This DFT of a signal $x(n)$ is said to exist if the signals are absolutely summable.

ie $$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

**DFT pair:**

$$x(n) \xleftrightarrow{DFT} X(k)$$

(of length L) (of Length N)

$$\boxed{N \geq L}$$

Note: Both $x(n)$ & $X(k)$ are discrete & periodic

so $x(n)$ is replaced by $x_p(n)$

$$x_p(n) = \begin{cases} x(n) \; ; & 0 \leq n \leq L-1 \\ 0 \; ; & L \leq n \leq N-1 \end{cases}$$

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

|||ly $$X(k) = \begin{cases} X_p(k) \rightarrow \text{periodic} \; ; & 0 \leq k \leq N-1 \\ 0 \; ; & \text{otherwise} \end{cases}$$

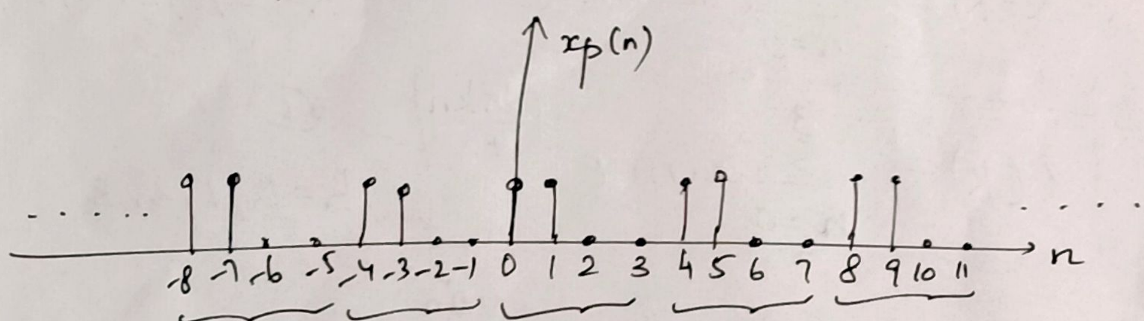
eg: if $x(n) = \{1, 1\} \implies L = 2$

then to compute N-point DFT. of $x(n)$ (let $N = 4$,) find the periodic extension of $x(n)$ (ie $x_p(n)$) by padding $(N-L)$ zeros to $x(n)$. at the end of the sequence.

$L = 2, \quad N = 4.$

$x(n) = \{1, 1\}$

$x_p(n) = \{1, 1, 0, 0\}$ and then compute DFT.




repeated with period N (ie $N=4$).

pbm: Find the DFT of a sequence

$$x(n) = \begin{cases} 1, & \text{for } 0 \le n \le 2 \\ 0, & \text{otherwise} \end{cases}$$

for i) $N = 4$ ii) $N = 8$.

plot $|X(k)|$ and $\angle X(k)$

$|X(k)| \Downarrow$ magnitude of $X(k)$

$\angle X(k) \Downarrow$ phase response of $X(k)$

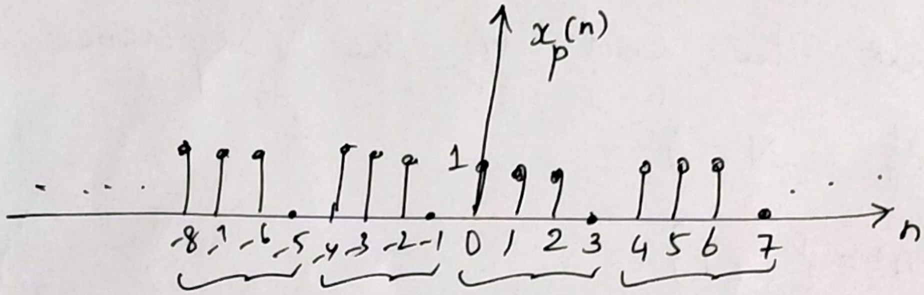
$x_p(n) = \{1, 1, 1, \underset{\Uparrow}{0}\}$ $\quad$ gn: $x(n) = \{1, 1, 1\}$

$L = 3$

for $\underline{\underline{N=4}}$: $\quad$ $N - L = 4 - 3 = 1$ zero added.

$$x(n) = x_p(n) = \{\underset{\uparrow}{1}, 1, 1, 0\}$$

ie. $x(0) = 1, \; x(1) = 1, \; x(2) = 1, \; x(3) = 0$




$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2\pi kn}{N}}, \quad k = 0, 1, 2 \dots N-1$$

$$\boxed{X(k) = \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi kn}{4_2}}} \rightarrow (1), \quad k = 0, 1, 2, 3$$

put $\underline{\underline{k=0}}$ in (1),

$$X(0) = \sum_{n=0}^{3} x(n)\, e^{-j\frac{\pi}{2}0n}$$

$$= \sum_{n=0}^{3} x(n) \cdot 1 \quad \left[\because e^{j0} = 1\right.$$

$$= x(0) + x(1) + x(2) + x(3)$$

$$\boxed{X(0) = 3} \text{ (or) } 3 + j0$$

$$|X(0)| = \sqrt{3^2 + 0^2} = 3, \quad \angle X(0) = \tan^{-1}\left(\frac{0}{3}\right) = 0$$

put $\underline{\underline{k=1}}$ in ①:

$$X(1) = \sum_{n=0}^{3} x(n) \cdot e^{-j\frac{\pi}{2}(1)n}$$

$$= x(0) \cdot e^{-j0} + x(1) \cdot e^{-j\pi/2(1)} + x(2) e^{-j\pi/2(2)} + x(3) e^{-j\pi/2(3)}$$

$$= 1 \cdot 1 + 1[\cos(\pi/2) - j\sin(\pi/2)] + 1 \cdot [\cos(\pi) - j\sin(\pi)] + 0 \cdot \underbrace{[\cos(\tfrac{3\pi}{2}) - j\sin(\tfrac{3\pi}{2})]}_{0}$$

$$= \cancel{1} - j \cancel{-1} = -j$$

$$\boxed{X(1) = -j} \text{ (or) } 0 - j1$$

$$|X(1)| = \sqrt{0^2 + 1^2} = 1, \quad \angle X(1) = \tan^{-1}\left(\frac{-1}{0}\right) = -\tan^{-1}\infty = -\pi/2$$

put $\underline{\underline{k=2}}$ in ①:

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j\pi/2(2)n}$$

$$= x(0) \cdot e^{-j0} + x(1) \cdot e^{-j\pi(1)} + x(2) e^{-j2\pi} + x(3) e^{-j3\pi}$$

$$= 1 \cdot 1 + 1 \cdot [\cos(\pi) - j\sin(\pi)] + 1[\cos 2\pi - j\sin 2\pi] + 0$$

$$= \cancel{1} \cancel{-1} + 1 = \boxed{1 = X(2)} \text{ (or) } 1 + j0$$

$$|X(2)| = 1, \quad \angle X(2) = 0.$$

put $\underline{k=3}$ in ①,

$$X(3) = \sum_{n=0}^{3} x(n) \cdot e^{j\frac{\pi}{2}3n}$$

$$\boxed{X(3) = +j} \quad (\text{or}) \; 0+j1$$

$|X(3)| = 1, \quad \angle X(3) = \pi/2$

$\therefore X(k) = \{3, -j, 1, j\}$

$|X(k)| = \{3, 1, 1, 1\}$

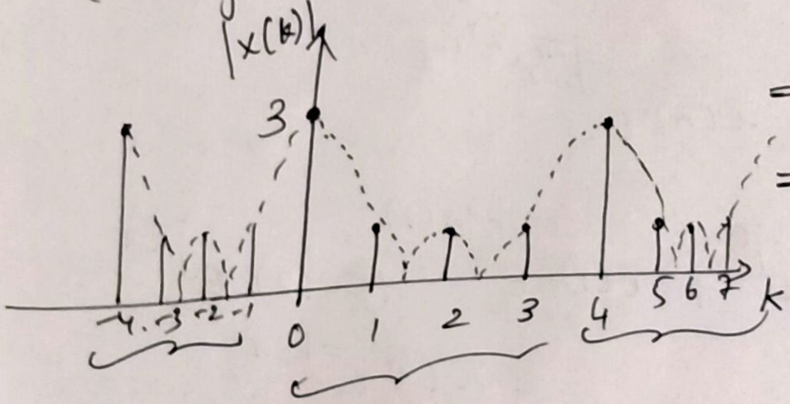
$\angle X(k) = \{0, -\pi/2, 0, \pi/2\}$

do the same procedure to calculate DFT for N=8

do it yourself

frequency response of $x(n)$:

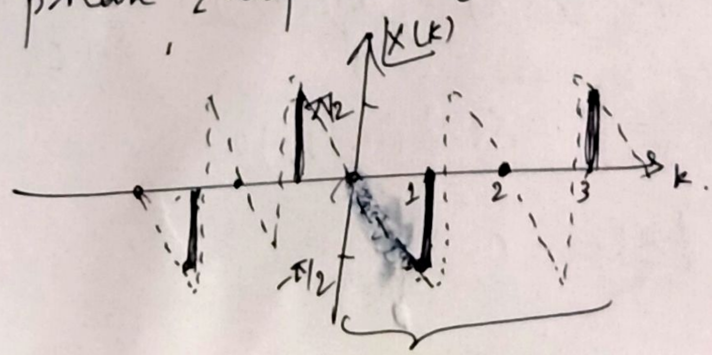
(i) magnitude response of $x(n)$




⇒ periodic function

⇒ even function

(ii) phase response of $x(n)$




⇒ periodic function

⇒ odd function

pbm: Find IDFT of $Y(k) = \{ \overset{Y(0)}{1}, \overset{Y(1)}{0}, \overset{Y(2)}{1}, \overset{Y(3)}{0} \}$

$$Y(k) \xleftrightarrow{DFT} y(n)$$

$$y(n) = IDFT\{Y(k)\} = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{j\frac{2\pi kn}{N}}, \quad n = 0, 1, \ldots (N-1)$$

N=4

$$\boxed{y(n) = \frac{1}{4} \sum_{k=0}^{3} Y(k) \cdot e^{j\frac{\cancel{2}\pi kn}{\cancel{4}2}}}, \quad n = 0,1,2,3 \rightarrow (1)$$

put $n=0$ in (1):

$$y(0) = \frac{1}{4} \sum_{k=0}^{3} Y(k) \cdot e^{j\frac{\pi}{2}(0)k}$$

$$= \frac{1}{4} \sum_{k=0}^{3} Y(k) \cdot 1 \quad [\because e^{j0} = 1$$

$$= \frac{1}{4} [Y(0) + Y(1) + Y(2) + Y(3)]$$

$$= \frac{1}{4}[1+0+1+0] = 2/4 = 1/2 = \boxed{0.5 = y(0)}$$

put $n=1$ in (1),

$$y(1) = \frac{1}{4} \sum_{k=0}^{3} Y(k) e^{j\frac{\pi}{2}k(1)}$$

$$= \frac{1}{4} \left[ Y(0) \cdot e^{j0} + Y(1) \cdot e^{j\frac{\pi}{2}(1)} + Y(2) \cdot e^{j\frac{\pi}{2}(2)} + Y(3) \cdot e^{j\frac{\pi}{2}(3)} \right]$$

$$= \frac{1}{4}\left[1.1 + 0.(\ ) + 1.e^{j\pi} + 0\right]$$

$$= \frac{1}{4}\left[1 + 0 + \cos\pi + j\underbrace{\sin\pi}_{0} + 0\right]$$

$$= \frac{1}{4}[1-1] = \boxed{0 = y(1)}$$

put $n=2$ in ①:

$$y(2) = \frac{1}{4}\sum_{k=0}^{3} Y(k)e^{j\frac{\pi}{2}k(2)}$$

$$= \frac{1}{4}\left[Y(0)e^{j0} + Y(1).e^{j\pi(1)} + Y(2).e^{j\pi(2)} + Y(3).e^{j\pi(3)}\right]$$

$$= \frac{1}{4}\left[1 + 0 + 1.\left[\cos 2\pi + j\underbrace{\sin 2\pi}_{0}\right] + 0\right]$$

$$= \frac{1}{4}[1+1] = 2/4 = 1/2 = \boxed{0.5 = y(2)}$$

Similarly, put $n=3$ in ①.

$$y(3) = \frac{1}{4}\sum_{k=0}^{3} Y(k).e^{j\frac{\pi}{2}k(3)}$$

$$\boxed{y(3) = 0}$$

$$y(n) = \{0.5, 0, 0.5, 0\}$$

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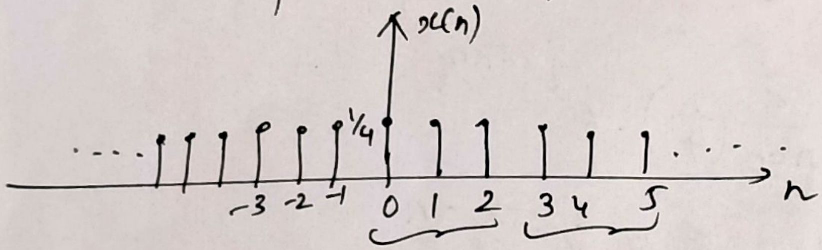
**pbm:** Determine the DFT of the sequence

$$x(n) = \begin{cases} 1/4, & 0 \le n \le 2 \quad \text{(or)} \quad n = \underbrace{0,1,2} \\ 0, & \text{otherwise} \end{cases}$$

$\boxed{N=3}$

**Solution:** here N value is not given.

So no need to add zeros to x(n) to get the periodic sequence.



$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots N-1$$

$$\boxed{X(k) = \sum_{n=0}^{2} x(n)\, e^{-j2\pi kn/3}, \quad k=0,1,2} \rightarrow (1)$$

put $k=0$ in (1):

$$X(0) = \sum_{n=0}^{2} x(n)\, e^{-j\frac{2\pi n}{3}(0)}$$

$$= x(0) + x(1) + x(2) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 3/4$$

$k=1$: $\Rightarrow$ $X(1) = \sum_{n=0}^{2} x(n)\, e^{-j\frac{2\pi n}{3}(1)}$

$$= x(0)\, e^{-j0} + x(1) \cdot e^{-j2\pi/3} + x(2)\, e^{-j4\pi/3}$$

$$= \frac{1}{4} + \frac{1}{4}\, e^{-j2\pi/3} \left[ 1 + e^{-j2\pi/3} \right]$$

$\underline{k=2}$: $$X(2) = \sum_{n=0}^{2} x(n) \cdot e^{-j\frac{2\pi n}{N}(2)} \qquad \boxed{N=3}$$

$$= x(0)\, e^{-j0} + x(1) \cdot e^{-j\frac{4\pi}{3}(1)} + x(2) \cdot e^{-j\frac{4\pi}{3}(2)}$$

$$= \frac{1}{4} + \frac{1}{4}\, e^{-j4\pi/3} \left[1 + e^{-j2\pi/3}\right]$$

(OR)

$$X(k) = \sum_{n=0}^{2} x(n)\, e^{-j2\pi kn/3}, \quad k = 0, 1, 2$$

$$= \sum_{n=0}^{2} \left(\frac{1}{4}\right) \cdot e^{-j2\pi kn/3}$$

$$X(k) = \frac{1}{4} \sum_{n=0}^{2} \left(e^{-j2\pi k/3}\right)^{n}$$

Using finite summation formula $$\boxed{\sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1-a}, \quad a \neq 1}$$ used when $N \to$ large

$$X(k) = \frac{1}{4}\left[e^{-j0} + e^{-j\frac{2\pi k}{3}(1)} + e^{-j\frac{2\pi k}{3}(2)}\right]$$

$$\cancel{= \frac{1}{4}\left[1 + e^{-j2\pi k/3}\right.}$$

$$= \frac{1}{4}\, e^{-j\frac{2\pi k}{3}} \left[e^{+j\frac{2\pi k}{3}} + 1 + e^{-j\frac{2\pi k}{3}}\right]$$

$$= \frac{1}{4}\, e^{-j\frac{2\pi k}{3}} \left[2\cos\left(\frac{2\pi k}{3}\right) + 1\right] \qquad \left[\because \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}\right.$$

Ans: $$\boxed{X(k) = \frac{1}{4}\, e^{-j\frac{2\pi k}{3}} \left[1 + 2\cos\left(\frac{2\pi k}{3}\right)\right]}$$

Home work:

① Find the DFT of the sequence

$$x(n) = \{2, 0, 0, 1\}$$

② Find the IDFT of the sequence

$$X(k) = \{3, (2+j1), 1, (2-j1)\}$$

③ Find the N-point DFT of the sequence

$$x(n) = a^n \text{ for } 0 < a < 1.$$

## Properties of DFT:

① Periodicity:

If $x(n)$ & $X(k)$ are N-point DFT pair,

then $x(n+N) = x(n)$, for all $n$

$X(k+N) = X(k)$, for all $k$.

② Linearity:

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$

and $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$,

then for any real (or) complex valued constants '$a_1$' & '$a_2$',

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow[N]{DFT} a_1 X_1(k) + a_2 X_2(k).$$

## (3) Circular Symmetries of a sequence:

$$\left.\begin{array}{l}\text{N-point DFT of a} \\ \text{finite duration} \\ \text{sequence } x(n), \text{ of} \\ \text{length } L \le N\end{array}\right\} \equiv \begin{array}{l}\text{N-point DFT of a} \\ \text{periodic sequence } x_p(n), \\ \text{of } \cancel{\text{length}} \text{ period } N.\end{array}$$

$x_p(n) =$ periodically extended sequence of $x(n)$

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

$$x_p(n) = \begin{cases} x(n); & 0 \le n \le L-1 \\ 0; & L \le n \le N-1 \end{cases}$$

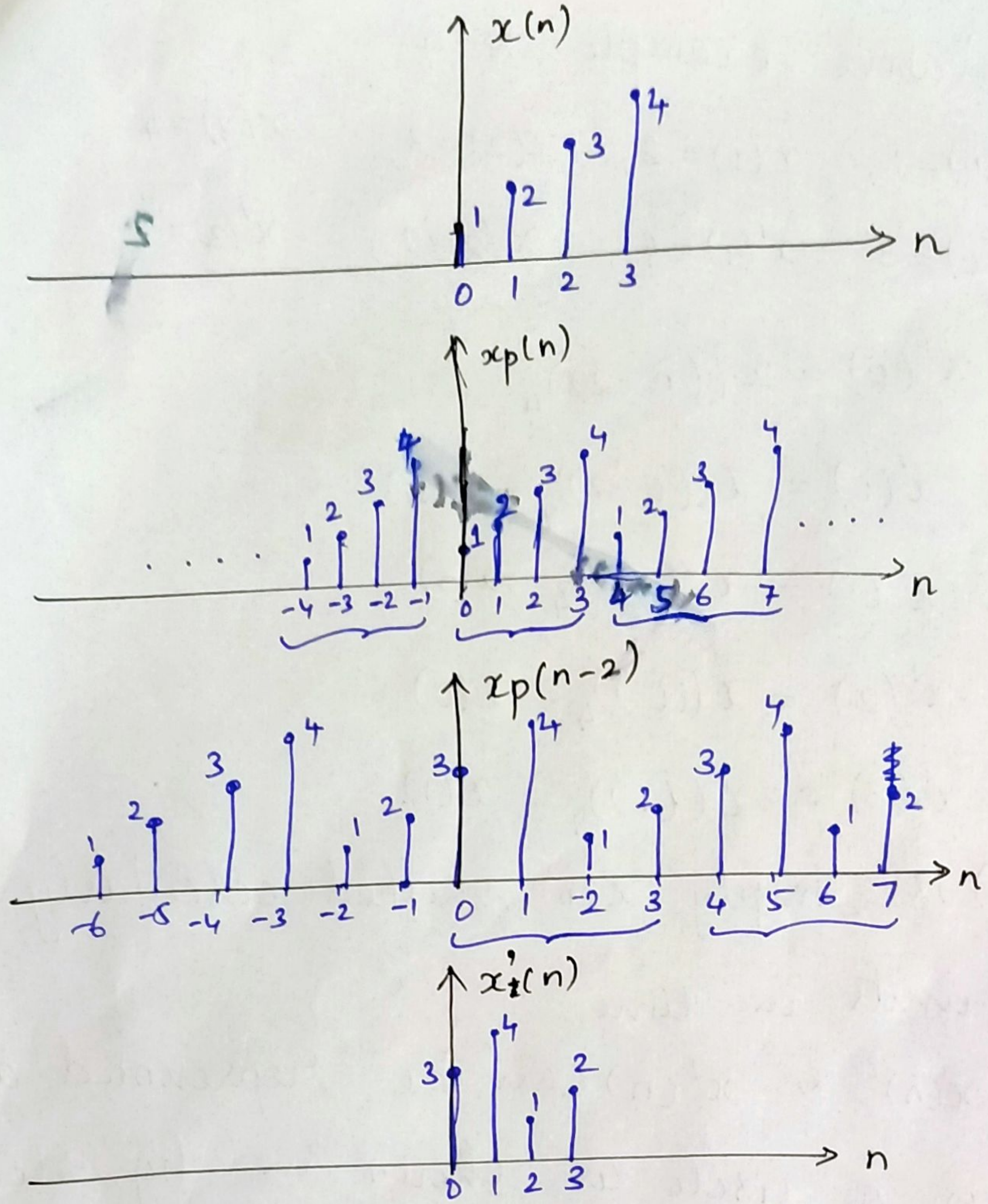
Suppose; shift $x_p(n)$ by $k$ units to the right.

another periodic sequence is obtained.

$$x_p'(n) = x_p(n-k) = \sum_{l=-\infty}^{\infty} x(n-k-lN)$$

$$x'(n) = \begin{cases} x_p'(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

$x'(n)$ is related to the original sequence by a circular shift.




The relationship between $x(n)$ and $x'(n)$ is shown below:

$$x'(n) = x(n-k, \text{modulo } N)$$

or

$$x'(n) = x(n-k, (\text{mod } N))$$

or

$$x'(n) = x((n-k))_N \longrightarrow (1)$$

from above example, $N=4$

$x(0)=1$, $x(1)=2$, $x(2)=3$, $x(3)=4$

$x'(0)=3$, $x'(1)=4$, $x'(2)=1$, $x'(3)=2$

Let $k=2$
$N=4$
in ①,

$$x'(n) = x((n-2))_4$$

$$x'(0) = x((0-2))_4 = x(2)$$

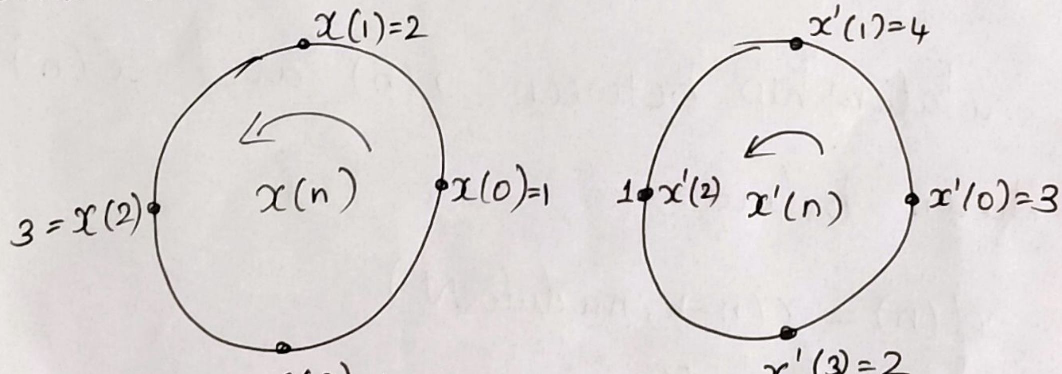
$$x'(1) = x((-1))_4 = x(3)$$

$$x'(2) = x((0))_4 = x(0)$$

$$x'(3) = x((1))_4 = x(1)$$

$x'(n)$ is simply $x(n)$ shifted circularly by 2 units in time.

So, $x(n)$ & $x'(n)$ can be represented as points on a circle as shown in fig. below.




Circular representation of $x(n)$ & $x'(n)$

| Linear Shift in time | Circular Shift in time |
| --- | --- |
| $x(n-2) \Rightarrow$ Shift $x(n)$ by 2 units right | $x((n-2))_4 \Rightarrow$ Shift $x(n)$ by 2 units in Anti-clockwise direction |
| $x(n+2) \Rightarrow$ " left | $x((n+2))_4 \Rightarrow$ " clockwise. |

Conclusion:

Circular shift of an N-point sequence is equivalent to a linear shift of its periodic extension and vice versa.

This periodicity results give rise to new definitions for even symmetry, odd symmetry and time reversal of a sequence.

An N-point sequence is called circularly even if it is symmetric about the point zero on the circle.

$$x(N-n) = x(n), \quad 1 \leq n \leq N-1$$

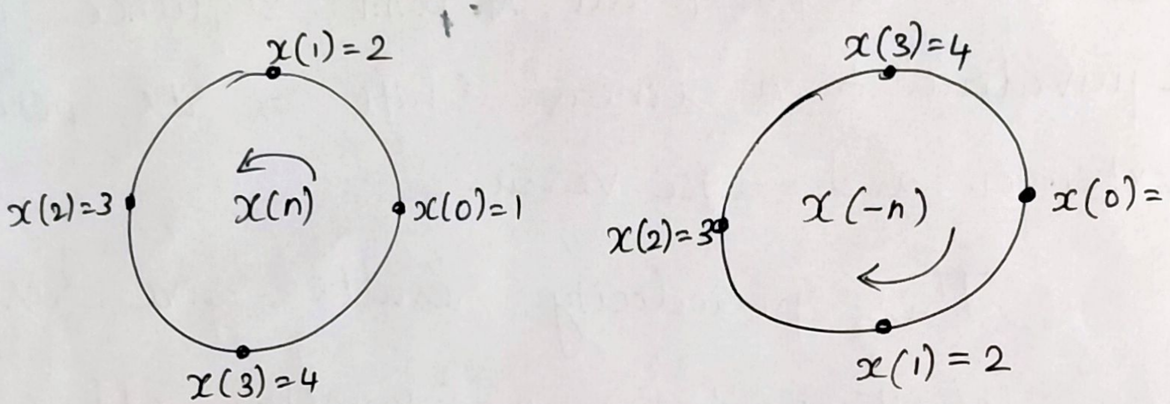
An N-point sequence is called circularly odd, if it is antisymmetric about the point zero on the circle.

$$x(N-n) = -x(n), \quad 1 \leq n \leq N-1$$

The time reversal of an N-point sequence is attained by reversing its samples about the point zero on the circle.

$$x((-n))_N = x(N-n), \quad 0 \le n \le N-1$$

i.e. folded sequence is equivalent to plotting $x(n)$ in a clockwise direction on a circle.




## Symmetry Properties of DFT:

i) Complex valued sequences:

$$\text{if complex} \Rightarrow x(n) \xleftrightarrow[N]{DFT} X(k) \ \text{(complex)}$$

$$x_R(n) + j\,x_I(n) \qquad\qquad X_R(k) + j\,X_I(k)$$

$$X_R(k) = \sum_{n=0}^{N-1} \left[ x_R(n)\cos\left(\frac{2\pi kn}{N}\right) + x_I(n)\sin\left(\frac{2\pi kn}{N}\right) \right]$$

$$X_I(k) = -\sum_{n=0}^{N-1} \left[ x_R(n)\sin\left(\frac{2\pi kn}{N}\right) - x_I(n)\cos\left(\frac{2\pi kn}{N}\right) \right]$$