DFT Properties

Thursday, September 3, 2020 11:46 AM

a) Linearity

$$x[n] \xrightarrow[N]{DFT} X(K)$$

$$g[n] \xrightarrow[N]{DFT} G(K)$$

$$\alpha\, x[n] + \beta\, g[n] \xrightarrow[N]{DFT} \alpha\, X(K) + \beta\, G(K)$$

b) Circular shifting (Time)

$$x[n] \xrightarrow[N]{DFT} X(K)$$

$$x[n-n_0] \xrightarrow[N]{DFT} W_N^{Kn_0} X(K)$$

c) circular shift [Frequency]

$$x[n] \xrightarrow[N]{DFT} X(K)$$

$$W_N^{-K_0 n}\, x[n] \xrightarrow[N]{DFT} X(K-K_0)$$

d) Convolution: circular

$x_1[n]$ & $x_2[n] \Rightarrow$ "N"

$$y[n] \Rightarrow x_1[n] \, (N) \, x_2[n]$$

$$y[n] = \sum_{m=0}^{N-1} x_1[m]\, x_2[(n-m)]_N$$

$$x_1[n] \circledN x_2[n] \xrightarrow[N]{DFT} X_1[K] \cdot X_2[K]$$

$$x_1[n] \cdot x_2[n] \xrightarrow[N]{DFT} X_1[K] \circledN X_2[K]$$

AM $\quad m(t) = A_m \sin(\omega_m t) \Rightarrow$ sum, diff

$c(t) = A_c \sin(\omega_c t)$.

e) Time reversal

$$x[n] \xrightarrow[N]{DFT} X(K)$$

$$x[-0] = x[4-0]_4 \Rightarrow x[4]_4 = x[0]$$

$$x[(-n)]_N \Rightarrow x[N-n]$$

$$x[n] = \{\underset{\substack{\uparrow \\ 0}}{a}, b, c, d\} \qquad x[-n] \Rightarrow \{ \qquad \}$$

$$x[-1]_4 = x[4-1]_4 = x[3]$$

$$x[-2] = x[4-2]_4 = x[2]$$

$$x[-3] = x[4-3] = x[1]$$

$$x[((-n))]_N \xrightarrow[N]{DFT} X((-k))_N$$

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$$X(N-n) \qquad\qquad\qquad X(N-k)$$

FAST FOURIER TRANSFORM

Thursday, September 3, 2020 11:46 AM

complexity of DFT

Number of multiplications $= N^2$

Number of additions $= N(N-1)$ $N^2 - N$

Fast Fourier Transform $(N) \Rightarrow 2^m$

→ provide an algorithm for computing DFT

→ complexity:

multiplication $\Rightarrow \frac{N}{2} \log_2 N$

Additions $\Rightarrow N \log_2 N$.

Friday, September 4, 2020 9:01 AM

Decimation in Time — FFT

$N = 2^M$

Algorithm

1. No. of samples $= N = 2^M$

   M - integer

2. Input sequence should be written in BIT reversal order. o/p sequence will be in normal order

3. No. of stage: $M = \log_2 N$

   $N = 8$, $M = 3$ $m = 1, 2, 3$

4. Each stage will $\frac{N}{2}$ butterfly

(5) Twiddle factor Exponent

$$W_N = e^{-j\frac{2\pi}{N}}$$

$$W_N^{K} \; ; \; K = \frac{Nt}{2^m} \; ;$$

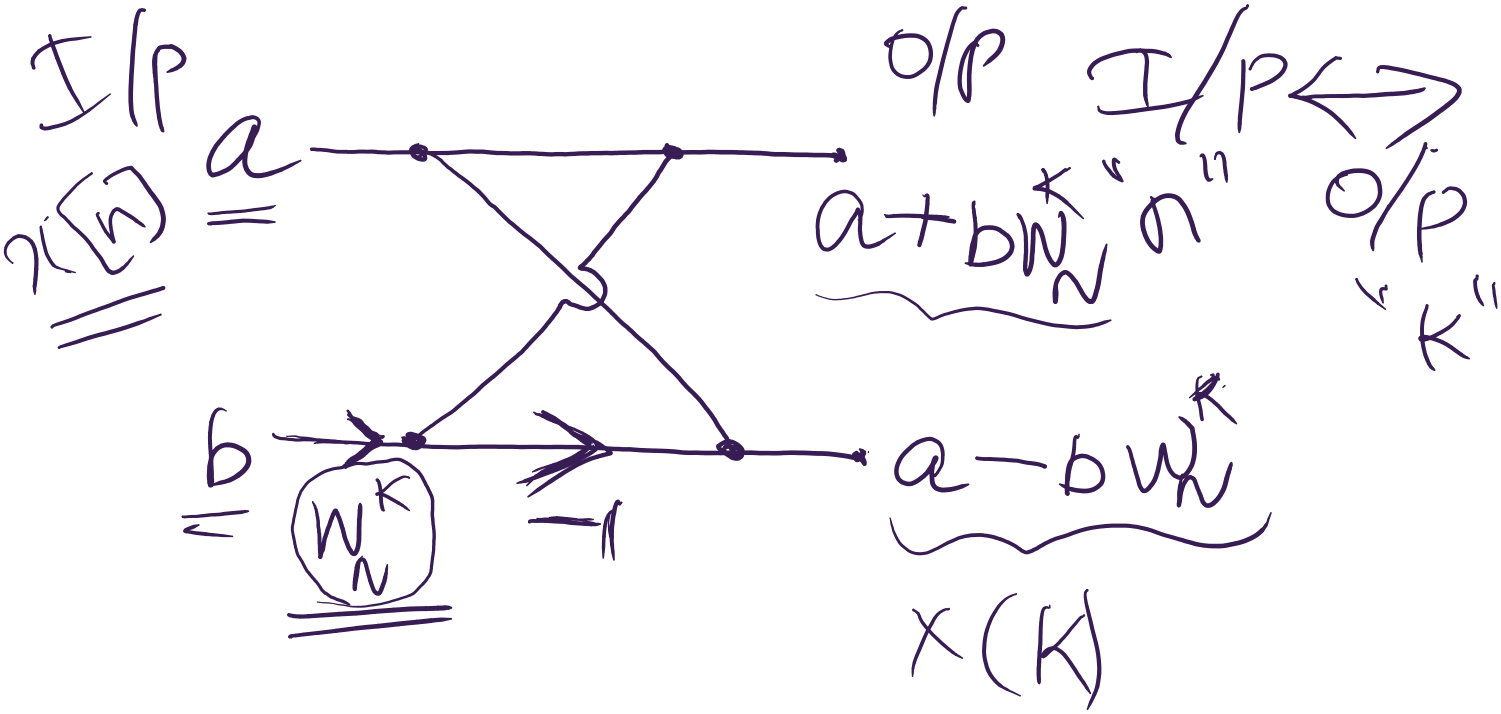
m - stage index

$$t = 0, 1, \ldots\ldots, 2^{(m-1)} - 1$$

Butterfly Diagram
Radix 2 FFT
Friday, September 4, 2020 9:11 AM

DIT — FFT

Radix-2 - Butterfly



Bit reversal

2 i/ps → 1 bit

4 i/ps ⟹ 2bits

8 i/ps ⟹ 3bits

2bits ⟹

$x[n] = \{1, -1, 0, 1\}$

0 1 2 3

$\rightarrow 0, 1, 2, 3$

2 bits

0 0
0 1
1 0
1 1

Bit reversal at i/p

00 $\Rightarrow$ 0
10 $\Rightarrow$ 2
01 $\Rightarrow$ 1
11 $\Rightarrow$ 3

DIT FFT problem

Friday, September 4, 2020 9:21 AM

① $x[n] = \{1, 2, 0, 1\}$

Find $X(K)$ using DIT-FFT

① $N = 2^M \Rightarrow \boxed{M = 2}$

$N = 4$

② Twiddle factor Exponent (K)

W_N^K ; $\boxed{K = \frac{Nt}{2^m}; \ t = 0, 1, \ldots, 2^{(m-1)} - 1}$

Stage 1

$m = 1$; $t = 0$;

$N = 4$ $K = \frac{4 \times 0}{2^1} = 0$

$\boxed{K = 0}$

Stage 2

$m = 2$; $t = 0, 1$;

$$\boxed{\begin{array}{l} t = 0; \ K = 0 \\ t = 1; \ K = 1 \end{array}}$$

$$\boxed{t=1;\ K=1}$$

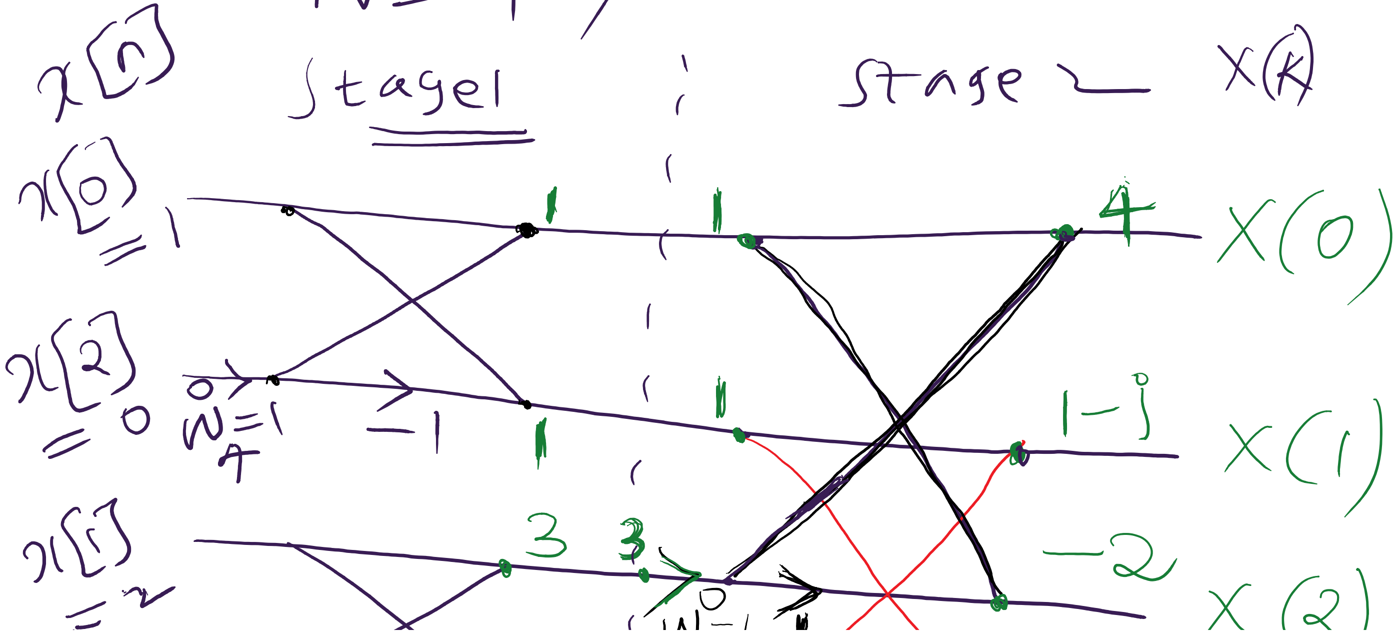
$$W_N \Rightarrow W_4 = e^{-j\frac{2\pi}{4}}$$
$$= e^{-j\pi/2} = -j$$

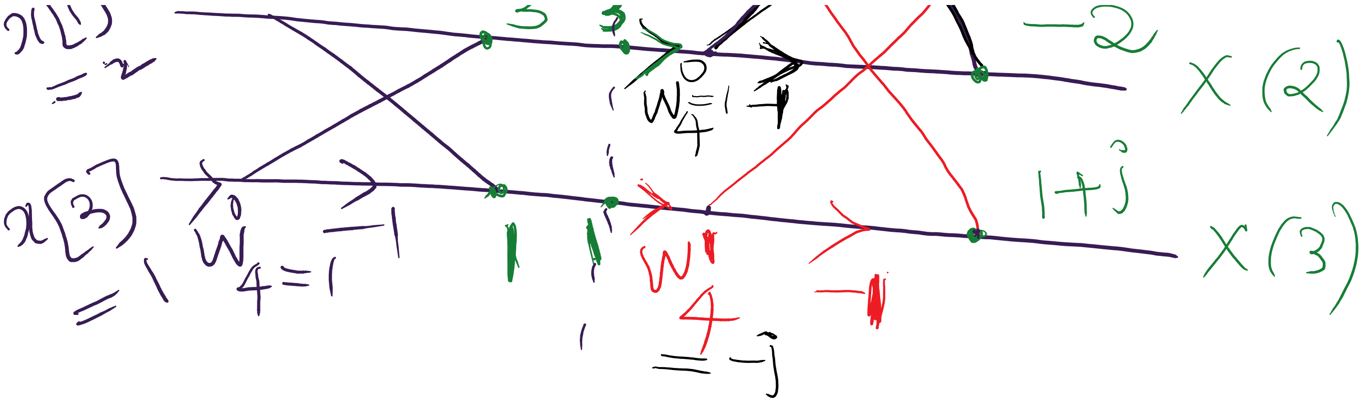
$$W_N^K \Rightarrow W_4^0 = (-j)^0 = \underline{\underline{1}}$$

$\underline{\underline{K=0,1}}$ $\quad W_4^1 = (-j)^1 = -j$

$$x[n] = \{1, 2, 0, 1\}$$

$$N=4,\ M=2$$





$$X(K) = \{4,\ 1-j,\ -2,\ 1+j\}.$$

multiplications: $\frac{N}{2}\log_2 N = 2\cdot(2) = 4$

Additions: $N\log_2 N = 4\times 2 = 8$

H(w)

$$h[n] = \{2, 2, 1, 1\}$$

DIT-FFT H(K)

8Point FFT

Friday, September 4, 2020 9:45 AM

① $x[n] = \{1, 2, 3, 4, 4, 3, 2, 1\}$.

DIT-FFT $X(K)$

① $N = 2^M \Rightarrow M = 3$

② Twiddle factor exponent (K)

W_N^K ; $K = \frac{Nt}{2^m}$; $t = 0, 1 \cdots 2^{m-1} - 1$

$M = 3$, $m = 1, 2, 3$ $\qquad \frac{8 \times 1}{2^2} = 2$

Stage 1 : $m = 1$, $\boxed{\begin{matrix} t = 0 \\ K = 0 \end{matrix}}$

Stage 2 : $m = 2$, $t = 0, 1$.

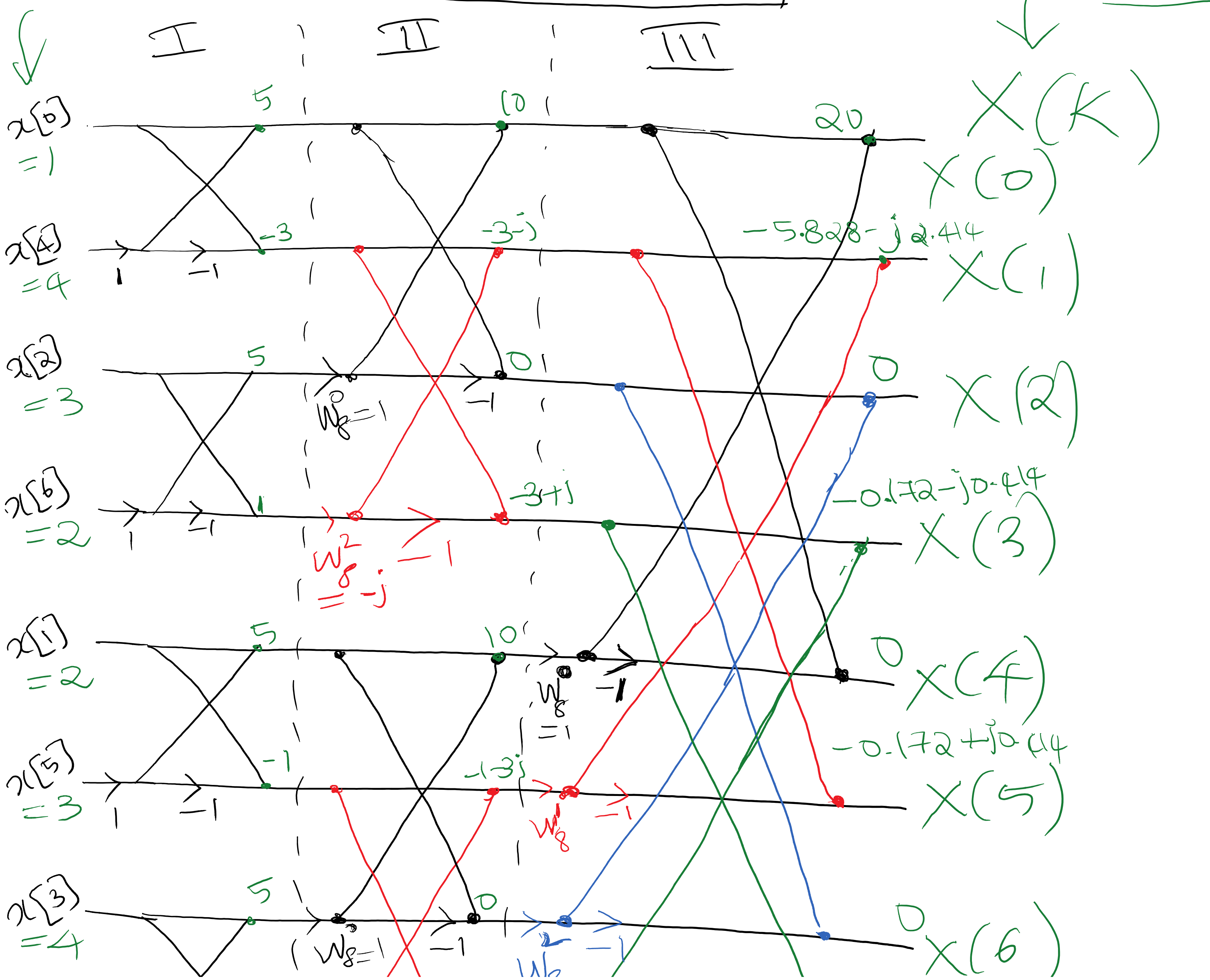
$K = 0, 2$.

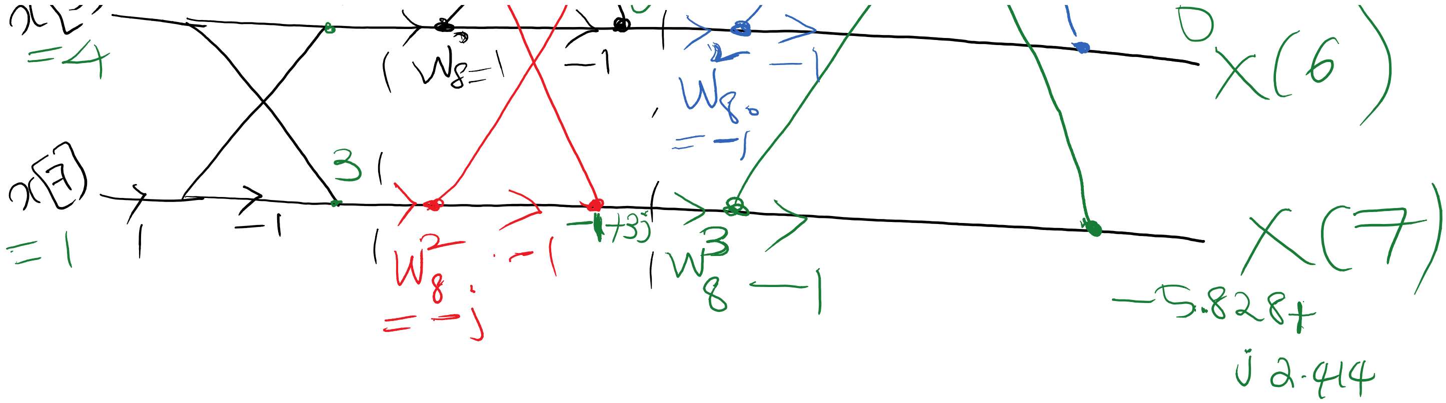
Stage 3 $m = 3$, $t = 0, 1, 2, 3$

$K = 0, 1, 2, 3$.

Twiddle factor

K	$W_N^K = W_8^K$
0	1
1	$\frac{1}{\sqrt{2}}(1-j)$
2	$-j$
3	$\frac{-1}{\sqrt{2}}(1+j)$





=4
x(7)
=1
1
−1
3
1
$W_8^0=1$
−1
W_8^2
$=-j$
−1
$-1+3j$
W_8^0
$=-1$
−1
W_8^3
−1
X(6)
X(7)
$-5.828+j2.414$

IFFT

Saturday, September 5, 2020 10:44 AM

$\Rightarrow$ DFT $\Rightarrow X(K) = \sum_{n=0}^{N-1} x[n] \cdot W_N^{Kn}$

$x[n]$

$\Rightarrow$ IDFT $\Rightarrow x[n] = \frac{1}{N} \sum_{K=0}^{N-1} X(K) \cdot W_N^{-Kn}$

$X(K)$

DIT: $K = \frac{Nt}{2^m}$; $t = 0, 1, \dots 2^{m-1} - 1$

$$x[n] = \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-Kn}$$

$$\boxed{x^*[n] = \frac{1}{N} \sum_{K=0}^{N-1} X^*(K) \cdot W_N^{Kn}}$$

$$x[n] = \frac{1}{N} \left[\sum_{K=0}^{N-1} X^*(K) W_N^{Kn} \right]^*$$

DIT-FFT

$N = 2^M$. M

{Bit reversal at i/p
normal order at o/p}

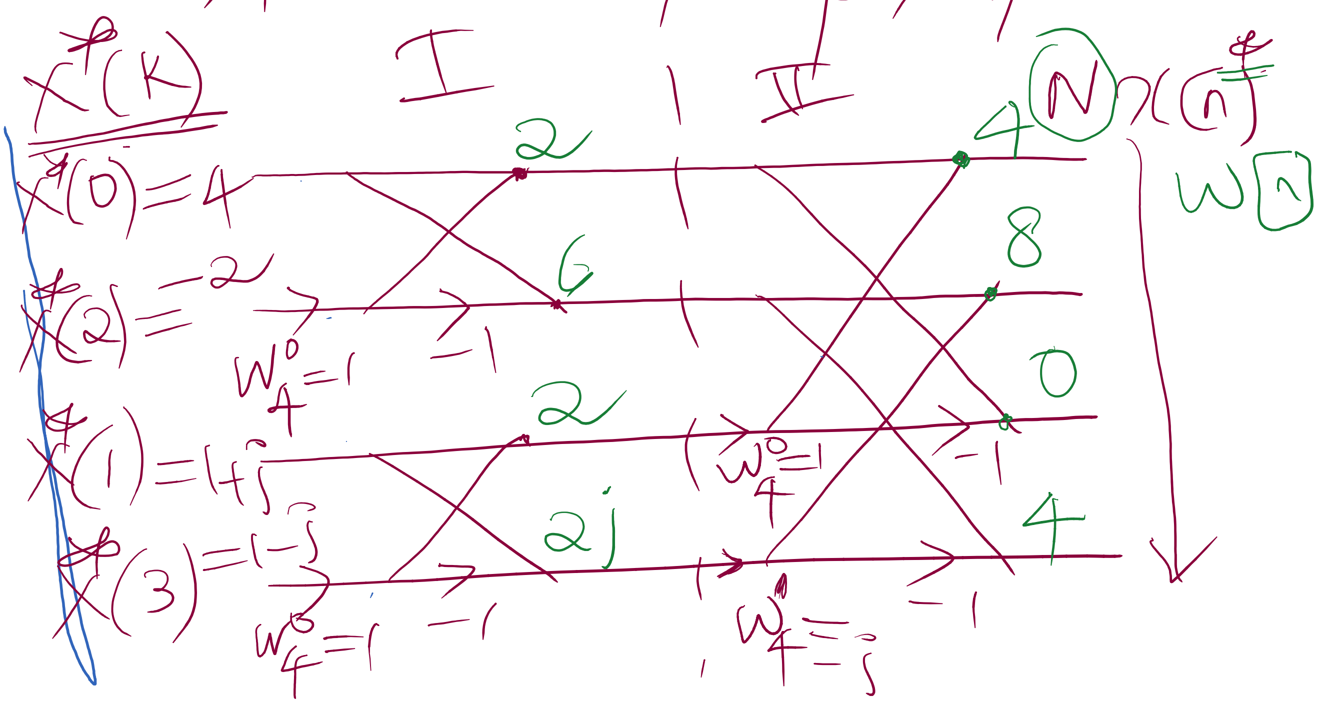
IFFT

$X(K) \longrightarrow x[n]$

$x[n] = \{1, 2, 0, 1\}$

$\rightarrow X(K) = \{4, 1-j, -2, 1+j\}$

$N=4 \quad M=2, \quad W_4^0 = 1, \quad W_4^1 = -j$



$x[n] \rightarrow$ | $w[n]^*$

$$x[n] = \frac{1}{N} \, w[n]'$$

$$x[n] = \{1, 2, 0, 1\}$$

$$h[n] = \{2, 2, 1, 1\}$$

$$H(k) = \{6, 1-j, 0, 1+j\}$$

FFT

IFFT $\rightarrow$ DFT

$h[n]$

$$X(k) = \{4, 1-j, -2, 1+j\}$$

$$Y(k) \longrightarrow y[n]$$

DIF FFT

Tuesday, September 8, 2020 10:28 AM

Algorithm

① $N = 2^M$; M — no. of stages

② i/p bits in normal order
o/p bits in BIT reverse order

③ Twiddle factor exponent (K)

DIT $\boxed{W_N^K \; ; \; K = \frac{Nt}{2^m} \quad t = 0, 1, 2, \ldots 2^{(m-1)} - 1}$

DIF n W_N^n

$n = \frac{Nt}{\cdot}$;

DIF $\begin{cases} n = \dfrac{Nt}{2^{M-m+1}}; \\ t = 0, 1, 2, \ldots\ldots 2^{(M-m)} - 1 \end{cases}$

$\Rightarrow$ no. of multiplications:

$$\frac{N}{2} \log_2 N$$

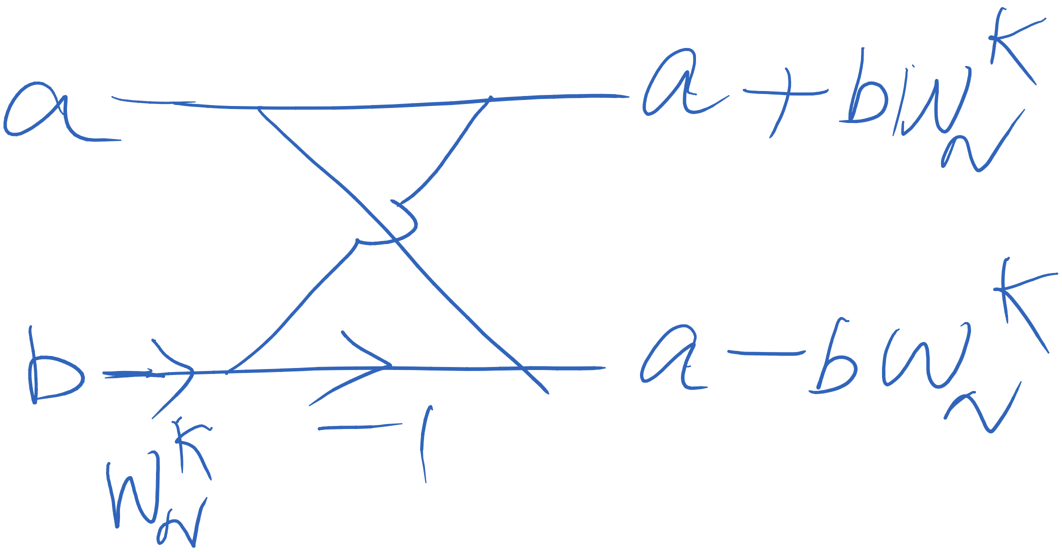
no. of additions

$$N \log_2 N$$

DIT-FFT

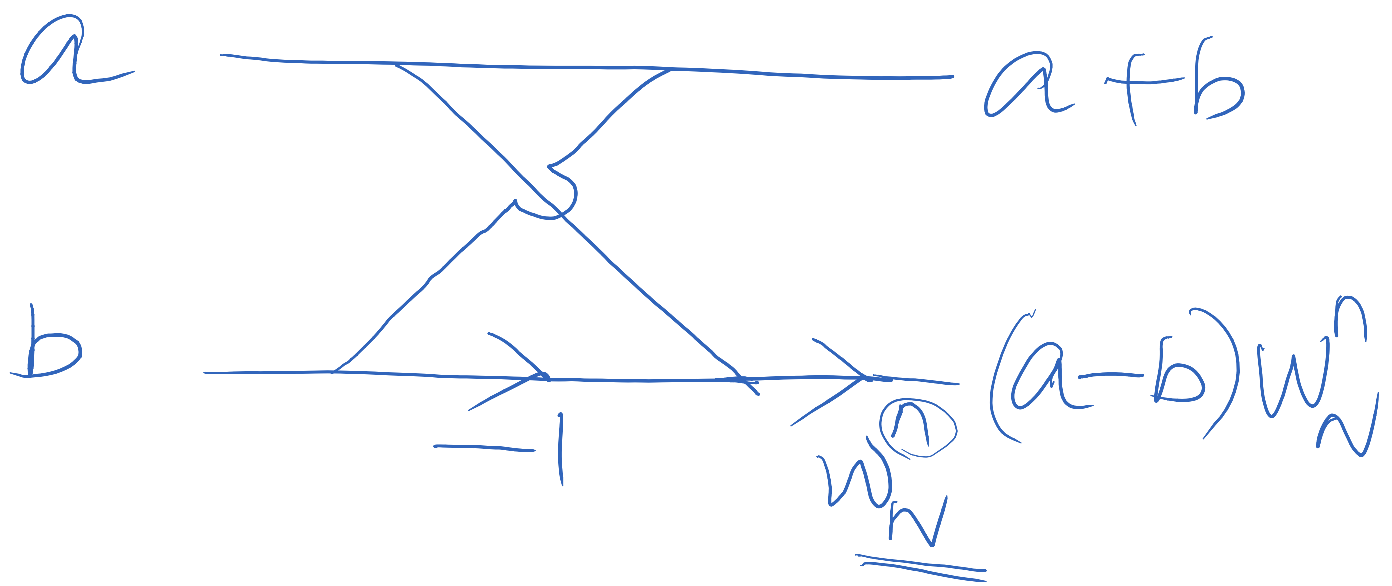
Radix-2

Butterfly diagram



DIF_FFT

Radix-2 B-Diagram.



DIF FFT Problem

Tuesday, September 8, 2020 10:39 AM

① 4 point DIF-FFT

$$x[n] = \{1, 2, 0, 1\}$$

Steps

① $N \Rightarrow 2^M \Rightarrow M = 2$

no. of stages

② Bitreversal $\Rightarrow$ o/p

③ Twiddle factor exponent (n)

$$W_N^n \Rightarrow n = \frac{Nt}{2^{M-m+1}}$$

$$t = 0, 1, 2, \ldots 2^{(M-m)} - 1$$

Stage I

$m = 1$; $t = 0, \ldots 2^{M-m} - 1$

$$\boxed{t = 0, 1}$$

$$n = \frac{Nt}{2^{M-m+1}}$$

$$\boxed{t=0;\ n=0}$$

$$t=1; \Rightarrow n = \frac{4\times 1}{2^{2-1+1}} = \frac{4}{4}$$

$$\boxed{t=1;\ n=1}$$

Stage I $\quad m=1; \quad \boxed{\begin{matrix} t=0,1 \\ n=0,1 \end{matrix}}$

Stage II $\quad t=0, \ldots 2^{M-m}-1$

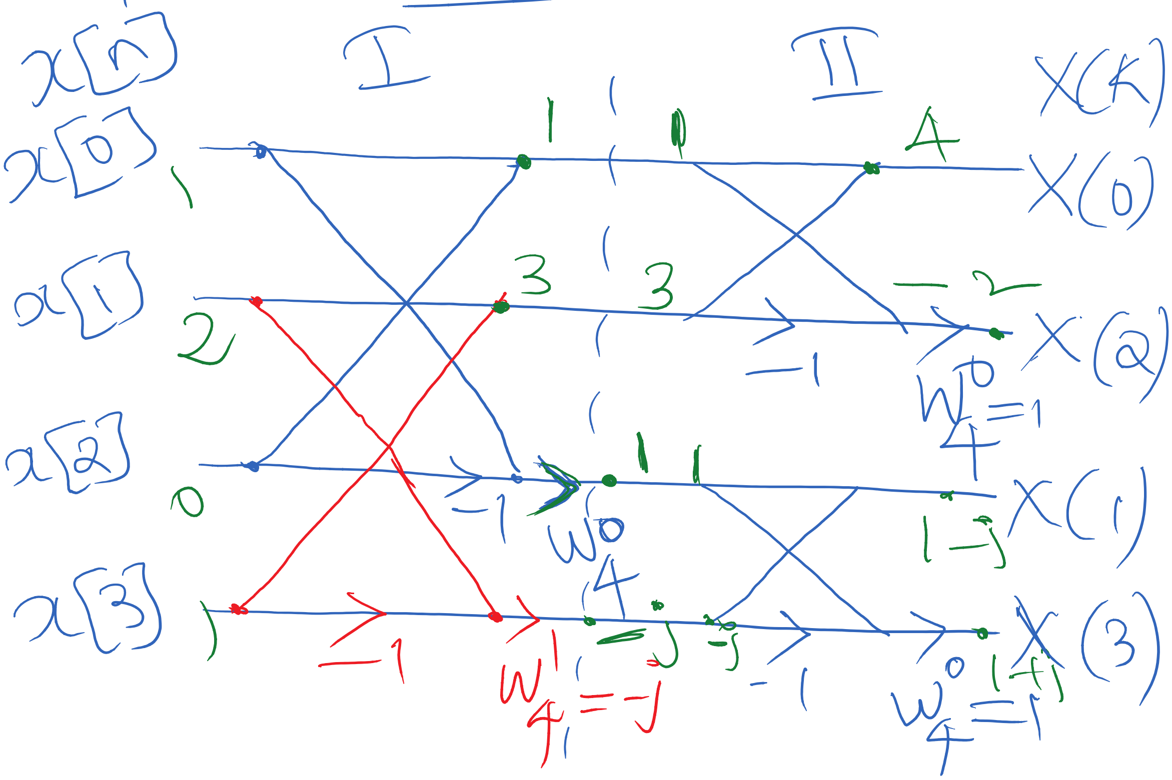
$m=2 \quad \boxed{t=0}; \boxed{n=0}$

$$W_N \Rightarrow e^{-j\frac{2\pi}{4}} = -j$$

$$W_N^n \Rightarrow W_4^0 = 1$$

$$W_4^1 = -j$$

4 Point DIF-FFT



$$X(K) = \{4,\ 1-j,\ -2,\ 1+j\}$$

$$H(K) = \{6,\ 1-j,\ 0,\ 1+j\}$$

$$\underline{\underline{Y(K)}} = X(K).H(K)$$

IFFT $y[n]$

(2) $x[n] = \{ \ldots$

(2) $x[n] = \{0, 1, 2, 3, 4, 5, 6, 7\}$.

$\Rightarrow N = 8,\ M = 3$

$\Rightarrow$ Bit reversal at o/p

$\Rightarrow$ "n" $\Rightarrow$ exponent

$$n = \frac{Nt}{2^{M-m+1}};$$

$$t = 0, 1, \ldots\ 2^{(M-m)} - 1.$$

Stage I	$m = 1$	$t = 0, 1, 2, 3$ $n = 0, 1, 2, 3$
Stage II	$m = 2$	$t = 0, 1$ $n = 0, 2$
stage III	$m = 3$	$t = 0$ $n = 0$

$$W_8^0 = 1$$

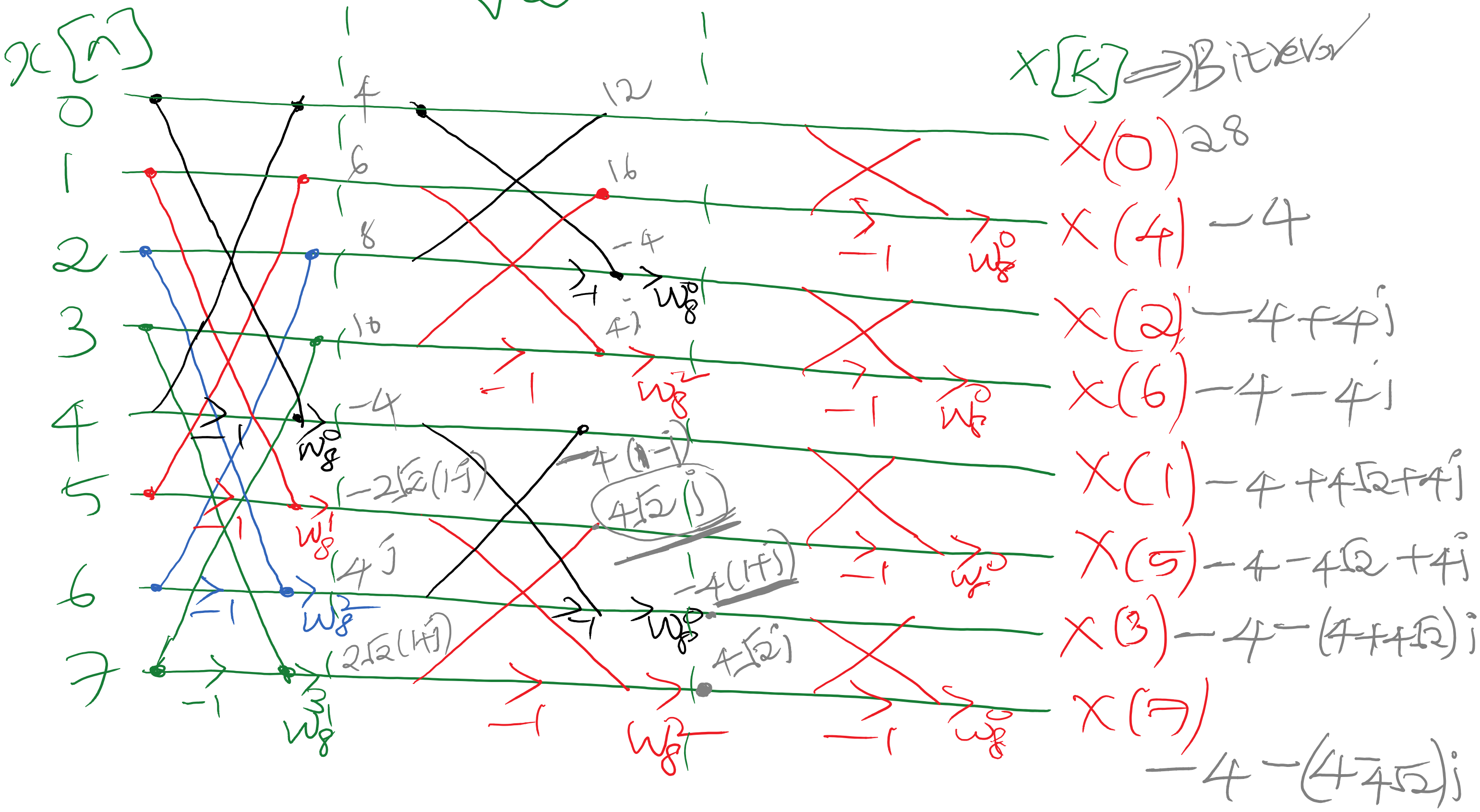
$$W_8^1 = e^{-j\frac{2\pi}{8}} = e^{-j\pi/4}$$

$$= e^{\overline{8}} = e$$

$$= \frac{1}{\sqrt{2}}(1-j)$$

$$W_8^2 = -j$$

$$W_8^3 = -\frac{1}{\sqrt{2}}(1+j)$$



$$X(k) = \{28, -4+9.656j, -4+4j, -4+1.656j, -4, -4-1.656j, -4-4j, -4-9.656j\}$$

DIF IFFT problem

Wednesday, September 9, 2020 5:10 PM

$x(n)$ O/P $\xrightarrow{\frac{1}{N} w[n]^*}$ Bit reversed

$X^*(K)$ I/P $\longrightarrow$ normal order

$$N = 2^M \; ; \; M = \text{no of stages}$$

<u>T.F.E</u> (n)

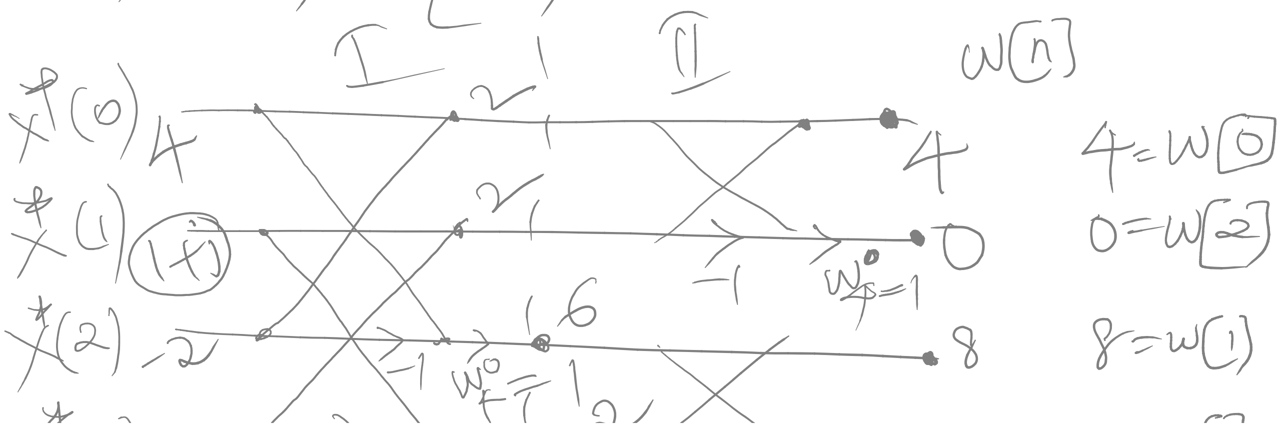
$$n = \frac{Nt}{2^{M-m+1}} \; ;$$

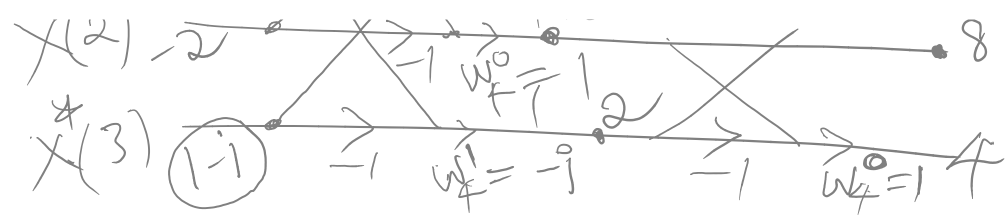
$$t = 0, 1, \dots\; 2^{M-m} - 1$$

① $X(K) = \{4, 1-j, -2, 1+j\}$.

DIF — IFFT

$X^*(K) = \{4, 1+j, -2, 1-j\}$.





8 = w(1)

4 = w[3]

$$x[n] = \frac{1}{N} W^*[n]$$

$$\boxed{x[n] = \{1, 2, 0, 1\}}$$

(2) 8 point — IFFT — DIF

$$X(K) = \{12, -1+2.414j, 0, -1+0.414j, 0, -1-0.414j, 0, -1-2.414j\}$$

x[n] DIF — IFFT

$$N = 2^M \; ; \; M = 3$$

TFE $n = \dfrac{Nt}{2^{M-m+1}}$; $t = 0, 1, \cdots 2^{M-m} - 1$

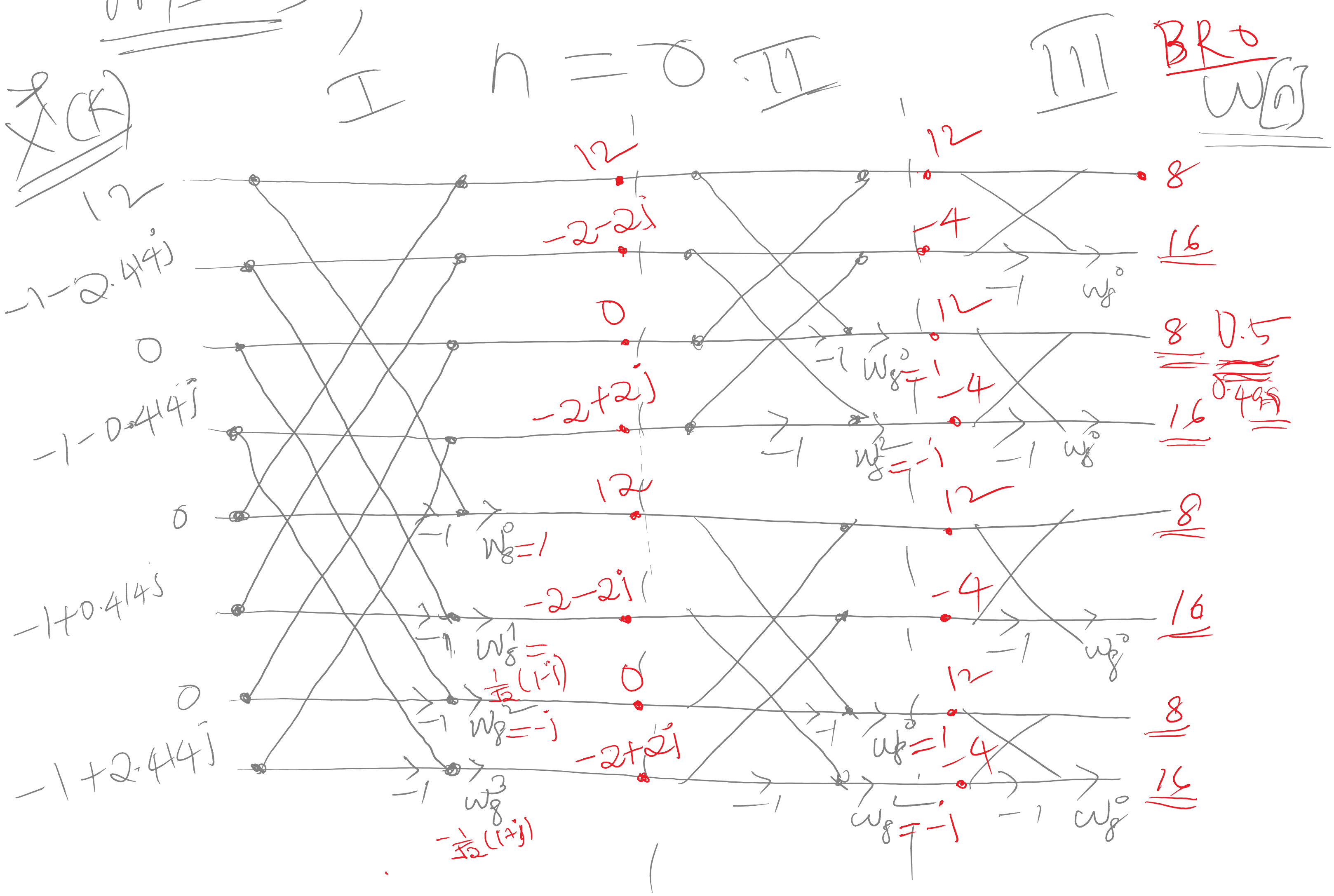
m=1

$t = 0, 1, 2, 3$

$n = 0, 1, 2, 3$

$m = 2$, $t = 0, 1$
$n = 0, 2$

$m = 3$, $t = 0$
$n = 0$



$$x[n] = \frac{1}{N}\left[W^{*}(n)\right]$$

$$x[n] = \{1, 1, 1, 1, 2, 2, 2, 2\}$$

$x[n] = \{1, 1, 1, 1, \alpha, \alpha, \alpha, \alpha\}$.

DIF — FFT, IFFT — 4 point, 8 point $\quad N \neq 2^M \quad N = 2^M$

DIT — FFT, IFFT → 16 → 32 $\quad 2^M$

Overlap add, Overlap save — FFT application

long duration sequence

OVERLAP ADD METHOD

Saturday, September 12, 2020 8:32 AM

DFT → N finite

i/p sequences → "N" length.

o/p sequence → "N" length

i/p sequences → non uniform in length.

→ shorter sequence

→ longer sequence

limitations

→ complexity — multiplication
— Additions

FFT

→ $N = 2^M$.

overlap add method

① $x[n] = \{1, 0, -1, 2, 5, 4, 3, 2, 1, 4\}$ N_1

$h[n] = \{1, 1, 1\}$ N_2

FFT

$N \Rightarrow 2^{M}$

1 2

4

8

16

New data block N_1^{p}

$N_1^{p} + N_2 - 1 = 2^{M} = N$

$\rightarrow 2 + 3 - 1 = 2^{M} = \boxed{4 = N}$

$$6 + 3 - 1 = \underline{\underline{8}}$$

$$\left\{ \begin{aligned} x_1[n] &= \{1, 0, 0, 0\} \\ x_2[n] &= \{-1, 2, 0, 0\} \\ x_3[n] &= \{5, 4, 0, 0\} \\ x_4[n] &= \{3, 2, 0, 0\} \\ x_5[n] &= \{1, 4, 0, 0\} \end{aligned} \right.$$

FFT

$$\text{FFT} \Rightarrow h[n] = \{1, 1, 1, 0\}$$

① FFT $\{x_1, x_2, x_3, \ldots x_5\}$

$X_1(k), X_2(k), \ldots X_5(k)$

$$FFT\{h(n)\} \Rightarrow H(K)$$

$$② \quad Y_1(K) = X_1(K) \cdot H(K)$$

$$\vdots$$

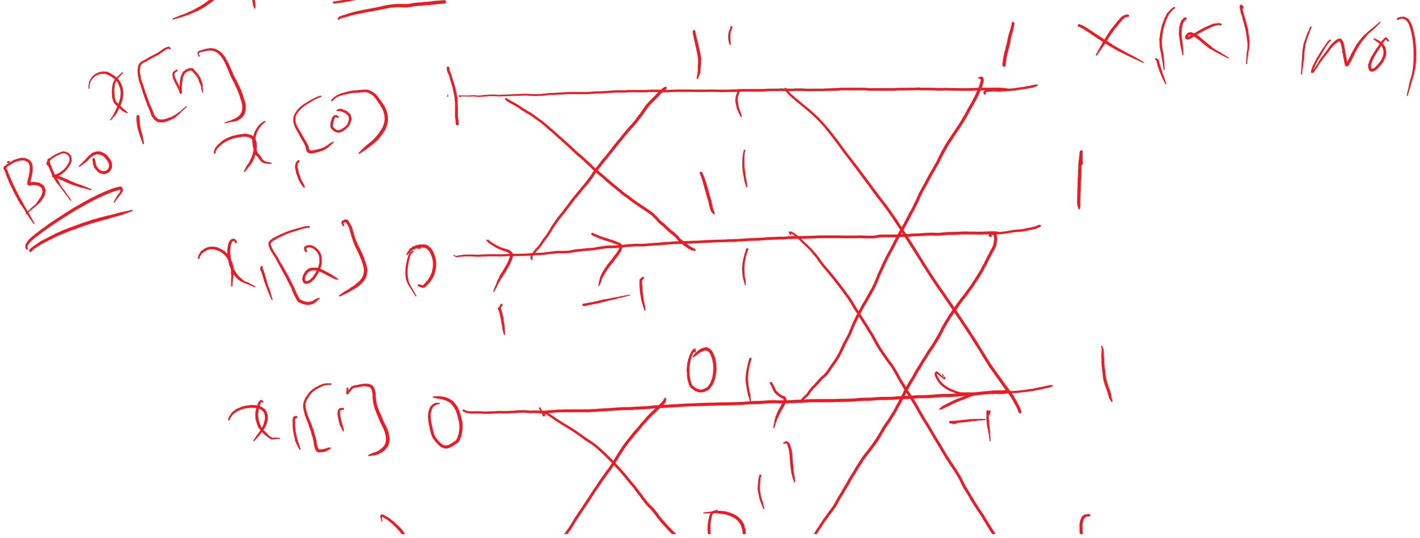
$$Y_5(K) = X_5(K) \cdot H(K)$$

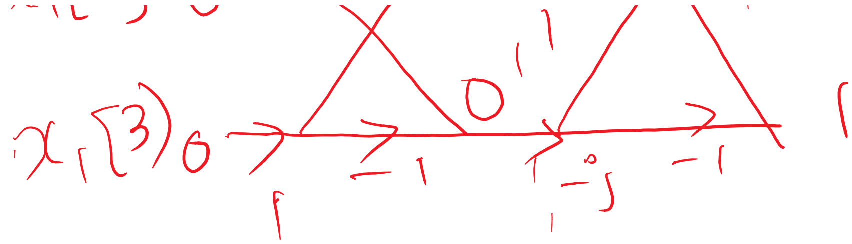
③ IFFT of step ②

$$\boxed{y_1(n), y_2(n) \ldots y_5(n)}$$

$$y[n]$$

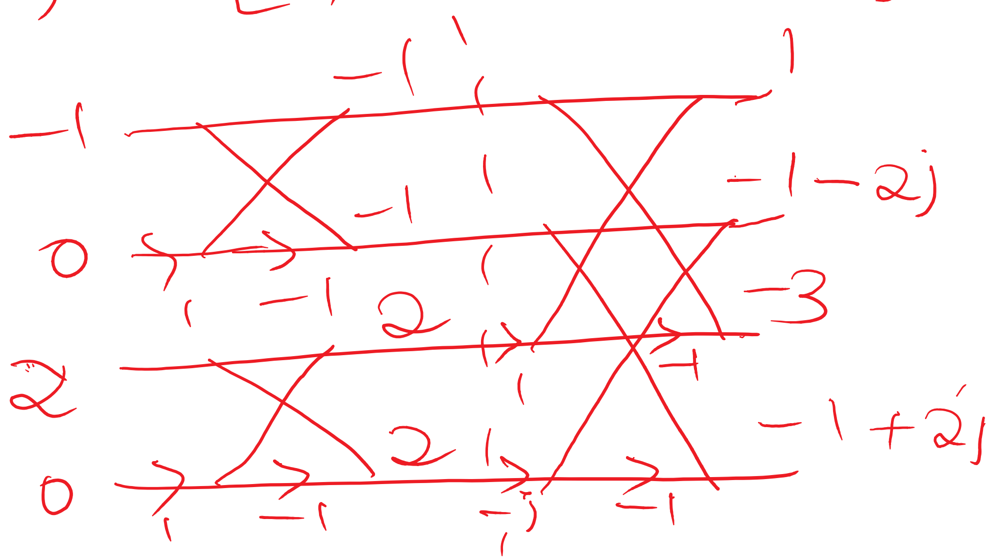
DIT FFT − 4 point FFT



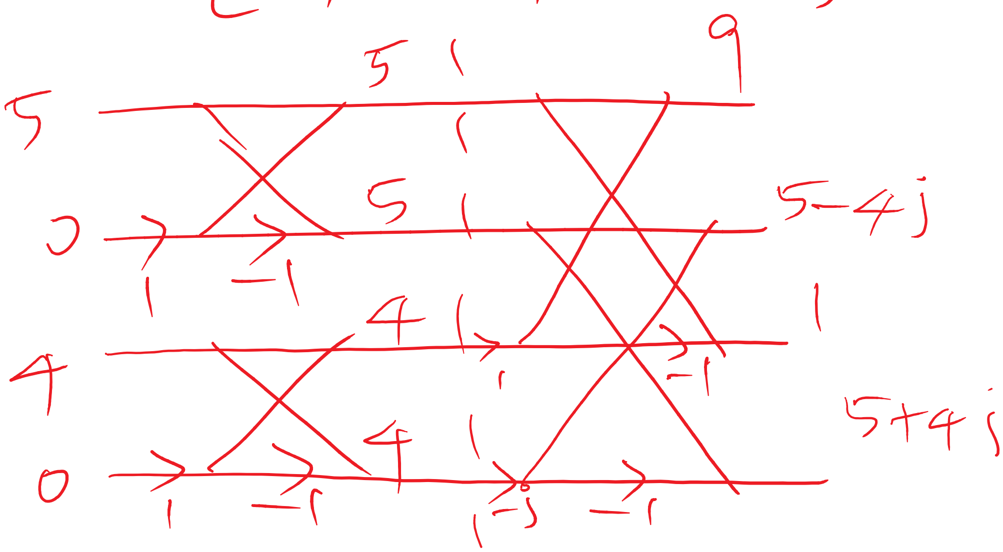


$$X_1(k) = \{1, 1, 1, 1\}$$

$$X_2(k) = \{1, -1-2j, -3, -1+2j\}$$



$$X_3(k) = \{9, 5-4j, 1, 5+4j\}$$



$$X_4(K) = \{5, 3-2j, 1, 3+2j\}$$

$$X_5(K) = \{5, 1-4j, -3, 1+4j\}$$

$$H(K) = \{3, -j, 1, j\}$$

<u>step 2</u> $Y_1(K) = X_1(K) \cdot H(K)$

$$Y_1(K) = \{3, -j, 1, +j\}$$

$$Y_2(K) = X_2(K) \cdot H(K)$$

$$= \{3, -2+j, -3, -2-j\}$$

$$Y_3(K) = X_3(K) \cdot H(K)$$

$$= \{27, -4-5j, 1, -4+5j\}$$

$$Y_4(K) = X_4(K) \cdot H(K)$$

$$= \{15, -2-3j, 1, -2+3j\}$$

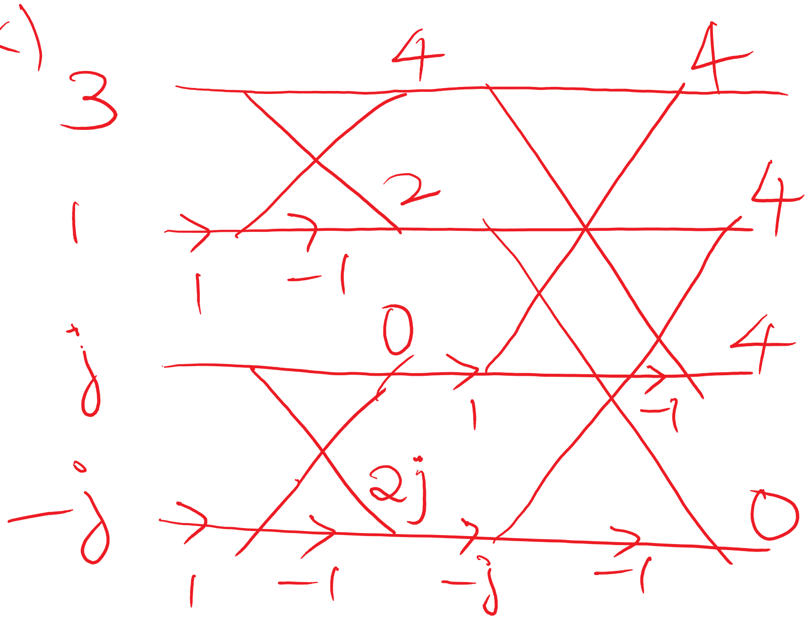
$$Y_5(K) = \{15, -4-j, -3, -4+j\}$$

Step 3 $\quad IFFT \Rightarrow Y_1(K), Y_2(K), Y_3(K), Y_4(K), Y_5(K)$

$$IFFT[Y_1(K)] = y_1[n]$$

DIT-IFFT

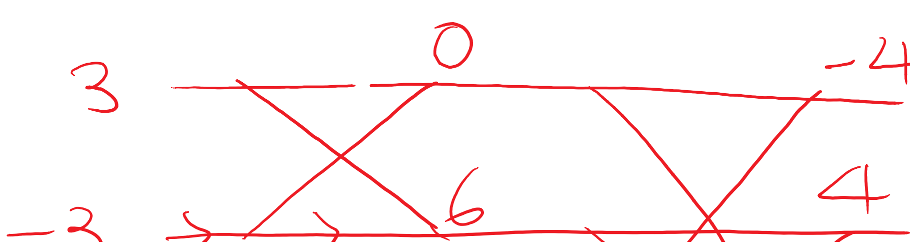
$Y_1^*(K)$ BRO

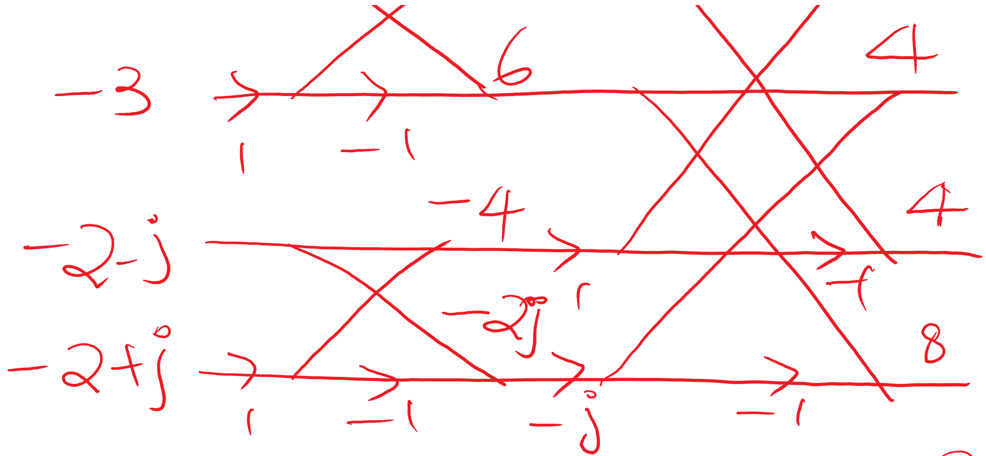


$w[n]$ (o/p)

$$y_1[n] = \frac{1}{N} w^*[n] = \{1, 1, 1, 0\}.$$

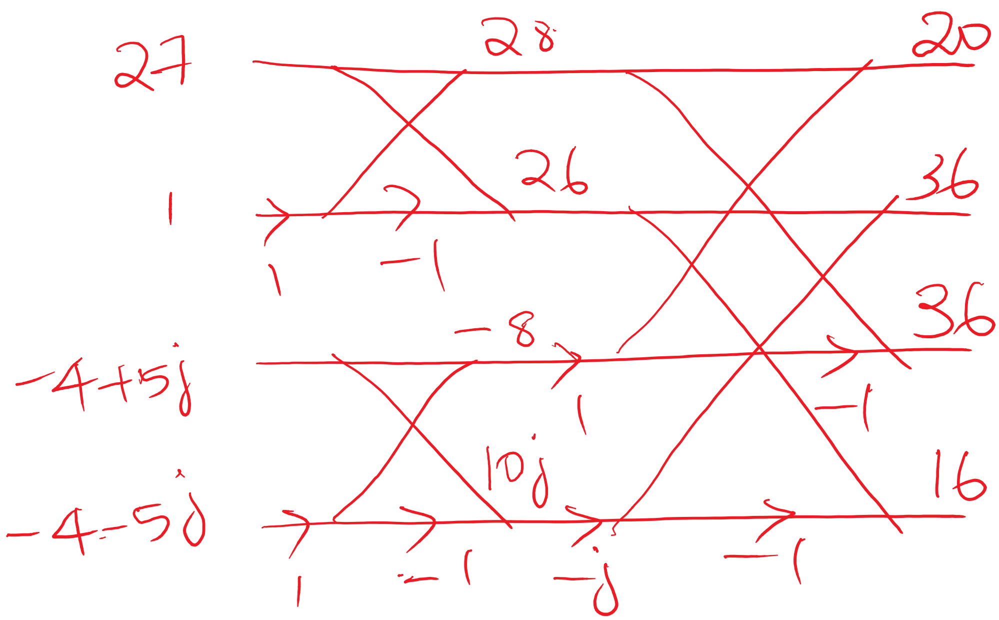
$$Y_2^*(K) = \{3, -2-j, -3, -2+j\}$$





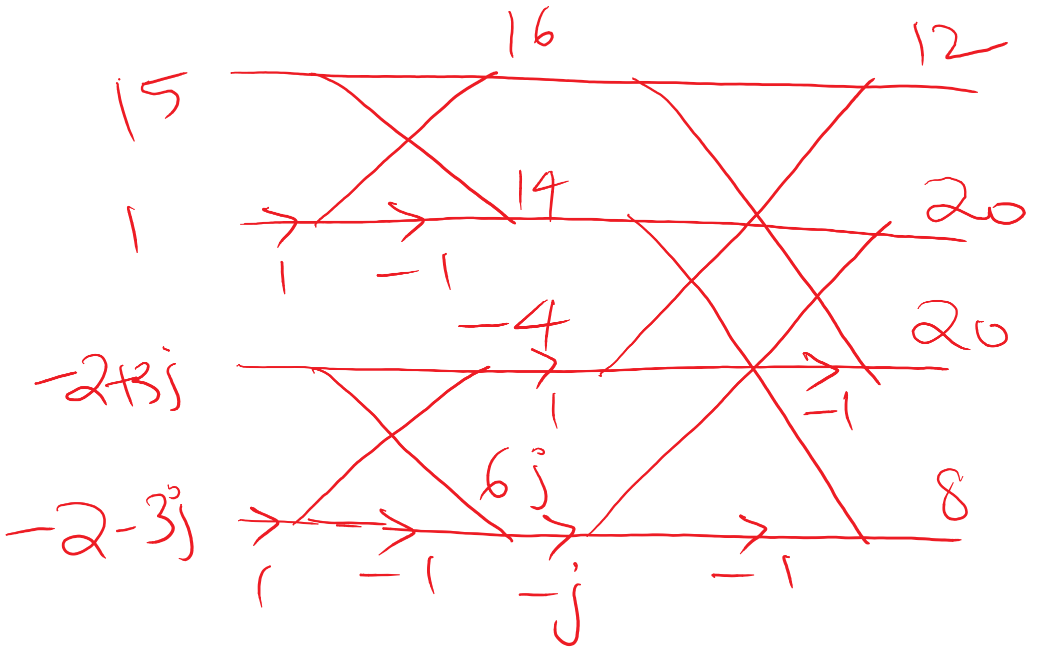
$$y_2[n] = \{-1, 1, 1, 2\}.$$

$$Y_3^p(K) = \{27, -4+5j, 1, -4-5j\}$$



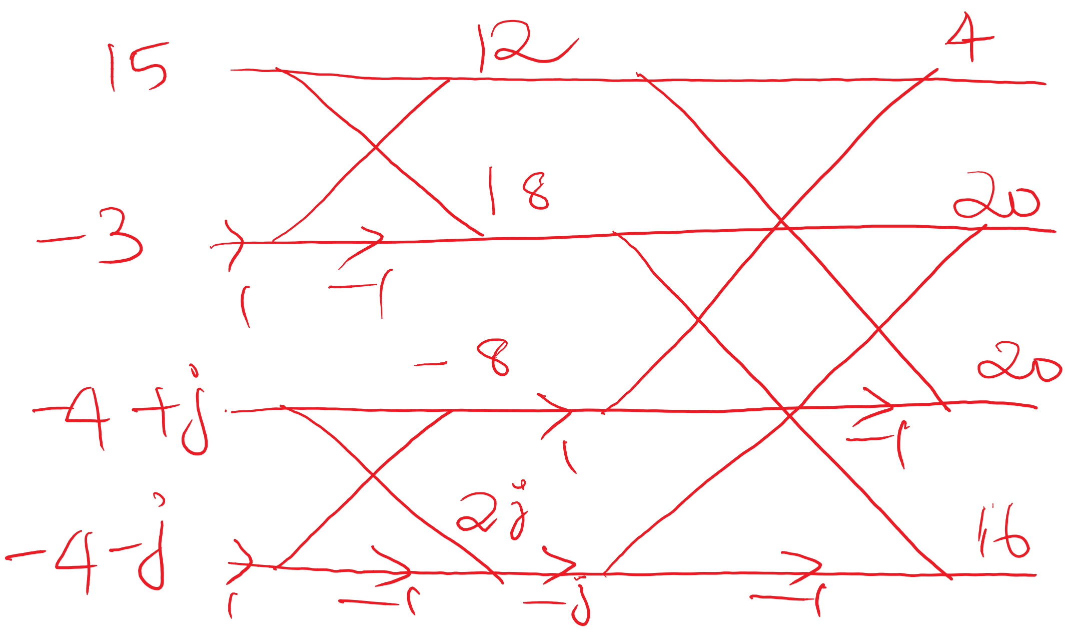
$$y_2[n] = \{5, 9, 9, 4\}.$$

$$Y_4^p(K) = \{15, -2+3j, 1, -2-3j\}$$



$$y_4[n] = \{3, 5, 5, 2\}$$

$$Y_5^*(K) = \{15, -4+j, -3, -4-j\}$$



$y_5[n] = \{1, 5, 5, 4\}$.

1	1	1	0								
		-1	1	1	2						
				5	9	9	4				
						3	5	5	2		
								1	5	5	4
1,	1,	0,	1,	6,	11,	12,	9,	6,	7,	5,	4

h[n] \ x(n)	1	0	-1	2	5	4	3	2	1	4
1	1	0	-1	2	5	4	3	2	1	4
1	1	0	-1	2	5	4	3	2	1	4
1	1	0	-1	2	5	4	3	2	1	4

$\{1, 1, 0, 1, 6, 11, 12, 9, 6, 7, 5, 4\}$

OVERLAP SAVE METHOD

Saturday, September 12, 2020 9:30 AM

① $x[n] = \{1, 0, -1, 2, 5, 4, 3, 2, 1, 4\}$. N_1

$h[n] = \{1, 1, 1\}$. N_2.

$N = 2^M = N_1 + N_2 - 1$

$= 2 + 3 - 1 = 4$

$x_1[n] = \{0, 0, 1, 0\}$

$x_2[n] = \{1, 0, -1, 2\}$

$x_3[n] = \{-1, 2, 5, 4\}$

$x_4[n] = \{5, 4, 3, 2\}$

$x_5[n] = \{3, 2, 1, 4\}$.

$x_6[n] = \{1, 4, 0, 0\}$.

④

4

$$h[n] = \{1, 1, 1, 0\}$$

$$X_1(K) = \{1, -1, 1, -1\}$$

$$X_2(K) = \{2, 2+2j, -2, 2-2j\}$$

$$X_3(K) = \{10, -6+2j, -2, -6-2j\}$$

$$X_4(K) = \{14, 2-2j, 2, 2+2j\}$$

$$X_5(K) = \{10, 2+2j, -2, 2-2j\}$$

$$X_6(K) = \{5, 1-4j, -3, 1+4j\}$$

$$H(K) = \{3, -j, 1, +j\}$$

$$Y_1(K) = \{3, j, 1, -j\}$$

$$Y_2(K) = \{6, 2-2j, -2, 2+2j\}$$

$$Y_3(K) = \{30, 2+6j, -2, 2-6j\}$$

$$Y_4(K) = \{42, -2-2j, 2, -2+2j\}$$

$$Y_5(K) = \{30, 2-2j, -2, 2+2j\}$$

$$Y_6(K) = \{15, -4-j, -3, -4+j\}$$

IFFT

$$y_1[n] = \{\cancel{1, 0}, 1, 1\}$$

$$y_2[n] = \{\cancel{2, 3}, 0, 1\}$$

$$y_3[n] = \{\cancel{8, 5}, 6, 1\}$$

$$y_4[n] = \{\cancel{10, 11}, 12, 9\}$$

$$y_5[n] = \{\cancel{8, 9}, 6, 7\}$$

$$y_6[n] = \{\cancel{1, 5}, 5, 4\}$$

$$y[n] = \{1, 1, 0, 1, 6, 11, 12, 9, 6, 7, 5, 4\}.$$