Steps to design digital IIR filter (LPF)

Step I: From the given specifications of digital filter (ω), find the analog frequencies (Ω) using the formula: ① (or) ②

(II) Impulse Invariance (or) (BT) Bilinear transformation

formula ①: $\Omega = \omega/T$ (or) formula ②: $\Omega = \frac{2}{T}\tan\frac{\omega}{2}$

Step II: Find $H_a(s)$ ~~for $\Omega_c = 1$ rad/sec.~~

$\Downarrow$

Step 1: $N = ?$ (order of the filter)

~~Step 2~~: using the formula given below.

Butterworth filter ~~II method~~:

$$N \geq \frac{\log\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log\left(\Omega_s/\Omega_p\right)}$$

(or)

$$N \geq \frac{\log(\lambda/\varepsilon)}{\log(1/k)}$$

Chebyshev filter ~~BT method~~:

$$N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\left(\Omega_s/\Omega_p\right)}$$

(or)

$$N \geq \frac{\cosh^{-1}(\lambda/\varepsilon)}{\cosh^{-1}(1/k)}$$

where $k = \Omega_p/\Omega_s$

$\varepsilon = \sqrt{10^{0.1\alpha_p}-1}$

$\lambda = \sqrt{10^{0.1\alpha_s}-1}$

Step 2: find poles:

Butterworth filter ~~II method~~

$$s_k = e^{j\phi_k}, \quad k = 1, 2, \ldots N$$

where,

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$$

(or)

$\boxed{\text{use the table to find denominator polynomial for Butterworth filter.}}$

$$H(s) = \frac{1}{(s-s_1)(s-s_2)\cdots(s-s_N)}$$

find $\Omega_c = ?$

$$\Omega_c = \Omega_p / \left(10^{0.1\alpha_p} - 1\right)^{1/2N}$$

Sub $s \longrightarrow s/\Omega_c$ in $H(s)$

$$H_a(s) = H(s)\Big/_{s = s/\Omega_c}$$

Chebyshev filter ~~BT method~~

$$s_k = r_1 \cos\phi_k + j r_2 \sin\phi_k, \quad k = 1, 2, \ldots N$$

where

$$r_1 = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right]$$

$$r_2 = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right]$$

$$\mu = \varepsilon^{-1} + \sqrt{1 + \varepsilon^{-2}}$$

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1}$$

$$\phi_k = \pi/2 + \frac{(2k-1)\pi}{2N}, \quad k = 1, 2, \ldots N$$

$$\left.\begin{array}{l}\text{denominator} \\ \text{of } H(s)\end{array}\right\} = (s-s_1)(s-s_2)\cdots(s-s_N) \quad \text{(D)}$$

$$\left.\begin{array}{l}\text{Numerator} \\ \text{of } H(s)\end{array}\right\} = ?$$

N-odd: Sub, $s = 0$ in (D)

N-even: Sub, $s = 0$ in (D) & divide by $\sqrt{1+\varepsilon^2}$

$$H_a(s) = \frac{\text{numerator of } H(s)}{\text{denominator polynomial of } H(s)}$$

Step III: Convert $H_a(s)$ into $H(z)$: $H(z) = ?$

Impulse Invariant Transformation	Bilinear Transformation
Write $Ha(s)$ as sum of poles. $\sum_{k=1}^{N} \frac{C_k}{s-p_k} \longrightarrow \sum_{k=1}^{N} \frac{C_k}{1-e^{p_k T} z^{-1}}$	$s \longrightarrow \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$

Steps to design digital IIR HPF, BPF or BSF:

method I
1. Step I, II, III same (to design LPF)
2. Convert digital LPF to digital HPF (or) BPF (or) BSF using digital transformation technique.

(or)

i.e.,) design of analog prototype LPF $H(s)$ $\xrightarrow{\text{Using Impulse Invariance (or) Bilinear Transformation method}}$ digital filter LPF $H(z)$ $\xrightarrow{\text{using digital frequency transformation technique}}$ digital filter HPF (or) BPF (or) BSF

(discussed in page no: 75)

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method II
1. perform Step I & II
2) Convert Analog LPF to Analog HPF (or) BPF (or) BSF using Analog transformation technique.
3) Transform $H(s)$ into $H(z)$ using Impulse Invariance (or) Bilinear Transf method.

design of analog prototype LPF $\xrightarrow[\text{using Analog frequency transformation technique}]{}$ Analog HPF (or) BPF (or) BSF $\xrightarrow[\text{using Impulse Invariance (or) Bilinear Transformation method}]{}$ digital filter HPF (or) BPF (or) BSF

Frequency Transformations in digital domain:
(or)
Digital transformation Technique:

LPF → LPF:

digital LP prototype filter / ω_p $H_p(z)$ → digital LP filter / ω_p' $H_l(z)$

$$z^{-1} \longrightarrow \frac{z^{-1}-a}{1-az^{-1}}$$

where $\omega_p' \to$ band edge frequency new filter

$$a = \frac{\sin[(\omega_p-\omega_p')/2]}{\sin[(\omega_p+\omega_p')/2]}$$

LPF → HPF:

digital LPF / ω_p → digital HPF / ω_p'

$$z^{-1} \longrightarrow \frac{-z^{-1}+a}{1+az^{-1}}$$

where $\omega_p' \to$ band edge frequency new filter

$$a = \frac{-\cos\left[(\omega_p + \omega_p')/2\right]}{\cos\left[(\omega_p - \omega_p')/2\right]}$$

LPF → BPF :

digital Low Pass filter / ω_p ⟶ digital BPF with
$\omega_l \rightarrow$ lower band edge freq.
$\omega_u \rightarrow$ upper " " "

$$z^{-1} \longrightarrow \frac{-z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}$$

where

$$a_1 = 2\alpha k/(k+1)$$

$$a_2 = (k-1)/(k+1)$$

$$\alpha = \frac{\cos\left[(\omega_u + \omega_l)/2\right]}{\cos\left[(\omega_u - \omega_l)/2\right]}$$

$$k = \cot \frac{\omega_u - \omega_l}{2} \tan \frac{\omega_p}{2}$$

LPF → BSF :

digital LPF/ω_p ⟶ digital BSF with ω_u & ω_l.

$$z^{-1} \longrightarrow \frac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-1} - a_1 z^{-1} + 1}$$

where

$$a_1 = 2\alpha/(k+1)$$

$$a_2 = (1-k)/(1+k)$$

$$\alpha = \frac{\cos\left[(\omega_u + \omega_l)/2\right]}{\cos\left[(\omega_u - \omega_l)/2\right]}$$

& $k = \tan \frac{\omega_u - \omega_l}{2} \tan \frac{\omega_p}{2}$

How to choose transformation technique ie either method 1 or method 2?

The frequency transformation may be accomplished in any of the available two techniques, caution must be taken to which technique to use. For example, the impulse invariant transformation is not suitable for HPF or BPF whose resonant frequencies are higher. In such case, suppose a LP prototype filter is converted into a HPF using analog frequency transformation and transformed later to a digital ~~frequency~~ filter using impulse invariant technique. This will result in aliasing problems. However, if the same prototype LPF is first transformed into a digital filter using impulse-invariant technique and later converted into a HPF using digital frequency transformation will not have any alaising problem. Whenever the bilinear transformation is used, it is of no significance whether analog frequency Transformation is used or digital frequency transformation. In this case, both analog and digital frequency transformation techniques will give same result.

Eg: Convert the single-pole LP Butterworth filter with system function $H(z) = \dfrac{0.245(1+z^{-1})}{1-0.509z^{-1}}$ into a BPF with upper and lower cutoff frequencies w_u & w_l respectively. The Low Pass filter has 3-dB bandwidth, $w_p = 0.2\pi$

Soln: LPF $\longrightarrow$ BPF (digital frequency Transformation)

$$z^{-1} \longrightarrow \frac{-(z^{-2} - a_1 z^{-1} + a_2)}{a_2 z^{-2} - a_1 z^{-1} + 1}$$

~~where~~

$$H(z) = \frac{0.245\left[1 - \dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}\right]}{1 + 0.509\left(\dfrac{z^{-2} - a_1 z^{-1} + a_2}{a_2 z^{-2} - a_1 z^{-1} + 1}\right)}$$

$$H(z) = \frac{0.245(1-a_2)(1-z^{-2})}{(1+0.509a_2) - 1.509\, a_1 z^{-1} + (a_2 + 0.509) z^{-2}}$$

For eg. $\omega_u = 3\pi/5$, $\omega_l = 2\pi/5$, $\omega_p = 0.2\pi$

$k = 1$, $a_2 = 0$, $a_1 = 0$

then $$H(z) = \frac{0.245(1-z^{-2})}{1+0.509z^{-2}}$$

This BPF has poles at $z = \pm 0.713j$ and hence resonates at $\omega = \pi/2$ $(\omega_r = \sqrt{\omega_u \cdot \omega_l})$.

Ex: Design a HPF, monotonic in passband with cutoff frequency of 1000 Hz and down 10dB at 350Hz. The sampling frequency is 5KHz

(i) Using Bilinear Transformation

(ii) Using Impulse Invariant transformation. home work

Soln:

(i) Using Bilinear:

Given: $\alpha_p = 3dB$, $\alpha_s = 10dB$

$f_p = 1000Hz$, $f_s = 350Hz$

$$\omega_p = 2\pi f_p = 2\pi \times 1000 = 2000\pi \text{ rad/sec}$$

$$\omega_s = 2\pi f_s = 2\pi \times 350 = 700\pi \text{ rad/sec}$$

$f_{samp} = 5KHz$, $T = \frac{1}{f_{samp}} = \frac{1}{5000} = 2 \times 10^{-4}$ sec method-II

Step I:

$$\Omega = \frac{2}{T} \tan \frac{\omega T}{2}$$

$T \rightarrow$ small

$$\Omega_p = \frac{2}{T} \tan \frac{\omega_p T}{2} = \frac{2}{2 \times 10^{-4}} \tan \left(\frac{2000\pi \times 2 \times 10^{-4}}{2} \right)$$

$$\Omega_p = 7265 \text{ rad/sec}$$

$$\Omega_s = \frac{2}{T} \tan \frac{\omega_s T}{2} = 2235 \text{ rad/sec.}$$

Step II:

Step 1:

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log \left(\Omega_s / \Omega_p \right)}$$

for HPF

$$\Longleftarrow (\Omega_p)_{HPF} = \Omega_c = (\Omega_s)_{LPF}$$

$$(\Omega_s)_{HPF} = (\Omega_p)_{LPF}$$

$$N \geq \frac{\log \sqrt{\frac{10^{0.1(10)} - 1}{10^{0.1 \times 3} - 1}}}{\log \left(\frac{7265}{2235} \right)}$$

$$\therefore \Omega_p = 2235 \text{ rad/sec}$$

$$\Omega_s = 7265 \text{ rad/sec}$$

$$N \geq \frac{\log 3}{\log 3.25} = \frac{0.4771}{0.5118} = 0.932$$

$$\boxed{N = 1}$$

Step 2: for $N=1$ using table of denominator polynomial of Butterworth filter,

$$H(s) = \frac{1}{s+1}$$

$\Uparrow$ Transfer function of analog prototype LPF.

Analog frequency Transformation technique used

Analog LPF $\longrightarrow$ HPF (Analog)

$$s \longrightarrow \Omega_c / s$$

$$\cancel{H_a(s)} = H_h(s) = H(s)\Big/_{s = \Omega_c/s}$$

$$\Omega_c = 7265 \text{ rad/sec}$$

$$H_h(s) = \frac{1}{\frac{7265}{s} + 1} = \boxed{\frac{s}{s + 7265} = H_h(s)}$$

Transfer function of Analog HPF.

Using Bilinear Transformation:

Analog HPF Transfer function $\xrightarrow{\text{to}}$ Digital HPF Transfer function.

$$s \longrightarrow \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H_h(z) = H_h(s)\Big/_{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H_h(z) = \frac{\frac{2}{2\times10^{-4}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}{\frac{2}{2\times10^{-4}}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 7265}$$

$$\boxed{H_h(z) = \frac{0.5792\,(1-z^{-1})}{1 - 0.1584\, z^{-1}}}$$

pbm: Design a digital Chebyshev type-I Band Reject Filter with the following Specifications

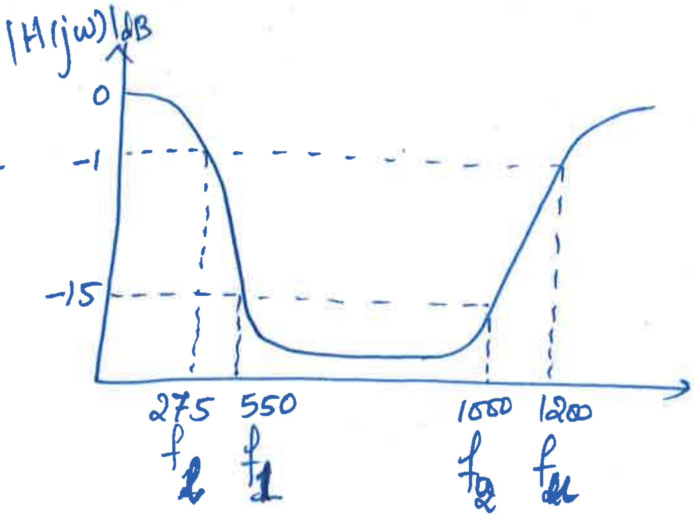
passband dc to 275Hz & 2KHz to ∞

Stop band 550 Hz to 1000 Hz

$\alpha_p = 1dB$, $\alpha_s = 15dB$, $F = 8KHz$

Soln:

Given



$\omega = 2\pi f$ rad/sec

$\alpha_p = 1dB$, $\alpha_s = 15dB$

$\omega_l = 2\pi \times f_l = 2\pi \times 275$ rad/sec

$\omega_1 = 2\pi f_1 = 2\pi \times 550$ rad/sec

$\omega_2 = 2\pi f_2 = 2\pi \times 1000$ rad/sec

$\omega_u = 2\pi f_u = 2\pi \times 1200$ rad/sec

$T = \frac{1}{8000}$ sec

Bilinear :

Step I:

$$\Omega = \frac{2}{T} \tan \frac{\omega T}{2}$$

$$\Omega_l = \frac{2}{T} \tan \frac{\omega_l T}{2} = 1.1084 \text{ rad/sec}$$

$$\Omega_1 = \frac{2}{T} \tan \frac{w_1 T}{2} = 0.2194 \text{ rad/sec}$$

$$\Omega_2 = 2/T \tan \frac{w_2 T}{2} = 0.4141 \text{ rad/sec}$$

$$\Omega_u = \frac{2}{T} \tan \frac{w_u T}{2} = 1 \text{ rad/sec}$$

Step II :

Step 1 : $N \geq \dfrac{\cosh^{-1}\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}(\Omega_s/\Omega_p)}$

$$\boxed{\frac{\Omega_s}{\Omega_p} = \Omega_r}$$

resonant frequency.

for BSF $\Rightarrow$

$$\Omega_r = \min\{|A|, |B|\}$$

$$A = \frac{\Omega_1(\Omega_u - \Omega_l)}{-\Omega_1^2 + \Omega_l \Omega_u} = 3.246$$

$$B = \frac{\Omega_2(\Omega_u - \Omega_l)}{-\Omega_2^2 + \Omega_l \Omega_u} = -5.847$$

for BPF

$$\Omega_r = \min\{|A|, |B|\}$$

$$A = \frac{-\Omega_1^2 + \Omega_l \Omega_u}{\Omega_1(\Omega_u - \Omega_l)}, \quad B = \frac{\Omega_2^2 - \Omega_l \Omega_u}{\Omega_2(\Omega_u - \Omega_l)}$$

$$\Omega_r = \min\{|3.246|, |-5.847|\}$$

$$= 3.246$$

$$N \geq 1.666$$

$$\boxed{N = 2}$$

Step 2: $\varepsilon = \sqrt{10^{0.1\alpha_p} - 1} = 0.508$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 4.17$$

$$r_1 = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2}\right] = 0.776$$

$$r_2 = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2}\right] = 1.266$$

$$\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}, \quad k = 1, 2$$

$$s_k = r_1 \cos\phi_k + j r_2 \sin\phi_k, \quad k = 1, 2$$

$$s_1 = r_1 \cos\phi_1 + j r_2 \sin\phi_2 = -0.5487 + j0.895$$

$$s_2 = r_1 \cos\phi_2 + j r_2 \sin\phi_2 = -0.5487 - j0.895$$

Denominator of $H(s) = \left[(s+0.5487) - j\,0.895\right]\left[(s+0.5487) + j0.895\right]$

$$= s^2 + 1.0974s + 1.102$$

$N \to$ even $\Leftarrow \boxed{N=2}$

$\therefore$ Numerator of $H(s)$ $= \dfrac{\text{denominator of } H(s)/_{s=0}}{\sqrt{1+\varepsilon^2}}$

$$= \frac{1.102}{\sqrt{1+(0.508)^2}} = 0.9825$$

$$\therefore \quad H_a(s) = \frac{0.9825}{s^2 + 1.0974s + 1.102}$$

Using Analog Frequency Transformation:

Analog LPF $\longrightarrow$ Analog BSF

$$s \longrightarrow \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$$

$$s \longrightarrow \frac{0.8916s}{s^2 + 0.1084}$$

$$H_{BSF}(s) = \frac{0.9825}{\left(\frac{0.8916s}{s^2+1.084}\right)^2 + 1.0974\left(\frac{0.8916}{s^2+1.084}\right) + 1.102}$$

$$= \frac{0.89156\,(s^2 + 0.2168s^2 + 0.01175)}{s^4 + 0.8878s^3 + 0.9382s^2 + 0.09618s + 0.01174}$$

Using Bilinear Transformation:

Analog BSF $\longrightarrow$ digital BSF

$$H_{BSF}(z) = H_{BSF}(s)\Big/_{s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$\boxed{H_{BSF}(z) = \frac{0.3732\,(1 - 3.2172z^{-1} + 4.5882z^{-2} - 3.2176z^{-3} + z^{-4})}{1 - 1.8869z^{-1} + 1.4292z^{-2} - 0.8077z^{-3} + 0.3292z^{-4}}}$$