IDFT:

$$X(k) \xleftrightarrow[N]{IDFT} x(n)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$X_R(k)+jX_I(k) \qquad x_R(n)+jx_I(n)$$

$$x_R(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\cos\left(\frac{2\pi kn}{N}\right) - X_I(k)\sin\left(\frac{2\pi kn}{N}\right)\right] \rightarrow (3)$$

$$x_I(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\sin\left(\frac{2\pi kn}{N}\right) + X_I(k)\cos\left(\frac{2\pi kn}{N}\right)\right] \rightarrow (4)$$

ii) Real-valued sequences:

$$x(n) = x_R(n) + j0 \qquad [\because x_I(n)=0.$$

a) Real and even:

If $x(n) = x(N-n)$, $0 \le n \le N-1$

$x(n) = x_R(n)$

then, $X(k) = \sum_{n=0}^{N-1} x(n)\cos\frac{2\pi kn}{N}$ → real, even

$= X_R(k)$

$\boxed{X_I(k)=0}$

|||ly $IDFT\{X_R(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\cos\frac{2\pi kn}{N}$, $0 \le n \le N-1$

$$x(n) \xrightarrow{DFT} X(k) \rightarrow X(k)=X_R(k)$$

$X_R(k) = X_R(N-k)$

$x(n)$ ⇓ real & even

$\boxed{\begin{array}{l} x(n) = x_R(n) \\ x(n) = x(-n) \\ = x(N-n) \end{array}}$

$X(k)$ ⇓ purely real & even

b) Real and odd:

If $x(n)=x_R(n)$ $\quad x(n)=-x(-n)=-x(N-n), \quad 0\le n\le N-1$

then, $X(k)=\sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi kn}{N}}, \quad k=0$ to $N-1$

$\boxed{X_R(k)=0}$

$$X(k)=X_I(k)=-j\sum_{n=0}^{N-1} x(n)\sin\frac{2\pi kn}{N}, \quad 0\le k\le N-1$$

$\Downarrow$

Purely imaginary and odd

$|||^{ly}$ $\quad IDFT\{X(k)\}=x(n)=j\frac{1}{N}\sum_{k=0}^{N-1} X(k)\sin\frac{2\pi kn}{N}, \quad 0\le n\le N-1$

$\boxed{\begin{matrix} x(n)=x_R(n) \\ x(n)=x(N-n)\end{matrix}}$ $\quad x(n)$ real & odd $\xrightarrow{DFT} X(k)$ $\Downarrow$ Purely imaginary & odd. $\quad \boxed{\begin{matrix} X_I(k)=-X_I(N-k) \\ X_I(k)=X(k)\end{matrix}}$

iii) Purely imaginary sequence:

$$x(n)=0+jx_I(n)$$

$$X_R(k)=\sum_{n=0}^{N-1} x_I(n)\sin\frac{2\pi kn}{N} \Rightarrow \text{odd function}$$

$$X_I(k)=\sum_{n=0}^{N-1} x_I(n)\cos\frac{2\pi kn}{N} \Rightarrow \text{even function.}$$

Conclusion:

Complex-conjugate property.

$$\boxed{\begin{matrix} x(n) & \overset{DFT}{\rightleftharpoons} & X(k) \\ x^*(n) & \rightleftharpoons & X^*(N-k)=X^*((-k))_N \end{matrix}}$$

4. Circular convolution:

If $DFT\{x_1(n)\} = X_1(k)$ & $DFT\{x_2(n)\} = X_2(k)$

then $DFT\{x_1(n) \underset{\Downarrow}{\overset{m \; N}{\circledast}} x_2(n)\} = X_1(k) \cdot X_2(k)$

circular convolution of 2 sequences in time domain is equivalent to multiplication of its DFTs in frequency domain

5. Time Reversal:

If $DFT\{x(n)\} = X(k)$,

then $\left. \begin{array}{c} DFT\{x(N-n)\} \\ (or) \\ DFT\{x((-n))_N\} \end{array} \right\} = \begin{array}{c} X(N-k) \\ (or) \\ X((-k))_N \end{array}$

i.e., Reversing the N-point sequence in time $\} \equiv$ Reversing DFT values.

6. Circular time-shift: (or) DFT of delayed sequence

If $x(n) \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k)$

then $x((n-l))_N \overset{DFT}{\underset{N}{\longleftrightarrow}} X(k) \cdot e^{-j\frac{2\pi kl}{N}}$

i.e., circular shift of the sequence $x(n)$ by 'l' units in time $\} \equiv \{$ Multiplication of DFT with complex exponential sequence $e^{-j(2\pi kl/N)}$.

7. Circular frequency-shift:

If $x(n) \xleftrightarrow[N]{DFT} X(k)$

then $x(n) \cdot e^{\frac{j2\pi ln}{N}} \xleftrightarrow[N]{DFT} X(k-l, (\text{mod } N))$

ie, multiplication of the sequence $x(n)$ with complex exponential sequence $e^{j2\pi ln/N}$ $\equiv$ circular shift of DFT by 'l' units in frequency

This property is dual to the circular time-shifting property.

8. Multiplication of 2 sequences:

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$ & $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$

$$x_1(n) \cdot x_2(n) \xleftrightarrow[N]{DFT} \frac{1}{N}\{X_1(k) \circledast X_2(k)\}$$

ie. multiplication of two sequences in time domain $\equiv$ circular convolution of its DFT in frequency domain

9. Circular Correlation:

If $x(n) \xleftrightarrow[N]{DFT} X(k)$ & $y(n) \xleftrightarrow[N]{DFT} Y(k)$

then $\tilde{r}_{xy}(l) \xleftrightarrow[N]{DFT} \tilde{R}_{xy}(k) \Leftarrow$ for cross correlation where, $\tilde{r}_{xy}(l) \rightarrow$ circular cross-~~corr~~ correlation sequence.

$$\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N = x(n) \circledast y^*(-n)$$

(or)

$$x(n) \; \textcircled{N} \; y^*(-n)$$

$\Downarrow$

circular convolution symbol.

$$\tilde{R}_{xy}(k) = X(k) \cdot Y^*(k).$$

for auto correlation sequence:

$$\tilde{r}_{xx}(l) \xleftrightarrow[N]{DFT} \tilde{R}_{xx}(k) = |X(k)|^2 = X(k) \cdot X^*(k)$$

10) Parseval's theorem:

If $x(n) \xleftrightarrow[N]{DFT} X(k)$ & $y(n) \xleftrightarrow[N]{DFT} Y(k)$

then $\underbrace{\sum_{n=0}^{N-1} x(n)\, y^*(n)}_{= \tilde{r}_{xy}(0)} \xleftrightarrow[N]{DFT} \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, Y^*(k)$

(=)

if $x(n) = y(n)$,

$$\sum_{n=0}^{N-1} |x(n)|^2 \xleftrightarrow[N]{DFT} \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$$

(or) (=)

Relationship between DFT & other transforms:

DFT & DTFT:

$$X(k) = \left\{ X(\omega) \right\}_{\omega = \frac{2\pi k}{N}}$$

where, $X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\omega n}$

DFT & ZT:

$$X(k) = X(z) \Big|_{z = e^{j2\pi k/N}}$$

where $X(z) = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$

Pbm: Consider $x(n) = \{2, 1, 1, 0, 3, 2, 0, 3, 4, 6\}$ whose DFT is $X(k)$. Evaluate the following without calculating $X(k)$.

(i) $X[0]$ $\sum_{n=0}^{N} x(n)$, (ii) $X[5]$, (iii) $\sum_{k=0}^{9} X[k]$, (iv) $\sum_{k=0}^{9} |X(k)|^2$

defn: $X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi kn/N}$

$N = 10$, $k = 0$ to 9. $n = 0$ to 9

Soln:

(i) $X[0] = X(k)\Big|_{k=0} = \sum_{n=0}^{9} x(n) \cdot 1$

$\boxed{X(0) = \sum_{n=0}^{N-1} x(n)}$

$= 2+1+1+0+3+2+0+3+4+6 = \boxed{22 = X[0]}$

(ii) $X(5) = X(k)\Big|_{k=5} = \sum_{n=0}^{9} x(n) \cdot e^{-j\frac{2\pi}{10} \times 5 \times n}$

$\left(e^{-j\pi}\right)^n = [\cos\pi - j\cancel{\sin\pi}^{\,0}]^n = (-1)^n$

$\boxed{X\left(\frac{N}{2}\right) = \sum_{n=0}^{N-1} x(n)(-1)^n}$ $= \sum_{n=0}^{9} x(n)(-1)^n$

$= 2-1+1-0+3-2+0-3+4-6 = \boxed{-2 = X(5)}$

iii) $\sum_{k=0}^{9} X(k) = x(n)\Big|_{n=0}$

IDFT: $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot e^{j2\pi kn/N}$, $n = 0$ to 9

$x(0) = \frac{1}{10} \underbrace{\sum_{k=0}^{9} X(k)}_{} \cdot 1$

$\boxed{x(0) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)}$

$\sum_{k=0}^{9} X(k) = 10 \cdot x(0) = 10 \times 2 = 20$

iv) $\sum_{k=0}^{9} |X(k)|^2 = ?$

Use Parseval's thm: $\sum_{n=0}^{N-1} |x(n)|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |X(k)|^2$

$$\sum_{k=0}^{9} |X(k)|^2 = 10 \times \sum_{n=0}^{9} |x(n)|^2$$

$$= 10\left[2^2+1^2+1^2+0^2+3^2+2^2+0^2+3^2+4^2+6^2\right]$$

$$= 800$$

Pbm: The DFT of a real value signal is given as

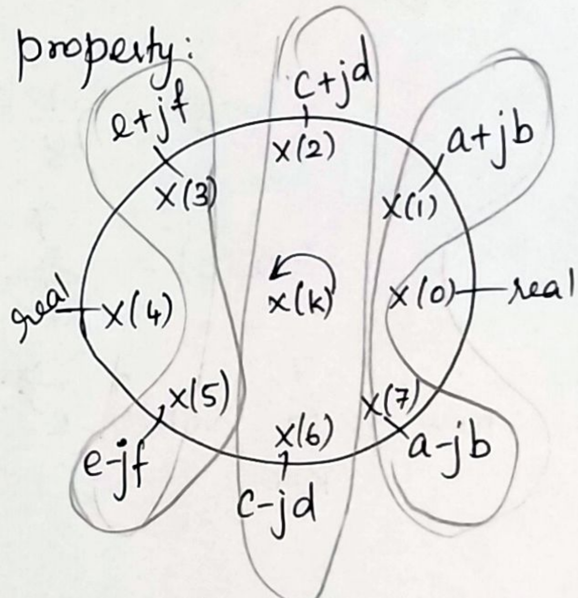
$$X(k) = \{\overset{X(0)}{10}, \overset{X(1)}{2+3j}, \overset{X(2)}{A}, \overset{X(3)}{3-2j}, \overset{X(4)}{-5}, \overset{X(5)}{B}, \overset{X(6)}{2+4j}, \overset{X(7)}{C}\}$$

$\boxed{N=8}$

Evaluate A, B, C values.

Soln: Use complex conjugate property:

$$X(k) = X^*(N-k)$$



N=8

N=8: $X(1) = X^*(8-1) = X^*(7)$

$X(2) = X^*(8-2) = X^*(6)$

$X(3) = X^*(5)$

and vice versa.

$A = X(2) = X^*(6) = 2-j4$

$B = X(5) = X^*(3) = 3+j2$

$C = X(7) = X^*(1) = 2-j3$

— x —

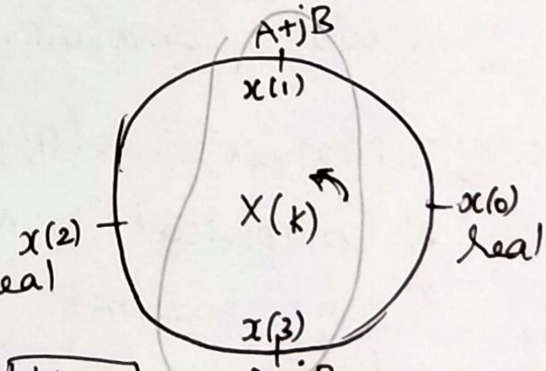
N=4 $X(k) = X^*(N-k)$

$X(1) = X^*(4-1) = X^*(3)$

or

$X(3) = X^*(4-3) = X^*(1)$

$\Leftarrow$ real



$\boxed{N=4}$

Pbm: consider $x(n) = \{\underset{\uparrow}{1}, 2, -3, 0, 1, -1, 4, 2\}$

find i) $e^{-j3\pi k/4} X(k)$. & ii) $\sum_{k=0}^{7} e^{-j3\pi k/4} X(k)$

Using Circular time-shifting property,

i) $$x((n-l))_N \xleftrightarrow[N]{DFT} e^{-j2\pi kl/N} X(k)$$

$$e^{-j2\pi kl/N} X(k) = e^{-j\frac{2\pi k \cdot 3}{8}} X(k) \longleftrightarrow x((n-3))_8 .$$

$$x((n-3))_8 = \{0, 1, -1, 4, 2, 1, 2, -3\}$$

ii) wkt, IDFT: $x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot e^{j2\pi kn/N}$, $n = 0$ to $N-1$

$$x(0) = \frac{1}{8} \sum_{k=0}^{7} X(k)$$

$$8 \cdot x(0) = \sum_{k=0}^{7} X(k)$$

now: $\left(\sum_{k=0}^{7} X(k) \right) . e^{-j3\pi k/4} \longleftrightarrow 8\, x((0-3))_8$

$$= 8 \cdot x((8-3))_8 = 8 \cdot x(5)$$

$$= 8(-1) = -8 .$$

Circular Convolution: (or) periodic Convolution:

methods: 1) Matrix multiplication method

2) Graphical method

3) DFT & IDFT "

4) Tabulation "

pbm: Compute circular periodic convolution of the two sequences $x_1(n) = \{1, 1, 2, 2\}$ & $x_2(n) = \{1, 2, 3, 4\}$

Soln: Note: $\boxed{\text{length of } x_1(n) \overset{\text{(should be)}}{=} \text{length of } x_2(n)}$

1) Matrix multiplication method:

$$\underbrace{\begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}}_{x_2(n)} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}}_{x_1(n)} = \begin{bmatrix} 1+4+6+4 \\ 2+1+8+6 \\ 3+2+2+8 \\ 4+3+4+2 \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \\ 15 \\ 13 \end{bmatrix}$$

(or)

$$\underbrace{\begin{bmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 \end{bmatrix}}_{x_1(n)} \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}_{x_2(n)} = \begin{bmatrix} 1+4+6+4 \\ 1+2+6+8 \\ 2+2+3+8 \\ 2+4+3+4 \end{bmatrix} = \begin{bmatrix} 15 \\ 17 \\ 15 \\ 13 \end{bmatrix}$$

$$y(n) = x_1(n) \circledast x_2(n) = x_1(n) \, Ⓝ \, x_2(n)$$

$$\boxed{y(n) = \{15, 17, 15, 13\}}$$

2) ~~Circular~~ Concentric circle method (or) Graphical method

$$y(m) = x_1(n) \circledast x_2(n) = \sum_{n=0}^{N-1} x_1(n)\, x_2(m-n, (\text{mod } N)), \quad m = 0, 1, 2 \ldots N-1$$

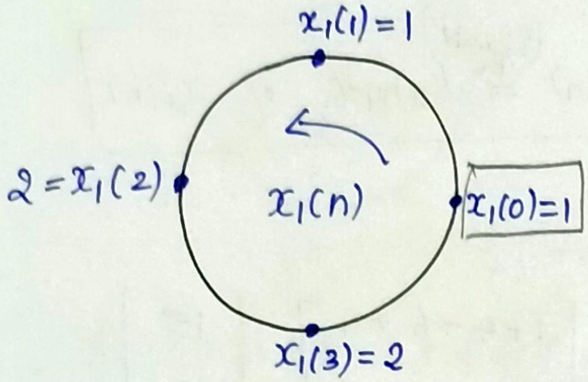
$x_2(m-n, (\text{mod } N))$ (or) $x_2((m-n))_N$

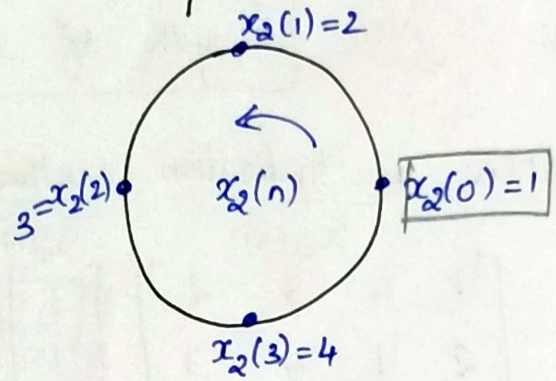
put $m=0$ in ①

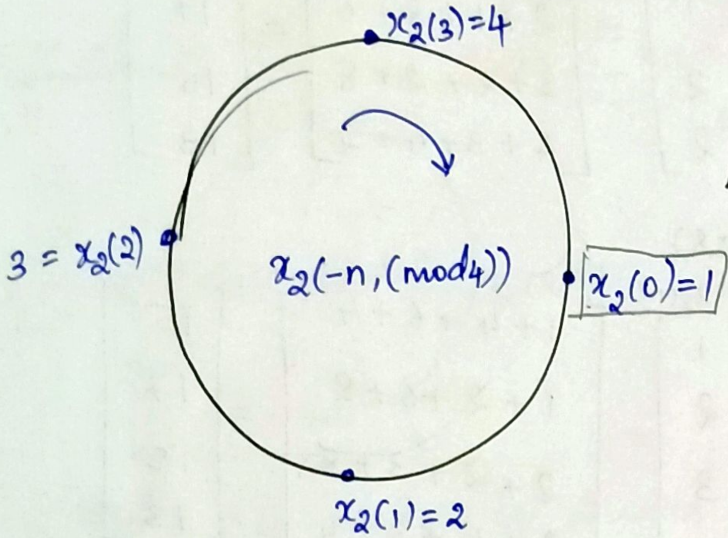
$$y(0) = \sum_{n=0}^{3} x_1(n)\, x_2(-n,(\mathrm{mod}\,4))$$

$\Downarrow$

folded sequence of $x_2(n)$



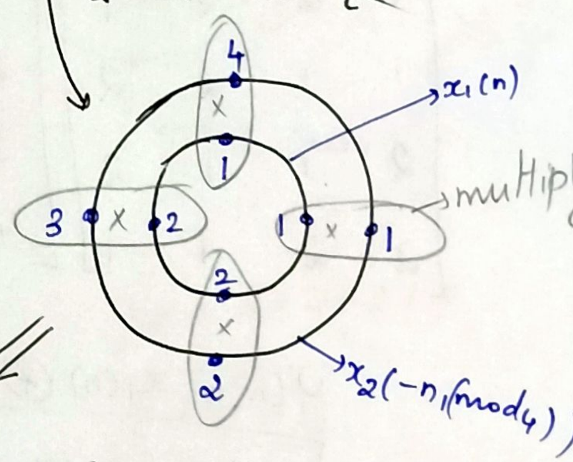


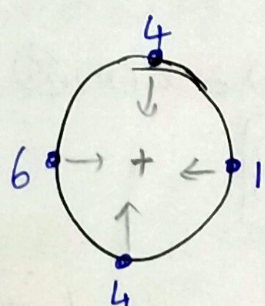


product sequence:

$x_1(n) = \{1, 1, 2, 2\}$

~~$x_2(-n,(\mathrm{mod}4)) = \{$~~

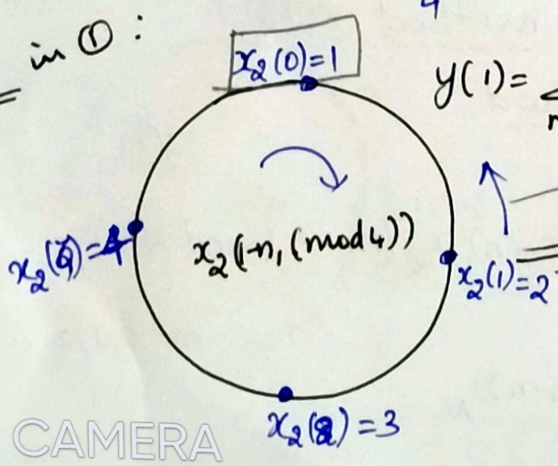




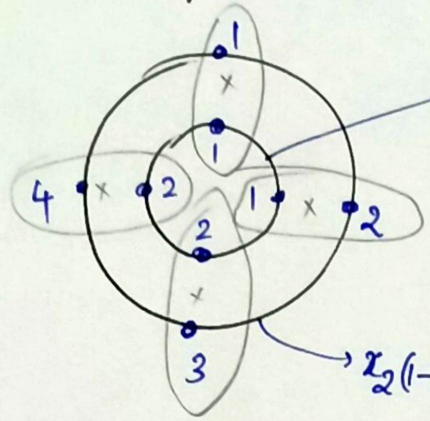
$y(0) = 6 + 4 + 1 + 4 = \boxed{15 = y(0)}$

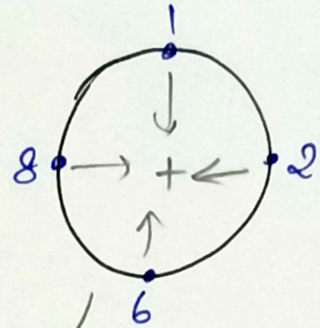
put $m=1$ in ①:

$$y(1) = \sum_{n=0}^{3} x_1(n)\, x_2(1-n,(\mathrm{mod}\,4))$$



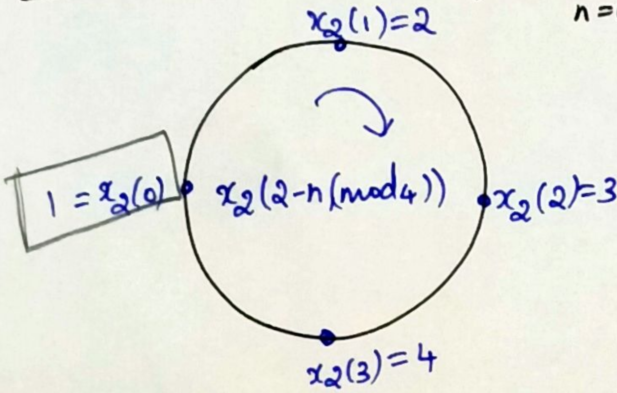
$\Longrightarrow$ Rotate folded sequence $x_2(-n,(\mathrm{mod}4))$ by '1' unit in time towards Anti-clockwise direction product sequence: $x_1(k)=$



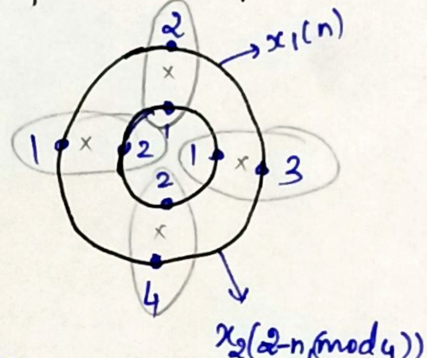


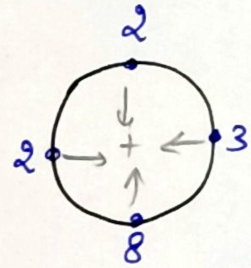
$$y(1) = 1+2+6+8 = \boxed{17 = y(1)}$$

put $m=2$ in ①: $\quad y(2) = \sum_{n=0}^{3} x_1(n)\, x_2(2-n,(\mathrm{mod}\,4))$.



product sequence



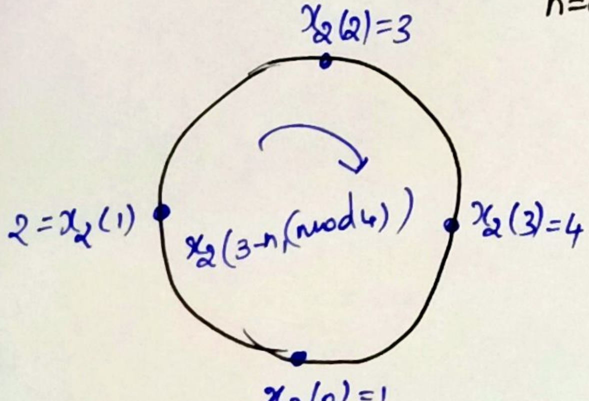


Rotate $x_2(-n,(\mathrm{mod}\,4))$ by 2 units
(or)
Rotate $x_2(1-n,(\mathrm{mod}\,4))$ by 1 unit towards counter clockwise direction

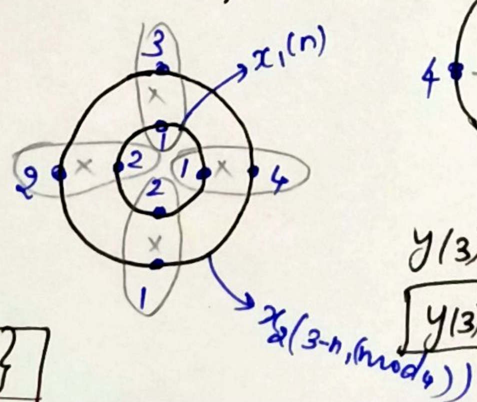
$$y(2) = 2+2+3+8$$

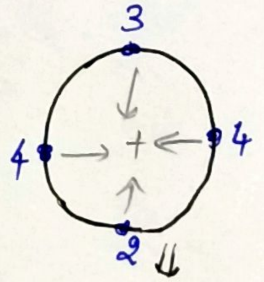
$$\boxed{y(2)=15}$$

put $m=3$ in ①, $\quad y(3) = \sum_{n=0}^{3} x_1(n)\, x_2(3-n,(\mathrm{mod}\,4))$



product sequence





$$y(3) = 3+4+2+4$$

$$\boxed{y(3)=13}$$

Ans: $\boxed{y(n) = \{15, 17, 15, 13\}}$