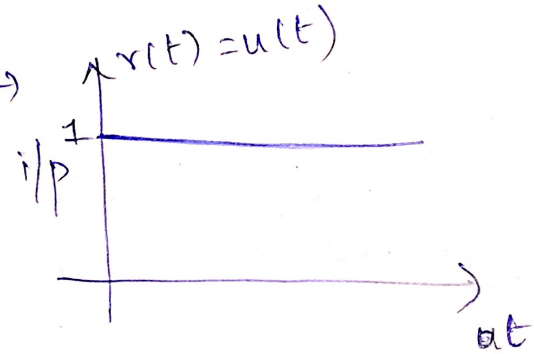
$\rightarrow$ $c(s) = \frac{1}{s} - \frac{s}{s^2+\omega_n^2}$

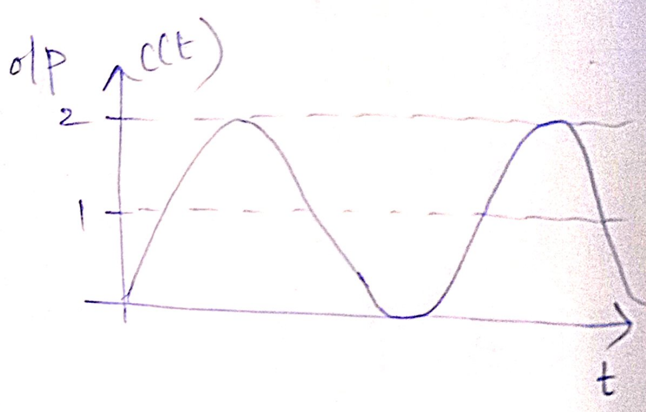
$\rightarrow$ $c(t) = L^{-1}\left[\frac{1}{s} - \frac{s}{s^2+\omega_n^2}\right]$

$LT(1) = 1/s$

$LT(\cos at) = \frac{s}{s^2+a^2}$

$$\boxed{c(t) = (1 - \cos\omega_n t)}$$

$\rightarrow$ 



$\rightarrow$ $\therefore$ For undamped second order s/m,

$$\boxed{c(t) = \begin{cases} (1-\cos\omega_n t), & \text{unit step i/p} \\ A(1-\cos\omega_n t), & \text{step i/p } (A\neq 1) \end{cases}}$$

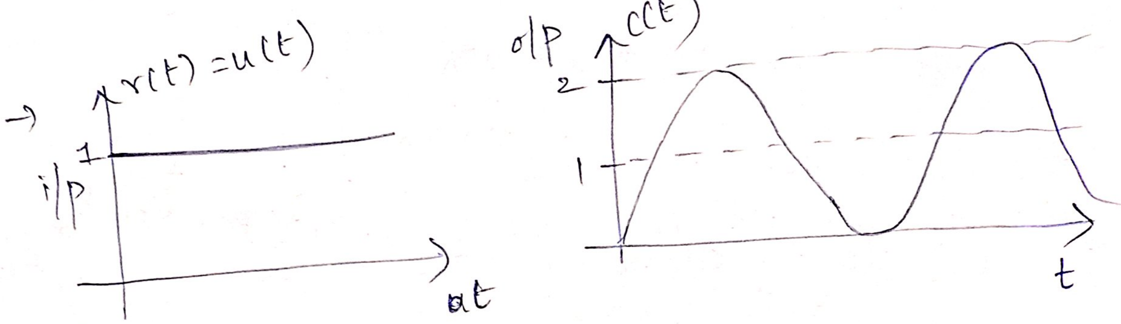
$$\rightarrow C(s) = \frac{1}{s} - \frac{s}{s^2+\omega_n^2}.$$

$$\rightarrow c(t) = L^{-1}\left[\frac{1}{s} - \frac{s}{s^2+\omega_n^2}\right]$$

$LT(1) = 1/s$

$LT(\cos at) = \frac{s}{s^2+a^2}$

$$\boxed{c(t) = (1 - \cos\omega_n t)}$$



$\rightarrow$ ∴ For undamped second order s/m,

$$\boxed{c(t) = \begin{cases} (1-\cos\omega_n t), & \text{unit step i/p} \\ A(1-\cos\omega_n t), & \text{step i/p } (A \neq 1) \end{cases}}$$

2) Response of Underdamped Second order s/m for Unit-Step I/p.

$\rightarrow$ Std. form of II^{nd} order CLTF is,

$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}}$$

→ For under-damped s/m, $0 < \xi < 1$ & the roots of the characteristic eqn. are complex conjugates.

$$s = -\xi\omega_n \pm j\omega_d .$$

→ unit step i/p ⇒ $r(t) = 1$ & $R(s) = 1/s$.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$C(s) = R(s)\left[\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}\right]$$

$$= \frac{\omega_n^2}{s\left[s^2 + 2\xi\omega_n s + \omega_n^2\right]}$$

→ By partial fraction,

$$\frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

$$A = \frac{\omega_n^2}{\cancel{s}(s^2 + 2\xi\omega_n s + \omega_n^2)} \times \cancel{s}\Bigg|_{s=0}$$

$$= \frac{\cancel{\omega_n^2}}{\cancel{\omega_n^2}} = 1$$

$$\boxed{A = 1}$$

$$\omega_n^2 = A\left(s^2 + 2\xi\omega_n s + \omega_n^2\right) + Bs^2 + Cs.$$

$$= s^2 + 2\xi\omega_n s + \omega_n^2 + Bs^2 + Cs.$$

$$\cancel{\omega_n^2} = s^2(1+B) + (2\xi\omega_n + C)s + \cancel{\omega_n^2}$$

$$1+B=0 \qquad 2\xi\omega_n + C = 0$$

$$\boxed{B=-1} \qquad \boxed{C=-2\xi\omega_n}$$

$$\rightarrow \therefore C(s) = \frac{1}{s} + \frac{(-s - 2\xi\omega_n)}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

$$C(s) = \frac{1}{s} - \frac{(s + 2\xi\omega_n)}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

$$= \frac{1}{s} - \frac{s + \xi\omega_n}{(s^2 + 2\xi\omega_n s + \omega_n^2)} - \frac{\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} - \frac{s + \xi\omega_n}{((s+\xi\omega_n) + j\omega_d)((s+\xi\omega_n) - j\omega_d)}$$

$$- \frac{\xi\omega_n}{((s+\xi\omega_n) + j\omega_d)((s+\xi\omega_n) - j\omega_d)}$$

$$= \frac{1}{s} - \frac{(s+\xi\omega_n)}{(s+\xi\omega_n)^2 + \omega_d^2} - \frac{\xi\omega_n}{(s+\xi\omega_n)^2 + \omega_d^2}.$$

$$C(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\omega_d(\zeta\omega_n)}{\omega_d\left((s+\zeta\omega_n)^2+\omega_d^2\right)}$$

$$c(t) = L^{-1}\left[C(s)\right]$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos(\omega_d t) - \frac{\zeta\omega_n}{\omega_d} e^{-\zeta\omega_n t}\sin(\omega_d t)$$

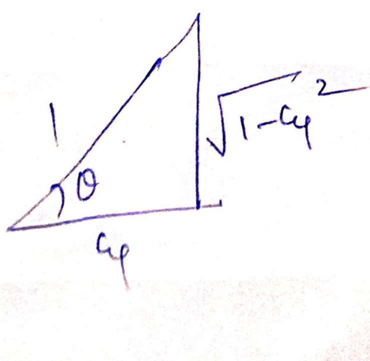
$$\because \; L\{e^{-at}\sin\omega t\} = \frac{\omega}{(s+a)^2+\omega^2}$$

$$L\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2+\omega^2}$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos(\omega_d t) + \frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\sin(\omega_d t)\right]$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right]$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\cos(\omega_d t)\times\sqrt{1-\zeta^2} + \zeta\sin(\omega_d t)\right]$$



$$\sin\theta = \sqrt{1-\zeta^2}$$

$$\cos\theta = \zeta$$

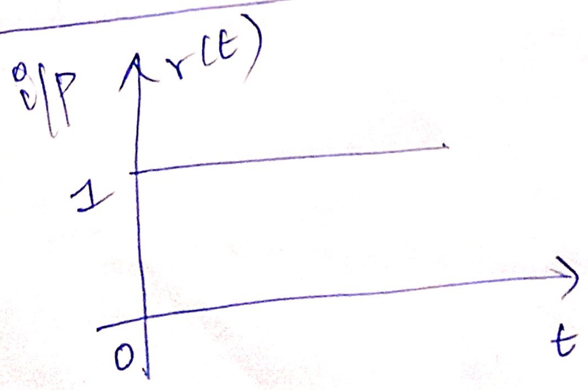
$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left[\cos(\omega_d t)\sin\theta + \sin(\omega_d t)\cos\theta\right]$$

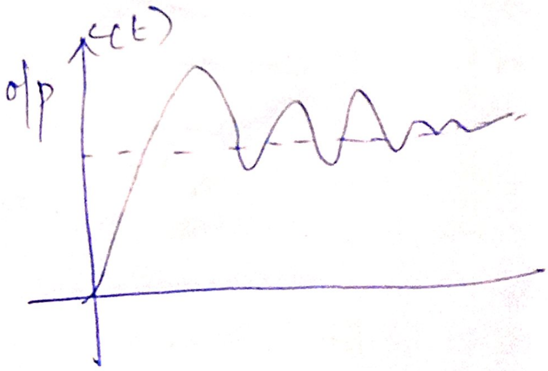
$$\boxed{c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)}$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

∴ For under-damped unit step Response for II nd order s/m,

$$\boxed{c(t) = \begin{cases} 1 - \dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta), & \text{unit step} \\ A\left(1 - \dfrac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)\right), & \text{Step.} \end{cases}}$$





#3) Critically damped Second-order S/m for unit Step - i/p

$\rightarrow \quad \dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

$\rightarrow$ For critical damping $\zeta = 1$

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\omega_n)^2}$$

$\rightarrow$ i/p $r(t) = 1$, $R(s) = 1/s$.

$$C(s) = R(s)\left(\frac{\omega_n^2}{(s+\omega_n)^2}\right)$$

$$= \frac{1}{s}\left(\frac{\omega_n^2}{(s+\omega_n)^2}\right)$$

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2}$$

$$\frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)} + \frac{C}{(s+\omega_n)^2}$$

$\rightarrow \quad A = \left.\dfrac{\omega_n^2}{(s+\omega_n)^2}\right|_{s=0} = 1$

$$B = \left.\frac{d}{ds}\left(\cancel{(s+\omega_n)^2} \cdot \frac{\omega_n^2}{s\cancel{(s+\omega_n)^2}}\right)\right|_{s=-\omega_n} = -\frac{\omega_n^2}{s^2} = -1$$

$$c = \frac{\omega_n^2}{s\cancel{(s+\omega_n)^2}} \times \cancel{(s+\omega_n)^2} \Bigg|_{s=-\omega_n}$$

$$c = \frac{\omega_n^2}{-\omega_n} = -\omega_n$$

$$c(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

$$c(t) = L^{-1}\left[c(s)\right]$$

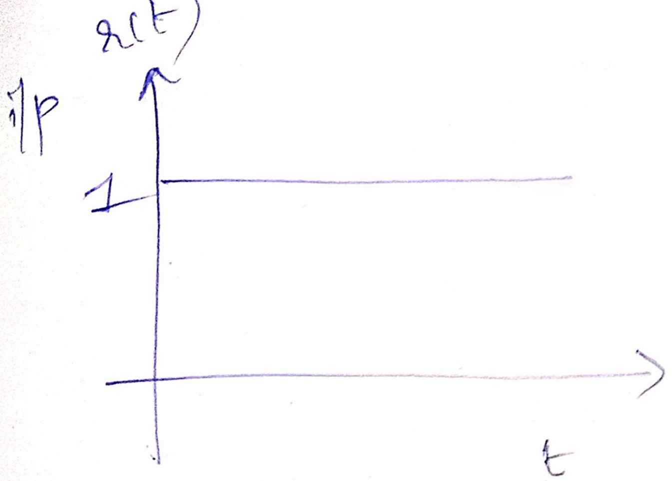
$$\boxed{c(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}}$$

$$\because L\left\{t e^{-at}\right\} = \frac{1}{(s+a)^2}.$$

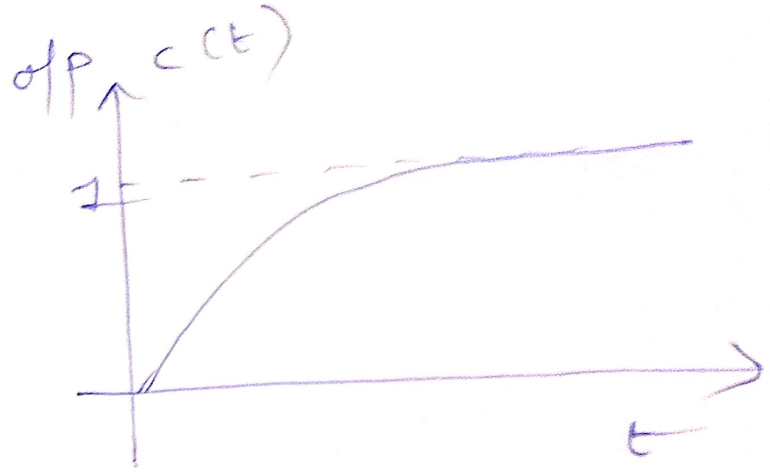
→ ∴ closed loop critically damped II-order s/m,

Response is

$$c(t) = \begin{cases} 1 - e^{-\omega_n t}(1+\omega_n t), & \text{unit step i/p} \\ A\left[1 - e^{-\omega_n t}(1+\omega_n t)\right], & \text{step i/p.} \end{cases}$$



i/p
r(t)
1
t



o/p
c(t)
1
t

#4). Response of over damped second order for unit step i/p.

→ second order s/m general tfr func.,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}.$$

→ over damped ⇒ $\zeta>1$, Roots of the polynomial are,

$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

$$\therefore s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} = -\left[\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]$$

& $$s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} = -\left[\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]$$

→ $$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{\omega_n^2}{(s+s_1)(s+s_2)}.$$

→ $$C(s) = R(s)\frac{\omega_n^2}{(s+s_1)(s+s_2)}$$

→ unit step i/p, $r(t) = u(t)$, $R(s) = 1/s$.

$$C(s) = \frac{\omega_n^2}{s(s+s_1)(s+s_2)}.$$

$$\frac{\omega_n^2}{s(s+s_1)(s+s_2)} = \frac{A}{s} + \frac{B}{s+s_1} + \frac{C}{s+s_2}.$$

$$A = s\times C(s)\Big|_{s=0} = \frac{\omega_n^2}{s_1 s_2} = \frac{\omega_n^2}{\zeta^2\omega_n^2 - \omega_n^2(\zeta^2-1)}$$

$$= \frac{\omega_n^2}{\omega_n^2}$$

$$\boxed{A = 1}$$

$$\rightarrow \quad B = (s+s_1)\, C(s)\Big|_{s=-s_1}$$

$$= \frac{\omega_n^2}{s(s+s_2)}\Big|_{s=-s_1}$$

$$= \frac{\omega_n^2}{-s_1(-s_1+s_2)}$$

$$= \frac{\omega_n^2}{\left[\xi\omega_n - \omega_n\sqrt{\xi^2-1}\right]\left[\xi\omega_n - \omega_n\sqrt{\xi^2-1} - \xi\omega_n - \omega_n\sqrt{\xi^2-1}\right]}$$

$$= \frac{\omega_n^2}{\left[\xi\omega_n - \omega_n\sqrt{\xi^2-1}\right]\left[-2\omega_n\sqrt{\xi^2-1}\right]}$$

$$= \frac{\omega_n^2}{-s_1\left(-2\omega_n\sqrt{\xi^2-1}\right)}$$

$$\boxed{B = \frac{\omega_n}{2s_1\sqrt{\xi^2-1}}}$$

$$\rightarrow \quad C = (s+s_2)\, C(s)\Big|_{s=-s_2}.$$

$$= \frac{\omega_n^2}{s(s+s_1)}\Big|_{s=-s_2}$$

$$= \frac{\omega_n^2}{-s_2(-s_2+s_1)} = \frac{\omega_n^2}{-s_2\left(\xi\omega_n + \omega_n\sqrt{\xi^2-1} - \xi\omega_n + \omega_n\sqrt{\xi^2-1}\right)}$$

$$= \frac{\omega_n^2}{-s_2\left(2\omega_n\sqrt{\xi^2-1}\right)}$$

$$\boxed{C = -\frac{\omega_n}{2s_2\sqrt{\xi^2-1}}}$$

$$\rightarrow C(s) = \frac{1}{s} + \frac{\omega_n}{2\sqrt{\xi^2-1}\,[s_1](s+s_1)} - \frac{\omega_n}{2\sqrt{\xi^2-1}\,(s_2)(s+s_2)}$$

$$c(t) = L^{-1}[C(s)]$$

$$= 1 + \frac{\omega_n}{2\sqrt{\xi^2-1}} \cdot \frac{1}{s_1} e^{-s_1 t} - \frac{\omega_n}{2\sqrt{\xi^2-1}} \cdot \frac{1}{s_2} e^{-(s_2 t)}$$

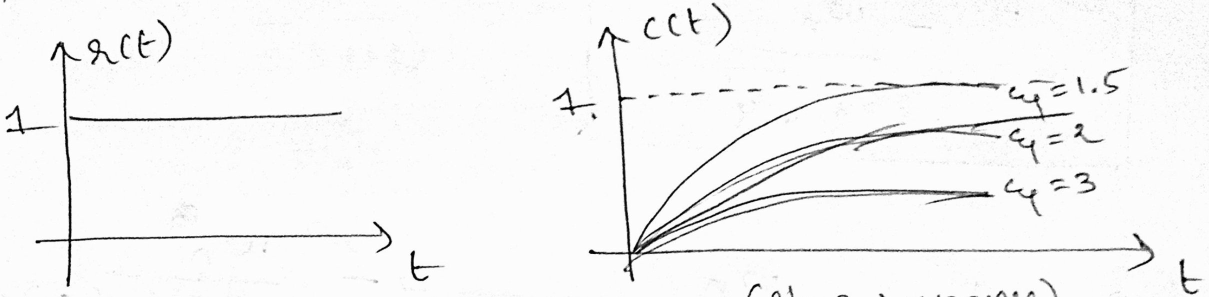
$$\boxed{c(t) = 1 + \frac{\omega_n}{2\sqrt{\xi^2-1}}\left[\frac{1}{s_1}e^{-s_1 t} - \frac{1}{s_2}e^{-s_2 t}\right]}$$

→ For overdamped II-nd order step Response,

$$c(t) = \begin{cases} 1 + \dfrac{\omega_n}{2\sqrt{\xi^2-1}}\left(\dfrac{1}{s_1}e^{-s_1 t} - \dfrac{1}{s_2}e^{-s_2 t}\right), & \text{unit step i/p} \\[2ex] A\left[1 + \dfrac{\omega_n}{2\sqrt{\xi^2-1}}\left(\dfrac{1}{s_1}e^{-s_1 t} - \dfrac{1}{s_2}e^{-s_2 t}\right)\right], & \text{step i/p} \end{cases}$$

$$s_1 = -\left[\xi\omega_n - \omega_n\sqrt{\xi^2-1}\right] \quad \& \quad s_2 = -\left[\xi\omega_n + \omega_n\sqrt{\xi^2-1}\right]$$

→ Response of over-damped II - order s/m, unit - step i/p



(slow response)

Time Domain Specifications :-

→ The desired performance characteristics of a s/m of any order may be specified in terms of the transient response to the unit step signal.

→ The transient response of a s/m to a unit step i/p depends on the initial conditions.

→ The transient response characteristics of a control s/m to a unit step i/p is specified in terms of the following time domain specifications.

(i) Delay time (t_d)

(ii) Rise time (t_r)

(iii) Peak time (t_p)

(iv) Maximum overshoot (M_p)

(v) Settling time (t_s).

(i) Delay time (t_d):- The time taken for response to reach 50% of the final value, for the first time.

(ii) Rise time (t_r):- The time taken for response to raise from 0 to 100% for the very first time.

But for over-damped s/m ⇒ 10% to 90%.

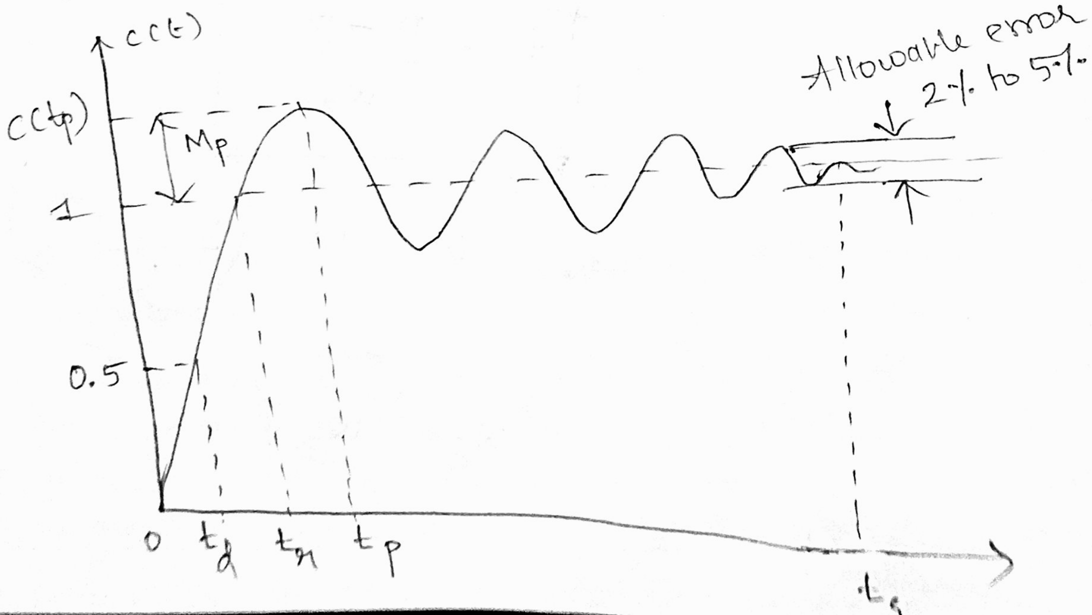
critically damped ⇒ 5% to 95%.

(iii) Peak time (t_p):- time taken for the response to reach the peak value, the very first time. It can also be defined as the time taken for the response to reach the peak overshoot (M_p).

(iv) Peak overshoot (M_p) :- it is defined as the ratio of the maximum peak value above the final value to the final value.

$$\% M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

(v) Settling time (t_s) :- It is defined as the time taken by the response to reach & stay within a specified error (2% or 5%).



Expressions for the time domain Specifications :-

(i) Rise time (t_r) :-

→ For under-damped s/m, unit step response is,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

→ $\boxed{\text{At } t = t_r, \quad c(t) = 1.}$

$$\frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 0.$$

→ $\because e^{-\zeta\omega_n t_r} \neq 0$

$$\sin(\omega_d t_r + \theta) = 0.$$

→ for $\phi = 0, \pi, 2\pi, 3\pi, \ldots$ $\sin(\phi) = 0$.

$$\omega_d t_r + \theta = \pi.$$

$$\omega_d t_r = \pi - \theta$$

$$\boxed{t_r = \frac{\pi - \theta}{\omega_d}}$$

where $\theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

ii) peak time (t_p) :-

→ maximum value at t_p.

$$\boxed{\therefore \frac{d}{dt} c(t) \Big|_{t=t_p} = 0.}$$

$$\rightarrow \frac{d}{dt}[c(t)] = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t).$$

$$\rightarrow \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_p} \sin(\omega_d t_p) = 0.$$

$$\rightarrow \sin(\omega_d t_p) = 0.$$

$$\omega_d t_p = \pi$$

$$\boxed{t_p = \frac{\pi}{\omega_d}}.$$

(iii) peak overshoot (M_p) :-

$$\% M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\ \%.$$

At $t = \infty$, $c(\infty) = 1$

At $t = t_p$, $c(t_p) = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$.

$$\% M_p = \frac{\left[1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\right] - 1}{1} \times 100\ \%.$$

$$\boxed{\% M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100}$$

(iv) Settling time (t_s)

→ The response of II-order s/m has two components. They are

(i) Decaying exponential, $\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}$

(ii) Sinusoidal component, $\sin(\omega_d t+\theta)$.

→ The settling time is decided by the Decaying exponential term since the sinusoidal signal is reduced by the Decaying exp. component.

→ For 2% tolerance, at $t=t_s$

$$\frac{e^{-\xi\omega_n t_s}}{\sqrt{1-\xi^2}} = 0.02$$

$$e^{-\xi\omega_n t_s} = 0.02 \qquad \because \sqrt{1-\xi^2} \simeq 1.$$

$$-\xi\omega_n t_s = \ln(0.02)$$

$$-\xi\omega_n t_s = -4$$

$$\boxed{t_s = \frac{4}{\xi\omega_n} = 4T} \rightarrow 2\% \text{ error}$$

→ Illy $\boxed{t_s = \frac{3}{\xi\omega_n} = 3T} \rightarrow 5\%$ error.

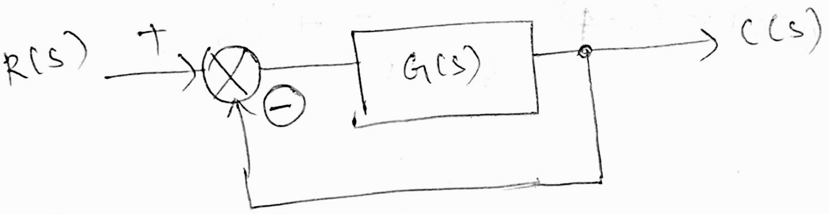
→ Generally, $\boxed{t_s = \dfrac{\ln(\%\,\text{error})}{\zeta\omega_n} = \dfrac{\ln(\%\,\text{error})}{T}}$

Eqs

i) obtain the response of unity fb s/m whose open loop transfer function is $\dfrac{4}{s(s+5)}$ and when the i/p is unit step.

solu.

OLTF, $G(s) = \dfrac{4}{s(s+5)}$



CLTF, $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)}$

$$= \frac{\dfrac{4}{s(s+5)}}{1+\dfrac{4}{s(s+5)}}$$

$$= \frac{4}{s(s+5)+4}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{4}{s^2+5s+4}} = \frac{4}{(s+1)(s+4)}$$

$$c(s) = R(s)\left[\frac{4}{\cancel{s}(s+1)(s+4)}\right]$$

$\rightarrow$ unit step i/p $\Rightarrow$ $r(t)=1$ & $R(s)=1/s$.

$$c(s) = \frac{\cancel{8}\ 4}{s(s+1)(s+4)} = \frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+4}.$$

$$\rightarrow A = c(s)\times s\Big|_{s=0} = 4/4 = 1.$$

$$B = c(s)\times(s+1)\Big|_{s=-1} = -4/3$$

$$C = c(s)\times(s+4)\Big|_{s=-4} = +1/3.$$

$$\rightarrow \therefore c(s) = \frac{\cancel{8}1}{s} - \frac{4}{3(s+1)} + \frac{1}{3(s+4)}$$

$$\rightarrow c(t) = L^{-1}[c(s)]$$

$$= 1\cancel{8} - \frac{4}{3}e^{-t} + \frac{1}{3}e^{-4t}.$$

$$\boxed{c(t) = 1 - \frac{1}{3}\left(4e^{-t} - e^{-4t}\right)}$$

2) The response of a servo-mechanism is,
$c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$ when subject to a unit step i/p. Obtain an expression for closed loop transfer function. Determine the undamped natural freq. & damping ratio.

Soln

$$c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$$

$$C(s) = \frac{1}{s} + 0.2\frac{1}{(s+60)} - \frac{1.2}{(s+10)}$$

$$= \frac{(s+60)(s+10) + (0.2)(s)(s+10) - 1.2s(s+60)}{s(s+60)(s+10)}$$

$$C(s) = \frac{600}{s(s+60)(s+10)}$$

$$C(s) = R(s)\frac{600}{(s+60)(s+10)}. \qquad \left[\because R(s) = 1/s\right].$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{600}{(s+60)(s+10)}} \Rightarrow \text{CLTF}.$$

$$\frac{C(s)}{R(s)} = \frac{600}{s^2 + 70s + 600}.$$

→ General II - order eqn. is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

→ ∴ $2\xi\omega_n s = 70s$ & $\omega_n^2 = 600$

$\omega_n = \sqrt{600}$

$= 10\sqrt{6}.$

$\boxed{\omega_n = 24.49 \text{ rad/sec.}}$

$2 \times \cancel{10}\sqrt{6} \times \xi = 7\cancel{0}$

$\xi = \frac{7}{2\sqrt{6}}.$

$= \frac{7}{4.90}$

$= 1.4285$

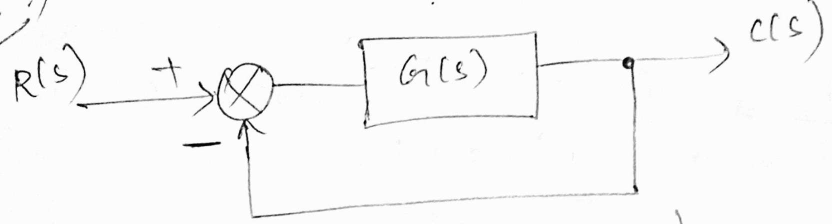
$\boxed{\xi \simeq 1.43.}$

Eg:

1) A unity feedback control s/m is characterized by the following open loop transfer function $G(s) = \frac{(0.4s+1)}{s(s+0.6)}$

Determine its transient response for unit step i/p & sketch the response. Evaluate the maximum overshoot and corresponding peak time.

solu :)



$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{\frac{(0.4s+1)}{s(s+0.6)}}{1+\frac{(0.4s+1)}{s(s+0.6)}}$$

$$= \frac{0.4s+1}{s(s+0.6)+0.4s+1}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{0.4s+1}{s^2+s+1}.}$$

$\rightarrow r(t) = u(t) \Rightarrow R(s) = 1/s.$

$$C(s) = R(s)\left(\frac{0.4s+1}{s^2+s+1}\right) = \frac{1}{s}\frac{(0.4s+1)}{(s^2+s+1)}$$

$$\frac{0.4s+1}{s(s^2+s+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1}.$$

$$A = C(s) \times s\Big|_{s=0} \Rightarrow \boxed{A=1}$$

$$\rightarrow \quad 0.4s + 1 = A(s^2+s+1) + (Bs+C)s$$

$$0.4s+1 = s^2+s+1 + Bs^2 + Cs.$$

$$1+0.4s = (1+B)s^2 + (1+C)s + 1$$

$\rightarrow$ equate $\Rightarrow$ $1+B = 0 \Rightarrow \boxed{B=-1}$

$1+C = 0.4 \Rightarrow \boxed{C = -0.6}$

$$\rightarrow c(s) = \frac{1}{s} + \frac{(-1)s + (-0.6)}{s^2+s+1}.$$

$$= \frac{1}{s} - \frac{(s+0.6)}{(s^2+s+1)}$$

$s^2+s+1 \Rightarrow$ to have a perfect square

$$s^2+s+1 + 0.25 - 0.25$$

$$(s^2+s+0.25) + 0.75$$

$$(s+0.5)^2 + 0.75$$

$$\rightarrow c(s) = \frac{1}{s} - \frac{(s+0.5)}{(s+0.5)^2+0.75} - \frac{0.1}{(s+0.5)^2+0.75}$$

$$= \frac{1}{s} - \frac{s+0.5}{(s+0.5)^2+0.75} - \frac{0.1 \times \sqrt{0.75}}{\sqrt{0.75}\left[(s+0.5)^2+0.75\right]}$$

$$= \frac{1}{s} - \frac{s+0.5}{(s+0.5)^2+0.75} - \frac{0.1}{\sqrt{0.75}} \frac{\sqrt{0.75}}{(s+0.5)^2+0.75}$$

$$c(t) = \mathcal{L}^{-1}[C(s)]$$

$$= 1 - e^{-0.5t} \cos(\sqrt{0.75}t) - 0.115 e^{-0.5t} \sin(\sqrt{0.75}t)$$

$$c(t) = 1 - e^{-0.5t}\left[\cos(\sqrt{0.75}\,t) + 0.115 \sin(\sqrt{0.75}\,t)\right]$$

→ Transient response is the response that vanishes (ie equal to zero) when $t \to \infty$.

In the above eqn, the 2nd term, as $t \to \infty$, becomes zero,

$$\therefore c(t)\Big|_{\text{transient}} = e^{-0.5t}\left[\cos(\sqrt{0.75}\,t) + 0.115 \sin(\sqrt{0.75}\,t)\right]$$

Note:

→ $c(t)$ ⇒ Total Response

Total Response = Transient Response + Steady state Response

$$c(t) = c_t(t) + c_{ss}(t).$$

$$\boxed{c(t)\Big|_{t\to\infty} = c_{ss}(t)}$$

since $\boxed{\underset{t\to\infty}{c_t(t)} = 0}$

→ Maximum overshoot (M_p)

$$\% M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} \times 100 \%$$

$$s^2+s+1 \Rightarrow s^2 + 2\xi\omega_n s + \omega_n^2$$

$$\omega_n^2 = 1 \Rightarrow \omega_n = 1 \text{ rad/sec}.$$

$$2\xi\omega_n = 1$$

$$\xi = \frac{1}{2} = 0.5 //$$

$$\% M_p = e^{-0.5\pi/\sqrt{1-0.5^2}} \times 100 \%$$

$$= e^{-1.813} \times 100 \%$$

$$\boxed{\% M_p = 16.32 \%}$$

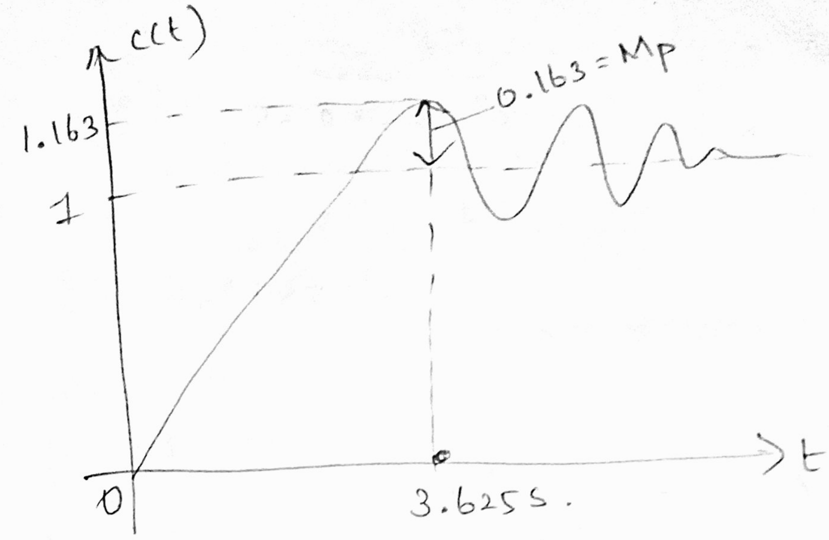
$$M_p = 0.163.$$

→ peak time $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$

$$= \frac{\pi}{(1)\sqrt{1-0.25}}$$

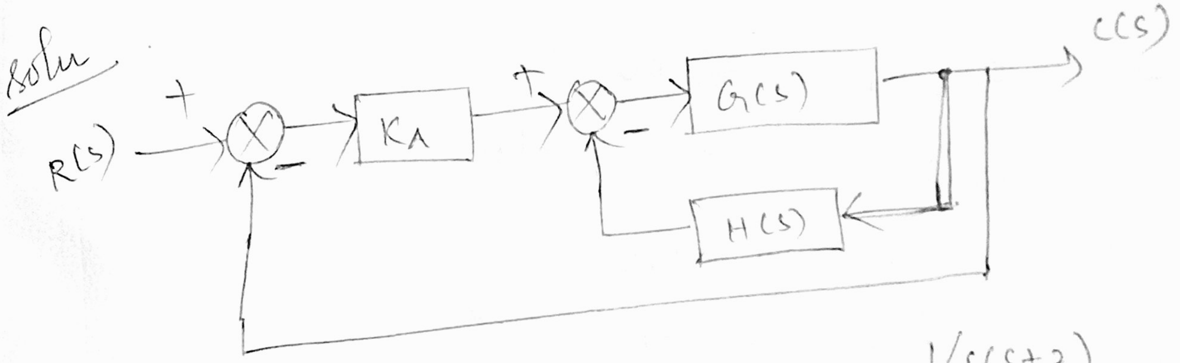
$$= \pi / 0.866$$

$$\boxed{t_p = 3.625 \text{ sec}}$$



2) A unity feedback control s/m has an amplifier with gain $K_A = 10$ & gain ratio $G(s) = 1/s(s+2)$ in the feed forward path. A derivative feedback, $H(s) = sK_o$ is introduced as a minor loop around G(s). Determine the derivative feedback constant, (K_o) so that the s/m damping factor is 0.6.

Soln



Inner fb loop $\Rightarrow \dfrac{G(s)}{1+G(s)H(s)} = \dfrac{1/s(s+2)}{1+\frac{1}{s(s+2)}\cdot sK_o}$

$$= \frac{1}{s(s+2)+sK_o}$$

$$= \frac{1}{s^2+2s+sK_o}$$

Multiplied with K_A $= \dfrac{K_A}{s^2+2s+sK_o}$

→ unity fb ⇒

$$\frac{C(s)}{R(s)} = \frac{\dfrac{K_A}{s^2+2s+K_o s}}{1+\dfrac{K_A}{s^2+2s+K_o s}}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{K_A}{s^2+(2+K_o)s+K_A}}$$

→ $K_A = 10$

$$\frac{C(s)}{R(s)} = \frac{10}{s^2+(2+K_o)s+10}$$

$$\omega_n^2 = 10 \Rightarrow \omega_n = \sqrt{10} = \underline{\underline{3.162 \text{ rad/sec}}}$$

→ $$2\zeta\omega_n = (2+K_o)$$

$$2 \times 0.6 \times 3.162 = 2 + K_o$$

$$K_o = 3.7944 - 2$$

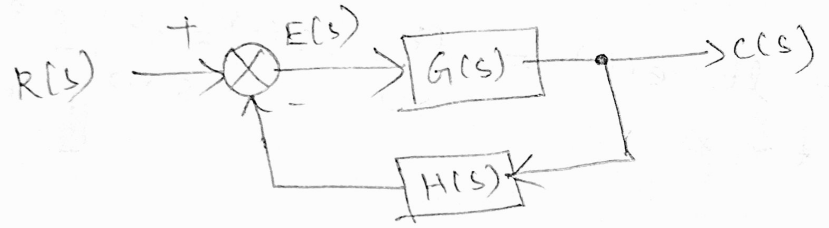
$$\boxed{K_o = 1.7944}$$

Steady State error :-

→ The steady state error is the value of error signal [e(t)], which t tends to infinity.

→ The error arise from the nature of i/ps, type of s/m & from non-linearity of s/m components.

→ The performance of the s/m is studied from the steady state error with step, ramp & parabolic i/ps.



→ $E(s) = R(s) - C(s)H(s)$.

& $C(s) = E(s)G(s)$.

→ ∴ $E(s) = R(s) - E(s)G(s)H(s)$

$E(s) + E(s)G(s)H(s) = R(s)$

$$\boxed{E(s) = \frac{R(s)}{1 + G(s)H(s)}}$$

→ e(t) → error signal in time domain.

∴ $e(t) = L^{-1}[E(s)]$

$$= L^{-1}\left[\frac{R(s)}{1 + G(s)H(s)}\right]$$

→ e_{ss} ⇒ Steady state error. is defined as the value of e(t) as 't' tends to infinity.

$$\boxed{e_{ss} = \lim_{t \to \infty} e(t)}$$

→ By final value theorem of Laplace transform,

$$\boxed{e_{ss} = \lim_{t \to \infty} e(t) \overset{LT}{\Longleftrightarrow} \lim_{s \to 0} sE(s)}$$

→ ∴ $$\boxed{e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \left[\frac{s\,R(s)}{1+G(s)H(s)}\right]}$$

Static Error constants:-

→ The steady state error value depends on the type number and the i/p signal.

→ Type-0 ⇒ constant steady state error (e_{ss}) with step i/p

Type-1 ⇒ constant e_{ss} for ramp i/p or velocity signal

Type-2 ⇒ constant e_{ss} for parabolic signal. or acceleration signal

→ errors:-

(i) positional error constant $(K_p) = \lim_{s \to 0} G(s)H(s)$

(ii) velocity error constant $(K_v) = \lim_{s \to 0} s\,G(s)H(s)$

(iii) Acceleration error constant $(K_a) = \lim_{s \to 0} s^2 G(s) H(s)$.

→ The error K_p, K_v & K_a are called as static error constants.

(1) I/p :- unit step signal.

$$e_{ss} = \lim_{s \to 0} s E(s)$$

$$= \lim_{s \to 0} \frac{s \cdot R(s)}{1 + G(s) H(s)}$$

$$= \lim_{s \to 0} \frac{\cancel{s} \cdot 1/\cancel{s}}{1 + G(s) H(s)}$$

$$= \frac{1}{1 + \lim_{s \to 0} G(s) H(s)}$$

$$e_{ss} = \frac{1}{1 + K_p}.$$

case (i) : Type '0' s/m

$$K_p = \lim_{s \to 0} G(s) H(s) = \lim_{s \to 0} \frac{K (s+z_1)(s+z_2) \ldots}{(s+P_1)(s+P_2) \ldots}$$

$$K_p = \frac{K (z_1)(z_2) \ldots}{(P_1)(P_2) \ldots}$$

$$= \text{constant}.$$

$$\therefore \boxed{e_{ss} = \frac{1}{1 + K_p}} \Rightarrow \text{constant}$$

case ii) Type - 1 s/m

$$K_p = \lim_{s \to 0} G(s)H(s)$$

$$= \lim_{s \to 0} K \frac{(s+z_1)(s+z_2)\ldots\ldots}{s(s+P_1)(s+P_2)\ldots\ldots}$$

$$K_p = \infty.$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$$

$$\boxed{\therefore e_{ss} = 0.}$$

→ ∴ the value of 'Kp' is infinity for Type '1' 4 above s/ms with unit step as i/p.

2) Ramp i/p

$$e_{ss} = \lim_{s \to 0} \frac{s R(s)}{1+G(s)H(s)}$$

$$r(t) = t \Rightarrow R(s) = 1/s^2.$$

$$e_{ss} = \lim_{s \to 0} \frac{s \cdot 1/s^2}{1+G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{1}{s[1+G(s)H(s)]}$$

$$e_{ss} = \lim_{s \to 0} \frac{1}{s + sG(s)H(s)}.$$

$$= \lim_{s \to 0} \frac{1}{sG(s)H(s)}$$

$$\boxed{e_{ss} = \lim_{s \to 0} \frac{1}{K_v}}$$

case (i) - Type 0 s/m.

$$K_v = \lim_{s \to 0} sG(s)H(s)$$

$$= \lim_{s \to 0} \frac{sK(s+z_1)(s+z_2)\ldots}{(s+P_1)(s+P_2)\ldots}$$

$$K_v = 0$$

$$\therefore \boxed{e_{ss} = \lim_{s \to 0} 1/K_v = 1/0 = \infty.}$$

case (ii) Type-1 s/m.

$$K_v = \lim_{s \to 0} sG(s)H(s)$$

$$= \lim_{s \to 0} \frac{\cancel{s} K(s+z_1)(s+z_2)\ldots}{\cancel{s}(s+P_1)(s+P_2)\ldots}$$

K_v = constant value

$$\boxed{e_{ss} = \frac{1}{K_v} \Rightarrow \text{constant}}$$

Case (iii) - Type - 2 s/m.

$$K_v = \lim_{s \to 0} s\, G(s) H(s)$$

$$= \lim_{s \to 0} \frac{\cancel{s} . K (s+z_1)(s+z_2) \ldots}{s^{\cancel{2}} (s+P_1)(s+P_2) \ldots}$$

$$= \infty .$$

$$\therefore \boxed{e_{ss} = 1/K_v = 1/\infty = 0 .}$$

Above Type 2 - the ess will be zero for ramp i/p.

3) Parabolic i/p.

$$\rightarrow \quad e_{ss} = \lim_{s\to 0} \frac{sR(s)}{1+G(s)H(s)}.$$

parabolic i/p $r(t) = t^2/2$, $R(s) = \frac{1}{s^3}$.

$$\rightarrow \therefore \; e_{ss} = \lim_{s\to 0} \frac{s \cdot 1/s^3}{1+G(s)H(s)} = \lim_{s\to 0} \frac{1}{s^2 + s^2 G(s)H(s)}$$

$$= \frac{1}{\lim s^2 G(s)H(s)}$$

$$e_{ss} = \frac{1}{K_a}$$

→ (i) Type 0 s/m :-

$$K_a = \lim_{s \to 0} s^2 G(s) H(s)$$

$$= \lim_{s \to 0} s^2 \frac{K (s+z_1)(s+z_2)(s+z_3)\ldots}{(s+P_1)(s+P_2)(s+P_3)}$$

$$= 0.$$

$$\therefore \boxed{e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty}$$

→ (ii) Type 1 - s/m

$$K_a = \lim_{s \to 0} s^2 G(s) H(s)$$

$$= \lim_{s \to 0} \frac{s^{\cancel{2}} K (s+z_1)(s+z_2)(s+z_3)\ldots}{\cancel{s}(s+P_1)(s+P_2)(s+P_3)\ldots}$$

$$= 0$$

$$\therefore \boxed{e_{ss} = \frac{1}{0} = \infty.}$$

→ (iii) Type 2 - s/m

$$K_a = \lim_{s \to 0} \frac{\cancel{s^2} K (s+z_1)(s+z_2)\ldots}{\cancel{s^2}(s+P_1)(s+P_2)\ldots}$$

⇒ constant

$$\therefore e_{ss} = \frac{1}{K_a} \Rightarrow \text{constant (finite)}.$$

(iv) Type 3 s/m

$$\rightarrow K_a = \lim_{s \to 0} \frac{\cancel{s^2}(s+z_1)(s+z_2)\dots}{s^{\cancel{3}}(s+P_1)(s+P_2)\dots}$$

$$K_a = \infty$$

$$\rightarrow \therefore e_{ss} = \frac{1}{K_a} = \frac{1}{\infty} = 0.$$

$$\boxed{e_{ss} = 0}$$

$\rightarrow \therefore$ for the type 3 & above, the unit parabolic i/p will have e_{ss} as 0.

Static error constants (K_p, K_v, K_a)

Error constant	Type no. of s/m: 0	1	2	3
K_p	constant	∞	∞	∞
K_v	0	constant	∞	∞
K_a	0	0	constant	∞

Steady state Error for various Types of i/p :-

I/p signal	Type no. of s/m 0	1	2	3
unit step	$\frac{1}{1+K_p}$	0	0	0
unit Ramp	∞	$\frac{1}{K_v}$	0	0
unit parabolic	∞	∞	$\frac{1}{K_a}$	0

Generalized Error Co-efficients / Dynamic Error co-efficients

$\rightarrow$ $$C_n = (-1)^n \int_0^t t^n f(t)\, dt$$

$$C_0 = (-1)^0 \int_0^t f(t)\, dt = \int_0^t f(t)\, dt$$

$$\boxed{F(s) = L[f(t)] = \int_0^t f(t) e^{-st} dt = \frac{1}{1+G(s)H(s)}}$$

$\rightarrow$ on taking $\lim s \to 0$ on both sides,

$$\lim_{s\to 0} F(s) = \lim_{s\to 0} \int_0^t f(t) e^{-st} dt$$

$$= \lim_{s\to 0} \int_0^t f(t)\, dt$$

$$\boxed{\lim_{s\to 0} F(s) = C_0} \Rightarrow C_0 = \lim_{s\to 0} F(s).$$

$\rightarrow c_1 = \lim_{s \to 0} \frac{d}{ds} F(s)$

$\rightarrow c_2 = \lim_{s \to 0} \frac{d^2}{ds^2} F(s)$

$\rightarrow$ $\therefore$ Generally, the dyanamic error-co-efficients are given by,

$$\boxed{C_n = \lim_{s \to 0} \frac{d^n}{ds^n} F(s)}$$

$\rightarrow$ correlation between static & Dynamic error co-efficients

$$C_0 = \frac{1}{1+K_p} \qquad \boxed{E(s) = \frac{R(s)}{1+G(s)H(s)}}$$

$$C_1 = \frac{1}{K_v}$$

$$C_2 = \frac{1}{K_a}$$

& $$\boxed{e(t) = C_0 r(t) + C_1 r'(t) + \frac{C_2}{2!} r''(t) + \cdots}$$

$r(t) \rightarrow$ i/p

$e(t) \rightarrow$ error

$C_n \rightarrow$ dynamic error constants.

pblm.

1) For a unity fb control s/m, the open loop transfer function, $G(s) = \frac{10(s+2)}{s^2(s+1)}$. Find

(i) the position, velocity & acceleration error constants.

(ii) the steady state error when the i/p is R(s), where $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$.

soln :)

(i) $K_p \rightarrow$ positional error.

$$K_p = \lim_{s \to 0} G(s) H(s)$$

$\because H(s) = 1$

$$= \lim_{s \to 0} \frac{10(s+2)}{s^2(s+1)}$$

$$= \frac{20}{0}$$

$$\boxed{K_p = \infty.}$$

$K_v \rightarrow$ velocity error

$$K_v = \lim_{s \to 0} s\,G(s) H(s)$$

$$= \lim_{s \to 0} \frac{10(s+2)}{s(s+1)}$$

$$= \frac{20}{0}$$

$$\boxed{K_v = \infty.}$$

$K_a \rightarrow$ Acceleration error constant.

$$K_a = \lim_{s \to 0} s^2 G(s) H(s)$$

$$= \frac{10(s+2)}{(s+1)}$$

$$= \frac{20}{1}$$

$$\boxed{K_a = 20}$$

(ii) Steady state error (e_{ss})

$$E(s) = \frac{R(s)}{1 + G(s) H(s)}.$$

$$\boxed{e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} s E(s)}$$

$$E(s) = \frac{R(s)}{1+G(s)\,H(s)}$$

$$= \frac{\left(\dfrac{9s^2 - 6s + 1}{3s^3}\right)}{\dfrac{s^3 + s + 10s + 20}{\cancel{s^2}(s+1)}}$$

$$= \frac{(9s^2 - 6s + 1)}{3s} \times \frac{(s+1)}{s^3 + s + 10s + 20}.$$

$$sE(s) = \frac{(9s^2 - 6s + 1)(s+1)}{3(s^3 + s + 10s + 20)}$$

$$e_{ss} = \lim_{s \to 0} sE(s)$$

$$= \frac{(1)(1)}{3(20)}$$

$$\boxed{e_{ss} = \frac{1}{60}}$$

2) The open loop transfer function of a servo s/m with unity fb is $G(s) = \frac{10}{s(0.1s+1)}$.

Evaluate the static error constants of the s/m. obtain the steady state error of the s/m, when subjected to an i/p given by the polynomial, $r(t) = a_0 + a_1 t + \frac{a_2}{2} t^2$.

Soln:) Static error constants => K_p, K_v & K_a.

$$K_p = \lim_{s \to 0} G(s) H(s)$$

$$= \lim_{s \to 0} \frac{10}{s(0.1s+1)} = \frac{1}{0}$$

$$\boxed{K_p = \infty}$$

$$K_v = \lim_{s \to 0} s\, G(s) H(s)$$

$$= \lim_{s \to 0} \frac{10}{(0.1s+1)} = 10$$

$$\boxed{K_v = 10}$$

$$K_a = \lim_{s \to 0} s^2 G(s) H(s)$$

$$= \lim_{s \to 0} \frac{10s}{(0.1s+1)}$$

$$\boxed{K_a = 0}$$

→ Steady state error :-

$$e(t) = c_0 r(t) + c_1 r'(t) + \frac{c_2}{2!} r''(t) + \cdots$$

$$r(t) = a_0 + a_1 t + \frac{a_2}{2} t^2 .$$

$$r'(t) = a_1 + \frac{a_2}{\cancel{2}} \cdot \cancel{2} t$$

$$r'(t) = a_1 + a_2 t$$

$$r''(t) = a_2 .$$

$$c_0 = \frac{1}{1+K_p} = \frac{1}{\infty} = 0 .$$

$$c_1 = \frac{1}{K_v} = \frac{1}{10} = 0.1$$

$$c_2 = \frac{1}{K_a} = \frac{1}{0} = \infty .$$

$$e(t) = 0 + \frac{1}{10}(a_1 + a_2 t) + \infty .$$

$$e_{ss} = \lim_{t \to \infty} e(t)$$

$$\boxed{e_{ss} = \infty .}$$

3) A unity fb s/m has the forward transfer function $G(s) = \frac{K_1(2s+1)}{s(5s+1)(1+s)^2}$. When the i/p $r(t) = 1+6t$ determine the minimum value of K_1, so that the steady state error is less than 0.1.

solu,

$$G(s) = \frac{K_1(2s+1)}{s(5s+1)(1+s)^2}$$

$$E(s) = \frac{R(s)}{1+G(s)H(s)}$$

$$r(t) = 1+6t$$

$$R(s) = \frac{1}{s} + \frac{6}{s^2} = \left(\frac{s+6}{s^2}\right)$$

$$E(s) = \frac{(s+6)}{s^2\left[1+\frac{K_1(2s+1)}{s[5s+1][1+s^2]}\right]}$$

$$= \frac{(s+6)(5s^2+s)(1+s^2)}{s^2\left[(5s^2+s)(1+s^2)+K_1(2s+1)\right]}$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{\cancel{s}(s+6)(5s^2+s)(1+s^2)}{\cancel{s^2}\left[(5s^2+s)(1+s^2)+K_1(2s+1)\right]}$$

$$= \lim_{s\to 0} \frac{6}{(K_1)}$$

$$e_{ss} = 0.1$$

$$\therefore \frac{6}{K_1} = 0.1$$

$$K_1 = \frac{6}{0.1}$$

$$\boxed{K_1 = 60}$$

---

controllers :-

→ A controller is a device introduced in the s/m to modify the error signal and to produce a control signal.

→ The controller modifies the transient response of the s/m.

→ Depending on the control actions provided the controllers can be classified as,

1) Two-position or ON-OFF controllers.
2) proportional (P) controller.
3) Integral (I) controller.
4) PI controller.
5) PD controller.
6) PID controller.

→ (i) proportional controllers :- (P)

→ It is a device that produces a control signal, p(t) proportional to the i/p error signal . e(t)

$$p(t) \propto e(t).$$

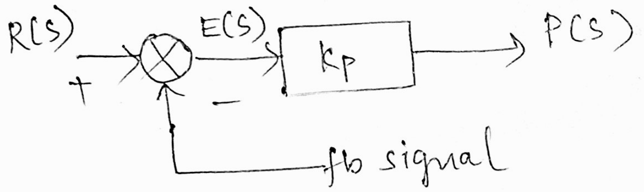
$$p(t) = K_p \, e(t).$$

$K_p \rightarrow$ proportional gain or constant.

→ on taking L.T.

$$P(s) = K_p \, E(s).$$

∴ Tfr function of p-controller, $\dfrac{P(s)}{E(s)} = K_p$.



→ P-controller amplifies the error signal by K_p and also increases the loop gain by the same amount of K_p.

→ ↑se in Loop gain improves the steady state tracking accuracy, disturbance signal rejection & makes the s/m less sensitive to parameter variations.

→ But increasing the gain to very large values may lead to instability of the s/m, which leads to constant steady state error.

ii) Integral-controllers :- (I)

→ The integral controller is a device which produces a control signal I(t) which is proportional to integral of the i/p error signal e(t).

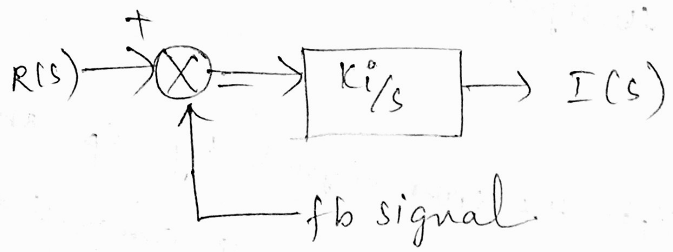
$$I(t) \propto \int e(t)\, dt$$

$$I(t) = K_i \int e(t)\, dt$$

K_i → Integral gain or constant.

→ LT ⇒ $I(s) = \frac{K_i}{s} E(s)$

$$\therefore \frac{I(s)}{E(s)} = \frac{K_i}{s}.$$



→ Integral controller removes or reduces the steady state error without the manual reset. Hence it is referred to as automatic reset.

→ This controller may lead to oscillatory response & hence the s/m may become unstable.

(iii) P-I controller :-

→ The proportional plus Integral controller (PI) produces an o/p signal; proportional to error signal and also proportional to the integral of error signal.

$$P_i(t) \propto \left[e(t) + \int e(t)\,dt \right]$$

$$P_i(t) = K_p\, e(t) + \frac{K_p}{T_i} \int e(t)\, dt.$$

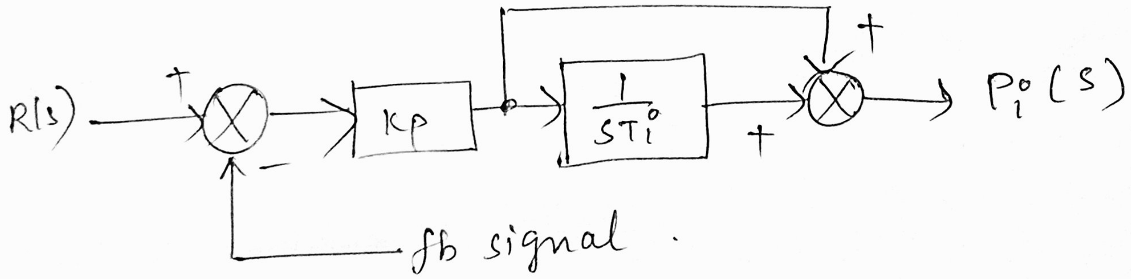
K_p → proportional gain

T_i → Integral time.

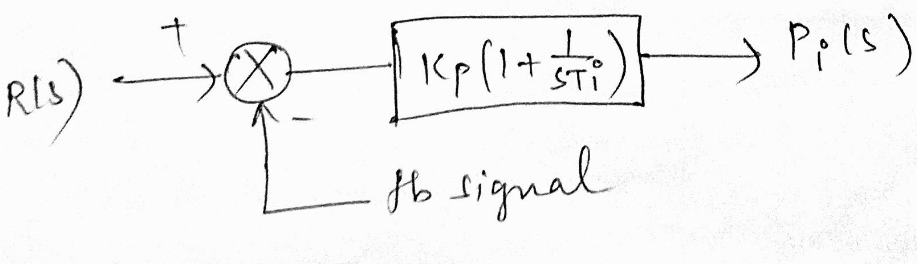
→ $$P_i(S) = K_p E(s) + \frac{K_p}{T_i} \frac{1}{S} E(s)$$

$$P_i(s) = E(s) \left[K_p + \frac{K_p}{S T_i} \right]$$

→ $$\frac{P_i(s)}{E(s)} = K_p \left(1 + \frac{1}{S T_i} \right)$$



⇓



→ The proportional action increases the loop gain & makes the s/m less sensitive to s/m parameters and the Integral action eliminates the steady state error.

→ The Inverse of Integral time T_i is called as reset rate.

(iv) P-D- controller. :-

→ The proportional plus derivative controller produces an o/p signal consisting of two terms.

(a) proportional to error signal

(b) proportional to the derivative of the i/p error signal.

→ $$P_d(t) \propto \left[e(t) + \frac{d}{dt}e(t)\right]$$

$$P_d(t) = K_p e(t) + K_p T_d \frac{d}{dt} e(t)$$

$K_p \rightarrow$ proportional gain

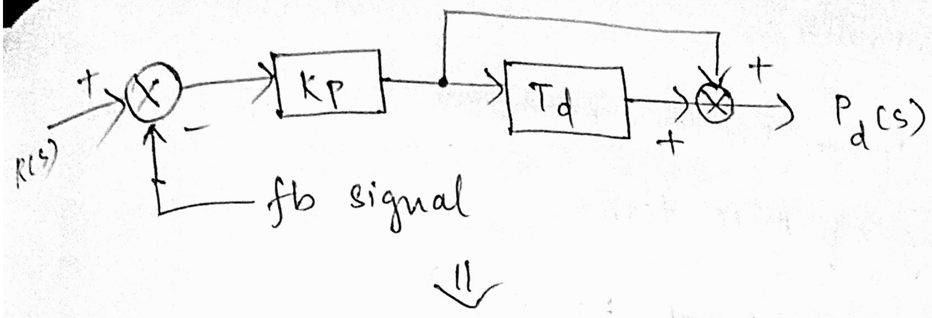
$T_d \rightarrow$ derivative time.

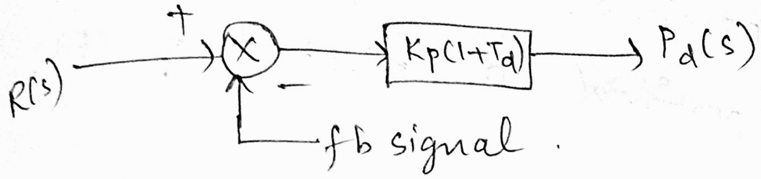
→ on, LT, $P_d(s) = K_p E(s) + K_p T_d \, s E(s)$

$$= E(s)\left[K_p + K_p T_d\right]$$

$$P_d(s) = E(s)\left[K_p(1+T_d)\right]$$

$$\frac{P_d(s)}{E(s)} = K_p(1+T_d).$$





→ The derivative control acts on the rate of change of error & its action is effective during transient periods and does not produce effective correction on constant error.

→ Hence the derivative controller is never used alone, but it is employed in association with proportional and integral controllers.

→ The derivative controller does not affect the steady state error directly but initiates an early correctione action and tends to increase the stability of the s/m.

→ But it amplifies noise signals & may cause a saturation effect in the actuator.

→ The derivative control action is adjusted by varying the derivative time, hence it is called as rate control

(v) P-I-D - controller :-

→ The PID controller produces an o/p signal consisting of three terms:

(a) proportional to e(t)

(b) proportional to integral of e(t)

(c) proportional to derivative of e(t).

→ $$P_{id}(t) \propto \left[e(t) + \int e(t)\,dt + \frac{d}{dt}e(t)\right]$$

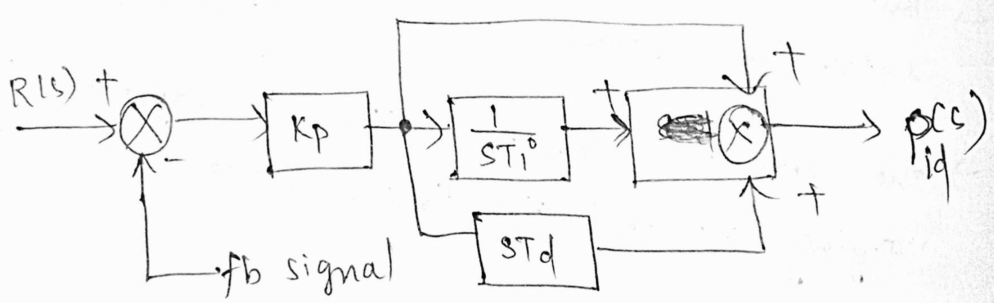
$$P_{id}(t) = K_p e(t) + \frac{K_p}{T_i}\int e(t)\,dt + K_p T_d \frac{d e(t)}{dt}$$

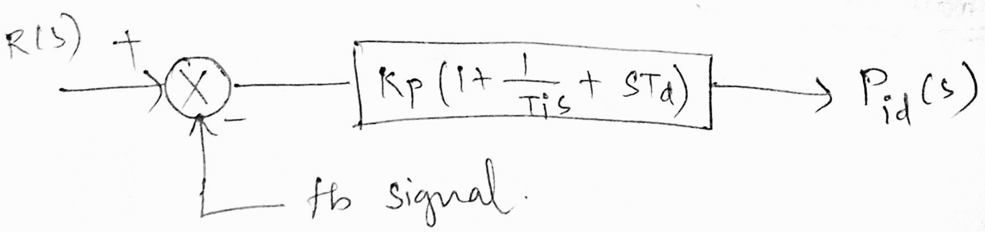
→ on, L.T

$$P_{id}(s) = K_p E(s) + \frac{K_p}{T_i}\frac{1}{s}E(s) + K_p T_d\, s E(s).$$

$$P_{id}(s) = E(s)\, K_p\left(1 + \frac{1}{sT_i} + sT_d\right)$$

$$\frac{P_{id}(s)}{E(s)} = K_p\left(1 + \frac{1}{sT_i} + sT_d\right)$$

→ 

→ 

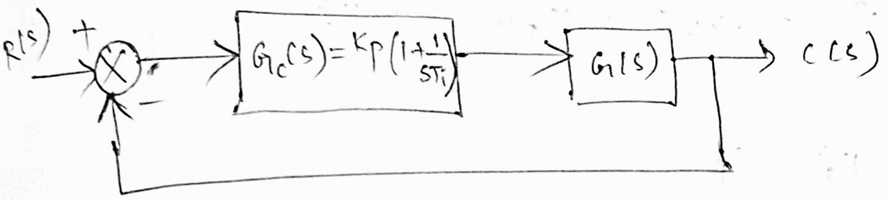
→ The proportional controller stabilizes the gain but produces a steady state error.

→ The Integral controller reduces or eliminates the steady state error.

→ The derivative controller reduces the rate of change of error.

Effect of PI & PD controllers :-

→ PI controller



$G_c(s) \rightarrow$ controller transfer func.

$G(s) \rightarrow$ open loop transfer func. $= \dfrac{\omega_n^2}{s(s+2\zeta\omega_n)}$.

$$CLTF \Rightarrow \frac{C(s)}{R(s)} = \frac{G_c(s)G(s)}{1+G_c(s)G(s)H(s)}$$

$$= \frac{K_p\left(1+\frac{1}{sT_i}\right)\left(\frac{\omega_n^2}{s^2+2\zeta\omega_n s}\right)}{1+K_p\left(\frac{sT_i+1}{sT_i}\right)\left(\frac{\omega_n^2}{s^2+2\zeta\omega_n s}\right)}$$

$$= \frac{K_p\left(\frac{sT_i+1}{sT_i}\right)\left(\frac{\omega_n^2}{s^2+2\zeta\omega_n s}\right)}{\dfrac{sT_i(s^2+2\zeta\omega_n s)+K_p(sT_i+1)(\omega_n^2)}{sT_i(s^2+2\zeta\omega_n s)}}$$

$$= \frac{K_p \omega_n^2 (1 + sT_i)}{s^3 T_i + 2\zeta\omega_n s^2 T_i + K_p T_i s \omega_n^2 + K_p \omega_n^2}$$

$$= \frac{K_p \omega_n^2 (1 + sT_i)}{T_i \left(s^3 + 2\zeta\omega_n s^2 + K_p \omega_n^2 s + \frac{K_p}{T_i} \omega_n^2 \right)}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{K_i \omega_n^2 (1 + sT_i)}{s^3 + 2\zeta\omega_n s^2 + K_p \omega_n^2 s + K_i \omega_n^2}}$$

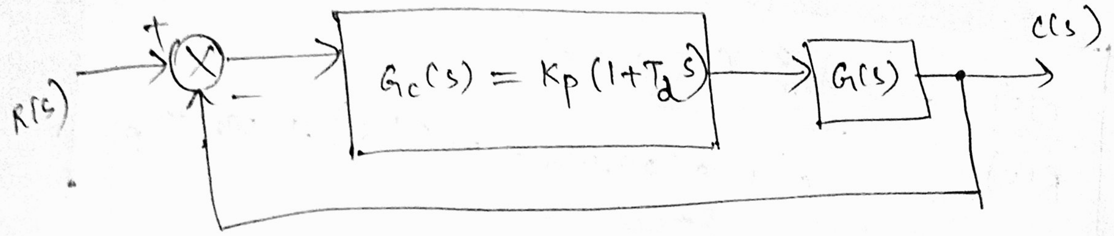
where $K_i = \frac{K_p}{T_i}$.

→ PI controller introduces one zero. ~~[illegible]~~ Also the order increases by one. Increase in the order of the s/m results in less stable s/m.

→ In $G_c(s)\,G(s)$, the PI controller increases the type number by one. The increase in type no. results in low steady state error.

PD controller :-

→ The proportional plus derivative controller, block diagram is,



$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)} \Rightarrow \text{OLTF}$$

→ $G_c(s)\,G(s) \Rightarrow$ Tf of PD

$$G_c(s)\,G(s) = K_p(1+T_d.s) \times \frac{\omega_n^2}{s(s+2\zeta\omega_n)}$$

$$= \frac{K_p\omega_n^2(1+T_d s)}{s(s+2\zeta\omega_n)}.$$

→ $$\text{CLTF} = \frac{G_c(s)\,G(s)}{1+G_c(s)\,G(s)}$$

$$= \frac{\left(K_p\omega_n^2(1+T_d s)\right)/s(s+2\zeta\omega_n)}{1+\dfrac{K_p\omega_n^2(1+T_d s)}{s(s+2\zeta\omega_n)}}$$

$$\text{CLTF} = \left[\frac{C(s)}{R(s)}\right] = \frac{K_p\omega_n^2(1+T_d s)}{s^2+2\zeta\omega_n s+K_p\omega_n^2(1+T_d s)}$$

$$= \frac{\omega_n^2(K_p+K_pT_d s)}{s^2+2\zeta\omega_n s+\omega_n^2(K_p+K_pT_d s)}$$

$$= \frac{\omega_n^2 (K_p + K_d s)}{s^2 + 2\zeta\omega_n s + \omega_n^2 (K_p + K_d s)}$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2 (K_p + K_d s)}{s^2 + \underbrace{s(2\zeta\omega_n + \omega_n^2 K_d)}_{2\zeta\omega_n s\left(\zeta + \frac{\omega_n K_d}{2}\right)} + \omega_n^2 K_p}}$$

where $K_d = K_p T_s$ & $\zeta' = \left(\zeta + \frac{\omega_n K_d}{2}\right)$

→ PD - controller introduces a zero in the s/m & increases the damping ratio. The addition of zero may increase the peak overshoot & reduce the rise time.

→ But ↑se in the damping ratio, ↓se the peak - overshoot & hence compensated.

$$\left[\because M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \right.$$
$$\left. \& \ t_p = \pi/\omega_d \right]$$

→ There is no increase in the type no. of the s/m in PD- controller. Hence PD- controller will not modify the steady state error.

pblms:

1) Find out the position, velocity and acceleration error co-efficients for the following unity fb s/ms having forward loop transfer function G(s) as

$$\frac{K}{s^2(s^2+8s+100)}.$$

soln

$$G(s) = \frac{K}{s^2(s^2+8s+100)}.$$

$$H(s) = 1$$

$$K_p = \lim_{s \to 0} G(s)H(s).$$

$$= \lim_{s \to 0} \frac{K}{s^2(s^2+8s+100)}$$

$$\boxed{K_p = \infty}$$

$$K_v = \lim_{s \to 0} s\,G(s)H(s)$$

$$= \lim_{s \to 0} \frac{K}{s(s^2+8s+100)}$$

$$\boxed{K_v = \infty.}$$

$$K_a = \lim_{s \to 0} \frac{K}{(s^2+8s+100)}$$

$$\boxed{K_a = K/100.}$$

2) A unity fb s/m has $G(s) = \frac{10}{(s+1)}$. Find the steady state error and the generalised error co-efficient for $r(t) = t$.

soln.

$$C_0 = \frac{1}{1+K_p}.$$

$$C_1 = \frac{1}{K_v}$$

$$C_2 = \frac{1}{K_a}.$$

$$\boxed{C_n = \underset{s \to 0}{lt} \frac{d^n}{ds^n} F(s)}$$

$$F(s) = \frac{1}{1 + G(s)H(s)}$$

$$= \frac{1}{1 + \left(\frac{10}{s+1}\right)}$$

$$F(s) = \frac{(s+1)}{(s+11)}.$$

$$K_p = \underset{s \to 0}{lt} \; G(s)H(s)$$

$$= \underset{s \to 0}{lt} \; \frac{10}{s+1}$$

$$\boxed{K_p = 10}$$

$$K_v = \underset{s \to 0}{lt} \; s\,G(s)H(s)$$

$$K_v = 0.$$

$$K_a = \underset{s \to 0}{lt} \; s^2 G(s)H(s)$$

$$\boxed{K_a = 0.}$$

$$C_0 = \underset{s \to 0}{lt} \; F(s) \Rightarrow \boxed{\frac{1}{11} = C_0}$$

$$C_1 = \underset{s \to 0}{lt} \; \frac{d}{ds}\left(\frac{s+1}{s+11}\right) = \underset{s \to 0}{lt} \frac{(s+11)(1) - (s+1)(1)}{(s+11)^2}$$

$$= \lim_{s \to 0} \frac{\cancel{s}+11-\cancel{s}-1}{(s+11)^2} = \lim_{s \to 0} \frac{10}{(s+11)^2}$$

$$\boxed{c_1 = \frac{10}{121}}$$

$$c_2 = \lim_{s \to 0} 10\left(-2 \times (s+11)^{-3}\right)$$

$$= \lim_{s \to 0} \frac{-20}{(s+11)^3}$$

$$= \frac{-20}{(11)^3}$$

$$\boxed{c_2 = -20/11^3.}$$

Steady state error.

$$e_{ss} = \lim_{s \to 0} s E(s), \qquad E(s) = \frac{R(s)}{1+G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{\cancel{s}(1/s^{\cancel{2}})}{1+\frac{10}{(s+1)}}$$

$$= \lim_{s \to 0} \frac{1\,(s+1)}{s(s+1+10)}$$

$$= \lim_{s \to 0} \frac{(s+1)}{s(s+11)}.$$

$$\boxed{e_{ss} = \infty}$$

3) The open loop Tfr. function of a unity fb ctrl s/m is $G(s) = \frac{9}{(S+1)}$, using the generalized error series determine the error signal and steady state error of the s/m when the s/m is excited by $3t^2/2$.

soln.

$$G(s) = \frac{9}{(S+1)}$$

$$H(s) = 1$$

$$r(t) = \frac{3t^2}{2} \Rightarrow R(s) = \frac{3}{2} \times \frac{2!}{s^3} = \frac{3}{s^3}.$$

$$\boxed{e(t) = c_0\, r(t) + c_1 r'(t) + \frac{c_2}{2!} r''(t) + \dots}$$

$$c_0 = \lim_{s \to 0} F(s) = \lim_{s \to 0} \frac{1}{1+G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{1}{1 + \left(\frac{9}{s+1}\right)}$$

$$= \lim_{s \to 0} \frac{s+1}{s+10}.$$

$$\boxed{c_0 = \frac{1}{10}.}$$

$$c_1 = \lim_{s \to 0} \frac{d}{ds} F(s) \Rightarrow \lim_{s \to 0} \frac{(s+10) - (s+1)}{(s+10)^2}$$

$$= \lim_{s \to 0} \frac{9}{(s+10)^2}$$

$$\boxed{c_1 = \frac{9}{100}.}$$

$$c_2 = \lim_{s \to 0} \frac{9(-2)}{(s+10)^3}$$

$$\boxed{c_2 = -\frac{18}{10^3}.}$$

$$r(t) = \frac{3t^2}{2}$$

$$r'(t) = \frac{3}{\cancel{2}} \times \cancel{2} t = 3t \quad 6t$$

$$r''(t) = 3.$$

$$e(t) = \frac{1}{10} \times \frac{3t^2}{2} + \frac{9}{100} \times 3t + \left(-\frac{\overset{9}{\cancel{18}}}{10^3} \times \frac{1}{\cancel{2}} \times 3\right)$$

$$e(t) = \frac{3}{20} t^2 + \frac{27}{100} t - \frac{27}{10^3}$$

$$\boxed{e(t) = 0.15t^2 + 0.27t - 0.027}$$

$$e_{ss} = \lim_{t \to \infty} e(t)$$

$$\boxed{e_{ss} = \infty}$$