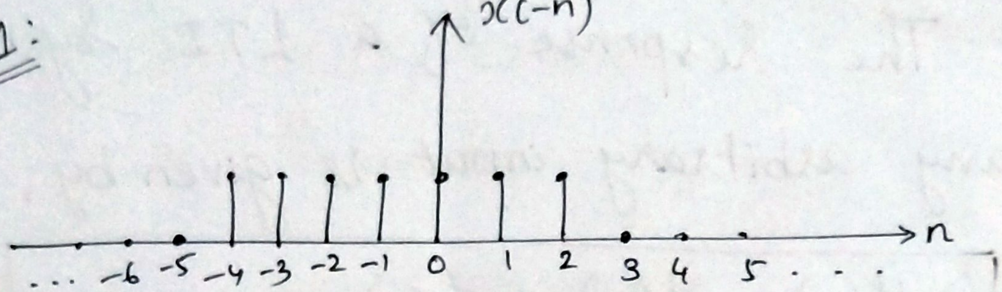
(or)

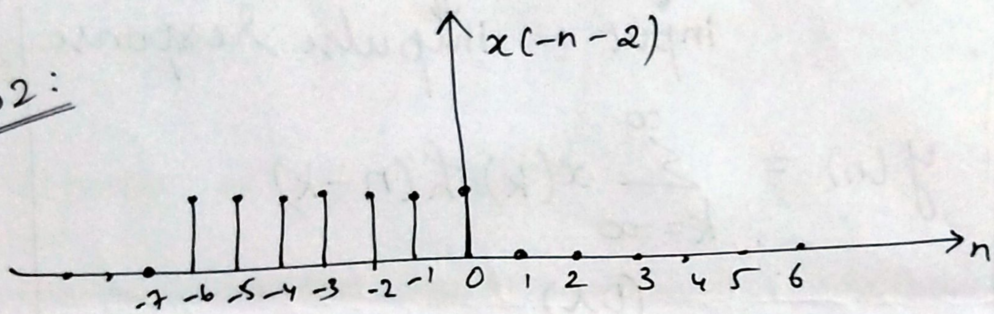
Step 1: fold $x(n)$ to get $x(-n)$

Step 2: Shift $x(-n)$ by 2 units towards left to get $x(-n-2)$ $\Longrightarrow$

Step 1:



Step 2:



Linear - Time Invariant System (LTI):

If a system satisfies both Linearity and Time - Invariance property, then it is said to be LTI.

Analysis of DT - LTI System:

Linear Convolution (or) Convolution Sum:

The response of a LTI System for any arbitrary input is given by,

$$\boxed{\begin{aligned} y(n) &= \underset{\text{input}}{\underset{\Downarrow}{x(n)}} * \underset{\text{impulse response}}{\underset{\Downarrow}{h(n)}} \\ y(n) &= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \\ &\text{(or)} \\ y(n) &= \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \end{aligned}}$$

If $N_1 \rightarrow$ no. of samples in $x(n)$

$N_2 \rightarrow$ " " " in $h(n)$

then, no. of samples in $y(n)$ (or) length of $y(n)$ $= N_1 + N_2 - 1$

The output $y(n)$ is a non-periodic sequence

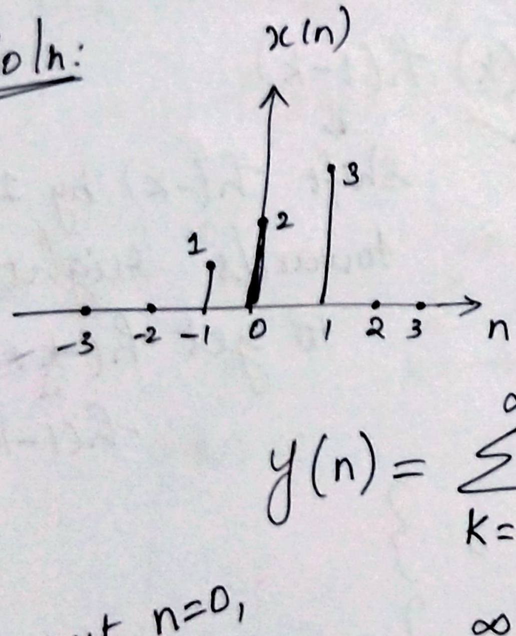
$\therefore$ this convolution is called as $\boxed{\text{aperiodic Convolution}}$

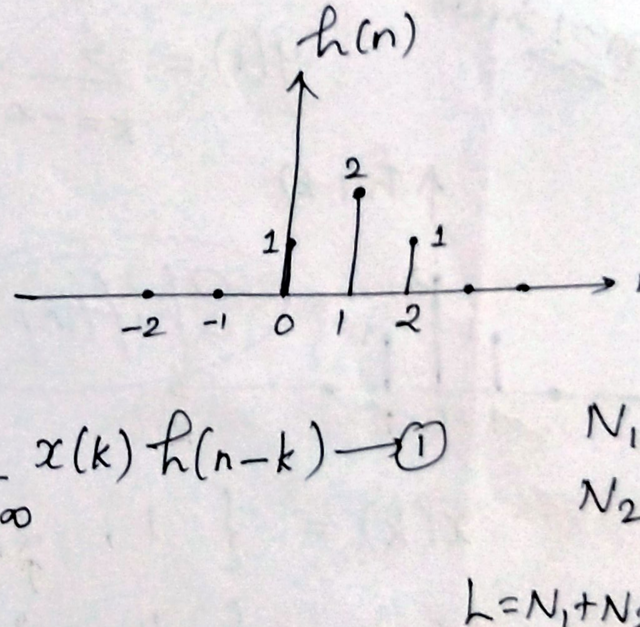
Steps to perform Convolution Sum:

1. Folding
2. Shifting
3. Multiplication
4. Summation.

pbm: The impulse response of a LTI system is $h(n) = \{ \underset{\uparrow}{1}, 2, 1 \}$, Determine the response of the system to i/p $x(n) = \{ 1, \underset{\uparrow}{2}, 3 \}$

Soln:





$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \longrightarrow (1)$$

$N_1 = 3$

$N_2 = 3$

$L = N_1 + N_2 - 1$

$= 3 + 3 - 1$

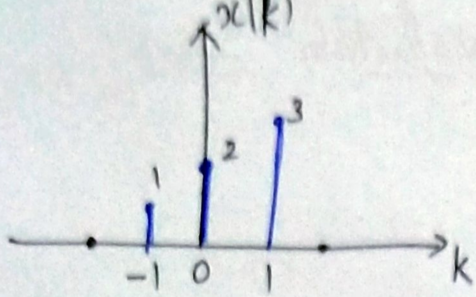
$L = 5$

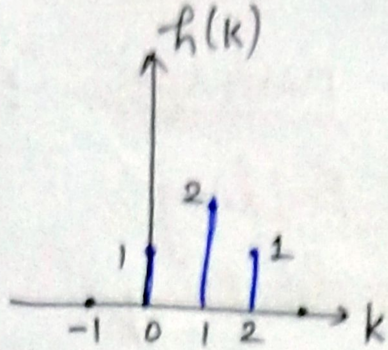
put $n=0$, in (1)

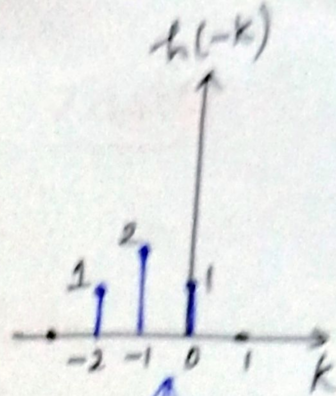
$$y(0) = \sum_{k=-\infty}^{\infty} x(k)\, h(0-k)$$

$x(k)$: put $n=k$ in $x(n)$

$h(0-k)$: fold $h(k)$; put $n=k$ in $h(n)$







folding

no shifting for n=0

$x(k) = \{1, \underset{\uparrow}{2}, 3\}$ multiplication

$h(-k) = \{1, 2, \underset{\uparrow}{1}\}$

$x(k) \cdot h(-k) = \{0, 2, \underset{\uparrow}{2}, 0\}$

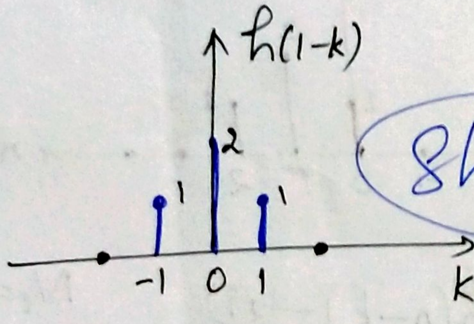
$$\sum_{k=-\infty}^{\infty} x(k)\, h(-k) = \underbrace{2+2}_{\text{addition}} = \boxed{4 = y(0)}$$

put $\underline{\underline{n=1}}$ in ①

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\, h(1-k)$$

$\Downarrow$

shift $h(-k)$ by 1 unit towards right to get $h(-k+1)$ ie $h(1-k)$



Shifting

$x(k) = \{1, \underset{\uparrow}{2}, 3\}$

$h(1-k) = \{1, \underset{\uparrow}{2}, 1\}$

$x(k)\, h(1-k) = \{1, 4, 3\}$

$$\sum_{k=-\infty}^{\infty} x(k)\, h(1-k) = 1+4+3 = \boxed{8 = y(1)}$$

put $\underline{n=2}$ in ①,

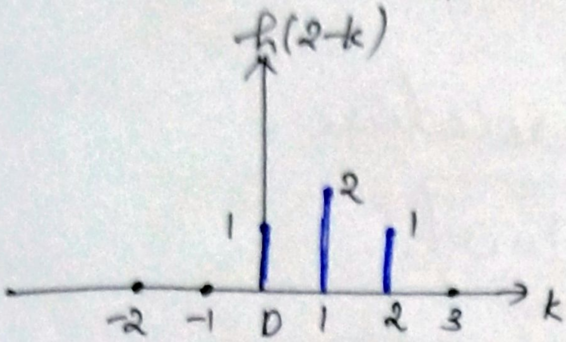
$$y(2)=\sum_{k=-\infty}^{\infty} x(k)\,h(2-k)$$

$\Downarrow$

Shift $h(1-k)$ by 1 unit towards right.

(or)

Shift $h(-k)$ by 2 units towards right.



$$x(k)=\{1, 2, 3\}$$

$$h(2-k)=\{1, 2, \cancel{1}\}$$

$$x(k)\,h(2-k)=\{2, 6\}$$

$$\sum x(k)\,h(2-k) = 2+6 = \boxed{8=y(2)}$$

put $\underline{n=3}$ in ①

$$y(3)=\sum_{k=-\infty}^{\infty} x(k)\,h(3-k)$$

$\Downarrow$

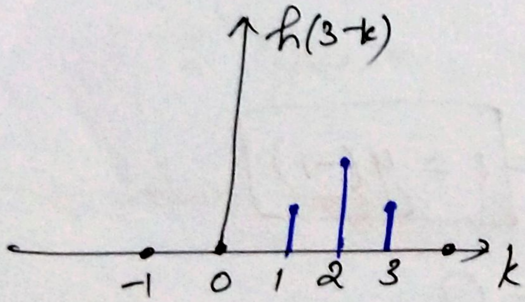
Shift $h(2-k)$ by 1 unit towards right.

(or)

Shift $h(1-k)$ by 2 units towards right

(or)

Shift $h(-k)$ by 3 units towards right



$$x(k)=\{1, 2, 3\}$$

$$h(3-k)=\{0, 1, 2, 3\}$$

$$x(k)\, h(3-k) = \{0, 3\}$$

$$\sum x(k)\, h(3-k) = 0+3 = \boxed{3 = y(3)}$$

|||ly

put $n=4$ in ①

do the same procedure.

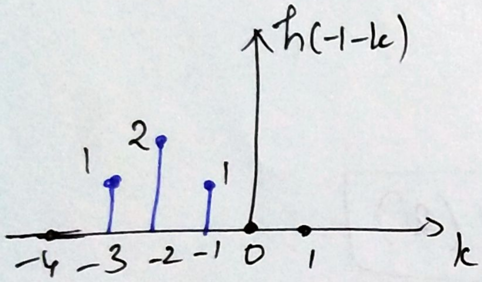
$\boxed{y(4) = 0}$. $\boxed{y(5)=0}$. . . . .

put $n=-1$ in ①

$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)\, h(-1-k)$$

$\Downarrow$

Shift $h(-k)$ by 1 unit towards left



$$x(k) = \{1, \underset{\uparrow}{2}, 3\}$$

$$h(-1-k) = \{1, 2, 1, \underset{\uparrow}{0}\}$$

$$x(k)\, h(-1-k) = \{1, \underset{\uparrow}{0}, 0\}$$

$$\sum x(k)\, h(-1-k) = 1+0 = \boxed{1 = y(-1)}$$

put $n=-2$,

$$y(-2) = y(-3) = \ldots = 0$$

$$\therefore \boxed{y(n) = \{\ldots 0, 1, \underset{\uparrow}{4}, 8, 8, 3, 0 \ldots\}}$$

(or)

Alternate method

$$x(n) = \{1, \underset{\uparrow}{2}, 3\}$$

$$h(n) = \{\underset{\uparrow}{1}, 2, 1\}$$

$$y(n) = x(n) * h(n)$$

	start $n=-1$: 1	$n=0$: 2	end $n=1$: 3
start $n=0$: 1	1	2	3
$n=1$: 2	2	4	6
end $n=2$: 1	1	2	3

+ + + +

$$y(n) = \{\dots 0, \underset{\underset{n=-1}{\uparrow}}{1}, \underset{\underset{n=0}{\uparrow}}{4}, 8, 8, \underset{\underset{n=3}{\uparrow}}{3}, 0 \dots\}$$

start + start   end + end

pbm Determine the output $y(n)$ of a relaxed LTI system with $h(n) = a^n u(n)$, where $|a| < 1$ when input is a step sequence.

gn: $x(n) = u(n)$

$h(n) = a^n u(n)$

$\because u(n) = 1, n \geq 0$
$\quad\quad 0, n < 0$

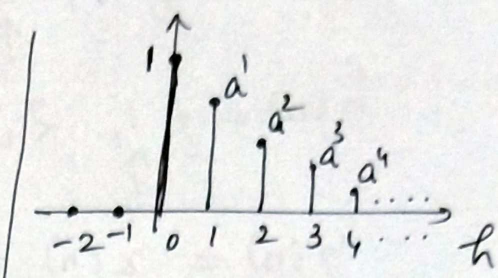
Soln:

$$y(n) = x(n) * h(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \quad \text{①}$$

$$h(k) = a^k u(k)$$

ie. $$h(k) = \begin{cases} a^k \cdot 1, & \text{for } k \geq 0 \\ 0, & \text{for } k < 0 \end{cases}$$



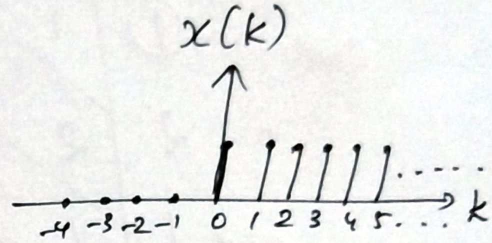
$x(n-k) = ?$

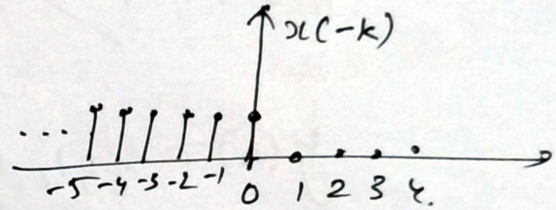
$$x(n) = u(n)$$

$$x(k) = u(k) = \begin{cases} 1, & k \geq 0 \\ 0, & k < 0 \end{cases}$$

$x(-k+n) = ?$

Shift $x(-k)$ by 'n' units towards right.





put $n=0$ in ①,

$$y(0) = \sum_{k=-\infty}^{\infty} h(k)\, x(-k)$$

$$h(k) = \{0, 0, \underset{\uparrow}{1}, a^1, a^2, a^3, a^4, a^5 \ldots\}$$

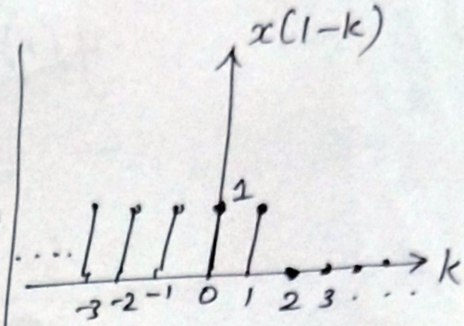
$$x(-k) = \{\ldots 1, 1, 1, \underset{\uparrow}{1}, 0, 0, 0 \ldots\}$$

$$h(k) \cdot x(-k) = \{0, 0, 0, \underset{\uparrow}{1}, 0, 0, \ldots\}$$

$$\sum h(k)\, x(-k) = 0 + 1 + 0 = \boxed{1 = y(0)}$$

put n=1 in ①,

$$y(1) = \sum_{k=-\infty}^{\infty} h(k)\, x(1-k)$$



$$h(k) = \{\ldots 0, 0, \underset{\uparrow}{1}, a^1, a^2, a^3, a^4, \ldots\}$$

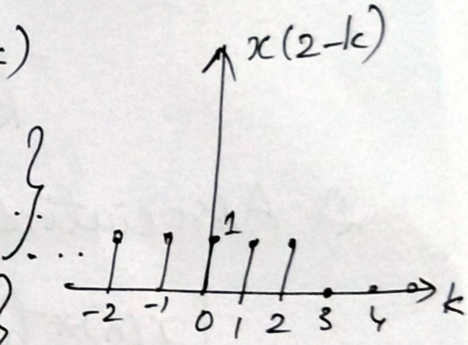
$$x(1-k) = \{\ldots, 1, \underset{\uparrow}{1}, 1, 0, 0, 0 \ldots\}$$

$$h(k)\, x(1-k) = \{\ldots 0, 0, \underset{\uparrow}{1}, a^1, 0, 0 \ldots\}$$

$$\sum h(k)\, x(1-k) = \boxed{1 + a^1 = y(1)}$$

put n=2 in ②

$$y(2) = \sum_{k=-\infty}^{\infty} h(k)\, x(2-k)$$



$$h(k) = \{\ldots 0, 0, \underset{\uparrow}{1}, a^1, a^2, a^3, a^4, a^5, \ldots\}$$

$$x(2-k) = \{\ldots 1, 1, \underset{\uparrow}{1}, 1, 1, 0, 0, 0, \ldots\}$$

$$h(k)\, x(2-k) = \{\ldots 0, 0, 1, a^1, a^2, 0, 0 \ldots\}$$

$$\sum h(k)\, x(2-k) = \boxed{1 + a + a^2 = y(2)}$$

Similary $y(3) = 1 + a + a^2 + a^3$

$y(4) = 1 + a + a^2 + a^3 + a^4$

$\vdots$

$$y(n) = 1+a+a^2+a^3+\cdots+a^n$$

$$= \sum_{m=0}^{n} a^m \quad \text{for } n \geq 0$$

$$\boxed{y(n) = \frac{1-a^{n+1}}{1-a}}, \quad n \geq 0.$$

Properties of Linear Convolution:

1) Commutative:

$$y(n) = x(n) * h(n) = h(n) * x(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k) h(n-k) = \sum_{k=-\infty}^{\infty} h(k) x(n-k)$$

2) Associative:

$$[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$$

3) Distributive:

$$x(n) * [h_1(n) + h_2(n)] = [x(n) * h_1(n)] + [x(n) * h_2(n)]$$

Causal LTI system:

If $h(n)$ of ~~a~~ LTI system is $h(n) = 0, n < 0$ then it is causal.

Discrete Fourier Transform (DFT) :

Intro:

In order to perform Spectral analysis (i.e. to find the frequency content) of the Signals and Systems, either Fourier Series (or) Fourier Transform is performed.

CT, periodic Signal (continuous in time) $x(t)$ $\xrightarrow{CTFS}$ Discrete frequency / Aperiodic Signal. C_n

DT, periodic Signal $x(n)$ $\xrightarrow{DTFS}$ Discrete freq & periodic Signal. C_k

Continuous time Aperiodic $x(t)$ $\xrightarrow{CTFT}$ Continuous freq. & Aperiodic Signal $X(\Omega)$ $(-\infty$ to $\infty)$

DT, Aperiodic $x(n)$ (finite or infinite) $\xrightarrow{DTFT}$ Continuous frequency periodic Signal $X(\omega)$ with period 2π. $(-\pi$ to π or 0 to $2\pi)$

DT, aperiodic $x(n)$ (finite) $\xrightarrow{DFT}$ Discrete frequency periodic Signal. $X(k)$

Fourier Transform (or) DTFT : $X(\omega)$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$$

$$x(n) \xleftrightarrow{FT} X(\omega)$$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) \cdot e^{j\omega n} \, d\omega.$$

$X(\omega) \rightarrow$ Continuous function of ω, so it cannot be processed by digital system.

So, we need to convert continuous function of ω into a discrete function of ω.

Why DFT?

→ to determine the frequency content of a signal, i.e to perform spectral analysis.

→ to perform filtering operations in the frequency domain.

→ plays important role in the implementation of many signal processing algorithms

Conversion of DTFT to DFT: (frequency-domain sampling) by

The DFT is obtained by sampling one period of the FT $X(\omega)$ of the signal $x(n)$ at a finite number of frequency points.

This sampling is conventionally performed at N equally spaced points in the period $0 \leq \omega \leq 2\pi$ (or) at $\omega_k = 2\pi k/N$, $0 \leq k \leq N-1$.

$$X(k) = X(\omega)\Big|_{\omega = \frac{2\pi k}{N}}, \quad \text{for } k = 0, 1, 2 \ldots (N-1)$$

$\downarrow$

Sequence consisting of N samples of $X(\omega)$.

$\therefore$ called as $\boxed{\text{N-point DFT}}$

Let

$\left.\begin{array}{l}\text{Length of finite duration} \\ \text{sequence } x(n)\end{array}\right\} \Rightarrow L$

No. of samples in $X(k) \Longrightarrow N$

condition: $N \geq L$

definition of DFT,

$$X(k) = DFT\{x(n)\} = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi kn}{N}}, \quad k = 0, 1, 2, \ldots N-1$$

IDFT,

$$x(n) = IDFT\{X(k)\} = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{+j\frac{2\pi kn}{N}}, \quad n = 0, 1, 2 \ldots N-1$$

This DFT of a signal $x(n)$ is said to exist if the signals are absolutely summable.

ie $$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

DFT pair:

$$x(n) \xleftrightarrow{DFT} X(k)$$

(of length L) (of Length N)

$$\boxed{N \geq L}$$