Signal:

Any physical quantity which can be a variable which is going to change w.r.t an independant parameter

ROCs - Stable and causal s/m

Unit impulse is applied - Impulse response of a system

DT signals - $r[n]$:- ramp $= \begin{cases} n, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$

$x[n] = C e^{\beta n}$

C ↓ complex; βn ↓ depending on this decay or increase is decided

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

$$rect[n] = \begin{cases} 1, & |n| < a \\ 0, & \text{otherwise} \end{cases}$$

Classification of signals:

(i) Periodic / Aperiodic $\rightarrow x(n) = x[n+N]$; N=m

(ii) Odd / Even

(iii) Causal / Non-Causal $\rightarrow$ i/p based on present & pas

(iv) Deterministic / Non-deterministic

(v) Energy / Power

(vi) Stable $\rightarrow h[n] = \sum_{n=-\infty}^{\infty} |h[n]| < \infty$

Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt}$$

$\omega_0 \rightarrow$ fundamental frequency
$k \rightarrow$ integral multiple

$k = 1 \rightarrow 1\omega_0 \rightarrow$ fundamental frequency
$k = 2 \rightarrow 2\omega_0 \rightarrow$ Second harmonic

Discrete time: $x[n] \rightleftharpoons n \rightarrow$ integer value

DTFS:

$$x[n] = \sum_{k=} a_k e^{jk\omega_0 n}$$

$$\omega_0 = 2\pi f_0 m = \frac{2\pi m}{N}$$

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow \boxed{y[n]}$$

i/p — Syst fn (impulse response) — o/p

$$y[n] = \sum_{k=-\infty}^{\infty} x[n]\, h[n-k]$$

Properties:

1) Static / Dynamic $\rightarrow$ depends on past as well as future
$\rightarrow$ depends only on current i/p

2) Time variant/invariant $\rightarrow x[n] \rightarrow y[n]$
$x[n \pm k] \rightarrow y[n \pm k]$

3) Linear / Non-linear $\rightarrow$ Superposition
$\Downarrow$
weighted sum of i/ps $\}$ = weighted sum of o/ps

4) Stability $\rightarrow \displaystyle\sum_{k=-\infty}^{\infty} \left| h[n] \right| < \infty$

5) Causality $\rightarrow$ $h[n]$: causal signal

causal s/m :- $h[n] = 0, \; n < 0$

$$y[n] = h[n] * x[n]$$

$$= \sum_{k=0}^{\infty} h[k]\,x[n-k]$$

$$= h[0]x[1] + x[1]h[0] + h[2]\cdot x[-1]$$

6) Invertible -

$$x[n] \rightarrow \boxed{h_1[n]} \xrightarrow{w[n]} \boxed{h_2[n]} \rightarrow y[n]$$

$$h_1[n] * h_2[n] = \delta[n]$$

Circular Convolution :

Finite length sequences :- $x_1[n] - N$

$x_2[n] - N$

$$y[n] = x_1[n] \; Ⓝ \; x_2[n] \longrightarrow \text{circular method}$$

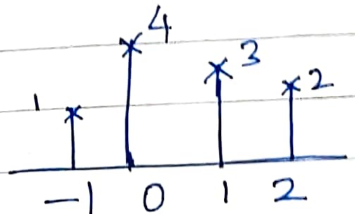
$$= \sum_{k=0}^{N-1} x_1[k]_N \, x_2[n-k]_N$$

Circular Convolution method —
- → Concentric circle method
- → Matrix method

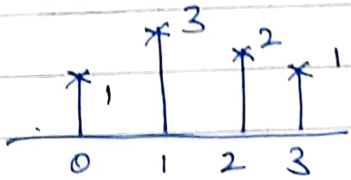
1) $x[n] = \{1, \underset{\uparrow}{4}, 3, 2\}$ $l = 4$

$h[n] = \{1, 3, 2, 1\}$ $m = 4$

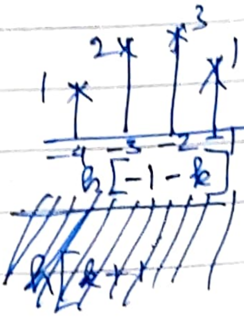
$y[n] = 4 + 4 - 1$
$= 7$

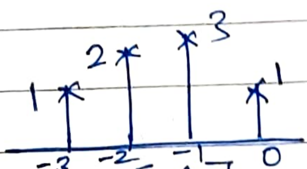


$x[k]$

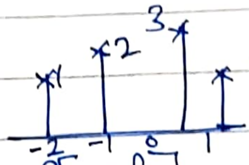


$h[k]$

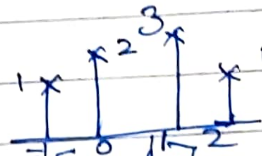




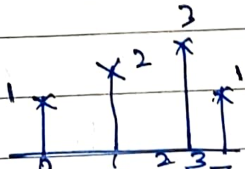
$h[-k]$



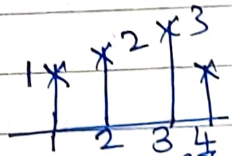
$h[1-k]$



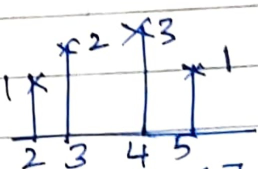
$h[2-k]$



$h[3-k]$



$h[4-k]$



$h[5-k]$

$y[-1] = 1 * 1 = 1$

$y[0] = 1 * 3 + 4 * 1 = 7$

$y[1] = 2 * 1 + 3 * 4 + 1 * 3 = 17$

$y[2] = 1 * 1 + 4 * 2 + 3 * 3 + 2 * 1 = 20$

$y[3] = 4 * 1 + 2 * 3 + 3 * 2 = 16$

$y[4] = 3 * 1 + 2 * 2 = 7$

$y[5] = 1 * 2 = 2$

$\Rightarrow y[n] = [1, \underset{\uparrow}{7}, 17, 20, 16, 7, 2]$

For C.C :+ $\begin{matrix} 1 & 7 & 17 & 20 \\ 16 & 7 & 2 & \end{matrix} \} = \{17, 14, 19, 20\}$

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

$$y[n] = x[n] * h[n]$$
$$= \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

$\Rightarrow l = 4$

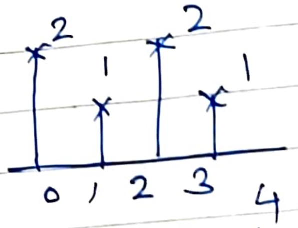
2) $x_1[n] = \{2, 1, 2, 1\}$

$y[n] = 4+4-$
$= 7$

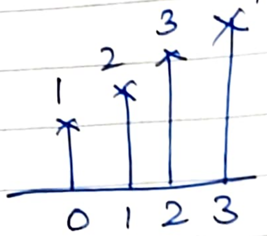
$x_2[n] = h[n] = \{1, 2, 3, 4\} \Rightarrow m = 4$

$x[n]$ and	$x[-n]$
$x[n+k]$ left	$x[-n+k]$ right
$x[n-k]$ right	$x[-n-k]$ left

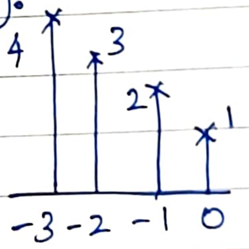
$\boxed{\text{LC:}}$ $x_1[k]$:



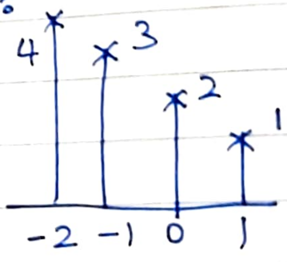
$h[k]$:



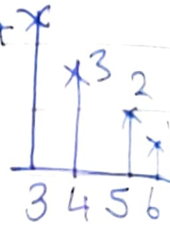
$h[-k]$:



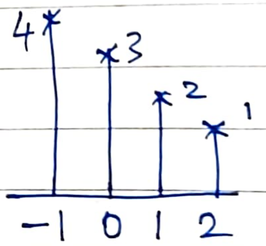
$h[1-k]$:



$h[6-k]$:

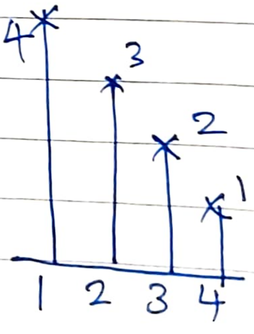


$h[2-k]$:

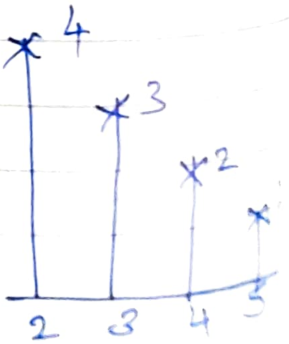


$h[3-k]$:

$h[4-k]$:



$h[5-k]$:



$y[0] = 2*1 = 2$

$y[1] = 2*2 + 1*1 = 5$

$y[2] = 2*3 + 2*1 + 2*1 = 10$

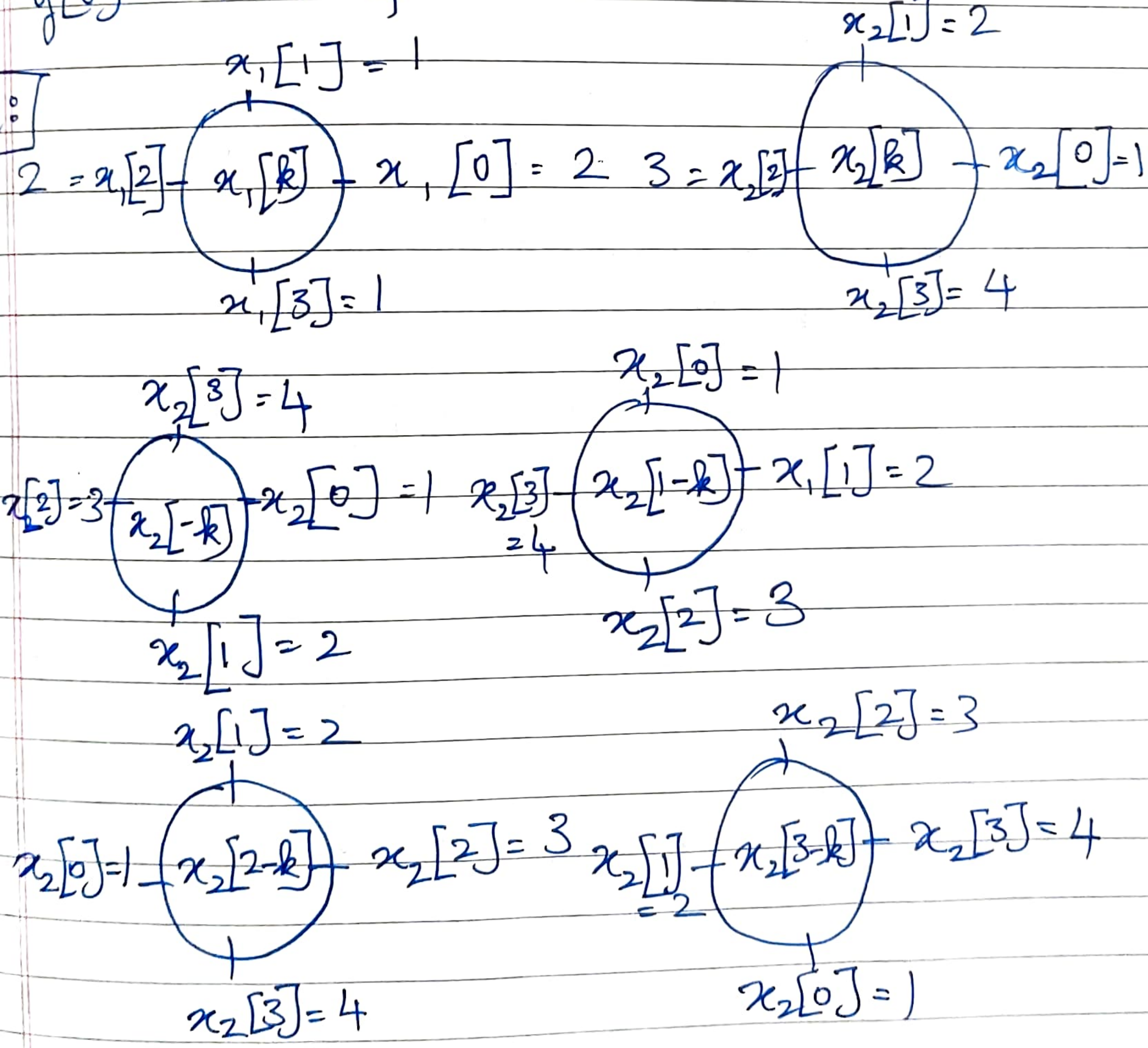
$y[3] = 4*2 + 1*3 + 2*2 + 1*1 = 16$

$y[4] = 4*1 + 3*2 + 2*1 = 12$

$y[5] = 2*4 + 1*3 = 11$

$y[6] = 4*1 = 4$

$\boxed{CC:}$



$y[0] = 2 + 2 + 4 + 6 = 14$

$y[1] = 1 + 8 + 3 + 4 = 16$

$y[2] = 2 + 2 + 4 + 6 = 14$

$y[3] = 1 + 4 + 3 + 8 = 16$

To Convert from L·C to C·C:

C·C – $y[n] = \{14, 16, 14, 16\}$

L·C – $y[n] = \{2, 5, 10, 16, \underline{12, 11, 4}\}$

$\Rightarrow$

2	5	10	16
12	11	4	
14	16	14	16

3) $x[n] = \{0.5, 2\} \Rightarrow x[n] = \{0.5, 2, 0\}$

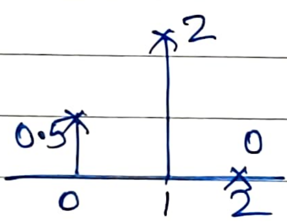
$h[n] = \{1, 1, 1\}$

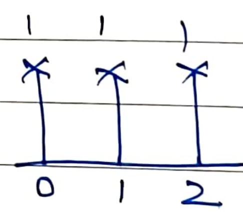
$\therefore y[n] = 3+3-1$
$= 5$

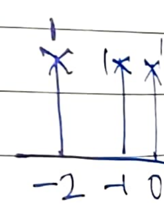
$y[n] = x[n] * h[n]$

$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$

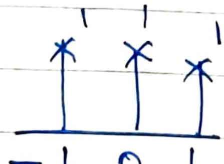
$\boxed{L.C:}$ $x[k] = \{0.5, 2, 0\}$ $h[k]$: $h[-k]$:

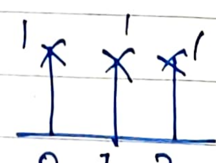


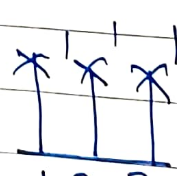


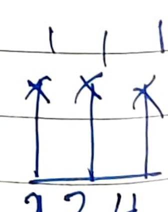


$h[1-k]$: $h[2-k]$: $h[3-k]$: $h[4-k]$:









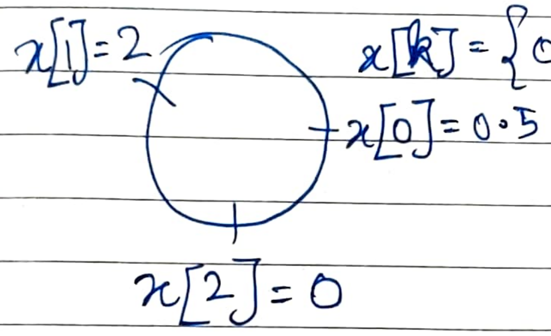
$$\left.\begin{aligned} y[0] &= 0.5 \\ y[1] &= 2.5 \\ y[2] &= 2.5 \\ y[3] &= 2 \\ y[4] &= 0 \end{aligned}\right\} \Rightarrow y[n] = \{0.5, 2.5, 2.5, 2, 0\}$$

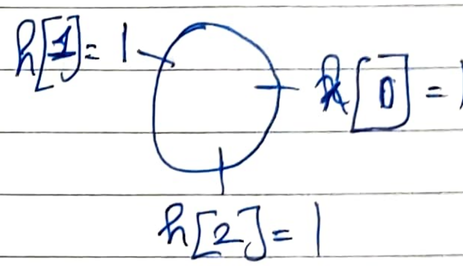
For circular convolution,

$$y[n] = \begin{matrix} 0.5 & 2.5 & 2.5 \\ 2 & 0 & \end{matrix}$$

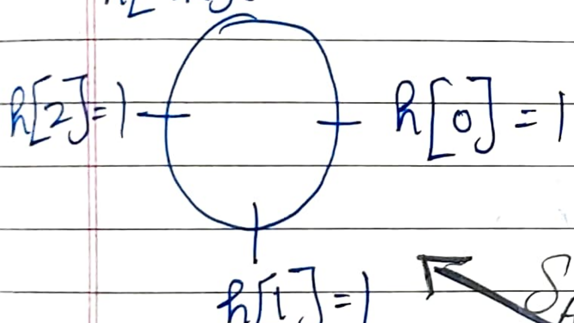
$$C.C \Leftrightarrow \{2.5 \quad 2.5 \quad 2.5\}$$

$\boxed{C.C}$: $x[k] = \{0.5, 2, 0\}$ $h[k]$:

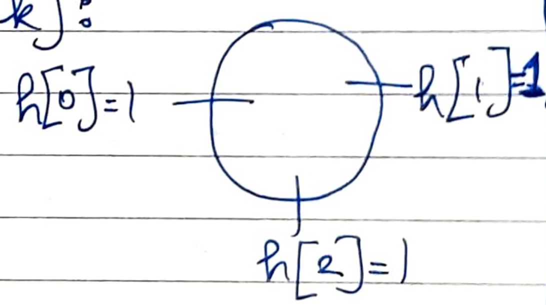




$h[-k]$:

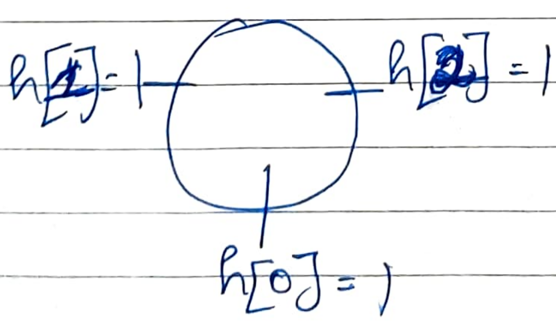


$h[1-k]$:

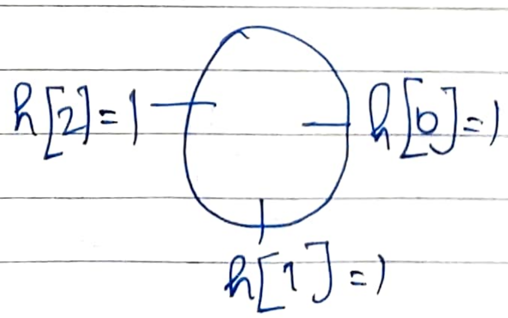


SAME

$h[2-k]$:



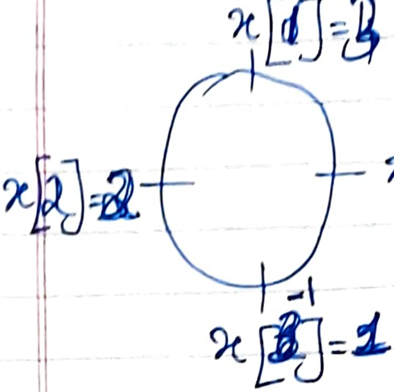
$h[3-k]$:

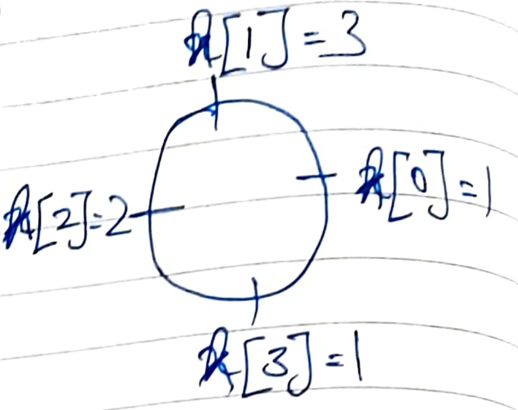


$$\left.\begin{aligned} y[0] &= \{0.5 + 2 + 0\} = 2.5 \\ y[1] &= \{0.5 + 2 + 0\} = 2.5 \\ y[2] &= \{0.5 + 2 + 0\} = 2.5 \end{aligned}\right\} \boxed{y[n] = \{2.5, 2.5, 2.5\}}$$

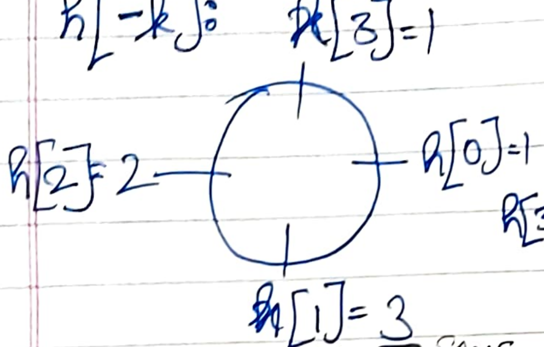
1) Check circular convolution for the same:-

$x[k] = \{1, \underset{\uparrow}{4}, 3, 2\}$ $\quad h[k] = \{1, 3, 2, 1\}$:



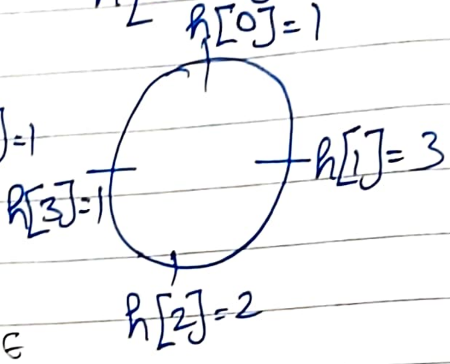


$h[-k]$:

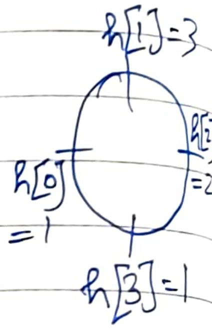


SAME

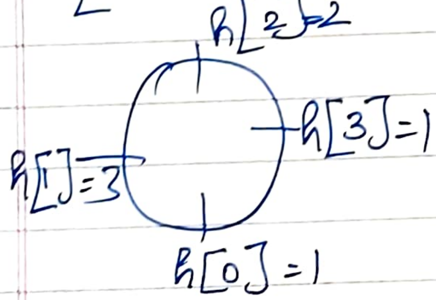
$h[1-k]$:



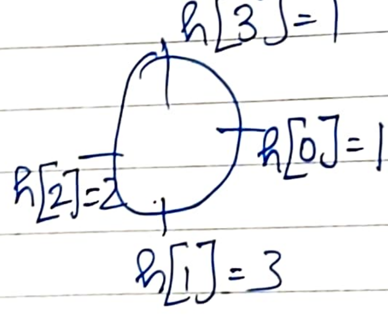
$h[2-k]$:



$h[3-k]$:



$h[4-k]$:



$y[0] = \{4+3+4+3\} = 14$

$y[1] = \{3+2+12\} = 19$
$+2$

$y[2] = 20 = \{9+2+1+8\}$

$y[3] = 17 = \{6+6+1+4\}$

$\boxed{y[n] = \{\cancel{17}, \underset{\uparrow}{14}, 19, 20}$

Fourier Series Representation:

C.T:- $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$ (only periodic)

$$a_k = \frac{1}{T} \int x(t) e^{-jk\omega_0 t} dt$$

D.T:-

$$x[n] = \sum_{k=0}^{N-1} a_k e^{+j\omega_0 n}$$

$$a_k = \frac{1}{N} \sum_{k=0}^{N-1} x[n] e^{-jk\omega_0 n}$$

Fourier Transform Representation:

C.T:- (aperiodic) $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$ (analysis)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} dt$$ (synthesis)

D.T:- (aperiodic) $X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$ (analysis eqn)

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$$ (synthesis eqn)

Convolution:

$$x_1[n] * x_2[n] \xrightarrow[T]{F} X_1[j\omega] \cdot X_2[j\omega]$$

DFT :- Discrete Fourier Transform :

→ holds for short sequences

$$X(j\omega) = X(e^{j\omega}) = X(\omega)$$

$$\boxed{X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}}$$ ← N points

$$\omega = 2\pi f_0 \Rightarrow \frac{2\pi K}{N}$$

$$\Rightarrow \omega = \frac{2\pi K}{N} \quad \text{where } K = 0 \ldots\ldots N-1$$

$$X\left(e^{j\left(\frac{2\pi}{N}\right)K}\right) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi K}{N}}$$

$$\Rightarrow \boxed{x(k) = \sum_{n=0}^{N-1} x[n] \underbrace{e^{-j\frac{2\pi}{N}k}}_{\Downarrow\, W_N}}$$

⇓ DFT of $x[n]$

W_N [Twiddle factor]

⇑

$$\Rightarrow \boxed{x(k) = \sum_{n=0}^{N-1} x[n] W_N^{kn}} \quad \text{—— (1)}$$

(k = 0 to N-1)

IDFT

⇓

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} x[k] e^{j\omega_0 kn}$$

Substituting $\omega_0 = \frac{2\pi K}{n}$ and $W_N = e^{-j\frac{2\pi}{N}}$

$$\boxed{x[n] = \frac{1}{N} \sum_{k=0}^{N-1} x[k] W_N^{-kn}}$$

where $k = 0 \ldots N-1$

NOTE:

No of addition : $N(N-1)$
No of multiplication : $N \cdot N = N^2$ } large ⟺ disadvantage

4) Find $y[n]$ using DFT and IDFT for
$x[n] = \{1, 2, 0, 1\}$
$h[n] = \{2, 2, 1, 1\}$

$$x[n] * h[n] \underset{IDFT}{\overset{DFT}{\rightleftharpoons}} X(k) \cdot H(k)$$

DFT of $x[n]$: $X(k) = \sum_{K=0}^{N-1} x[n] W_N^{kn}$

$$W_N = e^{-j\frac{2\pi}{N}} = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}}$$

$$W_N = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2}$$

$$W_N = -j$$

$$\Rightarrow X(k) = \sum_{K=0}^{N-1} x[n][-j]^{kn}$$

where $k = 0, 1, 2, 3$

$$x[0] = \sum_{n=0}^{3} x[n]$$

$= 1 + 2 + 0 + 1$
$= 4$

$$x[1] = \sum_{n=0}^{3} x[n](-j)^n$$

$= 1 \cdot 1 + 2 \cdot -j + 0 + j = 1 - j$

$$x[2] = \sum_{n=0}^{3} x[n](-1)^n$$

$= 1 \cdot 1 + 2 \cdot -1 + 0 + 1 \cdot -1 = -2$

$$x[3] = \sum_{n=0}^{3} x[n][-j]^{3n}$$

$$= \sum_{n=0}^{3} x[n][j]^{n}$$

$$= 1 + 2 \cdot j + 0 - j = 1 + j$$

$$\Rightarrow \boxed{X(k) = \{4, 1-j, -2, 1+j\}} \text{ — (1)}$$

$$\Rightarrow H(k) = \sum_{n=0}^{N-1} h[n][-j]^{kn}$$

where $k = 0, 1, 2, 3$

$$H(0) = \sum_{n=0}^{3} h[n][-j]^{0n}$$

$$= \sum_{n=0}^{3} h[n]$$

$$= 2 + 2 + 1 + 1 = 6$$

$$H(1) = \sum_{n=0}^{3} h[n][-j]^{1n}$$

$$= \sum_{n=0}^{3} h[n][-j]^{n}$$

$$= 2 + 2 \cdot -j + 1 \cdot -1 + 1 \cdot j$$

$$= 1 - j$$

$$H(2) = \sum_{n=0}^{3} h[n][-j]^{2n}$$

$$= \sum_{n=0}^{3} h[n][-1]^{n}$$

$$= 2 \cdot 1 + 2 \cdot -1 + 1 \cdot 1 + 1 \cdot -1 = 0$$

$$H(3) = \sum_{n=0}^{3} h[n][-j]^{3n}$$

$$= \sum_{n=0}^{3} h[n][j]^{n}$$

$$= 2 + 2\cdot j + 1\cdot -1 + 1\cdot -j$$

$$= 1 + j$$

$$\Rightarrow \boxed{H(k) = \{6, 1-j, 0, 1+j\}} \text{ — } (2)$$

$$Y(k) = X(k)\cdot Y(k)$$

Multiplying (1) and (2),

$$Y(k) = \{24, (1-j)^2, 0, (1+j)^2\}$$

$$\boxed{Y(k) = \{24, -2j, 0, 2j\}} \text{ — } (3)$$

Now applying IDFT for (3), we get $y[n]$,

$$y[n] = \frac{1}{N} \sum_{K=0}^{N-1} Y(k)\cdot W_N^{-kn}$$

$$= \frac{1}{4} \sum_{K=0}^{3} Y(k)(j)^{-kn}$$

where $n = 0$ to 3

For $y[0] = \frac{1}{4} \sum_{K=0}^{3} Y(k)(j)^{0} = \frac{1}{4} \sum_{K=0}^{3} Y(k)$

$$= \frac{1}{4}\cdot 24 = 6$$

$$y[1] = \frac{1}{4}\sum_{k=0}^{3} y[k)(-j)^{-k}$$

$$= \frac{1}{4}\left[24(1) + (-2j)(-j)^{-1} + 0 + 2j(-j)^{-3}\right]$$

$$= \frac{1}{4}\left[24 + 2 + 0 + 2\right]$$

$$= \frac{1}{4} \cdot 28 = 7$$

$$y[2] = \frac{1}{4}\sum_{k=0}^{3} y(k)(-j)^{-2k}$$

$$= \frac{1}{4}\left[24(1) + (-2j)(-1) + 0 + 2j(-1)\right]$$

$$= \frac{1}{4} \cdot 24 = 6$$

$$y[3] = \frac{1}{4}\sum_{k=0}^{3} y(k)(-j)^{-3k}$$

$$= \frac{1}{4}(24 + -2 + 0 - 2)$$

$$= \frac{1}{4} \cdot 20 = 5$$

$$\Rightarrow \boxed{y[n] = \{6, 7, 6, 5\}}$$

5) IDFT : $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn}$

S.T this expression converges to $x[n]$

$$X[K] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$

T.P: $x[n] = \frac{1}{N}\sum_{K=0}^{N-1} X(k) W_N^{-kn}$

<u>LHS:-</u> $\frac{1}{N}\sum_{K=0}^{N-1} X(k) W_N^{-kn}$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{K=0}^{N-1} x[n] W_N^{kn} \cdot W_N^{-kn}$$

$$= \frac{1}{N} x[n] \sum_{n=0}^{N-1} 1$$

$$= \frac{1}{N} x[n] \cdot N$$

$$= x[n]$$

$$= \underline{RHS}$$

<u>Therefore</u>, it is proved that

$$\boxed{x[n] = \frac{1}{N}\sum_{K=0}^{N-1} X(k) W_N^{-kn}}$$

DFT application - matrix representation :

DFT $\rightarrow$ $x(k) = \sum_{n=0}^{N-1} x[n] W_N^{kn}$ | Vector = (N×1) (column)

IDFT $\rightarrow$ $x[n] = \frac{1}{N} \sum_{K=0}^{N-1} x(K) W_N^{-Kn}$ | vector = (N×1) column

$\boxed{\text{where } W_N = e^{-j\frac{2\pi}{N}} \text{ (twiddle factor)}}$

(twiddle matrix) $D_N = (N \times N)$ vector where row $= K$ (0 to N-1) & column $= n$ (0 to N-1)

$$D_N = \begin{bmatrix} W_N^0=1 & W_N^0=1 & W_N^0=1 & W_N^0=1 & \cdots & W_N^0=1 \\ W_N^0=1 & W_N^1 & W_N^2 & W_N^3 & \cdots & W_N^{N-1} \\ W_N^0=1 & W_N^2 & W_N^4 & W_N^6 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ W_N^0=1 & W_N^{N-2} & W_N^{2(N-2)} & W_N^{3(N-2)} & \cdots & W_N^{(N-1)(N-2)} \\ W_N^0=1 & W_N^{N-1} & W_N^{2(N-1)} & W_N^{3(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & W_N^3 & \cdots & W_N^{N-1} \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & W_N^{3(N-1)} & & W_N^{(N-1)^2} \end{bmatrix}$$

N - number of data samples considered

6) For $x_N = \{1,2,0,1\}$ and $h_N = \{2,2,1,1\}$: $\boxed{\text{4 pt DFT}}$ calculate using matrix representation

$x(K) = D_N \cdot x_N$

$N = 4$; $D_N \rightarrow 4 \times 4$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \cdot x_N$$

where $W_4 = e^{-j\left(\frac{2\pi}{4}\right)}$

NOTE: When W_N ~~when~~ of multiples of N is given as / can be written as added terms

$$\Rightarrow X(K) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix} \cdot x_N$$

$$W_4^0 = 1$$
$$W_4^1 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}} = -j$$
$$W_4^2 = -1$$
$$W_4^3 = +j$$

$$\Rightarrow D_N = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$\Downarrow$

Symmetric $\rightarrow$ transpose will be same

For IDFT, take complex conjugate of D_N and replace it

$$\Rightarrow \cancel{x[n] =} \; x[k] = \underset{[N \times N]}{D_N^*} \cdot \underset{[N \times 1]}{X_K}$$

Steps: DFT $\rightarrow X(K)$: $X_K = D_N \cdot x_N$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} 4 \\ 2+-2j-2+2j=0 \\ 0 \\ -1 \end{bmatrix}} \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix}$$

$$H_K = H(K) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

$$Y(K) = x(K) \cdot H(K)$$

$$= \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix} \begin{bmatrix} 6 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

$1^2 + j^2 - 2j = -2j$

$1^2 + j^2 + 2j$

$= 2j$

$$= \begin{bmatrix} 24 \\ -2j \\ 0 \\ 2j \end{bmatrix}$$

IDFT: $Y_K \Rightarrow y[n] = \frac{1}{N} \sum_{K=0}^{N-1} Y(K) \cdot W_N^{-KN}$

$$= \frac{1}{4} \left[D_N^* \cdot Y_K \right]$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 24 \\ -2j \\ 0 \\ 2j \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 24 \\ 28 \\ 24 \\ 20 \end{bmatrix} = \begin{bmatrix} 6 \\ 7 \\ 6 \\ 5 \end{bmatrix} = Y(K)$$

$$\Rightarrow \boxed{y[n] = \{6, 7, 6, 5\}}$$

DFT Properties:

(a) Linearity

$$x[n] \xrightarrow[N]{DFT} X(K)$$

$$g[n] \xrightarrow[N]{DFT} G(K)$$

$$\alpha x[n] + \beta g[n] \xrightarrow[N]{DFT} \alpha X(k) + \beta G(k)$$

(b) Circular Shift (Time)

$$x[n] \xrightarrow[N]{DFT} X(K)$$

$$x[n-n_0] \xrightarrow[N]{DFT} W_N^{Kn} x[n]$$

(c) Circular Shift (Frequency)

$$x[n] \xrightarrow[N]{DFT} x(K)$$

$$W_N^{-K_0 n} x[n] \xrightarrow[N]{DFT} X[K-K_0]$$

(d) Convolution

$$x_1[n] \ \& \ x_2[n] \Rightarrow \text{"N"}$$

$$y[n] \Rightarrow x_1[n] \, N \, x_2[n]$$

$$y[n] = \sum_{m=0}^{N-1} x_1[m] \, x_2[(n-m)]_N$$

$$x_1[n] \, N \, x_2[n] \xrightarrow[N]{DFT} X_1(K) \cdot X_2[K]$$

$$x_1[n] \cdot x_2[n] \xrightarrow[N]{DFT} X_1[K] \; (N) \; X_2[K]$$

$\boxed{AM:-}$

$$\left.\begin{aligned} m(t) &= A_m \sin(\omega_m t) \\ c(t) &= A_c \sin(\omega_c t) \end{aligned}\right\} \text{sum \& diff}$$

(e) Time Reversal

$$x[n] \xrightarrow[N]{DFT} X(K)$$

$$x[(-n)]_N \xrightarrow[N]{DFT} \cancel{x[N-n]_N} \; x((-k))_N$$

$$x[N-n] \qquad\qquad [X(N-k)]$$

eg: $x[0] = x[4-0]_4 = x[4]_4$

$x[-1] = x[4-1]_4 = x[3]_4$

$x[-2] = x[4-2]_4 = x[2]_4$

$x[-3] = x[4-3]_4 = x[1]_4$

Fast Fourier Transform:

DFT has complexity and is time consuming because # addition $[N(N-1)]$ and # multiplication is more

Here, we use FFT as the multiplication involved is $\frac{N}{2}\log_2 N$ and addition is $N \log_2 N$

In FFT, N should be a multiple of 2^M and then only, it works and provides an algorithm for DFT

FFT → decimation in time (reducing / down sampling)
→ d ... frequency

For FFT, we use butterfly diag

(Decimation in time) DIT FFT:

Algorithm :- $N = 2^M$

(i) # Samples $N = 2^M \rightarrow$ integer

(ii) input sequence should be in BIT reversal order; output sequence should be in normal order

(iii) ~~each~~ # stages $= M = \log_2 N$

(iv) Each stage will have have $\frac{N}{2}$ butterfly

(v) Twiddle factor : $W_N = e^{-j\frac{2\pi}{N}}$

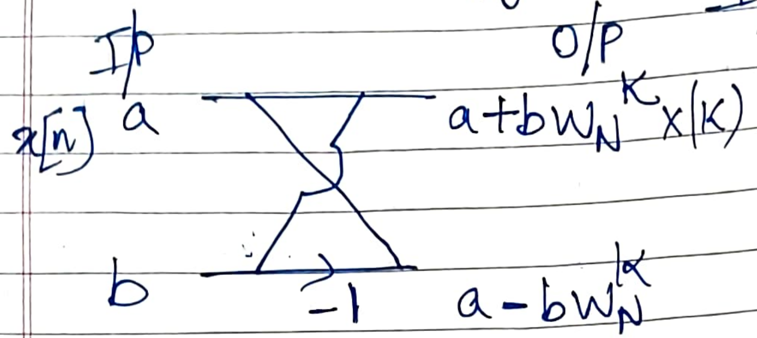
W_N^K will have $K = \frac{Nt}{2^m}$

where $m \rightarrow$ stage index
$t \rightarrow 0, 1, \ldots, 2^{(m-1)} - 1$

eg: $N = 8 \Rightarrow M = 3 \Rightarrow m =$ Stage 1, stage 2, stage 3 (index)

Radix-2-butterfly

I/p $\longleftrightarrow$ O/p
"n" "K"



Bit reversal :-

2 i/ps :- 1 bit
4 i/ps :- 2 bit
8 i/ps :- 3 bit

For 4 i/ps :- $x[n] = \{\underset{0}{-1}, \underset{1}{0}, \underset{2}{1}, \underset{3}{2}\}$

2 bits		Bit reversal
(0) 00	$\longrightarrow$	00 (0)
(1) 01	$\longrightarrow$	10 (2)
(2) 10	$\longrightarrow$	01 (1)
(3) 11	$\longrightarrow$	11 (3)

7) $x[n] = \{1, 2, 0, 1\}$

(i) $N = 2^M \Rightarrow \boxed{M = 2}$
$N = 4$

(ii) Twiddle factor exponent (K)

W_N^K ; $K = \frac{Nt}{2^m}$; $t = 0, 1 \ldots\ldots 2^{m-1} - 1$

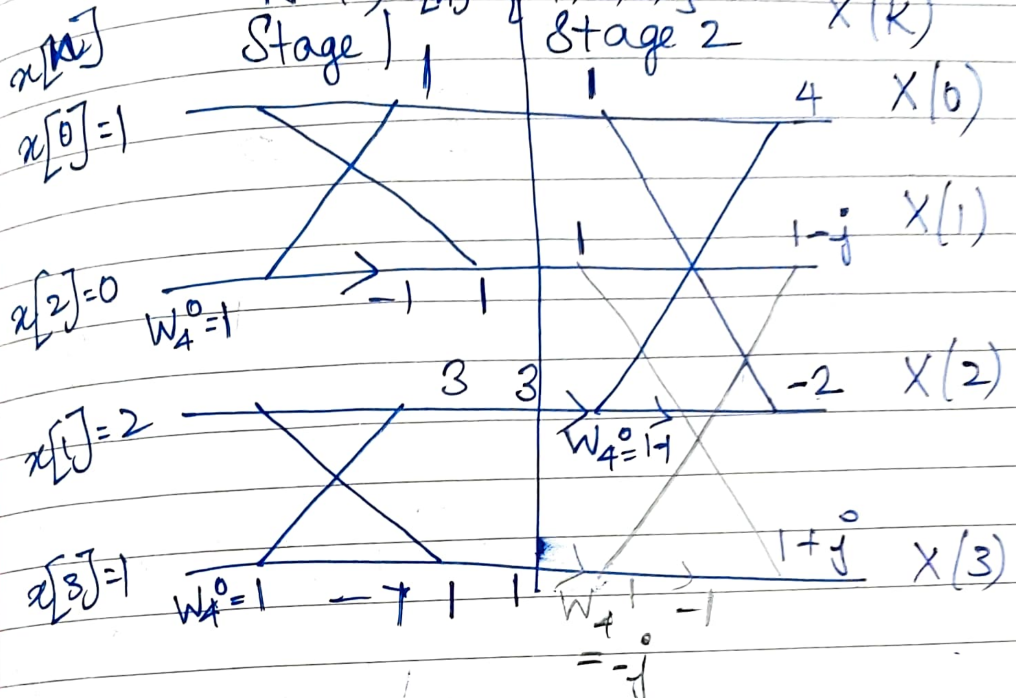
for $m = 1$; for $t = 0 \rightarrow K = 0$

for $m = 2$; for $t = 0, 1 \rightarrow K = 0, 1$

$W_N \Rightarrow W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}} = -j$

$$\left.\begin{aligned} W_4^K \Rightarrow W_4^0 &= (-j)^0 = 1 \\ W_4^1 &= (-j)^1 = -j \end{aligned}\right\} \text{for } k = 0, 1$$

$N = 4$; $x[n] = \{1, 2, 0, 1\}$ $M = 2$



$X(K) = \{4, 1-j, -2, 1+j\}$

multiplications : $\frac{N}{2}\log_2 N$: 4

additions : $N \log_2 N$: 8

8) $h[n] = \{2, 2, 1, 1\}$ Compute DIT-FFT and $H(K)$

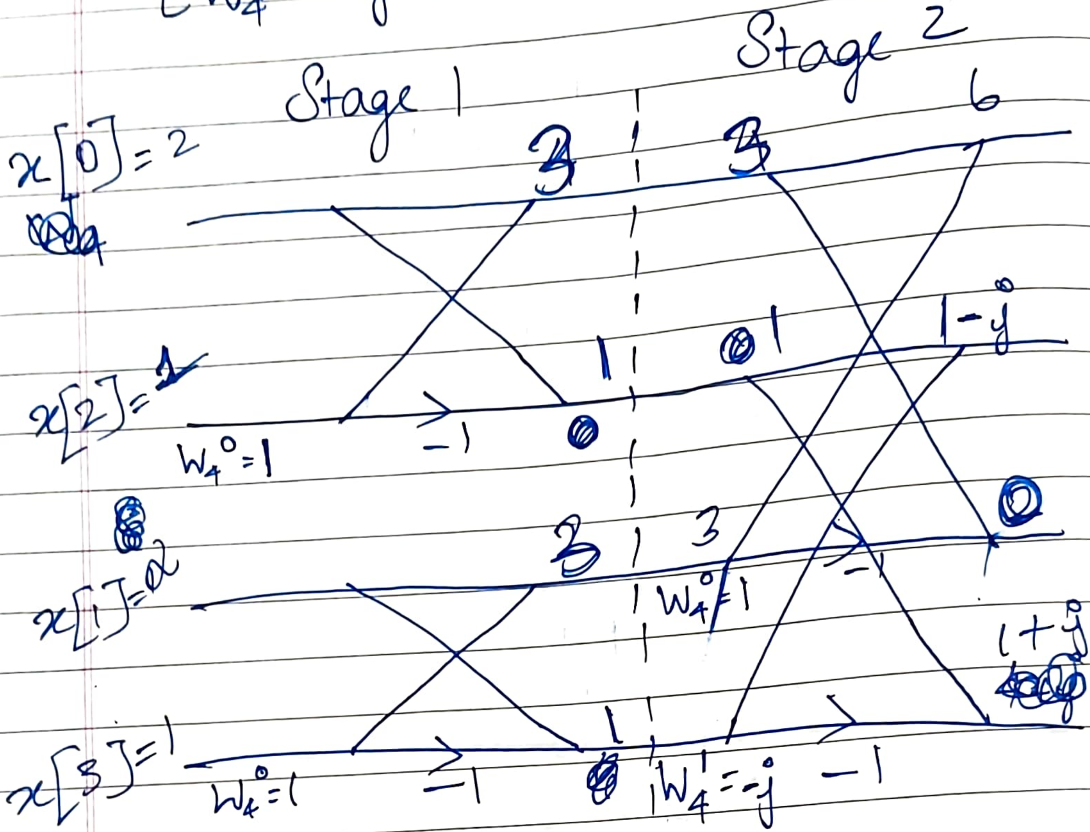
(i) $N = 4$ $\quad h[n] = \{2, 2, 1, 1\}$
$M = 2$

(ii) Twiddle factor exponent (K)
W_N^K ; $K = \frac{Nt}{2^m}$; $t = 0, 1, \ldots\ldots 2^{(m-1)} - 1$

for $m = 1$; $t = 0 \rightarrow K \rightarrow 0$
for $m = 2$; $t = 0, 1 \rightarrow K \rightarrow 0, 1$

$W_N = e^{-\frac{2\pi}{4}j} = -j$

$W_4^k = \begin{cases} W_4^0 = 1 \\ W_4^1 = -j \end{cases}$ for $k = 0, 1$



<u>8 pt FFT.</u>

9) $x[n] = \{1, 2, 3, 4, 4, 3, 2, 1\}$

(i) $N = 8$
$\Rightarrow M = 3$

(ii) $W_N^k ; K = \frac{Nt}{2^m} ; t = 0, 1, \ldots, 2^{m-1} - 1$

$M = 3 \Rightarrow m = 1, 2, 3$

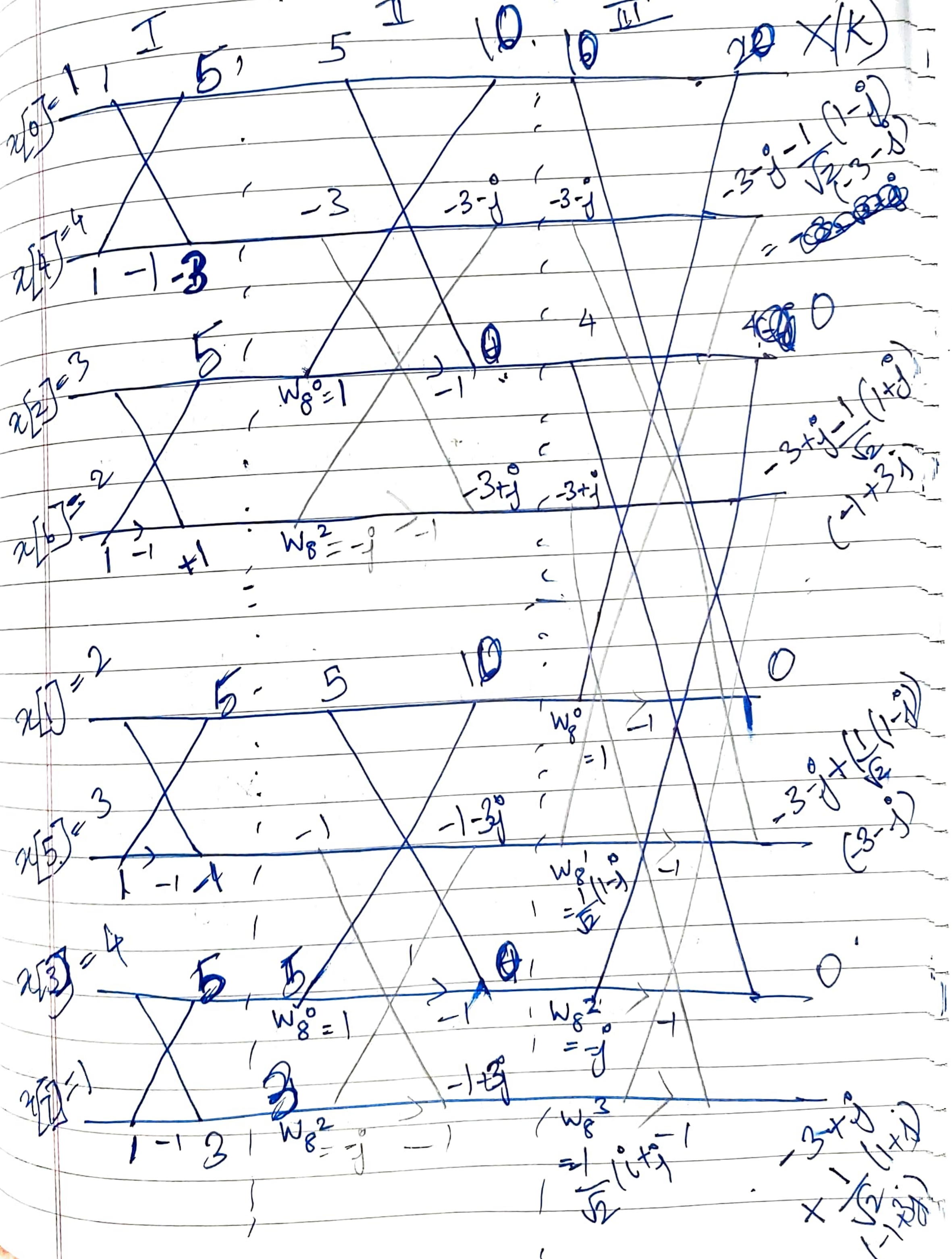
$t = 0 \quad K = 0$
$t = 0, 1 \quad K = 0, 2$
$t = 0, 1, 2, 3 \quad K = 0, 1, 2, 3$

Twiddle factor -

K	W_N^K
0	1
1	$\frac{1}{\sqrt{2}}(1-j)$
2	$-j$
3	$-\frac{1}{\sqrt{2}}(1+j)$

Bit reversal

000 (0)	(0)
001 (1)	(4)
010 (2)	(2)
011 (3)	(6)
100 (4)	(1)
101 (5)	(5)
110 (6)	(3)
111 (7)	(7)



IFFT (Inverse Fast Fourier Transform):

$$\text{DFT} \Rightarrow X(K) = \sum_{n=0}^{N-1} x[n] W_N^{Kn}$$
$x[n]$

$$\text{IDFT} \Rightarrow x[n] = \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-Kn}$$
$x(K)$

$$\text{DIT} \Rightarrow K = \frac{Nt}{2^M} ; t = 0, 1, \ldots 2^{m-1}-1$$

$$x[n] = \frac{1}{N} \sum_{K=0}^{N-1} x(K) W_N^{-Kn}$$

Taking complex conjugate, $x^*[n] = \frac{1}{N} \sum_{K=0}^{N-1} x^*(K) W_N^{Kn}$

Taking complex conjugate, $x[n] = \frac{1}{N} \left[\sum_{K=0}^{N-1} x^*(K) W_N^{Kn} \right]^*$

DIT-FFT: $N = 2^M$

{ Bit reversal at i/p
normal order at o/p }

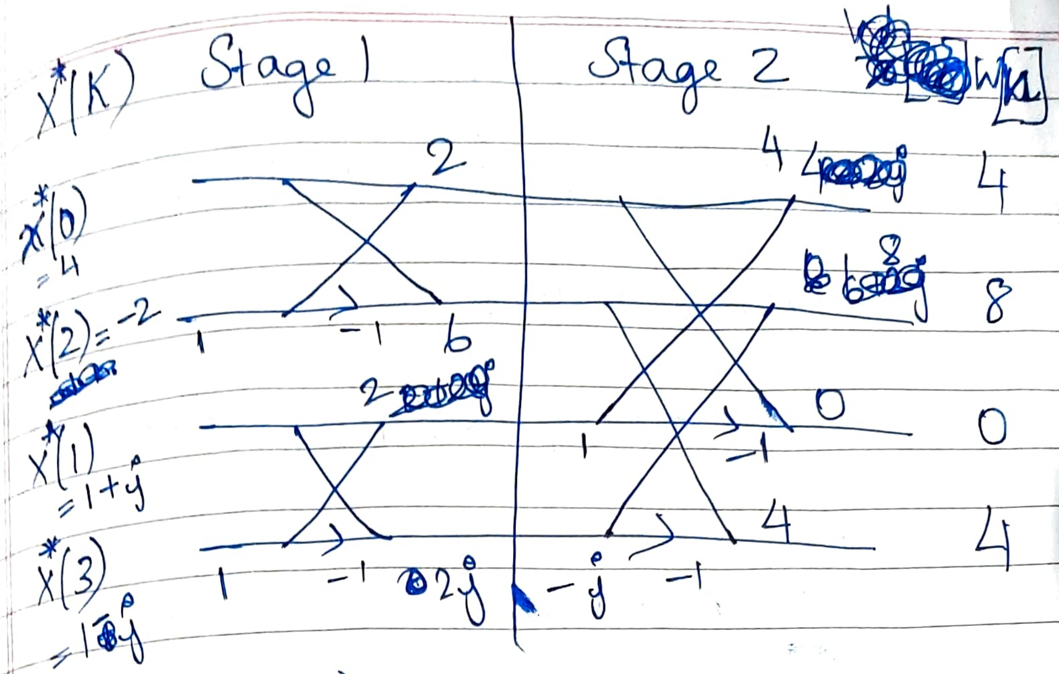
For IFFT, $x(K)$ - i/p ; $x[n]$ - o/p

1Q) $x(K) = \{4, 1-j, -2, 1+j\}$

$N = 4$ $\quad W_4^0 = 1$

$M = 2$ $\quad W_4^1 = -j$

Find $x[n]$ using IFFT



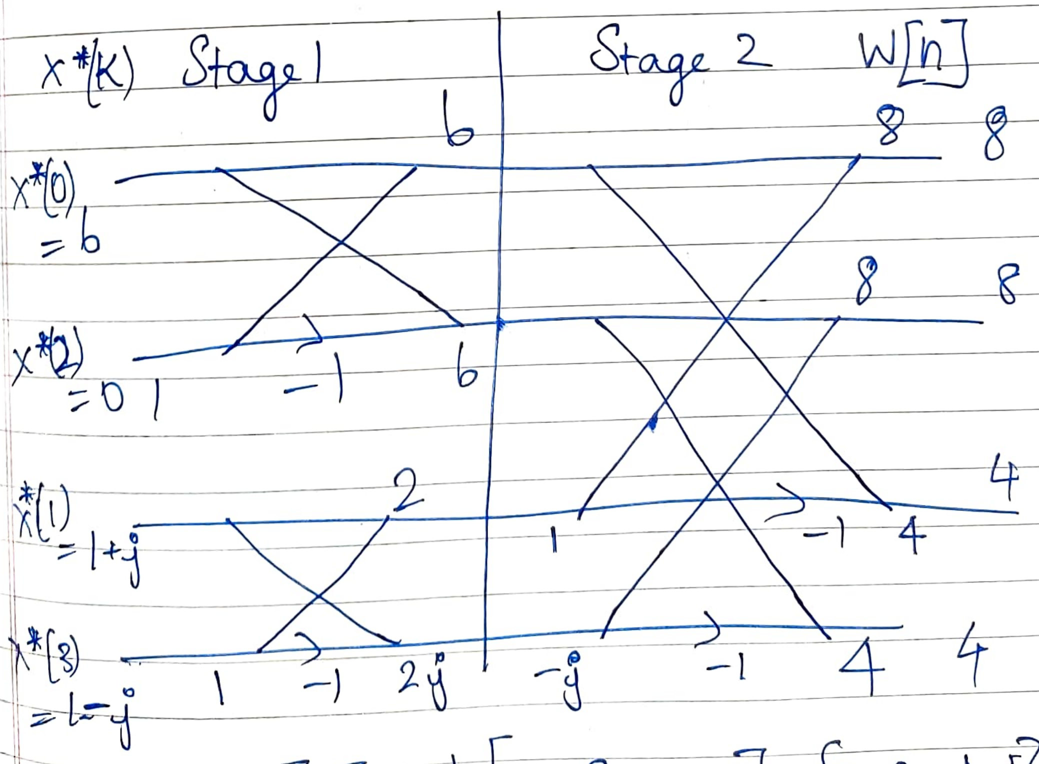
$x[n] = \{1, 2, 0, 1\}$

11) $H(K) = \{6, 1-j, 0, 1+j\}$

$N = 4$ $\qquad W_4^0 = 1$

$M = 2$ $\qquad W_4^1 = -j$

Find h[n] using IFFT



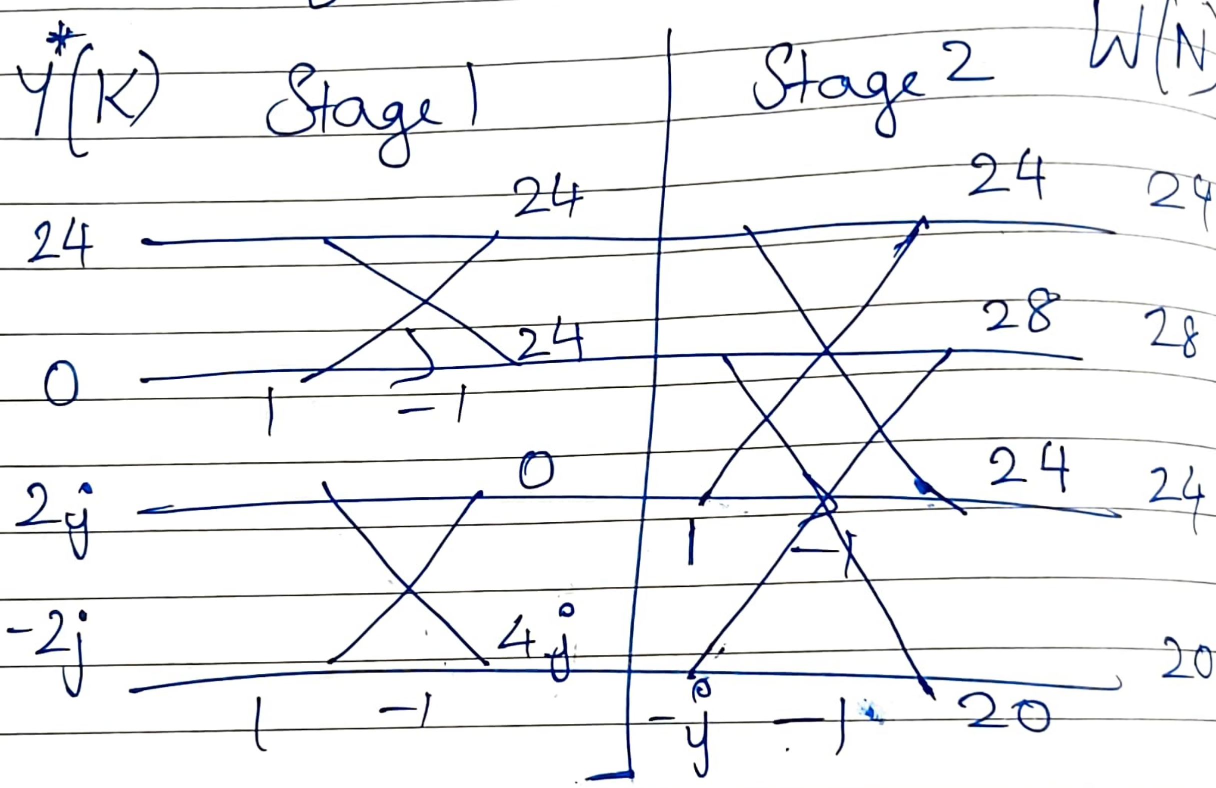
$\Rightarrow x[n] = \frac{1}{4}[8, 8, 4, 4] = \{2, 2, 1, 1\}$

12) $X(K) = \{4, 1-j, -2, 1+j\}$

$H(K) = \{6, 1-j, 0, 1+j\}$

$Y(K) = H(K) * X(K)$

$= \{24, -2j, 0, 2j\}$



$y[n] = \frac{1}{4}\{24, 28, 24, 20\}$

$y[n] = \{6, 7, 6, 5\}$

classmate

Date

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13) $X(K) = \{20, -5.828 - 2.414j, 0, -0.172 - 0.414j, 0, -0.172 + 0.414j, 0, -5.828 + j2.414\}$

$N = 8$

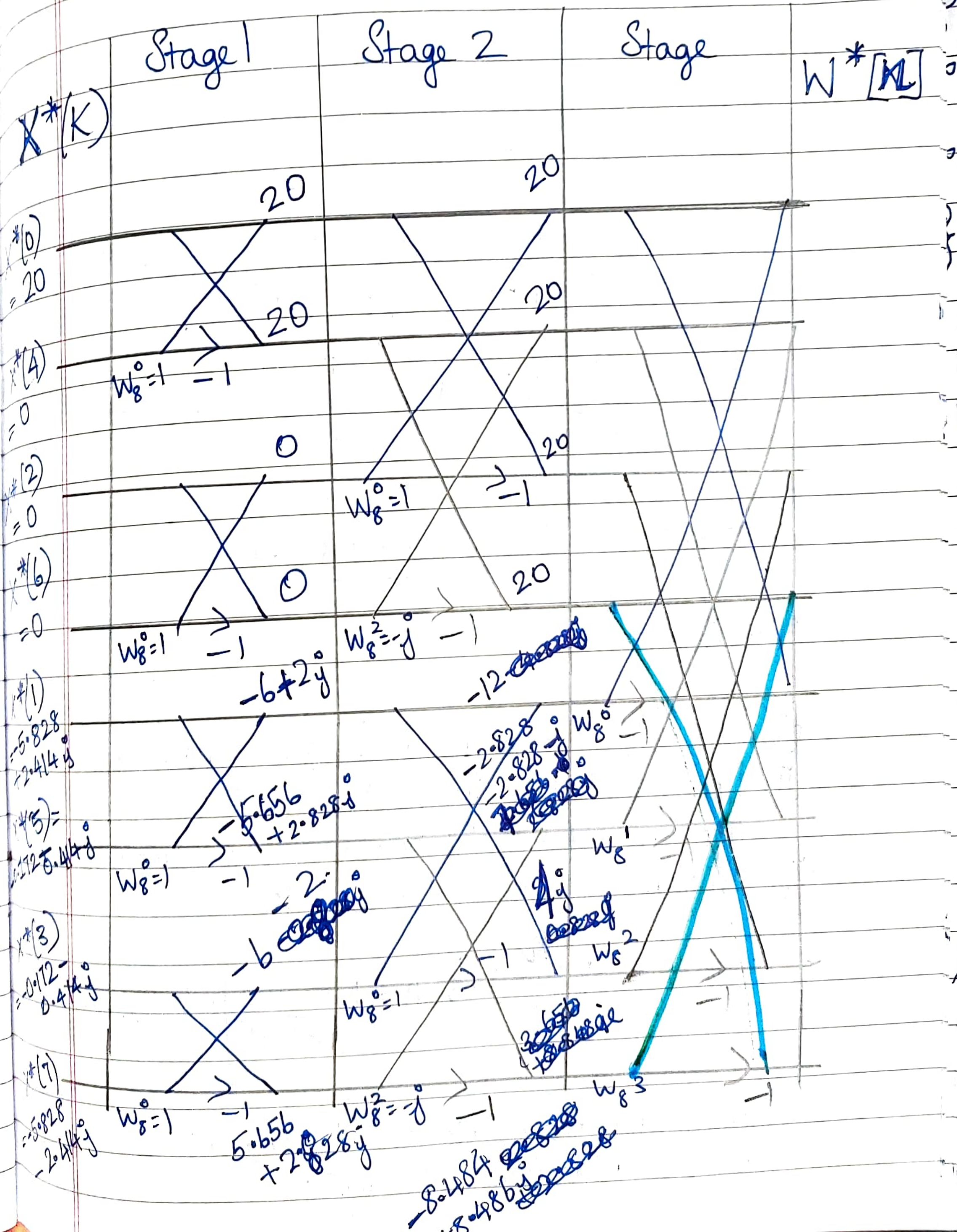
$M = 3$

$W_8^0 = 1$

$W_8^1 = \frac{1}{\sqrt{2}}(1 - j) = 0.707(1 - j)$

$W_8^2 = -j$

$W_8^3 = \frac{-1}{\sqrt{2}}(1 + j) = -0.707(1 + j)$



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$W[0] = 20 - 12 = 8$

$W[1] = 20 + (2.828 - 2.828j)(0.707 - 0.707j)$

$= 20 + (-2 - 2j + 2j - 2)$

$= 20 + (-4) = 16$

$W[5] = 20(-(-4)) = 24$

$W[3] = 20 + (8.484j - 8.484)(-0.707 - 0.707j)$

$= 20 + (12) = 32$

$W[7] = 20 - 12 = 8$

$W[n] = \{8, 16, 24, 32, 32, 24, 16, 8\}$

$x[n] = \frac{1}{N}[W[n]] = \{1, 2, 3, 4, 4, 3, 2, 1\}$

(Decimation in Frequency) DIF-FFT:

Algorithm -

(i) $N = 2^M$; M - no of stages; m - index of current stage

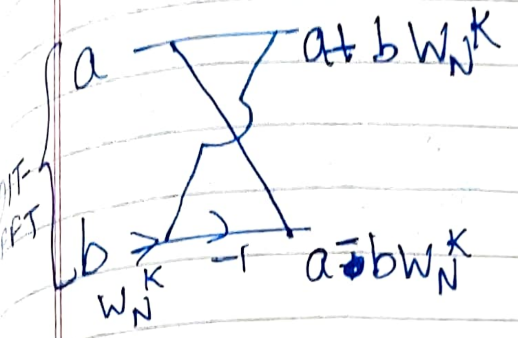
(ii) i/p would be in normal order
o/p would be in bit-reversal order.

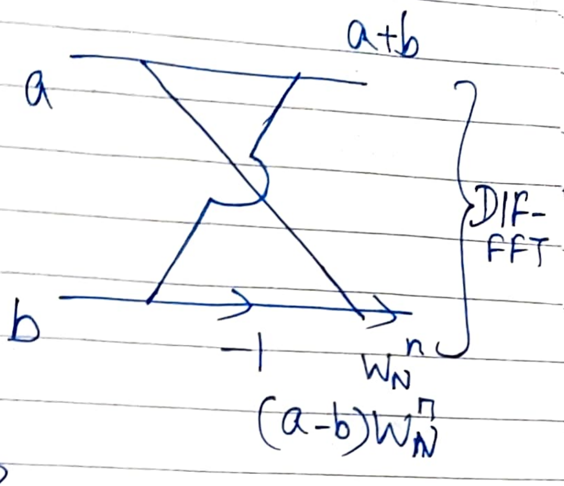
(iii) Twiddle factor exponent (k)

$$W_N^k \; ; \; \boxed{n = \frac{Nt}{2^{M-m+1}}} \; ; \; t = 0, 1, \ldots, 2^{(M-m)} - 1$$

no of multiplications - $\frac{N}{2}\log_2 N$
no of additions - $N\log_2 N$

Radix 2 - Butterfly diag





14) $x[n] = \{1, 2, 0, 1\}$

$N = 2^M \Rightarrow 4 = 2^M \Rightarrow M = 2$; $m = 1, 2$

i/p				o/p
00	(0)	(0)	00	
01	(1)	(2)	10	
10	(2)	(1)	01	
11	(3)	(3)	11	

$W_N^n \Rightarrow n = \frac{Nt}{2^{M-m+1}}$; $t = 0, 1, \ldots\ldots 2^{(M-m)} - 1$

$t = 0, 1, \ldots\ldots 2^{2-1} - 1$

for $m = 1$, $t = 0, 1$

$\Rightarrow n = 0$

$n = \frac{4}{2^{2-1+1}} = \frac{4}{4} = 1$

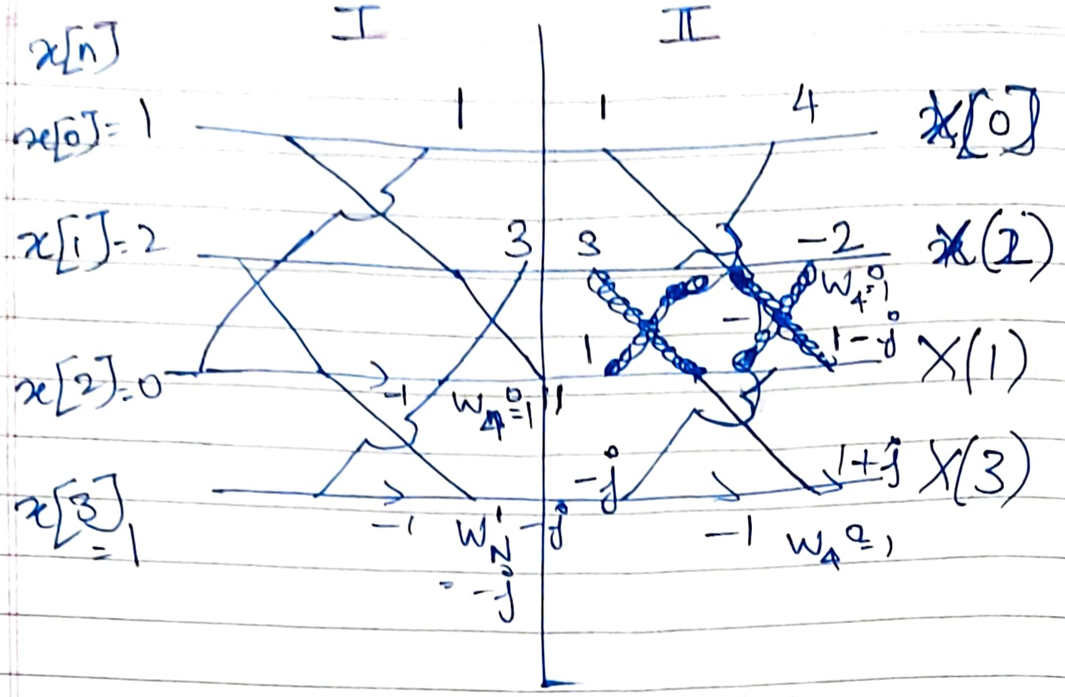
for $m = 2$, $t = 0, \ldots\ldots 2^{2-2} - 1$

$t = 0 \Rightarrow n = 0$

$(W_N)^n = e^{-j\frac{2\pi}{4}} = -j$

$(W_N)^0 = 1$

$(W_N)^1 = -j$



$$\Rightarrow X(K) = \{4,\ 1-j,\ -2,\ 1+j\}$$

15) $h[n] = \{2, 2, 1, 1\}$

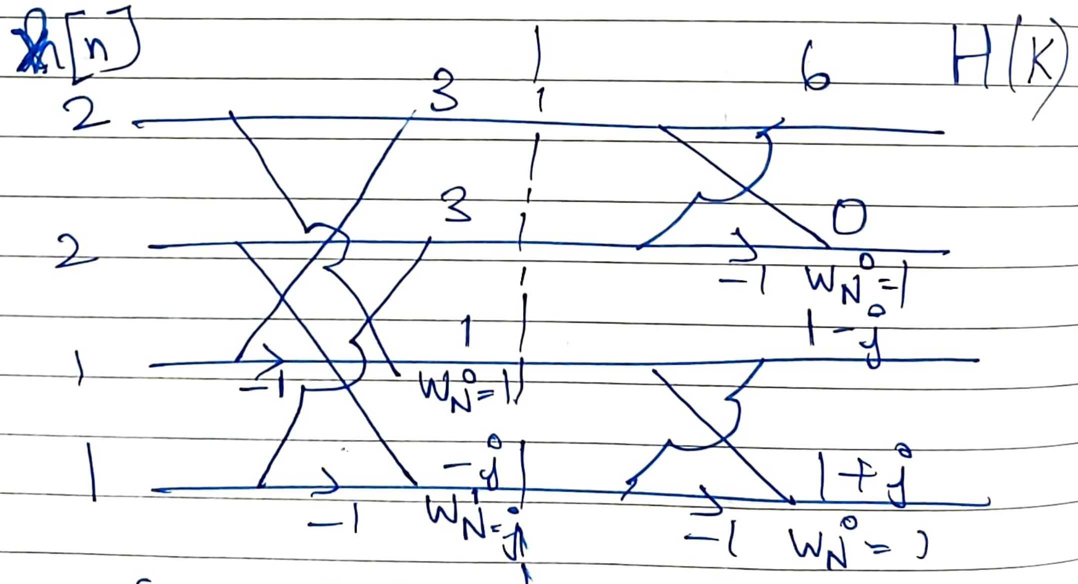
$N = 2^M \Rightarrow N = 4,\ M = 2 \Rightarrow m = 1, 2$

$t = 0, \ldots\ldots, 2^{2-1} - 1$

for $m = 1 \rightarrow \begin{cases} t = 0, 1 \\ n = 0, 1 \end{cases}$

for $m = 2 \rightarrow \begin{cases} t = 0 \\ n = 0 \end{cases}$

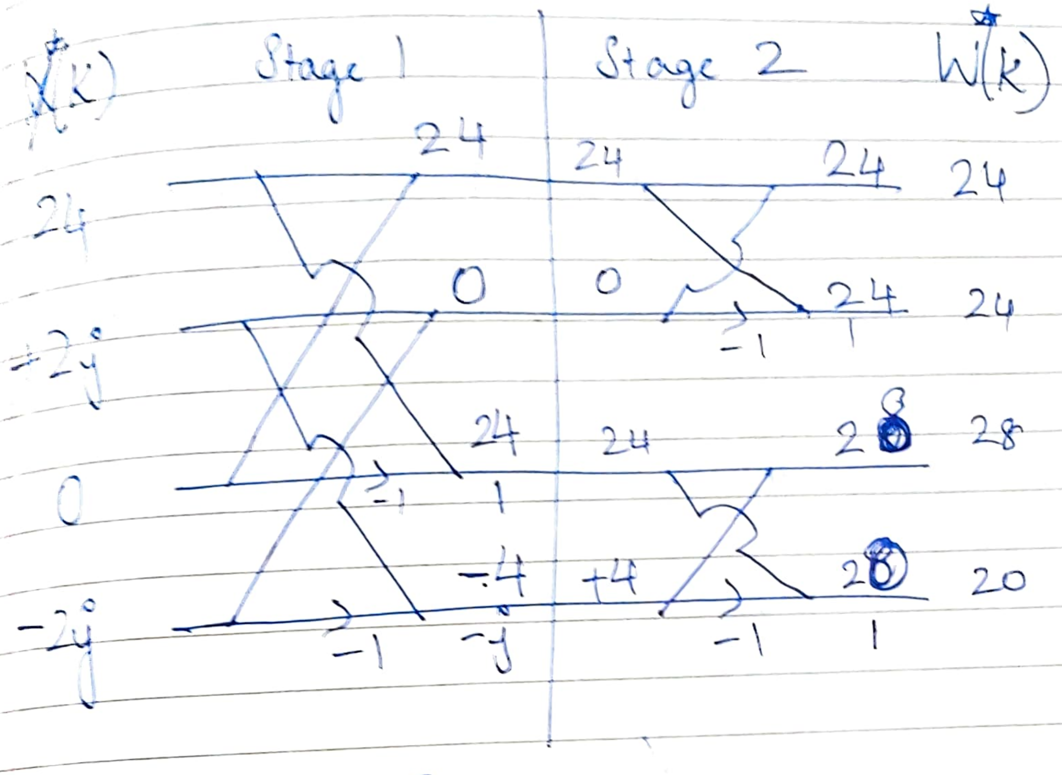
$W_N^0 = 1 \qquad W_N^1 = -j$



$$H(K) = \{6,\ 1-j,\ 0,\ 1+j\}$$

$\therefore Y(K) = H(K) \cdot X(K)$

$Y(K) = \{24, -2j, 0, 2j\}$



$W^*(K) = \{24, 28, 24, 20\}$

$x(K) = \frac{1}{N}\{24, 28, 24, 20\}$

$= \frac{1}{4}\{24, 28, 24, 20\}$

$$\boxed{x(K) = \{6, 7, 6, 5\}}$$

17) $x[n] = \{0, 1, 2, 3, 4, 5, 6, 7\}$

$N = 2^M \Rightarrow N = 8,\ M = 3,\quad m = 1 \Rightarrow t = 0,1,2,3$

$m = 2 \Rightarrow t = 0, 1$

$n = \dfrac{Nt}{2^{M-m+1}}$

$m = 3 \Rightarrow t = 0$

$\Rightarrow t = 0,\ n = 0$

$t = 1,\ n = 1 \qquad m = 1 \Rightarrow N = 0, 1, 2, 3$

$t = 2,\ n = 2 \qquad m = 2 \Rightarrow N = 0, 2$

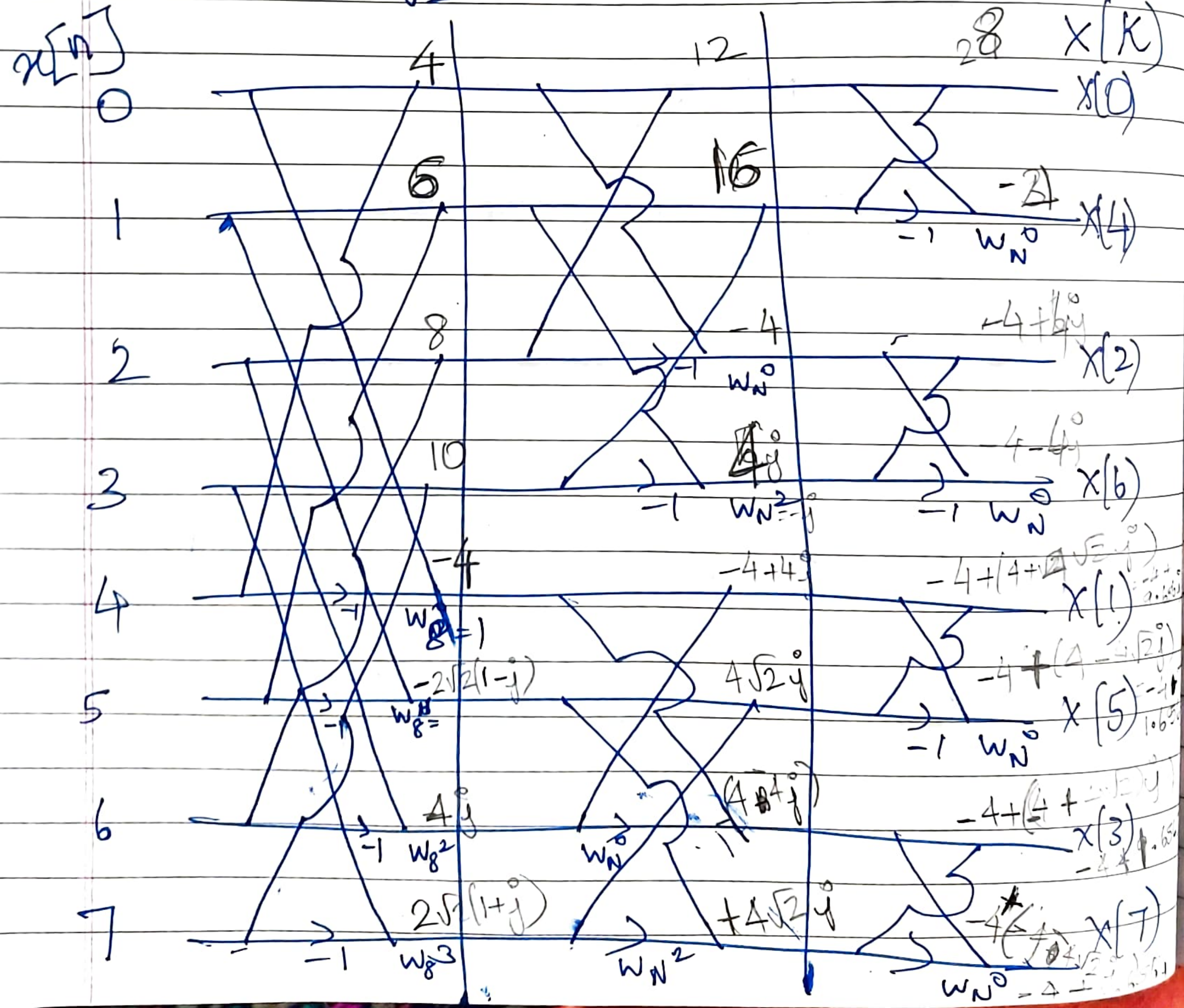
$t = 3,\ n = 3 \qquad m = 3 \Rightarrow N = 0$

$W_8^0 = 1$

$W_8^1 = \frac{1}{\sqrt{2}}(1-j)$

$W_8^2 = -j$

$W_8^3 = \frac{-1}{\sqrt{2}}(1+j)$



$$\Rightarrow X(K) = \cancel{\{27, -4+9.656j, -4+6j, -4-9.656j, -3, -4+1.656j, -4-6j, -4+1.656j\}}$$

$$\Rightarrow X(K) = \{28, -4+(4+4\sqrt{2})j, -4+4j, -4+(4-4\sqrt{2})j, -4, -4+(4-4\sqrt{2})j, -4-4j, -4+(4-4\sqrt{2})j\}$$

DIF - IFFT:

(Refer 16) also)

i/p - same normal order

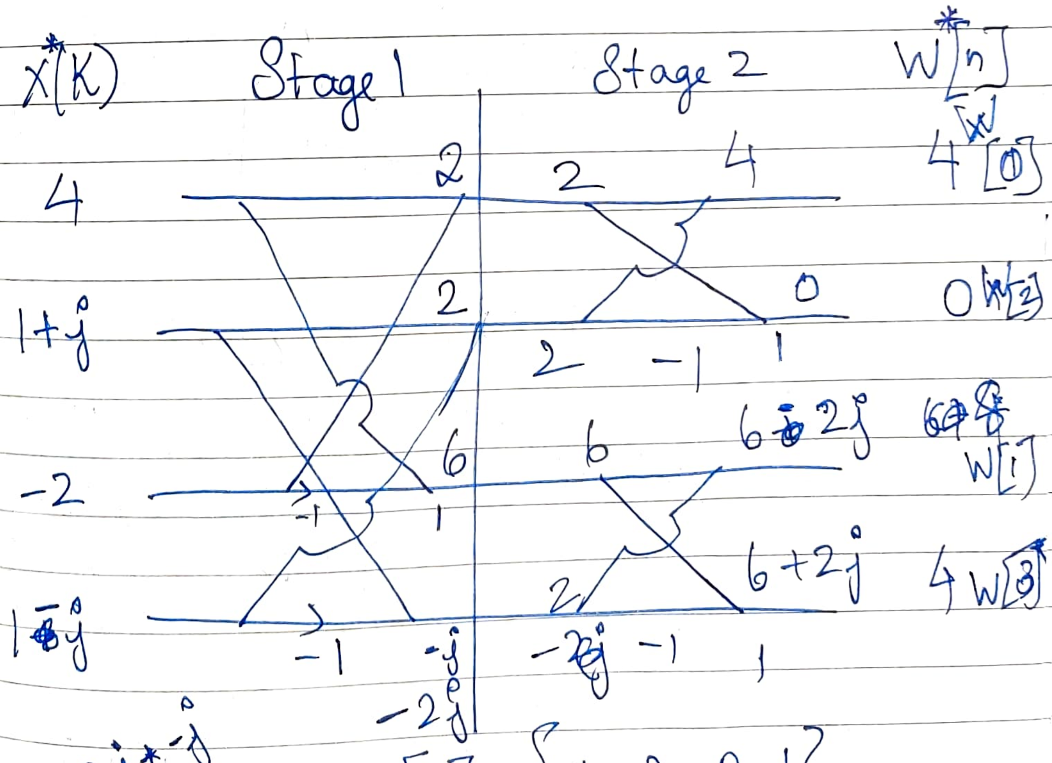
o/p - bit reversed order

$$x[n] = \frac{1}{N} W[n]^*$$

$X^*(K)$ — input → normal order

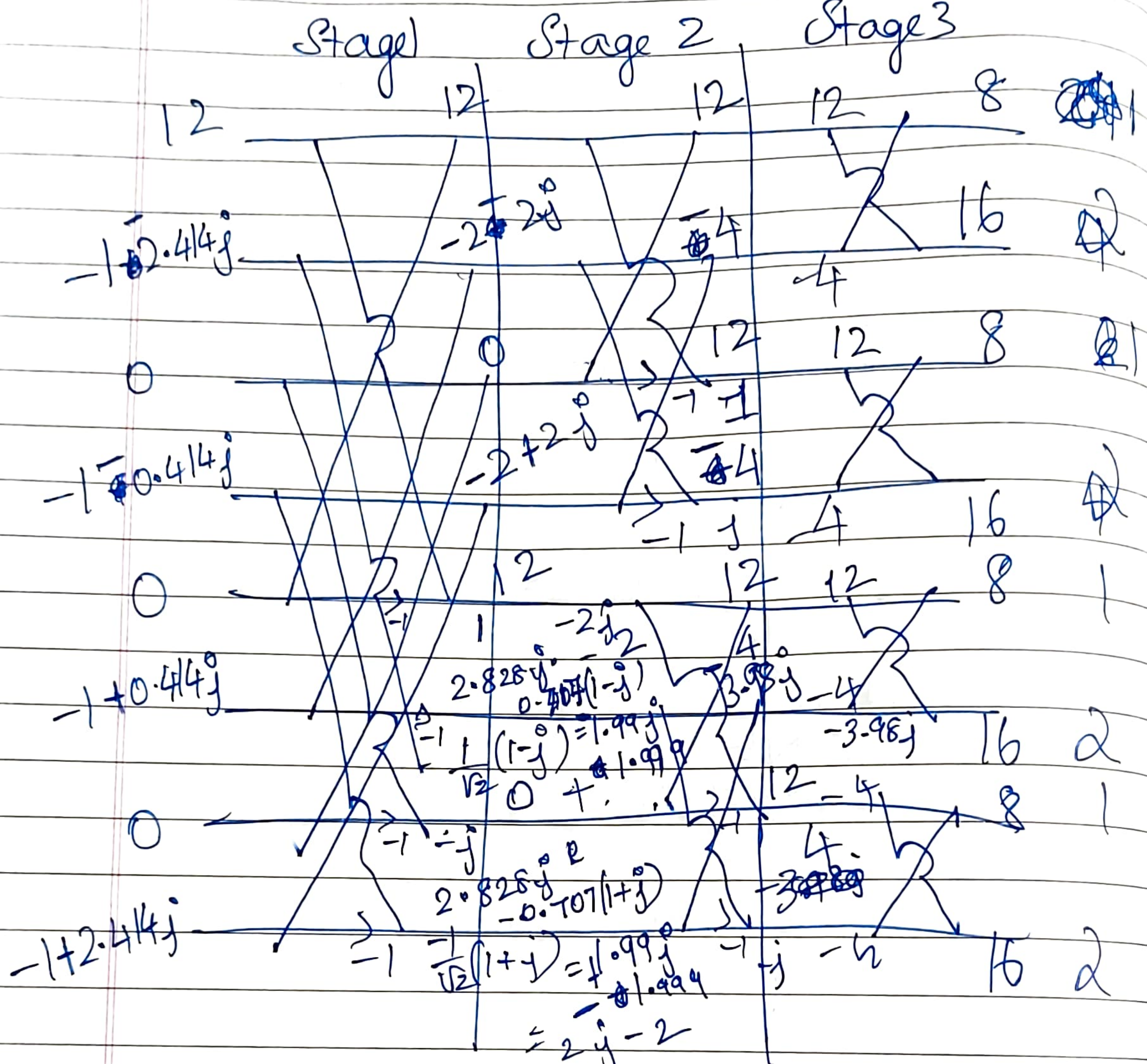
$N = 2^M$ — no of stages

$$K = \frac{Nt}{2^{M-m+1}} \; ; \; t = 0, 1 \ldots 2^{M-m} - 1$$

18) 

$$\Rightarrow x[n] = \{1, 2, 0, 1\}$$

19) $X(K) = \{12, -1+2.414j, 0, -1+0.414j, 0, -1-0.414j, 0, -1-2.414j\}$



$N = 2^M ; M = 3 \Rightarrow m = 1, 2, 3$

$\alpha = \frac{Nt}{2^{M-m+1}} \Rightarrow t = 0, 1, \ldots\ldots, 2^{M-m} - 1$

$t = 0, 1, 2, 3 \Rightarrow n = 0, 1, 2, 3$

$t = 0, 1 \Rightarrow n = 0, 2$

$t = 0, \Rightarrow n = 0$

$W_8^0 = 1$

$W_8^1 = \frac{1}{\sqrt{2}}(1-j)$

$W_8^2 = -j$

$W_8^3 = \frac{-1}{\sqrt{2}}(1+j)$

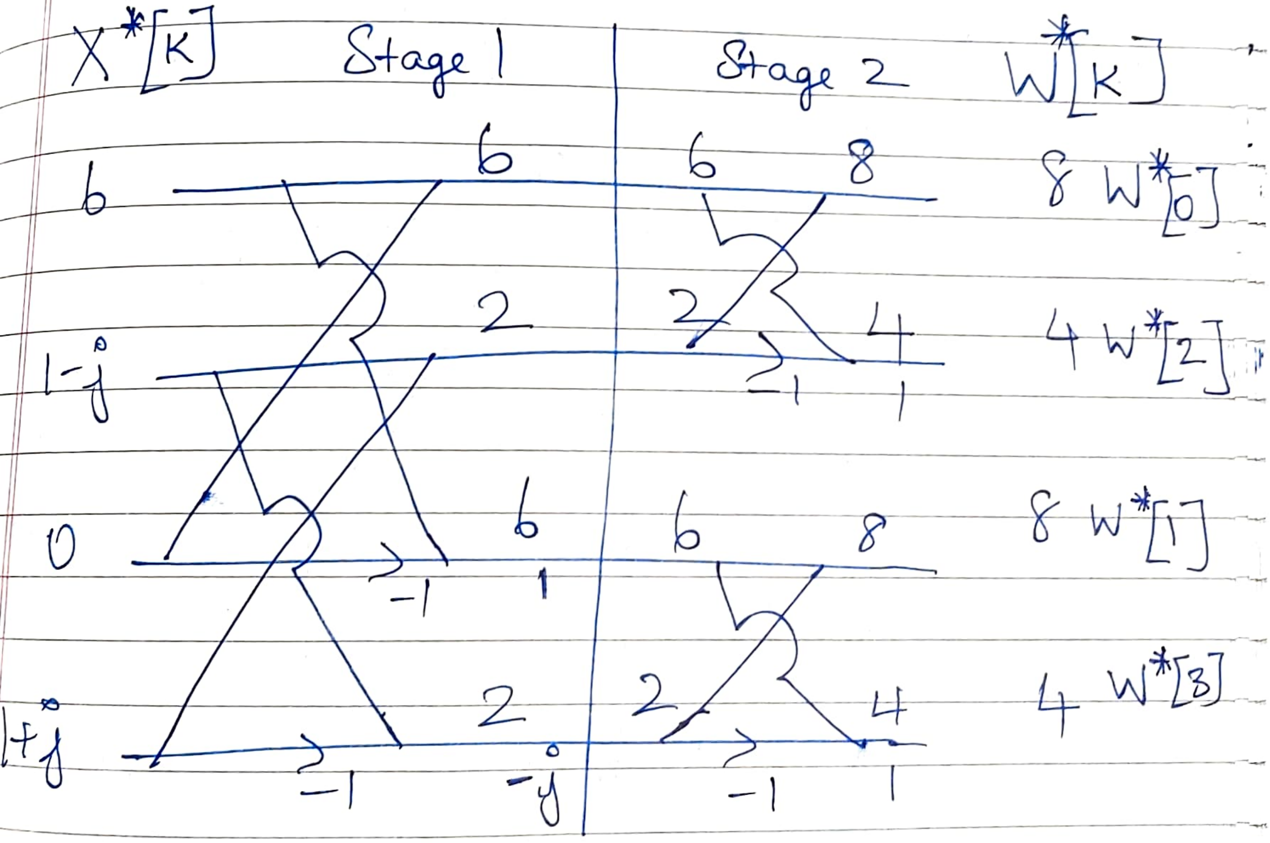
$x[n] = \{1, 1, 1, 2, 2, 2$

20) $H(K) = \{6, 1-j, 0, 1+j\}$

Sol: $N = 2^M \Rightarrow M = 3 \Rightarrow m = 1, 2, 3$

$n = \frac{Nt}{2^{M-m+1}}$; $t = 0, 1 \ldots\ldots 2^{M-m} - 1$

For $m=1$, $t = 0, 1, 2, 3 \Rightarrow n = 0, 1, 2, 3$
For $m=2$, $t = 0, 1 \Rightarrow n = 0, 2$
For $m=3$, $t = 0 \Rightarrow n = 0$



$w^*[k] = \{8, 8, 4, 4\}$

$H^*[n] = \frac{1}{N}\{w^*k\}$

$\boxed{H[n] = \{2, 2, 1, 1\}}$

~~Overlap Add method~~

Applications of FFT:

- DFT ⇒ circular (N)

i/p sequences are of same length and o/p is of that length too

If i/p sequences are not of uniform length, convolution is not possible

Limitations include complexity (in multiplication and addition) and in FFT, it has to be power of 2 is a restriction

I. Overlap Add method:

23) $x[n] = \{1, 0, -1, 2, 5, 4, 3, 2, 1, 4\} \rightarrow N_1$

$h[n] = \{1, 1, 1\} \rightarrow N_2$

FFT : $N = 2^M$ (not satisfied)

∴ New data block for 1st seq $\Big\} = N_1^* + N_2^* - 1 = 2^M = N$

eg. $2 + 3 - 1 = 2^2 = 4$

We cannot use $6 + 3 - 1 = 2^3 = 8$ ✗

$x_1[n] = \{1, 0, 0, 0\}$

$x_2[n] = \{-1, 2, 0, 0\}$ $h[n] = \{1, 1, 1, 0\}$

$x_3[n] = \{5, 4, 0, 0\}$

$x_4[n] = \{3, 2, 0, 0\}$

$x_5[n] = \{1, 4, 0, 0\}$

FFT for individual sequences $(x_1, x_2, \ldots\ldots x_5)$

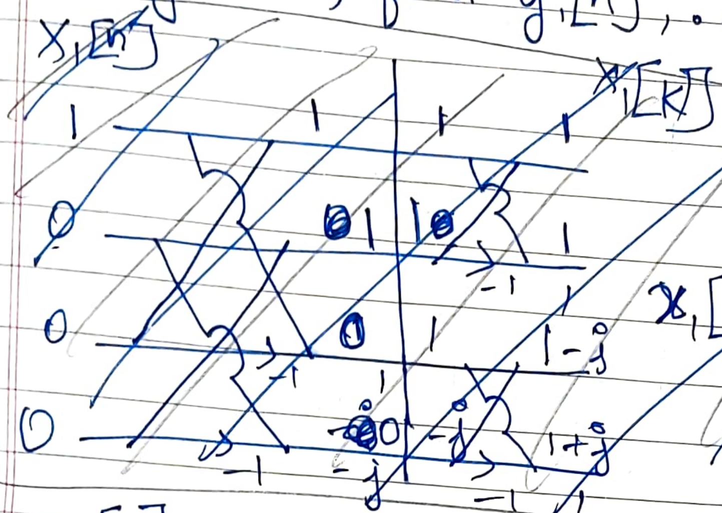
Find $X_1(K), X_2(K), \ldots\ldots X_5(K)$

$$y_1(K) = X_1(K)\,H(K)$$

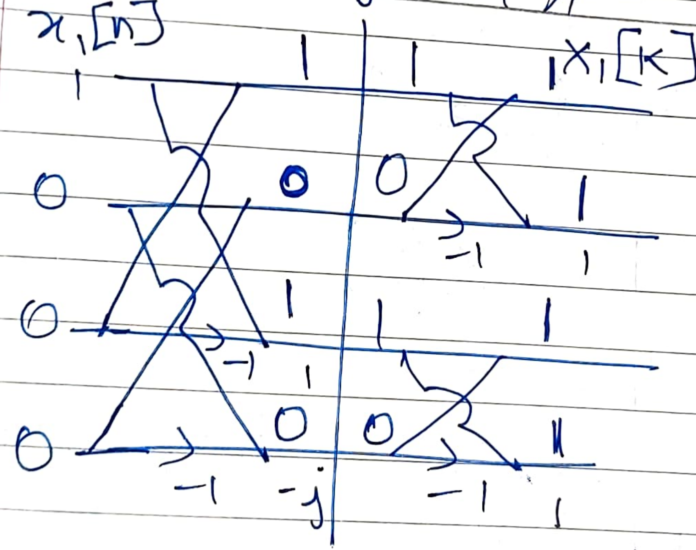
$$\vdots$$

$$y_5(K) = X_5(K)\,H(K)$$

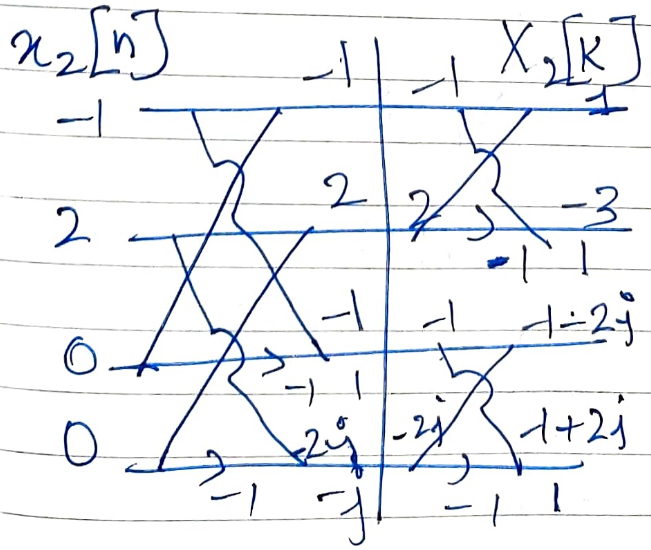
Using IFFT, find $y_1[n], \ldots\ldots, y_5[n]$



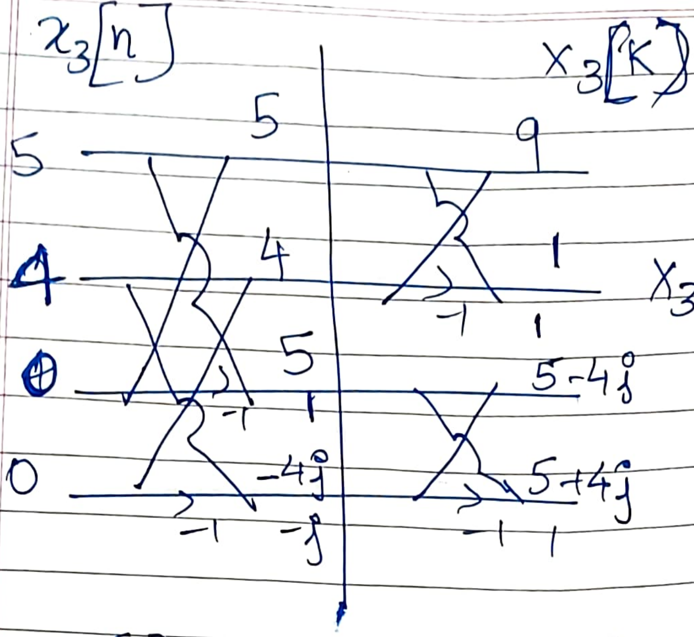
~~$x_1[n] = \{1, 1-j, 1, 1+j\}$~~



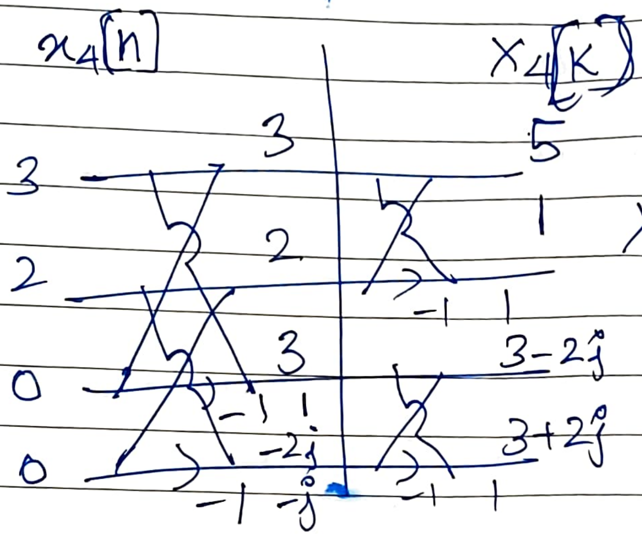
$x_1[n] = \{1, 1, 1, 1\}$



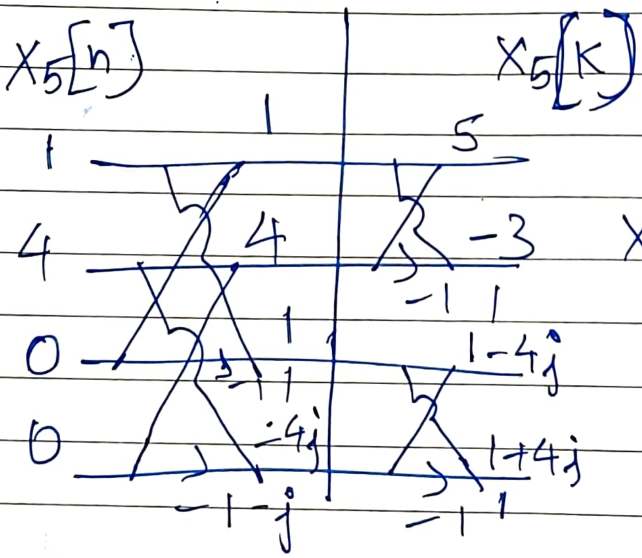
$x_2[n] = \{1, -1-2j, -3, -1+2j\}$



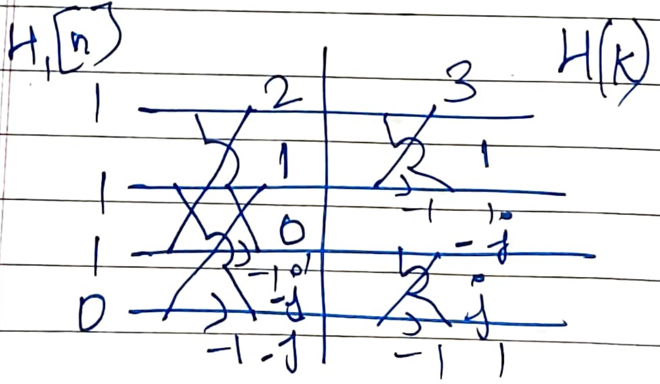
$X_3(K) = \{9, 5-4j, 1, 5+4j\}$



$X_4(K) = \{5, 3-2j, 1, 3+2$



$X_5(K) = \{5, 1-4j, -3, 1+4j\}$



$H(K) = \{3, -j, 1, j\}$

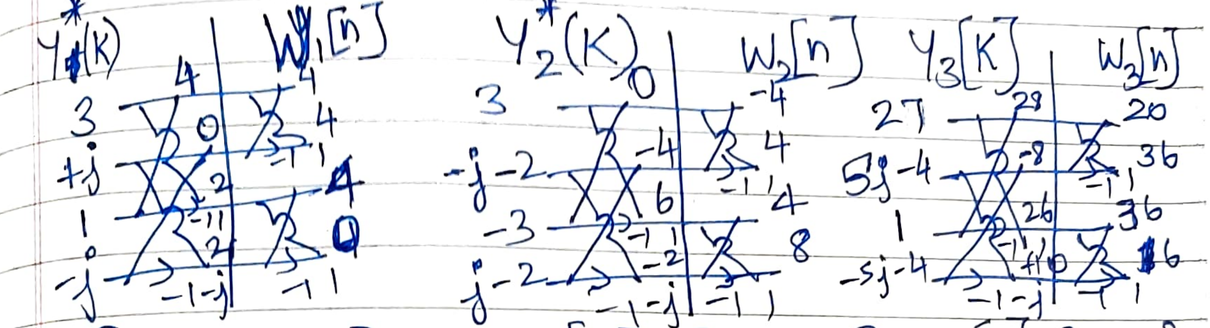
$Y_1(K) = \{3, -j, 1, +j\}$

$Y_2(K) = \{3, j-2, -3, -j-2\}$

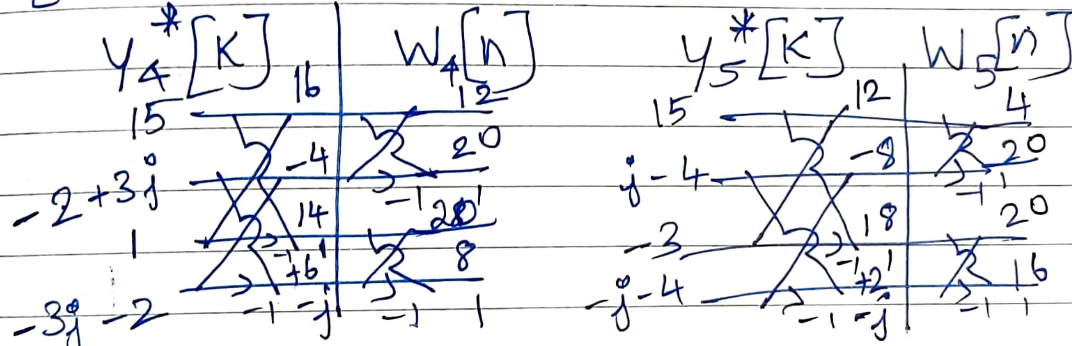
$Y_3(K) = \{27, -5j-4, 1, 5j-4\}$

$Y_4(K) = \{15, -3j-2, 1, 3j-2\}$

$Y_5(K) = \{15, -j-4, -3, j-4\}$



$y_1[n] = \{1, 1, 1, 0\}$ $y_2[n] = \{-1, 1, 1, 2\}$ $y_3[n] = \{5, 9, 9, 4\}$



$y_4[n] = \{3, 5, 5, 2\}$ $y_5[n] = \{1, 5, 5, 4\}$

Padding:

1 1 1 0
-1 1 1 2
5 9 9 4
3 5 5 2
1 5 5 4

$y[n] = 1, 1, 0, 1, 6, 11, 12, 9, 6, 7, 5, 4$

Verification:

	1	0	-1	2	5	4	3	2	1	4
1	1	0	-1	2	5	4	3	2	1	4
1	1	0	-1	2	5	4	3	2	1	4
1	1	0	-1	2	5	4	3	2	1	4
1	1	0	-1	2	5	4	3	2	1	4

$y[n] = \{1, 1, 0, 1, 6, 11, 12, 9, 6, 7, 5, 4\}$

Overlap Save Method:

1. $x[n] = \{1, 0, -1, 2, 5, 4, 3, 2, 1, 4\} \rightarrow N_1$

$h[n] = \{1, 1, 1\} \rightarrow N_2$

$$N = 2^M = N_1^* + N_2^* - 1$$
$$= 2 + 3 - 1$$
$$= 4$$

$x_1[n] = \{0, 0, 1, 0\} \Rightarrow X_1(k) = \{1, -1, 1, -1\}$

$x_2[n] = \{1, 0, -1, 2\} \Rightarrow X_2(k) = \{2, 2+j, -2, 2-j\}$

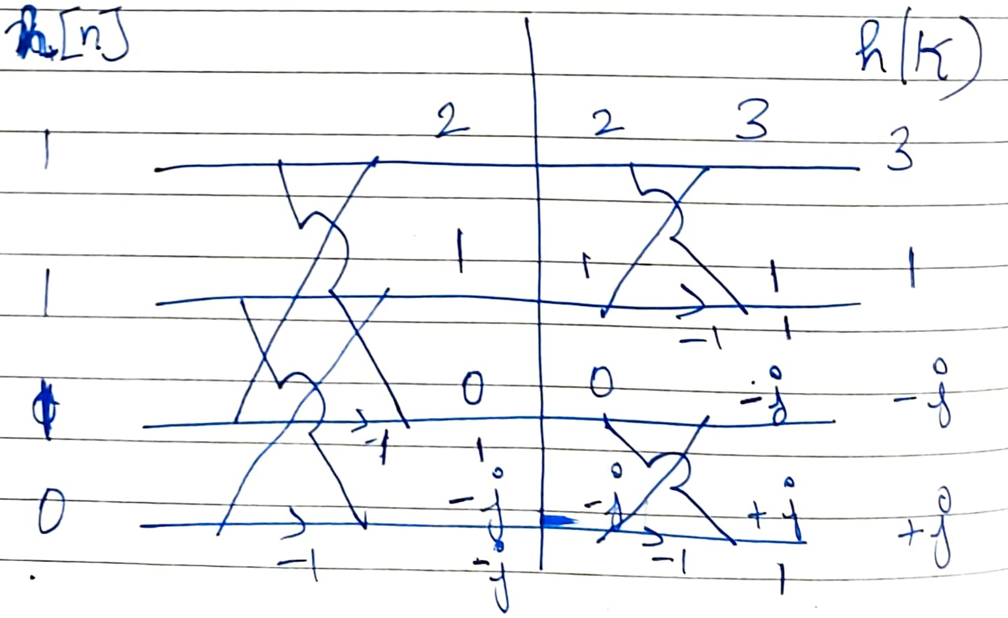
$x_3[n] = \{-1, 2, 5, 4\} \Rightarrow X_3(k) = \{10, -6+2j, -2, -6-2j\}$

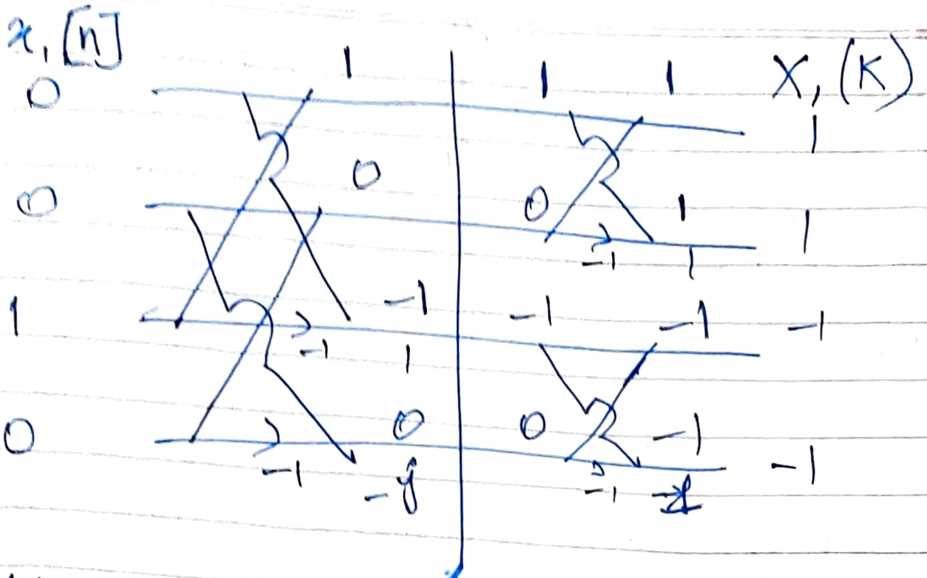
$x_4[n] = \{5, 4, 3, 2\} \Rightarrow X_4(k) = \{14, 2-j, 2, 2+2j\}$

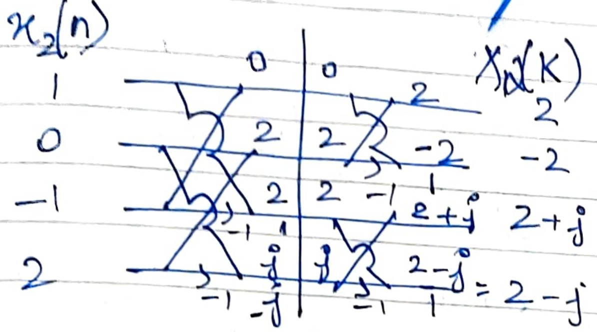
$x_5[n] = \{3, 2, 1, 4\} \Rightarrow X_5(k) = \{10, 2-2j, -2, 2+2j\}$

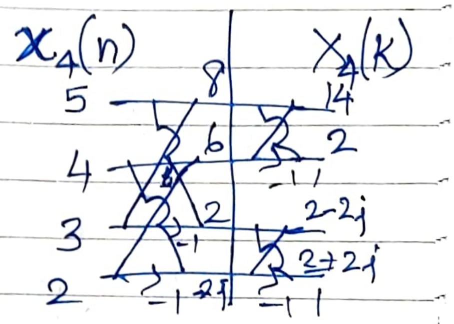
$x_6[n] = \{1, 4, 0, 0\} \Rightarrow X_6(k) = \{5, \cancel{4}\,1-4j, -3, \cancel{4}\,1+4j\}$

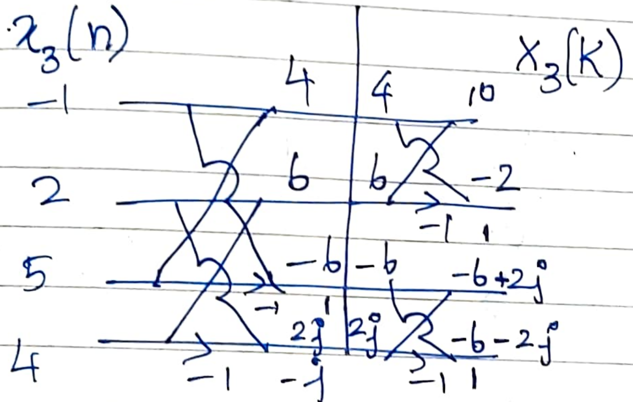
$h[n] = \{1, 1, 1, 0\} \Rightarrow H(k) = \{3, -j, 1, j\}$

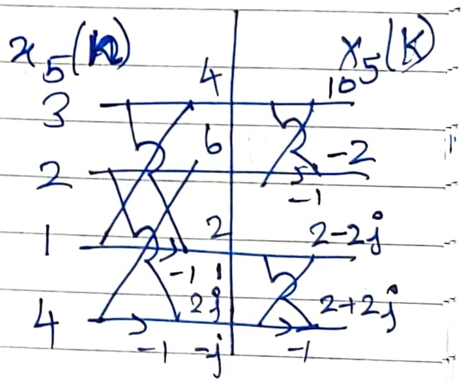


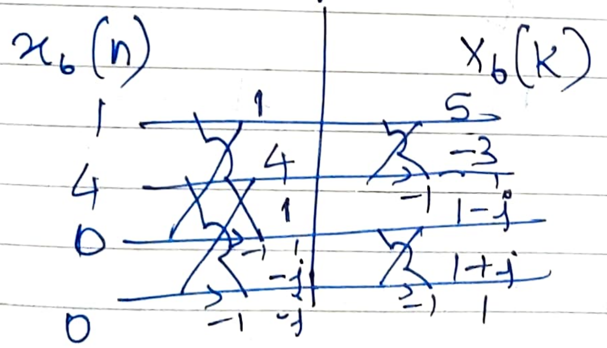












$Y_1(K) = \{3, j, 1, -j\}$

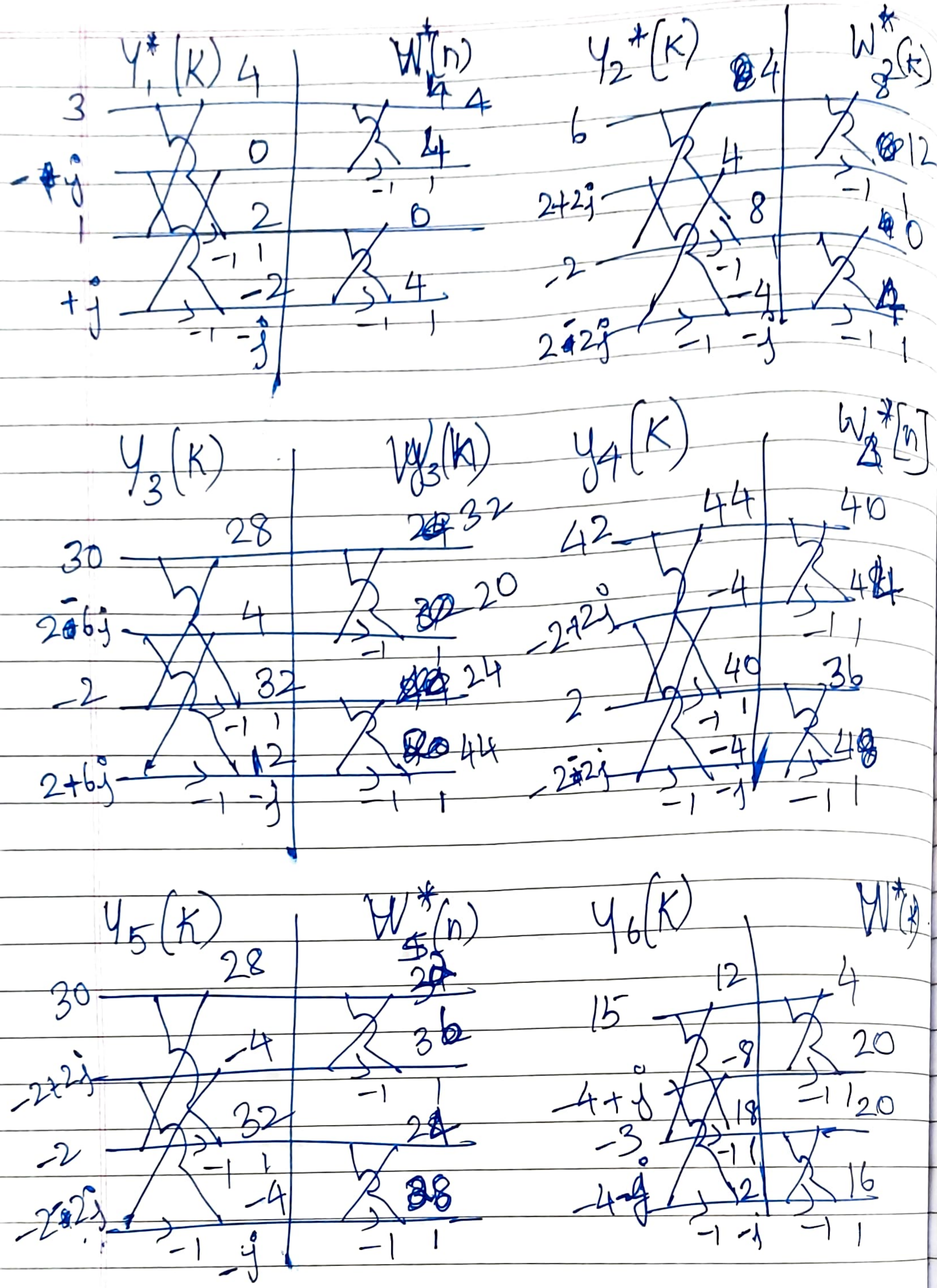
$Y_2(K) = \{6, (2-2j), -2, 2+2j\}$

$Y_3(K) = \{30, 2+6j, -2, 2-6j\}$

$Y_4(K) = \{42, -2-2j, 2, -2+2j\}$

$Y_5(K) = \{30, -2-2j, -2, 2+2j\}$

$Y_6(K) = \{15, -4-j, -3, -4+j\}$



$y_1[n] = \{1, 0, 1, 1\}$ $\quad$ $y_6[n] = \{1, 5, 5, 4\}$

$y_2[n] = \{2, 3, 0, 1\}$

$y_3[n] = \{8, 5, 6, 1\} \Rightarrow y[n] = \{1, 1, 0, 1,$

$y_4[n] = \{10, 11, 12, 9\}$ $\quad$ $6, 11, 12, 9,$

$y_5[n] = \{8, 9, 6, 9\}$ $\quad$ $6, 7, 5, 4\}$