Transformation of analog filters into equivalent digital filters:

(or)

Digitization of analog filter ~~into~~

1. Approximation of derivatives method (X)
2. Impulse Invariant method ✓
3. Bilinear Transformation method ✓
4. matched z-transform method. ✓

For the transformation (or) convertion to be effective, it should possess the following desirable properties:

1. The $j\Omega$ axis (i.e. imaginary axis) in the s-plane should map into the unit circle in the z-plane. Thus, there will be a direct relationship between the two frequency variables in the two domain.

2. The left-half of the s-plane should map into interior of the unit-circle in the z-plane. Thus, a stable analog filter will be converted to a stable digital filter.

s-plane	z-plane
on $j\Omega$ axis	on the unit circle
LH of $j\Omega$ axis	inside unit circle
RH of $j\Omega$ axis	outside unit circle

Impulse Invariant Transformation:

$$H_a(s) \Longrightarrow H(z)$$

In this method, IIR digital filter is designed such that the unit-impulse response $h(n)$ is the sampled version of the impulse response of analog filter. The main idea behind this technique is to preserve the frequency response characteristics of analog filter.

wkt $\quad H_a(s) = \int_{-\infty}^{\infty} h_a(t) \cdot e^{-st} \, dt$

$$H(z) = \sum_{n=0}^{\infty} h(n) \, z^{-n}$$

relationship b/w s & z-transform is,

$$\boxed{z = e^{sT}} \longrightarrow (1)$$

both s & z are complex quantities.

where $s = \sigma + j\Omega$, $z = r e^{jw}$ (polar form)

$\sigma \rightarrow$ real part $\qquad$ $r \rightarrow$ radius of the circle in z-plane

$\Omega \rightarrow$ imaginary part. $\qquad$ $w \rightarrow$ phase angle

sub s & z in (1), $\quad r e^{jw} = e^{(\sigma + j\Omega)T} = e^{\sigma T} \cdot e^{j\Omega T}$

$$r = e^{\sigma T}$$

$$\boxed{w = \Omega T}$$

$$\boxed{\begin{gathered} \omega = \Omega T \\ \text{or} \\ \Omega = \omega/T \end{gathered}} \longrightarrow (2)$$

$\Leftarrow$ Relationship b/w analog & digital freq.

$$z = r e^{j\omega} = |z| \angle z$$

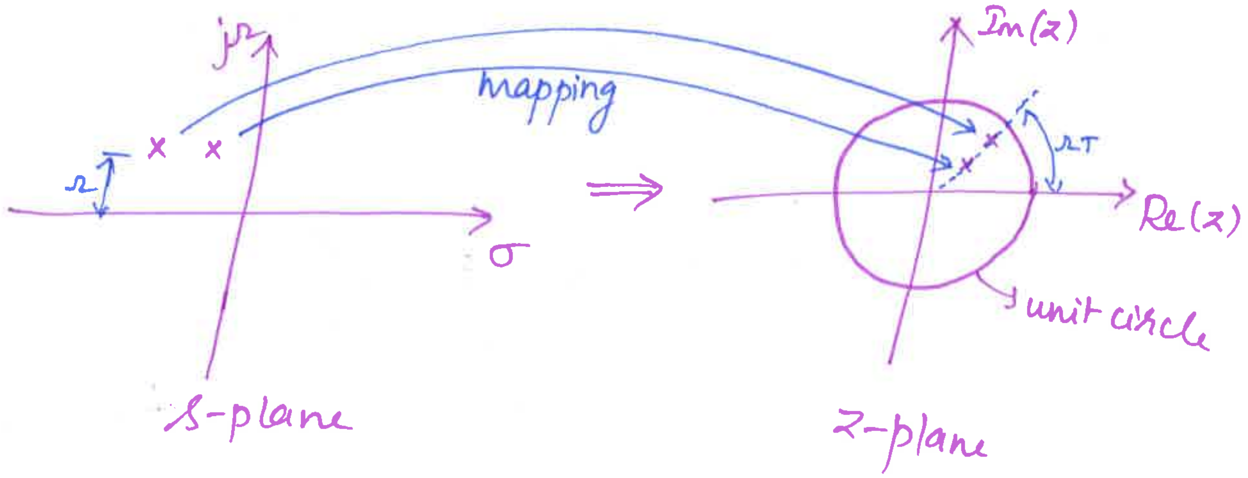
$$\left.\begin{aligned} |z| &= r = e^{\sigma T} \\ \angle z &= \Omega T \end{aligned}\right\} \longrightarrow (3)$$

$\Rightarrow$ Relationship b/w analog and digital filter poles.

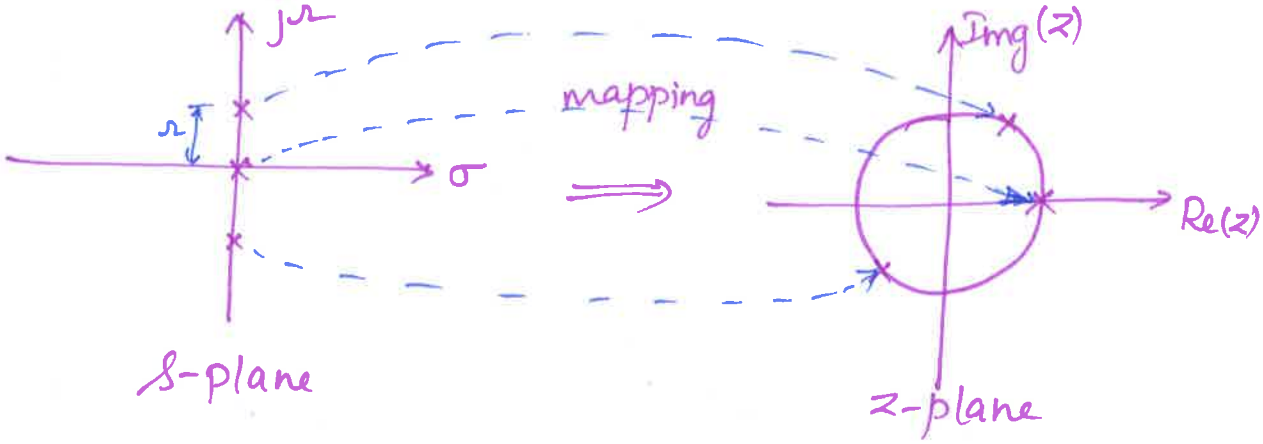
From the above equation (3), the following observations can be made:

1. If $\sigma < 0$ (i.e. σ is negative), then analog pole 's' lies on Left-half of $j\Omega$ axis of s-plane. In this case, $|z| < 1$, hence the corresponding digital pole 'z' will lie inside the unit-circle in z-plane.

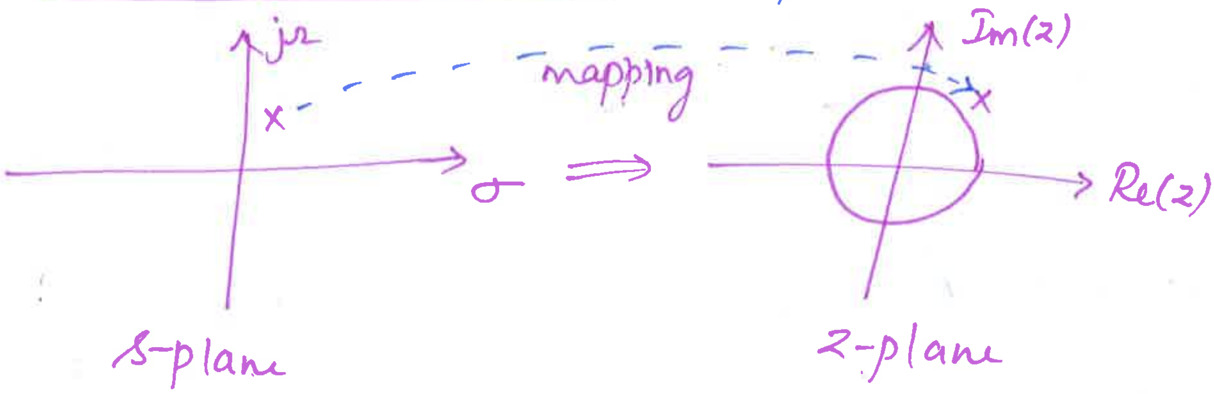
eg:



2) If $\sigma = 0$ (i.e real part is zero), then analog pole 's' lies on imaginary axis of s-plane. In this case $|z| = 1$, hence the corresponding digital pole 'z' will lie on the unit-circle in z-plane.



3. If $\sigma > 0$, (ie $\sigma \rightarrow$ positive), then analog pole 's' lies on Right half of $j\Omega$ axis of -s plane. In this case $|z| > 1$, hence the corresponding digital pole will lie outside the unitcircle in z-plane



Given

$$H_a(s) = LT\{h_a(t)\}$$

$$\longrightarrow h(n) = h_a(t)\Big|_{t=nT}$$

$$\longrightarrow T = 1\,sec, \quad H(z) = Z\text{-}T\{h(n)\}$$

Let $h_a(t) \rightarrow$ impulse response of analog filter

$H_a(s) \rightarrow$ Transfer function " " "

$h(n) \rightarrow$ Impulse response of digital filter

$H(z) \rightarrow$ Transfer function " " "

Let us consider $H_a(s)$ has distinct poles,

$$H_a(s) = \sum_{k=1}^{N} \frac{C_k}{s-P_k} = \frac{C_1}{s-P_1} + \frac{C_2}{s-P_2} + \dots \longrightarrow (4)$$

$P_k \rightarrow$ poles of analog filter

$C_k \rightarrow$ filter coefficients.

Take inverse laplace Transform of (4) eqn,

$$\left.\begin{aligned} h_a(t) &= \sum_{k=1}^{N} C_k\, e^{P_k t}, \quad t \geq 0 \\ &= C_1 e^{P_1 t} u_a(t) + C_2 e^{P_2 t} u_a(t) + \dots \end{aligned}\right\} \longrightarrow (5)$$

If we sample $h_a(t)$ periodically at $t = nT$,

$$h(n) = h_a(t)\big/_{t=nT} = h_a(nT)$$

where $T \rightarrow$ sampling period.

$$\left.\begin{aligned} h_a(nT) &= \sum_{k=1}^{N} C_k e^{P_k nT} \quad (6a) \\ &= C_1 e^{P_1 nT} u_a(nT) + C_2 e^{P_2 nT} u_a(nT) + \dots \quad (6b) \end{aligned}\right\} \longrightarrow (6)$$

Take z-transform of eqn (6),

$$H(z) = ZT\{h_a(nT)\}$$

$$H(z) = \sum_{n=0}^{\infty} h_a(nT)\, z^{-nT} \longrightarrow (7)$$

$$H(z) = \sum_{n=0}^{\infty} \sum_{k=1}^{N} c_k \cdot e^{p_k nT} z^{-n}$$

$$= \sum_{k=1}^{N} c_k \sum_{n=0}^{\infty} \left(e^{p_k T} z^{-1}\right)^n$$

$$H(z) = \sum_{k=1}^{N} \frac{c_k}{1 - e^{p_k T} z^{-1}} \quad \left[\because \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, |a|<1\right] \rightarrow (8)$$

(or)

ZT of eqn (6b)

$$H(z) = \frac{c_1}{1-e^{p_1 T} z^{-1}} + \frac{c_2}{1-e^{p_2 T} z^{-1}} + \dots$$

hence If $H_a(s) = \sum_{k=1}^{N} \frac{c_k}{s-p_k}$ then $H(z) = \sum_{k=1}^{N} \frac{c_k}{1-e^{p_k T} z^{-1}}$

$$\boxed{\frac{c_k}{s-p_k} \xrightarrow[\text{is transformed to}]{\text{(Mapped)}} \frac{c_k}{1-e^{p_k T} z^{-1}}}$$

For high sampling rate (small T), digital filter gain is high. Therefore, instead of (8), we can use

$$H(z) = \sum_{k=1}^{N} \frac{T \cdot c_k}{1-e^{p_k T} z^{-1}}.$$

drawback of Impulse Invariant method:

If analog poles are having same real parts and imaginary parts that differ by some integer multiple of $\frac{2\pi}{T}$, then they are mapped to a single ~~p~~ digital pole in z-plane. for eg: let us consider two poles

$$s_1 = \sigma + j\Omega$$

$$s_2 = \sigma + j\left(\Omega + \frac{2\pi}{T}\right)$$

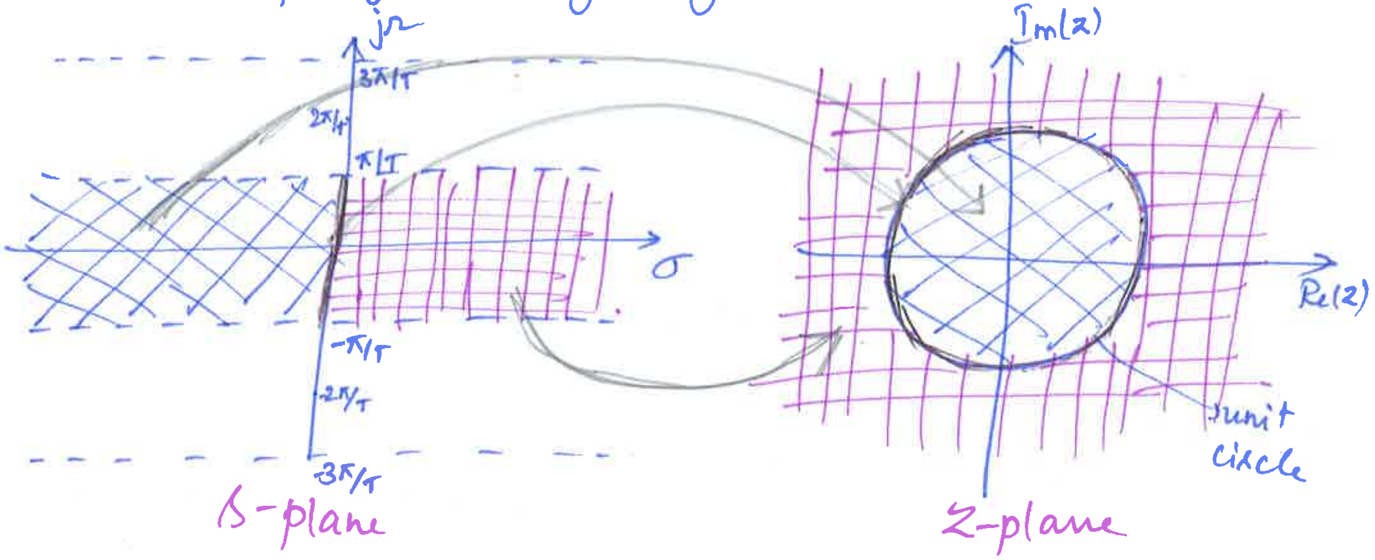
$$z_1 = e^{s_1 T} = e^{(\sigma + j\Omega)T} = e^{\sigma T} \cdot e^{j\Omega T}$$

$$z_2 = e^{s_2 T} = e^{\left[\sigma + j\left(\Omega + \frac{2\pi}{T}\right)\right]T} = e^{\sigma T} \cdot e^{j\Omega T} \cdot e^{j\frac{2\pi}{T}T}$$

$$= e^{\sigma T} \cdot e^{j\Omega T} \quad \left[\because e^{j2\pi} = 1\right.$$

$$\left.\begin{array}{l} s_1 \\ s_2 \end{array}\right\} \rightarrow z \qquad \boxed{z_2 = z_1} = z$$

hence, this impulse invariant technique results in many-to-one-mapping. This is due to aliasing when sampling analog signals.



From the above figure, we can say that, the strip of width $2\pi/T$ in the s-plane for values of s in the range $\frac{-\pi}{T} \leq \Omega \leq \pi/T$ is mapped into the entire z-plane. Similarly, the strip of width $2\pi/T$ in the s-plane for values of s in the range $\pi/T \leq \Omega \leq \frac{3\pi}{T}$ is also mapped into entire z-plane. Likewise, the strip of width $\frac{2\pi}{T}$ in the s-plane for values of s in the range $\frac{-3\pi}{T} \leq \Omega \leq \frac{-\pi}{T}$ is also mapped into the entire z-plane.

In general, any strip of width $2\pi/T$ in the s-plane for values of s in the range $(2k-1)\pi/T \leq \Omega \leq (2k+1)\pi/T$ (where 'k' is an integer) is mapped into the entire z-plane.

Due to the presence of aliasing, the impulse invariance method is appropriate for the design of LPF & BPF only. This method is unsuccessful for implementing digital filters such as a HPF.

pbm: For analog transfer function $H(s) = \frac{2}{(s+1)(s+2)}$, determine $H(z)$ using Impulse Invariant method for

i) $T = 1\,sec$ ii) $T = 0.1\,sec$

Soln: Given : $H_a(s) = \frac{2}{(s+1)(s+2)}$

By using Partial fraction Expansion method,

$$H_a(s) = \frac{A}{s+1} + \frac{B}{s+2}$$

$$A = (s+1)\, H_a(s) \Big|_{s=-1}$$

$$= \cancel{(s+1)} \cdot \frac{2}{\cancel{(s+1)}(s+2)}\Big|_{s=-1} = \frac{2}{-1+2} = \boxed{2 = A}$$

$$B = (s+2)\, H_a(s) \Big|_{s=-2}$$

$$= \cancel{(s+2)} \cdot \frac{2}{(s+1)\cancel{(s+2)}}\Big|_{s=-2} = \frac{2}{-2+1} = \boxed{-2 = B}$$

$$H_a(s) = \frac{2}{s+1} + \frac{-2}{s+2} = \frac{2}{s-\underset{P_1}{(-1)}} + \frac{-2}{s-\underset{P_2}{(-2)}}$$

By impulse invariant transformation,

$$\frac{C_k}{s-P_k} \xrightarrow[\text{to}]{\text{is transformed}} \frac{C_k}{1-e^{P_k T} z^{-1}}$$

$$H(z) = \frac{2}{1-e^{-T} z^{-1}} + \frac{-2}{1-e^{-2T} z^{-1}}$$

(i) $T = 1\,sec$

$$H(z) = \frac{2}{1-e^{-1} z^{-1}} + \frac{-2}{1-e^{-2} z^{-1}}$$

$$H(z) = \frac{2}{1-0.3678z^{-1}} + \frac{-2}{1-0.1353z^{-1}}$$

$$= \frac{2(1-0.1353z^{-1}) - 2(1-0.3678z^{-1})}{(1-0.3678z^{-1})(1-0.1353z^{-1})}$$

$$\boxed{H(z) = \frac{0.465z^{-1}}{(1-0.3678z^{-1})(1-0.1353z^{-1})}}$$ Answer.

homework 1: solve for $T = 0.1\,sec$.

Pbm2: Using impulse invariant method, determine $H(z)$ if $H(s) = \frac{1}{(s+1)(s^2+s+1)}$, Assume $T = 1\,sec$

Soln:

$$H(s) = \frac{A}{s+1} + \frac{B}{s+0.5+j0.866} + \frac{C}{s+0.5-j0.866}$$

s^2+s+1

$$s_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1}{2} \pm \frac{\sqrt{3j^2}}{2}$$

$$= \frac{-1}{2} \pm j\sqrt{3}/2$$

$$s_{1,2} = -0.5 \pm j0.866$$

$$A = \cancel{(s+1)} \cdot \frac{1}{\cancel{(s+1)}(s^2+s+1)} \Big|_{s=-1} = \boxed{1 = A}$$

$$B = \cancel{(s+0.5+j0.866)} \cdot \frac{1}{(s+1)\cancel{(s+0.5+j0.866)}(s+0.5-j0.866)} \Big|_{s=-0.5-j0.866}$$

$$= \frac{1}{(-\cancel{0.5}-j0.866+\cancel{0.5}-j0.866)(-0.5-j0.866+1)}$$

$$= \cancel{-0.5+} \frac{1}{(-j1.732)(0.5-j0.866)} = \frac{1}{-j0.866-1.5}$$

$$B = \frac{1}{-1.5-j0.866} \times \frac{-1.5+j0.866}{-1.5+j0.866}$$

$$= \frac{-1.5+j0.866}{(-1.5)^2+(0.866)^2} = \frac{-1.5+j0.866}{3}$$

$$\boxed{B = -0.5+j0.288}$$

$$\boxed{C = B^* = -0.5-j0.288}$$

$$H(s) = \frac{1}{(s+1)} + \frac{-0.5+j0.288}{(s+0.5+j0.866)} + \frac{-0.5-j0.288}{(s+0.5-j0.866)}$$

$$P_1 = -1, \quad P_2 = -0.5-j0.866, \quad P_3 = -0.5+j0.866$$

Using impulse invariant technique,

$$\frac{C_k}{s-P_k} \xrightarrow[\text{into}]{\text{is transformed}} \frac{C_k}{1-e^{P_kT}z^{-1}}$$

& T=1sec

$$H(z) = \frac{1}{1-e^{-1}z^{-1}} + \frac{-0.5+j0.288}{1-e^{-0.5-j0.866}z^{-1}} + \frac{-0.5-j0.288}{1-e^{-0.5+j0.866}z^{-1}}$$

$$= \qquad e^{-0.5}\, e^{-j0.866} \qquad e^{-0.5}\cdot e^{+j0.866}$$

$$\cos(0.866) - j\sin(0.866)$$

(in radian mode)

$$\boxed{H(z) = \frac{1}{1-0.368z^{-1}} + \frac{-1+0.66z^{-1}}{1-0.786z^{-1}+0.368z^{-2}}}$$ Answer

home work 2 & 3 & 4

2. Convert the analog filter with transfer function $H_a(s) = \dfrac{s+0.1}{(s+0.1)^2+9}$ into a digital IIR filter by means of impulse invariance method.

3. $H_a(s) = \dfrac{1}{s^2+\sqrt{2}s+1}$, $T = 1\,sec$

4. $H_a(s) = \dfrac{10}{s^2+7s+10}$, $T = 0.2\,sec$

Bilinear Transformation method:

The bilinear transformation is a conformal mapping that transforms the $j\Omega$ axis into the unit-circle in the z-plane only once, thus avoiding aliasing of frequency components.

Furthermore, all points in the LH of s-plane are mapped inside the unit circle in z-plane and all points in the RH of s-plane are mapped into corresponding points outside the unit-circle in z-plane.

Let us consider an analog linear filter with system function,

$$H(s) = \frac{b}{s+a} = \frac{Y(s)}{X(s)} \longrightarrow ①$$

$$s \cdot Y(s) + a \cdot Y(s) = b \cdot X(s) \longrightarrow ②$$

Taking Inverse Laplace transform of eqn ②,

$$\frac{dy(t)}{dt} + a \cdot y(t) = b \cdot x(t) \longrightarrow ③$$

If we approximate $y(t)$ by using trapezoidal formula,

i.e., $$y(t) = \int_{t_0}^{t} y'(\tau)\,d\tau + y(t_0) \longrightarrow ④$$

where $y'(t) \rightarrow$ derivative of $y(t)$.

The approximation of the integral in eqn ④ by the trapezoidal formula at $t = nT$ & $t_0 = nT - T$ yields,

$$y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT-T)\right] + y(nT-T) \longrightarrow ⑤$$

from eqn ③, at $t = nT$

$$y'(nT) = -a\,y(nT) + b\,x(nT) \longrightarrow ⑥$$

Substitute eqn ⑥ in eqn ⑤,

$$⑤ \Rightarrow y(nT) = \frac{T}{2}\left\{\underbrace{-ay(nT) + bx(nT)}_{y'(nT)} \overbrace{-ay(nT-T) + bx(nT-T)}^{y'(nT-T)}\right\} + y(nT-T) \longrightarrow ⑦$$

which implies,

$$y(nT) + \frac{aT}{2} y(nT) - \left[1 - \frac{aT}{2}\right] y(nT-T) = \frac{bT}{2}\left[x(nT) + x(nT-T)\right] \quad \text{(8)}$$

With $y(n) = y(nT)$ & $x(n) = x(nT)$

$y(nT-T) = y(n-1)$ & $x(nT-T) = x(n-1)$

we obtain,

$$\left(1 + \frac{aT}{2}\right) y(n) - \left(1 - \frac{aT}{2}\right) y(n-1) = \frac{bT}{2}\left[x(n) + x(n-1)\right] \quad \text{(9)}$$

Taking z-transform of eqn (9),

$$\left(1 + \frac{aT}{2}\right) Y(z) - \left(1 - \frac{aT}{2}\right) z^{-1} Y(z) = \frac{bT}{2}\left(1 + z^{-1}\right) X(z)$$

hence, the transfer function of digital filter is,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{bT/2\,(1+z^{-1})}{1 + \frac{aT}{2} - \left(1 - \frac{aT}{2}\right) z^{-1}}$$

$$H(z) = \frac{bT/2\,(1+z^{-1})}{(1-z^{-1}) + \frac{aT}{2}(1+z^{-1})} \longrightarrow (10)$$

Dividing numerator & denominator of eqn (10) by $T/2\,(1+z^{-1})$, we get

$$H(z) = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a} \longrightarrow (11)$$

If $H(s) = \frac{b}{s+a}$, then $H(z) = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + a}$

hence mapping is gn by,

$$\boxed{s \xrightarrow[\text{into}]{\text{is mapped}} \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} \rightarrow (12)$$

wkt, $z = re^{j\omega} = |z| \cdot \angle z \rightarrow (13a)$, $s = \sigma + j\Omega \rightarrow (13b)$

Substitute (13) a & b in eqn (12),

$$s = \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right] = \frac{2}{T}\,\frac{\cancel{z^{-1}}[z-1]}{\cancel{z^{-1}}[z+1]}$$

$$= \frac{2}{T}\left[\frac{z-1}{z+1}\right] = \frac{2}{T}\left[\frac{re^{j\omega}-1}{re^{j\omega}+1}\right]$$

$$= \frac{2}{T}\left[\frac{r\cos\omega - 1 + jr\sin\omega}{r\cos\omega + 1 + jr\sin\omega}\right]$$

$$= \frac{2}{T}\left[\frac{(r\cos\omega-1)+jr\sin\omega}{(r\cos\omega+1)+jr\sin\omega}\right] \times \left[\frac{(r\cos\omega+1)-jr\sin\omega}{(r\cos\omega+1)-jr\sin\omega}\right]$$

$$= \frac{2}{T}\left[\frac{r^2\cos^2\omega - 1 + r^2\sin^2\omega + j2r\sin\omega}{\underbrace{(r\cos\omega+1)^2}_{r^2\cos^2\omega+1+2r\cos\omega} + r^2\sin^2\omega}\right]$$

$$=$$

$$s = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega}\right] + j\,\frac{2}{T}\left[\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right] \rightarrow (14)$$

compare eqn ~~13b~~ ~~&~~ (13b) & (14)

$$\sigma = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega}\right], \longrightarrow (15)$$

$$\Omega = \frac{2}{T}\left[\frac{2r\sin w}{1+r^2+2r\cos w}\right] \rightarrow (16)$$

From the equations (15) & (16), we can say that

if $r<1$, then $\sigma<0$, where $r=|z|$ (radius of the circle in z-plane)

ie LH of $j\Omega$ axis of s-plane poles $\Big\}$ $\xrightarrow[\text{into}]{\text{are mapped}}$ $\Big\{$ inside the unit-circle in the z-plane

if $r>1$, ~~#~~ then $\sigma>0$,

i.e. RH of $j\Omega$ axis of s-plane poles $\Big\}$ $\xrightarrow[\text{into}]{\text{are mapped}}$ $\Big\{$ outside the unitcircle in the z-plane.

If $r=1$, then $\sigma=0$ and

$$\Omega = \frac{2}{T}\,\frac{2\sin w}{2+2\cos w} = \frac{2}{T}\,\frac{\sin w}{1+\cos w}$$

$$= \frac{2}{T}\,\frac{\cancel{2}\sin(w/2)\cancel{\cos(w/2)}}{\cancel{2}\cos^{\cancel{2}}(w/2)}$$

$$\boxed{\Omega = \frac{2}{T}\tan\left(\frac{w}{2}\right)} \rightarrow (17)$$

(or)

$$\boxed{w = 2\tan^{-1}\frac{\Omega T}{2}} \rightarrow (18)$$

gives the relationship between analog and digital filter frequencies.