From eqn (17), ie $\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right)$

For small values of ω,

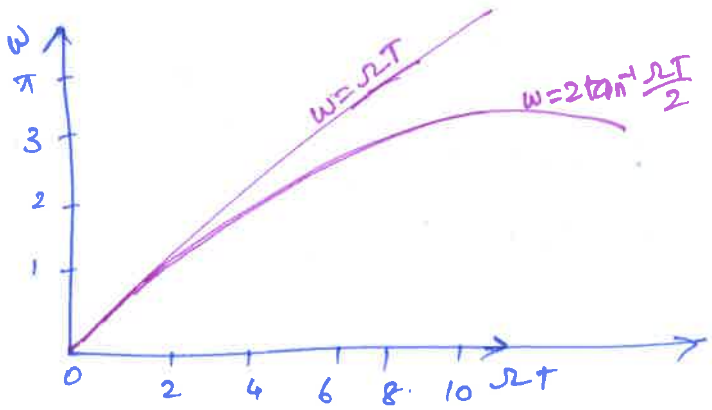
$$\Omega = \frac{2}{T}\cdot\frac{\omega}{2} = \frac{\omega}{T}.$$

[∵ for small values of θ, $\tan\theta = \theta$.

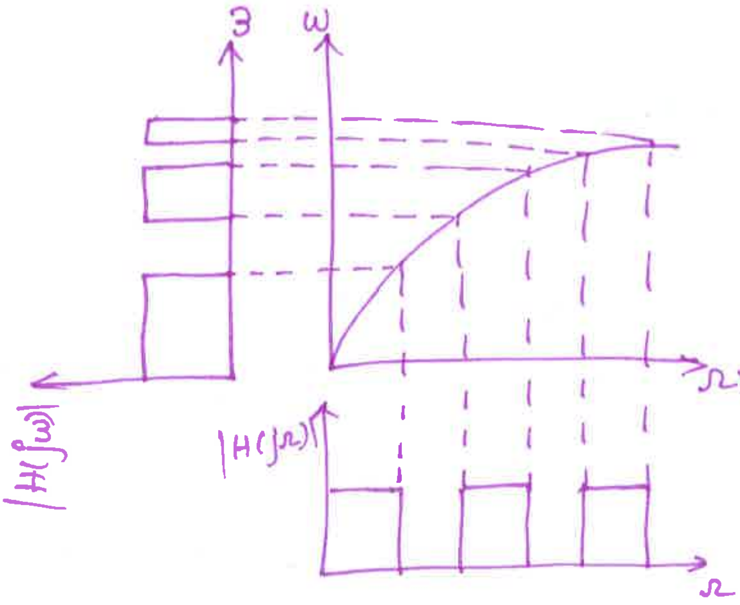
$\omega = \Omega T$

for small values of ω, the relationship between Ω and ω are linear. (ie digital filter have the same amplitude response as the analog filter)

For high frequencies, relationship is non-linear. hence distortion is introduced in the frequency scale of the digital filter to that of analog filter. This is known as warping effect.

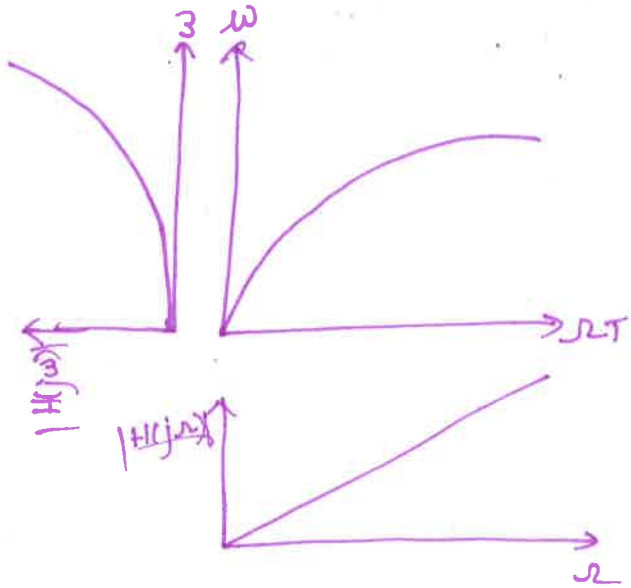


The effect of warping on the amplitude and phase response is shown below.



⇓

warping effect on magnitude response



warping effect on phase response.

Let us consider an analog filter with a no. of passbands centered at regular intervals. due to warping effect, after mapping digital filter will have same no. of Passbands. But centre frequencies and bandwidth of HF passband will tend to reduce disproportionately.

Let us consider an analog filter with linear phase response. due to warping effect, after mapping, phase response of digital filter will be non-linear.

Prewarping:

The warping effect can be eliminated by prewarping (or) (prescaling) the analog filter frequencies which are equivalent to digital frequencies using the formula $\Omega = \frac{2}{T} \tan \frac{\omega}{2}$. then analog filter transfer function is designed using the prewarp freq. and it is transformed to digital filter transfer function.

Advantages:

1) → provides one-to-one mapping
2) Stable analog filters are mapped into stable digital filters
3) → no aliasing.

disadvantages:

1) At high frequencies, the mapping is non-linear hence producing frequency compression.
2) Neither the impulse response nor the phase response of the analog filter is preserved in a digital filter obtained by bilinear transformation.

Pbm: Apply bilinear transformation to $H(s) = \frac{2}{(s+1)(s+2)}$ with T=1sec and find H(z)

soln:

$$H(s) = \frac{2}{(s+1)(s+2)}$$

~~Sub~~ bilinear mapping:

$$s \xrightarrow[\text{into}]{\text{is mapped}} \frac{2}{T}\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$H(z) = \frac{2}{\left[\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1\right]\left[\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+2\right]}$$

$$= \frac{2}{\left[\frac{2-2z^{-1}+T+Tz^{-1}}{T(1+z^{-1})}\right]\left[\frac{2-2z^{-1}+2T+2Tz^{-1}}{T(1+z^{-1})}\right]}$$

$$H(z) = \frac{2T^2(1+z^{-1})^2}{(2-2z^{-1}+T+Tz^{-1})(2-2z^{-1}+2T+2Tz^{-1})}$$

Put $T = 1\,sec$,

$$H(z) = \frac{2(1+z^{-1})^2}{(2-2z^{-1}+1+z^{-1})(2-2z^{-1}+2+2z^{-1})}$$

$$= \frac{2(1+z^{-1})^2}{(3-z^{-1})(4)}$$

$$= \frac{(1+z^{-1})^2}{2(3-z^{-1})} = \frac{(1+z^{-1})^2}{6-2z^{-1}}$$

$$\boxed{H(z) = \frac{0.166(1+z^{-1})^2}{(1-0.33z^{-1})}}$$

homework

5) Determine $H(z)$ from following $H_a(s)$ when

a) $T = 1\,sec$, $H_a(s) = \frac{2s}{s^2+0.2s+1}$

b) $T = 1\,sec$, $H_a(s) = \frac{s^3}{(s+1)(s^2+s+1)}$

c) $T = 1\,sec$, $H_a(s) = \frac{s+0.3}{(s+0.3)^2+16}$

pbm: Design a single-pole LP digital filter with a -3dB bandwidth of 0.2π, using bilinear transformation applied to the analog filter

$$H(s) = \frac{\Omega_c}{s+\Omega_c}$$

where Ω_c is the 3dB bandwidth of the analog filter.

Soln: Given $H(s) = \frac{\Omega_c}{s+\Omega_c}$, $\omega_c = 0.2\pi$ rad/sec

$$\Omega_c = \frac{2}{T} \tan^{-1}\left(\frac{\omega_c}{2}\right) = \frac{2}{T} \tan^{-1}(0.1\pi) = 0.65/T \text{ rad/sec}$$

$$H(s) = \frac{0.65/T}{\frac{0.65}{T} + s}$$

$$s \xrightarrow[\text{into}]{\text{is mapped}} \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = \frac{0.65/T}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + \frac{0.65}{T}}$$

$$= \frac{0.65(1+z^{-1})}{2(1-z^{-1}) + 0.65(1+z^{-1})}$$

$$= \frac{0.65(1+z^{-1})}{2-2z^{-1}+0.65+0.65z^{-1}} = \frac{0.65(1+z^{-1})}{2.65-1.35z^{-1}}$$

$$H(z) = \frac{0.245(1+z^{-1})}{1-0.509z^{-1}}$$

The frequency response of the filter is,

$$H(\omega) = \frac{0.245(1+e^{-j\omega})}{1-0.509\,e^{-j\omega}}$$

At $\omega=0$, $H(0)=1$

at $\omega=0.2\pi$, $|H(0.2\pi)|=0.707$, which is the desired response.

pbm: Convert the analog filter with system function

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+16}$$

into a digital IIR filter by means of bilinear transformation. The digital filter is to have a resonant frequency of $\omega_r = \pi/2$.

Soln:

from $H_a(s)$, $\Omega_c = 4$, $\omega_r = \pi/2 = \omega_c$

$$\Omega_c = \frac{2}{T}\tan\left(\frac{\omega_c}{2}\right)$$

$$T = \frac{2}{\Omega_c}\tan\left(\frac{\omega_c}{2}\right) = \frac{2}{4}\tan\left(\frac{\pi}{4}\right) = \frac{1}{2}$$

$$s \xrightarrow[\text{to}]{\text{is mapped}} 4\left[\frac{1-z^{-1}}{1+z^{-1}}\right]$$

$$H(z) = \frac{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.1}{\left[4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.1\right]^2+16} = \frac{0.128+0.006z^{-1}-0.122z^{-1}}{1+\underbrace{0.0006}_{=0}z^{-1}+0.975z^{-2}}$$

$$H(z) = \frac{0.128 + 0.006\,z^{-1} - 0.122\,z^{-2}}{1 + 0.975\,z^{-2}}$$

digital poles:

$$P_{1,2} = 0.987\,e^{\pm j\pi/2}$$

zeros at $z_{1,2} = -1, 0.95$

Thus a two-pole digital filter is designed that resonates near $w = \pi/2$.

The Matched-z transform method:

Another method for converting an analog filter into an equivalent digital filter is to map the poles and zeros of $H(s)$ directly into poles and zeros in the z-plane.

If $$H(s) = \frac{\prod_{k=1}^{M} (s - z_k)}{\prod_{k=1}^{N} (s - p_k)}$$

where $p_k \rightarrow$ poles of the filter
$z_k \rightarrow$ zeros " "

then, system function of the digital filter is

$$H(z) = \frac{\prod_{k=1}^{M} \left(1 - e^{z_k T} z^{-1}\right)}{\prod_{k=1}^{N} \left(1 - e^{p_k T} z^{-1}\right)}$$

$(s - a) \xrightarrow[\text{to}]{\text{is mapped}} (1 - e^{aT} z^{-1})$

where $T \rightarrow$ samping period.