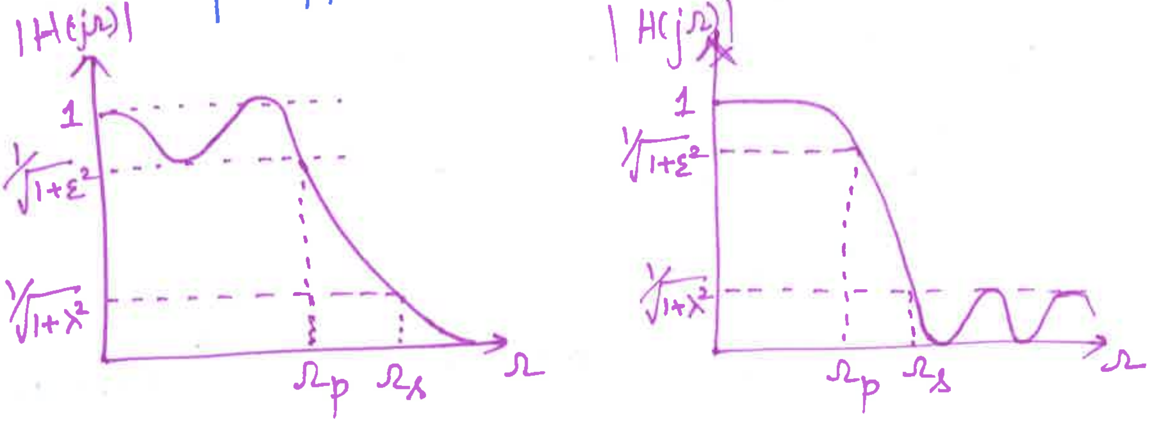
Analog Low Pass Chebyshev filters:

type I ✓ type II (inverse)

Type I chebyshev filters are all-pole filters that exhibit equiripple behaviour in the passband and a monotonic characteristics in the stop band.

Type II Chebyshev filters contain both poles and zeros and exhibits a monotonic behaviour in the passband and an equiripple behaviour in the stopband.



type - I type - II

Magnitude responses.

The magnitude squared response of N^{th} order type-I chebyshev filter can be expressed as,

$$\boxed{|H(j\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2\left(\frac{\Omega}{\Omega_p}\right)}} \quad \text{--- (1)}, \quad N = 1, 2, \ldots$$

where

$\varepsilon \rightarrow$ parameter related to ripples in passband

$T_N(x) \rightarrow N^{th}$ order Chebyshev polynomial.

The Chebyshev polynomial is defined as,

$$T_N(x) = \begin{cases} \cos(N \cos^{-1} x), & |x| \le 1 \\ \cosh(N \cosh^{-1} x), & |x| > 1 \end{cases} \longrightarrow (2)$$

The Chebyshev polynomials can be generated by the recursive equations

$$T_{N+1}(x) = 2x\, T_N(x) - T_{N-1}(x), \quad N = 1, 2, \dots \longrightarrow (3)$$

(or)

$$T_N(x) = 2x\, T_{N-1}(x) - T_{N-2}(x), \quad N = 2, 3 \dots$$

$$\left.\begin{aligned} &\text{where } T_0(x) = 1, \text{ and } T_1(x) = x \\ &\therefore\ T_2(x) = 2x \cdot x - 1 = 2x^2 - 1 \\ &T_3(x) = 2x(2x^2-1) - x = 4x^3 - 2x - x = 4x^3 - 3x \end{aligned}\right\} (4)$$

and so on.

Some of the properties of these polynomials are as follows:

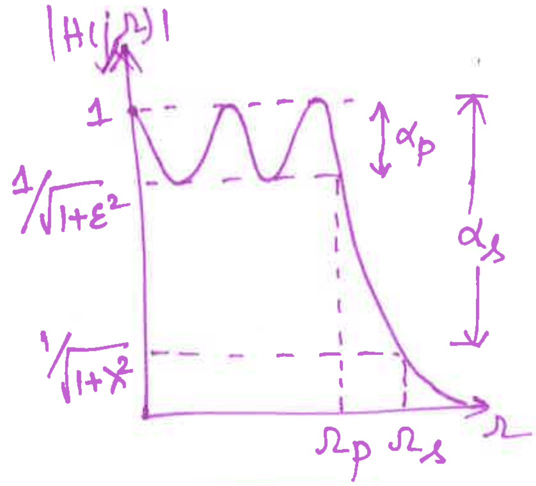
$$\left.\begin{aligned} 1.\ & T_N(x) = -T_N(-x), && \text{for } N \text{ odd} \\ & T_N(x) = T_N(-x), && \text{for } N \text{ even} \\ & T_N(0) = (-1)^{N/2}, && \text{for } N \text{ even} \\ & T_N(0) = 0, && \text{for } N \text{ odd} \\ & T_N(1) = 1, && \text{for all } N \\ & T_N(-1) = 1, && \text{for } N \text{ even} \\ & T_N(-1) = -1, && \text{for } N \text{ odd.} \end{aligned}\right\} \longrightarrow (5)$$

2. $T_N(x)$ oscillates with equal ripple between ± 1 for $|x| \le 1$ → (6)

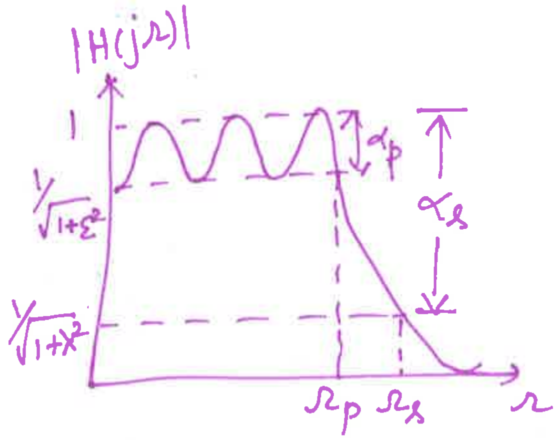
3. $|T_N(x)| \le 1$ for all $|x| \le 1$ → (7)

4. $T_N(x)$ is monotonically increasing for $|x| > 1$ for all N. → (8)

5. All the roots of the polynomial $T_N(x)$ occurs in the interval $-1 \le x \le 1$. → (9)



N-odd



N-even

The above figures show the equiripple characteristics of Chebyshev filter (type-I)

For odd values of N, the oscillatory curve starts from unity and for even values of N, the oscillatory curve starts from $1/\sqrt{1+\varepsilon^2}$.

Taking logarithm of eqn (1),

we get $$20 \log |H(j\Omega)| = 10 \underbrace{\log 1}_{=0} - 10 \log\left[1 + \varepsilon^2 T_N^2\left(\frac{\Omega}{\Omega_p}\right)\right] \rightarrow (10)$$

let α_p be attenuation in dB at passband freq. 'Ω_p'.

α_s be " " " Stopband " 'Ω_s'.

At $\Omega = \Omega_p$, eqn ⑩ becomes,

$$-10 \log (1+\varepsilon^2) = -\alpha_p \quad [\because T_N(1) = 1$$

$$\therefore \quad 0.1\alpha_p = \log(1+\varepsilon^2)$$

$$10^{0.1\alpha_p} = 1+\varepsilon^2$$

$$\varepsilon^2 = 10^{0.1\alpha_p} - 1$$

$$\boxed{\varepsilon = \sqrt{10^{0.1\alpha_p} - 1}} \longrightarrow (11)$$

At $\Omega = \Omega_s$, eqn ⑩ becomes,

$$-\alpha_s = -10\log\left[1+\varepsilon^2 T_N^2\left(\frac{\Omega_s}{\Omega_p}\right)\right] \longrightarrow (12)$$

Sub eqn (2) in eqn (12),

$$\alpha_s = 10\log\left\{1+\varepsilon^2\left[\cosh\left(N\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)\right)\right]^2\right\} \longrightarrow (13)$$

$\because \frac{\Omega_s}{\Omega_p} > 1$

always $\Omega_p < \Omega_s$.

Sub 'ε' value in eqn (13),

$$10^{0.1\alpha_s} - 1 = (10^{0.1\alpha_p} - 1)\cdot\left[\cosh\left(N\cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)\right)\right]^2 \longrightarrow (14)$$

$$\frac{\sqrt{10^{0.1\alpha_s}-1}}{\sqrt{10^{0.1\alpha_p}-1}} = \cosh\left[N \cosh^{-1}\left(\frac{\Omega_s}{\Omega_p}\right)\right]$$

$$\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}} = N \cdot \cosh^{-1}\left(\Omega_s/\Omega_p\right)$$

$$N = \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\left(\Omega_s/\Omega_p\right)} \longrightarrow (15)$$

$\Downarrow$

(may be a fractional number)

Round off 'N' to next higher integer,

$$\boxed{N \geq \frac{\cosh^{-1}\sqrt{\frac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\left(\Omega_s/\Omega_p\right)}} \longrightarrow (16)$$

wkt, $\varepsilon = \sqrt{10^{0.1\alpha_p}-1}$, & $\lambda = \sqrt{10^{0.1\alpha_s}-1}$, $A = \lambda/\varepsilon$

$$\boxed{N \geq \frac{\cosh^{-1} A}{\cosh^{-1}(1/k)}} \longrightarrow (17) \qquad k = \Omega_p/\Omega_s$$

In problem solving, $\cosh^{-1}(x)$ can be evaluated using the identity

$$\cosh^{-1}(x) = \ln\left[x + \sqrt{x^2-1}\right] \longrightarrow (18)$$

Pole locations for Chebyshev filter:

The poles of Chebyshev type-I filter are obtained by equating denominator of eqn ① to zero.

$$1+\varepsilon^2 T_N^2\left(\frac{\Omega}{\Omega_p}\right)=0 \longrightarrow (19)$$

$s = j\Omega$, $\Omega = s/j = -js$

$$1+\varepsilon^2 T_N^2\left(\frac{-js}{\Omega_p}\right)=0 \longrightarrow (20)$$

$$T_N^2\left(-js/\Omega_p\right) = -1/\varepsilon^2 = (j/\varepsilon)^2 \longrightarrow (21)$$

$$T_N\left(-js/\Omega_p\right) = \pm j/\varepsilon = \cos\left[N\cos^{-1}\left(\frac{-js}{\Omega_p}\right)\right] \longrightarrow (22)$$

let us define $\cos^{-1}\left(-js/\Omega_p\right) = \phi - j\theta \longrightarrow (23)$

Sub (23) in (22),

$$\pm j/\varepsilon = \cos\left[N(\phi - j\theta)\right] = \cos(N\phi - jN\theta)$$

$$= \cos(N\phi)\cos(jN\theta) + \sin(N\phi)\sin(jN\theta)$$

$$\pm j/\varepsilon = \cos(N\phi)\cosh(N\theta) + j\sin(N\phi)\sinh(N\theta) \rightarrow (24)$$

$\cos\theta = \frac{e^{j\theta}+e^{-j\theta}}{2}$, $\cos j\theta = \frac{e^{j(j\theta)}+e^{-j(j\theta)}}{2} = \frac{e^{-\theta}+e^{\theta}}{2} = \cosh\theta$

$\sin\theta = \frac{e^{j\theta}-e^{-j\theta}}{2j}$, $\sin j\theta = \frac{e^{-\theta}-e^{+\theta}}{2j} = \frac{-(e^{\theta}-e^{-\theta})}{2j} = j\,\frac{e^{\theta}-e^{-\theta}}{2}$

$= j\sinh\theta$

equate real & imaginary parts on both sides of eqn (24)

$$\left.\begin{aligned} \cos(N\phi)\cosh(N\theta) &= 0 \\ \sin(N\phi)\sinh(N\theta) &= \pm 1/\varepsilon \end{aligned}\right\} \quad 25 \text{ (a) \& (b)}$$

Since $\cosh(N\theta) > 0$ for θ-real,

to satisfy the equation (25a)

$$\cos(N\phi) = 0$$

$$\phi = \frac{(2k-1)\pi}{2N}, \quad k = 1, 2, \cdots N \longrightarrow (26)$$

Using this ϕ value, θ is calculated from (25b)

$$\underbrace{\sin\left(N\,\frac{(2k-1)\pi}{2N}\right)}_{=1} \sinh(N\theta) = \pm\frac{1}{\varepsilon}$$

$$N\theta = \pm\sinh^{-1}\left(\cancel{\pm}\,\frac{1}{\varepsilon}\right)$$

$$\theta = \pm\frac{1}{N}\sinh^{-1}(1/\varepsilon) \longrightarrow (27)$$

Now let us take eqn (23),

$$\cos^{-1}\left(\frac{-js}{\Omega_p}\right) = \phi - j\theta$$

$$-js/\Omega_p = \cos(\phi - j\theta)$$

$$s/j\Omega_p = \cos(\phi - j\theta)$$

$$s_k = j\Omega_p(\cos(\phi - j\theta))$$

$$s_K = j\Omega_p \left[\cos\phi \cosh\theta + j\sin\phi \sinh\theta\right]$$

$$s_K = \Omega_p \left[-\sin\phi \sinh\theta + j\cos\phi\cosh\theta\right] \longrightarrow (28)$$

$\left[\because \cos j\theta = \cosh\theta \right.$

$\left. \sin j\theta = j\sinh\theta \right]$

while solving s_K, use the identity

$$\sinh^{-1}\left(\tfrac{1}{\varepsilon}\right) = \ln\left(\tfrac{1}{\varepsilon} + \sqrt{1 + \tfrac{1}{\varepsilon^2}}\right) \quad \left[\because \sinh^{-1}(x) = \ln\left(x + \sqrt{1+x^2}\right)\right]$$

Take Antiln on both sides,

$$\mu = e^{\sinh^{-1}(\varepsilon^{-1})} = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} \longrightarrow (29)$$

and

'μ' is used in solving θ in eqn (27).

ie.

$$(27) \Rightarrow \sinh\theta = \sinh\left(\frac{1}{N}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)\right)$$

$$= \frac{e^{\left[\frac{1}{N}\sinh^{-1}(1/\varepsilon)\right]} - e^{-\left[\frac{1}{N}\sinh^{-1}(1/\varepsilon)\right]}}{2} \quad \left[\because \sinh x = \frac{e^{x} - e^{-x}}{2}\right]$$

$$= \frac{e^{\left[\sinh^{-1}(1/\varepsilon)\right]^{1/N}} - e^{\left[\sinh^{-1}(1/\varepsilon)\right]^{-1/N}}}{2}$$

$$\sinh\theta = \frac{\mu^{1/N} - \mu^{-1/N}}{2} \longrightarrow (30)$$

In the sameway,

$$\cosh\theta = \frac{\mu^{1/N} + \mu^{-1/N}}{2} \longrightarrow (31)$$

Substitute (30) in (28),

$$(28) \Rightarrow \quad s_k = \Omega_p\left[-\sin\phi\left(\frac{\mu^{1/N}-\mu^{-1/N}}{2}\right)+j\cos\phi\left(\frac{\mu^{1/N}+\mu^{-1/N}}{2}\right)\right] \longrightarrow (32)$$

let
$$r_1 = \Omega_p\left(\frac{\mu^{1/N}-\mu^{-1/N}}{2}\right) \longrightarrow (33)$$

$$r_2 = \Omega_p\left(\frac{\mu^{1/N}+\mu^{-1/N}}{2}\right) \longrightarrow (34)$$

& $\cancel{\phi_k} \ne \cancel{s_k = -a}$

$$s_k = -r_1\sin\phi + j r_2\cos\phi \longrightarrow (35)$$

Sub ϕ,
$$= -r_1 \sin\left[\frac{(2k-1)\pi}{2N}\right] + j\, r_2 \cos\left[\frac{(2k-1)\pi}{2N}\right]$$

$$s_k = r_1\cos\left[\frac{\pi}{2}+\frac{(2k-1)\pi}{2N}\right] + j r_2 \sin\left[\pi/2+\frac{(2k-1)\pi}{2N}\right]$$

$\left[\because \cos[\theta+\pi/2] = -\sin\theta \right.$ &
$\sin(\theta+\pi/2) = \cos\theta$

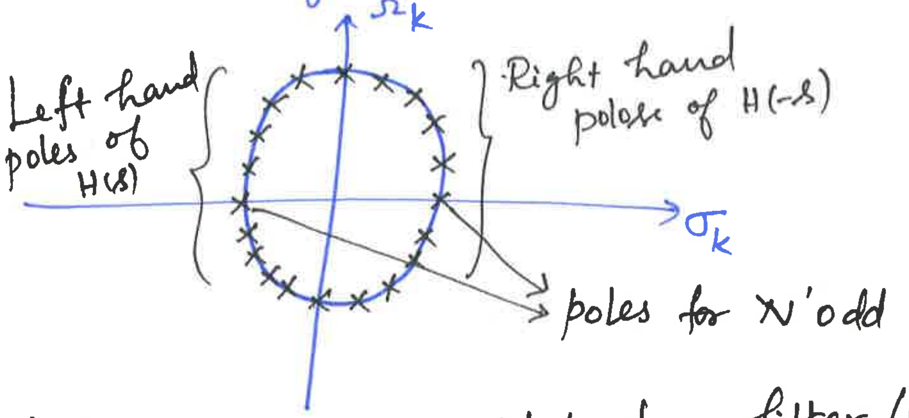
let $\phi_k = \pi/2 + \dfrac{(2k-1)\pi}{2N}$, $k = 1,2,3\ldots N \longrightarrow (36)$

$$s_k = r_1\cos\phi_k + j r_2\sin\phi_k \longrightarrow (37)$$

$$\boxed{s_k = \sigma_k + j\Omega_k,\ k=1,2\ldots N} \longrightarrow (38) \qquad [\because s = \sigma + j\Omega$$

The poles of a chebyshev filter (type I) can be determined by using the above equations (37) & (38)

The poles (s_k) are determined for $k = 1, 2, \dots N$ and finally they are located on an ellipse in the s-plane. as Shown in the figure below.



locus of poles of Chebyshev filter (type-I)

The equation of the ellipse is given by,

$$\frac{\sigma_k^2}{r_1^2} + \frac{\Omega_k^2}{r_2^2} = 1$$

where r_1, r_2 are minor and major axes of the ellipse respectively.

Comparison between Butterworth and Chebyshev filter:

1. Magnitude response of Butterworth filter decreases monotonically as the frequency 'Ω' increases from 0 to ∞, whereas in the chebyshev filter Magnitude response exhibits ripples either in passband or Stopband according to the type.

2. The transition band is more in Butterworth when compared to Chebyshev filter.

3. The poles of BW filter lie on a circle, whereas in the chebyshev, poles ~~a~~ lie on an ellipse.

4. For the same specifications, the number of poles in the Butterworth are more when compared to Chebyshev filters i.e. the order of the Cheb. filter is less than that of BW filter. This is a great adv of Cheby. filter because less number of discrete components will be necessary to construct the filter.

Steps to design an analog Chebyshev LPF:

Step 1: From the given specifications, find the order of the filter 'N' and round off it to next higher integer.

Step 2: Using the following formulae find the values of r_1 & r_2 which are minor & major axes of the ellipse respectively.

$$r_1 = \Omega_p \left[\frac{\mu^{1/N} - \mu^{-1/N}}{2} \right]$$

$$r_2 = \Omega_p \left[\frac{\mu^{1/N} + \mu^{-1/N}}{2} \right]$$

where $\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}}$

$$\varepsilon = \sqrt{10^{0.1\alpha_p} - 1}$$

$\Omega_p \rightarrow$ pass band edge frequency.

$\alpha_p \rightarrow$ Maximum allowable attenuation in Passband.

(For the normalized Chebyshev filter, $\Omega_p = 1$ rad/sec)

Step 3: Calculate the poles of Chebyshev filter which lie on an ellipse by using the formula

$$s_k = r_1 \cos\phi_k + j\, r_2 \sin\phi_k , \quad k = 1, 2 \ldots N$$

where $\phi_k = \dfrac{\pi}{2} + \dfrac{(2k-1)\pi}{2N}$, $k = 1, 2, \ldots N$

Step 4: Find the denominator polynomial of the transfer function using the above poles.

Step 5: Find the numerator of the transfer function based on the value of 'N'.

(i) For N odd : Substitute $s=0$ in the denominator polynomial and find the value. This value is equal to the numerator polynomial of the transfer function, because for odd value of 'N' the magnitude response $|H(j\Omega)|$ starts at 1.

(ii) For 'N' even: Substitute $s=0$ in the denominator polynomial and divide the result by $\sqrt{1+\varepsilon^2}$.

This value is equal to the numerator of the transfer function, because for even value of 'N', the magnitude response $|H(j\Omega)|$ starts at $1/\sqrt{1+\varepsilon^2}$.

pbm: Given the specifications $\alpha_p = 3dB$, $\alpha_s = 16dB$, $f_p = 1KHz$ & $f_s = 2KHz$. Determine the order of the filter using Chebyshev approximation. Find $H(s)$.

Solution:

Given: $\alpha_p = 3dB$, $\alpha_s = 16dB$

$f_p = 1KHz$, $\Omega_p = 2\pi f_p = 2000\pi$ rad/sec

$f_s = 2KHz$, $\Omega_s = 2\pi f_s = 4000\pi$ rad/sec.

Step 1: $N = ?$

$$N \geq \frac{\cosh^{-1}\sqrt{\dfrac{10^{0.1\alpha_s}-1}{10^{0.1\alpha_p}-1}}}{\cosh^{-1}\left(\dfrac{\Omega_s}{\Omega_p}\right)} \geq 1.91$$

$\boxed{N = 2}$ ⇒ oscillatory curve starts from $1/\sqrt{1+\varepsilon^2}$ in the passband.

Step 2: $r_1 = ?$, $r_2 = ?$

$$\varepsilon = \sqrt{10^{0.1\alpha_p}-1} = \sqrt{10^{0.3}-1} = 1$$

$$\mu = \varepsilon^{-1} + \sqrt{1+\varepsilon^{-2}} = 2.414$$

$$r_1 = \Omega_p \left(\frac{\mu^{1/N} - \mu^{-1/N}}{2} \right) = 2000\pi \left[\frac{(2.414)^{1/2} - (2.414)^{-1/2}}{2} \right] = 910\pi$$

$$r_2 = \Omega_p \left(\frac{\mu^{1/N} + \mu^{-1/N}}{2} \right) = 2000\pi \left[\frac{(2.414)^{1/2} + (2.414)^{-1/2}}{2} \right] = 2197\pi$$

Step 3: $s_k = ?$

$$s_k = r_1 \cos\phi_k + j r_2 \sin\phi_k \; , \; k = 1, 2, \text{—}$$

where, $\phi_k = \frac{\pi}{2} + \frac{(2k-1)\pi}{2N}$, $k = 1, 2$

$$\phi_1 = \frac{\pi}{2} + \frac{\pi}{4} = 135^\circ$$

$$\phi_2 = \frac{\pi}{2} + \frac{3\pi}{4} = 225^\circ$$

$$s_1 = 910\pi \times \cos(135^\circ) + j\, 2197\pi \times \sin(135^\circ)$$

$$= -643.46\pi + j\,1554\pi$$

$$s_2 = 910\pi \times \cos (225^\circ) + j\, 2197\, \sin(225^\circ)$$

$$= -643.46\pi - j\,1554\pi$$

Step 4: denominator of $H(s) = ?$

The denominator polynomial of $H(s)$ $\Big\} = (s + 643.46\pi - j\,1554\pi)(s + 643.46\pi + j\,1554\pi)$

$$= (s + 643.46\pi)^2 + (1554\pi)^2$$

Step 5: numerator of $H(s) = ?$ for $N \to$ even (i.e. $= 2$)

Sub $s = 0$ in step 4 answer & divide by $\sqrt{1+\varepsilon^2}$

numerator of $H(s) = (0 + 643.46\pi)^2 + (1554\pi)^2 / \sqrt{1+\varepsilon^2}$

$$= (643.46\pi)^2 + (1554\pi)^2 / \sqrt{1+1}$$

$$= 414040.77\pi^2 + 2414916\pi^2 / \sqrt{2}$$

$$= 2000374\pi^2$$

$$H(s) = \frac{2000374\pi^2}{(s + 643.46\pi)^2 + (1554\pi)^2}$$

$$\boxed{H(s) = \frac{2000374\pi^2}{s^2 + 1287\pi s + (1682)^2\pi^2}}$$

Pbm: Obtain an analog Chebyshev filter transfer function that satisfies the constraints

$$1/\sqrt{2} \leq |H(j\Omega)| \leq 1 \; ; \; 0 \leq \Omega \leq 2$$

$$|H(j\Omega)| < 0.1 \; ; \; \Omega \geq 4$$

Soln:

Given: $\frac{1}{\sqrt{1+\varepsilon^2}} = \frac{1}{\sqrt{2}} \Rightarrow \boxed{\varepsilon = 1}$

$\frac{1}{\sqrt{1+\lambda^2}} = 0.1 \Rightarrow \boxed{\lambda = 9.95} \Leftarrow \lambda = \sqrt{99}$

$\Omega_p = 2$ rad/sec, $\Omega_s = 4$ rad/sec

Step 1:

$$N \geq \frac{\cosh^{-1} \lambda/\varepsilon}{\cosh^{-1} \frac{\Omega_s}{\Omega_p}} \geq 2.269$$

$\boxed{N = 3} \Rightarrow$ oscillatory curve starts from 1 in the Passband

Step 2: $r_1 = ?$, $r_2 = ?$

$r_1 = 0.596$, $r_2 = 2.087$, $\mu = 2.414$

Step 3: $s_k = ?$ $k = 1, 2, 3$

$\phi_1 = 120^\circ$, $\phi_2 = 180^\circ$, $\phi_3 = 240^\circ$

$s_1 = -0.298 + j\,1.807$

$s_2 = -0.596$

$s_3 = -0.298 - j\,1.807$

Step 4: denominator polynomial of $H(s) = ?$

$= (s + 0.596)(s^2 + 0.596s + \cancel{3}.354)$

Step 5: numerator of $H(s) = ?$

put $s=0$ in step ④ answer.

$= (0.596)(\cancel{0.596}\,3.354) = 1.9989$

$= 2$

$$\boxed{H(s) = \frac{2}{(s+0.596)(s^2+0.596s+3.354)}}$$

homework:

1) Design a Chebyshev analog LPF that has a 1 dB ripple in the passband and pass band frequency $\Omega_p = 1$ rad/sec, and a stop band frequency of 20 rad/sec and an attenuation of 25 dB (or) more.

8) Find the pole locations of a normalized Chebyshev filter of order 3.

9) Determine the order and poles of a type I Chebyshev LPF that has a 1 dB ripple in passband and a cutoff frequency $\Omega_p = 1000\pi$, a stop band frequency of 2000π and an attenuation of 40dB or more for $\Omega \geq \Omega_s$.