Problem Set 2

Mathematics and Statistics for Economists

Due date: 11:59pm Tuesday 10th September 2024

Optimisation Theory

Question 1 For the implicit relation

$$x^3 - 3xy^2 = 1$$

1. Find the points where the slope of the tangent line is horizontal.
2. Determine if these points are maxima, minima, or saddle points.

Question 2 Solve the optimization problem where a firm wants to maximize its production function given a budget constraint.
Maximize:

$$F(x_1, x_2) = [\alpha x_1^\rho + \beta x_2^\rho]^{\frac{1}{\rho}}$$

Subject to the budget constraint:

$$p_1 x_1 + p_2 x_2 = M$$

where:

- the production function takes the form of the Constant Elasticity of Substitution (CES) function, ρ is related to the elasticity of substitution σ through $\rho = \frac{\sigma - 1}{\sigma}$.
- p_1 and p_2 are the prices of inputs x_1 and x_2, respectively.
- M is the total budget available.

Question 3 A firm produces goods using two inputs, x_1 and x_2, and aims to maximize its profit. The firm's profit is given by:

$$\Pi = pQ - (w_1 x_1 + w_2 x_2)$$

where p is the price of the output, w_1 and w_2 are the prices of the inputs, and Q is the output level. The production function Q is given by:

$$Q = F(x_1, x_2) = \sqrt{x_1 x_2}$$

The firm faces a constraint on its total spending on inputs, given by:

$$w_1 x_1 + w_2 x_2 = B$$

where B is the total budget available for inputs.

1. Formulate the Lagrangian for the firm's problem and derive the optimal input quantities x_1^* and x_2^*.

2. Determine the maximum profit function $\Pi^*(p, w_1, w_2, B)$ in terms of p, w_1, w_2, and B.

3. Use the Envelope Theorem to find how the maximum profit Π^* changes with respect to a change in the output price p.

Question 4 Consider a company that produces two products, x_1 and x_2. The company's objective is to maximize its profit, given by the following profit function:

$$\Pi = 5x_1x_2 - x_1^2 - x_2^2$$

The company faces the following constraints:
1. Budget Constraint: The total cost of production cannot exceed the budget B :

$$2x_1 + x_2 \leq 10$$

2. Production Requirement: The production of x_1 and x_2 must meet a minimum production level

$$x_1 + x_2 \geq 4$$

3. Non-Negativity Constraints: The quantities of the products must be non-negative:

$$x_1 \geq 0$$
$$x_2 \geq 0$$

Solve for the optimal input quantities, and verify that the Kuhn-Tucker conditions hold at the optimum.

Question 5 A small open economy produces two goods labelled 1 and 2 ; their prices p_1 and p_2 are determined in world markets. The economy has fixed supplies K and L if capital and labour. For $i = 1, 2$ let the economy's output of good i be $F_i(K_i, L_i)$, where K_i and L_i denote the quantities of capital and labour allocated to sector i, F_i is the production function for good i and subscripts do not indicate partial differentiation. Under perfect competition, the economy will choose K_1, K_2, L_1, L_2 so as to maximise

$$p_1F_1(K_1, L_1) + p_2F_2(K_2, L_2)$$

subject to the constraints $K_1 + K_2 = K, L_1 + L_2 = L$. The maximal value of the objective function is denoted $V(p_1, p_2, K, L)$ Using the envelope theorem, show that

1. $\frac{\partial V}{\partial p_i} = F_i(K_i, L_i)$ for $i = 1, 2$.
2. $\frac{\partial V}{\partial K} = p_1\frac{\partial F_1}{\partial K_1} = p_2\frac{\partial F_2}{\partial K_2}$
3. $\frac{\partial V}{\partial L} = p_1\frac{\partial F_1}{\partial L_1} = p_2\frac{\partial F_2}{\partial L_2}$

Probability and Statistics

Question 6 Define $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 - \exp(-x) & \text{if } x \geq 0 \end{cases}$

1. Show that $F(x)$ is a CDF.
2. Find the PDF $f(x)$.
3. Find $\mathbb{E}[X]$.
4. Find the PDF of $Y = X^{1/2}$.

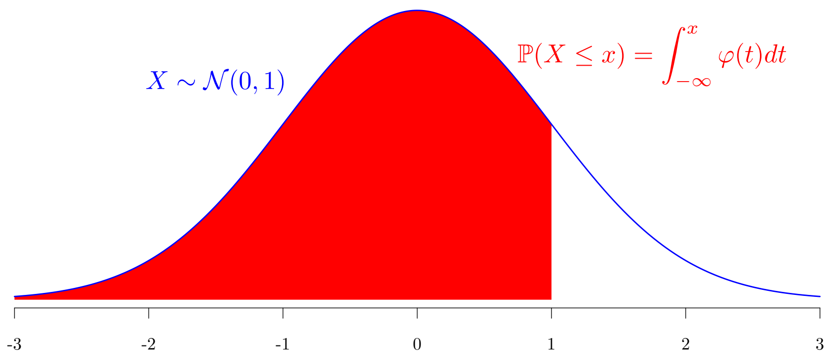
Question 7 Prove the Probability Integral Transformation Proposition in the class:
Let X have cdf F_X and define $Y = F_X(X)$. Then Y is uniformly distributed : $Y \sim \mathcal{U}[0,1]$, i.e. $F_Y(y) = y$ for $y \in [0,1]$

Question 8 If $X \sim \mathcal{N}(75, 100)$, find $P(X < 60)$ and $P(70 < X < 100$ by using the table on the normal distribution.

Question 9 Let $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$ so that $P(X < 89) = 0.90$ and $P(X < 94) = 0.95$. Find μ and σ^2.

Question 10 A packaging machine is supposed to produce 1 kg packs. The actual weight of a pack is modeled by a random variable following a normal distribution with a standard deviation of 20 g . It is possible to tune the mean weight of the packs. In order to check that the tuning is correct, a sample of 10 packs is weighed.

1. Let $\mathcal{H}_0$ be the hypothesis: "the mean weight is 1 kg ". Build a test at threshold 1%, of $\mathcal{H}_0$ against hypothesis $\mathcal{H}_1$: "the mean weight is different from 1 kg ". Find the p-value of that test, for a sample of average weight 1011 grams.
2. Same question for hypothesis $\mathcal{H}_1$: "the mean weight is larger than 1 kg ".
3. Answer again the two previous questions for a sample of 100 packs, with mean weight 1005 g .
4. On a sample of 10 packs, a mean weight of 1011 g has been observed, with an empirical standard deviation of 32 g. At threshold 1%, is this observation compatible with the value of 20 g for the theoretical standard deviation?
(Hint: The test statistic for testing variance is the chi-square statistic: $\chi^2 = \frac{(n-1)s^2}{\sigma^2}$, where s^2 is the empirical sd and σ^2 is the theoretical sd)
5. For the sample of the previous question, supposing the variance is unknown, can it be said that the packs are significantly too heavy on average at threshold 1% ?



x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990