Unit-II Time Response Analysis

Time Response Analysis

→ Time Response of the S/m is the o/p of the closed loop system as a function of time [c(t)]

→ The response c(t) can be obtained from the transfer function & the i/p to the S/m.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+H(s)G(s)} = M(s)$$

$$C(s) = R(s)\,M(s).$$

$$c(t) = L^{-1}\left[R(s)\,M(s)\right]$$

→ The time response of a control S/m consists of two parts.

(i) transient response

(ii) Steady state response.

→ Transient response: is the response of the S/m when the i/p changes from one state to another.

→ Steady state response: is the response as time (t) approaches infinity.

→ standard test signals :-

Name of the signal (i/p)	Time domain equation of signal $r(t)$	$L[r(t)]$ (ie) $R(s)$
step	A	A/s
unit step	1	$1/s$
Ramp	At	A/s^2
unit Ramp	t	$1/s^2$
Parabolic	$At^2/2$	A/s^3
unit parabolic	$t^2/2$	$1/s^3$
Impulse	$\delta(t)$	1

→ Impulse Response ⇒ with i/p as impulse signal

$$R(s) = 1$$

$$M(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{C(s)}{R(s)}$$

$$\therefore C(s) = R(s)\left[\frac{G(s)}{1+G(s)H(s)}\right]$$

$$C(s) = \frac{G(s)}{1+G(s)H(s)}$$

$$c(t) = L^{-1}\left[\frac{G(s)}{1+G(s)H(s)}\right]$$

∴ Impulse response is the inverse LT of transfer function.

Order of a s/m :-

→ The i/p & o/p relationship of a ctrl s/m can be expressed by n^{th} order differential equation,

$$a_0 \frac{d^n}{dt^n} p(t) + a_1 \frac{d^{n-1}}{dt^{n-1}} p(t) + \dots + a_n p(t) = b_0 \frac{d^m}{dt^m} q(t)$$

$$+ b_1 \frac{d^{m-1}}{dt^{m-1}} q(t) + \dots + b_m q(t).$$

$p(t) \rightarrow$ o/p / Response
$q(t) \rightarrow$ I/p / Excitation.

→ Also, order can be determined from the transfer function of the s/m.

$$T(s) = \frac{P(s)}{Q(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_{m-1} s + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_{n-1} s + a_n}.$$

→ $P(s) \Rightarrow$ Numerator polynomial.
$Q(s) \Rightarrow$ Denominator polynomial.

→ The order of the s/m is given by the maximum power of 's' in the denominator polynomial [Q(s)]

$\therefore Q(s) = a_0 s^n + a_1 s^{n-1} + \dots + a_n.$

$n \rightarrow$ order of the s/m.
$n = 0 \Rightarrow$ Zero order s/m
$n = 1 \Rightarrow 1^{st}$ order s/m
$n = 2 \Rightarrow 2^{nd}$ order s/m.

→ Type of the s/m : The numerator and denominator polynomial can be expressed in the factor form as shown in,

$$T(s) = \frac{P(s)}{Q(s)} = \frac{(s+z_1)(s+z_2)\ldots\ldots(s+z_m)}{(s+P_1)(s+P_2)\ldots\ldots(s+P_n)}.$$

→ here 'n' is the no. of poles. Therefore order of the s/m is given by the number of poles of the tfr function.

→ The No. of poles at the origin gives the type of the s/m.

Eg: $Q(s) = s^2(s+1)(s+2)$

no. of poles at origin ⇒ 2.

∴ It is type 2 s/m.

If no pole present at the origin, then it is type-0 s/m.

Recall : Partial Fraction Expansion.

Case 1 : Function with separate/distinct poles.

$$T(s) = \frac{K}{s(s+P_1)(s+P_2)}.$$

$$\frac{K}{s(s+P_1)(s+P_2)} = \frac{A}{s} + \frac{B}{s+P_1} + \frac{C}{s+P_2}.$$

$$A = T(s) \times s \Big|_{s=0}$$

$$B = T(s) \times (s+P_1) \Big|_{s=-P_1}$$

$$C = T(s) \times (s+P_2) \Big|_{s=-P_2}$$

Case 2: Tfr function with Multiple poles.

$$T(s) = \frac{K}{s(s+P_1)(s+P_2)^2}$$

$$\frac{K}{s(s+P_1)(s+P_2)^2} = \frac{A}{s} + \frac{B}{s+P_1} + \frac{C}{(s+P_2)} + \frac{D}{(s+P_2)^2}.$$

$$A = T(s) \times s \Big|_{s=0}$$

$$B = T(s) \times (s+P_1) \Big|_{s=-P_1}$$

$$C = \frac{d}{ds}\left[T(s) \times (s+P_2)^2\right]\Big|_{s=-P_2}.$$

$$D = T(s) \times (s+P_2)^2 \Big|_{s=-P_2}$$

Case 3: Tfr function with complex conjugate poles.

$$T(s) = \frac{K}{(s+P_1)(s^2+bs+c)}$$

$$\frac{K}{(s+P_1)(s^2+bs+c)} = \frac{A}{(s+P_1)} + \frac{Bs+C}{(s^2+bs+c)} \quad \text{---(1)}$$

$$A = T(s) \times (s+P_1) \Big|_{s=-P_1}$$

To solve for B & C, cross multiply the above equation (1), sub the value of (A) and the equate the like power of s.

eg: $1 = (s^2+s+1) + Bs^2 + 2Bs + Cs + 2C$

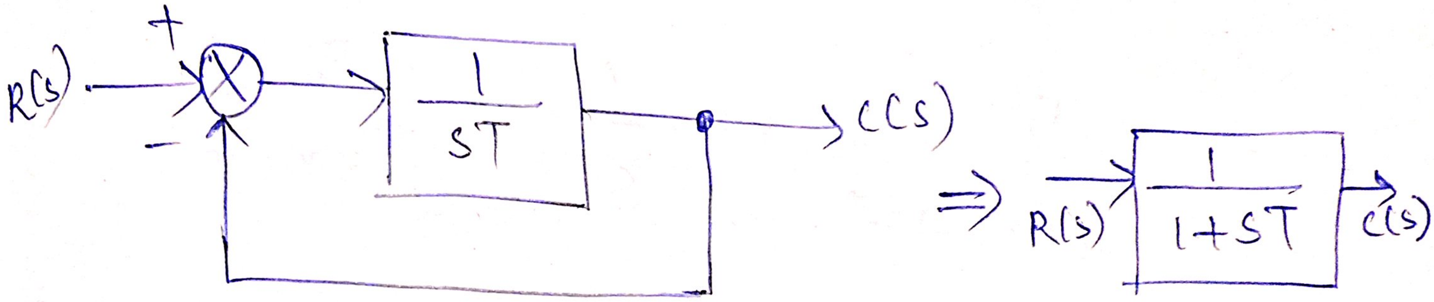
$1 = (1+B)s^2 + (1+2B+C)s + (1+2C)$

co-eff of s^2 $\Rightarrow$ $1+B = 0$

co-eff of s $\Rightarrow$ $1+2B+C = 0$

~~co-eff~~ constant $\Rightarrow$ $1+2C = 1$.

Response of First-order s/m for unit-step i/p



$\rightarrow \dfrac{C(s)}{R(s)} = \dfrac{1}{1+ST}$.

$\rightarrow$ unit step i/p, $r(t)=1 \Rightarrow R(s) = 1/s$.

$\rightarrow C(s) = R(s)\left[\dfrac{1}{1+ST}\right]$

$C(s) = \dfrac{1}{s}\left[\dfrac{1}{1+ST}\right]$

$C(s) = \dfrac{1}{s(1+ST)}$

$$\rightarrow \frac{1}{s(1+sT)} = \frac{1}{sT\left(s+\frac{1}{T}\right)} = \frac{(1/T)}{s(s+1/T)}$$

$$\rightarrow \frac{1/T}{s(s+1/T)} = \frac{A}{s} + \frac{B}{(s+1/T)}$$

$$\rightarrow A = \left.\frac{(1/T)}{\cancel{s}(s+1/T)} \times \cancel{s}\right|_{s=0} = \frac{(1/T)}{(1/T)} = 1$$

$$\rightarrow B = \left.\frac{(1/T)}{s\cancel{(s+1/T)}} \times \cancel{(s+1/T)}\right|_{s=-1/T} = \frac{(1/T)}{(-1/T)} = -1$$

$$\rightarrow C(s) = \frac{1}{s(1+sT)} = \frac{A}{s} + \frac{B}{(s+1/T)}$$

$$= \frac{1}{s} + \frac{-1}{s+1/T}$$

$$\rightarrow c(t) = L^{-1}\left[\frac{1}{s} + \frac{(-1)}{s+1/T}\right]$$

$$\boxed{c(t) = (1 - e^{-t/T})}$$

$$\boxed{LT(e^{-at}) = \frac{1}{s+a}}$$

$\rightarrow$ unit step response.

$\rightarrow$ For step Response, (Amplitude $\neq 1$)

$$\boxed{c(t) = A(1 - e^{-t/T})}$$, $T \rightarrow$ time constant

$$\boxed{c(t) = \begin{cases} (1 - e^{-t/T}), & \text{unit step i/p} \\ A(1 - e^{-t/T}), & \text{step i/p.} \end{cases}}$$

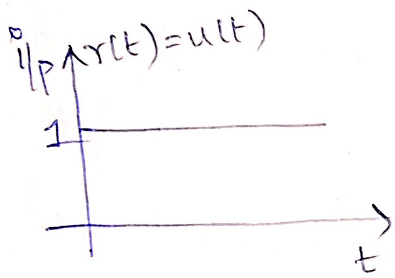
→ $t = 0$, $c(t) = 1 - 1 = 0$

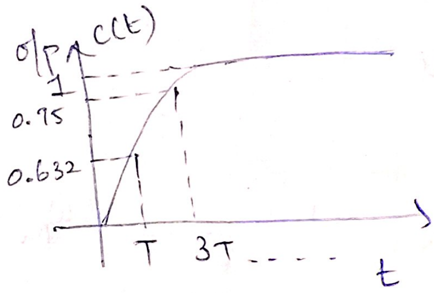
$t = 1T$, $c(t) = 1 - e^{-1} = 0.632$

$t = 2T$, $c(t) = 1 - e^{-2} = 0.865$

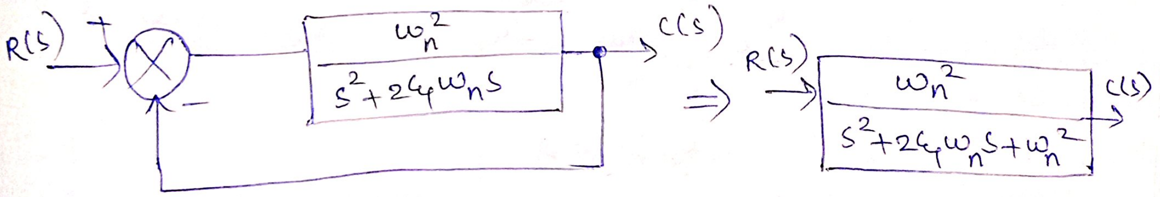
$t = 3T$, $c(t) = 1 - e^{-3} = 0.95$

$t = \infty$, $c(t) = 1 - e^{-\infty} = 1$





Second order s/m



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

ω_n → undamped natural frequency (rad/sec)

ζ → Damping Ratio. (ζ)

→ If $\zeta = 0$ ⇒ undamped

$0 < \zeta < 1$ ⇒ under damped s/m

$\zeta = 1$ ⇒ critically damped s/m

$\zeta > 1$ ⇒ overdamped s/m.

→ characteristic equation of IInd - order s/m

$$s^2 + 2\xi\omega_n s + \omega_n^2 = 0.$$

Roots $\boxed{s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}}$.

ξ	s_1, s_2	Roots & s/m
$\xi = 0$	$\pm j\omega_n$	Roots are purely imaginary. s/m → undamped.
$\xi = 1$	$-\omega_n$	Roots are Real & equal. s/m → critically damped.
$\xi > 1$	$-\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$	Roots are real & unequal. s/m → overdamped.
$0 < \xi < 1$	$-\xi\omega_n \pm \omega_n\sqrt{(-1)(1-\xi^2)}$ $-\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$ $-\xi\omega_n \pm j\omega_d$ where $\boxed{\omega_d = \omega_n\sqrt{1-\xi^2}}$ ↳ damped frequency of oscillation	Roots → complex conjugates s/m → underdamped

#1) Response of undamped II-order s/m for unit step i/p [$\zeta = 0$]

$\rightarrow$ $$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

$\rightarrow$ undamped s/m $\Rightarrow$ $\zeta = 0$.

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}.$$

$$C(s) = R(s)\left[\frac{\omega_n^2}{s^2+\omega_n^2}\right]$$

$\rightarrow$ i/p $\Rightarrow$ step i/p, $\therefore$ $r(t) = u(t)$ & $R(s) = 1/s$.

$$C(s) = \left(\frac{1}{s}\right)\left(\frac{\omega_n^2}{s^2+\omega_n^2}\right)$$

$$\frac{\omega_n^2}{s(s^2+\omega_n^2)} = \frac{A}{s} + \frac{Bs+C}{(s^2+\omega_n^2)}.$$

$\rightarrow$ $$A = \frac{\omega_n^2}{\cancel{s}(s^2+\omega_n^2)} \times \cancel{s}\Bigg|_{s=0} = \frac{\omega_n^2}{\omega_n^2} = 1$$

$\rightarrow$ $$\omega_n^2 = A(s^2+\omega_n^2) + Bs^2 + Cs.$$

$$\omega_n^2 = s^2+\omega_n^2 + Bs^2 + Cs.$$

$$\cancel{\omega_n^2} = s^2(1+B) + \cancel{\omega_n^2} + Cs$$

$1+B = 0 \Rightarrow B = -1$

$Cs = 0 \Rightarrow C = 0.$

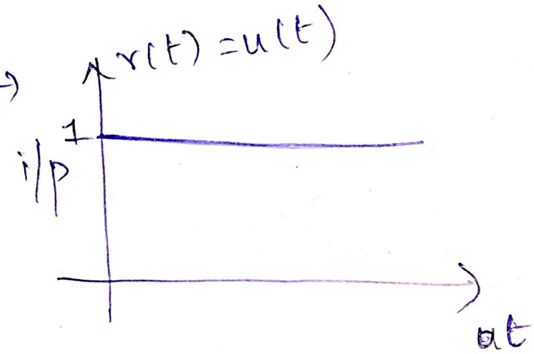
→ $c(s) = \frac{1}{s} - \frac{s}{s^2+\omega_n^2}$.

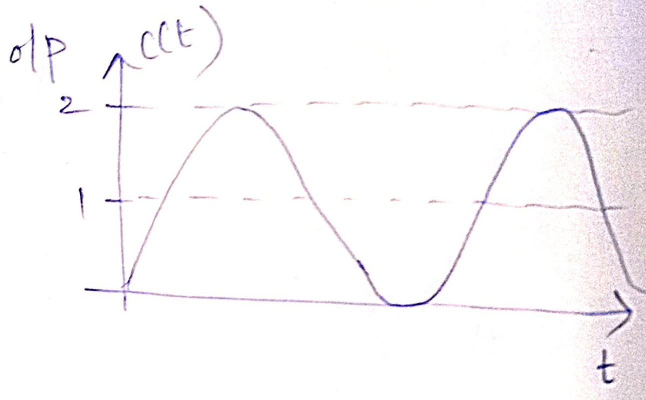
→ $c(t) = L^{-1}\left[\frac{1}{s} - \frac{s}{s^2+\omega_n^2}\right]$

$LT(1) = 1/s$

$LT(\cos at) = \frac{s}{s^2+a^2}$

$$\boxed{c(t) = (1 - \cos\omega_n t)}$$

→ 



→ ∴ For undamped second order s/m,

$$\boxed{c(t) = \begin{cases} (1-\cos\omega_n t), & \text{unit step i/p} \\ A(1-\cos\omega_n t), & \text{step i/p } (A \neq 1) \end{cases}}$$

$\rightarrow \quad C(s) = \frac{1}{s} - \frac{s}{s^2 + \omega_n^2}$

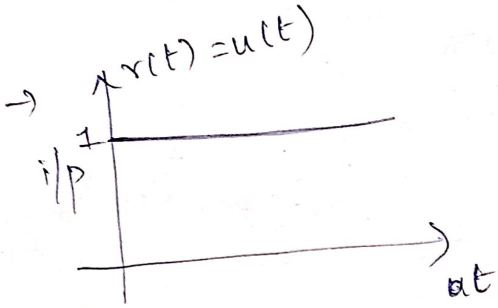
$\rightarrow \quad c(t) = L^{-1}\left[\frac{1}{s} - \frac{s}{s^2+\omega_n^2}\right]$

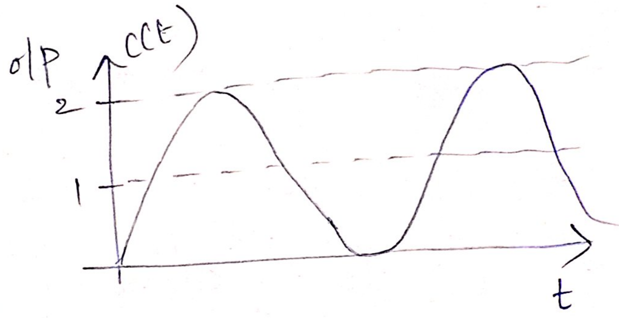
$LT(1) = 1/s$

$LT(\cos at) = \frac{s}{s^2+a^2}$

$$\boxed{c(t) = (1 - \cos \omega_n t)}$$

$\rightarrow$





$\rightarrow$ $\therefore$ For undamped second order s/m,

$$\boxed{c(t) = \begin{cases} (1 - \cos \omega_n t), & \text{unit step i/p} \\ A(1 - \cos \omega_n t), & \text{step i/p } (A \neq 1) \end{cases}}$$

2) Response of Underdamped Second Order s/m for Unit-Step I/p.

$\rightarrow$ Std. form of II^{nd} order CLTF is,

$$\boxed{\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}}$$

→ For under-damped s/m, $0 < \xi < 1$ & the roots of the characteristic eqn. are complex conjugates.

$$s = -\xi \omega_n \pm j \omega_d .$$

→ unit step i/p ⇒ $r(t) = 1$ & $R(s) = 1/s$.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$C(s) = R(s)\left[\frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}\right]$$

$$= \frac{\omega_n^2}{s\left[s^2 + 2\xi\omega_n s + \omega_n^2\right]}$$

→ By partial fraction,

$$\frac{\omega_n^2}{s(s^2 + 2\xi\omega_n s + \omega_n^2)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

$$A = \frac{\omega_n^2}{\cancel{s}(s^2 + 2\xi\omega_n s + \omega_n^2)} \times \cancel{s} \Bigg|_{s=0}$$

$$= \frac{\cancel{\omega_n^2}}{\cancel{\omega_n^2}} = 1$$

$$\boxed{A = 1}$$

$$\omega_n^2 = A(s^2 + 2\xi\omega_n s + \omega_n^2) + Bs^2 + Cs.$$

$$= s^2 + 2\xi\omega_n s + \omega_n^2 + Bs^2 + Cs.$$

$$\cancel{\omega_n^2} = s^2(1+B) + (2\xi\omega_n + C)s + \cancel{\omega_n^2}$$

$1 + B = 0$

$\boxed{B = -1}$

$2\xi\omega_n + C = 0$

$\boxed{C = -2\xi\omega_n}$

$$\rightarrow \therefore C(s) = \frac{1}{s} + \frac{(-s - 2\xi\omega_n)}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

$$C(s) = \frac{1}{s} - \frac{(s + 2\xi\omega_n)}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

$$= \frac{1}{s} - \frac{s + \xi\omega_n}{(s^2 + 2\xi\omega_n s + \omega_n^2)} - \frac{\xi\omega_n}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

$$= \frac{1}{s} - \frac{s + \xi\omega_n}{((s + \xi\omega_n) + j\omega_d)((s + \xi\omega_n) - j\omega_d)}$$

$$- \frac{\xi\omega_n}{((s + \xi\omega_n) + j\omega_d)((s + \xi\omega_n) - j\omega_d)}$$

$$= \frac{1}{s} - \frac{(s + \xi\omega_n)}{(s + \xi\omega_n)^2 + \omega_d^2} - \frac{\xi\omega_n}{(s + \xi\omega_n)^2 + \omega_d^2}.$$

$$c(s) = \frac{1}{s} - \frac{s+\zeta\omega_n}{(s+\zeta\omega_n)^2+\omega_d^2} - \frac{\omega_d(\zeta\omega_n)}{\omega_d\left((s+\zeta\omega_n)^2+\omega_d^2\right)}$$

$$c(t) = L^{-1}\left[C(s)\right]$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\cos(\omega_d t) - \frac{\zeta\omega_n}{\omega_d} e^{-\zeta\omega_n t}\sin(\omega_d t)$$

$$\because L\{e^{-at}\sin\omega t\} = \frac{\omega}{(s+a)^2+\omega^2}$$

$$L\{e^{-at}\cos\omega t\} = \frac{s+a}{(s+a)^2+\omega^2}$$

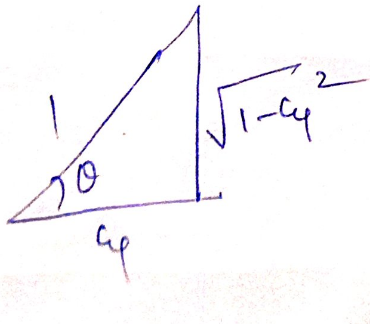
$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos(\omega_d t) + \frac{\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}}\sin(\omega_d t)\right]$$

$$c(t) = 1 - e^{-\zeta\omega_n t}\left[\cos(\omega_d t) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_d t)\right]$$

$$= 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\cos(\omega_d t)\times\sqrt{1-\zeta^2} + \zeta\sin(\omega_d t)\right]$$

$$\sin\theta = \sqrt{1-\zeta^2}$$

$$\cos\theta = \zeta$$



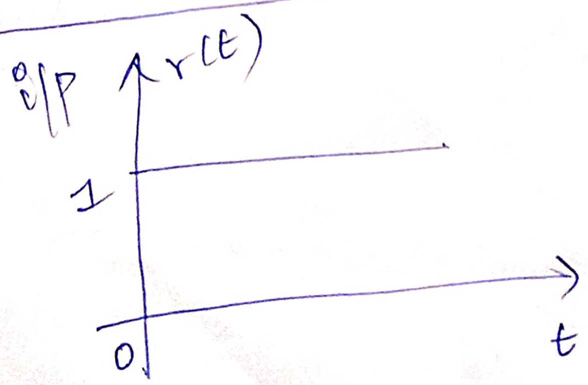
$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \left[\cos(\omega_d t)\sin\theta + \sin(\omega_d t)\cos\theta \right]$$

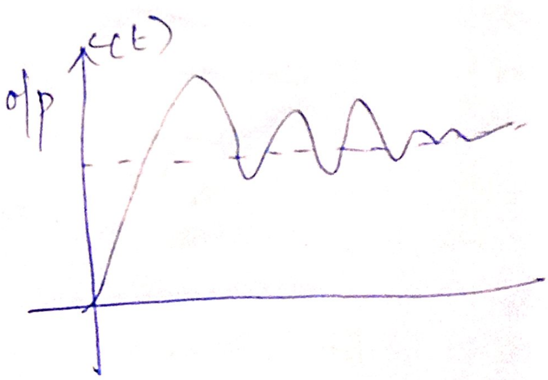
$$\boxed{c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)}$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$$

∴ For under-damped unit step Response for IInd order s/m,

$$\boxed{c(t) = \begin{cases} 1 - \dfrac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta), & \text{unit step} \\ A\left(1 - \dfrac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)\right), & \text{Step.} \end{cases}}$$





#3) Critically damped Second-order S/m for unit Step - i/p

$\rightarrow \quad \dfrac{C(s)}{R(s)} = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

$\rightarrow$ For critical damping $\zeta = 1$

$$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s+\omega_n)^2}$$

$\rightarrow$ i/p $r(t) = 1$, $R(s) = 1/s$.

$$C(s) = R(s)\left(\frac{\omega_n^2}{(s+\omega_n)^2}\right)$$

$$= \frac{1}{s}\left(\frac{\omega_n^2}{(s+\omega_n)^2}\right)$$

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2}$$

$$\frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)} + \frac{C}{(s+\omega_n)^2}$$

$\rightarrow \quad A = \left.\dfrac{\omega_n^2}{(s+\omega_n)^2}\right|_{s=0} = 1$

$$B = \frac{d}{ds}\left(\cancel{(s+\omega_n)^2} \cdot \frac{\omega_n^2}{s\cancel{(s+\omega_n)^2}}\right)\Bigg|_{s=-\omega_n} = -\frac{\omega_n^2}{s^2} = -1$$

S.No

$$c = \frac{\omega_n^2}{s\cancel{(s+\omega_n)^2}} \times \cancel{(s+\omega_n)^2} \Bigg|_{s=-\omega_n}$$

$$c = \frac{\omega_n^2}{-\omega_n} = -\omega_n$$

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

$$c(t) = L^{-1}\left[C(s)\right]$$

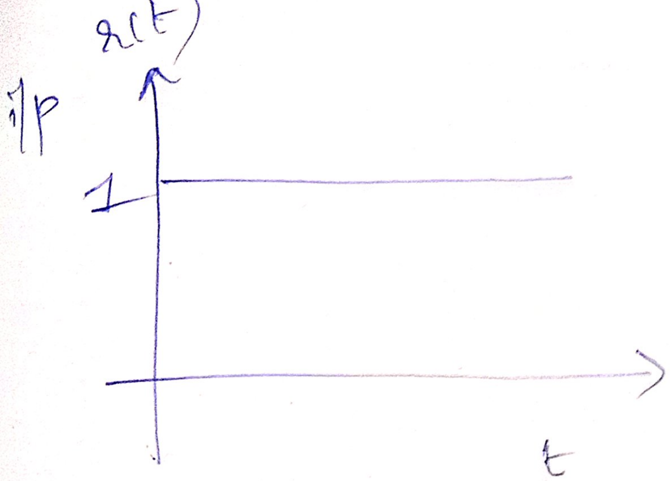
$$\boxed{c(t) = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}}$$

$$\because L\left\{te^{-at}\right\} = \frac{1}{(s+a)^2}.$$

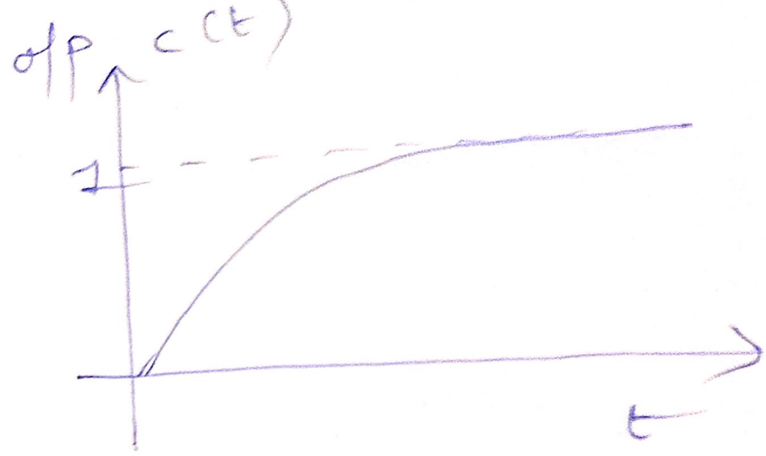
→ ∴ closed loop critically damped II-order s/m,

Response is

$$c(t) = \begin{cases} 1 - e^{-\omega_n t}(1+\omega_n t), & \text{unit step i/p} \\ A\left[1 - e^{-\omega_n t}(1+\omega_n t)\right], & \text{step i/p.} \end{cases}$$



r(t)
i/p
1
t



o/p
c(t)
1
t