

Lecture Notes: Macroeconomics A

Based on lectures by **Johannes Boehm** in Autumn semester, 2024

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These lecture notes were taken in the course *Macroeconomics A* taught by professor **Johannes Boehm** at Graduate of International and Development Studies, Geneva as part of the International Economics program (Semester I, 2024).

Currently, these are just drafts of the lecture notes. There can be typos and mistakes anywhere. So, if you find anything that needs to be corrected or improved, please inform at jingle.fu@graduateinstitute.ch.

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Lecture 1.

Solow Model & Development Accounting

1.1 Solow model

Mechanism of Solow model

- Firms: producing output using capital
- Consumers: consume output and save fraction of income → future capital → produce → save

Basic Assumptions

- Discrete time
- Closed economy, single good: for consumption and investment
- Two actors in the economy: firms and households
 - Firms: produce output using capital and Labor
 - Households: receives labor and capital income, and profits from owning the firms(now = 0); consume and save
- Three markets: labor, capital and final goods

1.1.1 Households(HH)

Note.

Households do not optimize, which is the main difference to Ramsey model.

Assuming the received income as below:

$$Y_t = \underbrace{w(t)L^s(t)}_{\text{labor income}} + \underbrace{R(t)K^s(t)}_{\text{capital income}} + \underbrace{\Pi(t)}_{\text{profits}}$$

Keynesian consumption function

$$S_t = sY_t \quad C_t = (1 - s)Y_t,$$

where s is the saving rate.

Technology

Firms produce according to the production function:

$$Y_t = F(K_t, L_t, A_t)$$

where L and K are labor and capital, A is the technology level.

Function F is the *neoclassical production function* with the following properties:

- Constant returns to scale:

$$F(\lambda K, \lambda L, A) = \lambda F(K, L, A)$$

- Diminishing marginal products for both K and L

$$\frac{\partial F}{\partial K} > 0, \quad \frac{\partial^2 F}{\partial^2 K} < 0$$

- Inada conditions (likewise for both L and K)

$$\lim_{K \rightarrow 0} \frac{\partial F}{\partial K} = \infty, \quad \lim_{K \rightarrow \infty} \frac{\partial F}{\partial K} = 0$$

Assumption 1.1.1.

- All markets are perfectly competitive;
- Labor market clearing: $L^s(t) = L^d(t) \equiv L_t$;
- Capital market clearing: $K^s(t) = K^d(t) \equiv K_t$;
- Goods market clearing: $Y_t = C_t + I_t$: total supply equals to consumption plus investment.

Theorem 1.1.1 (Law of motion of capital).

$$K_{t+1} = (1 - \delta)K_t + I_t$$

where δ is the depreciation rate, $K(0)$ is the initial capital endowment.

1.1.2 Firms

The firm's problem is to maximize profits:

$$\max_{K, L} \Pi(t) = F(K, L, A_t) - w(t)L_t - R(t)K_t.$$

Taking the FOCs:

$$\frac{\partial F}{\partial K} = R(t), \quad \frac{\partial F}{\partial L} = w(t).$$

Also, we know that

$$F(K, L, A) = \frac{\partial F}{\partial K} K + \frac{\partial F}{\partial L} L,$$

so we can get:

$$\Pi = F(K, L, A) - R(t)K - w(t)L = 0.$$

The equilibrium of the Solow Model is:

- $K_{t+1} = (1 - \delta)K_t + S_t$
- $C_t = (1 - s)Y_t$
- $S_t = sY_t$

From the Fundamental Law of Motion 1.1.1, we can get:

$$K_{t+1} = (1 - \delta)K_t + S_t \quad (1.1)$$

$$= (1 - \delta)K_t + sY_t \quad (1.2)$$

$$= (1 - \delta)K_t + sF(K_t, L_t, A_t) \quad (1.3)$$

$$(1.4)$$

1.1.3 The canonical Solow Model

Assumption 1.1.2.

- No population growth $L_t = L$
- No technological progress $A_t = A$

Then, we transform the equation (3) into per-capita terms:

$$k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = (1 - \delta) \frac{K_t}{L_t} + s \frac{F(K_t, L, A)}{L_t} \quad (1.5)$$

$$= (1 - \delta)k_t + sF(k_t, 1, A) \quad (1.6)$$

$$= (1 - \delta)k_t + sf(k_t) \quad (1.7)$$

where $k_t = \frac{K_t}{L_t}$ is the capital per capita, $f(k_t) = F(K_t/L_t, 1, A)$.

Similarly, we have the output per capita:

$$y(t) = \frac{Y_t}{L_t} = \frac{F(K_t, L, A)}{L} = f(k_t),$$

the rate of capital:

$$R(t) = F_K(K_t, L, A) = f'(k_t),$$

and the wage rate:

$$w(t) = F_L(K_t, L, A) = f(k_t) - k_t f'(k_t).$$

1.1.4 Dynamic evolution of the economy

Characterize everything in term of unique state variables k_t :

$$k_{t+1} = (1 - \delta)k_t + sf(k_t)$$

with $k(0)$ given.

Definition 1.1.1 (Steady state).

A steady state is a state of the economy where all variables are constant over time.

The concept corresponding to the steady state in the basic model is the *balanced growth path* (some researchers still prefer to use the name “steady state” for the balanced growth path, because the normalized variables are “steady” also in this case).

Steady state of the Solow Model

There are actually two steady states in the Solow Model:

- The trivial steady state: $k^* = 0$
- The non-trivial steady state: $k^* > 0$

But we can actually ignore the one at $k_t = 0$.

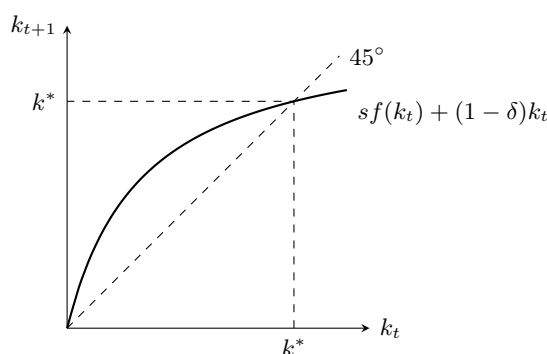


Figure 1.1: Steady state of Solow Model

Steady state analysis

At the steady-state, k^* satisfies:

$$\underbrace{\delta k^*}_{\text{Depreciation}} = \underbrace{sf(k^*)}_{\text{Investment}}$$

which means that the investment equals to the depreciation.

k^* defined by:

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}.$$

A different representation In the steady state, the amount of actual investment is exactly such that it compensates for depreciation.

Comparative statics

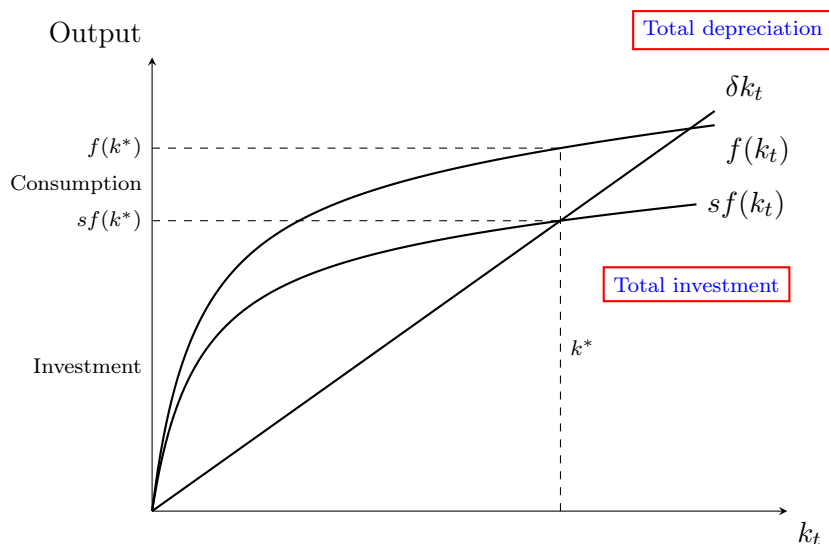


Figure 1.2: Alternative representation of Solow Model SS

At the steady state, k^* satisfies:

$$\frac{f(k^*(s, \delta, A), A)}{k^*(s, \delta, A)} = \frac{\delta}{s}$$

while the differential yields the comparative statics:

$$\begin{aligned} \frac{\partial k^*(s, \delta, A)}{\partial \delta} &< 0 \\ \frac{\partial k^*(s, \delta, A)}{\partial s} &> 0 \\ \frac{\partial k^*(s, \delta, A)}{\partial A} &> 0 \end{aligned}$$

The higher productivity A has the 'multiplier' effect on the output directly via k^* .

At the steady state, the consumption per capita is:

$$c^* = c(k^*(s, \delta, A)) = (1 - s)f(k^*(s, \delta, A)).$$

s has two opposing effects:

- Higher s leads to higher k^* , which leads to higher y^* ;
- Higher s leads to lower $1 - s$, which leads to lower fraction of output consumed.

Let \hat{s} be the savings rate that maximizes consumption, i.e.

$$\hat{s} = \arg \max_s (1 - s)f(k^*(s, \delta, A)) = \arg \max_s \{f(k^*(s, \delta, A)) - \delta k^*(s, \delta, A)\}.$$

Hence \hat{s} is implicitly defined by:

$$f'(k^*(\hat{s}, \delta)) = \delta.$$

Definition 1.1.2 (Golden Rule).

The Golden Rule is the savings rate that maximizes consumption in the steady state, which is \hat{s} in our model. And, we have $k^{gold} = f'^{-1}(\delta)$ as the golden rule capital stock.

Note.

- Savings are excessive if depreciation is higher than the marginal product of capital;
- By saving less, consumption go up;
- The economy is dynamically inefficient if $k > k^{gold}$.

1.1.5 Growth in the Solow Model

Dynamics is fully determined by the evolution of capital:

$$\begin{aligned}
 g_k(t) &= \frac{k_{t+1} - k_t}{k_t} \\
 &= \frac{(1 - \delta)k_t + sf(k_t) - k_t}{k_t} \\
 &= s \frac{f(k_t)}{k_t} - \delta \\
 &= g_k(k_t)
 \end{aligned}$$

So, in the long run, the growth rate of capital is constant and equal to $\frac{sf(k^*)}{k^*} - \delta$. As long as $k_t < k^*$, the growth rate of capital is positive, and vice versa.

1.1.6 The Growing Economy

Now we extend the model to the situation where A_t and L_t grow over time. In the previous section, the long-run outcome was the steady state where there is no growth. This extension is necessary for addressing the facts related to the growth issues described in later chapters.

We assume that the aggregate production function takes the form of

$$Y_t = F(K_t, A_t L_t).$$

There are two changes from the basic model: first, we allow the labor input (population times hours worked per person) L_t to grow over time. Second, and more importantly, we allow for technological progress. In this production function, the variable representing the technology level, A_t , is multiplied by labor input L_t . Technological progress thus takes a form of improving the labor input, and $A_t L_t$ is often referred to as the total number of efficiency units of labor (or effective labor).

This form of technological progress is labor-augmenting; it was introduced in the previous chapter. As was also asserted there, Uzawa (1961) proved that labor-augmenting technical change is the only form of technical progress that is consistent with exact balanced growth, that is, the growth path where aggregate variables such as output and capital grow at a constant rate.

We assume that the (net) growth rate of A_t is $\frac{\dot{A}_t}{A_t} = g$ and that the growth rate of L_t is $\frac{\dot{L}_t}{L_t} = n$. The same manipulations of equations as in the basic model yield

$$K_{t+1} = (1 - \delta)K_t + sF(K_t, A_t L_t).$$

The dynamic equation for capital in continuous time is

$$\dot{K}_t = sF(K_t, A_t L_t) - \delta K_t.$$

Let us define the efficient unit of capital k_t by

$$k_t = \frac{K_t}{A_t L_t}.$$

and take the log derivative w.r.t. time,

$$\frac{\dot{k}_t}{k_t} = \frac{\dot{K}_t}{K_t} - \frac{\dot{A}_t}{A_t} - \frac{\dot{L}_t}{L_t} = \frac{\dot{k}_t}{k_t} - g - n.$$

Output per efficiency unit of labor is:

$$\hat{y}_t = \frac{Y_t}{A_t L_t} = F(k_t, 1) = f(k_t).$$

Here, $f(k_t) = F(k_t, 1)$ as in the basic model. Hence, the above equation can be written as:

$$\begin{aligned} \frac{\dot{k}_t}{k_t} &= \frac{sF(K_t, A_t L_t) - \delta K_t}{K_t} - g - n \\ &= \frac{sF(K_t, A_t L_t)}{K_t} - (\delta + g + n) \\ &= \frac{sf(k_t)}{k_t} - (\delta + g + n). \end{aligned}$$

Therefore, we can get the dynamic of efficient capital k_t as

$$\dot{k}_t = sf(k_t) - (\delta + g + n)k_t.$$

After the capital stock K_t is normalized by $A_t L_t$, we thus obtain a very similar difference equation as the fundamental equation in the basic model, with effective depreciation rate $\delta + g + n$.

In this case, upon entering a balanced growth path, the rate of per capita output growth depends uniquely on the rate of technological progress g .

1.1.7 Growth effects of changes in the savings rate

Consider the consumption path c^* per unit of effective labor, especially in the context of returning to equilibrium and whether the consumption level per unit of effective labor will be above, below, or equal to the level during the previous equilibrium.

This analysis depends on a proposition: an increase in the savings rate persistently raises the average capital stock per unit of effective labor, as well as savings and output levels. The question then arises whether it also implies an increase or decrease in the consumption level per unit of

effective labor. Under what conditions does consumption per unit of effective labor reach its optimum?

Impact of Savings Rate Changes on Optimal Consumption per Unit of Effective Labor

The formula:

$$c^* = f(k^*) - sf(k^*) = f(k^*) - (n + g + \delta)k^*$$

$$\frac{\partial c^*}{\partial s} = [f'(k^*(s, n, g, \delta)) - (n + g + \delta)] \frac{\partial k^*}{\partial s}$$

reveals that whether an increase in the savings rate leads to a long-term rise in consumption per unit of effective labor depends on the comparison between the marginal product of capital at the steady-state per capita capital stock and the slope of the investment schedule.

This formula assists in addressing the query raised in the consumption path analysis diagram.

When the long-term steady-state capital stock per unit of effective labor satisfies the condition that $f'(k^*) = n + g + \delta$, the capital stock per unit of effective labor reaches the golden-rule(1.1.2) level, and the consumption per unit of effective labor is optimal.

Impact of Changes in the Savings Rate on the Output of Steady-State Efficient Labor, Capital Stocks

A rise in the savings rate can lead to a change in the amount of capital in steady state efficient labor:

$$sf(k^*(s, n, g, \delta)) = (n + g + \delta)k^*(s, n, g, \delta)$$

$$f(k^*(s, n, g, \delta)) + sf'(k^*) \frac{\partial k^*}{\partial s} = (n + g + \delta) \frac{\partial k^*}{\partial s}$$

$$\frac{\partial k^*}{\partial s} = \frac{f(k^*)}{(n + g + \delta) - sf'(k^*)} > 0$$

The sensitivity of the steady state capital per effective labor to the savings rate is crucial. Under the condition that the derivative $s'f(k^*)$ is smaller than $(n + g + \delta)$, implying that the change in the steady state capital stock $dk^*/ds < 0$, an increase in the savings rate will initially reduce the consumption per effective worker until new equilibrium is achieved.

Further, the change in production per effective labor y^* with respect to the savings rate is given by:

$$\frac{\partial y^*}{\partial s} = f'(k^*) \frac{\partial k^*(s, n, g, \delta)}{\partial s} = \frac{f'(k^*)f(k^*)}{(n + g + \delta) - sf'(k^*)} > 0$$

Finally, the response of production per effective labor to changes in the savings rate can be expressed as:

$$\begin{aligned} \frac{s}{y^*} \frac{\partial y^*}{\partial s} &= \frac{s}{f(k^*)} \frac{f'(k^*)f(k^*)}{(n + g + \delta) - sf'(k^*)} \\ &= \frac{(n + g + \delta)k^*f'(k^*)}{f(k^*)[(n + g + \delta) - (n + g + \delta)k^*f'(k^*)/f(k^*)]} \\ &= \frac{k^*f'(k^*)/f(k^*)}{1 - k^*f'(k^*)/f(k^*)} \end{aligned}$$

$$= \frac{\alpha_k(k^*)}{1 - \alpha_k(k^*)}$$

where $\alpha_k(k^*)$ represents the output elasticity of capital at the steady state. This equation helps in determining whether the increase in the savings rate will eventually increase the consumption level per unit of effective labor in the long run.

Summary.

- Baseline Model:
 - No long-run growth, i.e. capital accumulation alone does not make the economy grow in the long run
 - Conditional convergence: if countries have the same parameters s, δ, A and the same production function F , they will converge to the same steady state. Countries further from the steady state will grow faster.
 - Crucial mechanism: decreasing marginal product to capital
- Extended Model:
 - Long-run growth possible if productivity grows: $g > 0$;
 - Consistent with Kaldor facts if $g > 0$;

But s and g are exogenous.

1.2 Development Accounting

Development accounting asks whether the factors of production can explain income levels (See Caselli 2005[10]).

Assuming the production function is Cobb-Douglas in physical and human capital (more general than labor):

$$Y_j = A_j K_j^\alpha (L_j h_j)^{1-\alpha}$$

$$y_j = F(k_j, h_j, A_j) = A_j k_j^\alpha h_j^{1-\alpha} = A_j y_j^{KH}$$

Caselli(2005) use the following method to measure success:

$$\text{success} = \frac{\text{Var} [\log(y_j^{KH})]}{\text{Var} [\log(y_j)]}.$$

Then, if all countries had the same technology A_j , then the success would be 1.

How to measure human capital? Not directly observable. But: can see distribution of schooling in the population, and can see individual's wage and schooling.

From the production function

$$Y_j = K_j^\alpha (A_j H_j)^{1-\alpha}$$

we can calibrate the relative productivity differences relative to the US as

$$\frac{A_j}{A_{US}} = \left(\frac{Y_j}{Y_{US}} \right)^{\frac{1}{1-\alpha}} \left(\frac{K_{US}}{K_j} \right)^{\frac{\alpha}{1-\alpha}} \frac{H_{US}}{H_j}.$$

Neoclassical Growth Model

2.1 Neoclassical Growth Model

The same basic environment as Solow model without the assumption of the constant exogenous saving rate.

$$Y = F(K(t), L(t), Z(t))$$

As in basic general equilibrium theory, let us suppose that preference orderings can be represented by utility functions. In particular, suppose that there is a unique consumption good, and each household h has an *instantaneous utility function* given by: $u(c^h(t))$, where $c^h(t)$ is the consumption and $u : \mathbb{R}_+ \rightarrow \mathbb{R}$ is increasing and concave.

Note. Negative levels of consumption are not allowed.

We shall make 2 major assumptions:

1. Household does not derive any utility from the consumption of other households, so consumption externalities are ruled out.
2. Impose the condition that overall utility is *time-separable and stationary*; that is, instantaneous utility at time t is independent of the consumption levels at past or future dates and is represented by the same utility function u^h at all dates.

Representative household with preferences:

$$U^h(T) = \sum_{t=0}^T (\beta^h)^t u^h(c^h(t))$$

where $\beta^h \in (0, 1)$ is the time discount factor of household h .

A solution $\{x(t)\}_{t=1}^T$ to a dynamic optimization problem is *time-consistent* if the following is true: When $\{x(t)\}_{t=1}^T$ is a solution to the continuation dynamic optimization problem starting from $t = 0$, $\{x(t)\}_{t=t'}^T$ is a solution to the continuation dynamic optimization starting from the time $t = t' > 0$.

Let's consider the simplest case and suppose all households are infinitely-lived and identical. Then the demand side of the economy can be represented as the solution of the following maximization problem at time $t = 0$:

$$\max \sum_{t=0}^{\infty} \beta^t u(c(t))$$

where $\beta \in (0, 1)$.

Budget constraint:

$$C(t) + K(t+1) = (1 + r(t))K(t) + w(t)L(t)$$

$r(t)$ is the rate of return on lending capital to firms.

Note. Optimal Growth

If the economy consists of a number of identical households, then this problem corresponds to the Pareto optimal allocation giving the same (Pareto) weight to all households. Therefore the optimal growth problem in discrete time with no uncertainty, no population growth, and no technological progress can be written as follows:

$$\begin{aligned} \max_{\{c(t), k(t)\}_{t=1}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u(c(t)) \\ \text{s.t.} \quad & k(t+1) = f(k(t)) + (1 - \delta)k(t) - c(t) \end{aligned}$$

with $k(t) \geq 0$ and given $k(0) > 0$.

The constraint is straightforward to understand: *total output per capita produced with capital-labor ratio $k(t)$, $f(k(t))$, together with a fraction $(1 - \delta)$ of the capital that is undepreciated make up the total resources of the economy at date t .*

Assuming that the representative household has $L(t)$ unit of labor supplied inelastically and denoting its assets at time t by $a(t)$, this problem can be written as:

$$\begin{aligned} \max_{\{c(t), k(t)\}_{t=1}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u(c(t)) \\ \text{s.t.} \quad & a(t+1) = (1 + r(t))a(t) - c(t) + w(t)L(t) \end{aligned}$$

where $r(t)$ is the net rate of return on assets, so that $1 + r(t)$ is the gross rate of return, and $w(t)$ is the equilibrium wage rate. **Market clearing then requires $a(t) = k(t)$.**

And, we have the non-Ponzi condition:

$$\lim_{t \rightarrow \infty} \left\{ K(t) \left[\prod_{s=1}^{t-1} \left(\frac{1}{1 + r_s} \right) \right] \right\} \geq 0$$

which refers to: limit of the PDV of debt has to be nonnegative (you cannot die in debt)

Theorem 2.1.1 (Euler Equation).

$$\forall t : u'(c_t) = \beta(1 + r_{t+1})u'(c_{t+1})$$

With an infinite horizon ($T \rightarrow \infty$), the Euler equations still determine the relative consumption levels. The terminal condition becomes **transversality condition** is the limit of the terminal condition:

$$\lim_{T \rightarrow \infty} K_{T+1} \beta^T u'(c_T) = \lim_{T \rightarrow \infty} K_{T+1} \beta^{T+1} u'(c_{T+1})(1 + r_{T+1}) = 0$$

Hence

$$\lim_{t \rightarrow \infty} \beta^t u'(c_t)(1 + r_t)K_t = 0$$

Then, we have two conditions involve the infinity:

$$\lim_{t \rightarrow \infty} \beta^t u'(c_t)(1 + r_t)K_t = 0 \quad (2.1)$$

$$\lim_{t \rightarrow \infty} \left\{ K(t) \left[\prod_{s=1}^{t-1} \left(\frac{1}{1 + r_s} \right) \right] \right\} \geq 0 \quad (2.2)$$

The (2.1) is a condition for optimal behavior. The (2.2) is a condition to make sure that the flow budget constraints are consistent with a lifetime budget constraint and no perpetual debt.

Steady State (C^*, K^*)

From the Euler equation:

$$u'(C^*) = \beta(1 + F_K(K^*, L))u'(C^*)$$

hence

$$1 + r^* = 1 + F_K(k^*, 1) - \delta = \frac{1}{\beta}$$

which fully determines k^* .

Note. This shows that k^* does not depend on $u(\cdot)$.

Level of per-capita consumption is then given by the economy's resource constraint:

$$C^* + K^* = (1 - \delta)K^* + F(K^*, L)$$

hence,

$$c^* = F(k^*, 1) - \delta k^*$$

$$F_K(k^*, 1) = \delta + \left(\frac{1}{\beta} - 1 \right)$$

In the Solow model the golden-rule level of capital (which maximizes steady-state consumption) satisfies

$$F_K(k_S^{\text{GR}}, 1) = \delta$$

hence

$$k_{\text{NGM}}^* < k_S^{\text{GR}}$$

Note. Intuition:

In NGM model, people tend to consume more on the near future rather than in the far steady state.

The simple reason is discounting: in the Ramsey model the planner puts less weight on the long-run (anticipating that accumulating a lot of capital will require consumption sacrifices in the transition), while under the Golden Rule only the long-run outcome counts.

2.2 NGM in continuous time

Assume labor supply L growth at rate n :

$$L(t) = e^{nt}$$

Households choose consumption/saving to maximize

$$U = \int_0^\infty e^{-\rho t} e^{nt} u(c(t)) dt = \int_0^\infty e^{(n-\rho)t} u(c(t)) dt$$

Intertemporal budget constraint: as in the discrete time case, but treat $\Delta K \rightarrow 0$:

$$\dot{K}(t) = w(t)L(t) + (1 + r(t))K(t) - C(t)$$

Write in per-capita units:

$$\dot{k}(t) = w(t) + (1 + r(t))k(t) - c(t) - nk(t)$$

Non-Ponzi condition:

$$\lim_{t \rightarrow \infty} \left\{ k(t) \exp \left[- \int_0^t (r(v) - n) dv \right] \right\} \geq 0$$

To solve this, we define the **Hamiltonian**:

$$H(c, k, \mu) = e^{-(\rho-n)t} u(c(t)) + \mu(t) [w(t) + (1 + r(t) - n)k(t) - c(t)]$$

optimality conditions are

$$\begin{aligned} H_c(c, k, \mu) &= 0 \\ H_k(c, k, \mu) &= -\dot{\mu}(t) \\ H_\mu(c, k, \mu) &= \dot{k}(t) \\ \lim_{t \rightarrow \infty} \mu(t)k(t) &= 0 \end{aligned}$$

Hence:

$$\begin{aligned} e^{-(\rho-n)t} u'(c(t)) - \mu(t) &= 0 \\ \mu(t)(1 + r(t) - n) &= -\dot{\mu}(t) \end{aligned}$$

and we get:

$$e^{-(\rho-n)t} u'(c(t))(1 + r(t) - n) = -e^{-(\rho-n)t} u''(c(t)) \dot{c}(t) + (\rho - n) e^{-(\rho-n)t} u'(c(t))$$

Denote $\sigma(c) = -\frac{u'(c)}{u''(c)c}$, we have the Euler equation in continuous time:

$$\sigma(c)(1 + r(t) - \rho) = \frac{\dot{c}(t)}{c(t)}$$

$\sigma(c)$ is the intertemporal elasticity of substitution (IES).

2.3 From NGM to Endogenous growth

Key challenge: if capital is accumulated, and it has decreasing returns.

Then how can we keep marginal product high?

2.3.1 The AK model based on Arrow (1962) and Frankel (1962)

Arrow's (1962) approach to endogenous growth was to propose that technological progress is an unintended consequence of producing new capital goods, a phenomenon dubbed "learning by doing".

2.3.2 Romer (1980): increasing returns are a result of ideas being non-rival

- rival good: my consumption precludes you from consuming it
- non-rival good: all of us can consume it simultaneously without affecting others

Rivalry is a technological characteristic, whereas excludability is an institutional characteristic:
Non-excludable good: impossible to prevent it from being consumed/used by other agents.

Non-rivalry naturally suggests increasing returns to scale.

Example 1.

Cost of producing a movie is high and a one-time investment.

But once you produced it: constant returns to scale in DVD production,

There's IRS in "movies on DVD",

Movie itself is non-rivalrous!

Online piracy suggests that it may even not be (perfectly) excludable.

Hence, the natural structure of endogenous growth is:

$$Y_t = F(A_t, K_t, L_t)$$

with

$$\begin{aligned} F(A_t, \lambda K_t, \lambda L_t) &= \lambda F(A_t, K_t, L_t) \\ F(\lambda A_t, \lambda K_t, \lambda L_t) &> \lambda F(A_t, K_t, L_t) \end{aligned}$$

Constant returns in rivalrous factors, increasing returns in all factors.

Now, let's take the economy as a whole in a perfect competition setup with the previous production function. Then the profit in the economy is zero:

$$\Pi_t = F(A_t, K_t, L_t) - w_t L_t - R_t K_t = 0$$

as in Solow model/Ramsey model.

So how do we pay for the fixed cost of investment in ideas? Who pays the research labs?

There are two solutions:

1. Ideas accumulate as a by-product of production \rightarrow nobody has to pay (Romer 1986, Lucas 1988, learning-by-doing)
2. Give producers some market power \rightarrow positive ex-post profits \rightarrow can pay for research (Romer, 1990)

2.3.3 Models of R&D

Most simple models of endogenous technological change are those in which R&D expands the variety of inputs or machines used in production (e.g. Romer 1990) due to the increasing number of varieties, the division of labor (specialization) increases.

There's an alternative explanation: R&D improves the quality of inputs or machines used in production.

product innovation: increases the number of inputs or the quality of inputs.

process innovation: reduces the cost of existing products.

2.3.4 Sketch of Romer (1990) expanding varieties model

Romer assumed there are three decision makers in the model:

1. Households: maximize utility, subject to the usual budget constraint;
2. Final good producers: hire labor and purchase intermediate input, and combine them to produce the final output, sold at unit price
3. R&D firms:
 - Devote resources to invent new intermediate inputs
 - Once a product has been invented, the innovating firm obtains a perpetual patent (allows the firm to sell at the price it chooses)
 - price is chosen to maximize profits

So, the development intuition in this model is:

Higher willingness to save (lower rate of discounting ρ or higher elasticity of intertemporal substitution) leads to higher supply of capital, so more resources invested into R&D, resulting in higher rate of technological progress, thus, there will be higher growth rate in the steady state.

The fact that the new “idea” can be used in a nonrival manner is key for the R&D output affecting productivity (or more generally, the cost of production).

2.3.5 Schumpeterian endogenous growth models

Subsequent work used this mechanism to talk about the “industrial organization” of the model (Aghion-Howitt 1992)

- Stronger IP protection incentivizes firms to do R&D;
- But IP protection also causes a markup distortion: patents give market power
- And innovation potentially displaces an incumbent (“creative destruction”), which again somewhat lowers the returns to R&D.
- If factor reallocation is not working well, this can have additional costs.

2.3.6 Problems with endogenous growth models

At the heart of endogenous growth theories are mechanisms that are incredibly hard to observe and measure:

In the case of Romer it all relies on the R&D production function (and in Aghion-Howitt also on market structure/competition). But R&D is a process where we observe neither inputs nor outputs well.

1. Little of research output gets patented, and much of what gets patented is not economically valuable
2. Difficult to distinguish R&D input from production inputs with available data
3. Market structure/conduct: even more difficult to observe (see: antitrust/IO).

In the end, EGT use metaphors (“production of ideas”) that render the theory not falsifiable. And if it’s not falsifiable, it’s not scientific (Popper, 1934).

Real Business Cycle Model(1)

3.1 Introduction

Why do we have aggregate fluctuations (booms & recessions)?

What can we do against them?

3.1.1 Review of Undergrad Macro

Economy divided into three markets: labor market, money market, goods market.

- IS: $Y = \underbrace{A(Y, r, T, Y'^e, r'^e, T'^e)}_{\text{Demand for good}} + G$
- LM: $M = P \cdot L(Y, i)$
- PC: $\pi = \pi^e + \underbrace{f(Y, z)}_{\text{Deviation of output from MR level}}$

where Y'^e is the expected output of the next period, $r = i - \pi^e$ is the real expected interest rate.

3.1.2 Assumptions / implications of this model

In the short run the price level, P is given, hence IS and LM determines output. Demand matters and many factors affect demand.

In the medium run, $\pi^e = \pi$, so $f(Y, z) = 0$, output at the natural level.

money affects output in the short run, not in the medium run:

- money growth affects inflation and the nominal rate one-for-one in the medium run
- a fiscal expansion increases output in the short run, may decrease it in the medium run
- expectations matter: an anticipated fiscal contraction can be expansionary

3.1.3 Strength and weakness of these models

- Strengths:
 - provides a simple way of thinking about general equilibrium (complicated)
 - implicit micro-foundations: life cycle, q theory,...
 - time-tested shortcuts (but often not based on empirical evidence)
- Weakness:

- Static, not dynamic
- not really made for quantitative analysis (quantitative attempts very short-lived)
- need explicit micro-foundations: hard to do welfare analysis without that (Lucas critique: curves may not be invariant to policy)

3.1.4 Business Cycle Facts

Business Cycle Facts I

We'll study the **detrended macro times series**:

$$x_t = \log(X_t) - \log(X_t^*)$$

is the percentage deviation of variable X from its trend X^* .

How's the trend?

1. First linear,
2. more sophisticated filters: Baxter-King(Bankpass) filter, Hodrick-Prescott(HP) filter

Linear: just run OLS regression and use residuals.

Business Cycle Facts II: Look at the highest correlation with GDP:

$$\rho(X_t, Y_{t+k}) \quad k = -6, -5, \dots, 0, \dots, 5, 6$$

- if $\rho > 0$, then x is pro-cyclical.
- if $\rho < 0$, then x is counter-cyclical.
- if $k < 0$, then x lags behind output.
- if $k > 0$, then x leads output.

$\rho(X_t, X_{t+k})$ is called the autocorrelation function (as a function of k) of the stochastic process X_t .

Business Cycle Facts III:

Government spending, not strongly correlated with level of output, main driver of business cycles.

Real wage is mildly pro-cyclical: average wage evolves differently than the wage of a continuously employed worker.

Business Cycle Facts IV:

Standard deviations (unit of measure is percentage deviation from trend).

- GDP is more volatile than consumption
- investment is much more volatile than GDP
- government spending is pretty volatile

- working hours is almost exactly as volatile as GDP
- vast majority of the volatility of working hours is explained by the employment volatility
- TFP is very volatile

3.1.5 Towards the RBC Model

First, it's important to state the assumptions of our economic model:

- We will use a Walrasian model.
- The model assumes perfectly competitive markets.
- There are no externalities, asymmetric information, missing markets, or other imperfections.

Choice of Model

All the neoclassical models reviewed so far share these assumptions. A familiar model that could be built upon is the Ramsey model:

- The Ramsey model is known for converging to a balanced growth path in the absence of shocks, following which it exhibits smooth growth.

Model Extensions

It appears sensible to extend the Ramsey model to incorporate:

- Business cycle fluctuations.
- Real economic shocks.

These extensions will enhance the model's capability to reflect dynamic economic changes.

Factors to Consider

In extending the model, the following factors should be considered:

- Worker productivity.
- Government purchases.

Notably, since the Ramsey model does not incorporate monetary aspects, all introduced shocks will be in real terms, affecting the actual productive capacity of the economy.

The Real Business Cycle (RBC) Model

The modified version of the Ramsey model, incorporating real shocks and business cycle fluctuations, is known as the Real Business Cycle (RBC) model.

3.2 The Basic RBC Model

3.2.1 Preparation

We take the Cobb-Douglas production function as the basis for our model:

$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$$

Define the technological progress as:

$$A_t = A_t^* \tilde{A}_t$$

where $A_t^* = G^t \bar{A}$ is the long-run non-stochastic log-linear trend $G > 1$.

shock process is an autoregressive process of order 1

$$\log \tilde{A}_t = \rho \log \tilde{A}_{t-1} + \varepsilon_{A,t}$$

where $\varepsilon_{A,t} \sim N(0, \sigma^2)$.

The capital accumulation equation is:

$$K_{t+1} = (1 - \delta)K_t + I_t$$

where $I_t = Y_t - C_t$ is the investment.

Define the household's utility function as:

$$U = \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i U(C_{t+i}, 1 - L_{t+i})$$

where β is the discount factor, C is consumption, and L is fraction of time spent working.

$U_C(C, 1 - L) > 0$, $U_{CC}(C, 1 - L) < 0$ and $U_L(C, 1 - L) > 0$, $U_{LL}(C, 1 - L) < 0$

3.2.2 The Household's Problem

$$\max_{\{C_t, L_t, K_t, S_t\}} \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i U(C_{t+i}, 1 - L_{t+i})$$

$$\text{Subject to: } S_{t+i} + C_{t+i} = \tilde{R}_{t+i} K_{t+i} + W_{t+i} L_{t+i}$$

$$K_{t+i+1} = (1 - \delta)K_{t+i} + S_{t+i}$$

$$K_{t+i} \geq 0; \quad K_0 > 0$$

Define the Lagrangian as:

$$\mathcal{L} = \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i U(C_{t+i}, 1 - L_{t+i}) - \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \lambda_{t+i} (K_{t+i+1} - (1 - \delta)K_{t+i} - \tilde{R}_{t+i}K_{t+i} - W_{t+i}L_{t+i} + C_{t+i})$$

We want to take a snapshot of how the variables behave in the period in which we are optimising

in, t . Then, the period t variables appear as:

$$\mathcal{L}_t = U(C_t, 1 - L_t) - \lambda_t(K_{t+1} - (1 - \delta)K_t - \tilde{R}_t K_t - W_t L_t + C_t)$$

Household Behavior I

The inter-temporal Euler equation is:

$$U_1(X_t, 1 - L_t) = \beta \mathbb{E}_t (R_{t+1} U_1(C_{t+1}, 1 - L_{t+1}))$$

where $R_{t+1} = 1 + \tilde{R}_{t+1} - \delta$.

3.2.3 Effect of RBC

What are the effects of a positive technological shock?

It increases current and future R and W .

Consumption

- **Income effect:** People feel richer which leads to an increase in consumption.
- **Substitution effect:** Saving is worth more, thus consumption might decrease.
- **Net effect:** Consumption is probably increased.

Leisure

- **Income effect:** People feel richer, thus they want to enjoy more leisure, leading to an increase in leisure.
- **Substitution effect:** Higher wage leads to a decrease in leisure.
- **Net effect:** The overall impact on leisure depends on the relative strength of the two effects.

Type of Shock

- **Transitory shock:** Leads to a smaller wealth effect and a stronger substitution effect.
- **Permanent shock:** It is possible that consumption goes up and employment goes down.

3.2.4 Brief Summary

- Consumption is too pro-cyclical;
- investment $I_t = s_t Y_t = \alpha \beta Y_t$ also pro-cyclical;
- effects of R on saving? r increase, then inter-temporal substitution effect led C_t decreases, then the income effect increases C_t (in this model they cancel out)

- W_t is too pro-cyclical

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4.1 A simplified real-business cycle model with additive technology shocks

Problem 4.1.1 (1). Find the first-order condition (Euler equation) relating C_t and C_{t+1} in the following model:

Solution. The household's problem is to maximize

$$\max_{\{C_t\}} E_t \sum_{t=0}^{\infty} u(C_t)/(1+\rho)^t$$

subject to the budget constraint

$$K_{t+1} = K_t + Y_t - C_t$$

where $Y_t = AK_t + e_t$ and e_t is a random technology shock satisfying $e_t = \phi e_{t-1} + \varepsilon_t$, $-1 < \phi < 1$ and that ε_t is **i.i.d.**

Define the Lagrange function:

$$\mathcal{L} = E_t \sum_{t=0}^{\infty} [u(C_t)/(1+\rho)^t - \lambda_t(K_{t+1} - K_t - Y_t + C_t)] \quad (4.1)$$

The first-order condition with respect to C_t is:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \Rightarrow \frac{u'(C_t)}{(1+\rho)^t} - \lambda_t = 0 \quad (4.2)$$

The first-order condition with respect to K_{t+1} is:

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0 \Rightarrow -\lambda_t + E_t [\lambda_{t+1}] (1+\rho) = 0 \quad (4.3)$$

Combining (2) and (3) we get:

$$\frac{u'(C_t)}{(1+\rho)^t} = E_t [\lambda_{t+1}] (1+\rho) \quad (4.4)$$

We have the relationship between C_t and C_{t+1} :

$$u'(C_t) = E_t [u'(C_{t+1})] \quad (4.5)$$

Since $u(C_t) = C_t - \theta C_t^2$, we have $u'(C_t) = 1 - 2\theta C_t$. Thus, the Euler equation is:

$$1 - 2\theta C_t = E_t [1 - 2\theta C_{t+1}] \Rightarrow C_t = E_t [C_{t+1}] \quad (4.6)$$

Problem 4.1.2 (2). Guess that consumption takes the form $C_t = \alpha + \beta K_t + \gamma e_t$. What's K_t in this case?

Solution.

$$\begin{aligned} K_{t+1} &= K_t + Y_t - C_t \\ &= K_t + (AK_t + e_t) - \alpha - \beta K_t - \gamma e_t \\ &= (1 + A - \beta)K_t - \alpha + (1 - \gamma)e_t \end{aligned}$$

Problem 4.1.3 (3). What values must the parameters α , β , and γ have for the first-order condition in part 1 to be satisfied for all values of K_t and e_t ?

Solution. From the Euler equation, we have:

$$C_t = E_t [C_{t+1}] \quad (4.7)$$

Substituting $C_t = \alpha + \beta K_t + \gamma e_t$ and $C_{t+1} = \alpha + \beta K_{t+1} + \gamma e_{t+1}$ into (7), we get:

$$\alpha + \beta K_t + \gamma e_t = E_t [\alpha + \beta K_{t+1} + \gamma e_{t+1}] \quad (4.8)$$

Rearranging using

$$E_t [e_{t+1}] = E_t [\phi e_t + \varepsilon] = \phi e_t.$$

Substituting $K_{t+1} = (1 + A - \beta)K_t - \alpha - (1 - \gamma)e_t$ into (8), we get:

$$\alpha + \beta K_t + \gamma e_t = E_t [\alpha + \beta((1 + A - \beta)K_t - \alpha - (1 - \gamma)e_t) + \phi \gamma e_t + \gamma \varepsilon_t] \quad (4.9)$$

Simplifying (9) we get:

$$\begin{aligned} \alpha + \beta K_t + \gamma e_t &= \alpha + \beta(1 + A - \beta)K_t - \alpha\beta + \beta(1 - \gamma)e_t + \phi\gamma e_t + \gamma E_t [\varepsilon_t] \\ \Rightarrow \beta K_t + \gamma e_t &= \beta(1 + A - \beta)K_t - \alpha\beta + \beta(1 - \gamma)e_t + \phi\gamma e_t \end{aligned}$$

Hence, we have:

$$\begin{aligned} \alpha\beta &= 0 \\ \beta &= \beta(1 + A - \beta) \\ \gamma &= \beta(1 - \gamma) + \phi\gamma \end{aligned}$$

Solving the above equations, we get:

$$\begin{aligned} \alpha &= 0 \\ \beta &= A = \rho \\ \gamma &= \frac{\beta}{1 + \beta - \phi} = \frac{A}{1 + A - \phi} \end{aligned}$$

Problem 4.1.4 (4). What are the effects of a one-time shock to ε on the paths of Y , K , and C ?

Solution.

$$K_{t+1} = (1 + A - \beta)K_t - \alpha + (1 - \gamma)e_t = K_t + \frac{1 - \phi}{1 + A - \phi}e_t \quad (4.10)$$

Thus, we can have:

$$\begin{aligned} K_{t+T} &= K_t + \sum_{j=0}^{T-1} \frac{1 - \phi}{1 + A - \phi} e_{t+j} \\ &= K_t + \frac{1 - \phi}{1 + A - \phi} \sum_{j=0}^{T-1} \phi^j e_t \\ \Rightarrow \lim_{T \rightarrow \infty} K_{t+T} &= K_t + \frac{1}{1 + A - \phi} e_t \end{aligned}$$

For C_t :

$$\begin{aligned} C_t &= \alpha + \beta K_t + \gamma e_t \\ &= AK_t + \frac{A}{1 + A - \phi} e_t \\ C_{t+1} &= AK_{t+1} + \frac{A}{1 + A - \phi} e_{t+1} \\ &= A \left[K_t + \frac{1 - \phi}{1 + A - \phi} e_t \right] + \frac{A}{1 + A - \phi} e_{t+1} \\ &= AK_t + \frac{A(1 - \phi)}{1 + A - \phi} e_t + \frac{A}{1 + A - \phi} e_{t+1} \end{aligned}$$

Similarly,

$$\begin{aligned} C_{t+T} &= AK_{t+T} + \frac{A}{1 + A - \phi} e_{t+T} \\ &= AK_t + \frac{A}{1 + A - \phi} \sum_{t=0}^{T-1} \phi^j e_t \\ \Rightarrow \lim_{T \rightarrow \infty} C_{t+T} &= AK_t + \frac{A}{1 + A - \phi} \phi^T e_t + \frac{A}{1 + A - \phi} e_t \end{aligned}$$

4.2 Solving the RBC model by finding the social optimum

Problem 4.2.1 (1). Explain the Bellman equation:

$$V(K_t, A_t) = \max_{C_t, L_t} \{ \log C_t + b \log(1 - L_t) + \beta E_t [V(K_{t+1}, A_{t+1})] \} \quad (4.11)$$

Solution.

Problem 4.2.2 (2). Given the log-linear structure of the model, let us guess that V takes the form:

$$V(K_t, A_t) = \gamma_0 + \gamma_K \log K_t + \gamma_A \log A_t$$

where the values of the β 's are to be determined. Substituting this conjectured form and the facts that $K_{t+1} = Y_t - C_t$ and $E_t[\log A_{t+1}] = \rho \log A_t$ into the Bellman equation yields

$$V(K_t, A_t) = \max_{C_t, L_t} \{ \log C_t + b \log(1 - L_t) + \beta [\gamma_0 + \gamma_K \log(Y_t - C_t) + \gamma_A \log A_t] \}.$$

Find the first-order condition for C_t . Show that it implies that C_t/Y_t does not depend on K_t or A_t .

Solution. Based on our value function, the first-order condition for C_t is:

$$\frac{\partial V}{\partial C_t} = 0 \Rightarrow \frac{1}{C_t} = \beta \gamma_K \frac{1}{Y_t - C_t} \Rightarrow \frac{C_t}{Y_t} = \frac{1}{1 + \beta \gamma_K} \quad (4.12)$$

Hence, C_t/Y_t does not depend on K_t or A_t .

Problem 4.2.3 (3). Find the first-order condition for L_t . Use this condition and the result in part 2 to show that L_t does not depend on K_t or A_t .

Solution.

$$\frac{\partial V}{\partial L_t} = 0 \Rightarrow -\frac{b}{1 - L_t} + \beta \gamma_K \frac{1}{Y_t - C_t} (1 - \alpha) K_t^\alpha A_t^{1-\alpha} L_t^{-\alpha} = 0 \quad (4.13)$$

Simplifying (12) we get:

$$-\frac{b}{1 - L_t} + \frac{\beta \gamma_K}{Y_t - C_t} (1 - \alpha) \frac{Y_t}{L_t} = 0 \quad (4.14)$$

Replace $Y_t - C_t$ with $\beta \gamma_K C_t$:

$$\begin{aligned} -\frac{b}{1 - L_t} + \frac{\beta \gamma_K}{\beta \gamma_K C_t} (1 - \alpha) \frac{Y_t}{L_t} &= 0 \\ \Rightarrow \frac{1 - \alpha}{L_t} (1 + \beta \gamma_K) &= \frac{b}{1 - L_t} \\ \Rightarrow L_t &= \frac{(1 - \alpha)(1 + \beta \gamma_K)}{(1 - \alpha)(1 + \beta \gamma_K) + b} \end{aligned}$$

So, L_t does not depend on K_t or A_t .

Problem 4.2.4 (4). Substitute the production function and the results in questions 2 and 3 for the optimal C_t and L_t into the equation above for V , and show that the resulting expression has the form $V(K_t, A_t) = \gamma'_0 + \gamma'_K \log K_t + \gamma'_A \log A_t$.

Solution. As we know that

$$\begin{aligned} E_t[\log A_{t+1}] &= \rho \log A_t \\ E_t[K_{t+1}] &= \log(Y_t - C_t) \end{aligned}$$

So, by substituting the optimal C_t and L_t into the equation for V , we get:

$$V(K_t, A_t) = \log \frac{Y_t}{1 + \beta\gamma_K} + b \log \frac{b}{(1 - \alpha)(1 + \beta\gamma_K) + b} + \beta \left\{ \gamma_0 + \gamma_K \log \frac{\beta\gamma_K Y_t}{1 + \beta\gamma_K} + \gamma_A \rho \log A_t \right\}$$

Substitute $Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$, we have:

$$\begin{aligned} V(K_t, A_t) &= \alpha \log K_t + (1 - \alpha) \log A_t + (1 - \alpha) \log L_t - \log(1 + \beta\gamma_K) \\ &\quad + b \log \frac{b}{(1 - \alpha)(1 + \beta\gamma_K) + b} + \beta\gamma_0 \\ &\quad + \beta\gamma_K \{ \log(\beta\gamma_K) - \log(1 + \beta\gamma_K) + \alpha \log K_t + (1 - \alpha) \log A_t + (1 - \alpha) \log L_t \} \\ &\quad + \beta\gamma_A \rho \log A_t \end{aligned}$$

Rearranging the terms, we get:

$$\begin{aligned} V(K_t, A_t) &= (\alpha + \alpha\beta\gamma_K) \log K_t + (1 - \alpha + (1 - \alpha)\beta\gamma_K + \beta\gamma_A \rho) \log A_t \\ &\quad + (1 - \alpha + (1 - \alpha)\beta\gamma_K) \log L_t \\ &\quad - \log(1 + \beta\gamma_K) + b \log \frac{b}{(1 - \alpha)(1 + \beta\gamma_K) + b} + \beta\gamma_0 \\ &\quad + \beta\gamma_K \log(\beta\gamma_K) - \beta\gamma_K \log(1 + \beta\gamma_K) \end{aligned}$$

We define:

$$\begin{aligned} \gamma'_K &= \alpha(1 + \beta\gamma_K) \\ \gamma'_A &= (1 - \alpha)(\beta\gamma_K + 1) + \beta\gamma_A \rho \\ \gamma'_0 &= \beta\gamma_0 + b \log \frac{b}{(1 - \alpha)(1 + \beta\gamma_K) + b} - \log(1 + \beta\gamma_K)(1 + \beta\gamma_K) \\ &\quad + \beta\gamma_K \log(\beta\gamma_K) + (1 - \alpha + (1 - \alpha)\beta\gamma_K) \log L_t \end{aligned}$$

According to problem 3, we know that L_t does not depend on K_t or A_t , so, we could treat it as a constant in this expression about K_t and A_t . Thus, we have:

$$V(K_t, A_t) = \gamma'_0 + \gamma'_K \log K_t + \gamma'_A \log A_t.$$

Problem 4.2.5 (5). What must γ_K and γ_A be so that $\gamma'_K = \gamma_K$ and $\gamma'_A = \gamma_A$?

Solution. Let $\gamma'_K = \gamma_K$ and $\gamma'_A = \gamma_A$, we have:

$$\begin{aligned} \gamma_K &= \alpha(1 + \beta\gamma_K) \\ \gamma_A &= (1 - \alpha)(\beta\gamma_K + 1) + \beta\gamma_A \rho \\ \Rightarrow \gamma_K &= \frac{\alpha}{1 - \alpha\beta} \\ \Rightarrow \gamma_A &= \frac{1 - \alpha}{(1 - \alpha\beta)(1 - \beta\rho)} \end{aligned}$$

Problem 4.2.6 (6). What are the implied values of C_t/Y_t and L_t ? Are they the same as those found in the lecture?

Solution. Substitute γ_K into the expression for C_t/Y_t and L_t , we have:

$$\frac{C_t}{Y_t} = \frac{1}{1 + \beta\gamma_K} = \frac{1}{1 + \beta\frac{\alpha}{1-\alpha\beta}} = 1 - \alpha\beta$$
$$L_t = \frac{(1 - \alpha)(1 + \beta\gamma_K)}{(1 - \alpha)(1 + \beta\gamma_K) + b} = \frac{(1 - \alpha)}{(1 - \alpha) + b(1 - \alpha\beta)}$$

Lecture 5.

Problem Set 4

Question 1

Consider an RBC model with a representative household that maximizes utility

$$U_t = E_t \sum_{t=0}^{\infty} \beta^t (\log(C_t) + \nu(1 - L_t))$$

where ν is an increasing and concave function. The production function is

$$Y_t = A_t L_t^{1-\alpha} (z_t K_t)^\alpha$$

with $0 < \alpha < 1$ and z_t denotes a capital utilization choice. For example, low capital utilization may mean running the machines only at half speed or half of the time. Capital accumulation is described by

$$K_{t+1} = (1 - \bar{\delta} z_t^\phi) K_t + Y_t - C_t.$$

Problem (1). Explain how variable utilization of capital is modeled here. Should we assume that $\phi < 1$ or $\phi > 1$?

Solution. In the given model, capital utilization z_t directly affects both production and capital depreciation: in the production function, $z_t K_t$ is the effective capital input at time period t , higher z_t increases the effective capital services, thus boosting output. The depreciation rate is $\delta_t = \bar{\delta} z_t^\phi$, and higher z_t leads to a higher depreciation rate.

Depreciation Function $\delta_t = \bar{\delta} z_t^\phi$ determines how sensitively depreciation responds to changes in utilization z_t . The marginal increase in depreciation due to higher z_t is given by:

$$\frac{\partial \delta_t}{\partial z_t} = \bar{\delta} \phi z_t^{\phi-1}.$$

Intuitively, if utilization increases, the marginal cost of utilizing capital rises, reflecting higher incremental damage, meaning that the capital depreciates faster, which implies:

$$\frac{\partial^2 \delta_t}{\partial z_t^2} = \bar{\delta} \phi(\phi - 1) z_t^{\phi-2} > 0.$$

Hence, we should assume that $\phi > 1$.

Problem (2). Derive and explain the optimality conditions of the social planner's problem. The social planner's problem is to maximize total utility subject to the economy's physical resource constraints (i.e. production, and the law of motion for capital). The social planner can set the levels of consumption, investment, capacity utilization, and labor supply, without having to worry about prices or market clearing.

Solution. The social planner maximizes the expected discounted utility:

$$\begin{aligned} \max_{\{C_t, L_t, z_t\}} E_t \sum_{t=0}^{\infty} \beta^t (\log(C_t) + \nu(1 - L_t)) \\ \text{s.t. } Y_t = A_t L_t^{1-\alpha} (z_t K_t)^\alpha \\ K_{t+1} = (1 - \bar{\delta} z_t^\phi) K_t + Y_t - C_t \end{aligned}$$

Define the Lagrangian:

$$\mathcal{L} = E_t \sum_{t=0}^{\infty} \beta^t \left\{ \log(C_t) + \nu(1 - L_t) - \lambda_t \left[C_t + K_{t+1} - (1 - \bar{\delta} z_t^\phi) K_t - Y_t \right] \right\}$$

The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = \beta^t \left(\frac{1}{C_t} - \lambda_t \right) = 0 \Rightarrow \frac{1}{C_t} = \lambda_t \quad (5.1)$$

$$\frac{\partial \mathcal{L}}{\partial L_t} = \beta^t \left[-\nu'(1 - L_t) + \lambda_t A_t (1 - \alpha) L_t^{-\alpha} (z_t K_t)^\alpha \right] = 0 \Rightarrow \nu'(1 - L_t) = \lambda_t A_t (1 - \alpha) L_t^{-\alpha} (z_t K_t)^\alpha \quad (5.2)$$

$$\frac{\partial \mathcal{L}}{\partial z_t} = \beta^t \lambda_t \left[A_t \alpha L_t^{1-\alpha} z_t^{\alpha-1} K_t^\alpha - \bar{\delta} \phi z_t^{\phi-1} K_t \right] = 0 \Rightarrow \bar{\delta} \phi z_t^{\phi-1} = A_t \alpha L_t^{1-\alpha} z_t^{\alpha-1} K_t^{\alpha-1} \quad (5.3)$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} \left[(1 - \bar{\delta} z_{t+1}^\phi) + \alpha \frac{Y_{t+1}}{K_{t+1}} \right] = 0 \Rightarrow \lambda_t = \beta \lambda_{t+1} (1 - \bar{\delta} z_{t+1}^\phi + \alpha \frac{Y_{t+1}}{K_{t+1}}) \quad (5.4)$$

From the FOCs, we could tell that:

1. The shadow price of consumption equals to the marginal utility of consumption.
2. For the derivative with respect to L_t , we can transform it into the following format:

$$\nu'(1 - L_t) = \frac{\partial Y_t}{C_t \partial L_t}$$

The marginal utility of leisure equals the marginal utility gain from increased consumption made possible by supplying an extra unit of labor.

3. For the derivative with respect to z_t , we can transform it into the following format:

$$\frac{\partial \delta_t K_t}{\partial z_t} = \frac{\partial Y_t}{\partial z_t}$$

The marginal cost(depreciation) of capital equals to the marginal gain of production of increasing capital utilization.

4. The standard inter-temporal condition balancing current and future utility.

Problem (3). Derive an expression for the steady-state capital utilization z^* in terms of the steady-state capital-output ratio. If we want to normalize $z^* = 1$, how should we calibrate ϕ ?

Solution. At the steady state: $Y_t = Y$, $C_t = C$, $K_t = K$, $A_t = A$ and $L_t = L$. Since $C_t = C_{t+1}$, we know that $\lambda_t = \lambda_{t+1}$. From 5.3, we know that:

$$\begin{aligned}\alpha AL^{1-\alpha}(z^*K)^{\alpha-1} &= \phi\bar{\delta}z^{*\phi-1} = \frac{\alpha Y}{z^*K} \\ \Rightarrow \frac{Y}{K} &= \frac{\phi\bar{\delta}z^{*\phi}}{\alpha}\end{aligned}$$

Denote $\kappa = \frac{K}{Y}$, which is the output ratio, we would have:

$$z^* = \left(\frac{\alpha}{\kappa\phi\bar{\delta}} \right)^{\frac{1}{\phi}}.$$

Then, we use 5.4 and $\lambda_t = \lambda_{t+1}$, we'll have:

$$\begin{aligned}1 &= \beta(1 - \bar{\delta}z^{*\phi} + \alpha\frac{Y}{K}) = \beta(1 - \bar{\delta}z^{*\phi} + \alpha\frac{\phi\bar{\delta}z^{*\phi}}{\alpha}) = \beta(1 - \bar{\delta}z^{*\phi} + \phi\bar{\delta}z^{*\phi}) \\ \Rightarrow (\phi - 1)\bar{\delta}z^{*\phi} &= \frac{1 - \beta}{\beta} \\ \Rightarrow z^* &= \left(\frac{1 - \beta}{\beta(\phi - 1)\bar{\delta}} \right)^{\frac{1}{\phi}}\end{aligned}$$

Normalize $z^* = 1$, we simplify the equation into:

$$1 - \beta = \beta(\phi - 1)\bar{\delta},$$

which gives that

$$\phi = \frac{1 - \beta}{\beta\bar{\delta}} + 1 = \frac{1 - \beta + \beta\bar{\delta}}{\beta\bar{\delta}}$$

Or, we would simplify the expression of z^* with capital-output ratio κ :

$$\alpha = \kappa\phi\bar{\delta} \Rightarrow \phi = \frac{\alpha}{\kappa\bar{\delta}}$$

Problem (4). Compared to a standard RBC model (as in the lecture), do you expect this model to generate a stronger or weaker response of output to a TFP shock?

Solution. Compared to a standard RBC model, this model generates stronger response of output to a TFP shock.

In the Standard RBC Model:

Capital utilization is fixed ($z_t = 1$). Output response to TFP shocks relies solely on adjustments

in labor and capital.

In This Model with Variable Utilization:

Firms can adjust z_t in response to TFP shocks. A positive TFP shock allows firms to temporarily increase z_t , boosting output beyond what is possible in the standard model, which gives higher fluctuations.

Problem (5). Assuming the model is correct, how is the standard empirical Solow residual $\log Y_t - (1 - \alpha) \log L_t - \alpha \log K_t$ related to total factor productivity $\log A_t$?

Solution. If our model is correct, then we log-linearize the production function:

$$\begin{aligned}\log Y_t &= \log A_t + (1 - \alpha) \log L_t + \alpha \log K_t + \alpha \log z_t \\ \Rightarrow \text{Solow Residual} &= \log Y_t - (1 - \alpha) \log L_t - \alpha \log K_t = \log A_t + \alpha \log z_t\end{aligned}$$

Question 2

We study a variation of the simplified RBC model from the lecture. Consider an economy populated by a mass of representative households and a mass of representative firms which take output and factor prices as given. The production function is Cobb-Douglas:

$$Y_t = e^{z_t} K_t^\alpha L_t^{1-\alpha}$$

where e^{z_t} is total factor productivity, K is capital, and L is labor. Capital fully depreciates in every periods so that

$$K_{t+1} = I_t$$

where I_t is investment.

The representative household supplies labor L_t and consumes C_t . It maximizes

$$U = E_0 \left[\sum_{t=0}^{\infty} \beta^t [\log C_t - \chi_t L_t] \right]$$

where χ_t is a preference shock. Capital is accumulated by the household and rented to the firm, as usual. Any profits of the firms are rebated to the households. As usual, we have $0 < \alpha < 1, 0 < \beta < 1$.

Problem (1). Write down the budget constraint of the household and the profit function of firms.

Solution. Assume the rental rate of capital is r_t and the wage rate is w_t . The budget constraint of the household is:

$$C_t + K_{t+1} = r_t K_t + w_t L_t + \Pi_t$$

The profit function of firms is:

$$\Pi_t = Y_t - r_t K_t - w_t L_t = e^{z_t} K_t^\alpha L_t^{1-\alpha} - r_t K_t - w_t L_t$$

Problem (2). Derive first-order conditions of firms and households.

Solution. From problem (1), we established:

$$\Pi_t = Y_t - r_t K_t - w_t L_t \Rightarrow r_t K_t + w_t L_t + \Pi_t = Y_t.$$

Thus, our households budget constraint is:

$$C_t + K_{t+1} = Y_t.$$

We define the Lagrangian of the household problem as:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t [\log C_t - e^{\chi_t} L_t - \lambda_t (C_t + K_{t+1} - Y_t)].$$

The first-order conditions of households are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C_t} &= \beta^t \left(\frac{1}{C_t} - \lambda_t \right) = 0 \Rightarrow \frac{1}{C_t} = \lambda_t \\ \frac{\partial \mathcal{L}}{\partial L_t} &= \beta^t (-e^{\chi_t} + \lambda_t \frac{\partial Y_t}{\partial L_t}) = 0 \Rightarrow e^{\chi_t} = \lambda_t \frac{\partial Y_t}{\partial L_t} \\ \frac{\partial \mathcal{L}}{\partial K_{t+1}} &= -\beta^t \lambda_t + \beta^{t+1} E_t \lambda_{t+1} \frac{\partial Y_{t+1}}{\partial K_{t+1}} = 0 \Rightarrow \lambda_t = \beta \lambda_{t+1} E_t \frac{\partial Y_{t+1}}{\partial K_{t+1}} \end{aligned}$$

Let's compute $\frac{\partial Y_t}{\partial L_t}$ and $\frac{\partial Y_t}{\partial K_{t+1}}$:

$$\begin{aligned} \frac{\partial Y_t}{\partial L_t} &= e^{z_t} K_t^\alpha (1 - \alpha) L_t^{-\alpha} = (1 - \alpha) \frac{Y_t}{L_t} \\ \frac{\partial Y_{t+1}}{\partial K_{t+1}} &= \alpha e^{z_{t+1}} K_{t+1}^{\alpha-1} L_{t+1}^{1-\alpha} = \alpha \frac{Y_{t+1}}{K_{t+1}} \end{aligned}$$

So, our households FOCs are:

$$\lambda_t = \frac{1}{C_t} \tag{5.5}$$

$$e^{\chi_t} = \lambda_t (1 - \alpha) \frac{Y_t}{L_t} = \frac{(1 - \alpha) Y_t}{C_t L_t} \tag{5.6}$$

$$\lambda_t = \alpha \beta \lambda_{t+1} E_t \frac{Y_{t+1}}{K_{t+1}} \Rightarrow \frac{1}{C_t} = \alpha \beta E_t \frac{Y_{t+1}}{C_{t+1} K_{t+1}} \tag{5.7}$$

The Lagrangian of the firm problem is:

$$\max_{K_t, L_t} \Pi_t = e^{z_t} K_t^\alpha L_t^{1-\alpha} - r_t K_t - w_t L_t$$

The first-order conditions of firms are:

$$\frac{\partial \Pi_t}{\partial K_t} = \alpha e^{z_t} K_t^{\alpha-1} L_t^{1-\alpha} - r_t = 0 \Rightarrow r_t = \alpha e^{z_t} K_t^{\alpha-1} L_t^{1-\alpha} = \alpha \frac{Y_t}{K_t} \quad (5.8)$$

$$\frac{\partial \Pi_t}{\partial L_t} = (1 - \alpha) e^{z_t} K_t^\alpha L_t^{-\alpha} - w_t = 0 \Rightarrow w_t = (1 - \alpha) e^{z_t} K_t^\alpha L_t^{-\alpha} = (1 - \alpha) \frac{Y_t}{L_t} \quad (5.9)$$

Problem (3). Show that when factor markets and final good markets are competitive, firm profits are zero.

Solution. The competitive equilibrium requires that the rental rate of capital equals the marginal product of capital and the wage rate equals the marginal product of labor:

$$r_t = \alpha \frac{Y_t}{K_t}$$

$$w_t = (1 - \alpha) \frac{Y_t}{L_t}$$

In this case, the profit function of firms becomes:

$$\Pi_t = Y_t - \alpha \frac{Y_t}{K_t} K_t - (1 - \alpha) \frac{Y_t}{L_t} L_t = 0.$$

Problem (4). Solve the model and show that the process for output in a competitive equilibrium is

$$y_t = z_t + \alpha y_{t-1} - (1 - \alpha) \chi_t$$

(where lowercase letters denote the log of the uppercase letter).

Solution. From the production function FOC 5.7, we have: $\frac{1}{C_t} = \alpha \beta E_t \frac{Y_{t+1}}{K_{t+1} C_{t+1}}$. Since $Y_{t+1} = C_{t+1} + I_{t+1} = C_{t+1} + K_{t+2}$, We have:

$$\frac{1}{C_t} = \alpha \beta E_t \frac{C_{t+1} + K_{t+2}}{K_{t+1} C_{t+1}}$$

$$\Rightarrow \frac{K_{t+1}}{C_t} = \alpha \beta E_t \left(1 + \frac{K_{t+2}}{C_{t+1}} \right)$$

According to the Banach's Fixed-point Theorem, $x = \alpha \beta (1 + x)$, and we can solve

$$\frac{K_{t+1}}{C_t} = \frac{\alpha \beta}{1 - \alpha \beta}.$$

As $K_{t+1} = Y_t - C_t$, we have $K_{t+1} = \alpha \beta Y_t$ and $C_t = (1 - \alpha \beta) Y_t$. Then, by 5.6 and 5.9, we know that:

$$e^{x_t} = \frac{(1 - \alpha) Y_t}{C_t L_t}$$

$$\Rightarrow e^{x_t} = \frac{(1 - \alpha) Y_t}{(1 - \alpha \beta) Y_t L_t} = \frac{1 - \alpha}{1 - \alpha \beta} \frac{1}{L_t}$$

$$\Rightarrow L_t = \frac{1-\alpha}{1-\alpha\beta} e^{-\chi_t}$$

$$Y_t = e^{z_t} K_t^\alpha L_t^{1-\alpha} = e^{z_t} (\alpha\beta Y_{t-1})^\alpha \left(\frac{1-\alpha}{1-\alpha\beta} e^{-\chi_t} \right)^{1-\alpha}$$

The the log of both sides, we have:

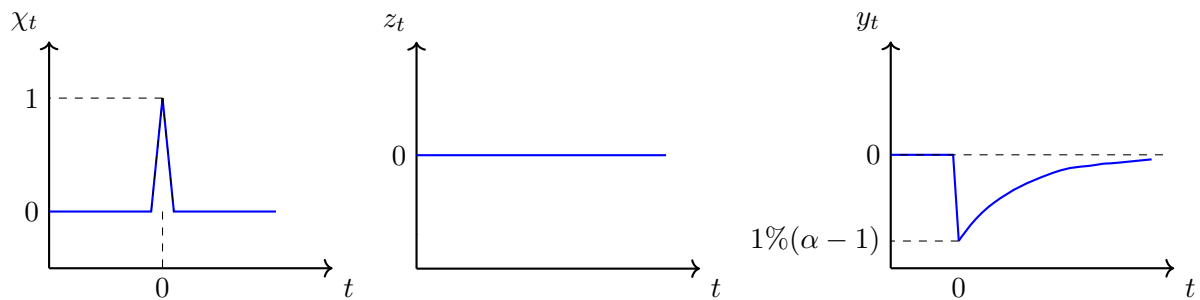
$$\begin{aligned} \log Y_t &= z_t + \alpha \log(\alpha\beta) + \alpha \log Y_{t-1} + (1-\alpha) \log \left(\frac{1-\alpha}{1-\alpha\beta} \right) - (1-\alpha)\chi_t \\ &= z_t + \alpha \log Y_{t-1} - (1-\alpha)\chi_t + \alpha \log(\alpha\beta) + (1-\alpha) \log \left(\frac{1-\alpha}{1-\alpha\beta} \right) \\ \Rightarrow y_t &= z_t + \alpha y_{t-1} - (1-\alpha)\chi_t + \alpha \log(\alpha\beta) + (1-\alpha) \log \left(\frac{1-\alpha}{1-\alpha\beta} \right) \end{aligned}$$

We drop the constant and gives: $y_t = z_t + \alpha y_{t-1} - (1-\alpha)\chi_t$.

Problem (5). Assume $y_{-1} = 0$ and $z_t = 0$ for all t , and $\chi_t = 0$ for all t except that $\chi_0 = 1\%$. Draw the time path for χ_t , z_t , and y_t . Explain why y_t is persistent.

Solution.

At time period 0, $y_0 = z_0 + \alpha y_{-1} - (1-\alpha)\chi_0 = 1\%(\alpha - 1)$, and after time period 0, we have $z_t = 0$ and $\chi_t = 0$, thus $y_t = \alpha y_{t-1} = 1\%\alpha^t(\alpha - 1)$. Then, we can draw the three graphs as below:



The initial preference shock χ_0 increases the disutility of labor, leading households to supply less labor at $t = 0$.

This reduces output at $t = 0$.

Since capital at $t + 1$ depends on savings from Y_t , the reduced output leads to lower capital K_{t+1} .

The lower capital stock in subsequent periods reduces output further, even though the preference shock is gone.

The capital accumulation process transmits the shock over time, causing persistence in output deviations from steady state.

But as the shock has ended, the output will finally return back to the steady state.

Lecture 6.

Real Business Cycle Model(2)

6.1 A more General Model

Assuming not full depreciation $0 < \delta < 1$. The household's problem is

$$U = E_t \sum_{t=0}^{\infty} \beta^t U(C_{t+i}, 1 - L_{t+i})$$

where the utility function is:

$$U(C_t, 1 - L_t) = \ln C_t + \theta \frac{(1 - L_t)^{1-\gamma_l}}{1 - \gamma_l}$$

log in consumption, CES is leisure. $\gamma_l = 1$ shows log in leisure as well.

6.1.1 The log-linearized version

Small letters denote the log-deviations from the non-stochastic BGP: $c_t = \ln C_t - \ln C_t^*$, $l_t = \ln L_t - \ln L_t^*$.

1. The production function The production function is:

$$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$$

rewrite as:

$$Y_t^* e^{y_t} = K_t^{*\alpha} e^{\alpha k_t} (A_t^* L_t^*)^{1-\alpha} e^{(a_t + l_t)(1-\alpha)}$$

and use the production function to get:

$$y_t = \alpha k_t + (a_t + l_t)(1 - \alpha).$$

2. Capital accumulation

$$K_{t+1} = (1 - \delta)K_t + Y_t - C_t$$

re-write as

$$K_{t+1}^* e^{k_{t+1}} = (1 - \delta)K_t^* e^{k_t} + Y_t^* e^{y_t} - C_t^* e^{c_t}$$

Now use the fact that for x small enough $e^x \approx 1 + x$ to get:

$$K_{t+1}^* + K_{t+1}^* k_{t+1} \approx (1 - \delta)K_t^* + (1 - \delta)K_t^* k_t + Y_t^* + Y_t^* y_t - C_t^* - C_t^* c_t$$

which simplifies to

$$\frac{K_{t+1}^*}{K_t^*} k_{t+1} \approx (1 - \delta)k_t + \frac{Y_t^*}{K_t^*} y_t - \frac{C_t^*}{K_t^*} c_t.$$

3. The rental rate

$$R_t = \alpha \left(\frac{A_t L_t}{K_t} \right)^{1-\alpha} + (1-\delta)$$

rewrite as

$$R_t^* e^{r_t} = \alpha \left(\frac{A_t^* L_t^*}{K_t^*} \right)^{1-\alpha} e^{(1-\alpha)(a_t + l_t - k_t)} + (1-\delta)$$

this is approximately

$$R_t^* r_t \approx \alpha(1-\alpha) \left(\frac{A_t^* L_t^*}{K_t^*} \right)^{1-\alpha} (a_t + I_t - k_t).$$

4. The wage rate

$$W_t = (1-\alpha) A_t^{1-\alpha} \left(\frac{K_t}{L_t} \right)^\alpha$$

re-write as

$$W_t^* e^{w_t} = (1-\alpha) A_t^{*1-\alpha} e^{(1-\alpha)a_t} \left(\frac{K_t^*}{L_t^*} \right)^\alpha e^{\alpha(k_t - I_t)}$$

which simplifies to

$$w_t = (1-\alpha)a_t + \alpha(k_t - I_t).$$

5. The intra-temporal FOC

$$\frac{W_t}{C_t} = \theta(1 - L_t)^{-\gamma_l}$$

re-write and approximate:

$$\frac{W_t^* e^{W_t}}{C_t^* e^{C_t}} = \theta(1 - L^* e^{I_t})^{-\gamma_l} \Rightarrow \frac{W_t^* + W_t^* w_t}{C_t^* + C_t^* c_t} \approx \theta(1 - L^* - L^* I_t)^{-\gamma_l}$$

we do a first order taylor approximation around $c_t = w_t = I_t = 0$:

$$\frac{W_t^*}{C_t^*} + \frac{W_t^*}{C_t^*} w_t - \frac{W_t^*}{C_t^*} c_t \approx \theta(1 - L^*)^{-\gamma_l} + \theta \gamma_l (1 - L^*)^{-\gamma_l - 1} L^* I_t \frac{W_t^*}{C_t^*} (w_t - c_t) \approx \theta \gamma_l (1 - L^*)^{-\gamma_l - 1} L^* I_t$$

which simplifies to

$$w_t - c_t \approx \gamma_l \frac{L^*}{1 - L^*} I_t.$$

6. The inter-temporal FOC

$$C_t = \beta E_t \left(\frac{R_{t+1}}{C_{t+1}} \right)$$

re-write as

$$C_t^* e^{C_t} = \beta E_t \left(\frac{R_{t+1}^* e^{r_{t+1}}}{C_{t+1}^* e^{c_{t+1}}} \right)$$

which is

$$e^{c_t} = E_t \left(\frac{e^{r_{t+1}}}{e^{c_{t+1}}} \right) \Rightarrow \frac{1}{1 + c_t} \approx E_t \left(\frac{1 + r_{t+1}}{1 + c_{t+1}} \right)$$

finally, do a first order Taylor approximation

$$-c_t \approx E_t(r_{t+1} - c_{t+1}).$$

Blanchard-Kahn conditions

6.2 Summary of the RBC model

6.2.1 What basic RBC models cannot do

endogenous persistence

We were expecting persistence to come from capital accumulation, but in RBC model, it goes up for a while, yet quantitatively not enough.

Output dynamics are essentially the same as the assumed dynamics of the shocks.

amplification problem

We were expecting this to come from the labor supply. But in the RBC model, only in extreme cases is the labor supply and/or capital accumulation response strong enough to generate multiplier like responses.

This means that slower than normal, but still positive technological growth can cause output to grow slower, but not to fall

For output to fall, technology must decline, for strongly pro-cyclical labor supply, we need strongly pro-cyclical wages.

possible solutions:

composition bias, implicit contracts, indivisible labor

Money and the New Keynesian Model

7.1 About Monetary Economics

Does an unanticipated shock to money affect real side of the economy (quantities)?

- Classical view: If prices are flexible, the shock will change wages and prices proportionally, but not change the quantities.

If the gov prints money and spend it, the prices increase and nominal assets lose in value, which is negative wealth effect.

Money supply increase acts like a tax on nominal asset holdings, some call it 'inflation tax'.

- Keynesian view: Prices are rigid, so the shock will give a positive wealth effect, this assumes that new money is distributed proportionally to existing money holdings.

How should we conduct monetary policy?

- Monetarists: steady money supply growth, in line with money demand growth.
- Keynesians: use expansionary MP to stimulate the economy in a recession.
- Inflation of 70's and subsequent successful tight MP (Volcker) led to Monetarists becoming very influential.

7.2 Lucas Critique

The Lucas critique (Lucas 1976[61]) is an important philosophical point that forms the basis of much of modern macroeconomics. From Keynes until the mid-1970s, macroeconomics looked very different to what it does now.

Theoretical side:

- used variants of a textbook IS-LM (investment- saving liquidity-money) model: not take agent optimisation, dynamics, or expectations formation very seriously

Empirical side:

- used large-scale macroeconometric models, systems of simultaneous equations featuring aggregate variables – many of the larger models would feature hundreds of variables

The design of these macroeconometric models was based on fit and forecasting - regressions, essentially - with little attention paid to any underlying theory or actual economics. There was

no microfoundation, and agents' behaviour was postulated to be based on adaptive expectations, which were essentially ad-hoc.

The essential gist of Lucas' Critique is that it is fraught with hazard to try and predict the effects of a policy change based on correlations (regression coefficients) based on historical data.

7.2.1 Is econometrics useful?

The conclusion of the Lucas Critique is that we need to take economic theory seriously – correlations (or regression coefficients) estimated in the data may not be policy-invariant, and therefore may not be useful in thinking about “counterfactuals” where we think of what would happen under alternative policy regimes.

The Lucas Critique tells us that we need to take theory seriously when doing econometrics; and when we do econometrics without theory (e.g. reduced form econometrics), be honest and open about the potential misgivings.

7.2.2 Empirical evidence on the effects of money supply shocks

Problem with identification of Monetary Policy: reverse causality.

The most widely accepted econometric methodology is the VAR (Vector Autoregression) model.

Example 2.

Example with detrended money stock M and detrended real output Y :

$$\begin{aligned} Y_t &= \phi_{YY}Y_{t-1} + \phi_{YM}M_t + u_t^Y \\ M_t &= \phi_{MY}Y_t + \phi_{MM}M_{t-1} + u_t^M \end{aligned}$$

where u^M are money supply shocks and u^Y are output supply (e.g. TFP) shocks (independent from other time t variables and from each other).

Then we consider another simple example:

Example 3.

$$\begin{aligned} Y_t &= \phi_1 Y_{t-1} + \phi_2 M_t + \phi_3 M_{t-1} + u_t^Y \\ M_t &= \phi_4 Y_t + \phi_5 M_{t-1} + \phi_6 Y_{t-1} + u_t^M \end{aligned}$$

The problem is: OLS will provide inconsistent estimate of ϕ . We could do it if we have direct measures of the shocks, or an instrument (but we don't).

We rearrange the equations to get:

$$\begin{aligned} Y_t &= \frac{1}{1 - \phi_2 \phi_4} ((\phi_1 + \phi_2 \phi_6) Y_{t-1} + (\phi_3 + \phi_2 \phi_5) M_{t-1} + u_t^Y + \phi_2 u_t^M) \\ M_t &= \frac{1}{1 - \phi_2 \phi_4} ((\phi_6 + \phi_4 \phi_1) Y_{t-1} + (\phi_5 + \phi_4 \phi_3) M_{t-1} + u_t^M + \phi_4 u_t^Y) \end{aligned}$$

Now, we can run OLS estimation! This gives us consistent estimates of coefficients (under

the assumption that u are \perp to variables at $t - 1$.)

Still, we estimate 4 coefficients, and have 6 unknowns.

Residuals we get are:

$$\varepsilon_t^Y = \frac{1}{1 - \phi_2\phi_4}(u_t^Y + \phi_2u_t^M)$$

$$\varepsilon_t^M = \frac{1}{1 - \phi_2\phi_4}(u_t^M + \phi_4u_t^Y)$$

Then, we need to make additional identifying assumptions! Currently, there are two alternatives:

1. $\phi_4 = 0$, monetary policy does not react to the contemporaneous level of output.

$$\varepsilon_t^Y = \phi_2\varepsilon_t^M + \frac{1}{1 - \phi_2\phi_4}u_t^Y$$

We can estimate ϕ_2 from regression of ε_t^Y on ε_t^M .

2. $\phi_2 = 0$, output does not react to change in monetary policy. We can estimate ϕ_4 from regression of ε_t^M on ε_t^Y .

Either assumption identifies all the parameters of the VAR.

Once we have all the ϕ parameters, we know the economy's response to a money shock u_t^M or to a real shock u_t^Y .

Technical aside: Granger causality

Granger(1969[53]) introduced the concept of Granger causality based on forecasting techniques in time series econometrics. A variable X is said to Granger-cause Y if and only if lagged values of X have marginal predictive content in a forecasting equation for Y . In other words, having some information about the future of Y is not enough for Granger causality, and that past observations of X should have more information about the future of Y than the past Y observations.

Consider the simple bivariate VAR(2) model:

$$Y_{1,t} = c_1 + \phi_{11}^{(1)}Y_{1,t-1} + \phi_{12}^{(1)}Y_{2,t-1} + \phi_{11}^{(2)}Y_{1,t-2} + \phi_{12}^{(2)}Y_{2,t-2} + \varepsilon_{1,t},$$

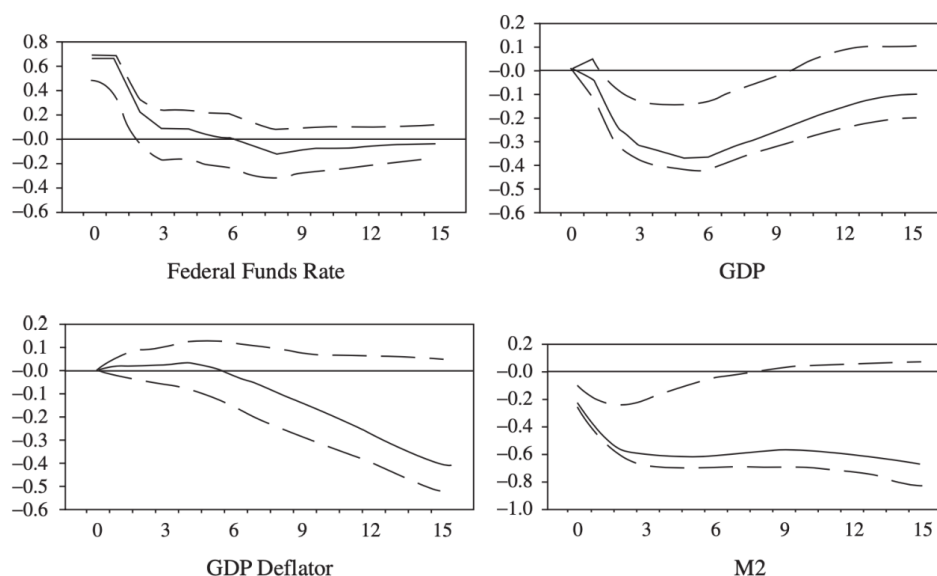
$$Y_{2,t} = c_2 + \phi_{21}^{(1)}Y_{1,t-1} + \phi_{22}^{(1)}Y_{2,t-1} + \phi_{21}^{(2)}Y_{1,t-2} + \phi_{22}^{(2)}Y_{2,t-2} + \varepsilon_{2,t}.$$

Y_2 does not Granger-cause Y_1 is analogous to saying that the coefficients of Y_2 in the Y_1 equation are all 0. In other words, $\phi_{12}^{(1)} = \phi_{12}^{(2)} = 0$. We can test for the Granger causality running from Y_2 to Y_1 by F-testing the null hypothesis that $\phi_{12}^{(1)} = \phi_{12}^{(2)} = 0$.

Christiano, Eichenbaum, Evans (1999)

Christiano et al.(1999[22]) studied the effect of a monetary shock as defined through the VAR. Their main findings were:

- Slow adjustment of output and prices (prices more sluggish than output)
- Short-run non-neutrality but long-run neutrality of money.



Source: Christiano et al. (1999)

Figure 7.1: Estimated Dynamic Response to a Monetary Policy Shock (Quarterly)

7.2.3 Criticisms of the VAR approach

- **Price Puzzle:** Observations show that after a contractionary policy shock, such as an increase in the Federal Funds Rate, the price level tends to rise, which contradicts typical economic expectations. This phenomenon, known as the 'price puzzle,' can be addressed by incorporating oil or commodity prices in the VAR system, which are critical inflation-sensitive indicators that a standard VAR might omit.
- **Lack of Forward-Looking Variables:**¹ Many VAR models do not include forward-looking variables, which are crucial as central banks often rely on forecasts of future economic conditions. Without these, VARs might misattribute the impact of policy actions on subsequent economic outcomes.
- **Limitations of VAR Analysis:**
 - VAR primarily captures the effects of monetary policy shocks – unanticipated changes in policy which are not a response to the economic conditions.
 - The majority of policy variations are endogenous, responding predictively to economic states rather than as exogenous shocks. This is often not captured well by VAR.
 - If monetary policy is seen solely as a feedback rule responding to the economy (with no exogenous shocks), VAR could inaccurately suggest that monetary policy does not influence economic outcomes.

¹Most, if not all, of what one thinks of in terms of policy and policy design represents the endogenous response of policy to the economy, and "most variation in monetary policy instruments is accounted for by responses of policy to the state of the economy, not by random disturbances to policy". (Sims and Zha 1998 [72])

- **Importance of Policy Response:** Despite VAR limitations, it's crucial to recognize that monetary policy's role is significant, especially how the economy responds to non-policy shocks which may be strongly influenced by the nature of policy adjustments.
- **Relation to Macroeconomic Theory:** These points highlight concerns similar to those raised by the Lucas Critique about the limitations of traditional econometric approaches in capturing the dynamic interplay between policy and economic outcomes.

Notably, as stated by Galí (2015[4]), the RBC model had little relevance for the analysis of macroeconomic policy. We could add monetary policy - as was done by papers such as Cooley and Hansen (1989[51]) - but, in the absence of any price or wage frictions, it would have no real effects.

To allow for a realistic model of business cycles and monetary policy, we need a framework in which prices don't simply follow the money supply, and nominal interest rates and inflation don't just move together one-for-one.

- Outcome from this program is the "New Keynesian Synthesis": combine RBC models with price rigidities.
- Money is non-neutral in the short run (Keynesian element), and neutral in the long run (classical element).

7.3 The Monetary Transmission Mechanism

Monetary policy should be a demand shock: loose MP should stimulate demand through increased consumption and investment.

We'll look into three questions and reach out to the New Keynesian model:

- What determines demand (consumption/investment)?
- What is monetary policy exactly?
- The role of nominal rigidities.

7.3.1 Households and nominal assets

Nominal assets are bonds B_t which pay interest at nominal rate i_t . Ignore the labor-leisure choice, the households maximize:

$$\begin{aligned} \max \quad & \mathbb{E}_t \sum_{i=0}^{\infty} U(C_{t+i}) \\ \text{s.t.} \quad & P_{t+i}C_{t+i} + P_{t+i}K_{t+i+1} + B_{t+i+1} = W_{t+i}L + P_{t+i}(1 + r_{t+i})K_{t+i} + (1 + i_{t+i})B_{t+i} \end{aligned}$$

where W is the nominal wage, P is the price of consumption goods, r_{t+i} is the rental rate of capital net of depreciation.

Define the Lagrangian:

$$\mathcal{L} = \mathbb{E}_t \sum_{i=0}^{\infty} [U(C_{t+i}) + \lambda_{t+i} (W_{t+i}L + P_{t+i}(1 + r_{t+i})K_{t+i} + (1 + i_{t+i})B_{t+i} - P_{t+i}C_{t+i} - P_{t+i}K_{t+i+1} - B_{t+i+1})]$$

The FOCs are:

$$\begin{aligned} U'(C_{t+i}) &= \mathbb{E}_t P_{t+i} \lambda_{t+i} \\ P_{t+i} \lambda_{t+i} &= \mathbb{E}_t \lambda_{t+i+1} P_{t+i+1} (1 + r_{t+i+1}) \\ \lambda_{t+i} &= \mathbb{E}_t \lambda_{t+i+1} (1 + r_{t+i+1}) \end{aligned}$$

Combining the first two, we get the standard Euler equation:

$$U'(C_t) = \mathbb{E}_t (U'(C_{t+1})(1 + r_{t+1}))$$

Combine the first and the third:

$$U'(C_t) = \mathbb{E}_t \left(\frac{1 + i_{t+1}}{1 + \pi_{t+1}} U'(C_{t+1}) \right)$$

where inflation $\pi = \frac{P_{t+1} - P_t}{P_t}$.

$$\begin{aligned} U'(C_t) &= \mathbb{E}_t \left(\frac{1 + i_{t+1}}{1 + \pi_{t+1}} U'(C_{t+1}) \right) \\ U'(C_t) &= \mathbb{E}_t ((1 + r_{t+1}) U'(C_{t+1})) \end{aligned}$$

- What matters for the intertemporal consumption decision is the real interest rate, not the nominal interest rate.
- The FOC are necessary conditions for having both positive B and positive K . By the two equations above, we get:

$$1 + r_{t+1} = \frac{1 + i_{t+1}}{1 + \pi_{t+1}}.$$

So, the **real return(interest rate)** of a nominal asset with **nominal interest rate** $1 + i_{t+1}$ is: $\frac{1 + i_{t+1}}{1 + \pi_{t+1}}$.

Corollary 7.3.1 (Fisher equation).

It's the relationship between asset returns that has to hold so that the household is willing to hold positive amounts of these assets in their portfolio (or: if you allow negative holdings, to clear asset markets).

This relationship between the real and nominal rates of interest is called the Fisher relationship, after Irving Fisher(1896)

When households are risk-neutral, the FOCs become:

$$\mathbb{E}_t(1 + r_{t+1}) = \mathbb{E}_t\left(\frac{1 + i_{t+1}}{1 + \pi_{t+1}}\right)$$

In a deterministic world, there's no E , which is often approxima.

If we note that $(1 + x)(1 + y) \approx 1 + x + y$, we can write the Fisher equation as:

$$r_{t+1} \approx i_{t+1} - \pi_{t+1}$$

7.3.2 Firms

The firms would like to maximize profits, with perfect competition and frictionless factor markets:

$$MPK = r$$

Again, *real interest rate matters*.

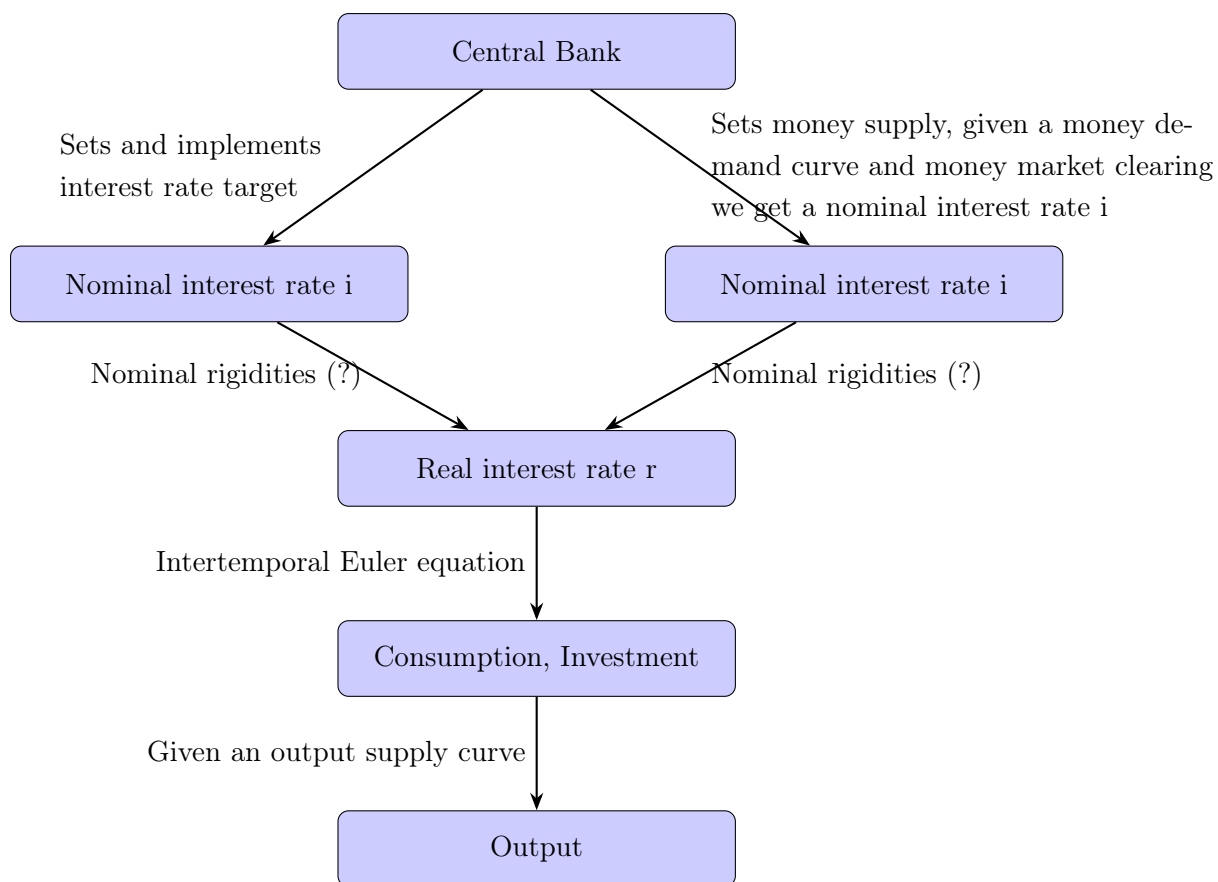
Conclusion: If monetary policy should affect demand, the real interest rate is the key price in the economy.

7.4 The tools of monetary policy

So, in order to stimulate demand, the central bank needs to change the real interest rate.

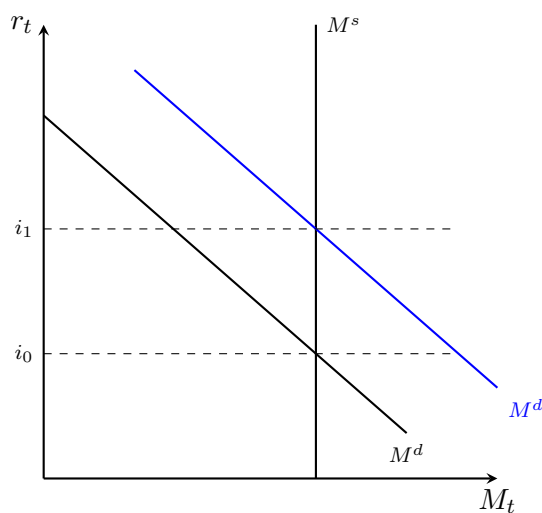
How?

- Change the money supply, with demand curve, this would implement a particular nominal interest rate.
- Change the nominal interest rate directly.

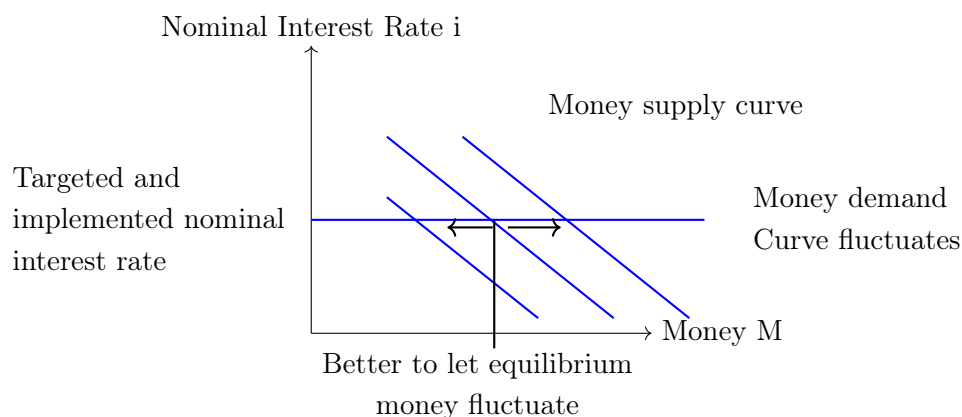


7.4.1 Money supply-demand equilibrium

Originally, the money market equilibrium is the IS-LM curve, if



If we have a interest rate target and implement particular interest rate,



7.5 A model of the interbank market

Then, we first build a model for interbank market,

- Market participants: n banks.
- Banks hold reserves at the central bank.
- Reserves are used for settling transfers between banks arising from the payment system.
- Banks are unsure of the transfers they will need to make.
- Banks can borrow reserves from one another in the interbank market, but do not know how much reserve they'll require at the point they participate in the interbank market.
- Banks can also acquire reserves through the repo (repurchase agreement) market. The central bank can be a participant in this market.

Note.

B_j = reserves of bank j at beginning-of-period.(after previous debts settles)

I_j = net borrowing in the interbank market (negative for lending) by bank j .

P_j = net bond repos(sales and repurchases) by bank j .

T_j = net payment bank j make to other banks.

R_j = balance of bank j at end-of-period.

Accounting: $R_j = B_j + I_j + P_j - T_j$.

7.5.1 Interbank market and repo market

Interbank market:

- Uncollateralized loans between banks.(potentially risky)
- Interest rate i .

Repo market:

- Sell bond now and agree to repurchase later.
- Effectively a collateralized loan, so risk-free if collateral is good.

Note (Central bank facilities).

Deposit Facility: Central bank offers an interest rate of i_d on deposit balances in banks' account at the end of time period.

Borrowing facility: Central bank charges an interest rate i_b on negative balances in banks' accounts.

- Positive spread over deposit rate: $i_b > i_d$.
- Equivalent to an offer to make loans at the end of period, with the loan credited to the bank's account.
- Collateral required – we should assume banks have enough collateral to use it.

7.5.2 Balance on account next period

Reserve balance of bank j at the end of period is:

$$B'_j = R_j - (1 + i)(I_j + P_j) + \begin{cases} i_d R_j & \text{if } R_j \geq 0 \\ i_b R_j & \text{if } R_j < 0 \end{cases}$$

Since $R_j = B_j + I_j + P_j - T_j$, we have:

$$B'_j = B_j - T_j - i(I_j + P_j) + \begin{cases} i_d(B_j + I_j + P_j - T_j) & \text{if } B_j + I_j + P_j - T_j \geq 0 \\ i_b(B_j + I_j + P_j - T_j) & \text{if } B_j + I_j + P_j - T_j < 0 \end{cases}$$

Assume banks are risk-neutral, and they aim to maximize their next period expected balance $\mathbb{E}(B'_j)$. Choice variables are: interbank transaction I_j , repo market transaction P_j .

$$\begin{aligned} \mathbb{E}(B'_j) = B_j - i(I_j + P_j) + i_d \int_{-\infty}^{B_j + I_j + P_j} (B_j + I_j + P_j - T_j) dF(T_j) \\ + i_b \int_{B_j + I_j + P_j}^{\infty} (B_j + I_j + P_j - T_j) dF(T_j) \end{aligned}$$

The FOCs are:

$$\begin{aligned} \frac{\partial \mathbb{E}(B'_j)}{\partial I_j} &= 0 \\ \frac{\partial \mathbb{E}(B'_j)}{\partial P_j} &= 0 \end{aligned}$$

Two equations lead to the same equation, so we only consider one of them.

Theorem 7.5.1 (Leibniz Rule).

Let $f(x, t)$ be a function such that both $f(x, t)$ and its partial derivative $f_x(x, t)$ are continuous in t and x in some region of the xt -plane, including $a(x) \leq t \leq b(x)$, $x_0 \leq x \leq x_1$. Also suppose that the functions $a(x)$ and $b(x)$ are both continuous and both have continuous derivatives for $x_0 \leq x \leq x_1$. Then, for $x_0 \leq x \leq x_1$,

$$\frac{d}{dx} \left(\int_{a(x)}^{b(x)} f(x, t) dt \right) = f(x, b(x)) \cdot \frac{d}{dx} b(x) - f(x, a(x)) \cdot \frac{d}{dx} a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) dt.$$

Use the Leibniz rule:

$$\begin{aligned} & \frac{\partial}{\partial I_j} \int_{-\infty}^{B_j + I_j + P_j} (B_j + I_j + P_j - T_j) dF(T_j) \\ &= \int_{-\infty}^{B_j + I_j + P_j} f(T_j) dT_j + ((B_j + I_j + P_j) - (B_j + I_j + P_j)) f(B_j + I_j + P_j) \\ &= F(B_j + I_j + P_j) \end{aligned}$$

Similarly,

$$\begin{aligned} & \frac{\partial}{\partial I_j} \int_{B_j + I_j + P_j}^{\infty} (B_j + I_j + P_j - T_j) dF(T_j) \\ &= 1 - F(B_j + I_j + P_j) \end{aligned}$$

Hence, the FOC reduces to:

$$-i + i_d F(B_j + I_j + P_j) + i_b (1 - F(B_j + I_j + P_j)) = 0$$

Equivalently,

$$(i - i_d) F(B_j + I_j + P_j) = (i_b - i) (1 - F(B_j + I_j + P_j))$$

The optimal demand for interbank and repo lending is characterized by the equation:

$$F(B_j + I_j + P_j) = \frac{i_b - i}{i_b - i_d}$$

Since we know that $0 \leq F(B_j + I_j + P_j) \leq 1$, the FOC implies that:

$$i_d \leq i \leq i_b.$$

So, the interest rate i lies in the channel $[i_d, i_b]$.

Note.

No one would lend at an interest rate lower than what you get from the central bank, and no one would borrow at an interest rate higher than what you have to pay to the central bank.

Since F is increasing, there is a negative relationship between i and $I_j + P_j$.

7.5.3 Aggregation and market clearing

$$\begin{aligned}
 B &= \sum_{j=1}^n B_j (\text{Aggregate beginning-of-period balances}) \\
 P &= \sum_{j=1}^n P_j (\text{Aggregate end-of-period balances}) \\
 T &= \sum_{j=1}^n T_j = 0 (\text{payments between banks cancel out}) \\
 R &= \sum_{j=1}^n R_j
 \end{aligned}$$

Since $R_j = B_j + I_j + P_j - T_j$, we have $R = B + P$ in the aggregate.

Since $B_j + I_j + P_j$ is the same for all banks: $B_j + I_j + P_j = \frac{R}{n}$.

7.5.4 Equilibrium interbank interest rate

$$\begin{aligned}
 F\left(\frac{R}{n}\right) &= \frac{i_b - i}{i_b - i_d} \\
 \frac{R}{n} &= F^{-1}\left(\frac{i_b - i}{i_b - i_d}\right) \\
 i &= i_d + \frac{i_b - i_d}{n} \int_{-\infty}^{F^{-1}\left(\frac{i_b - i}{i_b - i_d}\right)} x dF(x)
 \end{aligned}$$

The central bank directly controls:

- i_d and i_b .
- Open market operations P .

Indirectly determinem

- Interbank and repo rate i .
- Total end-of-period reserves $R = B + P$.

7.5.5 Interbank market with channel system

- **Demand:** Equation $F\left(\frac{R}{n}\right) = \frac{i_b - i}{i_b - i_d}$, giving a negative relationship between i and R .
- **Supply:** Central bank sets $R = B + P$ by open market operations.

If we hold all other policy instruments constant,

- An increase in total reserves R will shift supply curve to the right, decreasing i .

- An increase in i_d will shift demand curve upward, increasing i .
- An increase in i_b will shift demand curve upward, increasing i .

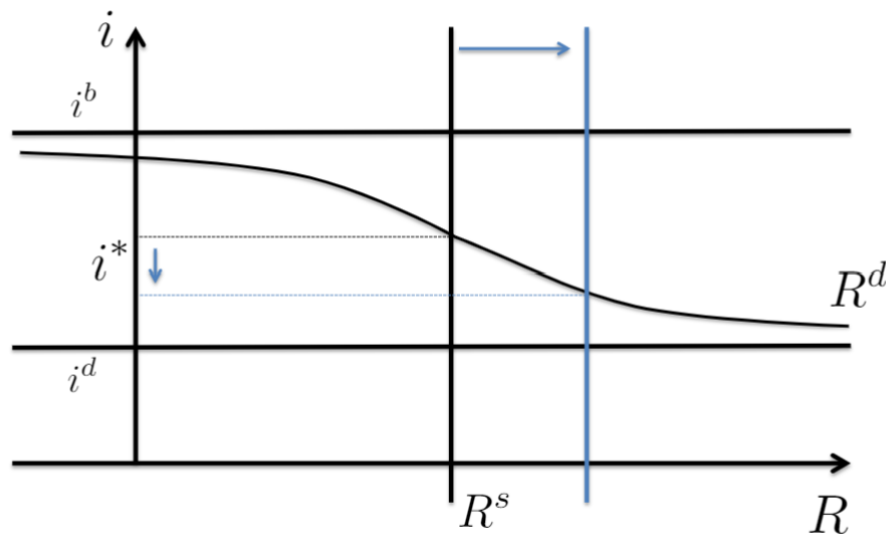


Figure 7.2: Increasing Open Market operations

Then, if the central bank would like to increase i by x , while both i_b and i_d increase by x , no new open market operations are needed.

Note.

A narrow channel has advantages:

- More precise control of i , less fluctuations.
- Less need for open market operations.

There are also disadvantages, e.g. $i_b = i_d$:

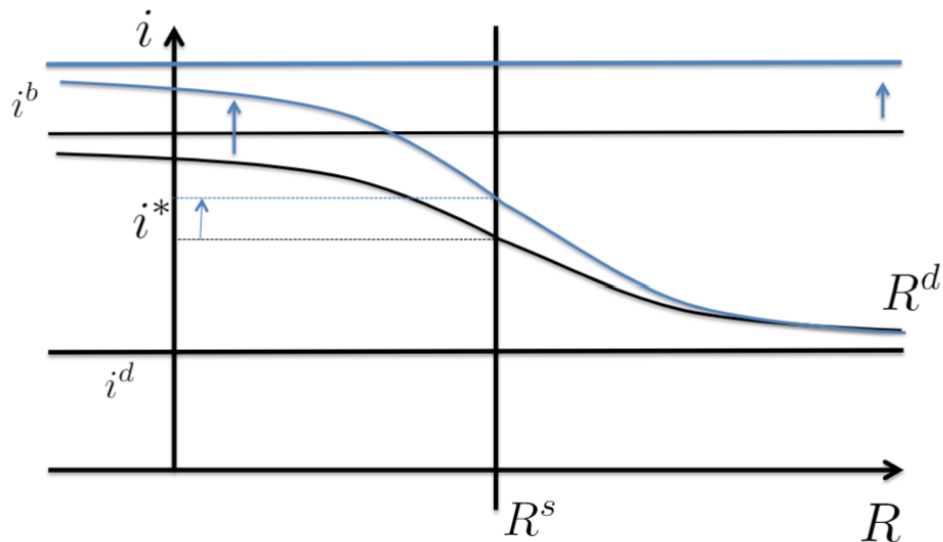
- Trading in interbank market dries up.
- CB becomes the intermediary of all borrowing and lending between banks, hence it incurs the cost of monitoring credit-worthiness of banks. **It's not the central bank's job.**

7.5.6 Interest rate maturity

Question.

How does this affect medium- and long-term interest rates?

How are interest rates of different maturities related?

Figure 7.3: Increasing i_b or i_d (similar)**Example 4.**

Imagine an investor invest for 2 years:

- Buy a 2-year bond, keep until it matures:

$$\text{return} = 1 + i_t^{(2y)}$$

- Buy a 1-year bond, then buy another 1-year bond:

$$\text{return} = (1 + i_t^{(1y)})(1 + i_{t+1}^{(1y)})$$

Expectations Hypothesis: Expectations of these two returns should be the same:

$$i_t^{(2y)} = \mathbb{E}_t(1 + i_t^{(1y)})(1 + i_{t+1}^{(1y)}).$$

This should hold exactly if agents are risk-neutral. With risk aversion, one would adjust the expected return for riskiness.

But, still, the current long-term rate is determined by the current short-term rate and the expected future short-term rates.

Note.

CB can use announcements of future policies to affect long-term rates.

Problem Set 5

Question 1

Consider the model of the interbank market presented in the lectures. i is the interbank interest rate, i_d is the interest rate offered by the central bank on positive balances on banks' reserve accounts (deposit rate), i_b is the interest rate charged by the central bank on negative reserve balances (borrowing rate). Banks face uncertainty about the payments they must make after the interbank market has closed. The probability that a bank's required payment is less than T is denoted by $F(T)$. Optimization by banks implies the following condition:

$$(i_b - i)(1 - F(R/n)) = (i - i_d)F(R/n)$$

or equivalently:

$$F(R/n) = \frac{i_b - i}{i_b - i_d}$$

where R is the total quantity of reserves and n is the number of banks.

Problem (1). Provide an intuitive explanation of the optimality condition for banks.

Solution.

Problem (2). Show how the equilibrium interbank interest rate is determined using a diagram.

Solution. The conditions gives that:

$$F(R/n) = \frac{i_b - i}{i_b - i_d} \in [0, 1] \Rightarrow i_d \leq i \leq i_b.$$

If the inter-bank rate

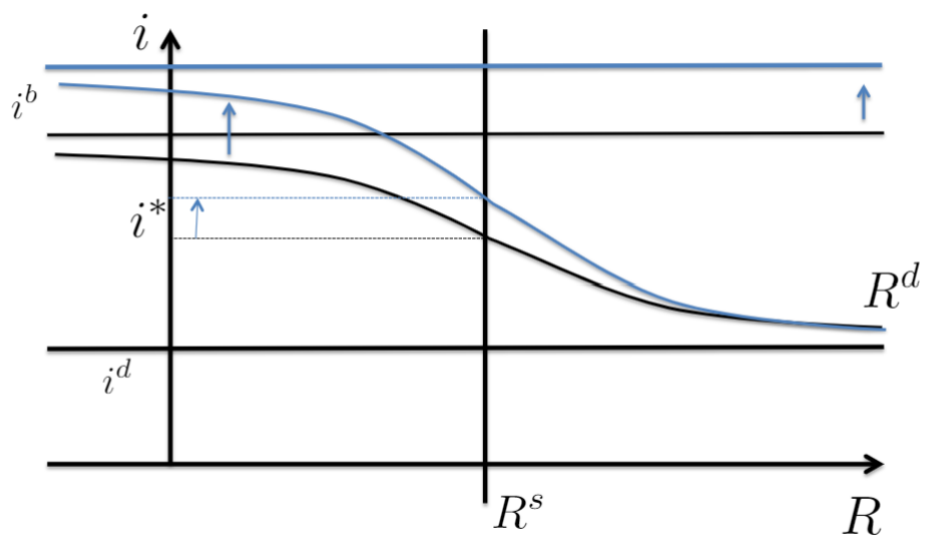
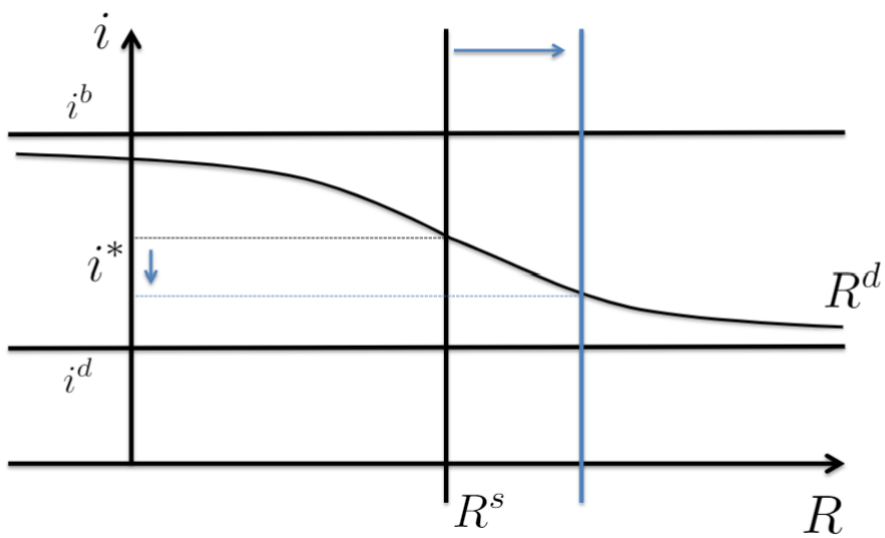
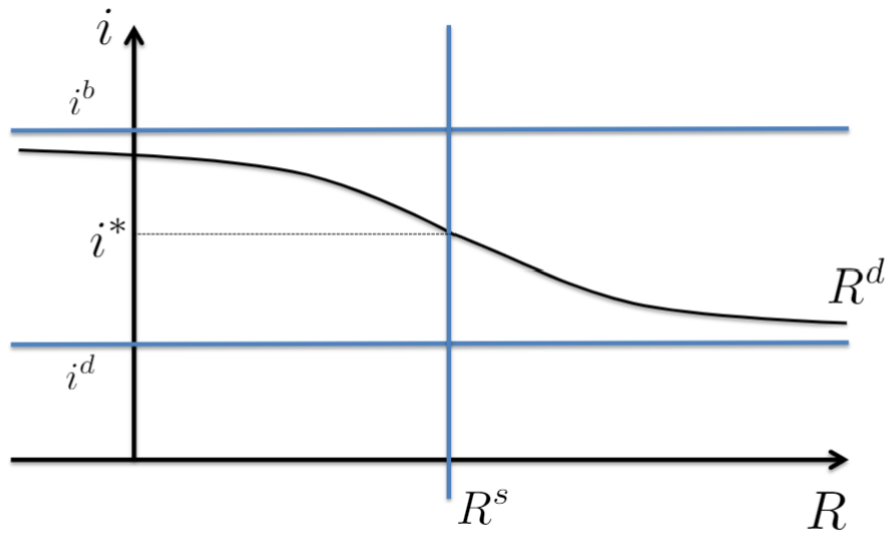
Problem (3). Suppose the central bank would like to raise the market interest rate. Explain how this can be done through:

- (a) Open market operations (only R is adjusted);
- (b) Adjusting standing facility terms (only i_b and i_d are adjusted)

Solution. Originally, the graph changes as the following graph:

Let's suppose the bank increase rate by x , say $i' = i + x$.

$$F(R/n) = \frac{(i_b + x) - (i + x)}{(i_b + x) - (i_d + x)} = \frac{i'_b - i'}{i'_b - i'_d}$$



Question 2

You are hired by the International Monetary Fund, and you're being assigned on a mission to provide technical assistance to Bank al-Maghrib, the central bank of Morocco. You're being asked to estimate the impact of monetary policy on unemployment. Knowing that your boss is not fond of DSGE models, but also distrusts the results from Christiano-Eichenbaum-Evans (because they were obtained with data from high-income countries), you decide to estimate a Vector Autoregression model (with interest rates and unemployment as the endogenous variables) yourself.

Problem (1). Write down the system of equations for a first-order Vector Autoregression where the nominal interest rate and unemployment depend on each other and a lagged variable for each. Both variables are also determined by a shock, η_t^i and η_t^u , which are orthogonal to past values of i and u .

Solution.

$$\begin{aligned} i_t &= \phi_1 i_{t-1} + \phi_2 u_t + \phi_3 u_{t-1} + \eta_t^i \\ u_t &= \phi_4 i_t + \phi_5 u_{t-1} + \phi_6 i_{t-1} + \eta_t^u \\ \text{cov}(x_{i_t}, \varepsilon_{i_t}) &= 0 \end{aligned}$$

$$\Rightarrow i_t = \phi_1 i_t + \phi_2 (\phi_4 i_{t-1} + \phi_5 u_{t-1} + \phi_6 i_{t-1} + \eta_t^u) + \phi_3 u_{t-1} + \eta_t^i$$

Solve the equation, we have:

$$\begin{aligned} i_t &= \frac{1}{1 - \phi_2 \phi_4} [(\phi_1 + \phi_2 \phi_6) i_{t-1} + (\phi_3 + \phi_2 \phi_5) u_{t-1} + \phi_2 \eta_t^u + \eta_t^i] \\ u_t &= \frac{1}{1 - \phi_2 \phi_4} [(\phi_1 \phi_4 + \phi_6) i_{t-1} + (\phi_3 \phi_4 + \phi_5) u_{t-1} + \phi_4 \eta_t^i + \eta_t^u] \\ \varepsilon_{i_t} &= \frac{1}{1 - \phi_2 \phi_4} [\phi_2 \eta_t^u + \eta_t^i] \\ \varepsilon_{u_t} &= \frac{1}{1 - \phi_2 \phi_4} [\phi_4 \eta_t^i + \eta_t^u] \end{aligned}$$

If we guess $\phi_4 = 0$, we have:

$$\begin{aligned} \varepsilon_t^i &= \eta_t^i + \phi_2 \eta_t^u \\ \varepsilon_t^u &= \eta_t^u \\ \Rightarrow \varepsilon_t^i &= \phi_2 \varepsilon_t^u + \eta_t^i \end{aligned}$$

New Keynesian Model

The NK model takes an RBC model as its backbone and adds some nominal rigidities - here, we will add sticky prices - which allows shocks to affect the real economy.

But price stickiness only makes sense when firms can set prices; hence, need some degree of market power. Special case of Rotemberg & Woodford.

Nominal rigidities: Calvo pricing: firm-specific shock that tells you whether you're allowed to reset your price. ¹

9.1 Households and Consumption

For representative household, they maximize the utility:

$$\begin{aligned} \max \quad & \mathbb{E}_0 \sum_{i=0}^{\infty} \beta^i [u(c_{t+i}) - \nu(l_{t+i})] \\ \text{s.t.} \quad & p_t c_t + b_t = (1 + i_{t-1})b_{t-1} + p_t w_t l_t + p_t d_t \end{aligned}$$

where

c_t is quantity of baskets of goods consumed, p_t is the price of each basket in terms of money,

d_t is the real wage, d_t are dividends (firms' profits) received from owning firms,

i_t is the nominal interest rate,

b_t is the quantity of nominal bonds held by the household.

The intertemporal optimality condition is as follows:

$$u'(c_t) = \beta \mathbb{E}_t (1 + r_t) u'(c_{t+1})$$

where r_t is the real interest rate given by: $1 + r_t = \frac{1+i_t}{1+\pi_{t+1}}$ and $\pi_{t+1} = \frac{p_{t+1}-p_t}{p_t}$ is the inflation rate between t and $t+1$.

Labor supply is given by:

$$MRS_{l,c} = \frac{\nu'(l_t)}{u'(c_t)} = w_t$$

¹Taylor's (1980) overlapping-contracts model adopts such a view of nominal rigidities and assumes that each price is fixed for some number n periods and therefore each period $1/n$ of the prices in the economy is updated. This assumption requires the modeler to keep track of n different prices. A simpler approach is to assume that a fraction $1 - \theta \in [0, 1]$ of the prices in the economy is updated each period, but unlike Taylor's model, the prices that are updated each period are randomly drawn from the existing prices. This assumption, which was introduced by Calvo (1983), leads to a much more tractable model because it is no longer necessary to keep track of the previously set prices as we will explain below. We'll discuss this later.

Introduce the basket of imperfectly substitutable goods through a consumption aggregator, a description of consumer preferences over the whole set of goods (indexed by i). Formula linking consumption $c(i)$ of each good $i \in [0, 1]$ to an 'amount' of total baskets c enjoyed by the household.

The most common aggregator used is the CES (constant elasticity of substitution; also called Dixit-Stiglitz):

$$c = \left(\int c(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

where ε is the elasticity of substitution between goods.

9.1.1 Expenditure minimization

Suppose that the household faces prices $p(i)$. The expenditure minimization problem is:

$$\begin{aligned} \min \quad & \int p(i)c(i)di \\ \text{s.t.} \quad & \left(\int c(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} = c \end{aligned}$$

Define the Lagrangian:

$$\mathcal{L} = \int p(i)c(i)di + \lambda \left(c - \left(\int c(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} \right)$$

The first order conditions are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c(i)} &= p(i) - \lambda c(i)^{-\frac{1}{\varepsilon}} \left(\int c(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}-1} = 0 \\ \Rightarrow p(i) &= \lambda c(i)^{-\frac{1}{\varepsilon}} c^{\frac{1}{\varepsilon}} \end{aligned}$$

From the first order conditions we get:

$$c(i) = c \left(\frac{p(i)}{\lambda} \right)^{-\varepsilon}$$

Substitute this into the FOC to get:

$$\begin{aligned} c &= \left(\int \left(\frac{p(i)}{\lambda} \right)^{1-\varepsilon} c^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} \\ \lambda &= \left(\int p(i)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}} \end{aligned}$$

Interpret λ as the shadow price: 'If you tighten the constraint by one unit, the objective function increases by λ units.' λ is the price of one basket, $p_t = \lambda$.

Hence, the demand for good i is:

$$c(i) = c \left(\frac{p(i)}{p} \right)^{-\varepsilon}$$

where

$$p = \left(\int p(i)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}}$$

Interpretation: demand depends on the relative price $p(i)/p$. Price elasticity of demand is now constant ε .

9.1.2 Production technology and market clearing

A typical intermediate firm produces output according a constant returns to scale technology in labour, with a common productivity shock, a_t : Output $y(i)$ is given by production function:

$$y_t(i) = a_t h_t(i)$$

where

a_t is the total factor productivity,

$h_t(i)$ is the hours of labor used in production of good i .

Market Clearing

There are three markets at each date: *the labor market, the goods market, and the bond market.*

Labor market clearing requires that the total labor supplied by households equals the total labor used in production. Interpret l_t as average labor supplied per good, the labor market clearing:

$$l_t = \int h_t(i) di$$

Goods market clearing assumes that there's no investment of gov spending, and as consumption is the only use for final goods, goods market clearing requires that production equals total consumption.

$$y_t(i) = c_t(i), \quad c_t = y_t$$

Bond market clearing requires that the net demand for bonds is zero.

Demand, revenues and costs

Firm i faces demand function:

$$y_t(i) = c_t(i) = y_t \left(\frac{p(i)}{p} \right)^{-\varepsilon}, \quad p_t = \left(\int p_t(i)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}}$$

Firm i has the power to set price $p_t(i)$, but it is too small to affect the general price level p_t or aggregate demand y_t (monopolistic competition).

In a competitive market, firms can hire labor at the wage rate w_t , let z_t denote the real cost of

production per unit of output. With linear production function for $y(i)$, we have²:

$$z_t = \frac{w_t}{a_t}$$

with total revenue: $\frac{p_t(i)}{p_t} y_t(i)$, total real cost of production: $z_t y_t(i)$.

Profits

The profits(in real terms) made by firm i at time t is:

$$d_t(i) = \frac{p_t(i)}{p_t} y_t(i) - z_t y_t(i) = \left(\frac{p_t(i)}{p_t} - z_t \right) \left(\frac{p_t(i)}{p_t} \right)^{-\varepsilon} y_t$$

Let $\mu_t(i) = \frac{p_t(i)}{p_t z_t}$ be the firm i 's (gross) markup of price over marginal cost.

The expression for profits can be rewritten as:

$$d_t(i) = z_t^{1-\varepsilon} y_t (\mu_t(i)^{1-\varepsilon} - \mu_t(i)^{-\varepsilon})$$

Let's first consider a world with flexible prices. To maximize profits, firm i chooses $\mu_t(i)$ to solve:

$$\max_{\mu_t(i)} d_t(i) = z_t^{1-\varepsilon} y_t (\mu_t(i)^{1-\varepsilon} - \mu_t(i)^{-\varepsilon})$$

The first order condition is:

$$\frac{\partial d_t(i)}{\partial \mu_t(i)} = z_t^{1-\varepsilon} y_t ((1-\varepsilon) \mu_t(i)^{-\varepsilon} + \varepsilon \mu_t(i)^{-\varepsilon-1}) = 0 \Rightarrow \mu_t^* = \frac{\varepsilon}{\varepsilon - 1}$$

So, a higher price elasticity ε will cause a lower markup.

9.1.3 General Equilibrium with flexible prices

Assuming the following utility function:

$$u(c_t) = \frac{c_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}, \quad \nu(l) = l$$

²The cost minimisation problem is to minimise total cost subject to the constraint producing enough to meet demand (see Blanchard and Kiyotaki (1987) for the derivation of this problem):

$$\begin{aligned} \min_{h_t(i)} \quad & w_t h_t(i) \\ \text{s.t.} \quad & a_t h_t(i) \geq y_t(i) = \left(\frac{p_t(i)}{p_t} \right)^{-\varepsilon} y_t \end{aligned}$$

define the Lagrangian(use z_t to replace λ) and take the FOC, we'll have:

$$\frac{\partial \mathcal{L}}{\partial h_t(i)} = w_t - z_t a_t = 0$$

The labor supply condition is:

$$w_t = \frac{\nu'(l_t)}{u'(c_t)} = y_t^{\frac{1}{\sigma}}$$

Assuming the market clearing condition holds, $c_t = y_t$. The firm's profit maximization implies that

$$w_t = \frac{a_t}{\mu_t^*}.$$

Wage equals to the marginal revenue product of capital.

9.1.4 Flexible-price world: natural level of output

With flexible prices, all firms set the same markup: $\mu_t(i) = \mu_t^*$. Hence the price would also be the same: $p_t(i) = p_t^*$ and the markups become:

$$\mu_t^* = \frac{1}{z_t}$$

The natural level of output is the level of output that would be produced if all prices were flexible.

Equating labor supply and labor demand with flexible prices:

$$y_t^{\frac{1}{\sigma}} = w_t = \frac{a_t}{\mu_t^*}$$

hence the natural level of output y_t^* is:

$$y_t^* = \left(\frac{a_t}{\mu_t^*} \right)^{\sigma}$$

- The natural output is higher when productivity a_t is higher.
- The natural output is lower when the markup(market power) μ_t^* is higher.

9.1.5 Natural real interest rate

The natural real interest rate is the real interest rate consistent with the natural level of output.

This requires $c_t = y_t = y_t^*$, which implies by Euler equation:

$$u'(y_t^*) = \beta(1 + r_t^*)u'(y_{t+1}^*)$$

with r_t^* being the natural real interest rate.

Using $u'(c) = c^{-\frac{1}{\sigma}}$, we get:

$$\begin{aligned} (y_t^*)^{-\frac{1}{\sigma}} &= \beta(1 + r_t^*) (y_{t+1}^*)^{-\frac{1}{\sigma}} \\ \frac{\mu_t^*}{a_t} &= \beta(1 + r_t^*) \frac{\mu_{t+1}^*}{a_{t+1}} \end{aligned}$$

The natural interest rate is given by:

$$1 + r_t^* = (1 + \bar{r}) \frac{a_{t+1}}{a_t} \frac{\mu_t^*}{\mu_{t+1}^*}$$

where \bar{r} is the average interest rate in the steady state with $\beta = \frac{1}{1+\bar{r}}$.

9.1.6 Efficiency of output

Imperfect competition makes market allocation of output not Pareto efficient.

Perfect competition is consistent with efficiency, so we can ask what level of output is efficient by finding the equilibrium in the special case of perfect competition:

- perfect competition: perfectly elastic demand curve because of perfect substitutability (no market power): $\varepsilon \rightarrow \infty$
- zero markup: $\mu_t^* \rightarrow 1$.

Efficient output \hat{y}_t found using formula for natural output, and setting $\mu_t^* = 1$:

$$\hat{y}_t = a_t^\sigma$$

Efficient interest rate \hat{r}_t is the real interest rate consistent with output as its efficient level:

$$1 + \hat{r}_t = (1 + \bar{r}) \frac{a_{t+1}}{a_t}.$$

9.2 Nominal Rigidities

Assumption 9.2.1 (Calvo pricing, Calvo and Yun).

Each period, a random selected fraction of $1 - \phi$ of firms has an opportunity to reset its price. All other firms must continue to use the same price as before.

This is a simple idea in a complicated approach:

- firms face a fixed physical cost of adjusting prices
- need to calculate the points in time when gains from adjusting prices exceed the costs
- hard to do in a dynamic model

9.2.1 Setting a new price

Suppose firm i set the new price s_t at time t . In each subsequent period, probability ϕ that the price won't be changed. Hence, the probability that the price is still used at time $T > t$ is ϕ^{T-t} .

The expected present value of profits from setting price s_t at time t is:

$$d_t(i) + \phi \frac{d_{t+1}(i)}{1 + r_t} + \phi^2 \frac{d_{t+2}(i)}{(1 + r_t)(1 + r_{t+1})} + \dots$$

where dividends(=profits) d_t, d_{t+1}, \dots are calculated assuming the price remains 'sticky' at s_t : $p_t = p_{t+1} = \dots = s_t$.

From the firms' profits equation, we have:

$$d_t(i) = \left(\frac{p_t(i)}{p_t} - z_t \right) \left(\frac{p_t(i)}{p_t} \right)^{-\varepsilon} y_t$$

we get the objective function to maximize:

$$\max \quad \mathbb{E}_t \sum_{i=0}^{\infty} \left(\prod_{j=1}^i \frac{1}{1+r_{t+j-1}} \right) \phi^i \left(\frac{p_t(i)}{p_{t+i}} - z_{t+i} \right) \left(\frac{p_t(i)}{p_{t+i}} \right)^{-\varepsilon} y_{t+i}$$

the first order condition with respect to ε is:

$$\mathbb{E}_t \sum_{i=0}^{\infty} \left(\prod_{j=1}^i \frac{1}{1+r_{t+j-1}} \right) \phi^i \left((1-\varepsilon) \frac{p_t(i)}{p_{t+i}} + \varepsilon z_{t+i} \right) \frac{1}{p_t(i)} \left(\frac{p_t(i)}{p_{t+i}} \right)^{-\varepsilon} y_{t+i} = 0$$

plug in the Euler equation to get rid of the interest rates:

$$\mathbb{E}_t \sum_{i=0}^{\infty} y_{t+i}^{1+\frac{1}{\sigma}} \beta^i \phi^i \left((1-\varepsilon) \frac{p_t(i)}{p_{t+i}} + \varepsilon z_{t+i} \right) \frac{1}{p_t(i)} \left(\frac{p_t(i)}{p_{t+i}} \right)^{-\varepsilon} = 0$$

The optimal reset price $p_t(i)$ doesn't depend on the current price level p_t , denote by s_t for simplicity. We have:

$$\mathbb{E}_t \sum_{i=0}^{\infty} y_{t+i}^{1+\frac{1}{\sigma}} (\beta\phi)^i \left(\frac{s_t}{p_{t+i}} \right)^{1-\varepsilon} = \frac{\varepsilon}{1-\varepsilon} \mathbb{E}_t \sum_{i=0}^{\infty} y_{t+i}^{1+\frac{1}{\sigma}} (\beta\phi)^i z_{t+i} \left(\frac{s_t}{p_{t+i}} \right)^{-\varepsilon}$$

then log-linearize the above equation to get:

$$\tilde{s}_t = (1-\beta\phi) \mathbb{E}_t \left(\sum_{i=0}^{\infty} (\beta\phi)^i (\tilde{p}_{t+i} + \tilde{z}_{t+i}) \right)$$

Note.

This indicates that firms set price in accordance with current and expected future price level and cost deviations.

Write the equation for time $t+1$ and subtract from this equation to get:

$$\tilde{s}_t - \beta\phi \mathbb{E}_t \tilde{s}_{t+1} = (1-\beta\phi)(\tilde{p}_t + \tilde{z}_t)$$

9.2.2 Overall price level

Recall the overall price level:

$$p_t = \left(\int p_t(i)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}}$$

The log-linearized version³ of the above equation is:

$$\tilde{p}_t = \int \tilde{p}_t(i) di$$

How old are the prices of the firms?

- $1 - \phi$ reset prices at time t ,
- $\phi(1 - \phi)$ reset prices at time $t - 1$,
- $\phi^2(1 - \phi)$ reset prices at time $t - 2$, etc.

Hence,

$$\tilde{p}_t = (1 - \phi)\tilde{s}_t + \phi(1 - \phi)\tilde{s}_{t-1} + \phi^2(1 - \phi)\tilde{s}_{t-2} + \dots$$

Write for $t - 1$ and subtract, we get:

$$\tilde{p}_t - \phi\tilde{p}_{t-1} = (1 - \phi)\tilde{s}_t$$

Note.

The price level today is a weighted average of past price level and the reset prices.

Log-linearize the inflation rate:

$$\tilde{\pi}_t = \tilde{p}_t - \tilde{p}_{t-1}$$

plug in the above equation, we get:

$$\phi\tilde{\pi}_t = (1 - \phi)(\tilde{s}_t - \tilde{p}_t).$$

Taking the firm's FOC,

$$\tilde{s}_t = (1 - \beta\phi)(\tilde{p}_t + \tilde{z}_t) + \beta\phi\mathbb{E}_t\tilde{s}_{t+1}$$

subtract \tilde{p}_t and times $1 - \phi$, we get:

$$(1 - \phi)(\tilde{s}_t - \tilde{p}_t) = (1 - \beta\phi)(1 - \phi)\tilde{z}_t + \beta\phi(1 - \phi)(\mathbb{E}_t\tilde{s}_{t+1} - \tilde{p}_t)$$

as we assumed prices are sticky, $\tilde{p}_t = \tilde{p}_{t+1}$, we have:

$$\phi\tilde{\pi}_t = \beta\phi\mathbb{E}_t\tilde{\pi}_{t+1} + (1 - \beta\phi)(1 - \phi)\tilde{z}_t$$

so that

$$\tilde{\pi}_t = \beta\mathbb{E}_t\tilde{\pi}_{t+1} + \underbrace{\frac{(1 - \beta\phi)(1 - \phi)}{\phi}}_{\equiv \gamma}\tilde{z}_t.$$

This expression is called the '*New Keynesian Phillips Curve*' (NKPC). It is 'new' because it is forward-looking unlike classic Phillips Curves, but it's a Phillips Curve in the sense that it captures a relationship between inflation and some real measure.

³We define $\tilde{x} = \frac{x_t - x^*}{x_t^*}$ for any variable x as the percent deviation under log-linearization.

9.2.3 Output gap

We define the output gap as the deviation of actual output from its efficient level:

$$x_t \equiv \frac{y_t}{\hat{y}_t}.$$

Remark.

Even in the nonstochastic steady state, $y_t < \hat{y}_t$. Why?

because output responds significantly less than the flexible price level of output to the productivity shock, we see a large negative output gap opening up following the positive productivity shock.

Recall the following:

- Real marginal cost: $z_t = \frac{w_t}{a_t}$;
- Labor supply: $w_t = y_t^{\frac{1}{\sigma}}$;
- Efficient output: $\hat{y}_t = a_t^\sigma$;

We can get:

$$z_t = \left(\frac{y_t}{\hat{y}_t} \right)^{\frac{1}{\sigma}} = x_t^{\frac{1}{\sigma}}$$

$$\Rightarrow \tilde{z}_t = \frac{1}{\sigma} \tilde{x}_t$$

We can re-write the NKPC(9.2.2) in terms of the output gap, we get:

$$\tilde{\pi}_t = \beta \mathbb{E}_t \tilde{\pi}_{t+1} + \underbrace{\frac{\gamma}{\sigma}}_{\equiv \kappa} \tilde{x}_t.$$

In our derivation we assumed that ε is constant.

9.2.4 The New Keynesian model with Rotemberg (1982) pricing

If instead we make ε stochastic and allow shocks to it (as done by Rotemberg 1982), would have shocks to markups even in a frictionless world:

$$\mu_t^* = \frac{\varepsilon_t}{\varepsilon_t - 1}.$$

Then, we use $\tilde{z}_t + \tilde{\mu}_t^*$ to replace \tilde{z}_t , and we get:

$$\tilde{\pi}_t = \beta \mathbb{E}_t \tilde{\pi}_{t+1} + \underbrace{\kappa \tilde{x}_t + \gamma \tilde{\mu}_t^*}_{\equiv e_t}. \quad (9.1)$$

9.3 The New Keynesian Phillips curve

The New Keynesian Phillips curve is a forward-looking relationship between inflation and the output gap. Inflation is forward-looking and increasing in the current output gap.

The slope of the Phillips curve, κ , is larger if prices are more flexible (lower ϕ) or if marginal cost is more sensitive to the level of production.

- Expected future inflation $\beta \mathbb{E}_t \tilde{\pi}_{t+1}$: firms try to anticipate future inflation and increase current price in line with expected future inflation;
- Current output gap $\kappa \tilde{x}_t$: higher real marginal cost means higher prices, $\phi = 0$ means $\kappa \rightarrow \infty$, gives a vertical NKPC, $\phi = 1$ means $\kappa = 0$, gives a horizontal NKPC;
- 'Cost-push shock' e_t : shock to magnitude of markups over real marginal cost.

9.3.1 The IS equation

The NKPC explains inflation in terms of the output gap, but how do we find the output gap?

Take consumption Euler equation with market clearing condition ($c_t = y_t$):

$$\begin{aligned} y_t^{-\frac{1}{\sigma}} &= \beta \mathbb{E}_t \left(y_{t+1}^{-\frac{1}{\sigma}} (1 + r_t) \right) \\ \Rightarrow y_t^{-\frac{1}{\sigma}} &= \mathbb{E}_t \left(y_{t+1}^{-\frac{1}{\sigma}} \frac{(1 + r_t)}{1 + \bar{r}} \right) \end{aligned}$$

where $\beta = \frac{1}{1 + \bar{r}}$, log-linearize it, we get:

$$\begin{aligned} -\frac{1}{\sigma} \log y_t &= \mathbb{E}_t \left(-\frac{1}{\sigma} \log y_{t+1} + \log(1 + r_t) - \log(1 + \bar{r}) \right) \\ \Rightarrow \tilde{y}_t &= \mathbb{E}_t (\tilde{y}_{t+1} - \sigma \tilde{r}_t) \end{aligned}$$

Plug in approximate Fisher equation $\tilde{r}_t = \tilde{i}_t - \tilde{\pi}_{t+1}$ to get:

$$\tilde{y}_t = \mathbb{E}_t (\tilde{y}_{t+1} - \sigma(\tilde{i}_t - \tilde{\pi}_{t+1}))$$

Similarly, for the efficient level of output:

$$\tilde{y}_t = \mathbb{E}_t (\tilde{y}_{t+1} - \sigma \tilde{r}_t).$$

Use the definition of the output gap: $\tilde{x}_t = \tilde{y}_t - \tilde{y}_t^*$, we have:

$$\tilde{x}_t = \mathbb{E}_t (\tilde{x}_{t+1} - \sigma(\tilde{i}_t - \tilde{\pi}_{t+1})) + \underbrace{\sigma \tilde{r}_t}_{\equiv v_t}$$

which is the *New Keynesian IS equation*, v_t is a shock to the efficient real interest rate.

9.3.2 The New Keynesian model

We can write the approximated dynamics of the model as:

$$\text{(NKPC)} \quad \tilde{\pi}_t = \beta \mathbb{E}_t \tilde{\pi}_{t+1} + \kappa \tilde{x}_t + e_t$$

$$\text{(NKIS)} \quad \tilde{x}_t = \mathbb{E}_t (\tilde{x}_{t+1} - \sigma(\tilde{i}_t - \tilde{\pi}_{t+1})) + v_t$$

$$\text{(Taylor rule)} \quad \tilde{i}_t = \alpha_1 \tilde{\pi}_t + \alpha_2 \tilde{x}_t + u_t$$

Sometimes researches specify that monetary policy (through the interest rate) follows the Taylor rule, i.e. interest rates respond to inflation and output gap deviations, subject to a shock u_t , which generally delivers a pretty good description of how MP is done.

9.3.3 Problem of the New Keynesian model

- Calvo pricing does not really make sense;
- Ideally price setting should be coming from firm's optimal responses to policy
- Large literature (even before NK model) where price rigidities are coming from adjustment costs to prices (menu costs)
- Alternative approach: imperfect information
- A large number of ad-hoc economic mechanisms designed mainly to fit persistence properties of the data rather than because economists have a strong belief in these particular stories;
- A large amount of unexplained shocks which are often highly persistent;
- Very limited modelling of policy tools or details of national accounts.

Problem Set 6

Question 1: The CES consumption aggregator

Consider the CES consumption aggregator $c = \left(\int c(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$.

Problem (1). Find the demand functions that minimize the expenditure $\int p(i)c(i)di$ needed to obtain a particular level of consumption c of the basket of goods.

Solution. The expenditure minimization problem is:

$$\begin{aligned} \min_{c(i)} \quad & \int p(i)c(i)di \\ \text{s.t.} \quad & \left(\int c(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} = c. \end{aligned}$$

The Lagrangian is:

$$\mathcal{L} = \int p(i)c(i)di + \lambda \left(c - \left(\int c(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} \right).$$

The first-order condition is:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c(i)} &= p(i) - \lambda c(i)^{-\frac{1}{\varepsilon}} \left(\int c(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}-1} = 0 \\ \Rightarrow p(i) &= \lambda c(i)^{-\frac{1}{\varepsilon}} c^{\frac{1}{\varepsilon}} \end{aligned}$$

From the first order conditions we get:

$$c(i) = c \left(\frac{p(i)}{\lambda} \right)^{-\varepsilon}$$

Substitute this into the FOC to get:

$$\begin{aligned} c &= \left(\int \left(\frac{p(i)}{\lambda} \right)^{1-\varepsilon} c^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} \\ &= c \left(\frac{1}{\lambda} \right)^{-\varepsilon} \left(\int p(i)^{1-\varepsilon} di \right)^{\frac{\varepsilon}{\varepsilon-1}} \\ \lambda &= \left(\int p(i)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}} \end{aligned}$$

Interpret λ as the shadow price: "If you tighten the constraint by one unit, the objective function increases by λ units." λ is the price of one basket, $p_t = \lambda$.

Hence, the demand for good i is:

$$c(i) = c \left(\frac{p(i)}{p} \right)^{-\varepsilon}$$

where

$$p = \left(\int p(i)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}}$$

Interpretation: demand depends on the relative price $p(i)/p$.

Problem (2). Show that the minimum expenditure required is $\int p(i)c(i)di = pc$ where $p = \left(\int p(i)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}}$ is the price index.

Solution. Plug in the demand function into the expenditure function:

$$\begin{aligned} \int p(i)c(i)di &= \int p(i)c \left(\frac{p(i)}{p} \right)^{-\varepsilon} di \\ &= c \int p(i) \left(\frac{p(i)}{p} \right)^{-\varepsilon} di \\ &= p^\varepsilon c \int p(i)^{1-\varepsilon} di \\ &= p^\varepsilon c \left[\left(\int p(i)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}} \right]^{1-\varepsilon} \\ &= p^\varepsilon c p^{1-\varepsilon} \\ &= pc \end{aligned}$$

Question 2: Deriving the IS curve

Consider the New Keynesian model. Define the output gap x_t as

$$x_t = \frac{y_t}{\hat{y}_t}$$

where \hat{y}_t denotes the efficient level of output. Recall the Euler equation for consumption (with $u'(c) = c_t^{-1/\sigma}$ and market clearing, $y_t = c_t$)

$$y_t^{-\frac{1}{\sigma}} = \mathbb{E}_t \left(\frac{1 + r_t}{1 + \bar{r}} y_{t+1}^{-\frac{1}{\sigma}} \right)$$

where \bar{r} is the real interest rate in steady state, $\beta = 1/(1 + \bar{r})$.

Problem (1). Log-linearize the Euler equation around the steady state.

Problem (2). Noting that the approximated Euler equation should also hold in an efficient world (i.e. for the efficient level of output and the efficient real interest rate), use approximations for the Fisher equation

$$r_t = i_t - \pi_{t+1}$$

and for the definition of the output gap (1) to show that

$$\tilde{x}_t = \mathbb{E}_t \left(\tilde{x}_{t+1} - \sigma(\tilde{i}_t - \tilde{\pi}_{t+1}) \right) + \sigma \tilde{r}_t.$$

Solution. In the steady state, the Euler equation is:

$$y_t^{-\frac{1}{\sigma}} = \beta \mathbb{E}_t \left((1 + r_t) y_{t+1}^{-\frac{1}{\sigma}} \right)$$

where $\beta = \frac{1}{1+i}$. Log-linearize the Euler equation around the steady state and take 1st order approximation:

$$\begin{aligned} \ln y_t^{-\frac{1}{\sigma}} &= \ln \beta + \mathbb{E}_t \left(\ln(1 + r_t) + \ln y_{t+1}^{-\frac{1}{\sigma}} \right) \\ -\frac{1}{\sigma} \ln y^* - \frac{1}{\sigma} \frac{1}{y^*} (y_t - y^*) &= \ln \beta + \ln(1 + r^*) + \frac{1}{1 + r^*} (1 + r_t - 1 - r^*) - \frac{1}{\sigma} \ln y^* \\ -\frac{1}{\sigma} \frac{1}{y^*} (y_t - y^*) &= \frac{r_t - r^*}{1 + r^*} \\ -\frac{1}{\sigma} \tilde{y}_t &= \mathbb{E}_t \left(\tilde{r}_t - \frac{1}{\sigma} \hat{y}_{t+1} \right) \\ \tilde{y}_t &= \mathbb{E}_t (\tilde{y}_{t+1} - \sigma \tilde{r}_t) \\ x_t = \frac{y_t}{\hat{y}_t} &\stackrel{approx}{\sim} \tilde{x}_t = \tilde{y}_t - \hat{y}_t \\ \Rightarrow \hat{y}_t &= \mathbb{E}_t (\tilde{y}_{t+1} - \sigma \tilde{r}_t) \\ \tilde{y}_t - \hat{y}_t &= \mathbb{E}_t (\tilde{y}_{t+1} - \sigma \tilde{r}_t - \tilde{y}_{t+1} + \sigma \tilde{r}_t) \\ \tilde{x}_t &= \mathbb{E}_t (\tilde{x}_{t+1} - \sigma(\tilde{i}_t - \tilde{\pi}_{t+1})) + \sigma \tilde{r}_t \end{aligned}$$

where $\tilde{r}_t = \tilde{i}_t - \tilde{\pi}_{t+1}$.

Question 3: The three-equation New Keynesian model

Consider the New Keynesian model, which consists of the New Keynesian Phillips curve,

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + e_t,$$

the IS curve

$$x_t = \mathbb{E}_t (x_{t+1} - \sigma(i_t - \pi_{t+1})) + v_t$$

where $v_t = \sigma \tilde{r}_t$ is a shock to the efficient real interest rate, and a monetary policy rule of the form

$$i_t = \alpha_1 \pi_t + u_t$$

where u_t is a shock (with the usual properties: mean zero, uncorrelated with all other variables). All variables in the three equations, except the shocks, are in logdeviations from the steady state (i.e. I dropped the \sim).

Note.

$$\begin{aligned} r_t &= \gamma \mu_t^* \\ \mu_t^* &= \frac{\varepsilon}{\varepsilon - 1} \\ v_t &= \sigma \hat{r}_t \\ \hat{r}_t &= (1 + \bar{r}) \frac{a_{t+1}}{a_t} - 1 \end{aligned}$$

Problem (1). Interpret the monetary policy rule. Which sign would you expect α to have?

Solution. The monetary policy rule is a Taylor rule where the interest rate is set as a function of inflation. The sign of α is positive since the central bank would want to raise interest rates when inflation is high.

Problem (2). State what effect the following have on e_t , v_t , and u_t :

1. A temporary rise in productivity (TFP) a_t
2. A permanent rise in productivity at (i.e. the percentage increase in a_t and a_{t+1} is the same)
3. A temporary rise in the competitiveness of markets as measured by the price elasticity ε_t
4. A shift in monetary policy caused by new personnel on the monetary policy committee

Solution. 1. A temporary rise in a_t would cause a temporary decrease in \hat{r}_t , thus a decrease in v_t .

2. A permanent rise in a_t and a_{t+1} makes no change to v_t , thus the three equations stay the same, but since the TFP has increased permanently, the economy will rise to a new higher level equilibrium.

3. A temporary rise in the competitiveness of markets would cause a decrease in μ_t^* , thus a decrease in e_t .

4. A shift in monetary policy

Problem (3). Consider a temporary shock to e_t and assume that this has no effect on expectations of the future, so that $\mathbb{E}_t \pi_{t+1} = 0$ and $\mathbb{E}_t x_{t+1} = 0$. No other shock occurs at the same time, thus $v_t = 0$ and $u_t = 0$. Find expressions for π_t and x_t in terms of e_t and interpret your results.

Solution. In this case, we have:

$$\pi_t = \kappa x_t + e_t$$

$$x_t = \sigma i_t$$

$$i_t = \alpha \pi_t$$

Thus,

$$\begin{aligned} x_t &= -\sigma \alpha_1 \pi_t \\ &= \frac{-\alpha_1 \sigma}{1 + \alpha \kappa \sigma} e_t \\ \pi_t &= \frac{e_t}{1 + \kappa \alpha \sigma} \\ i_t &= \frac{\alpha_1}{1 + \kappa \sigma \alpha} e_t \end{aligned}$$

Problem (4). Suppose that the monetary policy rule is modified so that $i_t = \hat{r}_t + \alpha \pi_t + u_t$. If this interest rate rule is used then what are the effects of a temporary shock to the natural interest rate (and hence v_t) on inflation and the output gap?

Solution. We know that:

$$x_t = \mathbb{E}_t x_{t+1} - \sigma(i_t - \mathbb{E}_t \pi_{t+1}) + v_t,$$

$$\pi_t = \mathbb{E}_t \pi_{t+1} + \kappa x_t + e_t.$$

and

$$i_t = \alpha \pi_t + \hat{r}_t + u_t.$$

We have:

$$\mathbb{E}_t x_{t+1} = \mathbb{E}_t \pi_{t+1} = 0,$$

and $u_t = e_t = 0$, $v_t = \sigma \hat{r}_t$. So we have:

$$x_t = -\sigma(\alpha \pi_t + \hat{r}_t) + \sigma \hat{r}_t.$$

Problem (5). Is it possible to redesign the monetary policy rule to achieve the same effect found in part (d) but now in response to a shock to e_t ? Explain why or why not.

Optimal Monetary Policy

11.1 Optimal MP Problem

In the previous section we developed the New Keynesian DSGE model, and explored its strengths and weaknesses. While it isn't perfect, it's the best tool we have when it comes to understanding macroeconomics fluctuations, and to explain the business cycle data that we observe. More importantly, the New Keynesian model allowed us to integrate a role for policy - most notably monetary policy - into a macroeconomic DSGE model framework.

11.1.1 What should monetary policy aim for?

In this section, we will explore the implications of the New Keynesian model for monetary policy. First, let's take a look at the considerations:

- Menu costs: keep inflation at 0 to minimize costs
- Incomplete indexation of taxes: when households have to pay taxes on nominal income (labor, capital), an increase in inflation might push them into a higher tax bracket, which distorts incentives. So, monetary policy should target zero inflation.
- Zero lower bound for nominal interest rates: targeting positive inflation rates leaves more space to decrease i during a recession.
- Measured inflation may overstate actual inflation due to incomplete adjustment for quality improvements, then the policy should target positive measured inflation.
- Nominal wages do not adjust downwards: target positive average inflation to allow real wages to fall in times of low TFP.

Output/Unemployment Stabilization Think of unemployment as inversely moving with output gap. More careful analysis of unemployment would require modeling the labor market.

Trying to bring output up to the efficient level \Leftrightarrow closing the output gap

But can we push output above the steady-state (= natural) level?

Note.

Natural level $y_t^* <$ Efficient level \hat{y}_t

11.1.2 Loss Function

Choice for CB's optimal MP policy would be to maximize welfare, i.e. the *net present value* of expected utility of households, subject to the optimal responses and market conditions of the

model.

Woodford (2003) showed that the optimal policy can be expressed as a loss function:

$$L_t = \frac{1}{2} ((\tilde{\pi}_t - \pi^*)^2 + a(\tilde{x}_t - x^*)^2)$$

where notation as NKPC: everything in pct deviations from steady state, π^* and x^* are inflation and output gap targets.

Assume that CB cannot commit to a policy, i.e. chooses current period monetary policy only.

Then, optimal MP problem is:

$$\begin{aligned} \min_{i_t} L_t &= \frac{1}{2} ((\tilde{\pi}_t - \pi^*)^2 + a(\tilde{x}_t - x^*)^2) \\ \text{s.t. } \tilde{\pi}_t &= \beta E_t \tilde{\pi}_{t+1} + \kappa \tilde{x}_t + u_t \\ \tilde{x}_t &= \mathbb{E}_t(\tilde{x}_{t+1} + \sigma \tilde{\pi}_{t+1}) - \sigma \tilde{i}_t + v_t \end{aligned}$$

Interest rate it only features in IS curve, hence this problem is equivalent to picking an output gap \tilde{x}_t and omitting IS curve.

Straight forward solve for optimal $\tilde{x}_t, \tilde{\pi}_t$, we get:

$$\kappa(\tilde{\pi}_t - \pi^*) + a(\tilde{x}_t - x^*) = 0$$

with NKPC to find \tilde{x}_t and $\tilde{\pi}_t$, and then use IS to find i_t .

Conclusions

- When the economy is hit by shocks to the efficient real interest rate (such as TFP shocks), the CB should just move nominal interest rate accordingly so that the shock cancels out in the IS curve, *There's no tradeoff for CB, it can stabilize both inflation and output gap.*
- When the economy is hit by *cost-push shocks*, the CB faces a tradeoff that is given by the NKPC. Tradeoff becomes worse when inflation expectations $\mathbb{E}_t \tilde{\pi}_t$ are high or cost-push shock e_t is high (NKPC shifts up)

11.1.3 Central Banks' Preferences

Parameter a governs the importance of output gap stabilization.

- Hawkish CB: $a = 0$, CB only cares about inflation; $\text{Var}(x)$ is large, $\text{Var}(\pi) = 0$
- Dovish CB: $a = \infty$, CB only cares about output gap; $\text{Var}(x) = 0$, $\text{Var}(\pi)$ is large

If $\text{Var}(x) = \mathbb{E}X^2$ if $\mathbb{E}X = 0$.

If the CB minimizes the loss function L_t in every period and implements the solution, this is called *optimal monetary policy under discretion*.

11.1.4 Inflation bias

Suppose CB chooses $\pi^* = 0$ but $x^* > 0$ in every period. Assume no shocks:

$$\min_{i_t} L_t = \frac{1}{2} (\pi_t^2 + a(\tilde{x}_t - x^*)^2)$$

the FOC is:

$$\tilde{x}_t = x^* - \frac{\kappa}{a} \pi_t,$$

the NKPC is:

$$\tilde{\pi}_t = \beta \mathbb{E}_t \tilde{\pi}_{t+1} + \kappa \tilde{x}_t.$$

Plug in the recursive NKPC,

$$\tilde{\pi}_t = \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \kappa \tilde{x}_{t+j}$$

we get:

$$\tilde{\pi}_t = \frac{\kappa}{1 - \beta} x^* - \mathbb{E}_t \sum_{j=0}^{\infty} \beta^j \frac{\kappa^2}{a} \pi_{t+j}.$$

In this case, we need to solve the expectations of future variables.

Note that in the next period, the CB's problem will look exactly like in this period. Hence optimal solution must be the same in all periods $\pi_t = \bar{\pi}$ for all t . Because of rational expectations:

$$\tilde{\pi}_t = \mathbb{E}_t \tilde{\pi}_{t+1} = \bar{\pi}.$$

Solving the π_t above, we get:

$$\tilde{\pi}_t = \bar{\pi} = \frac{\kappa}{1 - \beta + \frac{\kappa^2}{a}} x^*,$$

hence

$$\tilde{x}_t = x^* - \frac{\kappa}{a} \frac{\kappa}{1 - \beta + \frac{\kappa^2}{a}} x^*.$$

How large are these magnitudes? For $\beta \rightarrow 1$,

$$\lim_{\beta \rightarrow 1} \tilde{x}_t = 0, \quad \lim_{\beta \rightarrow 1} \tilde{\pi}_t = \frac{a}{\kappa} x^*$$

Hence \tilde{x}_t is close to 0 but π_t is positive, meaning that we are away from the target!

11.1.5 Optimal monetary policy with commitment

Instead of re-optimizing every period, the CB could plan a MP rule (a set of $\{i_t\}_t$ contingent on shocks). This would generally lead to a preferable outcome, called “commitment,” the CB solves the entire problem at the beginning of time and commits to its policy.

Here, the objective of the central bank is not just the current objective, but the present discounted value of the flow objective functions.

Let's consider a simple scenario: $x^* = \pi^* = 0$, with positive cost-push shock $e_t > 0$, but short-

lived $\mathbb{E}_t e_{t+j} = 0$ for $j > 0$, assume we are at time 0. The problem for the central bank is

$$\begin{aligned} \min_{\{\pi_t\}, \{\tilde{x}_t\}} L &= \frac{1}{2} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t (\tilde{\pi}_t^2 + a \tilde{x}_t^2) \\ \text{s.t. } \tilde{\pi}_t &= \beta \mathbb{E}_t \tilde{\pi}_{t+1} + \kappa \tilde{x}_t + e_t \text{ for every time period} \end{aligned}$$

Define the Lagrangian:

$$\mathcal{L} = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{1}{2} (\tilde{\pi}_t^2 + a \tilde{x}_t^2) + \lambda_t (\tilde{\pi}_t - \beta \mathbb{E}_t \tilde{\pi}_{t+1} - \kappa \tilde{x}_t - e_t) \right)$$

Take FOCs(w.r.t. $\tilde{\pi}_{t+1}, \tilde{x}_t$):

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \tilde{\pi}_{t+1}} &= \beta(\tilde{\pi}_{t+1} + \lambda_{t+1}) - \beta \lambda_t = 0 \\ \frac{\partial \mathcal{L}}{\partial \tilde{x}_t} &= a \tilde{x}_t - \kappa \lambda_t = 0 \end{aligned}$$

Note that the NKPC at time -1 is not a restriction, hence FOC for $t = 0$ is:

$$\tilde{\pi}_0 + \lambda_0 = 0$$

and we can eliminate λ_t to solve for \tilde{x}_t and $\tilde{\pi}_t$:

$$\begin{aligned} \tilde{x}_t - \tilde{x}_{t-1} &= -\frac{\kappa}{a} \tilde{\pi}_t \\ \tilde{x}_0 &= -\frac{\kappa}{a} \tilde{\pi}_0 \end{aligned}$$

11.1.6 Time consistency of the policy

- The optimal monetary policy with commitment features history dependence, i.e. policy decision for period t depends on what we want to implement in period $t - 1$.
- Exception: optimal policy for period 0 does not depend on past (why? economy is only forward-looking)
- The optimal policy with commitment is time inconsistent: at time 0, what we want to implement for period 1 is not what we will want to implement for period 1 at time 1.

Remark.

While the outcomes from the commitment policy are better than those from the discretionary policy at date 0, they are worse at future dates. Under the discretionary policy, there are no deviations in output or inflation at any date $t \geq 1$ while, under the commitment policy, output and inflation are too low relative to the targets.

This brings us to a *time-consistency problem*.

At date 0, the central bank would like to announce the commitment policy, but it would like to switch to the discretionary policy at date 1. If the private sector anticipates this switch

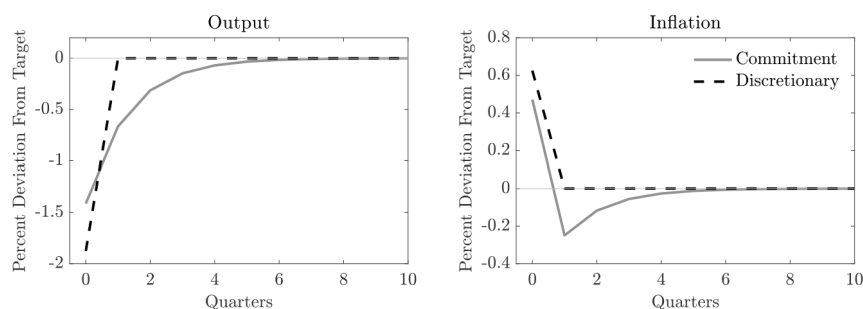


Figure 11.1: Response of output and inflation to a transitory cost push shock under commitment and discretionary policies.

in policy, then the benefits of the commitment policy at date 0 are unattainable because the central bank cannot convince the private sector that π_1 will be negative.

The central bank can only achieve the better outcomes if it is able to commit at date 0 to have tight monetary policy at future dates even though it will not want to do that when those future dates arrive.

Under the *discretionary policy*, the price-level jumps at date 0 and then remains constant thereafter.

Under the *commitment policy*, the price level rises, but then falls subsequently as inflation is negative for several periods. In fact, if you accumulate the inflation rates in the figure, you find that the long-run price level is unaffected.

This result demonstrates that long-run price stability is useful for a central bank even if it is pursuing an inflation targeting framework because it serves to stabilize inflation expectations and therefore helps stabilize inflation rates.

11.1.7 What if the CB cannot observe everything?

In reality, the CB faces uncertainties:

- Shock uncertainty: e_t is not directly observed;
- Parameter uncertainty: κ not directly observed.

Pretty straightforward to study optimal policy problem with these extensions in the linear-quadratic framework

Basic Outcomes:

- If there is uncertainty about something that enters additively (like shocks), the CB should ignore the uncertainty and act based on the expectation.
- If there is uncertainty about something that enters multiplicatively (like parameters in front of the variables), then the CB should do MP more cautiously (i.e. muted responses).

Problem Set 7

The zero lower bound: discretion and commitment

Suppose the central bank has the following loss function:

$$L_t = \frac{1}{2}x_t^2,$$

where x_t is the output gap (defined relative to the efficient level of output). The output gap is determined by the IS equation

$$x_t = E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1} - \hat{r}_t),$$

where i_t is the nominal interest rate (controlled by the central bank), π_t is the inflation rate, and \hat{r}_t is the efficient interest rate. Inflation is determined by the Phillips curve

$$\pi_t = E_t \pi_{t+1} + \kappa x_t$$

(assuming the discount factor $\beta = 1$, and ignoring cost-push shocks). Nominal interest rates are subject to a zero lower bound. Since all variables, including i_t , are given as percentage deviations from steady-state levels, and since the steady-state nominal interest rate is likely to be positive, the lower bound on i_t can be stated as $i_t \geq -b$ for some $b > 0$. (Note: i_t is the percentage deviation of the gross nominal interest rate from the steady-state level. The gross nominal interest rate is equal to one at the "zero lower bound".)

Problem (1). What factors would determine the size of b ?

Solution. As we know, $\tilde{i}_t = \frac{i_t - i^*}{i^*}$, where \tilde{i}_t is the percentage deviation of the nominal interest rate from the steady-state level, \hat{i}_t is the nominal interest rate, and i^* is the steady-state nominal interest rate. Since the steady-state nominal interest rate is likely to be positive, the lower bound on \tilde{i}_t can be stated as $\tilde{i}_t \geq -b$.

Problem (2). Now suppose at time t there is a temporary negative shock to the efficient interest rate \hat{r}_t , so that $\hat{r}_t = \hat{r} < 0$ and $\hat{r}_{t+1} = \hat{r}_{t+2} = \dots = 0$. Find the optimal interest rate i_t when the central bank acts with discretion, and the resulting value of the output gap x_t . Distinguish between the cases $\hat{r} \geq -b$ and $\hat{r} < -b$ in your answer.

Solution. There are two cases to consider: $\hat{r} \geq -b$ and $\hat{r} < -b$. If $\hat{r} \geq -b$, then the optimal interest rate i_t is given by

$$i_t = E_t \pi_{t+1} + \kappa x_t + \hat{r}.$$

Substituting this into the IS equation, we get

$$x_t = E_t x_{t+1} - \sigma(E_t \pi_{t+1} + \kappa x_t + \hat{r} - E_t \pi_{t+1} - \hat{r}) = E_t x_{t+1} - \sigma \kappa x_t.$$

Rearranging, we get

$$x_t = \frac{1}{1 + \sigma \kappa} E_t x_{t+1}.$$

Since $\hat{r}_{t+1} = \hat{r}_{t+2} = \dots = 0$, we have $x_{t+1} = 0$, so $x_t = 0$.

Problem (3). Suppose that $\hat{r} < -b$, and that the shock now lasts for two periods, hence $\hat{r}_t = \hat{r}_{t+1} = \hat{r}$ and $\hat{r}_{t+2} = \hat{r}_{t+3} = \dots = 0$. Find the level of the output gap x_t when the central bank follows the optimal policy with discretion. [Hint: work backwards, starting from period $t + 2$.] Compare your answer to part 2 (assuming \hat{r} is the same in both cases) and explain the intuition.

Solution. If $\hat{r} < -b$, the problem is:

$$\min_{i_t} L_t = \frac{1}{2} (-\sigma(i_t - r_t))^2,$$

take the first-order condition:

$$\sigma(i_t - \hat{r}_t) = 0 \Rightarrow i_t = \hat{r}_t = \hat{r}.$$

Substituting this into the IS equation, we get

$$x_t = -\sigma(i_t - \hat{r}) = -\sigma(-b - \hat{r}) = \sigma(b + \hat{r}).$$

Problem (4). Suppose again that the shock is only temporary ($\hat{r}_{t+1} = \hat{r}_{t+2} = \dots = 0$), and that $\hat{r}_t = \hat{r} < -b$. Assume the central bank is now able to commit to an interest-rate policy at time t for two periods, that is, it can choose both i_t and i_{t+1} at time t (but not influence outcomes further in the future). Find the optimal policy with commitment (minimizing the sum $L_t + L_{t+1}$, and assuming the shock is such that the zero lower bound will not bind at time $t + 1$). Compare the total loss $L_t + L_{t+1}$ under discretion and commitment, and comment on the behaviour of long-term interest rates in the two cases.

Solution. In stage $t + 2$, we have:

$$\mathbb{E}_t x_{t+2} = 0$$

$$\mathbb{E}_t \pi_{t+2} = 0$$

$$\mathbb{E}_t i_{t+2} = 0$$

We have:

$$x_{t+1} = \mathbb{E}_t x_{t+2} - \sigma(i_{t+1} - \mathbb{E}_t \pi_{t+2} - \hat{r}_{t+1}) = -\sigma(i_{t+1} - \hat{r}).$$

We aim to minimize the sum of losses $L_t + L_{t+1}$:

$$L_t + L_{t+1} = \frac{1}{2} x_t^2 + \frac{1}{2} x_{t+1}^2 = \frac{1}{2} \sigma^2 (b + \hat{r})^2 + \frac{1}{2} \sigma^2 (i_{t+1} - \hat{r})^2.$$

The first-order condition is:

$$i_{t+1} = -b$$

thus, $x_{t+1} = \sigma(b + \hat{r})$ and $\pi_{t+1} = \kappa\sigma(b + \hat{r})$. To solve:

$$\begin{aligned} \min & \frac{1}{2}x_t^2 \\ \text{s.t.} & x_t = E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1} - \hat{r}_t) \\ & \pi_{t+1} = \kappa\sigma x_{t+1} \end{aligned}$$

So, we have:

$$x_t = \sigma(b + \hat{r}) - \sigma(i_t - \kappa\sigma(b + \hat{r}) - \hat{r}) = (1 + \sigma\kappa)\sigma(b + \hat{r}) - \sigma(i_t - \hat{r}).$$

The problem is to solve:

$$\min_{i_t} \frac{1}{2} [(1 + \sigma\kappa)\sigma(b + \hat{r}) - \sigma(i_t - \hat{r})]^2.$$

The first-order condition is:

$$-\sigma [(1 + \sigma\kappa)\sigma(b + \hat{r}) - \sigma(i_t - \hat{r})] = 0$$

which implies:

$$i_t = (1 + \sigma\kappa)\sigma(b + \hat{r}) + \hat{r}.$$

The central bank could only set $i_t = -b$ in the commitment case, so that:

$$x_t = 2\sigma(b + \hat{r}) + \kappa\sigma^2(b + \hat{r}) < 0.$$

Then, we can find the total loss under discretion and commitment:

$$L_t + L_{t+1} = \frac{1}{2}\sigma^2(b + \hat{r})^2 + \frac{1}{2}\sigma^2(2\sigma(b + \hat{r}) + \kappa\sigma^2(b + \hat{r}))^2 = \frac{1}{2}\sigma^2(b + \hat{r})^2 + \frac{1}{2}\sigma^2(2\sigma(b + \hat{r}) + \kappa\sigma^2(b + \hat{r}))^2.$$

Problem (5). Describe the time-inconsistency problem* inherent in the optimal commitment found in part 4. Explain whether there is an analogy with the inflation bias problem*.

Solution. This time, we have: $\hat{r}_t = \hat{r} < -b$ and $\hat{r}_{t+1} = \hat{r}_{t+2} = \dots = 0$. We still need to solve:

$$\begin{aligned} \min_{i_t, i_{t+1}} & L_t + L_{t+1} \\ \text{s.t.} & x_t = E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1} - \hat{r}) \\ & x_{t+1} = \mathbb{E}_t x_{t+2} - \sigma(i_{t+1} - \mathbb{E}_t \pi_{t+2} - \hat{r}_{t+1}) = -\sigma i_{t+1} \\ & \pi_{t+1} = \mathbb{E}_t \pi_{t+2} = \kappa x_{t+1} \end{aligned}$$

the problem can be then rewrite as:

$$\begin{aligned} \min_{i_t, i_{t+1}} \quad & \frac{1}{2}x_t^2 + \frac{1}{2}\sigma^2 i_{t+1}^2 \\ \text{s.t.} \quad & x_t = -(1 + \sigma\kappa)\sigma i_{t+1} - \sigma(i_t - \hat{r}) \end{aligned}$$

implement the condition into the objective function:

$$\min_{i_t} \frac{1}{2} [(1 + \sigma\kappa)\sigma i_{t+1} + \sigma(i_t - \hat{r})]^2 + \frac{1}{2}\sigma^2 i_{t+1}^2.$$

the first-order condition with respect to i_t and i_{t+1} is:

$$\begin{aligned} \sigma [(1 + \sigma\kappa)\sigma i_{t+1} + \sigma(i_t - \hat{r})] &= 0 \\ [(1 + \sigma\kappa)\sigma i_{t+1} + \sigma(i_t - \hat{r})] (1 + \kappa\sigma) &+ \sigma^2 i_{t+1} = 0 \end{aligned}$$

the solution is:

$$\begin{aligned} i_t &= \hat{r} \\ i_{t+1} &= 0 \end{aligned}$$

but as $\hat{r} < -b$, we cannot reach this policy, we can only set $i_t = -b$: Then, we need to solve:

$$\begin{aligned} \min_{i_t} \quad & \frac{1}{2} [(1 + \sigma\kappa)\sigma i_{t+1} + \sigma(i_t - \hat{r})]^2 + \frac{1}{2}\sigma^2 i_{t+1}^2 \\ \text{s.t.} \quad & i_t = -b \end{aligned}$$

the FOC with respect to i_{t+1} is:

$$\sigma(1 + \kappa\sigma) [(1 + \sigma\kappa)\sigma i_{t+1} + \sigma(i_t - \hat{r})] + \sigma^2 i_{t+1} = 0$$

the solution is:

$$\begin{aligned} i_t &= -b \\ i_{t+1} &= \frac{1 + \kappa\sigma}{1 + (1 + \sigma\kappa)^2} (b + \hat{r}) < 0 \end{aligned}$$

Credit Frictions

The study of the role of financial market frictions for the dynamics of the macro-economy has a long tradition that started well before the 2008 financial crisis. Bernanke and Gertler (1989) and Kiyotaki and Moore (1997) are two well-known studies that incorporate financial frictions in general equilibrium models and became the standard references for this literature. However, before the 2008 financial crisis, the mainstream approach to the study of the macro-economy abstracted from financial frictions.

This was, in part, motivated by the view that, although markets are clearly incomplete, the importance of financial frictions for the dynamics of the macro-economy is somewhat negligible. Based on this view, it became preferable to use models with complete markets because they allow for simpler analytical characterization. These models featured many frictions such as sticky prices, sticky wages, adjustment costs in investment, variable capital utilization, matching frictions in the labor market, but not financial frictions.

Prior to the 2008 financial crisis, the main idea was that financial frictions could make the real impact of a shock bigger (amplification) or smaller (dampening). However, financial frictions are not the initial ‘cause’ of macroeconomic expansions and contractions: something else has to happen first in the nonfinancial sector. The analysis of financial shocks as a ‘source’ of macroeconomic fluctuations has received more attention after the 2008 financial crisis.

- The financial crisis can be analysed from many different angles (labor: banker’s pay and incentives, intl macro: savings glut, micro: coordination failures/bank runs, political economy).
- We are going to model one mechanism that gives a role for asset prices in the transmission of shocks.
- This mechanism gives an answer to the question: why were the real effects of the bursting of the housing bubble so big?

13.1 Credit markets and asymmetric information

Fundamental problem that borrowers have more information about (and more control over) the projects they undertake than the lenders who finance them.

WE assume that lender cannot verify the outcome of an investment project without some monitoring cost.

Known as the *costly state verification* problem.

Leads to a positive external finance premium.

13.1.1 Entrepreneurs

We assume they are *risk neutral*, and die with probability $1 - \gamma$, so that they never accumulate enough net worth to become self-financing.

They borrow money, in addition to using their net worth, to acquire capital.

They also supply some labor to the general labor market.

Idiosyncratic risk

Entrepreneurs can do a “project” in every period, which yields a total nominal payoff of

$$\omega R^k QK$$

if K units of capital are used. Q is the price of one unit of capital.

Hence, ωR^k is the gross nominal return on capital, where ω is the firm-specific component and R^k is the aggregate component (same for all firms).

ω is drawn from a continuous distribution $F(x)$, $f(x)$ and an expectation $\mathbb{E}[\omega] = 1$.

Ex ante, either the entrepreneur nor the lender observe ω before undertaking project; *ex post*, the entrepreneur knows ω , the lender would have to pay a monitoring cost, a friction μ of ex post project payoff to verify ω .

Borrowing decision and contract offered by lender

Suppose the entrepreneur has net worth N at the beginning, price of capital goods is Q .

For entrepreneurs to acquire K units of capital, they need to borrow $QK - N$ once net worth has been used.

If the entrepreneur's idiosyncratic shock is ω , the total project payoff is $\omega R^k QK$.

The financial contract we consider specifies a threshold $\bar{\omega}$ such that:

1. If $\omega \geq \bar{\omega}$, then a fixed amount $\bar{\omega} R^k QK$ is repaid.
2. If $\omega < \bar{\omega}$, then the borrower defaults. and after the lender determined ω , the borrower is seized $(1 - \mu)\omega R^k QK$.

This type of contract is known as *risky debt* and can be shown to contain a contract that is optimal from the borrower's perspective (see micro/mechanism design class).

Lender's problem

Suppose the lender is able to diversify lending across a large representative sample of entrepreneurs with net worth N .

As entrepreneurs' idiosyncratic draws are independent, the law of large numbers applies and the actual return on lending amount $QK - N$ is:

$$\left(\bar{\omega}(1 - F(\bar{\omega})) + (1 - \mu) \int_0^{\bar{\omega}} \omega f(\omega) d\omega \right) R^k QK$$

Suppose the lender's outside option for lending is to earn a risk-free interest R . Assuming perfect competition between lenders, we get that the above must equal $R(QK - N)$.

Division of gross profits

Define the lender's gross share of aggregate revenues (before monitoring costs) as

$$\Gamma(\bar{\omega}) = \int_0^{\bar{\omega}} \omega f(\omega) d\omega + \bar{\omega} \int_{\bar{\omega}}^{\infty} f(\omega) d\omega$$

and

$$G(\bar{\omega}) = \int_0^{\bar{\omega}} \omega f(\omega) d\omega.$$

Then the monitoring costs are $\mu G(\bar{\omega})$.

Lender's net share of revenues is $\Gamma(\bar{\omega}) - \mu G(\bar{\omega})$.

Borrower's net share of revenues is $1 - \Gamma(\bar{\omega})$.

13.1.2 Optimal contracting problem

Maximize borrower's payoff in the aggregate (or expectation of individual's payoff: is the same because of LLN) subject to lender wanting to participate (gets at least his outside option)

Choice of variable is the terms of loan: *default threshold* $\bar{\omega}$ and *amount lent* $B = QK - N$. For given Q and N , maximizing over K is equivalent to maximizing over B

Formally, we have the following maximization problem:

$$\begin{aligned} \max_{K, \bar{\omega}} \quad & (1 - \Gamma(\bar{\omega})) R^k QK \\ \text{s.t.} \quad & (\Gamma(\bar{\omega}) - \mu G(\bar{\omega})) R^k QK = R(QK - N) \end{aligned}$$

We define the external finance premium $s \equiv \frac{R^k}{R}$ and $k \equiv \frac{QK}{N}$, the optimal problem can be rewritten as:

$$\begin{aligned} \max_{k, \bar{\omega}} \quad & (1 - \Gamma(\bar{\omega})) sk \\ \text{s.t.} \quad & (\Gamma(\bar{\omega}) - \mu G(\bar{\omega})) sk = k - 1 \end{aligned}$$

Take the FOCs, we get:

$$\begin{aligned} \Gamma'(\bar{\omega}) &= \lambda(\Gamma'(\bar{\omega}) - \mu G'(\bar{\omega})) \\ [(1 - \Gamma(\bar{\omega})) + \lambda(\Gamma(\bar{\omega}) - \mu G(\bar{\omega}))] s &= \lambda \\ (\Gamma(\bar{\omega}) - \mu G(\bar{\omega})) sk &= k - 1 \end{aligned}$$

Under some regularity conditions, you can show that these equations imply a solution of the form

$$k = \psi(s)$$

where $\psi(s)$ is an increasing function of s .

Rewrite as $QK = \psi(s)N$.

Interpretation

The borrower can raise more capital K when:

1. The average return on project R^k increase compared to lender's outside option R (increase in s);
2. The borrower's net worth N increases: higher collateral;
3. When the price of capital Q_t decreases

The full model makes R^k stochastic, and allows for contracts contingent on R^k , results are similar, with $s = \mathbb{E} \left[\frac{R^k}{R} \right]$.

Putting this into general equilibrium

Entrepreneurs' aggregate production function is:

$$Y_t = A_t K_t^\alpha L_t^{1-\alpha}$$

(Individual production function includes idiosyncratic ω component)

Expected gross return to holding one more unit of capital is

$$\mathbb{E}[R_{t+1}^k] = \mathbb{E} \left[\frac{MPK_{t+1} + Q_{t+1}(1 - \delta)}{Q_t} \right]$$

which is the supply curve for capital.

Demand curve for capital is found by inverting the result from the optimal contract:

$$\mathbb{E}[R_{t+1}^k] = s \left(\frac{N_{t+1}}{Q_t K_{t+1}} \right) R_{t+1}$$

Note.

In "standard" RBC model: supply of capital from households consumption-saving decision, demand from firms.

Here: households lend to banks, who then lend to entrepreneurs.

Entrepreneurs' net worth is:

$$N_{t+1} = \gamma \int (\Pi_i)_t di + W_t^e$$

where Π_i is entrepreneur i 's profits, W_t^e is the entrepreneurs' labor income, γ is the probability of not dying.

For HHs, they are like in the standard RBC model, with capital (return R_{t+1}).

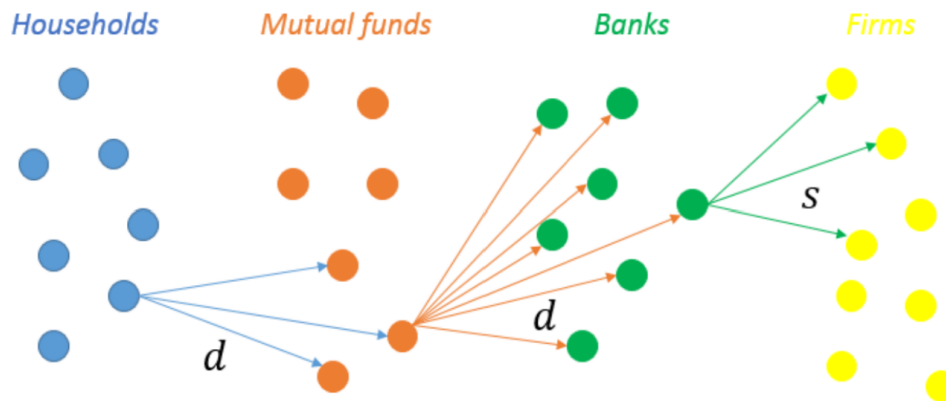


Figure 13.1: Household lending to banks

Entrepreneurs' output Y_t is sold to the retail sector firms, which are like the monopolistically competitive firms in the NK model (with Calvo prices etc) and who turn this into the households' consumption goods $c(i)$. Households as in NK model.

13.1.3 What do we get from all this?

Remark (Financial accelerator mechanism).

Consider a negative demand shock to the economy,

- reduces the quantity (or price Q_t) of output firms can sell
- reduces the MPK and Firm i 's asset value of holding capital K
- reduces the net worth of entrepreneurs N
- reduces the amount of capital entrepreneurs can borrow $QK - N$
- reduces the amount of capital entrepreneurs can use in future investments
- increases the external finance premium s
- reduces investment and aggregate demand

Further analysis

If you have a channel of how an asset price shock will be amplified, you get even more amplification:

Bank capital channel

Banks have to maintain a particular maximum leverage ratio (size of the balance sheet vs. core capital), which is Basel I, II, III.

If the value of your assets decrease, you have to write it off; capital requirements will not be met anymore.

You have to sell off assets. This, in turn, reduces asset prices (pecuniary externality).

Then the value of your assets decreases again.

Definition 13.1.1 (Margin/Haircut spiral).

In times of crisis, banks increase 'haircuts', i.e. crisis \Rightarrow value assets as a collateral less \Rightarrow tightens borrowing constraint further

Problem Set 8

Monopolistically competitive firm

Consider a monopolistically competitive firm with real unit cost of z_t per unit sold at time t (marginal cost is constant at the firm level). The firm (firm i) sells output at price $p_t(i)$ and faces the following demand function for its output $y_t(i)$:

$$y_t(i) = \left(\frac{p_t(i)}{p_t} \right)^{-\varepsilon_t} y_t$$

where p_t is the general price level, y_t is aggregate output, and ε_t is the price elasticity of demand. The firm's price $p_t(i)$ implies a (gross) markup of $\mu_t(i) = \frac{p_t(i)}{p_t z_t}$ on marginal cost. Profits (in real terms) made by the firm at time t are therefore given by

$$\text{Profits}_{i,t} = z_t^{1-\varepsilon_t} y_t (\mu_t(i)^{1-\varepsilon_t} - \mu_t(i)^{-\varepsilon_t})$$

Problem (a). Show that the profit-maximizing markup for a firm with flexible prices is $\mu_t^* = \frac{\varepsilon_t}{\varepsilon_t - 1}$.

Solution.

Take the first order derivative with respect to $\mu_t(i)$ and set it to zero, we have:

$$\begin{aligned} \frac{\partial \text{Profits}_{i,t}}{\partial \mu_t(i)} &= \frac{\partial}{\partial \mu_t(i)} (z_t^{1-\varepsilon_t} y_t (\mu_t(i)^{1-\varepsilon_t} - \mu_t(i)^{-\varepsilon_t})) = 0 \\ \Rightarrow z_t^{1-\varepsilon_t} y_t ((1 - \varepsilon_t) \mu_t(i)^{-\varepsilon_t} + \varepsilon_t \mu_t(i)^{-\varepsilon_t - 1}) &= 0 \\ \Rightarrow (1 - \varepsilon_t) \mu_t(i)^{-\varepsilon_t} + \varepsilon_t \mu_t(i)^{-\varepsilon_t - 1} &= 0 \\ \Rightarrow \mu_t^* &= \frac{\varepsilon_t}{\varepsilon_t - 1} \end{aligned}$$

The firm hires labor in a perfectly competitive market at wage w_t and each unit of labor has productivity a_t . Thus $z_t = w_t/a_t$. The wage is determined by the household labor supply condition

$$w_t = \frac{\nu'(l_t)}{u'(c_t)}$$

where l_t is labor supply and c_t is consumption.

Problem (b). Assume that $u(c_t) = 1 - c_t^{-1}$ and $\nu(l_t) = l_t$. Show that the natural level of output y_t^* is

$$y_t^* = \sqrt{\frac{a_t}{\mu_t^*}}$$

and explain why this means that the efficient level of output is $\hat{y}_t = \sqrt{a_t}$.

Solution. From the household's labor supply condition:

$$w_t = \frac{\nu'(l_t)}{u'(c_t)} = \frac{1}{c_t^{-2}} = c_t^2.$$

In the flexible price equilibrium, firms set prices to maximize profits, leading to

$$\mu_t(i) = \frac{\varepsilon_t}{\varepsilon_t - 1} = \mu_t^*.$$

Hence the price would also be the same: $p_t(i) = p_t^*$ and the markups become:

$$\mu_t^* = \frac{1}{z_t}.$$

In a perfectly competitive market, the market clearing condition is: $y_t = c_t$. So, we have:

$$\begin{aligned} z_t &= \frac{w_t}{a_t} = \frac{c_t^2}{a_t} = \frac{y_t^2}{a_t} = \frac{1}{\mu_t^*} \\ \Rightarrow y_t^* &= \sqrt{\frac{a_t}{\mu_t^*}} \end{aligned}$$

The efficient level of output occurs when there are no distortions, i.e. when the markup is eliminated: $\mu_t^* = 1$. Thus, $\hat{y}_t = \sqrt{a_t}$. The natural level of output y_t^* is lower than the efficient level \hat{y}_t due to the presence of monopolistic competition, which introduces a markup over marginal cost ($\mu_t^* > 1$). This markup leads to higher prices and reduced output compared to the efficient (perfect competition) case. The efficient level of output is achieved when $\mu_t^* = 1$, eliminating the distortion caused by the markup, and thus maximizing output given the technology level a_t .

When firms can adjust their prices at random staggered intervals (Calvo pricing), inflation is determined by the New Keynesian Phillips curve:

$$\tilde{\pi}_t = \beta E_t \tilde{\pi}_{t+1} + \kappa \tilde{x}_t + \gamma \tilde{\mu}_t^*$$

where $\tilde{\pi}_t$ is inflation and \tilde{x}_t is the output gap (defined relative to efficient output). The notation \tilde{x}_t indicates the percentage deviation of x_t from its steady-state value.

Problem (c). In this environment, should monetary policy aim to stabilize the output gap \tilde{x}_t completely following an unexpected temporary decrease in a_t ? Why or why not?

Solution.

Yes. The monetary policy should aim to stabilize the output gap.

\tilde{x}_t is defined as the difference between actual output y_t and efficient level of output \hat{y}_t :

$$\tilde{x}_t = \frac{y_t}{\hat{y}_t}.$$

Recall $z_t = \frac{w_t}{a_t}$, $w_t = y_t^2$ and $\hat{y}_t = \sqrt{a_t}$, we know that:

$$z_t = \tilde{x}_t^2$$

As a_t shock is exogenous, it won't affect the markup term $\tilde{\mu}_t^*$, so just cancel it out in this case, we have:

$$\tilde{\pi}_t = \beta \mathbb{E}_t \tilde{\pi}_{t+1} + \kappa \tilde{x}_t$$

If we stabilize the output gap completely in response to a temporary decrease in a_t , we have:

$$\tilde{x}_t = 0 \Rightarrow \tilde{\pi}_{t+1} = 0 \Rightarrow \tilde{\pi}_t = \beta \mathbb{E}_t \tilde{\pi}_{t+1} = 0$$

In this case, output is kept equal to the flexible-price equilibrium level of output. This also guarantees inflation equals zero, thereby eliminating the costly dispersion of relative prices that arises with inflation. When firms do not need to adjust their prices, the fact that prices are sticky is no longer relevant.

Now suppose that markets become more competitive when output is higher (perhaps because of more competition for new customers). In particular, assume that $\varepsilon_t = 1 + \frac{1}{2}y_t$. Price adjustment is staggered according to the Calvo model.

Problem (d). Derive an expression for the desired flexible-price markup μ_t^* in terms of the output gap $x_t \equiv y_t/\hat{y}_t$ and productivity a_t . (Hint: the efficient level of output is still $\hat{y}_t = \sqrt{a_t}$.) Find a log-linear approximation of this equation in terms of $\tilde{\mu}_t^*$, \tilde{a}_t , and \tilde{x}_t , and interpret it.

Solution. As given, $\varepsilon_t = 1 + \frac{1}{2}y_t$. The profit-maximizing markup is $\mu_t^* = \frac{\varepsilon_t}{\varepsilon_t - 1}$. So, we have:

$$\mu_t^* = \frac{1 + \frac{1}{2}y_t}{\frac{1}{2}y_t} = 1 + \frac{2}{y_t} = 1 + \frac{2}{x_t \sqrt{a_t}}.$$

We also know that $y_t = x_t \hat{y}_t = x_t \sqrt{a_t}$. Substitute this into the equation, we have:

$$\mu_t^* = 1 + \frac{2}{y_t} \Rightarrow y_t \mu_t^* = y_t + 2.$$

Take the log-linear approximation, we get:

$$\begin{aligned} \tilde{y}_t &\approx \tilde{x}_t + \frac{1}{2}\tilde{a}_t \\ \tilde{y}_t + \tilde{\mu}_t^* &\approx \frac{\bar{y}}{2 + \bar{y}} \tilde{y}_t \\ \Rightarrow \tilde{\mu}_t^* &\approx -\frac{2}{2 + \bar{y}} \left(\tilde{x}_t + \frac{1}{2}\tilde{a}_t \right) \\ &= -\frac{2}{2 + \sqrt{\bar{a}}} \left(\tilde{x}_t + \frac{1}{2}\tilde{a}_t \right) \\ &\approx -\tilde{x}_t - \frac{1}{2}\tilde{a}_t \end{aligned}$$

This indicates that the desired markup $\tilde{\mu}_t^*$ decreases when the output gap increases. This reflects the assumption that markets become more competitive when output is higher.

Higher productivity can enhance competition by enabling firms to produce more efficiently, which may encourage them to lower prices to gain market share.

Problem (e). In this environment, should monetary policy aim to stabilize the output gap \tilde{x}_t completely following an unexpected temporary decrease in a_t ? Why or why not?

Solution.

No, in this case, the monetary policy should not aim to stabilize the output gap completely following an unexpected temporary decrease in a_t .

As we have a negative coefficient, if we stabilize the output gap completely in response to a temporary decrease in a_t , we have:

$$\tilde{\mu}_t^* = -\frac{1}{2}\tilde{a}_t > 0.$$

Then, if we still expect the future inflation to be zero, we have:

$$\tilde{\pi}_t = \beta\mathbb{E}_t\tilde{\pi}_{t+1} + \kappa\tilde{x}_t + \gamma\tilde{\mu}_t^* = -\frac{\gamma}{2}\tilde{a}_t > 0.$$

This shows that we have a tradeoff between stabilizing inflation and closing the output gap.

Problem (f). Compare the size of the response of inflation $\tilde{\pi}_t$ to the output gap \tilde{x}_t in the two cases where the price elasticity ε_t is exogenous and where it depends positively on output y_t . Assuming the central bank is minimizing the same standard loss function in both cases, what implications does the dependence of ε_t on y_t have for the optimal balance between stabilizing inflation and stabilizing the output gap (when shocks shift the New Keynesian Phillips curve)? Why?

Solution.

With exogenous ε_t , we have:

$$\tilde{\pi}_t = \beta\mathbb{E}_t\tilde{\pi}_{t+1} + \kappa\tilde{x}_t.$$

With $\varepsilon_t = 1 + \frac{1}{2}y_t$, we have:

$$\tilde{\pi}_t = \beta\mathbb{E}_t\tilde{\pi}_{t+1} + \kappa\tilde{x}_t + \gamma(-\tilde{x}_t - \frac{1}{2}\tilde{a}_t) = \beta\mathbb{E}_t\tilde{\pi}_{t+1} + (\kappa - \gamma)\tilde{x}_t - \frac{1}{2}\gamma\tilde{a}_t.$$

Thus, the response of inflation to the output gap is smaller when ε_t depends positively on output y_t .

Assuming the loss function is:

$$L_t = \frac{1}{2}(\tilde{\pi}_t^2 + a\tilde{x}_t^2).$$

The optimal balance problem is:

$$\begin{aligned} \min_{\pi_t, x_t} \quad & L_t = \frac{1}{2}(\tilde{\pi}_t^2 + a\tilde{x}_t^2) \\ \text{s.t.} \quad & \tilde{\pi}_t = \beta\mathbb{E}_t\tilde{\pi}_{t+1} + \kappa\tilde{x}_t + \gamma(-\tilde{x}_t - \frac{1}{2}\tilde{a}_t) \\ & \tilde{\pi}_t = \beta\mathbb{E}_t\tilde{\pi}_{t+1} + \kappa\tilde{x}_t \end{aligned}$$

Take the first order condition with respect to $\tilde{\pi}_t$ and \tilde{x}_t , we have:

$$\tilde{\pi}_t = -\frac{a}{\kappa}\tilde{x}_t \quad \text{under exogenous } \varepsilon_t;$$

$$\tilde{\pi}_t = -\frac{a}{\kappa - \gamma} \tilde{x}_t \quad \text{under } \varepsilon_t = 1 + \frac{1}{2}y_t.$$

So, the optimal balance between stabilizing inflation and stabilizing the output gap is affected by the dependence of ε_t on y_t . When ε_t depends positively on output y_t , the inflation rate is more volatile to the output gap. The monetary policy will prioritize inflation control over output gap stabilization.

Short Questions

Problem (2). Explain the channel system of conducting monetary policy, and illustrate its advantages and disadvantages.

Solution.

In a channel system, the central bank sets lower and upper bounds for the interest rate.

1. **Deposit Facility Rate (Floor):** The interest rate the central bank pays on excess reserves deposited by commercial banks.
2. **Lending Facility Rate (Ceiling):** The interest rate at which the central bank lends to commercial banks (typically through overnight loans).

The target policy rate is kept within this corridor, and the central bank uses open market operations to manage liquidity and guide the market rate towards the target.

Advantages:

- **Interest Rate Control:** The corridor provides clear boundaries for short-term interest rates, enhancing the central bank's ability to control market rates.
- **Liquidity Management:** By providing standing facilities, banks have predictable access to liquidity, reducing uncertainty in interbank markets.
- **Stability:** The system can mitigate volatility in interest rates by absorbing shocks through the corridor's floor and ceiling rates.

Disadvantages:

- **Interest Rate Volatility:** If the corridor is wide, market rates can fluctuate significantly within it, potentially leading to uncertainty.
- **Ineffective Transmission:** In times of excess reserves (e.g., during quantitative easing), the floor may become the de facto policy rate, weakening the central bank's control over other interest rates.
- **Dependency on Market Operations:** The system requires active management of liquidity through frequent open market operations, which can be resource-intensive.

Problem (3). Discuss whether (and in which way) optimal monetary policy with commitment is preferable over discretionary optimal monetary policy. Which assumption(s) of the New Keynesian model generate the dynamic externality that drives the differences between discretion and commitment?

Solution.

- **Commitment Policy:**

- Under commitment, the central bank commits to future policy actions, influencing expectations and economic outcomes today.
- It can implement time-inconsistent policies that are optimal in the long run but may not be optimal in the short run.

- **Discretionary Policy:**

- The central bank optimizes policy period by period without committing to future actions.
- This can lead to suboptimal outcomes due to the inability to influence expectations effectively.

If we suppose a temporary positive cost-push shock at time $t = 0$, then for the commitment policy, the central bank will increase the interest rate to reduce inflation, while for the discretionary policy, x_{t+j} would be negative for a while and the inflation expectations will fall $\Rightarrow \pi_0$ increases by less than under discretion without large decrease in x_0 .

Hence, outcome leads to higher welfare.

Assumptions Generating the Dynamic Externality:

- **Forward-Looking Expectations:**

- In the New Keynesian model, prices and wages are set based on expectations of future economic conditions.
- This creates a dynamic externality where current policy affects future expectations and, therefore, current economic outcomes.

- **Sticky Prices and Wages:**

- Price and wage rigidities cause adjustments to be gradual, making expectations about future policy more impactful.
- The slow adjustment amplifies the importance of commitment in shaping expectations.

- **Monopolistic Competition and Price Setting:**

- Firms set prices considering future marginal costs and the anticipated policy path.
- Commitment allows the central bank to influence these expectations, improving allocation efficiency.

Labor Market

Central question for macro and labor:

What determines the level of employment and unemployment in the economy?

In neoclassical model, theoretically cannot deal with unemployment, because there is no involuntary unemployment.

1. Given prices, some agents optimally choose to work zero hours.
2. There is supply and demand only; demand determined by technology or by demand for output; supply driven by inter- or intratemporal substitution
3. There can be under-employment due to wage stickiness, but there is no unemployment in equilibrium

15.1 Some facts about the labor market

- unemployment is a persistent phenomenon
- large flows of workers between employment, unemployment and non-participation states
- $\Delta u = \text{inflow} - \text{outflow}$
- employed workers often change jobs - with a wage gain or wage reduction

15.1.1 How should we model labor friction?

- incentive problems, efficiency wages
- wage rigidity, bargaining, non-market clearing prices
- search Frictions

Definition 15.1.1 (Search and matching).

It is costly process for workers (firms) to find the right jobs (workers)

Facts about job flows

1. job creation is mildly pro-cyclical
2. job destruction is strongly counter-cyclical

3. job destruction leads job creation, it is the driving force of the business cycle - especially in economies with flexible labor markets
4. job creation seems to be the main cause of long-run changes in unemployment
5. worker turnover about three times as large as job turnover
6. worker quits are strongly pro-cyclical \Leftrightarrow offset by the counter-cyclical job destruction rate
recession \rightarrow job destruction increases \rightarrow voluntary quitting decreases \Rightarrow inflow to unemployment increases, but less
7. unemployment changes driven mainly by the outflow from unemployment
8. in monthly data: employment \leftrightarrow non-participation flows \approx employment \leftrightarrow unemployment flows

15.2 A simple model of unemployment

(Shimer's exercise) Let the unemployment rate at period t be u_t , the change in the unemployment rate is

$$u_{t+1} - u_t = s_t(1 - u_t) - f_t u_t$$

where s_t is the job separation rate (probability of employed workers move into unemployment), f_t is the job finding rate. We ignore exit from the labor force, and entry from out of labor force.

Denote average rates by:

$$\bar{s} = \frac{1}{T} \sum_{t=1}^T s_t, \quad \bar{f} = \frac{1}{T} \sum_{t=1}^T f_t$$

Compare the actual unemployment rate with

1. A hypothetical unemployment rate constructed using the average (a constant) separation rate:

$$u_{t+1} - u_t = \bar{s}(1 - u_t) - f_t u_t$$

2. A hypothetical unemployment rate constructed using the average (a constant) job finding rate:

$$u_{t+1} - u_t = s_t(1 - u_t) - \bar{f} u_t$$

15.2.1 Lessons from Shimer's exercise

- separation rate not so important in the evolution of unemployment
- job finding rate is a more important determinant of unemployment

Why?

separation rates do increase during recessions BUT the job finding rate is high in the US, even during recessions, even if more workers get laid off, they find a job quickly \rightarrow job separation rate not so important.

15.3 Search and matching models

15.3.1 First generation search models (PE, McCall)

Let's start right away with continuous time.

Assuming workers start out unemployed, receive unemployment benefits $b \cdot \delta t$ in every infinitesimal time interval δt .

Job offers arrive according to a Poisson process with rate $a > 0$, i.e. in each period of length δt a worker receives $n = 0, 1, 2, \dots$ offers with probability $a(n, \delta t)$, where

$$a(n, \delta t) = \frac{(a\delta t)^n}{n!} e^{-a\delta t}$$

and the number of offers is independent across time.

Remark.

Assume a constant discount rate r . Constant discount rate r and value function of unemployed worker U .

If you stay unemployed, you can never be worse off than rU , so unless someone pays at least this, you won't start working

15.4 Second-generation model (Mortensen and Pissarides, 1994)

15.4.1 Two-sided matching

To close the one-sided search model, to derive a proper equilibrium of the economy, we need to

1. Make arrival date a endogenous, to specify decision of firms whether or not to offer jobs
2. ensure that employment is not an absorbing state, exogenous job destruction, at rate λ
interpretation: negative shocks arrive to existing matches, that destroy the match
3. the wage is the result of bargaining between the matched worker and firm

15.4.2 The matching function

Key assumption

The aggregate flow depends on an aggregate matching function.

Vacancy posting is a (risky) investment for firms: it is costly to post a vacancy, but if firms successfully hire workers, they can enjoy the profit arising from production in the future. Unemployed workers search for firms to work for. The matching process between firms (vacancies) and unemployed workers is summarized by the *matching function*:

$$m = m(u, v)$$

where $m(\cdot)$ is increasing in both terms and exhibits constant returns to scale.

15.4.3 The matching function

The empirical literature (Petrongolo & Pissarides 2001) found that a Cobb-Douglas function matches the data well:

$$m = Au^\eta v^{1-\eta}$$

where usually $\eta = 0.5$.

Job matching is pairwise: $m = au = qv$, where: arrival rate of workers to vacant jobs

$$q = \frac{m(u, v)}{v} = m\left(\frac{u}{v}, 1\right) = m(\theta^{-1}, 1) = q(\theta)$$

where q is a decreasing function of θ , and the elasticity of q w.r.t θ is $\frac{\partial q}{\partial \theta} \frac{\theta}{q} = -\eta \in (-1, 0)$.

arrival rate of jobs to workers

$$a = \frac{m(u, v)}{u} = m\left(1, \frac{v}{u}\right) = a(\theta) = \theta q(\theta)$$

$\theta = \frac{u}{v}$ is a measure of *market tightness*.

u is a state, v is the firm's control, which drives unemployment.

However, probably both firms and workers ignore their effect on θ when they make their search choices → **search externalities**.

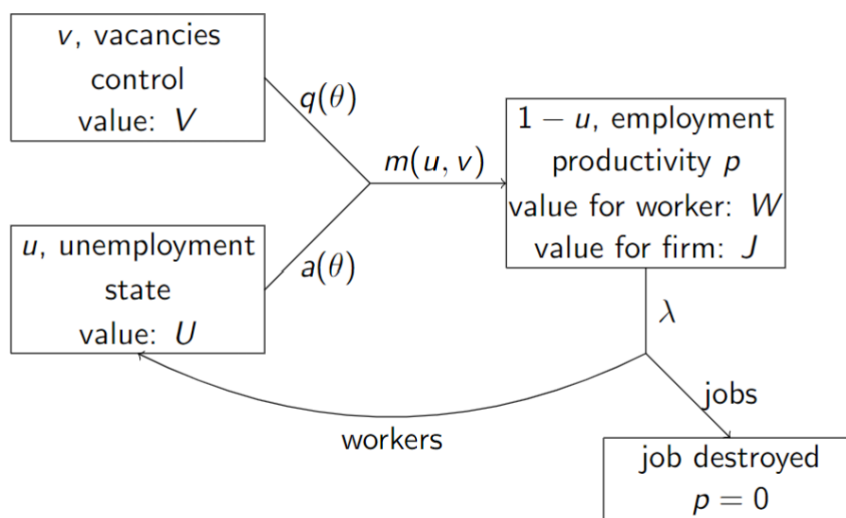


Figure 15.1: Match and searching model

15.4.4 Value of vacancis and jobs

We define

- V : value of a vacancy

- J : value of a job

A job is an asset owned by the firm, and its value is determined by arbitrage equations:

$$\begin{aligned} rV &= -pc + q(\theta)(J - V) \\ rJ &= p - w - \lambda J \end{aligned}$$

where r is the discount rate, p is the value of product, pc is the cost of maintaining a vacancy, w is the wage, λ is the job destruction rate.

Firm creates a job vacancy when there are gains from entering the market and there is free entry.

Zero profits $V = 0$ means that $J = \frac{pc}{q(\theta)}$.

$\frac{1}{q(\theta)}$ is the expected duration of a vacancy.

$\frac{pc}{q(\theta)}$ is the expected total cost of finding a worker.

15.4.5 Job creation

The value of a filled job is:

$$J = \frac{p - w}{r + \lambda}$$

Note.

For the firm to accept a wage w , the value of the job must be at least w ($p \geq w$).

Combine the two functions of the value of filled jobs:

$$\underbrace{p - w}_{\text{profit flow}} - \underbrace{\frac{(r + \lambda)pc}{q(\theta)}}_{\text{expected cost of finding a worker}} = 0$$

Job creation condition: generalization of the labor demand condition, downward sloping in the $\theta - w$ space.

15.4.6 Wage determination

Wages are determined through a process of bargaining between firm and worker.

Using Nash bargaining, each party gets outside option plus a fraction, β for worker, $1 - \beta$ for firm, of the difference between total surplus and outside options.

This process yields a wage equation:

$$w = (1 - \beta)z + \beta p(1 + c\theta)$$

which is:

- increasing in the worker outside option z
- increasing in the worker marginal product p

- increasing in the bargaining power of the worker β
- increasing in the market tightness θ

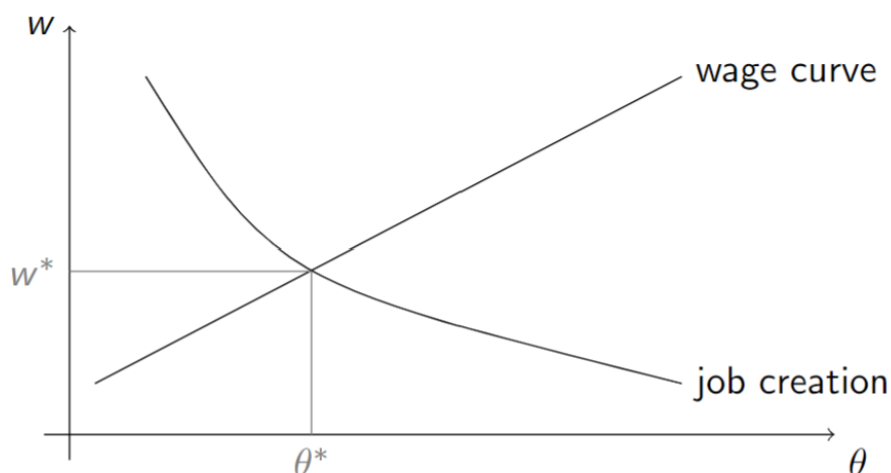


Figure 15.2: Equilibrium wages and market tightness

In this economy, the equilibrium is a path for

The controls:

- Wages – from the wage equation: $J \geq 0, W \geq U$
- Job vacancies (or tightness) – from job creation $V = 0$

The state:

- Employment – *condition for the evolution of unemployment*

$$\dot{u} = \lambda(1 - u) - m(u, v) = \lambda - (\lambda + \theta q(\theta))u$$

Given θ^* , this has a unique stable solution, the *natural rate of unemployment*:

$$u = \frac{\lambda}{\lambda + \theta^* q(\theta^*)}$$

The above is the *Beveridge curve*.

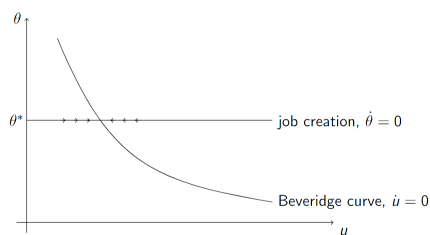


Figure 15.3: Constant Job Creation Beveridge curve

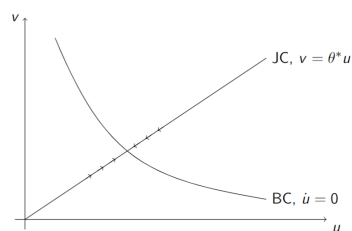


Figure 15.4: Beveridge curve

Assuming a permanent increase in $A(MPL)$, the job-wage curve would shift up right,

For the Beveridge curve, because u cannot jump, v immediately decreases so that $\frac{v}{u}$ becomes the new value at point B . After that, the economy gradually converges to the new steady state of C on the new job creation curve.

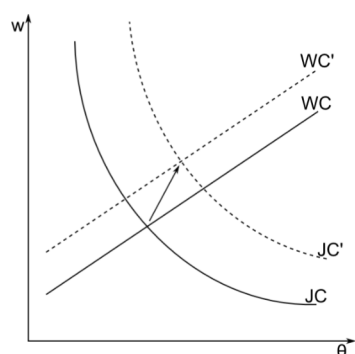


Figure 15.5: Shift in job-wage curve

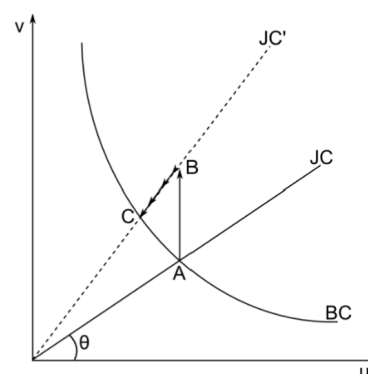


Figure 15.6: Shift in Beveridge curve

For a temporary positive TFP shock, JC curve will rotate back to the original position over time, and response to u would be U-shaped.

If unemployment flow utility b is high enough, elasticity of wage is low.

15.4.7 The unemployment volatility puzzle

The empirical implementations of this model

1. can match contemporaneous correlations between unemployment and vacancies
2. cannot match the volatility of unemployment, vacancies, tightness, job finding rate (no amplification)
3. cannot match the correlations with labor productivity

This discrepancy is often referred to as the *unemployment volatility puzzle* or the labor market volatility puzzle (or the **Shimer puzzle**, after Shimer(2005)).

Remark (Reasons for the Shimer's puzzle).

Intuitively, there are two reasons, corresponding to benefits and costs of hiring, for the quantitatively small response.

First, the benefit of hiring a worker is procyclical, but the magnitude is not large. One reason is that the wage increases in booms, and it weakens the response of profit to the productivity shock.

Second, the cost of hiring a worker moves together with θ . In booms, θ increases, and this increase in cost restricts the firm's response to a positive productivity shock.

What happens after a TFP shock?

- Wage increases, but not as much as the MPL

- Hence firms want to increase employment. More vacancies get posted.
- Over time, this reduces the unemployment rate.

Hence:

- Since the intensive margin of labor supply is fixed, higher wages do not increase labor supply
- Any effect on u must go through vacancy posting
- But wage is too elastic: $MPL - w$ increases by too little, few vacancies are being posted

Possible solutions

- Sticky wages;
- Make unemployment not a big deal, low vacancy posting costs;
- Fixed costs for vacancy posting.

Consumption

16.1 Introduction and Motivation

Consumption is the largest part of GDP.

So, probably utility and welfare depend the most heavily on consumption, and consumption is a part of a dynamic decision problem.

16.2 Benchmark consumption model without uncertainty

Assume perfect foresight and no uncertainty. Consumers can save in a risk-free one period bond, the budget constraint in period t is:

$$c_t + b_{t+1} \leq (1 + r_t)b_t + y_t.$$

Imagine they can also 'buy' period t consumption in period 0: the price of period t consumption relative to period 0 consumption, for a given price of period 0 consumption $q_0 > 0$, is:

$$q_t = \frac{q_0}{(1 + r_1)(1 + r_2) \cdots (1 + r_t)}.$$

Using this assumption, we can aggregate the per period budget constraint until T :

$$\sum_{t=0}^T q_t c_t + q_T b_{T+1} \leq q_0(1 + r_0)b_0 + \sum_{t=1}^T q_t y_t$$

this inter-temporal BC is equivalent to the stream of per-period BCs.

If the agent dies in period T , then it is optimal to have $b_{T+1} = 0$;

or if the agent is immortal, then we impose $T \rightarrow \infty$, $q_T b_{T+1} \rightarrow 0$.

The inter-temporal BC is given by:

$$\sum_{t=0}^{\infty} q_t c_t \leq q_0(1 + r_0)b_0 + \sum_{t=1}^{\infty} q_t y_t = q_0((1 + r_0)b_0 + h_0) = q_0 x_0.$$

where: $(1 + r_0)b_0$ is the *financial wealth* of the agent at time 0,

$h_0 = \sum_{t=1}^{\infty} \frac{q_t}{q_0} y_t$ is the *human wealth* of the agent at time 0,

x_0 is the *effective wealth* of the agent at time 0.

The agent's problem can then be re-written as:

$$\begin{aligned} \max_{c_t} \quad & \sum_{t=0}^T \beta^t U(c_t) \\ \text{s.t.} \quad & \sum_{t=0}^T q_t c_t \leq q_0 x_0 \end{aligned}$$

Note.

This is equivalent to cake-eating problem, as $q_0 x_0$ is given at 0, due to the fact that we are not looking at general equilibrium, i.e. the consumption-saving choice does not affect the interest rates.

This can be viewed as a static consumption problem where we can interpret c_t as different consumption goods, and q_t as the price of these goods.

16.3 Introducing uncertainty

What is the optimal path of consumption if there is uncertainty about the realization of income, y_t ?

We introduce the idiosyncratic shocks.

Some notation:

- $s_t \in S_t$ is the current state of the economy.
- $s^t = \{s_0, s_1, s_2, \dots, s_t\}$ is the history of the economy up to time t , where $s^t \in S^t = S_0 \times S_1 \times \dots \times S_t$.
- $\pi(s^t)$ is the probability of history s^t occurring.
- $y_t^i(s^t)$ is the income of individual i in period t if history s^t occurred, and $\sum_{i \in I} y_t^i(s^t) = Y_t(s^t)$ is the aggregate income in the economy upon the realization of s^t .

16.3.1 Benchmark 1 - Autarky

Assuming in an endowment economy, where agents cannot trade with each other, and there's no storage tech to transfer resources across time. ($b_t = 0$)

In such a setup it is optimal for the agent to consume all his income in each period:

$$c_t^i(s^t) = y_t^i(s^t)$$

16.3.2 Benchmark 2 - Complete Markets

Still in an endowment economy, but with full set of insurance markets, where agents can trade assets that pay off one unit of consumption if a particular history s^t occurs. (*Arrow-Debreu securities*)

With S states in each period, T periods, there are $1 + S + S^2 + \dots + S^T$ markets.

Let $p_t(s^t)$ be the price in terms of period 0 consumption of a unit of income (or consumption, i.e. the price of the AD-security) in period after history s^t .

For every individual $i \in I$, the budget constraint at time 0 is:

$$\sum_{t=0}^T \sum_{s^t \in S^t} p_t(s^t) (c_t^i(s^t) - y_t^i(s^t)) = 0.$$

every transfer of income across states and time is possible, as long as the expected discounted consumption expenditures equal (or do not exceed) the expected discounted income for each individual.

16.3.3 The AD competitive equilibrium

The equilibrium is an allocation of consumption and prices of AD-securities:

$$\{c_t^i(s^t)\}_{t=0}^T, \{p_t(s^t)\}_{t=0}^T, \forall i \in I, \forall s^t \in S^t$$

such that:

given $p_t(s^t)$ for every i , $c_t^i(s^t)$ solves the utility maximization problem:

$$\begin{aligned} \max_{c_t^i(s^t)} \quad & \sum_{t=0}^{\infty} \beta^t \sum_{s^t} \pi(s^t) U(c_t^i(s^t)) \\ \text{s.t.} \quad & \sum_{t=0}^T \sum_{s^t \in S^t} p_t(s^t) (c_t^i(s^t) - y_t^i(s^t)) = 0 \end{aligned}$$

while the markets clear:

$$\sum_i c_t^i(s^t) = Y_t(s^t) \forall s^t \in S^t, t \geq 0.$$

Take the FOC, we get:

$$\beta^t \pi(s^t) U'(c_t^i(s^t)) = \lambda^i p_t(s^t) \forall i, s^t, t.$$

1. Perfect risk sharing:

$$\frac{U'(c_t^i(s^t))}{U'(c_t^j(s^t))} = \frac{\lambda^i}{\lambda^j}$$

The relative marginal utility does not depend on state/history.

Given that

$$\sum_i (U')^{-1} \left(\frac{\lambda^i}{\lambda^j} U'(c_t^j(s^t)) \right) = \sum_i c_t^i(s^t) = \sum_i y_t^i(s^t) = Y_t(s^t)$$

only aggregate income matters for individual consumption

Hence, agents fully insure against idiosyncratic risk (but cannot insure against aggregate risk!)

2. Consumption smoothing:

$$\beta^{t-\tau} \frac{\pi(s^t) U'(c_t^i(s^t))}{\pi(s^\tau) U'(c_\tau^j(s^\tau))} = \frac{p_t(s^t)}{p_\tau(s^\tau)}$$

Desire to smooth consumption across time and states attenuated by prices and probabilities.

Marginal rate of substitution = Marginal rate of transformation

For conditions of the First Welfare Theorem hold, the competitive equilibrium is Pareto efficient.

Hence, it can be modeled as the outcome of a social planner's problem:

$$\begin{aligned} \max_{\{c_t^i(s^t)\}} \quad & \sum_{t=0}^T \sum_{s^t} \pi(s^t) \beta^t \sum_{i \in I} \alpha^i U(c_t^i(s^t)) \\ \text{s.t.} \quad & \sum_i c_t^i(s^t) = Y_t(s^t) \forall s^t \in S^t, t \geq 0 \end{aligned}$$

yields identical outcome to decentralized Arrow-Debreu model when $\alpha^i = (\lambda^i)^{-1}$.

16.4 empirical implications

Both autarky and complete markets benchmarks have empirically testable implications.

We estimate the following form micro data:

$$\Delta \log c_t^i = \beta_1 \Delta \log C_t + \beta_2 \Delta \log y_t^i + \varepsilon_t$$

Autarky implies that $\beta_1 = 0, \beta_2 = 1$.

Arrow-Debreu implies that $\beta_1 = 1, \beta_2 = 0$.

Both hypotheses are rejected, we have to look for a model of partial consumption insurance.

16.5 The permanent income hypothesis

Assume instead of complete markets that the household can only trade in a non state-contingent asset, i.e. they can only buy one-period bonds (which have return r).

The consumer has an incentive to hold some bonds in order to smooth consumption, with the budget constraint:

$$b_t + y_t = c_t + \frac{b_{t+1}}{1+r}$$

which is the 'standard' setup in the RBC model.

Euler equation:

$$u'(c_t) = \beta(1+r) \mathbb{E}_t(u'(c_{t+1}))$$

Make two additional assumptions:

1. quadratic preferences, $U(c) = a_1 c - \frac{1}{2} a_2 c^2$.

2. the interest rate on the bond equals the inverse of the discount rate $\beta(1+r) = 1$.

Then, the Euler equation becomes:

$$a_1 - a_2 c_t = \beta(1+r) \mathbb{E}_t(a_1 - a_2 c_{t+1}) \Rightarrow \mathbb{E}_t c_{t+1} = c_t$$

which implies that consumption is a random walk.

Law of iterated expectations yields $\mathbb{E}_t c_{t+j} = c_t \forall j \geq 0$.

Using the budget constraint, we can get:

$$\begin{aligned} \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j \mathbb{E}_t c_{t+j} &= b_t + \underbrace{\sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j \mathbb{E}_t y_{t+j}}_{\equiv h_t} \\ \Rightarrow c_t &= \frac{r}{1+r} (b_t + h_t) \end{aligned}$$

consumption today only depends on the annuity value of income, on the *permanent income*.

16.5.1 Consumption dynamics

$$\begin{aligned} \Delta c_t &= c_t - c_{t-1} = c_t - \mathbb{E}_{t-1} c_t \\ &= \frac{r}{1+r} \left(b_t - \mathbb{E}_{t-1} b_t + \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j (\mathbb{E}_t y_{t+j} - \mathbb{E}_{t-1}(\mathbb{E}_t y_{t+j})) \right) \\ &= \frac{r}{1+r} \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^j (\mathbb{E}_t y_{t+j} - \mathbb{E}_{t-1} y_{t+j}) \end{aligned}$$

the change in consumption between period $t-1$ and t is due to the revision of permanent income, and it is proportional to the revision of expected earnings due to information that arrived in the same period.

16.6 Implications

1. Consumption is a random walk:

$$\mathbb{E}_{t-1} \Delta c_t = 0$$

there is no insurance, just consumption smoothing.

2. The marginal propensity to consume from wealth is

$$\frac{r}{1+r}$$

the marginal propensity to consume from an innovation to current income depends on the persistence of the income process.

Lower persistence \Rightarrow smaller reaction in consumption to current income changes.

Recommended Resources

Books

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