Unit - IV

Rectangular W.G's

Waves between $\|^{d}$ planes, characteristics of TE, TM and TEM waves. velocities of propagation, Solution of wave equation in Rectangular guides TE and TM modes, Dominent Mode, attenuation, Mode excitation, Dielectric slab wave guides - Numerical Examples.

Guided waves

Waves that are guided along or over conducting or dielectric surfaces.

Eg:- * waves along ordinary $\|^{d}$ wire &

* co-axial transmission lines.

* waves in wave guides.

The study of such guided waves will now be undertaken.

Waves Between parallel planes:

* consider an Electromagnetic wave, propagating b/w a pair of $\|^{d}$ perfectly conducting planes of infinite extent in Y and Z - directions, as shown below.

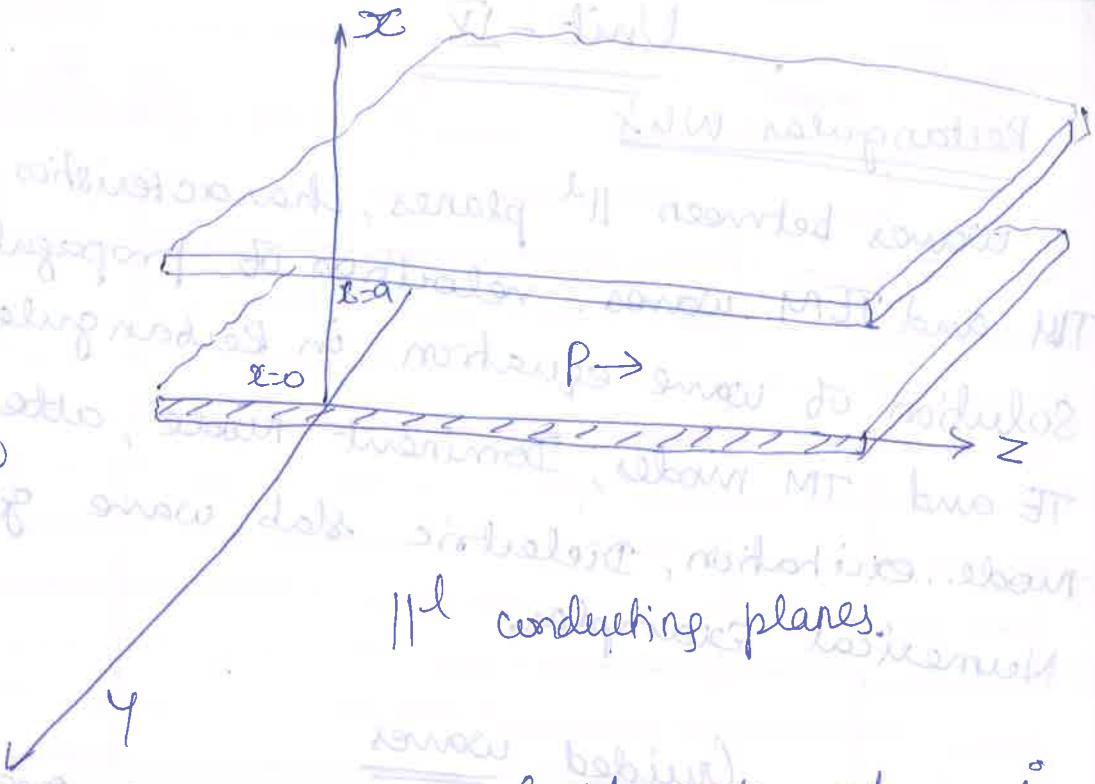


Fig ①

||l conducting planes.

* To determine the EM field configurations in the region between the planes, Maxwell's equ's will be solved with appropiate boundary conditions.

* for perfectly conducting planes, the boundary conditions are, $E_{tangential} = 0$ & $H_{normal} = 0$ *

The Maxwell's curl equation is,

$$\left.\begin{aligned} \nabla \times H &= (\sigma + j\omega\epsilon)E \\ \nabla \times E &= -j\omega\mu H \end{aligned}\right\} \quad (1)$$

Wave Equ's are,

$$\left.\begin{aligned} \nabla^2 E &= \gamma^2 E \\ \nabla^2 H &= \gamma^2 H \end{aligned}\right\} \quad (2)$$

where $\gamma = \sqrt{(\sigma + j\omega\varepsilon)(j\omega\mu)}$ ——— (3)

In rectangular co-ordinates, & non-conducting region between the planes, these eqn's becomes,

$$\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = j\omega\varepsilon E_x \quad ; \quad \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = -j\omega\mu H_x$$

$$\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = j\omega\varepsilon E_y \quad ; \quad \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z \quad ; \quad \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -j\omega\mu H_z$$

$$\frac{\partial^2 E}{\partial x^2} + \frac{\partial^2 E}{\partial y^2} + \frac{\partial^2 E}{\partial z^2} = -\omega^2\mu\varepsilon E \quad \text{——— (4)}$$

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} + \frac{\partial^2 H}{\partial z^2} = -\omega^2\mu\varepsilon H \quad \text{——— (5)}$$

It is assumed that,

* Propagation is in z-direction
* Variation of all fields in this direction may be expressed in the form of $e^{-\bar{\gamma} z}$

where $\bar{\gamma} = \bar{\alpha} + j\bar{\beta}$ ——— (6)

$\bar{\gamma} \rightarrow$ complex propagation constant.

* when the time variation factor is combined with the z variation factor, it is seen that,

$$e^{j\omega t} \cdot e^{-\bar{\gamma} z} = e^{(j\omega t - \bar{\gamma} z)} = e^{-\bar{\alpha} z} e^{j(\omega t - \bar{\beta} z)} \quad (7)$$

Equ (7) represents a wave propagating in the z-direction.

(1) when $\bar{\alpha} = 0$, $\bar{\gamma}$ = imaginary number.
then Equ (4) represents wave without attn.

(2) when $\bar{\beta} = 0$ $\bar{\gamma}$ = real number
then, there is no wave motion, but only an exponential decrease in amplitude.

* The space b/w the planes is infinite in y-direction, there is no boundary condn. to be met in this direction.

∴ it is assumed that the field is uniform or const. in y direction.

ie the derivative w.r. to y in (4) is zero.

* In x direction there are certain boundary condn's which must be met.

* when the variation in the z-direction of each of the field components is,

$$H_y = H_y^0 e^{-\bar{\gamma} z}$$

$$\frac{\partial H_y}{\partial z} = -\bar{\gamma} \cdot H_y^0 e^{-\bar{\gamma} z} = -\bar{\gamma} H_y$$

Similar results for z derivative of the other components. & making y derivative of any component is zero, equ (4) & (5) becomes,

$$-\frac{\partial H_y}{\partial z} = \bar{\gamma} H_y = j\omega\varepsilon_1 E_x$$

$$\bar{\gamma} E_y = -j\omega\mu H_x \qquad E_y = -\frac{j\omega\mu H_x}{\bar{\gamma}}$$

$$-\bar{\gamma} H_x - \frac{\partial H_z}{\partial x} = j\omega\varepsilon_1 E_y = j\omega\varepsilon_1 \cdot \left(\frac{-j\omega\mu H_x}{\bar{\gamma}}\right)$$

$$-\frac{\partial H_z}{\partial x}\bar{\gamma} = (\omega^2\mu\varepsilon_1 + \bar{\gamma}^2) H_x$$

$$-\bar{\gamma} E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu H_y$$

$$\frac{\partial H_y}{\partial x} = j\omega\varepsilon_1 E_z$$

$$\frac{\partial E_y}{\partial x} = -j\omega\mu H_z \quad \text{—— (8)}$$

E_x, E_y, H_y, H_x

$$\frac{\partial^2 E}{\partial x^2} + \bar{\gamma}^2 E = -\omega^2\mu\varepsilon E \quad \text{—— (9)}$$

$$\frac{\partial^2 H}{\partial x^2} + \bar{\gamma}^2 H = -\omega^2\mu\varepsilon H$$

from (8), we have,

$$* \quad H_x = \frac{-\bar{\gamma}\,\partial H_z}{\partial x}\frac{1}{(\omega^2\mu\varepsilon+\bar{\gamma}^2)}$$

$$H_y = \frac{j\omega\varepsilon E_x}{\bar{\gamma}}$$

$$-\bar{\gamma} E_x - \frac{\partial E_z}{\partial x} = -j\omega\mu\left(\frac{j\omega\varepsilon E_x}{\bar{\gamma}}\right)$$

$$-\frac{\partial E_z}{\partial x} = \frac{\omega^2\mu\varepsilon E_x}{\bar{\gamma}} + \bar{\gamma} E_x$$

$$-\bar{\gamma}\frac{\partial E_z}{\partial x} = E_x[\omega^2\mu\varepsilon + \bar{\gamma}^2]$$

$$E_x = \frac{-\bar{\gamma}}{h^2}\frac{\partial E_x}{\partial x}$$

* $H_x = \frac{-\bar{\gamma}}{h^2} \cdot \frac{\partial H_z}{\partial x}$ $\qquad E_x = -\frac{\bar{\gamma}}{h^2} \frac{\partial E_x}{\partial x}$

Substitute Ex + Hx, we have,

$$H_y = -\frac{j\omega\varepsilon}{h^2} \cdot \frac{\partial E_z}{\partial x} \qquad E_y = +\frac{j\omega\mu}{h^2} \cdot \frac{\partial H_z}{\partial x} \quad \text{—— (10)}$$

$$h^2 = \bar{\gamma}^2 + \omega^2\mu\varepsilon \quad \text{—— (11)}$$

Equ (10) shows various components of Electric and magnetic field strengths are expressed in terms of E_z and H_z. Also there must be z component of either E or H.

* In general E_z + H_z could be present.

* It is convenient & desirable to divide the soln's into two sets.

Set 1 :- There is a component of E in the direction of propagation E_z, but no component of H in this direction.

Such waves are called E waves or Transverse Magnetic (TM) waves. ∴ Magnetic field strength H is entirely transverse. $\boxed{H_z = 0 \quad E_z \neq 0}$

Set : 2 component of H in the direction of

Magnetic flux lines are ⊥r to the axis of the WG propagation, but no E_z component.

* Such waves are called Transverse Electric (TE) waves. $E_z = 0$ elect. lines of flux are ⊥r to the axis of the WG

Since the differential equations are linear, the sum of these two sets of solutions yields the most general solution.

Transverse Electromagnetic $\boxed{\text{For TEM} \Rightarrow E_z = 0 \quad H_z = 0}$

Transverse Electric Waves ($E_z = 0$)

Equ (10) shows that, when $\boxed{E_z = 0 \text{ but } H_z \neq 0}$, the field components H_y & E_x is also $= 0$. & non zero values for H_x & E_y.

* Since each of the field components obeys the wave equ., given by equ (9), the ~~at~~ for, the component wave equation can be written

E_y,
$$\frac{\partial^2 E_y}{\partial x^2} + \bar{\gamma}^2 E_y = -\omega^2 \mu \varepsilon E_y .$$

$$\frac{\partial^2 E_y}{\partial x^2} = -(\omega^2 \mu \varepsilon E_y + \bar{\gamma}^2 E_y)$$
$$= -E_y(\omega^2 \mu \varepsilon + \bar{\gamma}^2) = -h^2 E_y \quad \text{— (12)}$$

Recalling that, $E_y = E_y^0(x) \cdot e^{-\bar{\gamma} z}$, ∴ (12) becomes,

$$\frac{\partial^2 E_y^0}{\partial x^2} = -h^2 E_y^0 \quad \text{— (12) a.}$$

(12)a → is the differential equation of simple harmonic motion.

$D^2 - a^2 = 0$
$D^2 = \pm a$

* $E_y^0 = C_1 \sinh x + C_2 \cosh x$. —— (13)

where C_1 & C_2 are arbitrary constants.

Showing the variation with time & in the z-direction the expression for E_y is,

$$E_y = (C_1 \sinh hx + C_2 \cosh hx)e^{-\bar{\gamma} z} \quad \text{—— (13a)}$$

From Fig (1) & boundary conditions,

$$\left.\begin{array}{l} E_y = 0 \text{ at } x = 0 \\ E_y = 0 \text{ at } x = a \end{array}\right\} \text{For all values of } z$$

From the 1st of these condn's to be true, $C_2 = 0$,

$$\therefore E_y = C_1 \sin hx\, e^{-\bar{\gamma} z}$$

* Application of 2nd Boundary condn, imposes a restriction on h.

$\therefore$ E_y to be zero at $x = a$ for all values of z and t, it is necessary that,

* $$h = \frac{m\pi}{a} \quad \text{—— (14)}. \quad \text{where } m = 1, 2, 3 \ldots$$

$$\therefore E_y = C_1 \sin\left(\frac{m\pi}{a}x\right) e^{-\bar{\gamma} z} \quad \text{—— (15)}$$

The other components of E & H can be

obtained by sub (15) into (8), we have,

$\frac{\partial E_y}{\partial x} = C_1 \cos\left(\frac{m\pi}{a}\right) x e^{-\bar{\gamma} z} \cdot \left(\frac{m\pi}{a}\right)$

$$* E_y = C_1 \sin\left(\frac{m\pi}{a} x\right) e^{-\bar{\gamma} z}$$

$$H_z = \frac{-1}{j\omega\mu} \frac{\partial E_y}{\partial x}$$

$$H_z = \frac{-m\pi}{j\omega\mu a} C_1 \cos\left(\frac{m\pi}{a} \cdot x\right) e^{-\bar{\gamma} z}$$

$$H_x = \bar{\gamma} E_y \left(\frac{-1}{j\omega\mu}\right)$$

$$H_x = \frac{-\bar{\gamma}}{j\omega\mu} C_1 \sin\left(\frac{m\pi}{a} \cdot x\right) e^{-\bar{\gamma} z} \quad \text{--- (16)}$$

$m \rightarrow$ particular field configuration or mode

* The wave associated with integer m is TE_{mo} wave or TE_{mo} mode.

* The 2nd subscript varies with y

when $m=0$, all the fields identically zero.

$\therefore$ use $m=1$ in eqn (16). ie the lowest order mode that can exist in this case is TE_{10} mode.

Rewrite eqn (16). Rather than carry the factor $e^{-\bar{\gamma} z}$, use zero superscript notation,

$$E_y^0 = C_1 \sin \frac{\pi m}{a} \cdot x$$

$$H_z^0 = -\frac{m\pi}{j\omega\mu a} C_1 \cos \frac{\pi m}{a} \cdot x$$

$$H_x^0 = \frac{-\bar{\gamma}}{j\omega\mu} C_1 \sin \frac{m\pi}{a} \cdot x \quad \text{--- (17)}$$

$\bar{\gamma} \rightarrow$ propagation const. $= \bar{\alpha} + j\bar{\beta}$

$\bar{\alpha}$ = Attenuation constant

$\bar{\beta}$ = phase-shift constant.

When $\bar{\gamma} \rightarrow$ real, $\bar{\beta} = 0$, $\bar{\alpha}$ has value.

ie there is attn. but no phase shift.

$\therefore$ no wave motion.

when $\bar{\gamma} \rightarrow$ imaginary $\bar{\alpha} = 0$ $\bar{\beta}$ has value

ie there is propagation by wave motion without attenuation.

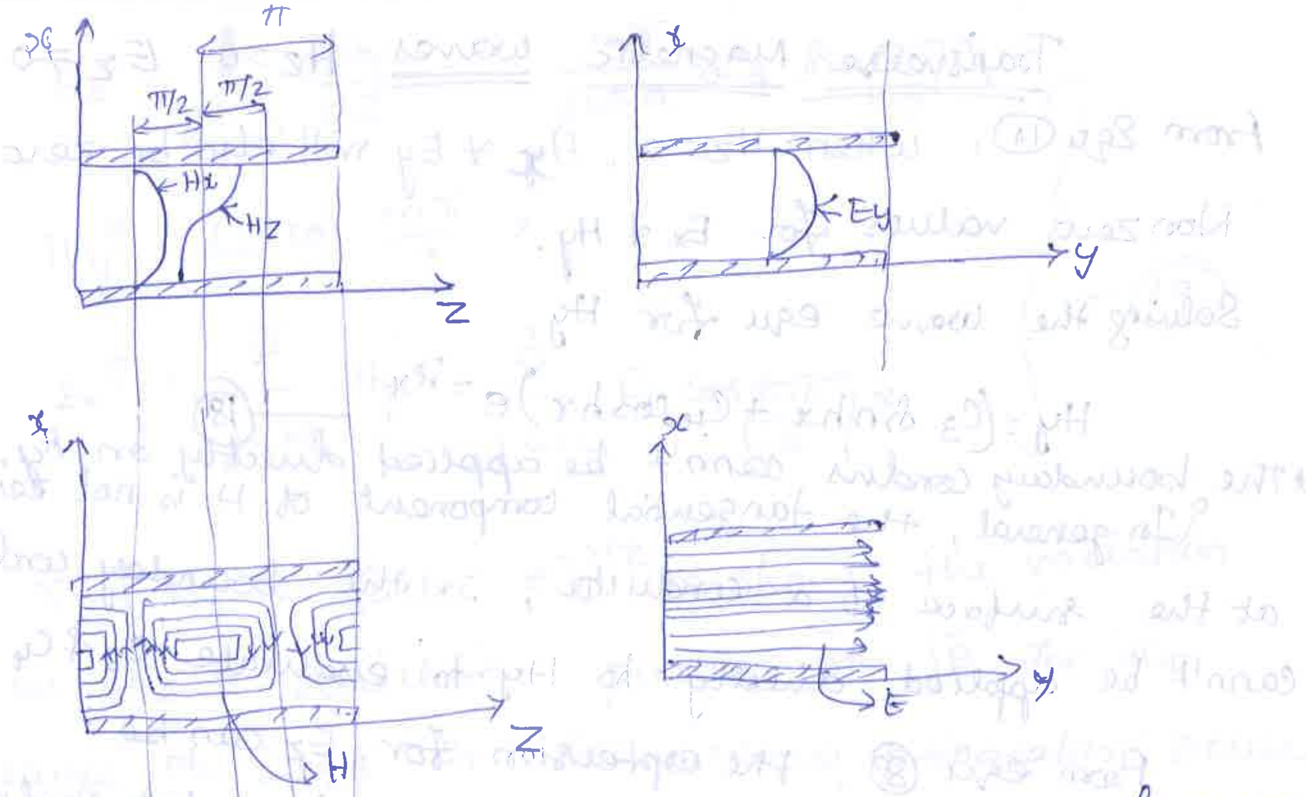
Sub. $\bar{\gamma} = j\bar{\beta}$ in Eqn (16) for TE_{m0} waves in the propagation range may be,

$$E_y = C_1 \sin \frac{m\pi}{a} x \cdot e^{-j\bar{\beta} z}$$

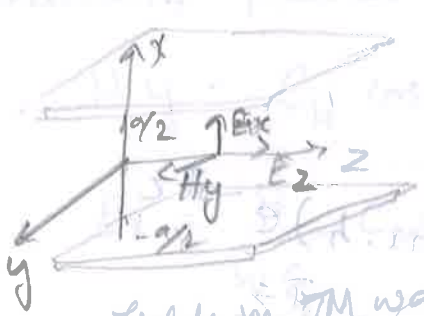
$$H_z = -\frac{m\pi}{j\omega\mu a} C_1 \cos\left(\frac{m\pi}{a} x\right) \cdot e^{-j\bar{\beta} z}$$

$$= \frac{jm\pi}{\omega\mu a} \cdot C_1 \cos\left(\frac{m\pi}{a} \cdot x\right) \cdot e^{-j\bar{\beta} z}$$

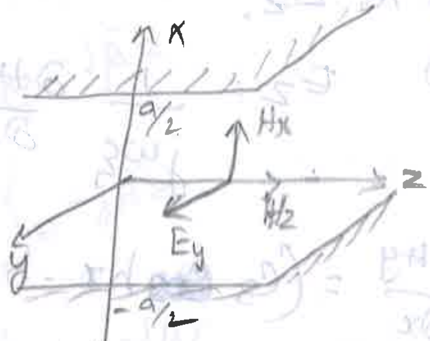
$$H_x = \frac{-\bar{\beta}}{\omega\mu} \cdot C_1 \sin\left(\frac{m\pi}{a} \cdot x\right) e^{-j\bar{\beta} z} \qquad (16)a$$



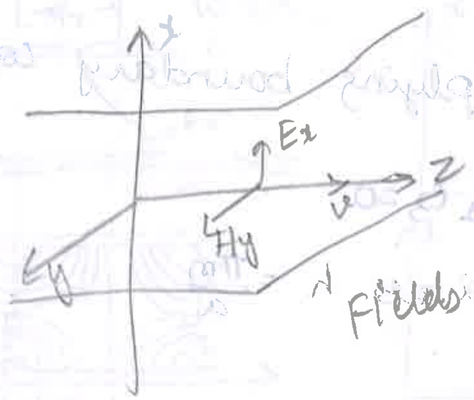
Electric and Magnetic fields between ||l planes for TE_{10} mode/wave.



Fields in TM wave



Fields in TE wave



Fields in TEM waves

Transverse Magnetic waves $H_z = 0 \quad E_z \neq 0$

From Equ (10), when $H_z = 0$, H_x & E_y will also be zero.

Non zero values for E_x & H_y.

Solving the wave equ for H_y,

$$H_y = (C_3 \sin hx + C_4 \cos hx) e^{-\bar{\gamma} x} \quad \text{——— (18)}$$

*The boundary condn's cann't be applied directly on H_y.

(In general, the tangential component of H is not zero at the surface of a conductor, ∴ the boundary condns cann't be applied directly to H_y to evaluate C_3 & C_4.

From equ (8), the expression for E_z can be obtained in terms of H_y, then the boundary condn's applied to E_z.

From (8) $\quad E_z = \frac{1}{j\omega\varepsilon} \frac{\partial H_y}{\partial x}$

$$\frac{\partial H_y}{\partial x} = (C_3 \cos hx \cdot h - C_4 \sin hx \cdot h) e^{-\bar{\gamma} z}$$

$$= h(C_3 \cos hx - C_4 \sin hx) e^{-\bar{\gamma} z}$$

$$\therefore E_z = \frac{h}{j\omega\varepsilon} \frac{\partial H_y}{\partial x}.$$

Applying boundary condn's,

ie $E_z = 0$ at $x = 0 \rightarrow C_3 = 0$

$E_z = 0$ at $x = 0$, requires $h = \frac{\pi m}{a}$

$$E_z^0 = \frac{1}{j\omega\varepsilon}\frac{\partial H_y}{\partial z} = \cancel{-}\frac{m\pi}{j\omega\varepsilon a} \cdot C_4 \sin\left(\frac{m\pi}{a}\right)x$$

$$H_y^0 = C_4 \cos\frac{m\pi}{a} \cdot x$$

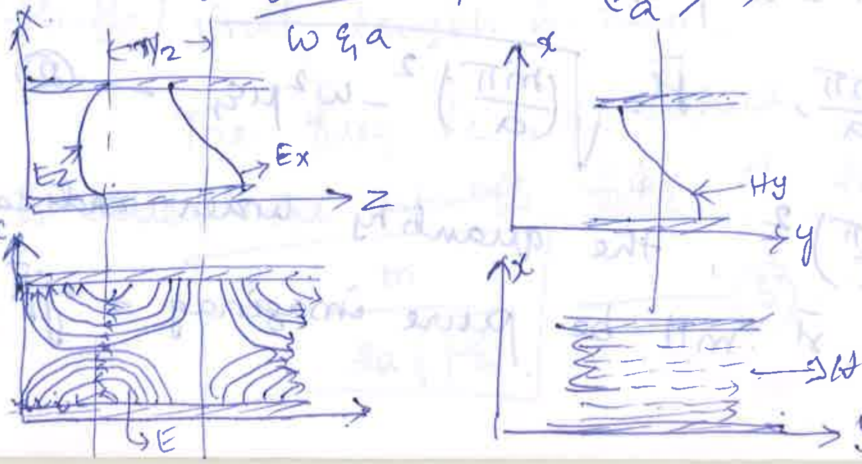
$$E_x^0 = \frac{\bar\gamma}{j\omega\varepsilon} \cdot H_y = \frac{\bar\gamma}{j\omega\varepsilon} \cdot C_4 \cos\left(\frac{m\pi}{a}\right)x \qquad \text{(19)}$$

X ing by the factor $e^{-\gamma z}$ to show the variation in the z-direction. putting $\bar\gamma = j\bar\beta$ for the range of freq's in which wave propagation occurs, the expressions for TM waves b/w ||el perfectly conducting planes are,

$$H_y = C_4 \cos\left(\left(\frac{\pi m}{a}\right)x\right)e^{-j\beta z}$$

$$E_x = \frac{\beta}{\omega\varepsilon} C_4 \cos\left(\left(\frac{m\pi}{a}\right)x\right)e^{-j\beta z}$$

$$E_z = \frac{j m\pi}{\omega\varepsilon a} C_4 \cdot \sin\left(\left(\frac{m\pi}{a}\right)x\right)e^{-j\beta z} \qquad \text{(19 a)}$$



TE_{10} b/w parallel planes

Characteristics of TE and TM waves

The properties can be studied by investigating the prop. const $\bar{\gamma}$ for these waves.

Examine the equ's 16 & 19 for TE & TM waves, shows that for each of the components of E or H. there is a sinusoidal or cosinusoidal standing wave distribution across the guide in x direction.

✓ Each of these components varies in magnitude, but not in phase, in the z direction.

* In y direction, by assumption, there is no variation of either magnitude or phase of any field components.

In the z direction, the velocity $\bar{v} = \frac{\omega}{\beta}$

where $\beta \rightarrow$ phase shift const in $\bar{\gamma}$

from 11 $h^2 = \bar{\gamma}^2 + \omega^2\mu\varepsilon_r$

$$\bar{\gamma} = \sqrt{h^2 - \omega^2\mu\varepsilon_1} \quad \text{--- (1)}$$

where $h = \frac{m\pi}{a}$, (14)→

$$\bar{\gamma} = \sqrt{\left(\frac{m\pi}{a}\right)^2 - \omega^2\mu\varepsilon_1} \quad \text{--- (2)}$$

$\omega^2\mu\varepsilon_1 > \left(\frac{m\pi}{a}\right)^2$, the quantity under radical will be -ve & $\bar{\gamma}$ will be pure imaginary $= +j\bar{\beta}$

$$\beta = \sqrt{\omega^2 \mu \epsilon_1 - \left(\frac{m\pi}{a}\right)^2} \quad \text{--- (3)}$$

Under these condn's the fields will progress in the z direction as waves, & the attenuation of these waves will be zero.

As freq ↓sed, the critical freq $f_c = \frac{\omega_c}{2\pi}$ will be reached at which, $4\pi^2 f_c^2 \mu \epsilon_1 = \frac{m^2\pi^2}{a^2}$

$$\omega_c^2 \mu \epsilon_1 = \left(\frac{m\pi}{a}\right)^2 \quad \text{--- (4)}$$

$f_c^2 = \frac{m^2\pi^2}{a^2 \cdot 4\pi^2 \cdot \mu\epsilon_1}$

$f_c = \frac{m}{2a\sqrt{\mu\epsilon_1}}$

* For all freq's $< f_c$, the quantity under the radical will be +ve & the prop. const γ will be a real no.

ie α will have value & $\beta = 0$.

ie the fields will be attenuated exponentially in the z direction

* There will be no wave motion, since phase shift / unit length is zero.

The freq f_c, at which wave motion ceases ~~causes~~ is called the cut-off freq of the guide.

$$\boxed{f_c = \frac{m}{2a\sqrt{\mu\epsilon_1}}} \quad \text{--- (5)}$$

* for each value of m, there is a corresponding cut-off freq, below which wave propagation cann't occur.

$\boxed{f_c \text{ defined as the freq. at which prop. const. changes from real to imaginary}}$

* Above the cut-off freq, wave prop does occur & attn is zero. (for perfectly conducting planes)

* β varies from 0 ~~at~~ f_c upto value of $\omega\sqrt{\mu\varepsilon}$ as the freq approaches infinity.

* The distance required ~~for~~ the phase to shift thro' 2π radians is a wavelength.

$$\therefore \lambda = \frac{2\pi}{\beta} \quad \text{———} \; (6)$$

* The wave or phase velocity is,

$$\bar{u} = \lambda f = \frac{2\pi f}{\beta} = \frac{\omega}{\beta} \quad \text{———} \; (7) \qquad \beta = \frac{\omega}{u}$$

(3)d

Sub (7) into (6),

$$\bar{\lambda} = \frac{2\pi}{\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2}} \quad \text{———} \; (8)$$

$$\bar{u} = \frac{\omega}{\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2}} \quad \text{———} \; (9)$$

* It is seen that at ~~ab~~ cut-off freq both $\bar{\lambda}$ & $\bar{u}$ are infinitely large.

As freq rises above f_c, $\bar{v}$ decreases from this very large value, it approaches lower limit,

$$\bar{v} = v_0 = \frac{1}{\sqrt{\mu \varepsilon}} \quad \text{——— (10)}$$

* The freq becomes high, so that $\left(\frac{m\pi}{a}\right)^2$ is negligible compared with $\omega^2 \mu \varepsilon$.

* when the dielectric medium b/w the plates is air, μ & ε have their free space values μ_0 & ε_0, the lower limit velocity is just free space velocity **c**.

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \simeq 3 \times 10^8 \text{ m/sec} \quad \text{——— (11)}$$

$\therefore$ The phase velocity of waves varies from velocity of light c in free space to an infinitely large value as the freq is tried from extremely high to cut-off freq f_c.

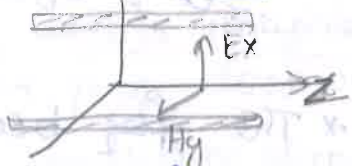
The wave velocity or phase velocity is different from the velocity with which the energy propagates.

* EF is totally along the x axis ie $E_y = E_z = 0$

Transverse Electromagnetic waves (TEM)

* MF is totally along the y axis ie $H_z = H_x = 0$
* The Electromagnetic field is entirely transverse

$E_z = 0 \quad H_z = 0$



* Also called principal wave.

The $\bar{\gamma}$, $\bar{\beta}$, $\bar{v}$ and $\bar{\lambda}$ reduce to, (for m=0)

$\bar{\gamma} = \sqrt{\left(\frac{m\pi}{a}\right)^2 - \omega^2\mu_0\varepsilon_0}$

$\bar{\gamma} = \gamma = j\omega\sqrt{\mu_u\varepsilon_u}$

$\alpha = 0 \quad \bar{\beta} = \beta = \omega\sqrt{\mu_u\varepsilon_u}$

velo. of propagation $\bar{v} = v = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu_u\varepsilon_u}} = C$

$\frac{2\pi}{\beta} = \bar{\lambda} = \lambda = \frac{2\pi}{\omega\sqrt{\mu_u\varepsilon_u}} = C/f$

(19a) becomes,

$H_y = C_4 e^{-j\bar{\beta}z}$

$E_x = \frac{\bar{\beta}}{\omega\varepsilon_u} C_4 \cdot e^{-j\bar{\beta}z}$ (2)

$E_z = 0$ (1)

const. in ampl. across a cross section normal to the direction of propagation.

velocity of TEM wave is independent of freq. f is familiar to $C = 3\times10^8$ m/sec. ie only when the planes are perfectly conducting and the space b/w them is vaccum.

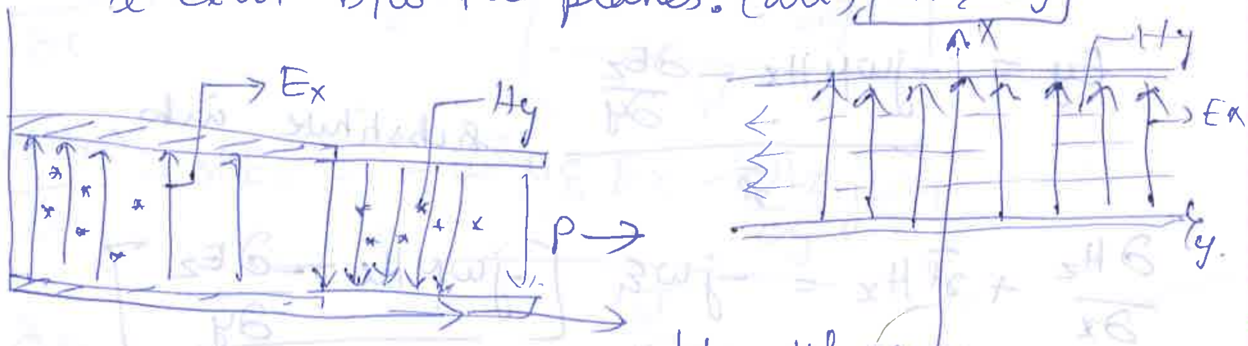
* From equ (5), The cut-off freq. for the TEM waves is zero. $f_c = \frac{m}{2a\sqrt{\mu\varepsilon}}$ $m=0$

ie for TEM waves all freq's down to zero can propagate along the guide.

* The ratio of E to H b/w $\parallel^{l}$ planes for traveling wave is, characteristics wave imp

$$\left.\frac{E_x}{H_y}\right) = \frac{E_x}{H_y} = \frac{\bar{\beta}}{\omega\varepsilon_u} = \sqrt{\frac{\mu_u}{\varepsilon_u}} = \eta \quad (2)$$

$\eta \rightarrow$ intrinsic impedance of free space.
ie exist b/w the planes. (air), $\boxed{E_x = \eta H_y}$



TEM waves b/w $||^l$ planes

Rectangular Guides

For rectangular guides, Maxwell's eqn's and the wave equations are expressed in Rectangular co-ordinates and the soln's exactly for waves b/w $||^l$ planes.

The variation in z direction is expressed as $e^{-\bar{\gamma} z}$ where $\bar{\gamma} = \bar{\alpha} + j\bar{\beta}$, Maxwell's eqn's becomes,

from (4),
$$\frac{\partial H_z}{\partial y} + \bar{\gamma} H_y = j\omega\varepsilon E_x \qquad \frac{\partial E_z}{\partial y} + \bar{\gamma} E_y = -j\omega\mu H_x$$

$$\frac{\partial H_z}{\partial x} + \bar{\gamma} H_x = -j\omega\varepsilon E_y \qquad \frac{\partial E_z}{\partial x} + \bar{\gamma} E_x = j\omega\mu H_y$$

$$\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = j\omega\varepsilon E_z \qquad \frac{\partial E_y}{\partial x} - \frac{\partial E_z}{\partial y} = -j\omega\mu H_z \quad \text{— (1)}$$

and the wave eqn's for E_z & H_z are,

$$\left.\begin{aligned} \frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \bar{\gamma}^2 E_z &= -\omega^2\mu\varepsilon E_z \\ \partial^2 H_z/\partial x^2 + \frac{\partial^2 H_z}{\partial y^2} + \bar{\gamma}^2 H_z &= -\omega^2\mu\varepsilon H_z \end{aligned}\right\} \text{— (2)}$$

Combine equ ①, we have,

$$E_y = \frac{-j\omega\mu H_x - \frac{\partial E_z}{\partial y}}{\bar{\gamma}}$$ substitute into,

$$\frac{\partial H_z}{\partial x} + \bar{\gamma} H_x = -j\omega\varepsilon_1 \left[\frac{-j\omega\mu H_x - \frac{\partial E_z}{\partial y}}{\bar{\gamma}}\right]$$

$$= \frac{-\omega^2\mu\varepsilon_1 H_x + j\omega\varepsilon_1 \frac{\partial E_z}{\partial y}}{\bar{\gamma}}$$

$$\frac{\partial H_z}{\partial x} = \frac{-\omega^2\mu\varepsilon_1 H_x + j\omega\varepsilon_1\frac{\partial E_z}{\partial y} - \bar{\gamma}^2 H_x}{\bar{\gamma}}$$

$$= \frac{-H_x\left[\bar{\gamma}^2 + \omega^2\mu\varepsilon_1\right] + j\omega\varepsilon_1\frac{\partial E_z}{\partial y}}{\bar{\gamma}}$$

$$\bar{\gamma}\frac{\partial H_z}{\partial x} - j\omega\varepsilon_1\frac{\partial E_z}{\partial y} = -H_x h^2$$

$$H_x h^2 = j\omega\varepsilon_1\frac{\partial E_z}{\partial y} - \bar{\gamma}\frac{\partial H_z}{\partial x}$$

$$\boxed{H_x = j\frac{\omega\varepsilon_1}{h^2}\frac{\partial E_z}{\partial y} - \frac{\bar{\gamma}}{h^2}\frac{\partial H_z}{\partial x}} \quad \text{— ③ a}$$

$$H_y = \frac{j\omega\varepsilon_1 E_x - \frac{\partial H_z}{\partial y}}{\bar{\gamma}}$$ substitute into,

$$\frac{\partial E_x}{\partial x} + \bar{\gamma} E_x = j\omega\mu \left(\frac{j\omega\varepsilon E_x - \frac{\partial H_z}{\partial y}}{\bar{\gamma}} \right)$$

$$\frac{\partial E_x}{\partial x} + \bar{\gamma} E_x = \frac{-\omega^2 \mu\varepsilon E_x - \frac{\partial H_z}{\partial y} j\omega\mu}{\bar{\gamma}}$$

$$\frac{\partial E_x}{\partial x} = \frac{-\omega^2 \mu\varepsilon E_x - \frac{\partial H_z}{\partial y} j\omega\mu - \bar{\gamma}^2 E_x}{\bar{\gamma}}$$

$$\bar{\gamma}\frac{\partial E_x}{\partial x} = \boxed{-E_x\left[\bar{\gamma}^2 + \omega^2\mu\varepsilon\right] - j\omega\mu\frac{\partial H_z}{\partial y}}$$

$$\bar{\gamma}\frac{\partial E_x}{\partial x} = -E_x h^2 - j\omega\mu\frac{\partial H_z}{\partial y}$$

$$E_x h^2 = -\bar{\gamma}\frac{\partial E_x}{\partial x} - j\omega\mu\frac{\partial H_z}{\partial y}$$

$$\boxed{E_x = \frac{-\bar{\gamma}}{h^2}\frac{\partial E_x}{\partial x} - j\frac{\omega\mu}{h^2}\frac{\partial H_z}{\partial y}} \longrightarrow 3b$$

$$\bar{\gamma} E_x = j\omega\mu H_y - \frac{\partial E_z}{\partial x}$$

$$E_x = \frac{j\omega\mu H_y - \frac{\partial E_z}{\partial x}}{\bar{\gamma}}$$

Substitute into

$$\frac{\partial H_z}{\partial y} + \bar{\gamma} H_y = j\omega\varepsilon \left(\frac{j\omega\mu H_y - \frac{\partial E_z}{\partial x}}{\bar{\gamma}} \right)$$

$$\frac{\partial H_z}{\partial y} = \frac{-\omega^2 \mu \varepsilon H_y - j\omega\varepsilon \cdot \frac{\partial E_z}{\partial x} - \bar{\gamma}^2 H_y}{\bar{\gamma}}$$

$$\bar{\gamma}\frac{\partial H_z}{\partial y} = -H_y(\bar{\gamma}^2 + \omega^2\mu\varepsilon) - j\omega\varepsilon \cdot \frac{\partial E_z}{\partial x}$$

$$H_y h^2 = -\bar{\gamma}\frac{\partial H_z}{\partial y} - j\omega\varepsilon \cdot \frac{\partial E_z}{\partial x}$$

$$\boxed{H_y = \frac{-\bar{\gamma}}{h^2}\frac{\partial H_z}{\partial y} - j\frac{\omega\varepsilon}{h^2}\frac{\partial E_z}{\partial x}} \quad \text{--- } 3c$$

$$\bar{\gamma} H_x = -j\omega\varepsilon E_y - \frac{\partial H_z}{\partial x}$$

$$H_x = \frac{-j\omega\varepsilon E_y - \frac{\partial H_z}{\partial x}}{\bar{\gamma}}$$ substitute into,

$$\frac{\partial E_z}{\partial y} + \bar{\gamma} E_y = -j\omega\mu\left(\frac{-j\omega\varepsilon E_y - \frac{\partial H_z}{\partial x}}{\bar{\gamma}}\right)$$

$$= \frac{-\omega^2\mu\varepsilon E_y + \frac{\partial H_z}{\partial x} j\omega\mu}{\bar{\gamma}}$$

$$\frac{\partial E_z}{\partial y} = \frac{-\omega^2\mu\varepsilon E_y + \frac{\partial H_z}{\partial x} j\omega\mu - \bar{\gamma}^2 E_y}{\bar{\gamma}}$$

$$\frac{\partial E_z}{\partial y} = \frac{-E_y\left[\bar{\gamma}^2 + \omega^2\mu\varepsilon\right] + j\omega\mu\frac{\partial H_z}{\partial x}}{\bar{\gamma}}$$

$$\bar{\gamma}\frac{\partial E_z}{\partial y} = -E_y h^2 + j\omega\mu\frac{\partial H_z}{\partial x}$$

$$\boxed{E_y = -\frac{\bar{\gamma}}{h^2}\frac{\partial E_z}{\partial y} + j\frac{\omega\mu}{h^2}\frac{\partial H_z}{\partial x}} \quad \text{--- (3)d}$$

where $h^2 = \bar{\gamma}^2 + \omega^2\mu\varepsilon$

These eqns give the relationships among the fields within the guide.

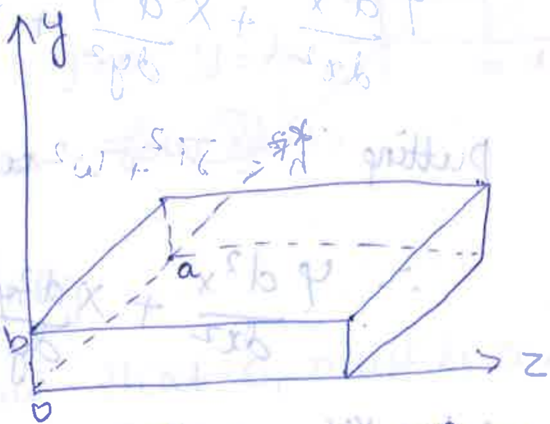
If E_z & H_z are zero, all the fields within the guide will vanish.

∴ For waveguide tr-ion, there must exist either E_z or H_z component.

The boundary condn's are,

$E_x = E_z = 0$ at $y=0$ & $y=b$

$E_y = E_z = 0$ at $x=0$ & $x=a$



Rectangular Guide

Transverse Magnetic Waves in Rectangular Guides TM

The wave eqn (2) are partial differential eqns, that can be solved by the technique of product solution.

This procedure leads to two ordinary differential eqns

$$E_z(x,y,z) = E_z^{\circ}(x,y)\, e^{-\bar{\gamma} z}$$

Let $E_z^{\circ} = XY$ ——— (4)

where X is a fn of x alone

Y is a fn. of y alone.

Sub (4) into (2) we have,

$$Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} + \bar{\gamma}^2 XY = -\omega^2\mu\varepsilon XY$$

putting $h^2 = \bar{\gamma}^2 + \omega^2\mu\varepsilon$,

$$\therefore\ Y\frac{d^2X}{dx^2} + X\frac{d^2Y}{dy^2} + h^2 XY = 0$$

$\div$ by XY, $\frac{1}{X}\frac{d^2X}{dx^2} + h^2 = -\frac{1}{Y}\frac{d^2Y}{dy^2}$ ——— (5)

Equ (5) equates a fn. of x alone to a fn. of y alone. For all values of x & y is to have each of these fns equal to some const, say A^2

$$\frac{1}{X}\frac{d^2X}{dx^2} + h^2 = A^2 \quad \text{--- (6)}$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -A^2 \quad \text{--- (7)}$$

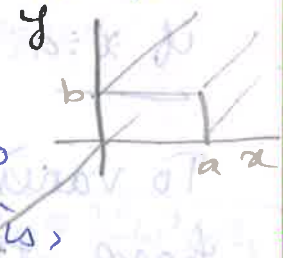
Soln. of (6) becomes, $X = C_1 \cos Bx + C_2 \sin Bx$

where $B^2 = h^2 - A^2$

Soln. of Equ (7) is, $Y = C_3 \cos Ay + C_4 \sin Ay$

This gives $E_z^0 = XY = C_1C_3 \cos Bx . \cos Ay + C_1C_4 \sin Bx \sin Ay$
$+ C_2C_3 \sin Bx \cos Ay + C_2C_4 \sin Bx . \sin Ay$ --- (8)

$C_1\ C_2\ C_3\ C_4$, A, & B must be selected to fit the boundary condns,



$E_z^0 = 0$, when $x = 0$, $x = a$, $y = 0$, $y = b$

if $x = 0$, the general expression for (8) is,

$$E_z^0 = C_1C_3 \cos Ay + C_1C_4 \sin Ay$$

* For E_z^0 to vanish, it is evident that C_1 must be zero.

The general Expression for E_z^0 will be,

(A) $E_z^0 = C_2\ C_3 \sin Bx . \cos Ay + C_2C_4 \sin Bx . \sin Ay$ --- (9)

when $y = 0$, equ (9) becos to

$E_z^0 = C_2C_3 \sin Bx$

* For this to be zero, either C_2 or $C_3 = 0$ for all values of x.

* Putting $C_2 = 0$ in (9) E_z^0 is identically zero. So instead $C_3 = 0$.

Then (9) for E_z^0 reduces to,

$$E_z^0 = C_2 C_4 \sin Bx \sin Ay \quad \text{—— (10)}$$

$$= C \sin Bx \cdot \sin Ay \qquad C - \text{amp. const.}$$

there are still two consts A & B.

∴ two more boundary condn's to be applied.

if $x = a$ $\quad E_z^0 = C \sin Ba \sin Ay$

To vanish all values of y, the const B must have the value $B = \frac{m\pi}{a}$, when $m = 1, 2, 3 \ldots$

Again if $y = b$ $\quad E_z^0 = C \sin \frac{m\pi}{a} x \sin Ab$.

for this to vanish for all values of x, A must have the value, $A = \frac{n\pi}{b}$, $n = 1, 2, 3 \ldots$

∴ The final Expression for $E_z^0 = C \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y$ —— (11)

Using Eqn (3) & $\bar{\gamma} = j\bar{\beta}$, for freq's above the cut-off freq, following expressions are obtained.

$$E_x^0 = \frac{-j\bar{\beta} C}{h^2} B \cos Bx \sin Ay$$

$$E_y^0 = \frac{-j\bar{\beta} C}{h^2} A \sin Bx \cos Ay$$

$$H_x^0 = \frac{j\omega \varepsilon_1 C}{h^2} A \sin Bx \cos Ay$$

$$H_y^0 = -\frac{j\omega \varepsilon_1 C}{h^2} B \cos Bx \cdot \sin Ay \longrightarrow (12)$$

where $B = \frac{m\pi}{a}$ & $A = \frac{n\pi}{b}$ — (13)

(12) & (13) show how each of the components of E & M field strength varies with x and y.

a & b → width & height m & n → integers.

By definition

$A^2 + B^2 = h^2$ $h^2 = \bar{\gamma}^2 + \omega^2 \mu \varepsilon_1$

$$\bar{\gamma} = \sqrt{h^2 - \omega^2 \mu \varepsilon_1} = \sqrt{A^2 + B^2 - \omega^2 \mu \varepsilon_1} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 - \omega^2 \mu \varepsilon_1} \quad (14)$$

(14) → prop. constant for rectangular wg. for TM waves.

For low freq's, $\omega^2 \mu \varepsilon_1$ is small $\bar{\gamma}$ → real no. ie $j\beta = 0$

* ∴ no wave motion along the tube for LF's.

* When freq rises, $\omega \geq \omega_c$, $\bar{\gamma}$ will be imaginary ie $\bar{\alpha} = 0$

for these freq's, $\bar{\beta} = \sqrt{\omega^2 \mu \varepsilon_1 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$ — (15)

$$\omega_c = \frac{1}{\sqrt{\mu \varepsilon_1}} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \quad (16)$$

The cut-off freq, ie the freq below which wave propagation will not occur.

$$f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2} \quad \text{---} (17)$$

Cut-off wavelength is, $\lambda_c = \dfrac{2}{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$ — (18)

ie $f_c \lambda_c = v_0$ $\qquad f = \dfrac{c}{\lambda_c}$

* The velocity of wave propagation, ie

$$\bar{v} = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \quad \text{---} (19)$$

* From (19), the velocity of prop. of wave in the guide > the phase velocity ~~of~~ in free space.

* As freq ↑ses, above cut-off, the phase velocity ↓ses from infinitely large value, & approaches c.

* WL in the guide,

$$\bar{\lambda} = \frac{2\pi}{\sqrt{\omega^2\mu\varepsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \quad \text{---} (20)$$

From (12) & (13) Either m or n = 0, fields will be identically zero. *A

∴ the lowest possible value for m or n (for TM) is unity. ie TM_{11} wave. Higher order waves require higher freq's to propagate along a guide of given dimensions.

Transverse Electric waves in Rectangular Guides

$H_z^0 \rightarrow$ Same general form as (8)

Applying boundary conditions;

$$H_z^0 = C \cos Bx \cos Ay$$

$$H_x^0 = \frac{j\bar{\beta}}{h^2} CB \sin Bx \cos Ay$$

$$H_y^0 = \frac{j\bar{\beta}}{h^2} CA \cos Bx \sin Ay$$

$$E_x^0 = \frac{j\omega\mu}{h^2} CA \cos Bx \sin Ay$$

$$E_y^0 = \frac{-j\omega\mu}{h^2} CB \sin Bx \cdot \cos Ay \qquad \text{—— (21)}$$

$$B = \frac{m\pi}{a} \qquad A = \frac{n\pi}{b}$$

$\bar{\gamma} = j\bar{\beta} \rightarrow$ For above cut-off.

For TE waves, $\bar{\beta}, f_c, \lambda_c, \bar{v}$ & λ are found to be identical to TM waves.

* found possible m & n values without causing all the fields to vanish.

The lowest order TE wave in Rectangular WG is $TE_{10} \rightarrow$ which has lowest cut off freq is called dominent (mode) wave.

for TE_{10} wave, sub. $m=1$ $n=0$

$m, n \rightarrow$ no. of half period variations of the field along x, & y coordinates.

$$H_z^0 = C\cos\frac{\pi x}{a}$$

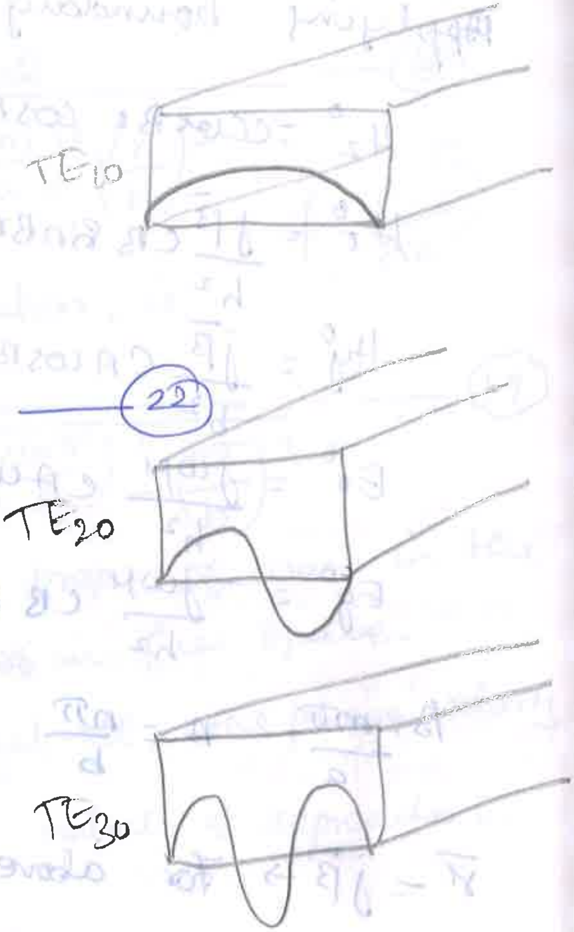
$$H_x^0 = \frac{j\bar{\beta} a C}{\pi}\sin\frac{\pi x}{a}$$

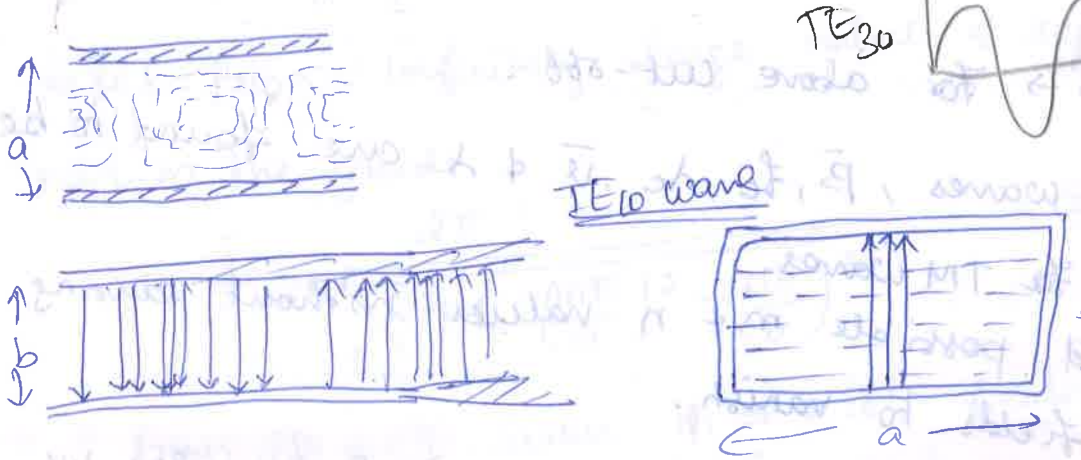
$$E_y^0 = \frac{-j\omega\mu a C}{\pi}\sin\frac{\pi x}{a}$$

$$E_x^0 = H_y = 0$$

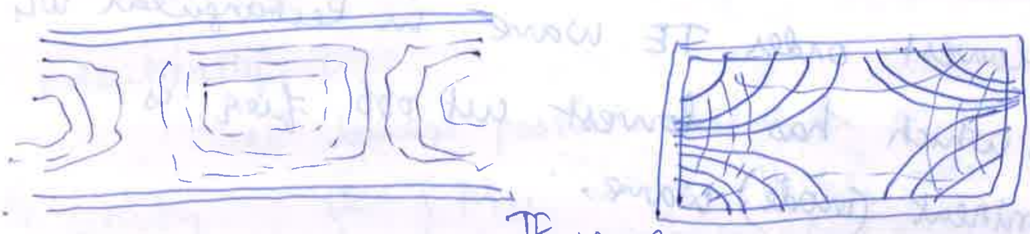
$$\bar{\beta} = \sqrt{\omega^2\mu\varepsilon - (\pi/a)^2}$$

$$f_c = \frac{c}{2a} \qquad \lambda_c = 2a \qquad h = \pi/a$$

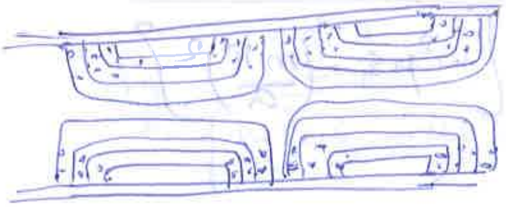




TE_{10} wave



TE_{11} wave



— Electric field
¦-- Mag field

TM_{11} wave

***
Cut-off freq :- At which prop. const. changes from real to imaginary

$$\gamma = \sqrt{\left(\frac{m\pi}{a}\right)^2 - \omega^2\mu\varepsilon} = \frac{m\pi}{a}\sqrt{1 - \frac{\omega^2\mu\varepsilon}{\left(\frac{m\pi}{a}\right)^2}}$$

$$= \frac{m\pi}{a}\sqrt{1 - \frac{\omega\mu\varepsilon}{\omega_c^2\mu\varepsilon}} = \frac{m\pi}{a}\sqrt{1 - \frac{f^2}{f_c^2}}$$

$$= \omega_c\sqrt{\mu\varepsilon}\sqrt{1 - \frac{f}{f_c^2}}$$

~~Under~~ f_c, $\gamma = \alpha$, $\beta = 0$.

where $f > f_c$, γ = imaginary $= j\beta \Rightarrow \alpha = 0$

$$\gamma = j\beta = \frac{m\pi}{a}\sqrt{-\left(\left(\frac{f}{f_c}\right)^2 - 1\right)} = j\frac{m\pi}{a}\sqrt{\frac{f^2}{f_c^2} - 1}$$

$$= \frac{jm\pi}{a}\cdot\sqrt{\frac{f^2}{f_c^2}} = j\,\omega_c\sqrt{\mu\varepsilon}\sqrt{\frac{f^2}{f_c^2} - 1}$$

$$\beta = \omega_c\sqrt{\mu\varepsilon}\sqrt{\frac{f^2}{f_c^2} - 1} \qquad \boxed{\beta = 2\pi\sqrt{\mu\varepsilon}\sqrt{f^2 - f_c^2}}$$

$$\gamma = j\beta = \sqrt{\left(\frac{m\pi}{a}\right)^2 - \omega^2\mu\epsilon} = \sqrt{-\left[\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2\right]}$$

$$\gamma = j\beta = j\sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2}$$

phase shift } $\boxed{\beta = \sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2}}$

cut-off freq $= f_c = \frac{m}{2a\sqrt{\mu\epsilon}} = \boxed{\frac{mu}{2a} = f_c}$

$$\boxed{u = \frac{1}{\mu\epsilon} = 3\times10^8 \text{ m/sec}}$$

Wavelength:- The distance travelled by a wave to under go a phase shift of 2π radians is called wavelength. Also it is in the direction of prop. so called as guided wavelength.

$$\lambda = \frac{2\pi}{\beta} = \lambda g = \boxed{\frac{2\pi}{\sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2}} = \lambda g}$$

cut-off wavelength (λc) wavelength at cut-off freq. is called as cut-off wavelength.

$$\lambda c = \frac{v}{f_c} = \frac{u}{\frac{m}{2a}\cdot v} = \boxed{\frac{2a}{m} = \lambda c}$$

$$\lambda_g = \frac{2\pi}{\sqrt{\omega^2\mu\varepsilon - \omega_c^2\mu\varepsilon}} = \frac{2\pi}{\omega\sqrt{\mu\varepsilon}\cdot\sqrt{1-\frac{\omega_c^2}{\omega^2}}}$$

$$\lambda_g = \frac{v}{f\cdot\sqrt{1-\frac{f_c^2}{f^2}}} = \frac{\lambda}{\sqrt{1-\frac{f_c^2}{f^2}}}$$

$$\boxed{\lambda = v/f}$$

$$\frac{\lambda}{\sqrt{1-(\lambda/\lambda_c)^2}}$$

squaring on both sides,

$$\lambda_g^2 = \frac{\lambda^2}{1-\frac{f_c^2}{f^2}} = \frac{\lambda^2}{1-(\lambda/\lambda_c)^2}$$

$$1-\left(\frac{\lambda}{\lambda_c}\right)^2 = \left(\lambda/\lambda_g\right)^2 \Rightarrow 1 = \lambda^2\left[\frac{1}{\lambda_c^2}+\frac{1}{\lambda_g^2}\right]$$

$$\boxed{\frac{1}{\lambda^2} = \frac{1}{\lambda_c^2}+\frac{1}{\lambda_g^2}}$$

λ – free space WL

λ_c – cut-off WL

λ_g = Guide WL.

× Velocities of the waves

Velocities of propagation

* Except for the TEM wave, the velocities with which a EM wave propagates b/w a pair of ||l planes is always greater than c.

* In both Rectangular and circular w.g's, TEM wave can't exist.

* The wave or phase velocity is > free space velocity.

Attenuation in Parallel-plane Guides

TE_{mo} & TM_{mo} also TEM modes are represented by 16a, 19a & (2) respectively.

* In wu's the conductivity of walls is usually very large & some losses.

* These losses, will modify the results obtained for the lossless case by the introduction of x-ing factor $e^{-\alpha z}$ in 16a, 19a & (2) respectively.

* The problem to determine the attn. factor α caused by losses in the walls of the guide.

* Consider any two conductor tr-ion line.

* The voltage and current phasors are,

$$V = V_0 e^{-\alpha z} \cdot e^{-j\beta z} \quad \text{--- (1)}$$

$$I = I_0 e^{-\alpha z} \cdot e^{-j\beta z} \quad \text{--- (2)}$$

The avg power tr-ed is,

$$W_{av} = \frac{1}{2} Re \{ V I^* \}$$

$$= \frac{1}{2} Re \{ V_0 I_0^* \} e^{-2\alpha z} \quad \text{--- (3)}$$

The rate of ↓se of tr-ed power along the line is,

$$-\frac{\partial W_{av}}{\partial z} = +2\alpha W_{av} \quad \text{—— (4)}$$

The dse of tr-ed power per unit length of line is,

$$-\Delta W_{av} = 2\alpha W_{av}$$

This is equal to the power lost or dissipated power / unit length.

$$\frac{\text{Power lost / unit length}}{\text{Power tr-ed}} = \frac{2\alpha W_{av}}{W_{av}} = 2\alpha$$

$$\therefore \alpha = \frac{\text{Power lost / unit length}}{2\alpha \text{ power tr-ed}} \quad \text{—— (5)}$$

The value of current & Resistances of the walls are known, the losses are computed.

Attenuation factor for TEM wave

$$H_y = C_4 e^{-j\beta z}$$

$$E_x = \eta C_4 e^{-j\beta z} \quad \text{—— (6)}$$

The linear ct. density of each conducting planes, will be $J_s = \hat{n} \times H$.

* The amplitude of the linear current density is,

$J_s = C_4$

* The loss per sq. meter in each conducting plane is, $\frac{1}{2} J_s^2 R_s = \frac{1}{2} C_4^2 R_s$

Where,

$R_s = \sqrt{\frac{\omega \mu_m}{2\sigma_m}}$ is the real resistive component of the Surface impedance

which is $Z_s = \sqrt{\frac{j\omega\mu_m}{\sigma_m}}$

* The total loss in the upper & lower conducting surface / unit meter is for width b meter is, $C_4^2 R_s b$.

* The power transferred down the guide / unit cross sectional area is

$\frac{1}{2} Re (E \times H^*)_z$ ——— (7)

$E_x \Rightarrow \eta H_y$. ∴ 7 becomes,

$\frac{1}{2} \eta C_4^2$

The spacing b/w the planes is a,

∴ Power transferred is $= \frac{1}{2} \eta C_4^2 ba$.

The attenuation factor $\alpha = \dfrac{C_4^2 R_s b}{2 \times \frac{1}{2} \times \eta C_4^2 b \cdot a}$

$$= \frac{R_s}{\eta a} = \frac{1}{\eta a}\sqrt{\frac{\omega \mu_m}{2\sigma_m}} \text{ nepers / meter} \qquad (8)$$

This is compared with attn factor of ordinary tr. line $\alpha = \dfrac{R}{2 Z_0}$.

R - resistance / unit length of the line.

Attenuation of TM waves

The expressions for E & H for the TE modes b/w perfectly conducting ∥l planes are,

$$\left.\begin{aligned} E_y &= C_1 \sin\left(\frac{m\pi}{a}\right)x \, e^{-j\beta z} \\ H_x &= \frac{-\beta}{\omega\mu} C_1 \sin\left(\frac{m\pi}{a}\right)x \cdot e^{-j\beta z} \\ H_z &= \frac{j m\pi}{\omega\mu a} C_1 \cos\left(\frac{m\pi}{a}\right)x \cdot e^{-j\beta z} \end{aligned}\right\} \quad (16.a)$$

Solution of the Field Equations

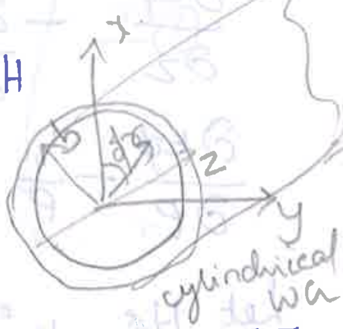
cylindrical co-ordinates

In cylindrical co-ordinates in a nonconducting region (Assuming the variation with z to be given by $e^{-\gamma z}$).

Maxwell's Equations are,

$$\nabla \times H = j\omega\varepsilon E \quad ; \quad \nabla \times E = -j\omega\mu H$$

Expanding in cylindrical ~~equations~~ co-ordinates,



$$\nabla \times H = \begin{vmatrix} \hat{\rho}_r & \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho_r} & \frac{1}{\rho}\frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ H_{\rho_r} & H_\phi & H_z \end{vmatrix} = j\omega\varepsilon\left[E_{\rho_r}\hat{\rho}_r + E_\phi\hat{\phi} + E_z\hat{z}\right]$$

$$\left.\begin{aligned} \frac{1}{\rho_r}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} &= j\omega\varepsilon E_{\rho_r} \\ \frac{\partial H_\phi}{\partial \rho_r} - \frac{1}{\rho_r}\frac{\partial H_{\rho_r}}{\partial \phi} &= j\omega\varepsilon E_z \\ \frac{\partial H_{\rho_r}}{\partial z} - \frac{\partial H_z}{\partial \rho_r} &= j\omega\varepsilon E_\phi \end{aligned}\right\} \quad \text{—(1)}$$

$\rho = r$

$r = \sqrt{x^2+y^2}$

$x = r\cos\phi$

$y = r\sin\phi$

$z = z$

$$\nabla \times E = \begin{vmatrix} \hat{\rho}_r & \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho_r} & \frac{1}{\rho}\frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ E_{\rho_r} & E_\phi & E_z \end{vmatrix} = -j\omega\mu\left(H_{\rho_r}\hat{\rho} + H_\phi\hat{\phi} + H_z\hat{z}\right)$$

Expanding

$$\left.\begin{aligned}
\frac{1}{\rho}\frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} &= -j\omega\mu H_\rho \\
\frac{\partial E_\rho}{\partial z} - \frac{\partial E_z}{\partial \rho} &= -j\omega\mu H_\phi \\
\frac{\partial E_\phi}{\partial \rho} - \frac{1}{\rho}\frac{\partial E_\rho}{\partial \phi} &= -j\omega\mu H_z
\end{aligned}\right\} \text{— (2)}$$

Let $H_\phi = H_\phi^0 e^{-\gamma z}$

$$\left.\begin{aligned}
&\frac{\partial H_\phi}{\partial z} = -\gamma H_\phi^0 e^{-\gamma z} = -\gamma H_\phi \\
&H_\rho = H_\rho^0 e^{-\gamma z} \; ; \; \frac{\partial H_\rho}{\partial z} = -\gamma H_\rho \\
&\text{||ly } E_\phi = E_\phi^0 e^{-\gamma z} \; ; \; \frac{\partial E_\phi}{\partial z} = -\gamma E_\phi \\
&E_\rho = E_\rho^0 e^{-\gamma z} \; ; \; \frac{\partial E_\rho}{\partial z} = -\gamma E_\rho
\end{aligned}\right\} \text{— (3)}$$

Substituting (3) into (1) & (2), we have,

$$\left.\begin{aligned}
\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} + \gamma H_\phi &= j\omega\varepsilon E_\rho \\
-\gamma H_\rho - \frac{\partial H_z}{\partial \rho} &= j\omega\varepsilon E_\phi \\
\frac{\partial H_z}{\partial \rho} + \gamma H_\rho &= -j\omega\varepsilon E_\phi
\end{aligned}\right\} \text{— (1)}$$

$$\frac{1}{\rho}\left[\frac{\partial(\rho H_\phi)}{\partial\rho} - \frac{\partial H_\rho}{\partial\phi}\right] = j\omega\varepsilon E_z$$

$$\frac{1}{\rho}\frac{\partial E_z}{\partial\phi} + \gamma E_\phi = -j\omega\mu H_\rho$$

$$\frac{\partial E_z}{\partial\rho} + \gamma E_\rho = j\omega\mu H_\phi$$ — ①

$$\frac{1}{\rho}\left[\frac{\partial(\rho E_\phi)}{\partial\rho} - \frac{\partial E_\rho}{\partial\phi}\right] = -j\omega\mu H_z$$

$$E_\phi = \frac{-\gamma H_\rho - \frac{\partial H_z}{\partial\rho}}{j\omega\varepsilon}$$ sub's into,

$$\frac{\partial E_z}{\rho\,\partial\phi} + \gamma\left(\frac{-\gamma H_\rho - \frac{\partial H_z}{\partial\rho}}{j\omega\varepsilon}\right) = -j\omega\mu H_\rho$$

$$\frac{j\omega\varepsilon}{\rho}\frac{\partial E_z}{\partial\phi} - \gamma^2 H_\rho - \gamma\frac{\partial H_z}{\partial\rho} = -j^2\omega^2\mu\varepsilon H_\rho$$

$$\frac{j\omega\varepsilon}{\rho}\frac{\partial E_z}{\partial\phi} - \gamma\frac{\partial H_z}{\partial\rho} = H_\rho(\gamma^2 + j^2\omega^2\mu\varepsilon)$$

$$h^2 H_\rho = -j\frac{\omega\varepsilon}{\rho}\frac{\partial E_z}{\partial\phi} + \gamma\frac{\partial H_z}{\partial\rho}$$ — ② a

$$E_\rho = \frac{\frac{\partial H_z}{\rho\,\partial\phi} + \gamma H_\phi}{j\omega\varepsilon}$$ sub's into,

$$-\bar{\gamma}\left(\frac{\frac{\partial H_z}{\rho\partial\phi}+\bar{\gamma}H_\phi}{j\omega\varepsilon_1}\right)-\frac{\partial E_z}{\partial\rho}=-j\omega\mu H_\phi$$

$$-\bar{\gamma}\frac{\partial H_z}{\rho\partial\phi}-\bar{\gamma}^2H_\phi-j\omega\varepsilon_1\frac{\partial E_z}{\partial\rho}=\omega^2\mu\varepsilon_1 H_\phi$$

$$-\bar{\gamma}\frac{\partial H_z}{\rho\partial\phi}-j\omega\varepsilon_1\frac{\partial E_z}{\partial\rho}=\left(\bar{\gamma}^2+\omega^2\mu\varepsilon_1\right)H_\phi$$

$$h^2H_\phi=-j\omega\varepsilon_1\frac{\partial E_z}{\partial\rho}-\frac{\bar{\gamma}}{\rho}\cdot\frac{\partial H_z}{\partial\phi} \quad \text{--- (2-b)}$$

$$H_\phi=\frac{-\bar{\gamma}E_\rho-\frac{\partial E_z}{\partial\rho}}{-j\omega\mu}$$ subs into,

$$\frac{\partial H_z}{\rho\partial\phi}+\bar{\gamma}\left(\frac{\bar{\gamma}E_\rho+\frac{\partial E_z}{\partial\rho}}{j\omega\mu}\right)=j\omega\varepsilon_1 E_\rho$$

$$j\omega\mu\frac{\partial H_z}{\rho\partial\phi}+\bar{\gamma}^2E_\rho+\frac{\partial E_z}{\partial\rho}\bar{\gamma}=-\omega^2\mu\varepsilon_1 E_\rho$$

$$\left(\bar{\gamma}^2+\omega^2\mu\varepsilon_1\right)E_\rho=-\frac{j\omega\mu}{\rho}\frac{\partial H_z}{\partial\phi}-\bar{\gamma}\frac{\partial E_z}{\partial\rho}$$

$$h^2E_\rho=-\bar{\gamma}\frac{\partial E_z}{\partial\rho}-\frac{j\omega\mu}{\rho}\frac{\partial H_z}{\partial\phi} \quad \text{--- (2c)}$$

$$H_\rho = \frac{\dfrac{\partial E_z}{\rho\,\partial\phi} + \bar{\gamma} E_\phi}{-j\omega\mu}$$ sub.'s into,

$$\frac{-\bar{\gamma}\left(\dfrac{\partial E_z}{\rho\,\partial\phi} + \bar{\gamma} E_\phi\right)}{-j\omega\mu} - \frac{\partial H_z}{\partial\rho} = j\omega\varepsilon E_\phi$$

$$-\bar{\gamma}^2 E_\phi - \bar{\gamma}\frac{\partial E_z}{\rho\,\partial\phi} + j\omega\mu\frac{\partial H_z}{\partial\rho} = \omega^2\mu\varepsilon E_\phi$$

$$-\frac{\bar{\gamma}}{\rho}\frac{\partial E_z}{\partial\phi} + j\omega\mu\frac{\partial H_z}{\partial\rho} = \left(\bar{\gamma}^2 + \omega^2\mu\varepsilon\right) E_\phi$$

$$h^2 E_\phi = \frac{-\bar{\gamma}}{\rho}\cdot\frac{\partial E_z}{\partial\phi} + j\omega\mu\frac{\partial H_z}{\partial\rho} \quad \text{— (2)d.}$$

$$h^2 = \bar{\gamma}^2 + \omega^2\mu\varepsilon$$

The wave equation in cylindrical co-ordinates

for E_z is,

$$\frac{\partial^2 E_z}{\partial\rho^2} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial\phi^2} + \frac{\partial^2 E_z}{\partial z^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial\rho} = -\omega^2\mu\varepsilon E_z$$

for H_z is,

$$\frac{\partial^2 H_z}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial H_z}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2 H_z}{\partial\phi^2} + \frac{\partial^2 H_z}{\partial z^2} = \omega^2\mu\varepsilon H_z \quad \text{— (3)}$$

$$E_z = E_z^{\circ} e^{-\gamma z} = P(\rho)\, Q(\phi)\, e^{-\gamma z} \quad \text{--- (4)}$$

$P(\rho) \rightarrow$ is a fn. of ρ alone.

$Q(\phi)$ = is a fn of ϕ alone.

Substitute the expression for E_z in the wave equ.

From (3), we have, the wave equ.

$$Q\frac{d^2P}{d\rho^2} + \frac{1}{\rho P}\frac{dP}{d\rho} + \frac{P}{\rho^2}\cdot\frac{d^2Q}{d\phi^2} + PQ\gamma^2 + \omega^2\mu\varepsilon PQ = 0$$

$\div$ by PQ

$$\frac{1}{P}\cdot\frac{d^2P}{d\rho^2} + \frac{1}{\rho P}\frac{dP}{d\rho} + \frac{1}{Q\rho^2}\cdot\frac{d^2Q}{d\phi^2} + h^2 = 0. \quad \text{--- (5)}$$

Equ (5) can be broken up into two ordinary differential equs,

$$\frac{d^2P}{d\rho^2} + \frac{1}{\rho}\cdot\frac{dP}{d\rho} + \left(1 - \frac{n^2}{\rho^2}\right)P = 0$$

$$\frac{1}{Q}\frac{d^2Q}{d\phi^2} = -n^2$$

$$\frac{d^2Q}{d\phi^2} = -n^2 Q \quad \text{--- (6)}$$

$\times$ by P (5) $\therefore$

$$\frac{d^2P}{d\rho^2} + \frac{1}{\rho}\cdot\frac{dP}{d\rho} + (h^2 P) + \frac{P}{Q\rho^2}\cdot(-h^2 Q) = 0$$

$$\frac{d^2P}{d\rho^2} + \frac{1}{\rho}\frac{dP}{d\rho} + \left(h^2 - \frac{n^2}{\rho^2}\right)P = 0 \quad \text{--- (7)}$$

where n is const, the soln of equ (6) is,

$$Q = A_n \cos n\phi + B_n \sin n\phi \quad \text{—— (8)}$$

$\div$ by h^2, Equ (7) becomes,

$$\frac{d^2P}{d(\rho h)^2} + \frac{1}{\rho h}\frac{dP}{d(\rho h)} + P\left[1 - \frac{n^2}{(\rho h)^2}\right] = 0 \quad \text{—— (9)}$$

This is a std. form of Bessels equ in terms of (ρh).

Using only the soln. ie finite at $\rho h = 0$,

$$P(\rho h) = J_n(\rho h) \quad \text{—— (10)}$$

where, $J_n(\rho h)$ is → Bessel fn. of the 1st kind of order n. Sub (8) & (10) into (4), we have,

$$E_z = J_n(\rho h)(A_n \cos n\phi + B_n \sin n\phi)e^{-\gamma z} \quad \text{—— (11)}$$

The soln for H_z will have exactly the same form as for E_z,

$$H_z = J_n(\rho h)(C_n \cos n\phi + D_n \sin n\phi)e^{-\gamma z} \quad \text{—— (12)}$$

The field components for TM waves can be obtained by sub. ing 11 & into (2)
for TE waves (12) into (2).

TM and TE waves in circular waveguide

Boundary condn: $E_\phi = 0$, at $\rho = a$

* for TM waves H_z is identically zero.

* ∴ wave eqn for E_z is used.

The wave equation for E_z for cylindrical co-ordinates is from (8)

$$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \phi^2} + \frac{\partial^2 E_z}{\partial z^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial \rho} = -\omega^2 \mu \epsilon E_z$$

$$E_z = E_z^0 e^{-\bar{\gamma} z}$$

$$\frac{\partial E_z}{\partial z} = -\bar{\gamma} E_z \quad ; \quad \frac{\partial^2 E_z}{\partial z^2} = -\bar{\gamma} \cdot \frac{\partial E_z}{\partial z}$$

$$\frac{\partial^2 E_z}{\partial z^2} = +\bar{\gamma}^2 E_z$$

$$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \phi^2} + \bar{\gamma}^2 E_z + \frac{1}{\rho}\cdot\frac{\partial E_z}{\partial \rho} = -\omega^2 \mu \epsilon E_z$$

$$\frac{\partial^2 E_z}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 E_z}{\partial \phi^2} + \frac{1}{\rho}\frac{\partial E_z}{\partial \rho} + E_z h^2 = 0$$

The Boundary condn's require that E_z must vanish at the surface of the guide.

∴ from (11) $J_n(ha) = 0$ ——— (13)

where $a \rightarrow$ radius of the guide.

There is an infinite no of possible TM waves corresponding to infinite no. of roots of (13)

$h^2 = \gamma^2 + \omega^2\mu\varepsilon_r$

In case of rectangular guides, $h^2 < \omega^2\mu\varepsilon_r$ — to-conv occur

ie h is small or extremely high freq's will be required. ie only first few roots of (13) will be of practical interest.

The 1st few roots are,

$$\left.\begin{array}{ll} (ha)_{01} = 2.405 & (ha)_{11} = 3.85 \\ (ha)_{02} = 5.52 & (ha)_{12} = 7.02 \end{array}\right\} \text{— (14)}$$

1st subscript → value of n

2nd subscript → roots of their order of magnitude.

$H_n\ \rho =$

The various TM waves referred as, $TM_{01}, TM_{12} \ldots$

From equ (2)

$$\left.\begin{aligned} h^2 H_\rho &= \frac{j\omega\varepsilon_r}{\rho}\frac{\partial E_z}{\partial\phi} - \bar{\gamma}\frac{\partial H_z}{\partial\rho} \\ h^2 H_\phi &= -j\omega\varepsilon_r\frac{\partial E_z}{\partial\rho} - \frac{\bar{\gamma}}{\rho}\frac{\partial H_z}{\partial\phi} \\ h^2 E_\rho &= -\bar{\gamma}\frac{\partial E_z}{\partial\rho} - j\frac{\omega\mu}{\rho}\frac{\partial H_z}{\partial\phi} \\ h^2 E_\phi &= -\frac{\bar{\gamma}}{\rho}\cdot\frac{\partial E_z}{\partial\phi} + j\omega\mu\cdot\frac{\partial H_z}{\partial\rho} \end{aligned}\right\} \text{— (2)}$$

The various components of TM waves can be computed in terms of E_z.

The expressions for TM waves in $\odot$lar guides are,

$$E_z^\circ = A_n J_n(h\rho)\cos n\phi$$

$$H_\rho^\circ = -j\frac{A_n \omega \varepsilon n}{h^2\rho} J_n(\rho h)\sin n\phi$$

$$H_\phi^\circ = \frac{-jA_n\omega\varepsilon}{h} J_n'(\rho h)\cos n\phi \quad -(15)$$

$$E_\rho^\circ = \frac{\bar\beta}{\omega\varepsilon}\cdot H_\phi^\circ$$

$$E_\phi^\circ = \frac{-\bar\beta}{\omega\varepsilon}\cdot H_\rho^\circ$$

$$\Rightarrow E_z = J_n(\rho h)\left[A_n\cos n\phi + B_n\sin n\phi\right]e^{-\bar\gamma z}$$

B_n can be put to 0, $\because$ The relative amplitudes of B_n & A_n determines only the orientation of the field in the guide because both components are periodic. For $\odot^r$ guide, any value of n, $\phi = 0$ axis can always oriented to make either A_n or $B_n = 0$.

$$E_z = J_n(\rho h)A_n\cos n\phi e^{-\bar\gamma z}, \quad \frac{\partial E_z}{\partial\rho} =$$

$$E_\rho = \frac{-\bar\gamma}{h^2}\frac{\partial E_z}{\partial\rho} - \frac{j\omega\mu}{\rho h^2}\cdot\frac{\partial H_z}{\partial\phi} \qquad \alpha = 0, \quad \bar\gamma = j\bar\beta$$

For TM waves, $H_z = 0$,

$$E_\rho = \frac{-\bar\gamma}{h^2}\frac{\partial E_z}{\partial\rho} = \frac{-\bar\gamma}{h^2}\cdot J_n'(\rho h)\,h\,.A_n\cos n\phi\, e^{-\bar\gamma z}$$

$$E_\rho = \frac{-\bar\gamma}{h^2}\frac{\partial E}{\partial\rho} = \frac{-j\beta}{h^2} J_n'(\rho h).A_n\cos n\phi\, e^{-j\beta z}$$

$$E_\phi = \frac{-\bar{\gamma}}{\rho h^2}\frac{\partial E_z}{\partial \phi} = \frac{+j\bar{\beta}}{\rho h^2} J_n(\rho h)\, A_n \sin n\phi \cdot n\, e^{-j\bar{\beta} z}$$

$$E_\phi = \frac{j\beta}{\rho h^2} \cdot J_n(\rho h) A_n \sin n\phi \cdot n \cdot e^{-j\beta z}.$$

$$H_\rho = \frac{j\omega\varepsilon_1}{\rho h^2} \cdot \frac{\partial E_z}{\partial \phi} = \frac{j\omega\varepsilon_1}{\rho h^2} n \cdot A_n J_n(\rho h)(-\sin n\phi) e^{-j\beta z}$$

$$= \frac{-j\omega\varepsilon_1}{\rho h^2} \cdot n A_n \cdot J_n(\rho h) \sin n\phi \cdot e^{-j\beta z}$$

$$H_\phi = \frac{-j\omega\varepsilon_1}{h^2} \frac{\partial E_z}{\partial \rho}$$

$$H_\phi = \frac{-j\omega\varepsilon_1}{h^2} \cdot J_n'(\rho h)\, h\, A_n \cos n\phi \; e^{-j\beta z}$$

$$= \frac{-j\omega\varepsilon_1}{h^2} \cdot A_n \cdot J_n'(\rho h) \cos n\phi \; e^{-j\beta z}$$

Transverse electric waves:- $E_z = 0$.

H_z is given by (12)

$$H_z = J_n(\rho h)(C_n \cos n\phi + D_n \sin n\phi) e^{-\bar{\gamma} z}$$

Sub (12) into (2), we have, the remaining field components can be found.

From (3), $\frac{\partial^2 H_z}{\partial \rho^2} + \frac{1}{\rho^2} \cdot \frac{\partial^2 H_z}{\partial \phi^2} + \frac{\partial^2 H_z}{\partial z^2} + \frac{1}{\rho}\frac{\partial H_z}{\partial \rho} = -\omega^2\mu\varepsilon H_z$

$$\frac{\partial^2 H_z}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 H_z}{\partial \phi^2} + \bar{\gamma}^2 H_z + \frac{1}{\rho}\cdot\frac{\partial H_z}{\partial \rho} = -\omega^2\mu\varepsilon_1 H_z$$

$$\frac{\partial^2 H_z}{\partial \rho^2} + \frac{1}{\rho^2}\cdot\frac{\partial^2 H_z}{\partial \phi^2} + \frac{1}{\rho}\frac{\partial H_z}{\partial \rho} + h^2 H_z = 0 \quad \text{——— (16)}$$

Let $H_z = H_z^0 \, e^{-\gamma z}$

$$H_z^0 = P(\rho)Q(\phi)e^{-\gamma z}$$ — (17)

Sub (17) into (15) we have,

$$Q\frac{d^2P}{d\rho^2} + \frac{P}{\rho^2}\frac{d^2Q}{d\phi^2} + \frac{Q}{\rho}\frac{dP}{d\rho} + PQh^2 = 0 \quad \text{— (18)}$$

$\div$ by PQ throughout,

$$\frac{1}{P}\cdot\frac{d^2P}{d\rho^2} + \frac{1}{Q\rho^2}\cdot\frac{d^2Q}{d\phi^2} + \frac{1}{P\rho}\frac{dP}{d\rho} + h^2 = 0 \quad \text{— (19)}$$

Let $\frac{1}{Q}\cdot\frac{d^2Q}{d\phi^2} = -n^2$

$$\frac{d^2Q}{d\phi^2} = -n^2Q \quad \text{— (20)}$$

The Soln. of the above equ is,

$$Q = C_n \cos n\phi + D_n \sin n\phi \quad \text{— (21)}$$

Sub (20) into (19),

$$\frac{1}{P}\cdot\frac{d^2P}{d\rho^2} + \frac{1}{Q\rho^2}(-n^2Q) + \frac{1}{P\rho}\cdot\frac{dP}{d\rho} + h^2 = 0$$

$$\frac{1}{P}\cdot\frac{d^2P}{d\rho^2} - \frac{n^2}{\rho^2} + \frac{1}{P\rho}\frac{dP}{d\rho} + h^2 = 0$$

$\times$ by P, $\frac{d^2P}{d\rho^2} + \frac{1}{\rho}\frac{dP}{d\rho} + \left(h^2 - \frac{n^2}{\rho^2}\right)P = 0$

$\div h^2$, $\frac{d^2P}{d(\rho h)^2} + \frac{1}{\rho h}\frac{dP}{d(\rho h)} + \left(1 - \frac{n^2}{(\rho h)^2}\right)P = 0$ — (22)

This is the std. form of Bessels eqn, & has a soln, which is a Bessels fn,

$$P(\rho h) = J_n(\rho h) \longrightarrow (23)$$

Combine (21) & (23),

$$H_z = P Q \cdot e^{-\gamma z}$$

$$H_z = J_n(\rho h)\left[C_n \cos n\phi + D_n \sin n\phi\right] e^{-\gamma z} \quad (24)$$

D_n is taken as zero, ∴ both terms are periodic.

& C_n & D_n refers only the diff. value of the signal.

$$H_z = J_n(\rho h) \cdot C_n \cos n\phi \cdot e^{-\gamma z} \quad (25)$$

$$H_z = J_n(\rho h) C_n \cos n\phi \; e^{-j\beta z} \qquad \gamma = j\beta$$

The other field components are, From (2),

$$E_\rho = \frac{-j\omega\mu}{h^2}\frac{1}{\rho}\cdot\frac{\partial H_z}{\partial\phi} = \frac{-j\omega\mu}{h^2\rho} J_n(h\rho)(-\sin n\phi)\, n\, e^{-j\beta z}$$

$$\boxed{E_\rho = \frac{j\omega\mu}{h^2\rho}\cdot n C_n J_n(h\rho)\sin n\phi\, e^{-j\beta z}}$$

$$E_\phi = \frac{j\omega\mu}{h^2}\frac{\partial H_z}{\partial\rho} = \frac{j\omega\mu}{h^2} C_n\, h\, J_n'(h\rho)\cos n\phi \cdot e^{-j\beta z}$$

$$\boxed{E_\phi = \frac{j\omega\mu}{h} C_n J_n'(h\rho)\cos n\phi\, e^{-j\beta z}}$$

$$H_\rho = \frac{-j\beta}{h^2}\left(\frac{\partial H_z}{\partial \rho}\right) = \frac{-j\beta}{h} C_n h J_n'(h\rho)\cos n\phi \cdot e^{-j\beta z}$$

$$\boxed{H_\rho = \frac{j\beta}{h} C_n J_n'(h\rho)\cos n\phi \cdot e^{-j\beta z}}$$

$$H_\phi = \frac{-j\beta}{\rho h^2} \cdot \frac{\partial H_z}{\partial \phi} = \frac{-j\beta}{\rho h^2} C_n \cdot n(-\sin n\phi) J_n(h\rho) e^{-j\beta z}$$

$$H_\phi = \frac{j\beta}{\rho h^2} \cdot n \cdot C_n J_n(h\rho)\sin n\phi e^{-j\beta z}$$

Boundary condn for TE waves are -

$E_\phi = 0$, at $\rho = a$.

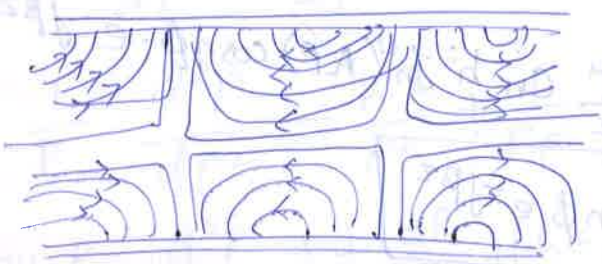
$$E_\phi = \frac{j\omega\mu}{h} C_n J_n'(ah) \cdot \cos n\phi \cdot e^{-j\beta z} = 0$$

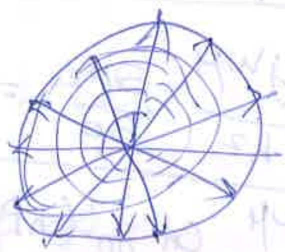
above eqn is true only if, $J_n'(ha) = 0$

∴ The first few of these roots are,

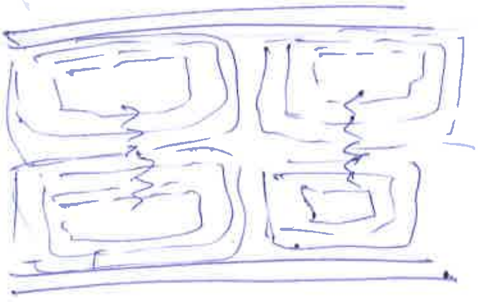
$(ha)'_{01} = 3.83$ $\quad (ha)'_{11} = 1.84$

$(ha)'_{02} = 7.02$ $\quad (ha)'_{12} = 5.33$ → 26





TM_{01}



TE_{01}

Dominent Mode:- The mode is having the lowest cut-off frequency is called dominent mode.

For TM waves, the lowest order mode is TM_{11}

For TE " " " " is TE_{10} or TE_{01}.

$$f_c < TE_{10} < f_{c(TE_{01})} < f_{c(TM_{11})}$$

Degenerate Mode:- Higher order modes having the same cut-off freq are called degenerate mode.

TE_{11} & TM_{11} are degenerate modes as they have the same cut-off freq.

Wave impedances and characteristics impedances

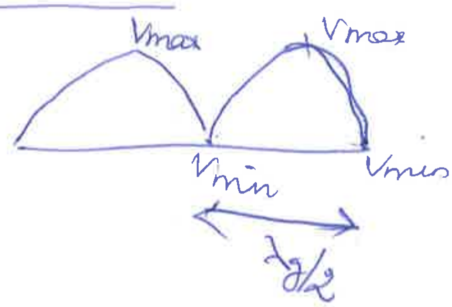
Resonant Cavities

* when one end of the WG is terminated in a shorting plate, there will be complete reflection of waves.

* when one more shorting plate is kept at a distance of multiple of $\lambda g/2$, from 1st shorting plate, then a hallow space is formed & not support a signal which bounces back and forth b/w the two shorting plates.

* The waves appear as stationary & hence called standing waves.

The distance b/w successive min. is $\lambda g/2$.



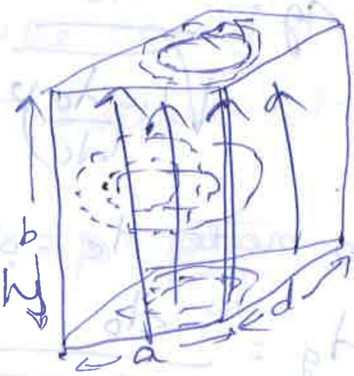
Applns:- *1) Used in klystron Amplifier for amplifying µwave signals

2) In Klystron oscillators or reflex klystron for generation of µwaves signals.

3) Cavity resonators are used in Magnetrons for µwave signal generation.

4. Used in cavity wave meters for the measurement of freq. of μwave signals.

Rectangular cavity resonator (CR)



Hollow metal cavity resonators are usually made of metals that have a high electrical conductivity silver & copper or else the inner surface of the resonator is coated with a layer of silver or gold.

Extremely high figure of merit is obtained by using super conducting metals.

* CR can be tuned to a given freq. by changing the volume of the cavity by moving the walls or by inserting metal plungers, plates or other tuning elements into the cavity.

* Coupling to external ckt is usually carried out thro' apertures in the walls of the cavity with the aid of loops, probs and other coupling components.

* Used in the freq. range of 10^9 to 10^{11} Hz

Guided wavelength is,

$$\lambda_g = \frac{\lambda_0}{\sqrt{1-\left(\frac{\lambda_0}{\lambda_c}\right)^2}}$$; The dominant mode is TE_{10} mode.

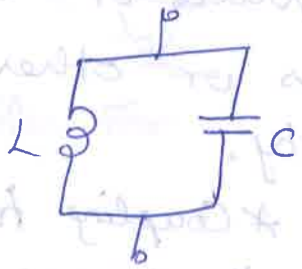
For TE_{10} mode $\lambda_c = 2a$,

$$\lambda_g = \frac{\lambda_0}{\sqrt{1-\left(\frac{\lambda_0}{2a}\right)^2}}$$

where, $\lambda_0 \rightarrow$ Free space wavelength

$$f_0 = \frac{C}{\lambda_0}$$ $C \rightarrow$ wave velocity.

The hollow space cavity resonators can support only one freq for a given mode called resonant freq. ∴ a cavity resonator can be replaced by an "Equivalent tank circuit"



Expression for resonant freq f_0 for rectangular cavity

$$\gamma^2 + \omega^2 \mu \varepsilon = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2 \quad \text{--- (1)}$$

$$\omega^2 \mu \varepsilon = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2 - \gamma^2 \quad \text{--- (2)}$$

For wave prop. inside the wg with no attn. is,

$$\gamma = j\beta \quad \text{--- (3)}$$

Sub (3) into (2), $\omega^2 \mu \varepsilon_r = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{a}\right)^2 + \beta^2$

But, the condn. is, the cavity is formed by an integer multiple of halfwave length at resonant frequency. $\therefore \beta = \frac{p\pi}{d}$

p is an integer. = 1, 2, 3 - - -

* depending on the value of 'p', the mode is denoted by TM_{mnp} for Transverse ~~Electric~~ magnetic waves & TE_{mnp} for transverse Electric waves

To have a resonator at a fixed freq, f_0 or ω_0

$$\underset{2\pi f_0}{\omega_0^2} \mu \varepsilon_r = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2$$

$$f_0 = \frac{1}{2\pi \mu \varepsilon} \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$$

$$= \frac{1}{2\sqrt{\mu\varepsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}$$

If the hollow dielectric space inside the resonator is filled with air dielectric,

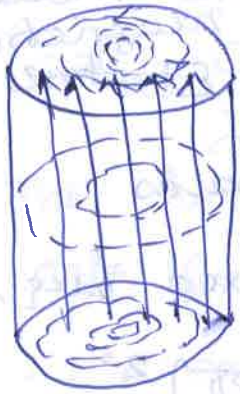
$\sqrt{\mu\varepsilon_1} = \sqrt{\mu_0 \varepsilon_{10}} = \frac{1}{v}$; $v = 3\times 10^8 m/s$ velocity of light.

$$\therefore f_0 = \frac{v}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2}$$

$$= \frac{c}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2} \quad \text{—— (4)}$$

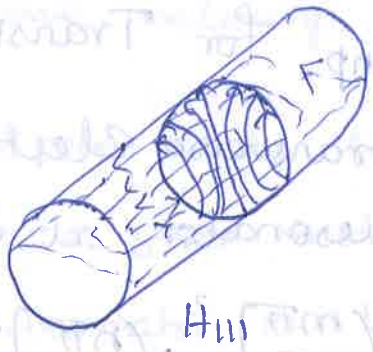
f_0 is for both TM_{mnp} and TE_{mnp} modes.

Circular Cavity or cylindrical cavity resonators

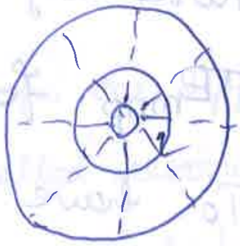


E_{010}

(a) cylindrical hallow resonator



H_{111}



H_{011}

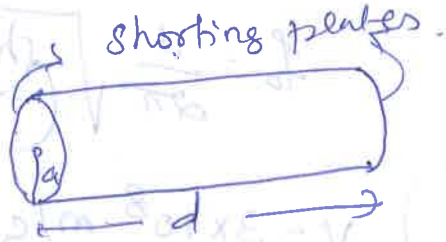
Only ct. along the lateral surface of the cylinder to the axis.

* Cylindrical resonators are most widely used type of cavity resonators.

* The types of oscillation is characterized by the subscripts m, n and p that correspond to the no. of halfwaves ~~of~~ the Electric or magnetic field that fit along the diameter, circumference

and length of the resonator. ie E_{mnp} or H_{mnp}.

* Cylindrical resonator is formed when a olden waveguide of length d + radius 'a' is terminated in a shorting plates at both ends.



$$\gamma^2 + \omega^2 \mu \epsilon_r = h^2$$

$$\gamma = \sqrt{h_{mn}^2 - \omega^2 \mu \epsilon_r}$$

where $h_{mn} = \frac{(ha)_{mn}}{a}$

$$\gamma = \left[\frac{(ha)_{mn}}{a}\right]^2 - \omega^2 \mu \epsilon_r$$

For wave propagation, $\gamma = j\beta$; $\gamma^2 = -\beta^2$

condn. for resonator is, $\beta = \frac{p\pi}{a}$,

$$\gamma^2 = -\beta^2 = -\left(\frac{p\pi}{d}\right)^2$$

At a fixed freq ω_0 or f_0, ,

$$-\left(\frac{p\pi}{d}\right)^2 = \left(\frac{(ha)_{mn}}{a}\right)^2 - \omega_0^2 \mu \epsilon_r$$

$$\omega_0^2 \mu \epsilon_r = \left(\frac{(ha)_{mn}}{a}\right)^2 + \left(\frac{p\pi}{d}\right)^2$$

$$f_0 = \frac{1}{2\pi\sqrt{\mu\epsilon}}\sqrt{\left(\frac{(ha)_{mn}}{a}\right)^2+\left(\frac{p\pi}{d}\right)^2}$$

if the WG is filled with air, dielectric then,

$$f_0 = \frac{v}{2\pi}\sqrt{\left(\frac{(ha)_{mn}}{a}\right)^2+\left(\frac{p\pi}{d}\right)^2}$$

$v = 3\times10^8 \; m/s = c$.

$$\therefore \; f_0 = \frac{c}{2\pi}\sqrt{\left[\frac{(ha)_{mn}}{a}\right]^2+\left(\frac{p\pi}{d}\right)^2}$$

for both TE_{mnp} & TM_{mnp}.

Co-axial cavity

Radial cavity.

Tunable cavity

Toroidal cavity

Butterfly cavity.