ADC Quantization noise :

⇒ error due to conversion of analog signal into digital value.

quantization error $\boxed{e(n) = x_q(n) - x(n)}$

where $x_q(n)$ → Sampled quantized value

$x(n)$ → Sampled unquantized value

Quantization may be rounding (or) truncation.

If rounding process is performed, then error range is given by,

$$\frac{-q}{2} \le e(n) \le \frac{q}{2}$$

where $q = 2^{-b}$ and $e(n) = x_R(n) - x(n)$

The probability density function $p(e)$ for round off error and quantization characteristics with rounding is shown in fig 1 & fig 2 respectively.

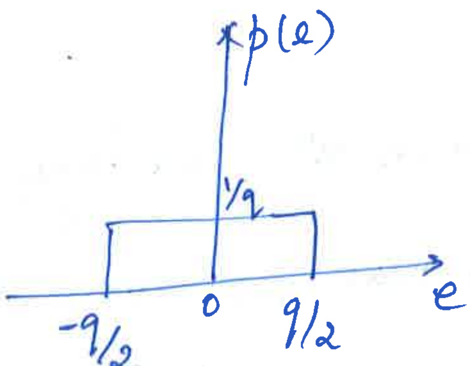


fig 1

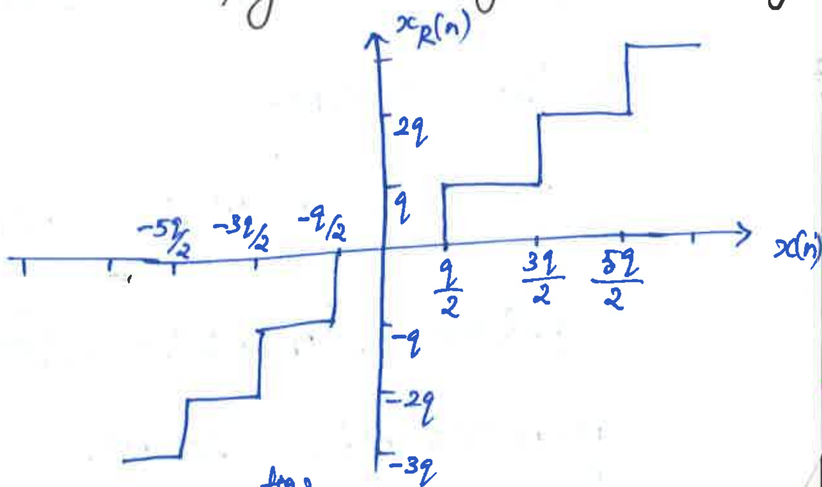


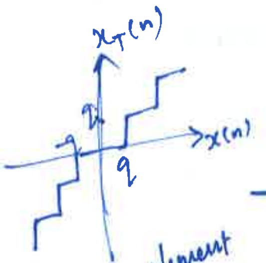
fig 2

If 2's complement truncation is performed, then error range is given by

$$-q \le e(n) < 0$$

where $q = 2^{-b}$ and $e(n) = x_T(n) - x(n)$

The probability density function $p(e)$ for truncation error and quantizer characteristics with 2's complement truncation is shown in fig 1 & 2 respectively.



1's complement & sign-mag. truncation characteristics.

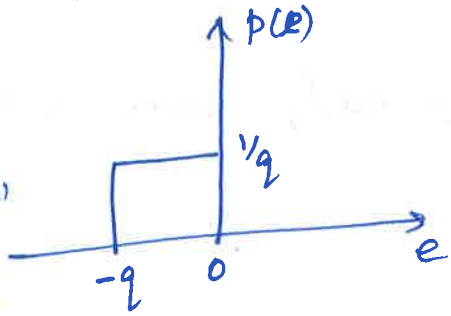


fig 1

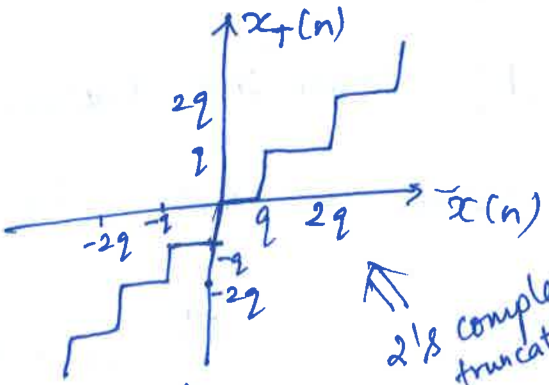


fig. 2

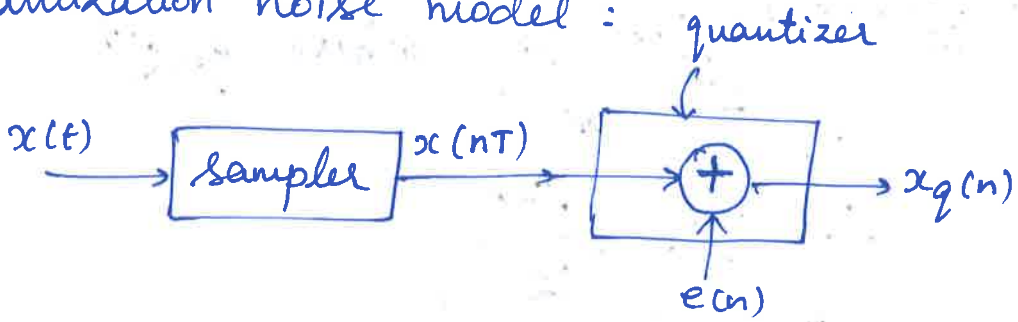
In most cases, this ADC noise $e(n)$ has the following properties:

1. $e(n)$ is a sample sequence of a stationary random process.

2. $e(n)$ is uncorrelated with $x(n)$ & other signals in the system.

3. $e(n)$ is a white noise process with uniform amplitude probability distribution over the range of quantization error.

Quantization noise model :



$$x_q(n) = x(n) + e(n)$$

Due to input quantization, steady state noise power is obtained by finding variance of error signal $e(n)$, (ie. σ_e^2)

$$\sigma_e^2 = E[e^2(n)] - E^2[e(n)]$$

where $E[e^2(n)] \rightarrow$ average of $e^2(n)$

$E[e(n)] \rightarrow$ mean value of $e(n)$

for rounding process,

$$E[e(n)] = \frac{1}{q}\int_{-q/2}^{q/2} e(n) \cdot de$$

$$= \frac{1}{q}\left[\frac{e^2(n)}{2}\right]_{-q/2}^{q/2}$$

$$E[e(n)] = \frac{1}{2q}\left[\frac{q^2}{4} - \frac{q^2}{4}\right] = 0 \rightarrow \text{zero mean.}$$

$$E[e^2(n)] = \frac{1}{q}\int_{-q/2}^{q/2} e^2(n) de = \frac{1}{q}\left[\frac{e^3(n)}{3}\right]_{-q/2}^{q/2}$$

$$= \frac{1}{3q}\left[\frac{q^3}{8} + \frac{q^3}{8}\right] = \frac{2q^2}{24} = q^2/12$$

$$\sigma_e^2 = \frac{q^2}{12} - 0 = \boxed{\frac{q^2}{12} = \sigma_{e^2}} \quad \text{where } q = 2^{-b}$$

(or)

$$\therefore \boxed{\sigma_e^2 = \frac{2^{-2b}}{12}}$$

for 2's complement truncation process,

$$E[e(n)] = \frac{1}{q}\int_{-q}^{0} e(n)\,de$$

$$= \frac{1}{q}\left[\frac{e^2(n)}{2}\right]_{-q}^{0} = \frac{-q}{2}$$

$$E[e^2(n)] = \frac{1}{q}\int_{-q}^{0} e^2(n)\,de$$

$$= \frac{1}{q}\left[\frac{e^3(n)}{3}\right]_{-q}^{0} = \frac{q^2}{3}$$

$$\sigma_e^2 = \frac{q^2}{3} - \left[\frac{-q}{2}\right]^2 = \frac{q^2}{3} - \frac{q^2}{4} = \frac{q^2}{12} \text{ (or) } \frac{2^{-2b}}{12}$$

$$\boxed{\sigma_e^2 = \frac{q^2}{12} \text{ (or) } \frac{2^{-2b}}{12}}$$

$\sigma_e^2 \Rightarrow$ Steady State input quantization noise power.

here 'q' is the quantization step size and 'b' is the number of ADC bits. hence, the level of noise can be reduced by increasing the no. of bits (or) it is also possible to reduce it using multirate technique.

In general for values of 'b' above 12 bits, the noise due to quantization error is insignificant, except for applications such as professional audio where atleast 16 bits are required for acceptable performance.

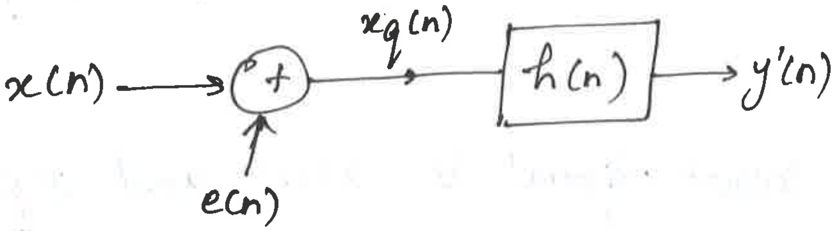
If the input signal is $x(n)$, and it's variance is σ_x^2 (i.e. signal power), then the ratio of signal power to noise power is known as SNR (Signal to Noise Ratio) which is given by,

$$\frac{\sigma_x^2}{\sigma_e^2} = \frac{\sigma_x^2}{2^{-2b}/12} = 12 \times 2^{2b} \times \sigma_x^2$$

$$(SNR)_{dB} = \underbrace{10.79}_{10\log 12} + \underbrace{6.02b}_{10\log 2^{2b}} + 10\log \sigma_x^2$$

From the above equation, we can say that SNR increases approximately 6dB for each bit added to register length.

The noise due to ADC quantization is fed into DSP system as an irreversible error. The noise power at the output of the DSP system, due to ADC, is known as ADC quantization noise at system output (or) steady state ~~noise~~ output noise power.



Representation of quantization noise in an LTI system.

$y'(n) = x_q(n) * h(n)$

$= [x(n) + e(n)] * h(n)$

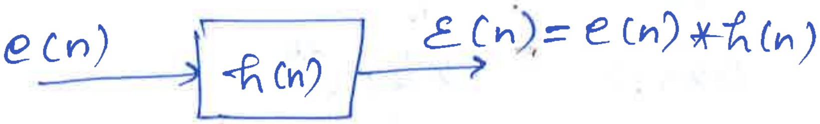
$= [x(n) * h(n)] + [e(n) * h(n)]$

$y'(n) = y(n) + \mathcal{E}(n)$

where $y'(n) \rightarrow$ o/p of the system due to i/p & error signal

$y(n) \rightarrow$ o/p due to input.

$\mathcal{E}(n) \rightarrow$ o/p due to error



$$\varepsilon(n) = e(n) * h(n)$$

$$\varepsilon(n) = \sum_{k=0}^{n} h(k) \cdot e(n-k)$$

The variance of any term in the above sum is $\sigma_e^2 \cdot h^2(n)$.

According to the property of variance,
The variance of the sum of independent random variable is the sum of their variances.

If the quantization errors are assumed to be independent at different sampling instances, the the variance of the output is given by,

$$\text{var}[\varepsilon(n)] = \text{var}\left[\sum_{k=0}^{n} h(k) \cdot e(n-k)\right]$$

$$\sigma_\varepsilon^2 = \sigma_e^2 \sum_{n=0}^{k} h^2(n)$$

To find the steady state variance, extend the limit k upto infinity, then

$$\boxed{\sigma_\varepsilon^2 = \sigma_e^2 \sum_{n=0}^{\infty} h^2(n)}$$

ADC quantization noise at the system output.
(or)
Steady state output noise power.

$\sum_{n=0}^{\infty} h^2(n)$ can be evaluated using Parseval's theorem

$$\sigma_{\varepsilon}^2 = \sigma_{OA}^2 = \sigma_e^2 \sum_{n=0}^{\infty} h^2(n)$$

$$\boxed{\sigma_{\varepsilon}^2 = \sigma_e^2 \left[\frac{1}{2\pi j} \oint_c H(z) \cdot H(z^{-1}) z^{-1} \, dz \right]}$$

$\Downarrow$

represents "system power gain" which amplifies (or alters) the ADC noise, depending on the characteristics of the DSP system.

The closed contour integration of above equation can be evaluated using residue theorem in Z-transform.

$$\sigma_{\varepsilon}^2 = \sigma_e^2 \sum_{i=1}^{N} \text{Residue}\left[H(z) \cdot H(z^{-1}) z^{-1}\right] \Big/_{z=p_i}$$

where $p_i \rightarrow$ ith pole of $H(z) \cdot H(z^{-1}) \cdot z^{-1}$

$N \rightarrow$ no. of poles in the system.

Only residues of the poles that lie inside the unit circle are considered.

pbm: Find the steady state variance of the noise in the output due to quantization of input for the first order filter

$$y(n) = a\,y(n-1) + x(n)$$

Solution:

$$\sigma_\varepsilon^2 = \sigma_e^2 \frac{1}{2\pi j} \oint_c H(z)\, H(z^{-1})\, z^{-1}\, dz$$

$$= \sigma_e^2 \sum_{i=1}^{N} \text{residue}\left[H(z)\,H(z^{-1})\,z^{-1}\right]\Big/_{z=p_i}$$

$H(z) = ?$

$$H(z) = \frac{Y(z)}{X(z)}$$

Take z transform of given difference equation,

$$Y(z) = a\, z^{-1}\, Y(z) + X(z)$$

$$Y(z)\left[1 - a z^{-1}\right] = X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

$$H(z^{-1}) = \frac{z^{-1}}{z^{-1}-a}$$

poles of $H(z) \cdot H(z^{-1}) \cdot z^{-1} = \frac{\cancel{z}}{(z-a)} \cdot \frac{\cancel{z^{-1}}}{(z^{-1}-a)} \cdot z^{-1}$

poles are: $z = a$ & $z^{-1} - a = 0$

$z^{-1} = a, \quad z = 1/a$

let $a < 1$, $\frac{1}{a} > 1$

$\boxed{P_1 = a}$ $\boxed{P_2 = 1/a}$

$$\text{residue}\left[H(z)\,H(z^{-1})\,z^{-1}\right]\Big/_{z=P_1=a} = \frac{z^{-1}\cdot(z-a)}{(z-a)\,(z^{-1}-a)}\Big/_{z=a}$$

$$= \frac{\cancel{(z-a)}\;z^{-1}}{\cancel{(z-a)}\,(z^{-1}-a)}\Big/_{z=a} = \frac{z^{-1}}{z^{-1}-a}$$

$$= \frac{a^{-1}}{a^{-1}-a}$$

Since $P_2 = 1/a > 1$

$$\text{residue}\left[H(z)\,H(z^{-1})\,z^{-1}\right]\Big/_{z=P_2=1/a} = 0$$

$$\sigma_{\varepsilon}^2 = \sigma_e^2\cdot\cancel{\frac{1}{2\pi j}}\;\frac{a^{-1}}{a^{-1}-a}$$

Answer: $\boxed{\sigma_{\varepsilon^2} = \sigma_e^2\,\frac{1}{1-a^2}}$ where $\sigma_e^2 = \dfrac{2^{-2b}}{12}$

(or)

$$\sigma_{\varepsilon}^2 = \sigma_e^2\cdot\sum_{n=0}^{\infty} h^2(n)$$

Gn: $y(n) = a\,y(n-1) + x(n)$

~~if $x(n)=\delta(n)$, $h(n) \Leftarrow y(n)$~~ $H(z) = \dfrac{z}{z-a}$ (or) $\dfrac{1}{1-az^{-1}}$

~~$h(n)=1/a$~~ $h(n) = Z^{-1}\{H(z)\} = a^n u(n)$

$$\sigma_{\varepsilon}^2 = \sigma_e^2\sum_{n=0}^{\infty}\left(a^n u(n)\right)^2$$

$$= \sigma_e^2\cdot\sum_{n=0}^{\infty} a^{2n} = \sigma_e^2\sum_{n=0}^{\infty}(a^2)^n = \sigma_e^2\cdot\frac{1}{1-a^2}$$

Answer: $\boxed{\sigma_{\varepsilon^2} = \dfrac{2^{-2b}}{12}\times\dfrac{1}{1-a^2}}$

Pbm: The output signal of an ADC is passed through a first order LPF, with transfer function given by

$$H(z) = \frac{(1-a)z}{z-a} \quad \text{for } 0<a<1$$

Find the steady state output noise power due to quantization at the output of the digital filter.

Solution: $$\sigma_\varepsilon^2 = \sigma_e^2 \left\{ \frac{1}{2\pi j} \oint_C H(z) \cdot H(z^{-1}) z^{-1} dz \right\}$$

Gn: $H(z) = \dfrac{(1-a)z}{z-a}$, $H(z^{-1}) = \dfrac{(1-a)z^{-1}}{z^{-1}-a}$

$$H(z) \cdot H(z^{-1}) \cdot z^{-1} = \frac{(1-a)\cancel{z}}{(z-a)} \frac{(1-a)\cancel{z^{-1}}}{(z^{-1}-a)} \cdot z^{-1}$$

$$= \frac{(1-a)^2 z^{-1}}{(z-a)(z^{-1}-a)}$$

Poles are $z=a$, $z=1/a$

$P_1 = a < 1$, $P_2 = \frac{1}{a} > 1$

$\therefore$ Residue at $P_2 = 0$

Residue of $H(z) \cdot H(z^{-1}) \cdot z^{-1}$ at $z=a$ is $= \left. \dfrac{\cancel{(z-a)}(1-a)^2 z^{-1}}{\cancel{(z-a)}(z^{-1}-a)} \right|_{z=a}$

$$= \frac{(1-a)^2 a^{-1}}{(a^{-1}-a)}$$

$$= \frac{(1-a)^2 \cancel{a^{-1}}}{\cancel{a^{-1}}(1-a^2)} = \frac{(1-a)^2}{1-a^2}$$

$$\sigma_\varepsilon^2 = \sigma_e^2 \times \frac{(1-a)^2}{1-a^2} = \sigma_e^2 \frac{(1-a)\cancel{(1-a)}}{(1+a)\cancel{(1-a)}}$$

$$= \sigma_e^2 \frac{(1-a)}{(1+a)} = \boxed{\frac{2^{-2b}}{12} \times \frac{1-a}{1+a} = \sigma_\varepsilon^2}$$