for $N=4$,

$$s_1 = e^{j\left(\pi/2+\frac{\pi}{8}\right)} = e^{j\left(\frac{5\pi}{8}\right)} = -0.3827 + j0.9239$$

$$s_2 = e^{j7\pi/8} = -0.9239 + j0.3827$$

$$s_3 = e^{j9\pi/8} = -0.9239 - j0.3827$$

$$s_4 = e^{j11\pi/8} = -0.3827 - j0.9239$$

Now, the denominator polynomial of $H(s)$ is

$$\left\{(s+0.3827)^2+(0.9237)^2\right\}\left\{(s+0.9239)^2+(0.3827)^2\right\}=0$$

$$(s^2+1.84776s+1)(s^2+0.76536s+1)=0$$

$\therefore$ for 4^{th} order Butterworth filter with $\Omega_c = 1\,rad/sec$,

$$H(s) = \frac{1}{(s^2+0.76536s+1)(s^2+1.847768s+1)}$$

All the poles obtained from equations (4), (5) & (6) are normalized poles (i.e. for $\Omega_c = 1\,rad/sec$)

The unnormalized poles are given by

$$s_k' = \Omega_c \cdot s_k$$

$\therefore$ The transfer function of un-normalized Butterworth filter is obtained by Substituting

$s \rightarrow s/\Omega_c$ in the transfer function of normalized Butterworth filter.

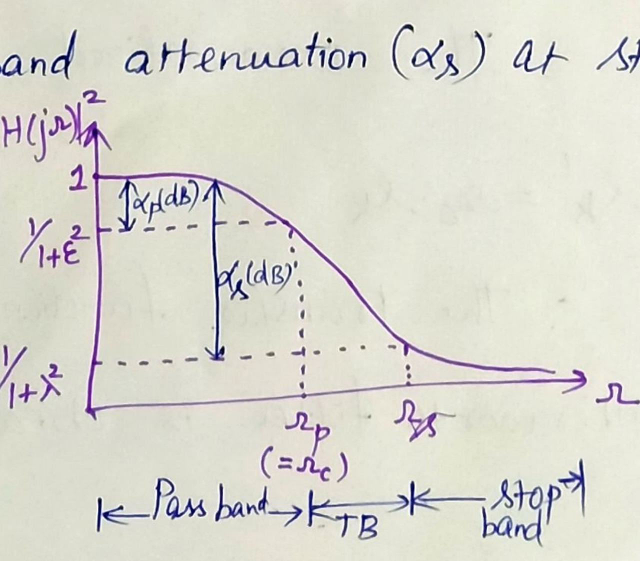
List of Butterworth Polynomials:

N	denominator of H(s)
1	$s+1$
2	$s^2+\sqrt{2}\,s+1$
3	$(s+1)(s^2+s+1)$
4	$(s^2+0.76537s+1)(s^2+1.84776s+1)$
5.	$(s+1)(s^2+0.618034s+1)(s^2+1.618034s+1)$

upto 3rd order filter (Syllabus)

Order of Butterworth filter (N):

Let us consider approximate magnitude response of Butterworth filter which has maximum passband attenuation (α_p) at passband frequency (Ω_p) and minimum stop band attenuation (α_s) at stopband frequency (Ω_s) as shown in fig.



Butterworth approximation of squared magnitude response

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}} \longrightarrow ①a$$

(or)

$$|H(j\Omega)| = \frac{1}{\sqrt{1+\varepsilon^2\left(\Omega/\Omega_p\right)^{2N}}} \longrightarrow ①b$$

Take logarithm on both sides of eqn ①a,

$$20\log|H(j\Omega)| = \underbrace{10\log 1}_{(=0)} - 10\log\left[1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}\right]$$

$$20\log|H(j\Omega)| = -10\log\left[1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}\right] \longrightarrow ②$$

$\alpha_p \rightarrow$ max. passband attenuation at $\Omega = \Omega_p$

To find α_p, sub $\Omega = \Omega_p$ in equation ②

$$20\log|H(j\Omega_p)| = -10\log(1+\varepsilon^2) = -\alpha_p \text{ (dB)}$$

$$\boxed{\alpha_p = 10\log(1+\varepsilon^2)} \longrightarrow ③$$

$$0.1\,\alpha_p = \log(1+\varepsilon^2)$$

Take antilog on both sides

$$10^{0.1\alpha_p} = 1+\varepsilon^2$$

$$\boxed{\varepsilon = \sqrt{10^{0.1\alpha_p} - 1}} \longrightarrow ④$$

$\alpha_s \rightarrow$ minimum stop band attenuation at $\Omega = \Omega_s$,

to find α_s, sub $\Omega = \Omega_s$ in eqn ②,

$$20 \log |H(j\Omega_s)| = \underbrace{10 \log 1}_{=0} - 10 \log \left[1 + \varepsilon^2 \left(\frac{\Omega_s}{\Omega_p} \right)^{2N} \right] = -\alpha_s$$

$$\boxed{\alpha_s = 10 \log \left[1 + \varepsilon^2 \left(\frac{\Omega_s}{\Omega_p} \right)^{2N} \right]} \rightarrow ⑤$$

$$0.1 \alpha_s = \log \left[1 + \varepsilon^2 \left(\frac{\Omega_s}{\Omega_p} \right)^{2N} \right]$$

Take antilog,

$$10^{0.1\alpha_s} = 1 + \varepsilon^2 \left(\frac{\Omega_s}{\Omega_p} \right)^{2N}$$

$$\varepsilon^2 \left(\frac{\Omega_s}{\Omega_p} \right)^{2N} = 10^{\alpha_s(0.1)} - 1 \rightarrow ⑥$$

Sub ε^2 in eqn ⑥,

$$10^{0.1\alpha_p} - 1 \left(\frac{\Omega_s}{\Omega_p} \right)^{2N} = 10^{0.1\alpha_s} - 1$$

$$\boxed{\left(\frac{\Omega_s}{\Omega_p} \right)^{\cancel{2}N} = \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}} \rightarrow ⑥a$$

Take log on both sides,

$$\log \left(\frac{\Omega_s}{\Omega_p} \right)^{\cancel{2}N} = \log \left\{ \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}} \right\}$$

$$\boxed{N = \frac{\log \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log (\Omega_s / \Omega_p)}} \rightarrow ⑦$$

$N \rightarrow$ order of the filter (integer)

equation (7) result will not be an integer value.

so roundoff N to next higher integer.

ie. $$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\log(\Omega_s/\Omega_p)}$$

$$\boxed{N \geq \frac{\log(\lambda/\varepsilon)}{\log(\Omega_s/\Omega_p)}} \longrightarrow (8)$$

where $\lambda = \sqrt{10^{0.1\alpha_s}-1}$

$\varepsilon = \sqrt{10^{0.1\alpha_p}-1}$

let $\frac{\Omega_p}{\Omega_s} = k$ (transition ratio), & $A = \lambda/\varepsilon$

$$\boxed{N \geq \frac{\log A}{\log(1/k)}} \longrightarrow (9)$$

Cut-off frequency (Ω_c):

magnitude squared response of ideal BWorth filter,

$$|H(j\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \longrightarrow (1)$$

Magnitude squared response of practical filter,

$$|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}} \longrightarrow (2)$$

Compare (1) & (2),

$$\cancel{1}+\left(\frac{\Omega}{\Omega_c}\right)^{2N} = \cancel{1}+\varepsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}$$

$$\varepsilon^2 = \left(\frac{\Omega_p}{\Omega_c}\right)^{2N} = 10^{0.1\alpha_p}-1$$

$$\frac{\Omega_p}{\Omega_c} = \left[10^{0.1\alpha_p}-1\right]^{1/2N}$$

$$\boxed{\Omega_c = \frac{\Omega_p}{\left[10^{0.1\alpha_p}-1\right]^{1/2N}}} \longrightarrow (3)$$

$$\Omega_c = \frac{\Omega_p}{(\varepsilon^2)^{1/2N}} = \boxed{\frac{\Omega_p}{\varepsilon^{1/N}} = \Omega_c} \longrightarrow (4)$$

Refer eqn (6a) in page no 18.

$$\left(\frac{\Omega_s}{\Omega_p}\right)^{2N} = \frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}$$

$$\Omega_s = (\Omega_p)\left[\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}\right]^{1/2N}$$

Sub Ω_p from (3)

$$\Omega_s = \Omega_c \cancel{\left[10^{0.1\alpha_p}-1\right]^{1/2N}} \cdot \frac{\left[10^{0.1\alpha_s}-1\right]^{1/2N}}{\cancel{\left[10^{0.1\alpha_p}-1\right]^{1/2N}}}$$

$$\boxed{\Omega_c = \frac{\Omega_s}{\left[10^{0.1\alpha_s}-1\right]^{1/2N}}} \longrightarrow (5)$$

wkt $\lambda^2 = 10^{0.1\alpha_s}-1$

$$\Omega_c = \frac{\Omega_s}{(\lambda^2)^{1/2N}} = \boxed{\frac{\Omega_s}{\lambda^{1/N}} = \Omega_c} \longrightarrow (6)$$

$$\boxed{\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p}-1\right)^{1/2N}} = \frac{\Omega_s}{\left[10^{0.1\alpha_s}-1\right]^{1/2N}}} \longrightarrow (7)$$

At $\alpha_p = 3dB$, $\boxed{\Omega_c = \Omega_p}$

Pbm Given the specification $\alpha_p = 1dB$, $\alpha_s = 30dB$, $\Omega_p = 200$ rad/sec, $\Omega_s = 600$ rad/sec, find 'N' (order of the filter)

$$N \geq \frac{\log A}{\log(1/k)}$$

$$A = \frac{\lambda}{\varepsilon} = \sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}} = \left[\frac{10^3-1}{10^{0.1}-1}\right]^{0.5}$$

$A = 62.115$

$$k = \frac{\Omega_p}{\Omega_s} = \frac{200}{600} = 1/3$$

$$N \geq \frac{\log(62.115)}{\log(3)}$$

$N \geq 3.758$ (round off to next higher integer)

$$\boxed{N = 4}$$

Pbm Determine the order and poles of Low Pass Butterworth filter that has a +3dB attenuation at 500 Hz (or −3dB Bandwidth) and attenuation of 40dB at 1000 Hz.

Given: $\alpha_p = 3dB$, $\alpha_s = 40dB$

$f_p = 500 Hz$ $f_s = 1000 Hz$

$\Omega_p = 2\pi f_p = 1000\pi$ rad/sec, $\Omega_s = 2\pi f_s = 2000\pi$ rad/sec

$$N \geq \frac{\log A}{\log(1/k)}$$

$N \geq 6.6$

$$\boxed{N = 7}$$

The poles of Butterworth filter are given by,

$$s_k' = \Omega_c \cdot e^{j\phi_k}$$

where $\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$, $k = 1, 2, \ldots 7$

$$s_k' = 1000\pi \cdot e^{j\phi_k} = 1000\pi \cdot s_k$$

$$k=1,\ e^{j\left(\frac{\pi}{2}+\frac{\pi}{14}\right)} = e^{j\left(\frac{\cancel{8}^{4}\pi}{\cancel{14}_{7}}\right)} = -0.22 + j0.974 = s_1$$

$$k=2,\ e^{j\frac{10\pi}{14}} = -0.623 + j0.781 = s_2$$

$$k=3,\ e^{j\frac{12\pi}{14}} = -0.9009 + j0.433 = s_3$$

$$k=4,\ e^{j\pi} = -1 = s_4$$

$$k=5,\ e^{j\frac{\cancel{16}^{8}\pi}{\cancel{14}_{7}}} = -0.901 - j0.433 = s_5$$

$$k=6,\ e^{j\frac{18\pi}{14}} = -0.623 - j0.781 = s_6$$

$$k=7,\ e^{j\frac{20\pi}{14}} = -0.22 - j0.974 = s_7$$

Design of an analog Low Pass Butterworth filter:

Procedure:

Step 1: From the given Specifications, find the order of the filter (N). ← integer value

Step 2: Find the transfer function $H(s)$ for $\Omega_c = 1\,\text{rad/sec}$ for the value of N.

Step 3: Calculate the value of cut off frequency (Ω_c)

Step 4: Find the transfer function $H_a(s)$ for the above value of Ω_c by substituting $s \to \frac{s}{\Omega_c}$ in $H(s)$.

Pbm Design an analog Butterworth filter that has a $-2dB$ passband attenuation at a frequency of 20 rad/sec and atleast $-10dB$ stopband attenuation at 30 rad/sec.

Given: $\alpha_p = 2dB$, $\alpha_s = 10dB$

$\Omega_p = 20$ rad/sec, $\Omega_s = 30$ rad/sec.

Solution:

Step 1: $N = ?$

$$N \geq \frac{\log \sqrt{\frac{10^{0.1\alpha_s} - 1}{10^{0.1\alpha_p} - 1}}}{\log (\Omega_s/\Omega_p)}$$

$N \geq 3.37$

$\boxed{N = 4}$

Step 2: $H(s) = ?$ for $\Omega_c = 1$ rad/sec

from table,

Using denominator of $H(s)$ for $N=4$,

$$H(s) = \frac{1}{(s^2 + 0.76537s + 1)(s^2 + 1.84773s + 1)}$$

Step 3: $\Omega_c = ?$

$$\Omega_c = \frac{\Omega_p}{\left(10^{0.1\alpha_p} - 1\right)^{1/2N}} = \frac{20}{\left(10^{0.2} - 1\right)^{1/8}} = 21.3868$$

Step 4:

$$H_a(s) = H(s)\Big|_{s = s/\Omega_c}$$

$$H_a(s) = \frac{1}{\left\{\left(\frac{s}{21.3868}\right)^2 + 0.76537\left(\frac{s}{21.3868}\right) + 1\right\}\left\{\left(\frac{s}{21.3868}\right)^2 + 1.8477\left(\frac{s}{21.3868}\right) + 1\right\}}$$

$$H_a(s) = \frac{0.20921 \times 10^6}{(s^2 + 16.3686\,s + 457.394)(s^2 + 39.5176\,s + 457.394)}$$

Pbm For the given specifications design an analog Butterworth filter.

$$0.9 \le |H(j\Omega)| \le 1 \,, \quad 0 \le \Omega \le 0.2\pi \rightarrow \Omega_p$$

$$|H(j\Omega)| \le 0.2 \,, \quad 0.4\pi \le \Omega \le \pi$$

Ω_s (for 0.4π)

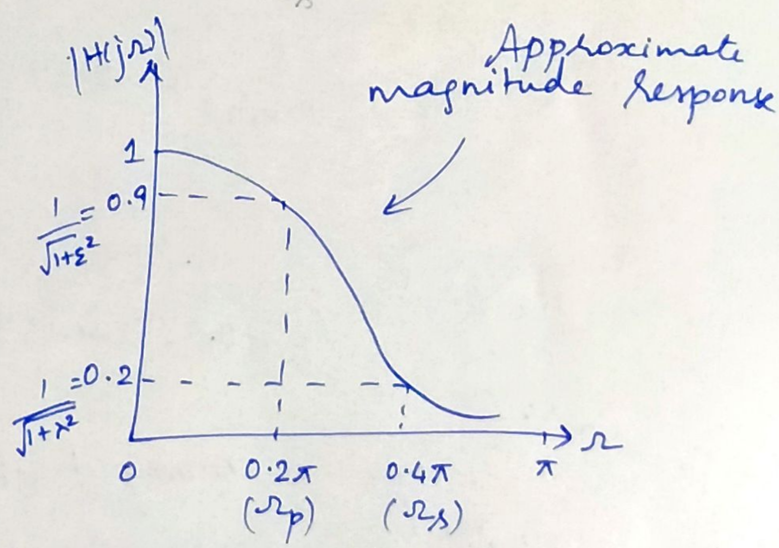
Given:

$$\Omega_p = 0.2\pi$$

$$\Omega_s = 0.4\pi$$

$$\frac{1}{\sqrt{1+\varepsilon^2}} = 0.9$$

$$\frac{1}{\sqrt{1+\lambda^2}} = 0.2$$



$$\varepsilon = 0.484, \quad \lambda = 4.898$$

Step 1: N=?

$$N \geq \frac{\log(\lambda/\varepsilon)}{\log(\Omega_s/\Omega_p)}$$

$$N \geq 3.34 \qquad \boxed{N=4}$$

Step 2: H(s)=? at $\Omega_c = 1$ rad/sec

$$H(s) = \frac{1}{(s^2+0.76537s+1)(s^2+1.8477s+1)}$$

Step 3: $\Omega_c = ?$

$$\Omega_c = \frac{\Omega_p}{(10^{0.1\alpha_p}-1)^{1/2N}} = \frac{\Omega_p}{\varepsilon^{1/N}}$$

$$= \frac{0.2\pi}{(0.484)^{1/4}} = 0.24\pi$$

Step 4: Ha(s) = ?

$$H_a(s) = H(s)\Big/_{s = s/0.24\pi}$$

$$H_a(s) = \frac{1}{\left\{\left(\frac{s}{0.24\pi}\right)^2 + 0.76537\left(\frac{s}{0.24\pi}\right)+1\right\}\left\{\left(\frac{s}{0.24\pi}\right)^2+1.8477\left(\frac{s}{0.24\pi}\right)+1\right\}}$$

$$H_a(s) = \frac{0.323}{(s^2+0.577s+0.0576\pi^2)(s^2+1.393s+0.0576\pi^2)}$$

Homework:
Design an analog Butterworth LPF for the following specifications.

$\alpha_p = 0.5$ dB, $\alpha_s = 22$ dB, $f_p = 10$ KHz, $f_s = 25$ KHz