$\boxed{\text{EX 1.}}$

a) $\min \int p(i)\, c(i)\, di$

s.t. $\left(\int c(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} = c$

$$\mathcal{L} = \int p(i) c(i) di - \lambda \left(\left(\int c(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} - c \right)$$

$$\frac{d\mathcal{L}}{d c(i)} = p(i) - \lambda \frac{\varepsilon}{\cancel{\varepsilon-1}} \left(\int c(i)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{\varepsilon}{\varepsilon-1}-1} \cdot \cancel{\frac{\varepsilon-1}{\varepsilon}} c(i)^{\frac{\varepsilon-1}{\varepsilon}-1} = 0$$

$$p(i) = \lambda c(i)^{-\frac{1}{\varepsilon}} \underbrace{\left(\int c(i)^{\frac{\varepsilon-1}{\varepsilon}}\right)^{\frac{1}{\varepsilon-1}}}_{c^{\frac{1}{\varepsilon}} \text{ from above}} = \lambda c(i)^{-\frac{1}{\varepsilon}} c^{\frac{1}{\varepsilon}}$$

$$\Downarrow$$

(1) $c(i) = \left(\frac{p(i)}{\lambda}\right)^{-\varepsilon} c$

We need to find λ. Plug (1) back into the constraint

$$\left\{ \int \left[\left(\frac{p(i)}{\lambda}\right)^{-\varepsilon} c \right]^{\frac{\varepsilon-1}{\varepsilon}} di \right\}^{\frac{\varepsilon}{\varepsilon-1}} = c$$

$\Rightarrow$ c can be brought out of the integral since it's constant

$$\cancel{c}^{\cancel{\frac{\varepsilon-1}{\varepsilon}} \cancel{\frac{\varepsilon}{\varepsilon-1}}} \cdot \left[\int \left(\frac{p(i)}{\lambda}\right)^{1-\varepsilon} di \right]^{\frac{\varepsilon}{\varepsilon-1}} = \cancel{c}$$

Also λ is constant so I can bring it out of the integral

$$\left[\left(\frac{1}{\lambda}\right)^{1-\varepsilon}\right]^{\frac{\varepsilon}{\varepsilon-1}} \cdot \left[\int p(i)^{1-\varepsilon} di\right]^{\frac{\varepsilon}{\varepsilon-1}} = 1$$

$$\lambda^{-\varepsilon} = \left[\int p(i)^{1-\varepsilon} di\right]^{\frac{\varepsilon}{\varepsilon-1}} = \left[\int p(i)^{1-\varepsilon} di\right]^{\frac{1}{1-\varepsilon}}$$

λ, as alway, is the shadow price => If you tighten the constraint by 1-unit (i.e. 1 unit more of the consumption basket), the objective function (i.e. expenditure) increases by λ

→ For 1-unit more you need to pay λ more

λ => PRICE OF THE CONSUMPTION BASKET

$$P_t = \left(\int p(i)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}}$$

$$C(i) = \left(\frac{p(i)}{p} \right)^{-\varepsilon} C$$ => DEMAND FUNCTION

b) Plug the demand function into the expenditure function $\int p(i) C(i) di$

$$\int p(i) \left(\frac{p(i)}{p} \right)^{-\varepsilon} C di = \int p(i)^{1-\varepsilon} p^{\varepsilon} C =$$

From $p = \left[\int p(i)^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}} di \Rightarrow p^{1-\varepsilon} = \int p(i)^{1-\varepsilon} di$

$$\int p(i)^{1-\varepsilon} p^{\varepsilon} C = p^{1-\varepsilon} p^{\varepsilon} C = pC \checkmark$$

Exercise 2

Euler Equation:

$$y_t^{-\frac{1}{\sigma}} = \beta E_t\left((1+r_t)\ y_{t+1}^{-\frac{1}{\sigma}}\right)$$

Log-Linearize Steps:

1) Take Logs
2) Do a first order Taylor expansion around a point (SS)
3) Simplify so that everything is expressed in percentage dev. from SS.

1) $-\frac{1}{\sigma}\log y_t = \log\beta + \log(1+r_t) - \frac{1}{\sigma}\log y_{t+1}$

2) $-\frac{1}{\sigma}\log y^* - \frac{1}{\sigma}\frac{1}{y^*}(y_t - y^*) = \log\beta + \frac{(1+r_t-(1+r^*))}{1+r^*}$
$+ \log(1+r^*) - \frac{1}{\sigma}\log y^* - \frac{1}{\sigma}\frac{y_{t+1}-y^*}{y^*}$

$$\hat{x}_t = \frac{\tilde{x}_t - x^*}{x^*}$$

$$-\frac{1}{\sigma}\hat{y}_t = E_t\left(\hat{r}_t - \frac{1}{\sigma}\hat{y}_{t+1}\right)$$

$$(2)\quad \hat{y}_t = E_t\left(\hat{y}_{t+1} - \sigma\hat{r}_t\right) \Rightarrow \text{IS EQUATION}$$

Log-Lin Output Gap: $\tilde{x}_t = \tilde{y}_t - \hat{y}_t$

Where $\hat{y}_t$ efficient level of output

$$\hat{y}_t = E_t(\hat{y}_{t+1} - \sigma \hat{r}_t) \quad (3)$$

Take (2) and subtract (3)

$$\tilde{y}_t - \hat{y}_t = E_t\left[\hat{y}_{t+1} - \sigma \tilde{r}_t - \hat{y}_{t+1} + \sigma \tilde{\hat{r}}_t\right]$$

$$\tilde{x}_t = E_t\left[\tilde{x}_{t+1} - \sigma \tilde{r}_t\right] + \sigma \tilde{\hat{r}}_t$$

From Fisher Equation $r_t = i_t - \hat{\pi}_{t+1}$

$$\Downarrow \text{LOG-LIN}$$

$$\ln(r^*) + \frac{r_t - r^*}{r^*} = \ln(i^* - \hat{\pi}^*_t) + \frac{1}{i^* - \hat{\pi}^*} \cdot (i_t - i^*) - \frac{1}{i^* - \hat{\pi}^*}(\hat{\pi}_{t+1} - \hat{\pi}^*)$$

$$\frac{r_t - r^*}{r^*} = \frac{1}{i^* - \hat{\pi}^*}\left(i_t - i^* - (\hat{\pi}_{t+1} - \hat{\pi}^*)\right)$$

$$\tilde{r}_t = \tilde{i}_t - \tilde{\hat{\pi}}_{t+1}$$

$$\text{IS: } \hat{x}_t = E_t\left[\hat{x}_{t+1} - \sigma(i_t - \hat{\pi}_{t+1})\right] + v_t$$