ECON 39: Undergraduate International Trade
Practice Problems, Week #2
Winter 2017, Professor Treb Allen

Setting

Consider a world with a single country (say, the United States) which is inhabited with L workers. Suppose the workers each have one unit of time, with which they can either produce good 1 or good 2. Suppose it takes α_1 units of time to produce good 1 and α_2 units of time to produce good 2. Finally, suppose the workers have preferences expressed by the following utility function:

$$U = \min\{\beta_1 c_1, \beta_2 c_2\},$$

where c_1 and c_2 are the quantities that each worker consumes of good 1 and good 2, respectively, and $\beta_1 > 0$ and $\beta_2 > 0$. (These preferences are known as *Leontief* preferences). Let q_1 and q_2 denote the quantity produced by each worker of good 1 and 2, respectively. Finally, let p_1 and p_2 be the price of goods 1 and 2, respectively.

Questions

1. What are the exogenous model parameters in this setting?
   - Answer: $\{L, \alpha_1, \alpha_2, \beta_1, \beta_2\}$.
2. What are the endogenous model outcomes in this setting?
   - Answer: $\{c_1, c_2, q_1, q_2, p_1, p_2\}$.
3. Let us first consider the problem of the worker *producing* things.
   (a) What does a worker try to maximize in her production decision?
   - Answer: A worker tries to maximize her income.

   (b) What endogenous outcomes does the worker choose in her production decision?
   - Answer: A worker chooses how to split her time between producing different goods, which in turn determines the quantity produced.

   (c) What endogenous outcomes does the worker take as given in her production decision?
   - Answer: A worker takes as given the price of the two goods.

   (d) What constraints does the worker face in her production decision?
   - Answer: a worker cannot work more than her one unit of time.

   (e) Write down the mathematical expression of the worker's production decision problem:
   - Answer: A worker chooses what fraction of her time she spends on producing good 1,θ, which leaves fraction $(1-\theta)$ of her time to the production of good 2 in order to maximize her income:
   $$\max_{\theta \in [0,1]} \theta \frac{p_1}{\alpha_1} + (1-\theta)\frac{p_2}{\alpha_2} \tag{1}$$

   (f) In equilibrium, will the worker ever spend all of her time producing just one good? Why or why not?
   - Answer: As long as $\beta_1 > 0$ and $\beta_2 > 0$, she will always spend some time producing each good, as otherwise her utility will be zero.

   (g) What are the first order conditions of the mathematical expression in 3(d)? What is their intuition?

- Answer: 3(e) tells us that we can ignore the constraint $\theta \in [0,1]$. Hence, the Lagrangian of equation (1) is:

$$L : \theta \frac{p_1}{\alpha_1} + (1-\theta)\frac{p_2}{\alpha_2}$$

The first order conditions are:

$$\frac{\partial L}{\partial \theta} = 0 \iff \frac{p_1}{\alpha_1} = \frac{p_2}{\alpha_2} \iff \frac{p_1}{p_2} = \frac{\alpha_1}{\alpha_2}. \tag{2}$$

Intuitively, for a worker to be willing to produce both goods, her pay per unit time must be the same for both goods.

(h) Can we determine how much of each good is produced from the worker's production decision? Why or why not?

- Answer: We cannot, because if $\frac{p_1}{\alpha_1} = \frac{p_2}{\alpha_2}$ then the worker is indifferent between working in each sector and hence would be willing to spend any fraction of time producing either good.

(i) Can we determine what a workers income is?

- Answer: Yes we can. As long as $\frac{p_1}{\alpha_1} = \frac{p_2}{\alpha_2}$, a worker's income y will be $\frac{p_1}{\alpha_1}$ (measured in units of good 1) or equivalently, $\frac{p_2}{\alpha_2}$ (measured in units of good 2):

$$y = \frac{p_1}{\alpha_1} = \frac{p_2}{\alpha_2} \tag{3}$$

4. Now let us consider the problem of the worker *consuming* things.

(a) What does a worker try to maximize in her consumption decision?

- Answer: A worker tries to maximize her utility.

(b) What endogenous outcomes does the worker choose in her consumption decision?

- Answer: A worker chooses how much to consume of each of the goods.

(c) What endogenous outcomes does the worker take as given in her consumption decision?

- Answer: A worker takes as given the price of the two goods and her income.

(d) What constraints does the worker face in her consumption decision?

- Answer: a worker cannot spend more on the goods she consumes than her income.

(e) Write down the mathematical expression of the worker's consumption decision problem:

- Answer: A worker chooses how much to consume of each good subject to her budget constraint in order to maximize her utility:

$$\max_{c_1 \geq 0, c_2 \geq 0} \{\min\{\beta_1 c_1, \beta_2 c_2\}\} \text{ subject to } p_1 c_1 + p_2 c_2 \leq y \tag{4}$$

(f) Solve for the equilibrium consumption as a function of β_1, β_2, p_1, p_2, and y.

- Answer: This problem is not differentiable, so calculus will not help us! Instead, we proceed by common sense. A worker is trying to maximize $\min\{\beta_1 c_1, \beta_2 c_2\}$. Suppose she chooses to consume some quantity c_1: how much of c_2 will she then choose to consume? If she chooses an amount of c_2 such that $\beta_2 c_2 < \beta_1 c_1$, she will only get utility $\beta_2 c_2$. This cannot be optimal, because she would be spending money on c_1 that did not give her any utility – she would be throwing money away! Conversely, if she chooses an amount of c_2 such that $\beta_2 c_2 > \beta_1 c_1$, she would only get utility $\beta_1 c_1$: now she would be throwing money away on c_2! Hence, for her to not throw money away, it must be that she sets

$$\beta_1 c_1 = \beta_2 c_2 \iff c_2 = \frac{\beta_1}{\beta_2} c_1. \tag{5}$$

Let us then substitute equation (5) into equation (4) to get a much simpler utility maximization problem:

$$\max \beta_1 c_1 \text{ subject to } \left(p_1 + p_2\frac{\beta_1}{\beta_2}\right) c_1 \leq y$$

Since $\beta_1 > 0$, the worker will simply want to consume as much as she can up until she reaches her budget constraint, so that:

$$c_1 = \frac{y}{p_1 + p_2\frac{\beta_1}{\beta_2}} \text{ and } c_2 = \frac{y}{p_1 + p_2\frac{\beta_1}{\beta_2}}\frac{\beta_1}{\beta_2} \tag{6}$$

5. Let us now calculate the market equilibrium.

   (a) Define the market equilibrium:

   - Answer: For a given set of parameters $\{L, \alpha_1, \alpha_2, \beta_1, \beta_2\}$, market equilibrium is defined by the set of endogenous market outcomes $\{c_1, c_2, q_1, q_2, p_1, p_2\}$ such that (1) Given prices, each worker allocates her time to maximize her income; (2) Given prices and income, each worker chooses her consumption to maximize utility; and (3) consumption is equal to production of both goods.

   (b) Solve for the equilibrium quantity produced / consumed of each good solely as a function of exogenous model parameters.

   - Answer: To solve this, we combine our expressions for the optimal consumption (i.e. equation (6)) with our expressions for optimal production (equations (2) and (3)). Let us start with the expression for c_1. Substitute in $y = \frac{p_1}{\alpha_1}$ yields:

$$\begin{aligned} c_1 &= \frac{y}{p_1 + p_2\frac{\beta_1}{\beta_2}} \iff \\ c_1 &= \frac{\frac{p_1}{\alpha_1}}{p_1 + p_2\frac{\beta_1}{\beta_2}} \iff \\ c_1 &= \frac{\frac{1}{\alpha_1}}{1 + \frac{p_2}{p_1}\frac{\beta_1}{\beta_2}}, \end{aligned}$$

   where in the last line we divided everything by p_1. From equation (2), we know $\frac{p_1}{p_2} = \frac{\alpha_1}{\alpha_2}$ or equivalently $\frac{p_2}{p_1} = \frac{\alpha_2}{\alpha_1}$ so we get:

$$\begin{aligned} c_1 &= \frac{\frac{1}{\alpha_1}}{1 + \left(\frac{\alpha_2}{\alpha_1}\right)\frac{\beta_1}{\beta_2}} \iff \\ c_1 &= \frac{1}{\alpha_1 + \alpha_2\frac{\beta_1}{\beta_2}} = q_1, \end{aligned} \tag{7}$$

   where the last line multiplied everything by α_1 to make it look pretty. Since $c_2 = \frac{\beta_2}{\beta_1}c_1$ we then have:

$$c_2 = \frac{\frac{\beta_1}{\beta_2}}{\alpha_1 + \alpha_2\frac{\beta_1}{\beta_2}} = q_2 \tag{8}$$

   (c) Does L appear in your expressions? Why or why not?

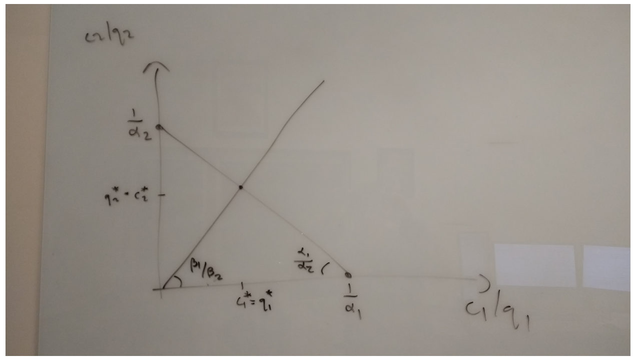
   - Answer: it does not. Intuitively, every worker is identical, so each worker just consumes what she produces: there is no trade. L will be important, however, when we introduce trade!

   (d) If we multiplied β_1 and β_2 by the same positive number, would it change the equilibrium consumption and production? Why or why not?

- Answer: It would not change the equilibrium consumption. This is because utility is ordinal: it just provides a preference ordering of the choices that a consumer can make. Changing the units of utility does not change the choices of the workers.

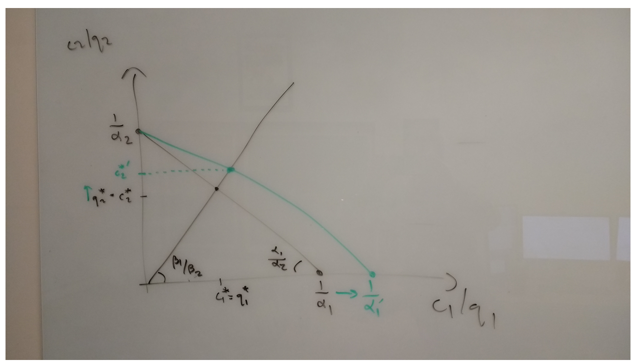
(e) Draw a picture of the equilibrium (i.e. the production possibilities frontier and the indifference curve). Carefully label any intersections of curves and the axis.

- Answer:



6. Finally, let us do a counterfactual: what happens to the consumption of good 2 if workers become more efficient at producing good 1? Show the answer on the board and derive $-\frac{\partial c_2}{\partial \alpha_1}$.

- Answer:



- The derivation is straightforward:

$$-\frac{\partial c_2}{\partial \alpha_1} = -\frac{\partial}{\partial \alpha_1}\left(\frac{\frac{\beta_1}{\beta_2}}{\alpha_1 + \alpha_2 \frac{\beta_1}{\beta_2}}\right) = \frac{\frac{\beta_1}{\beta_2}}{\left(\alpha_1 + \alpha_2 \frac{\beta_1}{\beta_2}\right)^2}$$