11). Response of over damped second order for unit step i/p.

→ second order s/m general tfr func.,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}.$$

→ over damped ⇒ $\zeta > 1$, Roots of the polynomial are,

$$s_1, s_2 = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

$$\therefore s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} = -\left[\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]$$

$$s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} = -\left[\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]$$

→ $$\therefore \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} = \frac{\omega_n^2}{(s+s_1)(s+s_2)}.$$

→ $$C(s) = R(s)\frac{\omega_n^2}{(s+s_1)(s+s_2)}$$

→ unit step i/p, $r(t) = u(t)$, $R(s) = 1/s$.

$$C(s) = \frac{\omega_n^2}{s(s+s_1)(s+s_2)}.$$

$$\frac{\omega_n^2}{s(s+s_1)(s+s_2)} = \frac{A}{s} + \frac{B}{s+s_1} + \frac{C}{s+s_2}.$$

$$A = s \times C(s)\Big|_{s=0} = \frac{\omega_n^2}{s_1 s_2} = \frac{\omega_n^2}{\zeta^2\omega_n^2 - \omega_n^2(\zeta^2-1)}$$

$$= \frac{\omega_n^2}{\omega_n^2}$$

$$\boxed{A = 1}$$

$$\rightarrow \quad B = (s+s_1)\, C(s) \Big|_{s=-s_1}$$

$$= \frac{\omega_n^2}{s(s+s_2)} \Big|_{s=-s_1}$$

$$= \frac{\omega_n^2}{-s_1(-s_1+s_2)}$$

$$= \frac{\omega_n^2}{\left[\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]\left[\zeta\omega_n - \omega_n\sqrt{\zeta^2-1} - \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]}$$

$$= \frac{\omega_n^2}{\left[\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]\left[-2\omega_n\sqrt{\zeta^2-1}\right]}$$

$$= \frac{\omega_n^2}{-s_1\left(-2\omega_n\sqrt{\zeta^2-1}\right)}$$

$$\boxed{B = \frac{\omega_n}{2s_1\sqrt{\zeta^2-1}}}$$

$$\rightarrow \quad C = (s+s_2)\, C(s) \Big|_{s=-s_2}.$$

$$= \frac{\omega_n^2}{s(s+s_1)} \Big|_{s=-s_2}$$

$$= \frac{\omega_n^2}{-s_2(-s_2+s_1)} = \frac{\omega_n^2}{-s_2\left(\zeta\omega_n + \omega_n\sqrt{\zeta^2-1} - \zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)}$$

$$= \frac{\omega_n^2}{-s_2\left(2\omega_n\sqrt{\xi^2-1}\right)}$$

$$\boxed{c = -\frac{\omega_n}{2 s_2\sqrt{\xi^2-1}}}$$

$$\rightarrow c(s) = \frac{1}{s} + \frac{\omega_n}{2\sqrt{\xi^2-1}\,[s_1](s+s_1)} - \frac{\omega_n}{2\sqrt{\xi^2-1}\,(s_2)(s+s_2)}$$

$$c(t) = L^{-1}\left[c(s)\right]$$

$$= 1 + \frac{\omega_n}{2\sqrt{\xi^2-1}}\cdot\frac{1}{s_1}e^{-s_1 t} - \frac{\omega_n}{2\sqrt{\xi^2-1}}\cdot\frac{1}{s_2}e^{-(s_2 t)}$$

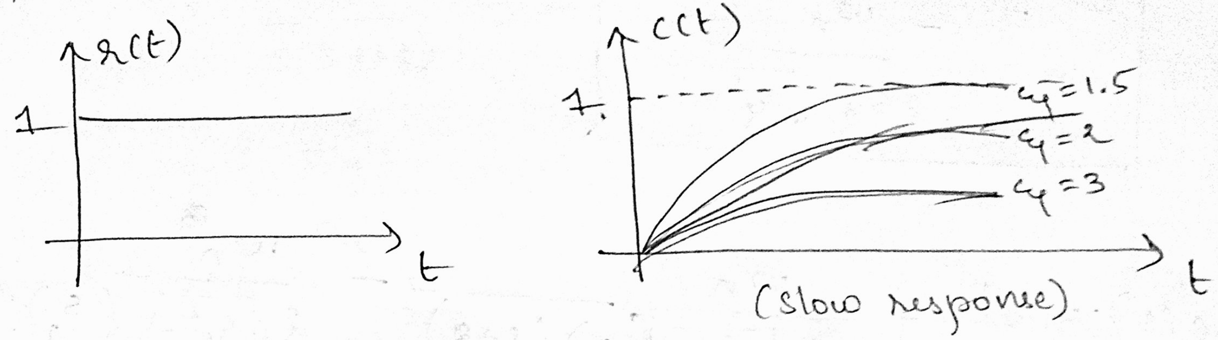
$$\boxed{c(t) = 1 + \frac{\omega_n}{2\sqrt{\xi^2-1}}\left[\frac{1}{s_1}e^{-s_1 t} - \frac{1}{s_2}e^{-s_2 t}\right]}$$

→ For overdamped II-nd order step Response,

$$c(t) = \begin{cases} 1 + \dfrac{\omega_n}{2\sqrt{\xi^2-1}}\left(\dfrac{1}{s_1}e^{-s_1 t} - \dfrac{1}{s_2}e^{-s_2 t}\right), & \text{unit step i/p} \\[2ex] A\left[1 + \dfrac{\omega_n}{2\sqrt{\xi^2-1}}\left(\dfrac{1}{s_1}e^{-s_1 t} - \dfrac{1}{s_2}e^{-s_2 t}\right)\right], & \text{step i/p} \end{cases}$$

$$s_1 = -\left[\xi\omega_n - \omega_n\sqrt{\xi^2-1}\right] \quad \& \quad s_2 = -\left[\xi\omega_n + \omega_n\sqrt{\xi^2-1}\right]$$

→ Response of over-damped II - order s/m, unit-step i/p



Time Domain Specifications :-

→ The desired performance characteristics of a s/m of any order may be specified in terms of the transient response to the unit step signal.

→ The transient response of a s/m to a unit step i/p depends on the initial conditions.

→ The transient response characteristics of a control s/m to a unit step i/p is specified in terms of the following time domain specifications.

(i) Delay time (t_d)

(ii) Rise time (t_r)

(iii) Peak time (t_p)

(iv) Maximum overshoot (M_p)

(v) Settling time (t_s).

(i) Delay time (t_d):- The time taken for response to reach 50% of the final value, for the first time.

(ii) Rise time (t_r):- The time taken for response to raise from 0 to 100% for the very first time.

But for over-damped s/m ⇒ 10% to 90%.

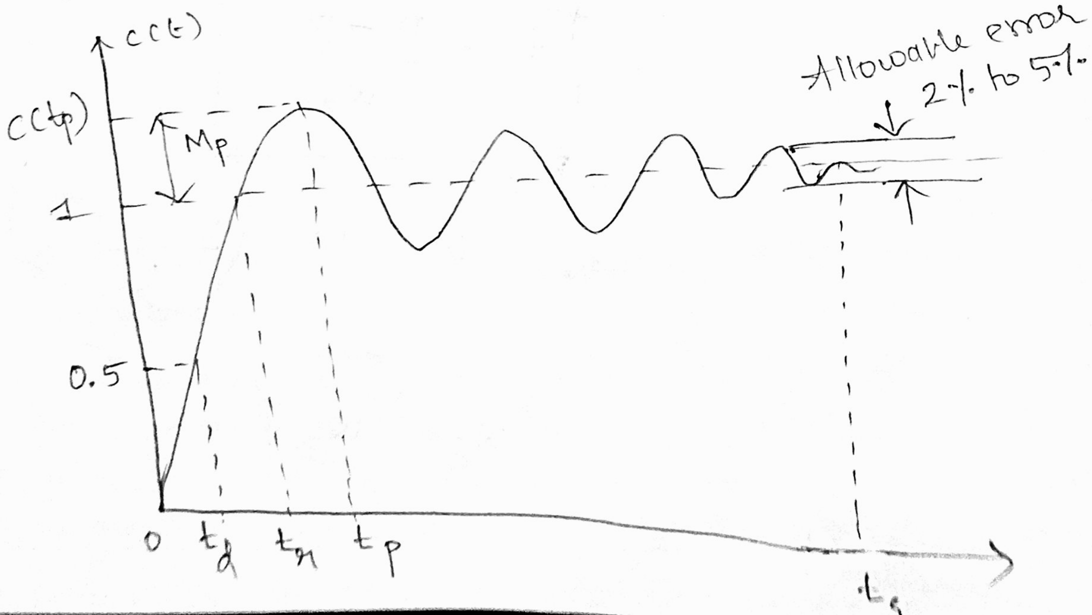
critically damped ⇒ 5% to 95%.

(iii) Peak time (t_p):- time taken for the response to reach the peak value, the very first time. It can also be defined as the time taken for the response to reach the peak overshoot (M_p).

(iv) Peak overshoot (M_p) :- it is defined as the ratio of the maximum peak value above the final value to the final value.

$$\% M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%$$

(v) Settling time (t_s) :- It is defined as the time taken by the response to reach & stay within a specified error (2% or 5%).



Expressions for the time domain specifications :-

(i) Rise time (t_r) :-

→ For under-damped s/m, unit step response is,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

→ $\boxed{\text{At } t = t_r, \quad c(t) = 1.}$

$$\frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_r + \theta) = 0.$$

→ $\because e^{-\zeta\omega_n t_r} \neq 0$

$$\sin(\omega_d t_r + \theta) = 0.$$

→ for $\phi = 0, \pi, 2\pi, 3\pi, \ldots$ $\sin(\phi) = 0$.

$$\omega_d t_r + \theta = \pi.$$

$$\omega_d t_r = \pi - \theta$$

$$\boxed{t_r = \frac{\pi - \theta}{\omega_d}}$$

where $\theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

ii) peak time (t_p) :-

→ maximum value at t_p.

$$\boxed{\therefore \frac{d}{dt} c(t) \Big|_{t=t_p} = 0.}$$

$$\rightarrow \frac{d}{dt}[c(t)] = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t).$$

$$\rightarrow \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_p} \sin(\omega_d t_p) = 0.$$

$$\rightarrow \sin(\omega_d t_p) = 0.$$

$$\omega_d t_p = \pi$$

$$\boxed{t_p = \frac{\pi}{\omega_d}}.$$

(iii) peak overshoot (M_p) :-

$$\% M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%.$$

At $t = \infty$, $c(\infty) = 1$

At $t = t_p$, $c(t_p) = 1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}$.

$$\% M_p = \frac{\left[1 + e^{-\frac{\zeta\pi}{\sqrt{1-\zeta^2}}}\right] - 1}{1} \times 100\%.$$

$$\boxed{\% M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} \times 100}$$

(iv) Settling time (t_s)

→ The response of II-order s/m has two components. They are

(i) Decaying exponential, $\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}$

(ii) Sinusoidal component, $\sin(\omega_d t+\theta)$.

→ The settling time is decided by the Decaying exponential term since the sinusoidal signal is reduced by the Decaying exp. component.

→ For 2% tolerance, at $t = t_s$

$$\frac{e^{-\xi\omega_n t_s}}{\sqrt{1-\xi^2}} = 0.02$$

$$e^{-\xi\omega_n t_s} = 0.02 \qquad \because \sqrt{1-\xi^2} \simeq 1.$$

$$-\xi\omega_n t_s = \ln(0.02)$$

$$-\xi\omega_n t_s = -4$$

$$\boxed{t_s = \frac{4}{\xi\omega_n} = 4T} \rightarrow 2\% \text{ error}$$

→ Illy $$\boxed{t_s = \frac{3}{\xi\omega_n} = 3T} \rightarrow 5\% \text{ error.}$$

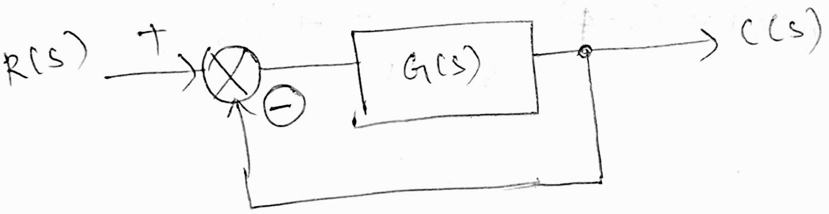
→ Generally, $\boxed{t_s = \dfrac{\ln(\%\text{error})}{\zeta\omega_n} = \dfrac{\ln(\%\text{error})}{T}}$

Eqs

i) obtain the response of unity fb s/m whose open loop transfer function is $\dfrac{4}{s(s+5)}$ and when the i/p is unit step.

soln.

OLTF, $G(s) = \dfrac{4}{s(s+5)}$.



CLTF, $\dfrac{C(s)}{R(s)} = \dfrac{G(s)}{1+G(s)}$

$$= \frac{\dfrac{4}{s(s+5)}}{1+\dfrac{4}{s(s+5)}}$$

$$= \frac{4}{s(s+5)+4}.$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{4}{s^2+5s+4}.} = \frac{4}{(s+1)(s+4)}$$

$$c(s) = R(s)\left[\frac{4}{s(s+1)(s+4)}\right]$$

$\rightarrow$ unit step i/p $\Rightarrow$ $r(t) = 1$ & $R(s) = 1/s$.

$$c(s) = \frac{4}{s(s+1)(s+4)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}.$$

$$\rightarrow \quad A = c(s) \times s \Big|_{s=0} = 4/4 = 1.$$

$$B = c(s) \times (s+1) \Big|_{s=-1} = -4/3$$

$$C = c(s) \times (s+4) \Big|_{s=-4} = +1/3.$$

$$\rightarrow \therefore \quad c(s) = \frac{1}{s} - \frac{4}{3(s+1)} + \frac{1}{3(s+4)}.$$

$$\rightarrow \quad c(t) = L^{-1}\left[c(s)\right]$$

$$= 1 - \frac{4}{3} e^{-t} + \frac{1}{3} e^{-4t}.$$

$$\boxed{c(t) = 1 - \frac{1}{3}\left(4e^{-t} - e^{-4t}\right)}$$

2) The response of a servo-mechanism is,

$c(t) = 1 + 0.2\,e^{-60t} - 1.2\,e^{-10t}$ when subject to a unit step i/p. Obtain an expression for closed loop transfer function. Determine the undamped natural freq. & damping ratio.

Soln

$$c(t) = 1 + 0.2\,e^{-60t} - 1.2\,e^{-10t}$$

$$C(s) = \frac{1}{s} + 0.2\frac{1}{(s+60)} - \frac{1.2}{(s+10)}$$

$$= \frac{(s+60)(s+10) + (0.2)(s)(s+10) - 1.2s(s+60)}{s(s+60)(s+10)}$$

$$C(s) = \frac{600}{s(s+60)(s+10)}$$

$$C(s) = R(s)\,\frac{600}{(s+60)(s+10)}. \qquad \left[\because R(s) = 1/s\right].$$

$$\boxed{\frac{C(s)}{R(s)} = \frac{600}{(s+60)(s+10)}} \Rightarrow \text{CLTF}.$$

$$\frac{C(s)}{R(s)} = \frac{600}{s^2 + 70s + 600}.$$

→ General II - order eqn. is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

→ ∴ $2\xi\omega_n s = 70s$ & $\omega_n^2 = 600$

$\omega_n = \sqrt{600}$

$= 10\sqrt{6}.$

$\boxed{\omega_n = 24.49 \text{ rad/sec}}$

$2 \times \cancel{10}\sqrt{6} \times \xi = 7\cancel{0}$

$$\xi = \frac{7}{2\sqrt{6}}.$$

$$= \frac{7}{4.90}$$

$= 1.4285$

$\boxed{\xi \simeq 1.43.}$