Chapter Six

Microwave Resonators

Microwave resonators are used in a variety of applications, including filters, oscillators, frequency meters, and tuned amplifiers. Since the operation of microwave resonators is very similar to that of the lumped-element resonators of circuit theory, we will begin by reviewing the basic characteristics of series and parallel *RLC* resonant circuits. We will then discuss various implementations of resonators at microwave frequencies using distributed elements such as transmission lines, rectangular and circular waveguide, and dielectric cavities. We will also discuss the excitation of resonators using apertures and current sheets.

6.1 SERIES AND PARALLEL RESONANT CIRCUITS

Near resonance, a microwave resonator can usually be modeled by either a series or parallel *RLC* lumped-element equivalent circuit, and so we will derive some of the basic properties of such circuits below.

Series Resonant Circuit

A series *RLC* resonant circuit is shown in Figure 6.1a. The input impedance is

$$Z_{\text{in}} = R + j\omega L - j\frac{1}{\omega C}, \tag{6.1}$$

and the complex power delivered to the resonator is

$$\begin{aligned} P_{\text{in}} &= \frac{1}{2}VI^* = \frac{1}{2}Z_{\text{in}}|I|^2 = \frac{1}{2}Z_{\text{in}}\left|\frac{V}{Z_{\text{in}}}\right|^2 \\ &= \frac{1}{2}|I|^2\left(R + j\omega L - j\frac{1}{\omega C}\right). \end{aligned} \tag{6.2}$$

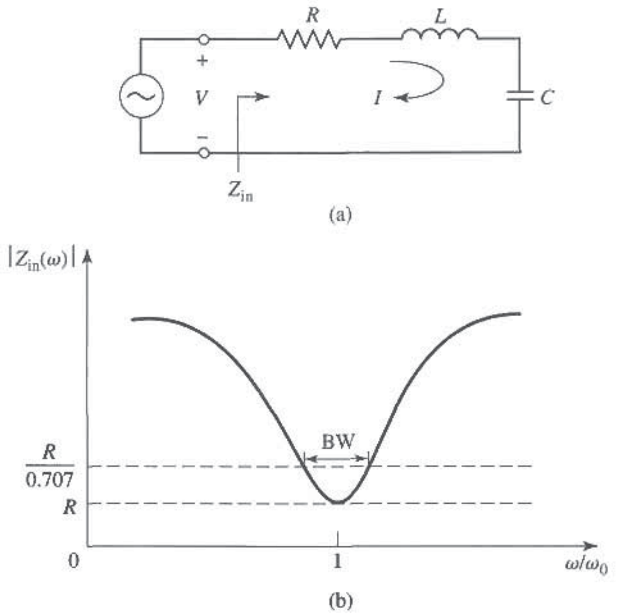


FIGURE 6.1 A series *RLC* resonator and its response. (a) The series *RLC* circuit. (b) The input impedance magnitude versus frequency.

The power dissipated by the resistor, R, is

$$P_{\text{loss}} = \frac{1}{2}|I|^2 R, \tag{6.3a}$$

the average magnetic energy stored in the inductor, L, is

$$W_m = \frac{1}{4}|I|^2 L, \tag{6.3b}$$

and the average electric energy stored in the capacitor, C, is

$$W_e = \frac{1}{4}|V_c|^2 C = \frac{1}{4}|I|^2 \frac{1}{\omega^2 C}, \tag{6.3c}$$

where V_c is the voltage across the capacitor. Then the complex power of (6.2) can be rewritten as

$$P_{\text{in}} = P_{\text{loss}} + 2j\omega(W_m - W_e), \tag{6.4}$$

and the input impedance of (6.1) can be rewritten as

$$Z_{\text{in}} = \frac{2P_{\text{in}}}{|I|^2} = \frac{P_{\text{loss}} + 2j\omega(W_m - W_e)}{\frac{1}{2}|I|^2}. \tag{6.5}$$

Resonance occurs when the average stored magnetic and electric energies are equal, or $W_m = W_e$. Then from (6.5) and (6.3a), the input impedance at resonance is

$$Z_{\text{in}} = \frac{P_{\text{loss}}}{\frac{1}{2}|I|^2} = R,$$

which is purely real. From (6.3b,c), $W_m = W_e$ implies that the resonant frequency, ω_0, must be defined as

$$\omega_0 = \frac{1}{\sqrt{LC}}. \tag{6.6}$$

Another important parameter of a resonant circuit is its Q, or quality factor, which is defined as

$$\begin{aligned} Q &= \omega \frac{\text{(average energy stored)}}{\text{(energy loss/second)}} \\ &= \omega \frac{W_m + W_e}{P_\ell}. \end{aligned} \tag{6.7}$$

Thus Q is a measure of the loss of a resonant circuit—lower loss implies a higher Q. For the series resonant circuit of Figure 6.1a, the Q can be evaluated from (6.7) using (6.3), and the fact that $W_m = W_e$ at resonance, to give

$$Q = \omega_0 \frac{2W_m}{P_{\text{loss}}} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}, \tag{6.8}$$

which shows that Q increases as R decreases.

Now consider the behavior of the input impedance of this resonator near its resonant frequency [1]. We let $\omega = \omega_0 + \Delta\omega$, where $\Delta\omega$ is small. The input impedance can then be rewritten from (6.1) as

$$\begin{aligned} Z_{\text{in}} &= R + j\omega L \left(1 - \frac{1}{\omega^2 LC}\right) \\ &= R + j\omega L \left(\frac{\omega^2 - \omega_0^2}{\omega^2}\right), \end{aligned}$$

since $\omega_0^2 = 1/LC$. Now $\omega^2 - \omega_0^2 = (\omega - \omega_0)(\omega + \omega_0) = \Delta\omega(2\omega - \Delta\omega) \simeq 2\omega\Delta\omega$ for small $\Delta\omega$. Thus,

$$\begin{aligned} Z_{\text{in}} &\simeq R + j2L\Delta\omega \\ &\simeq R + j\frac{2RQ\Delta\omega}{\omega_0}. \end{aligned} \tag{6.9}$$

This form will be useful for identifying equivalent circuits with distributed element resonators.

Alternatively, a resonator with loss can be modeled as a lossless resonator whose resonant frequency ω_0 has been replaced with a complex effective resonant frequency:

$$\omega_0 \longleftarrow \omega_0 \left(1 + \frac{j}{2Q}\right). \tag{6.10}$$

This can be seen by considering the input impedance of a series resonator with no loss, as given by (6.9) with $R = 0$:

$$Z_{\text{in}} = j2L(\omega - \omega_0).$$

Then substituting the complex frequency of (6.10) for ω_0 gives

$$\begin{aligned} Z_{\text{in}} &= j2L\left(\omega - \omega_0 - j\frac{\omega_0}{2Q}\right) \\ &= \frac{\omega_0 L}{Q} + j2L(\omega - \omega_0) = R + j2L\Delta\omega, \end{aligned}$$

which is identical to (6.9). This is a useful procedure because for most practical resonators the loss is very small, so the Q can be found using the perturbation method, beginning with the solution for the lossless case. Then the effect of loss can be added to the input impedance by replacing ω_0 with the complex resonant frequency given in (6.10).

Finally, consider the half-power fractional bandwidth of the resonator. Figure 6.1b shows the variation of the magnitude of the input impedance versus frequency. When the frequency is such that $|Z_{\text{in}}|^2 = 2R^2$, then by (6.2) the average (real) power delivered to the circuit is one-half that delivered at resonance. If BW is the fractional bandwidth, then $\Delta\omega/\omega_0 = \text{BW}/2$ at the upper band edge. Then using (6.9) gives

$$|R + jRQ(\text{BW})|^2 = 2R^2,$$

or

$$\text{BW} = \frac{1}{Q}. \tag{6.11}$$

Parallel Resonant Circuit

The parallel *RLC* resonant circuit, shown in Figure 6.2a, is the dual of the series *RLC* circuit. The input impedance is

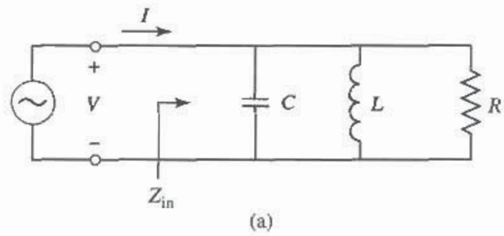
$$Z_{\text{in}} = \left(\frac{1}{R} + \frac{1}{j\omega L} + j\omega C\right)^{-1}, \tag{6.12}$$

and the complex power delivered to the resonator is

$$\begin{aligned} P_{\text{in}} &= \frac{1}{2}VI^* = \frac{1}{2}Z_{\text{in}}|I|^2 = \frac{1}{2}|V|^2\frac{1}{Z_{\text{in}}^*} \\ &= \frac{1}{2}|V|^2\left(\frac{1}{R} + \frac{j}{\omega L} - j\omega C\right). \end{aligned} \tag{6.13}$$

The power dissipated by the resistor, R, is

$$P_{\text{loss}} = \frac{1}{2}\frac{|V|^2}{R}, \tag{6.14a}$$



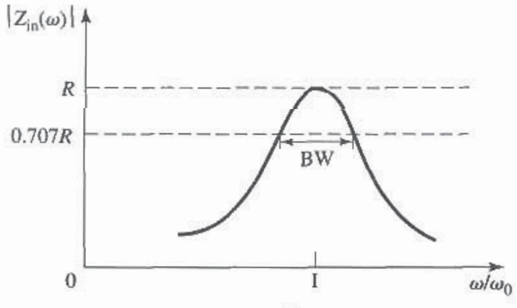


FIGURE 6.2 A parallel *RLC* resonator and its response. (a) The parallel *RLC* circuit. (b) The input impedance magnitude versus frequency.

the average electric energy stored in the capacitor, C, is

$$W_e = \frac{1}{4}|V|^2 C, \tag{6.14b}$$

and the average magnetic energy stored in the inductor, L, is

$$W_m = \frac{1}{4}|I_L|^2 L = \frac{1}{4}|V|^2 \frac{1}{\omega^2 L}, \tag{6.14c}$$

where I_L is the current through the inductor. Then the complex power of (6.13) can be rewritten as

$$P_{\text{in}} = P_{\text{loss}} + 2j\omega(W_m - W_e), \tag{6.15}$$

which is identical to (6.4). Similarly, the input impedance can be expressed as

$$Z_{\text{in}} = \frac{2P_{\text{in}}}{|I|^2} = \frac{P_{\text{loss}} + 2j\omega(W_m - W_e)}{\frac{1}{2}|I|^2}, \tag{6.16}$$

which is identical to (6.5).

As in the series case, resonance occurs when $W_m = W_e$. Then from (6.16) and (6.14a) the input impedance at resonance is

$$Z_{\text{in}} = \frac{P_{\text{loss}}}{\frac{1}{2}|I|^2} = R,$$

which is a purely real impedance. From (6.14b,c), $W_m = W_e$ implies that the resonant frequency, ω_0, should be defined as

$$\omega_0 = \frac{1}{\sqrt{LC}}, \tag{6.17}$$

which again is identical to the series resonant circuit case.

From the definition of (6.7), and the results in (6.14), the Q of the parallel resonant circuit can be expressed as

$$Q = \omega_0 \frac{2W_m}{P_{\text{loss}}} = \frac{R}{\omega_0 L} = \omega_0 RC, \tag{6.18}$$

since $W_m = W_e$ at resonance. This result shows that the Q of the parallel resonant circuit increases as R increases.

Near resonance, the input impedance of (6.12) can be simplified using the result that

$$\frac{1}{1+x} \simeq 1 - x + \cdots.$$

Letting $\omega = \omega_0 + \Delta\omega$, where $\Delta\omega$ is small, (6.12) can be rewritten as [1]

$$\begin{aligned}
Z_{\text{in}} &\simeq \left(\frac{1}{R} + \frac{1 - \Delta\omega/\omega_0}{j\omega_0 L} + j\omega_0 C + j\Delta\omega C\right)^{-1} \\
&\simeq \left(\frac{1}{R} + j\frac{\Delta\omega}{\omega_0^2 L} + j\Delta\omega C\right)^{-1} \\
&\simeq \left(\frac{1}{R} + 2j\Delta\omega C\right)^{-1} \\
&\simeq \frac{R}{1 + 2j\Delta\omega RC} = \frac{R}{1 + 2jQ\Delta\omega/\omega_0},
\end{aligned} \tag{6.19}$$

since $\omega_0^2 = 1/LC$. When $R = \infty$ (6.19) reduces to

$$Z_{\text{in}} = \frac{1}{j2C(\omega - \omega_0)}.$$

As in the series resonator case, the effect of loss can be accounted for by replacing ω_0 in this expression with a complex effective resonant frequency:

$$\omega_0 \longleftarrow \omega_0 \left(1 + \frac{j}{2Q}\right) \tag{6.20}$$

Figure 6.2b shows the behavior of the magnitude of the input impedance versus frequency. The half-power bandwidth edges occur at frequencies $(\Delta\omega/\omega_0 = \text{BW}/2)$, such that

$$|Z_{\text{in}}|^2 = \frac{R^2}{2},$$

which, from (6.19), implies that

$$\text{BW} = \frac{1}{Q}, \tag{6.21}$$

as in the series resonance case.

Loaded and Unloaded Q

The Q defined in the preceding sections is a characteristic of the resonant circuit itself, in the absence of any loading effects caused by external circuitry, and so is called the unloaded Q. In practice, however, a resonant circuit is invariably coupled to other circuitry, which will always have the effect of lowering the overall, or loaded Q, Q_L, of the circuit. Figure 6.3 depicts a resonator coupled to an external load resistor, R_L. If the resonator is a series *RLC* circuit, the load resistor R_L adds in series with R so that the effective resistance in (6.8) is $R + R_L$. If the resonator is a parallel *RLC* circuit, the load resistor R_L combines in parallel with R so that the effective resistance in (6.18) is $RR_L/(R + R_L)$. If we define an external Q, Q_e, as

$$Q_e = \begin{cases} \dfrac{\omega_0 L}{R_L} & \text{for series circuits} \\ \dfrac{R_L}{\omega_0 L} & \text{for parallel circuits,} \end{cases} \tag{6.22}$$

then the loaded Q can be expressed as

$$\frac{1}{Q_L} = \frac{1}{Q_e} + \frac{1}{Q}. \tag{6.23}$$

Table 6.1 summarizes the above results for series and parallel resonant circuits.

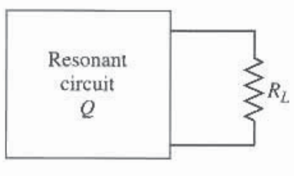


FIGURE 6.3 A resonant circuit connected to an external load, R_L.

TABLE 6.1 Summary of Results for Series and Parallel Resonators

Quantity	Series Resonator	Parallel Resonator
Input Impedance/admittance	$Z_{in} = R + j\omega L - j\dfrac{1}{\omega C}$ $\simeq R + j\dfrac{2RQ\Delta\omega}{\omega_0}$	$Y_{in} = \dfrac{1}{R} + j\omega C - j\dfrac{1}{\omega L}$ $\simeq \dfrac{1}{R} + j\dfrac{2Q\Delta\omega}{R\omega_0}$
Power loss	$P_{loss} = \dfrac{1}{2}\|I\|^2 R$	$P_{loss} = \dfrac{1}{2}\dfrac{\|V\|^2}{R}$
Stored magnetic energy	$W_m = \dfrac{1}{4}\|I\|^2 L$	$W_m = \dfrac{1}{4}\|V\|^2 \dfrac{1}{\omega^2 L}$
Stored electric energy	$W_e = \dfrac{1}{4}\|I\|^2 \dfrac{1}{\omega^2 C}$	$W_e = \dfrac{1}{4}\|V\|^2 C$
Resonant frequency	$\omega_0 = \dfrac{1}{\sqrt{LC}}$	$\omega_0 = \dfrac{1}{\sqrt{LC}}$
Unloaded Q	$Q = \dfrac{\omega_0 L}{R} = \dfrac{1}{\omega_0 RC}$	$Q = \omega_0 RC = \dfrac{R}{\omega_0 L}$
External Q	$Q_e = \dfrac{\omega_0 L}{R_L}$	$Q_e = \dfrac{R_L}{\omega_0 L}$

6.2 TRANSMISSION LINE RESONATORS

As we have seen, ideal lumped circuit elements are usually unattainable at microwave frequencies, so distributed elements are more commonly used. In this section we will study the use of transmission line sections with various lengths and terminations (usually open or short circuited) to form resonators. Since we will be interested in the Q of these resonators, we must consider lossy transmission lines.

Short-Circuited $\lambda/2$ Line

Consider a length of lossy transmission line, short circuited at one end, as shown in Figure 6.4. The line has a characteristic impedance Z_0, propagation constant β, and attenuation constant α. At the frequency $\omega = \omega_0$, the length of the line is $\ell = \lambda/2$, where $\lambda = 2\pi/\beta$. From (2.91), the input impedance is

$$Z_{in} = Z_0 \tanh(\alpha + j\beta)\ell.$$

Using an identity for the hyperbolic tangent gives

$$Z_{in} = Z_0 \frac{\tanh\alpha\ell + j\tan\beta\ell}{1 + j\tan\beta\ell\tanh\alpha\ell}. \tag{6.24}$$

Observe that $Z_{in} = jZ_0 \tan\beta\ell$ if $\alpha = 0$ (no loss).

In practice, most transmission lines have small loss, so we can assume that $\alpha\ell \ll 1$, and so $\tanh\alpha\ell \simeq \alpha\ell$. Now let $\omega = \omega_0 + \Delta\omega$, where $\Delta\omega$ is small. Then, assuming a TEM line,

$$\beta\ell = \frac{\omega\ell}{v_p} = \frac{\omega_0\ell}{v_p} + \frac{\Delta\omega\ell}{v_p},$$

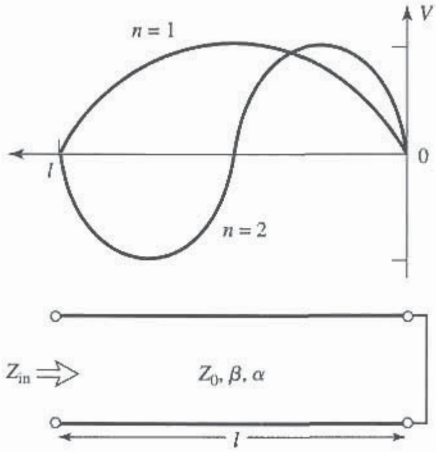


FIGURE 6.4 A short-circuited length of lossy transmission line, and the voltage distributions for $n = 1$ ($\ell = \lambda/2$) and $n = 2$ ($\ell = \lambda$) resonators.

where v_p is the phase velocity of the transmission line. Since $\ell = \lambda/2 = \pi v_p/\omega_0$ for $\omega = \omega_0$, we have

$$\beta\ell = \pi + \frac{\Delta\omega\pi}{\omega_0},$$

and then

$$\tan\beta\ell = \tan\left(\pi + \frac{\Delta\omega\pi}{\omega_0}\right) = \tan\frac{\Delta\omega\pi}{\omega_0} \simeq \frac{\Delta\omega\pi}{\omega_0}.$$

Using these results in (6.24) gives

$$Z_{\text{in}} \simeq Z_0 \frac{\alpha\ell + j(\Delta\omega\pi/\omega_0)}{1 + j(\Delta\omega\pi/\omega_0)\alpha\ell} \simeq Z_0\left(\alpha\ell + j\frac{\Delta\omega\pi}{\omega_0}\right), \tag{6.25}$$

since $\Delta\omega\alpha\ell/\omega_0 \ll 1$.

Equation (6.25) is of the form

$$Z_{\text{in}} = R + 2jL\Delta\omega,$$

which is the input impedance of a series *RLC* resonant circuit, as given by (6.9). We can then identify the resistance of the equivalent circuit as

$$R = Z_0\alpha\ell, \tag{6.26a}$$

and the inductance of the equivalent circuit as

$$L = \frac{Z_0\pi}{2\omega_0}. \tag{6.26b}$$

The capacitance of the equivalent circuit can be found from (6.6) as

$$C = \frac{1}{\omega_0^2 L}. \tag{6.26c}$$

The resonator of Figure 6.4 thus resonates for $\Delta\omega = 0$ ($\ell = \lambda/2$), and its input impedance at this frequency is $Z_{\text{in}} = R = Z_0\alpha\ell$. Resonance also occurs for $\ell = n\lambda/2, n = 1, 2, 3, \ldots$. The voltage distributions for the $n = 1$ and $n = 2$ resonant modes are shown in Figure 6.4.

The Q of this resonator can be found from (6.8) and (6.26) as

$$Q = \frac{\omega_0 L}{R} = \frac{\pi}{2\alpha\ell} = \frac{\beta}{2\alpha}, \tag{6.27}$$

since $\beta\ell = \pi$ at the first resonance. This result shows that the Q decreases as the attenuation of the line increases, as expected.

EXAMPLE 6.1 *Q* OF HALF-WAVE COAXIAL LINE RESONATORS

A $\lambda/2$ resonator is made from a piece of copper coaxial line, with an inner conductor radius of 1 mm and an outer conductor radius of 4 mm. If the resonant frequency is 5 GHz, compare the Q of an air-filled coaxial line resonator to that of a Teflon-filled coaxial line resonator.

Solution

We must first compute the attenuation of the coaxial line, which can be done using the results of Example 2.6 or 2.7. From Appendix F, the conductivity of copper is $\sigma = 5.813 \times 10^7$ S/m. Then the surface resistivity is

$$R_s = \sqrt{\frac{\omega\mu_0}{2\sigma}} = 1.84 \times 10^{-2}\ \Omega,$$

and the attenuation due to conductor loss for the air-filled line is

$$\begin{aligned}\alpha_c &= \frac{R_s}{2\eta \ln b/a}\left(\frac{1}{a} + \frac{1}{b}\right)\\ &= \frac{1.84 \times 10^{-2}}{2(377)\ln(0.004/0.001)}\left(\frac{1}{0.001} + \frac{1}{0.004}\right) = 0.022 \text{ Np/m.}\end{aligned}$$

For Teflon, $\epsilon_r = 2.08$ and $\tan\delta = 0.0004$, so the attenuation due to conductor loss for the Teflon-filled line is

$$\alpha_c = \frac{1.84 \times 10^{-2}\sqrt{2.08}}{2(377)\ln(0.004/0.001)}\left(\frac{1}{0.001} + \frac{1}{0.004}\right) = 0.032 \text{ Np/m.}$$

The dielectric loss of the air-filled line is zero, but the dielectric loss of the Teflon-filled line is

$$\begin{aligned}\alpha_d &= k_0\frac{\sqrt{\epsilon_r}}{2}\tan\delta\\ &= \frac{(104.7)\sqrt{2.08}(0.0004)}{2} = 0.030 \text{ Np/m.}\end{aligned}$$

Finally, from (6.27), the Qs can be computed as

$$Q_{\text{air}} = \frac{\beta}{2\alpha} = \frac{104.7}{2(0.022)} = 2380,$$

$$Q_{\text{Teflon}} = \frac{\beta}{2\alpha} = \frac{104.7\sqrt{2.08}}{2(0.032 + 0.030)} = 1218.$$

Thus it is seen that the Q of the air-filled line is almost twice that of the Teflon-filled line. The Q can be further increased by using silver-plated conductors. ■

Short-Circuited $\lambda/4$ Line

A parallel type of resonance (antiresonance) can be achieved using a short-circuited transmission line of length $\lambda/4$. The input impedance of the shorted line of length ℓ is

$$\begin{aligned} Z_{\text{in}} &= Z_0 \tanh(\alpha + j\beta)\ell \\ &= Z_0 \frac{\tanh \alpha\ell + j \tan \beta\ell}{1 + j \tan \beta\ell \tanh \alpha\ell} \\ &= Z_0 \frac{1 - j \tanh \alpha\ell \cot \beta\ell}{\tanh \alpha\ell - j \cot \beta\ell}, \end{aligned} \tag{6.28}$$

where the last result was obtained by multiplying both numerator and denominator by $-j \cot \beta\ell$. Now assume that $\ell = \lambda/4$ at $\omega = \omega_0$, and let $\omega = \omega_0 + \Delta\omega$. Then, for a TEM line,

$$\beta\ell = \frac{\omega_0 \ell}{v_p} + \frac{\Delta\omega\ell}{v_p} = \frac{\pi}{2} + \frac{\pi\,\Delta\omega}{2\omega_0},$$

and so
$$\cot \beta\ell = \cot\left(\frac{\pi}{2} + \frac{\pi\,\Delta\omega}{2\omega_0}\right) = -\tan \frac{\pi\,\Delta\omega}{2\omega_0} \simeq \frac{-\pi\,\Delta\omega}{2\omega_0}.$$

Also, as before, $\tanh \alpha\ell \simeq \alpha\ell$ for small loss. Using these results in (6.28) gives

$$Z_{\text{in}} = Z_0 \frac{1 + j\alpha\ell\pi\,\Delta\omega/2\omega_0}{\alpha\ell + j\pi\,\Delta\omega/2\omega_0} \simeq \frac{Z_0}{\alpha\ell + j\pi\,\Delta\omega/2\omega_0}, \tag{6.29}$$

since $\alpha\ell\pi\,\Delta\omega/2\omega_0 \ll 1$. This result is of the same form as the impedance of a parallel RLC circuit, as given in (6.19):

$$Z_{\text{in}} = \frac{1}{(1/R) + 2j\Delta\omega C}.$$

Then we can identify the resistance of the equivalent circuit as

$$R = \frac{Z_0}{\alpha\ell} \tag{6.30a}$$

and the capacitance of the equivalent circuit as

$$C = \frac{\pi}{4\omega_0 Z_0}. \tag{6.30b}$$

The inductance of the equivalent circuit can be found as

$$L = \frac{1}{\omega_0^2 C}. \tag{6.30c}$$

The resonator of Figure 6.4 thus has a parallel type resonance for $\ell = \lambda/4$, with an input impedance at resonance of $Z_{in} = R = Z_0/\alpha\ell$. From (6.18) and (6.30) the Q of this resonator is

$$Q = \omega_0 RC = \frac{\pi}{4\alpha\ell} = \frac{\beta}{2\alpha}, \tag{6.31}$$

since $\ell = \pi/2\beta$ at resonance.

Open-Circuited $\lambda/2$ Line

A practical resonator that is often used in microstrip circuits consists of an open-circuited length of transmission line, as shown in Figure 6.5. Such a resonator will behave as a parallel resonant circuit when the length is $\lambda/2$, or multiples of $\lambda/2$.

The input impedance of an open-circuited line of length ℓ is

$$Z_{in} = Z_0 \coth(\alpha + j\beta)\ell = Z_0 \frac{1 + j \tan\beta\ell \tanh\alpha\ell}{\tanh\alpha\ell + j\tan\beta\ell}. \tag{6.32}$$

As before, assume that $\ell = \lambda/2$ at $\omega = \omega_0$, and let $\omega = \omega_0 + \Delta\omega$. Then,

$$\beta\ell = \pi + \frac{\pi\Delta\omega}{\omega_0},$$

and so

$$\tan\beta\ell = \tan\frac{\Delta\omega\pi}{\omega} \simeq \frac{\Delta\omega\pi}{\omega_0},$$

and $\tanh\alpha\ell \simeq \alpha\ell$. Using these results in (6.32) gives

$$Z_{in} = \frac{Z_0}{\alpha\ell + j(\Delta\omega\pi/\omega_0)}. \tag{6.33}$$

Comparing with the input impedance of a parallel resonant circuit as given by (6.19) suggests that the resistance of the equivalent RLC circuit is

$$R = \frac{Z_0}{\alpha\ell}, \tag{6.34a}$$

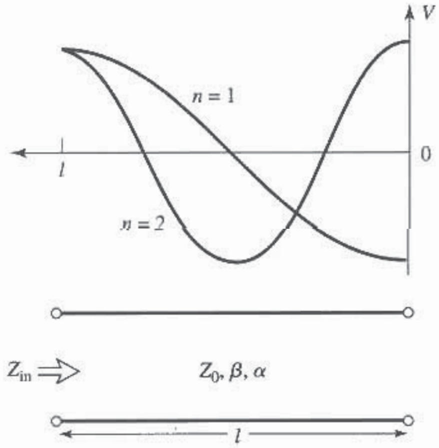


FIGURE 6.5 An open-circuited length of lossy transmission line, and the voltage distributions for $n = 1$ ($\ell = \lambda/2$) and $n = 2$ ($\ell = \lambda$) resonators.

and the capacitance of the equivalent circuit is

$$C = \frac{\pi}{2\omega_0 Z_0}. \tag{6.34b}$$

The inductance of the equivalent circuit is

$$L = \frac{1}{\omega_0^2 C}. \tag{6.34c}$$

From (6.18) and (6.34) the Q is

$$Q = \omega_0 RC = \frac{\pi}{2\alpha\ell} = \frac{\beta}{2\alpha}, \tag{6.35}$$

since $\ell = \pi/\beta$ at resonance.

EXAMPLE 6.2 A HALF-WAVE MICROSTRIP RESONATOR

Consider a microstrip resonator constructed from a $\lambda/2$ length of 50 Ω open-circuited microstrip line. The substrate is Teflon ($\epsilon_r = 2.08$, $\tan\delta = 0.0004$), with a thickness of 0.159 cm. The conductors are copper. Compute the length of the line for resonance at 5 GHz, and the Q of the resonator. Ignore fringing fields at the end of the line.

Solution

From (3.197), the width of a 50 Ω microstrip line on this substrate is found to be

$$W = 0.508 \text{ cm},$$

and the effective permittivity is

$$\epsilon_e = 1.80.$$

Then the resonant length can be calculated as

$$\ell = \frac{\lambda}{2} = \frac{v_p}{2f} = \frac{c}{2f\sqrt{\epsilon_e}} = \frac{3\times 10^8}{2(5\times 10^9)\sqrt{1.80}} = 2.24 \text{ cm}.$$

The propagation constant is

$$\beta = \frac{2\pi f}{v_p} = \frac{2\pi f\sqrt{\epsilon_e}}{c} = \frac{2\pi(5\times 10^9)\sqrt{1.80}}{3\times 10^8} = 151.0 \text{ rad/m}.$$

From (3.199), the attenuation due to conductor loss is

$$\alpha_c = \frac{R_s}{Z_0 W} = \frac{1.84\times 10^{-2}}{50(0.00508)} = 0.0724 \text{ Np/m},$$

where we used R_s from Example 6.1. From (3.198), the attenuation due to dielectric loss is

$$\alpha_d = \frac{k_0\epsilon_r(\epsilon_e - 1)\tan\delta}{2\sqrt{\epsilon_e}(\epsilon_r - 1)} = \frac{(104.7)(2.08)(0.80)(0.0004)}{2\sqrt{1.80}(1.08)} = 0.024 \text{ Np/m}.$$

Then from (6.35) the Q is

$$Q = \frac{\beta}{2\alpha} = \frac{151.0}{2(0.0724 + 0.024)} = 783.$$

■

6.3 RECTANGULAR WAVEGUIDE CAVITIES

Resonators can also be constructed from closed sections of waveguide, which should not be surprising since waveguides are a type of transmission line. Because of radiation loss from open-ended waveguide, waveguide resonators are usually short circuited at both ends, thus forming a closed box or cavity. Electric and magnetic energy is stored within the cavity, and power can be dissipated in the metallic walls of the cavity as well as in the dielectric filling the cavity. Coupling to the resonator can be by a small aperture or a small probe or loop.

We will first derive the resonant frequencies for a general TE or TM resonant mode, and then derive an expression for the Q of the $\mathrm{TE}_{10\ell}$ mode. A complete treatment of the Q for arbitrary TE and TM modes can be made using the same procedure, but is not included here because of its length and complexity.

Resonant Frequencies

The geometry of a rectangular cavity is shown in Figure 6.6. It consists of a length d of rectangular waveguide shorted at both ends ($z = 0, d$). We first find the resonant frequencies of this cavity under the assumption that the cavity is lossless, then we determine the Q using the perturbation method outlined in Section 2.7. We could begin with the wave equations and use the method of separation of variables to solve for the electric and magnetic fields that satisfy the boundary conditions of the cavity, but it is easier to start with the TE and TM waveguide fields, which already satisfy the necessary boundary conditions on the side walls ($x = 0, a$ and $y = 0, b$) of the cavity. Then it is only necessary to enforce the boundary conditions that $E_x = E_y = 0$ on the end walls at $z = 0, d$.

From Table 3.2 the transverse electric fields (E_x, E_y) of the TE_{mn} or TM_{mn} rectangular waveguide mode can be written as

$$\bar{E}_t(x, y, z) = \bar{e}(x, y)[A^+ e^{-j\beta_{mn}z} + A^- e^{j\beta_{mn}z}], \tag{6.36}$$

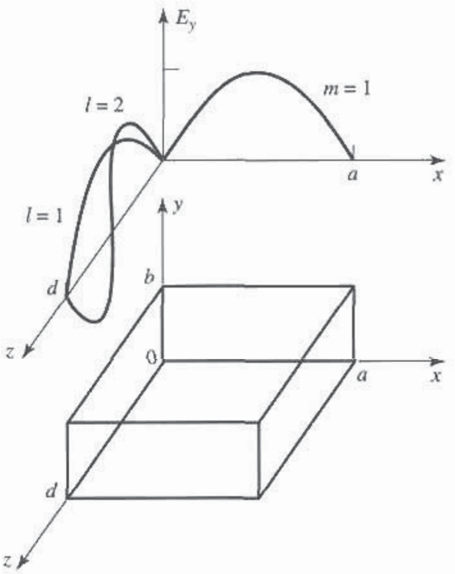


FIGURE 6.6 A rectangular resonant cavity, and the electric field distributions for the TE_{101} and TE_{102} resonant modes.

where $\bar{e}(x, y)$ is the transverse variation of the mode, and A^+, A^- are arbitrary amplitudes of the forward and backward traveling waves. The propagation constant of the m, nth TE or TM mode is

$$\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}, \tag{6.37}$$

where $k = \omega\sqrt{\mu\epsilon}$, and μ, ϵ are the permeability of permittivity of the material filling the cavity.

Applying the condition that $\bar{E}_t = 0$ at $z = 0$ to (6.36) implies that $A^+ = -A^-$ (as we should expect for reflection from a perfectly conducting wall). Then the condition that $\bar{E}_t = 0$ at $z = d$ leads to the equation

$$\bar{E}_t(x, y, d) = -\bar{e}(x, y)A^+ 2j \sin \beta_{mn} d = 0.$$

The only nontrivial ($A^+ \neq 0$) solution thus occurs for

$$\beta_{mn} d = \ell\pi, \qquad \ell = 1, 2, 3, \ldots, \tag{6.38}$$

which implies that the cavity must be an integer multiple of a half-guide wavelength long at the resonant frequency. No nontrivial solutions are possible for other lengths, or for frequencies other than the resonant frequencies. The rectangular cavity is thus a waveguide version of the short-circuited $\lambda/2$ transmission line resonator.

A resonant wavenumber for the rectangular cavity can be defined as

$$k_{mn\ell} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{\ell\pi}{d}\right)^2}. \tag{6.39}$$

Then we can refer to the $\mathrm{TE}_{mn\ell}$ or $\mathrm{TM}_{mn\ell}$ resonant mode of the cavity, where the indices m, n, ℓ refer to the number of variations in the standing wave pattern in the x, y, z directions, respectively. The resonant frequency of the $\mathrm{TE}_{mn\ell}$ or $\mathrm{TM}_{mn\ell}$ mode is then given by

$$f_{mn\ell} = \frac{ck_{mn\ell}}{2\pi\sqrt{\mu_r\epsilon_r}} = \frac{c}{2\pi\sqrt{\mu_r\epsilon_r}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{\ell\pi}{d}\right)^2}. \tag{6.40}$$

If $b < a < d$, the dominant resonant mode (lowest resonant frequency) will be the TE_{101} mode, corresponding to the TE_{10} dominant waveguide mode in a shorted guide of length $\lambda_g/2$. The dominant TM mode is the TM_{110} mode.

Q of the $\mathrm{TE}_{10\ell}$ Mode

From Table 3.2, (6.36), and the fact that $A^- = -A^+$, the total fields for the $\mathrm{TE}_{10\ell}$ resonant mode can be written as

$$E_y = A^+ \sin\frac{\pi x}{a}[e^{-j\beta z} - e^{j\beta z}], \tag{6.41a}$$

$$H_x = \frac{-A^+}{Z_{\mathrm{TE}}} \sin\frac{\pi x}{a}[e^{-j\beta z} + e^{j\beta z}], \tag{6.41b}$$

$$H_z = \frac{j\pi A^+}{k\eta a} \cos\frac{\pi x}{a}[e^{-j\beta z} - e^{j\beta z}]. \tag{6.41c}$$

Letting $E_0 = -2jA^+$ and using (6.38) allows these expressions to be reduced to

$$E_y = E_0 \sin\frac{\pi x}{a}\sin\frac{\ell\pi z}{d}, \tag{6.42a}$$

$$H_x = \frac{-jE_0}{Z_{\rm TE}}\sin\frac{\pi x}{a}\cos\frac{\ell\pi z}{d}, \tag{6.42b}$$

$$H_z = \frac{j\pi E_0}{k\eta a}\cos\frac{\pi x}{a}\sin\frac{\ell\pi z}{d}, \tag{6.42c}$$

which clearly show that the fields form standing waves inside the cavity. We can now compute the Q of this mode by finding the stored electric and magnetic energies, and the power lost in the conducting walls and the dielectric.

The stored electric energy is, from (1.84),

$$W_e = \frac{\epsilon}{4}\int_V E_y E_y^* dv = \frac{\epsilon abd}{16}E_0^2, \tag{6.43a}$$

while the stored magnetic energy is, from (1.86),

$$\begin{aligned} W_m &= \frac{\mu}{4}\int_V (H_x H_x^* + H_z H_z^*) dv \\ &= \frac{\mu abd}{16}E_0^2\left(\frac{1}{Z_{\rm TE}^2} + \frac{\pi^2}{k^2\eta^2 a^2}\right). \end{aligned} \tag{6.43b}$$

Since $Z_{\rm TE} = k\eta/\beta$, and $\beta = \beta_{10} = \sqrt{k^2 - (\pi/a)^2}$, the quantity in parentheses in (6.43b) can be reduced to

$$\left(\frac{1}{Z_{\rm TE}^2} + \frac{\pi^2}{k^2\eta^2 a^2}\right) = \frac{\beta^2 + (\pi/a)^2}{k^2\eta^2} = \frac{1}{\eta^2} = \frac{\epsilon}{\mu},$$

which shows that $W_e = W_m$. Thus, the stored electric and magnetic energies are equal at resonance, analogous to the *RLC* resonant circuits of Section 6.1.

For small losses we can find the power dissipated in the cavity walls using the perturbation method of Section 2.7. Thus, the power lost in the conducting walls is given by (1.131) as

$$P_c = \frac{R_s}{2}\int_{\rm walls} |H_t|^2 ds, \tag{6.44}$$

where $R_s = \sqrt{\omega\mu_o/2\sigma}$ is the surface resistivity of the metallic walls, and H_t is the tangential magnetic field at the surface of the walls. Using (6.42b,c) in (6.44) gives

$$\begin{aligned} P_c &= \frac{R_s}{2}\left\{2\int_{y=0}^{b}\int_{x=0}^{a} |H_x(z=0)|^2 dx\,dy + 2\int_{z=0}^{d}\int_{y=0}^{b} |H_z(x=0)|^2\,dy\,dz\right. \\ &\quad \left. + 2\int_{z=0}^{d}\int_{x=0}^{a}\left[|H_x(y=0)|^2 + |H_z(y=0)|^2\right] dx\,dz\right\} \\ &= \frac{R_s E_0^2\lambda^2}{8\eta^2}\left(\frac{\ell^2 ab}{d^2} + \frac{bd}{a^2} + \frac{\ell^2 a}{2d} + \frac{d}{2a}\right), \end{aligned} \tag{6.45}$$

where use has been made of the symmetry of the cavity in doubling the contributions from the walls at $x = 0$, $y = 0$, and $z = 0$ to account for the contributions from the walls at $x = a$, $y = b$, and $z = d$, respectively. The relations that $k = 2\pi/\lambda$ and $Z_{\rm TE} = k\eta/\beta =$

$2d\eta/\ell\lambda$ were also used in simplifying (6.45). Then, from (6.7), the Q of the cavity with lossy conducting walls but lossless dielectric can be found as

$$\begin{aligned} Q_c &= \frac{2\omega_0 W_e}{P_c} \\ &= \frac{k^3 ab\, d\eta}{4\pi^2 R_s} \frac{1}{[(\ell^2 ab/d^2) + (bd/a^2) + (\ell^2 a/2d) + (d/2a)]} \\ &= \frac{(kad)^3 b\eta}{2\pi^2 R_s} \frac{1}{(2\ell^2 a^3 b + 2bd^3 + \ell^2 a^3 d + ad^3)}. \end{aligned} \tag{6.46}$$

We now compute the power lost in the dielectric. As discussed in Chapter 1, a lossy dielectric has an effective conductivity $\sigma = \omega\epsilon'' = \omega\epsilon_r\epsilon_0 \tan\delta$, where $\epsilon = \epsilon' - j\epsilon'' = \epsilon_r\epsilon_0(1 - j\tan\delta)$, and $\tan\delta$ is the loss tangent of the material. Then the power dissipated in the dielectric is, from (1.92),

$$P_d = \frac{1}{2}\int_V \bar{J} \cdot \bar{E}^* dv = \frac{\omega\epsilon''}{2}\int_V |\bar{E}|^2 dv = \frac{abd\omega\epsilon''|E_0|^2}{8}, \tag{6.47}$$

where $\bar{E}$ is given by (6.42a). Then from (6.7) the Q of the cavity with a lossy dielectric filling, but with perfectly conducting walls, is

$$Q_d = \frac{2\omega W_e}{P_d} = \frac{\epsilon'}{\epsilon''} = \frac{1}{\tan\delta}. \tag{6.48}$$

The simplicity of this result is due to the fact that the integral in (6.43a) for W_e cancels with the identical integral in (6.47) for P_d. This result thus applies to Q_d for an arbitrary resonant cavity mode. When both wall losses and dielectric losses are present, the total power loss is $P_c + P_d$, so (6.7) gives the total Q as

$$Q = \left(\frac{1}{Q_c} + \frac{1}{Q_d}\right)^{-1}. \tag{6.49}$$

EXAMPLE 6.3 DESIGN OF A RECTANGULAR WAVEGUIDE CAVITY

A rectangular waveguide cavity is made from a piece of copper WR-187 H-band waveguide, with $a = 4.755$ cm and $b = 2.215$ cm. The cavity is filled with polyethylene ($\epsilon_r = 2.25$, $\tan\delta = 0.0004$). If resonance is to occur at $f = 5$ GHz, find the required length, d, and the resulting Q for the $\ell = 1$ and $\ell = 2$ resonant modes.

Solution

The wavenumber k is

$$k = \frac{2\pi f\sqrt{\epsilon_r}}{c} = 157.08\ \mathrm{m}^{-1}.$$

From (6.40) the required length for resonance can be found as ($m = 1,\ n = 0$)

$$d = \frac{\ell\pi}{\sqrt{k^2 - (\pi/a)^2}},$$

$$\text{for } \ell = 1, \qquad d = \frac{\pi}{\sqrt{(157.08)^2 - (\pi/0.04755)^2}} = 2.20\ \text{cm},$$

$$\text{for } \ell = 2, \qquad d = 2(2.20) = 4.40\ \text{cm}.$$

From Example 6.1, the surface resistivity of copper at 5 GHz is $R_s = 1.84 \times 10^{-2}\ \Omega$. The intrinsic impedance is

$$\eta = \frac{377}{\sqrt{\epsilon_r}} = 251.3\ \Omega.$$

Then from (6.46) the Q due to conductor loss only is

$$\text{for } \ell = 1, \qquad Q_c = 8403,$$
$$\text{for } \ell = 2, \qquad Q_c = 11{,}898.$$

From (6.48) the Q due to dielectric loss only is, for both $\ell = 1$ and $\ell = 2$,

$$Q_d = \frac{1}{\tan\delta} = \frac{1}{0.0004} = 2500.$$

So the total Qs are, from (6.49)

$$\text{for } \ell = 1, \qquad Q = \left(\frac{1}{8403} + \frac{1}{2500}\right)^{-1} = 1927,$$

$$\text{for } \ell = 2, \qquad Q = \left(\frac{1}{11{,}898} + \frac{1}{2500}\right)^{-1} = 3065.$$

Note that the dielectric loss has the dominant effect on the Q; higher Q could thus be obtained using an air-filled cavity. These results can be compared to those of Examples 6.1 and 6.2, which used similar types of materials at the same frequency. ■

6.4 CIRCULAR WAVEGUIDE CAVITIES

A cylindrical cavity resonator can be constructed from a section of circular waveguide shorted at both ends, similar to rectangular cavities. Since the dominant circular waveguide mode is the TE_{11} mode, the dominant cylindrical cavity mode is the TE_{111} mode. We will derive the resonant frequencies for the $TE_{nm\ell}$ and $TM_{nm\ell}$ circular cavity modes, and the expression for the Q of the $TE_{nm\ell}$ mode.

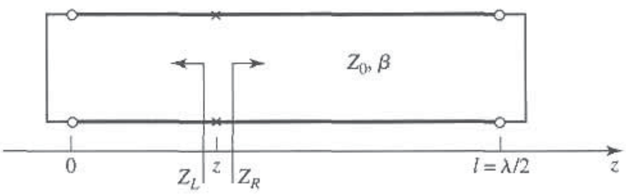
Circular cavities are often used for microwave frequency meters. The cavity is constructed with a movable top wall to allow mechanical tuning of the resonant frequency, and the cavity is loosely coupled to a waveguide with a small aperture. In operation, power will be absorbed by the cavity as it is tuned to the operating frequency of the system; this absorption can be monitored with a power meter elsewhere in the system. The tuning dial is usually directly calibrated in frequency, as in the model shown in Figure 6.7. Since frequency resolution is determined by the Q of the resonator, the TE_{011} mode is often used for frequency meters because its Q is much higher than the Q of the dominant circular cavity mode. This is also the reason for a loose coupling to the cavity.

Resonant Frequencies

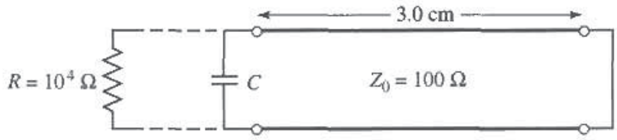
The geometry of a cylindrical cavity is shown in Figure 6.8. As in the case of the rectangular cavity, the solution is simplified by beginning with the circular waveguide modes, which already satisfy the necessary boundary conditions on the circular waveguide wall. From Table 3.5, the transverse electric fields (E_ρ, E_ϕ) of the TE_{nm} or TM_{nm} circular waveguide mode can be written as

$$\bar{E}_t(\rho, \phi, z) = \bar{e}(\rho, \phi)[A^+ e^{-j\beta_{nm}z} + A^- e^{j\beta_{nm}z}], \tag{6.50}$$

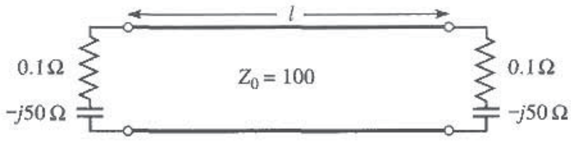
6.4 Consider the resonator shown below, consisting of a $\lambda/2$ length of lossless transmission line shorted at both ends. At an arbitrary point z on the line, compute the impedances Z_L and Z_R seen looking to the left and to the right, and show that $Z_L = Z_R^*$. (This condition holds true for any lossless resonator and is the basis for the transverse resonance technique discussed in Section 3.9.)



6.5 A resonator is constructed from a 3.0 cm length of 100 Ω air-filled coaxial line, shorted at one end and terminated with a capacitor at the other end, as shown. **(a)** Determine the capacitor value to achieve the lowest-order resonance at 6.0 GHz. **(b)** Now assume that loss is introduced by placing a 10,000 Ω resistor in parallel with the capacitor. Calculate the Q.

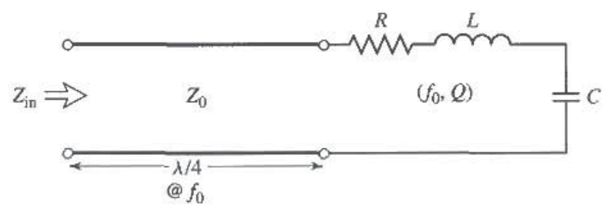


6.6 A transmission line resonator is made from a length ℓ of lossless transmission line of characteristic impedance $Z_0 = 100\ \Omega$. If the line is terminated at both ends as shown, find ℓ/λ for the first resonance, and the Q of this resonator.



6.7 Write the expressions for the $\bar{E}$ and $\bar{H}$ fields for a short-circuited $\lambda/2$ coaxial line resonator, and show that the time-average stored electric and magnetic energies are equal.

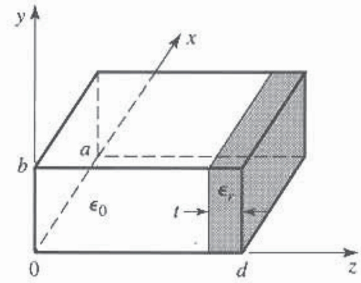
6.8 A series *RLC* resonant circuit is connected to a length of transmission line that is $\lambda/4$ long at its resonant frequency. Show that, in the vicinity of resonance, the input impedance behaves like that of a parallel *RLC* circuit.



6.9 An air-filled, brass-plated rectangular waveguide cavity has dimensions $a = 4$ cm, $b = 2$ cm, $d = 5$ cm. Find the resonant frequency and Q of the TE_{101} and TE_{102} modes.

6.10 Derive the Q for the TM_{111} mode of a rectangular cavity, assuming lossy conducting walls and lossless dielectric.

6.11 Consider the following rectangular cavity resonator partially filled with dielectric. Derive a transcendental equation for the resonant frequency of the dominant mode by writing the fields in the air- and dielectric-filled regions in terms of TE_{10} waveguide modes, and enforcing boundary conditions at $z = 0,\ d - t$, and d.



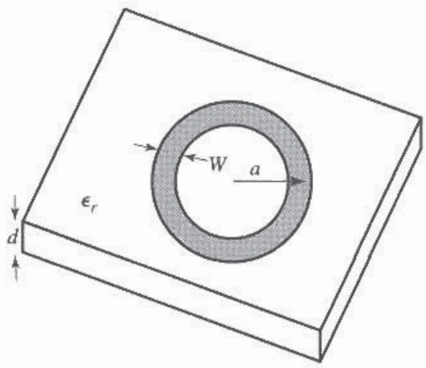
6.12 Determine the resonant frequencies of a rectangular cavity by carrying out a full separation of variables solution to the wave equation for E_z (for TM modes) and H_z (for TE modes), subject to the appropriate boundary conditions of the cavity. (Assume a solution of the form $X(x)Y(y)Z(z)$.)

6.13 Find the Q for the TM_{nm0} resonant mode of a circular cavity. Consider both conductor and dielectric losses.

6.14 Design a circular cavity resonator to operate in the TE_{111} mode with maximum Q at a frequency of 6 GHz. The cavity is gold plated, and filled with a dielectric material having $\epsilon_r = 1.5$ and $\tan\delta = 0.0005$. Find the cavity dimensions and the resulting Q.

6.15 An air-filled rectangular cavity resonator has its first three resonant modes at the frequencies 5.2 GHz, 6.5 GHz, and 7.2 GHz. Find the dimensions of the cavity.

6.16 Consider the microstrip ring resonator shown below. If the effective dielectric constant of the microstrip line is ϵ_e, find an equation for the frequency of the first resonance. Suggest some methods of coupling to this resonator.



6.17 A circular microstrip disk resonator is shown below. Solve the wave equation for TM_{nm0} modes for this structure, using the magnetic wall approximation that $H_\phi = 0$ at $\rho = a$. If fringing fields are neglected, show that the resonant frequency of the dominant mode is given by

$$f_{110} = \frac{1.841c}{2\pi a\sqrt{\epsilon_r}}$$