PS 1 due 30 September 2025 – Questions

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1 RBC model

Consider first a slightly simplified version of the basic RBC model, in which σ is set equal to 1, as is the overall price level P.

The budget constraint for the representative consumer is given by:

$$C_t + I_t = W_t L_t + R_t K_t \cdots (A)$$

The capital stock moves according to the standard law of motion (where the capital stock is defined as of the beginning of the period - not the end as in Dynare):

$$K_{t+1} = (1-\delta) K_t + I_t$$

where δ is the depreciation rate of capital and where I_t is the gross investment. Substituting the capital accumulation equation in the individual's budget constraint one obtains:

$$C_t + K_{t+1} = W_t L_t + (R_t + 1 - \delta) K_t$$

where C_t is the consumption of goods, K_t is the private capital stock (owned by the household), W_t is the wage rate for workers and R_t is the (net) real interest rate.

The utility function to be maximized by consumers has the following form:

$$U(C_t, L_t) = \gamma \log C_t + (1-\gamma) \log (1 - L_t)$$

The problem facing the consumer is to select trajectories of

$$\{C_t, L_t, I_t\}$$

that maximize the value of her utility throughout her life cycle:

$$\underset{\{C_t, L_t, I_t\}_t^{\infty}}{Max} \mathcal{L} = \sum_{t=0}^{\infty} \beta^t [\gamma \log C_t + (1-\gamma) \log (1 - L_t)] \cdots (B)$$

subject to the restrictions given by the budget constraint and by the production function of domestic goods given the initial capital stock K_0 and where $\beta \in (0,1)$, is the consumer discount rate.

Assume that the production function in the goods market is represented by a Cobb-Douglas type function. The production of goods and services, Y_t, requires the use of labour services, L_t, and of capital services, K_t. Assume that all capital is used in the production of goods. The technology is given by:

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$

where A_t measures total factor productivity and α is the ratio of capital income to total output.

The firm's decision then consists of maximizing the profit function, defined by:

$$\max_{K_t, L_t} \left[A_t K_t^{\alpha} L_t^{1-\alpha} - R_t K_t - W_t L_t \right]$$

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Q1. Show that the first order conditions associated with the firm's problem lead to the following definitions in terms of the remuneration to each of the productive factors:

$$R_t K_t = \alpha Y_t$$
$$W_t L_t = (1-\alpha) Y_t$$

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Putting together equations A and B, the Lagrangian associated with the consumer problem is defined by:

$$\max_{(C_t, K_t, L_t)} \mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left(\begin{array}{c} [\gamma \log C_t + (1-\gamma) \log (1 - L_t)] \\ -\lambda_t [C_t + K_{t+1} - W_t L_t - (R_t + 1 - \delta) K_t] \end{array} \right)$$

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Q2. Show that the first order conditions of the above maximization problem are as follows:

$$\frac{\partial \mathcal{L}}{\partial C} : \gamma \frac{1}{C_t} - \lambda_t = 0 \cdots (C)$$
$$\frac{\partial \mathcal{L}}{\partial L} : -\frac{1-\gamma}{1-L_t} + \lambda_t W_t = 0 \cdots (D)$$
$$\frac{\partial \mathcal{L}}{\partial K} : \beta^t \lambda_t (R_t + 1 - \delta) - \beta^{t-1} \lambda_{t-1} = 0 \cdots (E)$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} : C_t + K_{t+1} - (R_t + 1 - \delta) K_t - W_t L_t = 0 \cdots (F)$$

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From the first order condition (C), the shadow price of consumption of market goods is given by:

$$\lambda_t = \frac{\gamma}{C_t}$$

Combining the expression (C) with the expression (D), the equilibrium condition for the labour supply is obtained as:

$$\frac{1-\gamma}{\gamma}\frac{C_t}{1-L_t} = W_t$$

Combining the expression (C) with the expression (E), intertemporal consumption decisions of the individual are given by:

$$\frac{C_t}{C_{t-1}} = \beta\left[R_t + 1 - \delta\right]$$

The competitive equilibrium of the model is thus given by a set of seven equations that represent the behaviour of the seven endogenous variables, $Y_t, C_t, I_t, K_t, L_t, R_t, W_t$ plus the exogenous variable, A_t (which in the model is endogenised through the definition of a (logarithmic) stochastic process).

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Q3. Construct the complete model (in non-linear form) as follows, noting that for Dynare, capital stock must be lagged one period:

$$\begin{aligned}
&EulerEquation_t = \cdots \\
&ProductionFunction_t = \cdots \\
&LabourSupplyFunction_t = \cdots \\
&ResourceConstraint_t = \cdots \\
&K_t = (1-\delta)K_{t-1} + I_t \\
&W_t = \cdots \\
&R_t = \cdots \\
&\ln A_t = (1-\rho_A)\ln \bar{A} + \rho_A \ln A_{t-1} + \varepsilon_t^A
\end{aligned}$$

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To calibrate the model, it is necessary to determine the value of the following parameters:

$$\Omega = \{\alpha, \beta, \gamma, \delta, \rho_A, \sigma_A\}$$

Table 1 shows the calibrated values of the parameters that you will use to simulate the model.

Table 1: Calibrated parameters

Parameter Value	Definition	
α	Capital share	0.350
β	Discount factor	0.970
γ	Preferences parameter	0.400
δ	Depreciation rate	0.060
ρ_A	TFP autoregressive parameter	0.950
σ_A	TFP standard deviation	0.010

In order to implement this (non-linear) model, the steady-state needs to be found. To do so, it is necessary to make an assumption about $\bar{A}$ - assume it is = 1. Most calculations are straightforward, but for completeness, note the following: the production function is given by

$$Y_t = AK_t^{\alpha}L_t^{1-\alpha}$$

Hence in steady-state

$$\bar{Y} = \bar{A}\bar{K}^{\alpha}\bar{L_m}^{1-\alpha}$$

But, given the definition of R_t in Q1 and the Euler equation found below Q2, it is easy to see that

$$1 = \beta\left[\alpha\frac{\bar{Y}}{\bar{K}} + 1 - \delta\right]$$

Hence

$$\bar{K} = \frac{\alpha\beta}{1 - \beta + \beta\delta}\bar{Y}$$

Hence

$$\bar{Y} = \bar{A}\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\alpha}\bar{Y}^{\alpha}\bar{L}^{1-\alpha}$$

Hence

$$\bar{Y}^{1-\alpha} = \bar{A}\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\alpha}\bar{L}^{1-\alpha}$$

Hence

$$\bar{Y} = \bar{A}^{\frac{1}{1-\alpha}}\left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{\alpha}{1-\alpha}}\bar{L}$$

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Q4. Calculate the steady-state values of the other model variables and check that your values are the same as those found in Table 2 below.

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Table 2: Steady-state values

Variable	Value	Ratio w.r.t. $\bar{Y}$
$\bar{Y}$	0.74469	1.000
$\bar{C}$	0.57270	0.769
$\bar{I}$	0.17199	0.231
$\bar{K}$	2.86649	3.849
$\bar{L}$	0.36039	–
$\bar{R}$	0.09092	–
$\bar{W}$	1.34312	–
$\bar{A}$	1.00000	–

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Q5. Using the attached skeleton of a Dynare .mod file for this model, simulate it with the parameters indicated in Table 1 and comment on your results. $Note$ that for obscure technical reasons, Dynare will not solve the model unless you $lead$ the variables C and R of the Euler equation by one period each (try it without so doing to see this).

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2 RBC model with housework

Now consider an economy otherwise identical to that in Part 1, but in which explicit recognition is made of domestic activities, hitherto ignored.

Assume that total consumption is composed of consumption of goods and services produced in the market and consumption of domestic goods produced by the individual herself. Additionally, assume that the goods do not have a unit elasticity of substitution, and therefore use a CES type function to aggregate. Therefore, the total consumption of the individual is given by:

$$C_t = \left[\omega C_{m,t}^{\eta} + (1-\omega) C_{h,t}^{\eta}\right]^{1/\eta}$$

where $C_{m,t}$ represents the consumption of market-produced goods and services, $C_{h,t}$ is the consumption of (non-market) domestic ("home") activities and where η is the parameter that determines the agents' desire to substitute one good for another, owing to variations in relative prices, ω being the proportion that market goods represent in the total consumption of the individual.

The parameter η determines the relationship between market activities and domestic activities. Thus the elasticity of substitution between consumption of market goods and consumption of domestic goods is defined as $1/(1-\eta)$. (If η were equal to 1, both goods would be perfect substitutes and total consumption would simply be the sum of consumption of market goods and household goods. If, on the contrary, η were equal to 0, total consumption would be a function of the Cobb-Douglas type and the elasticity of substitution between both goods would be unitary.)

The household's total allocation of time is divided into three activities: time dedicated to work, $L_{m,t}$, time dedicated to carrying out domestic activities, $L_{h,t}$, and leisure, defined by $1 - L_{m,t} - L_{h,t}$. Thus, the total time not dedicated to leisure activities is defined as the sum of the time spent working plus the time dedicated to domestic activities:

$$L_t = L_{m,t} + L_{h,t}$$

Therefore, the utility function of the individual is defined as:

$$U\left(C_{m,t}, C_{h,t}, L_{m,t}, L_{h,t}\right)$$

The market budget constraint for the representative consumer is given by:

$$C_{m,t} + I_t = W_t L_{m,t} + R_t K_t \cdots (A2)$$

where it is assumed that saving is only possible in the market sector; there is no saving in the domestic sector, since all domestic goods are consumed.

As previously, the capital stock moves according to the standard law of motion:

$$K_{t+1} = (1-\delta) K_t + I_t$$

where δ is the depreciation rate of capital and where I_t is gross investment. Substituting the capital accumulation equation in the individual's budget constraint one obtains:

$$C_{m,t} + K_{t+1} = W_t L_{m,t} + (R_t + 1 - \delta) K_t$$

where $C_{m,t}$ is the consumption of market goods, K_t is the private capital stock (owned by the household), $W_{m,t}$ is the market wage rate for workers and R_t is the real interest rate.

The utility function to be maximized by consumers thus now has the following form:

$$U\left(C_{m,t}, C_{h,t}, L_{m,t}, L_{h,t}\right) = \gamma \log C_t + (1-\gamma) \log\left(1 - L_{m,t} - L_{h,t}\right)$$

The problem facing the consumer is therefore to select trajectories of

$$\left\{C_{m,t}, C_{h,t}, L_{m,t}, L_{h,t}, I_t\right\}$$

that maximize the value of her utility throughout her life cycle:

$$\underset{\left\{C_t, L_{m,t}, L_{h,t}, I_t\right\}_t^{\infty}}{Max} \mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\gamma \log C_t + (1-\gamma) \log\left(1 - L_{m,t} - L_{h,t}\right)\right] \cdots (B2)$$

subject to the restrictions given by the budget constraint and by the production function of domestic goods given the initial capital stock K_0 and where $\beta \in (0,1)$, is the consumer discount rate.

Assume as previously that the production function in the goods market is represented by a Cobb-Douglas type function. The production of market goods and services, Y_t, requires the use of marketed labour services, $L_{m,t}$, and of capital services, K_t. Assume that all capital is used in the production of market goods. The technology is given by:

$$Y_t = A_t K_t^{\alpha} L_{m,t}^{1-\alpha}$$

where A_t measures total factor productivity and α is the ratio of capital income to total output.

The firm's decision consists of maximizing the profit function, defined now by:

$$\max_{K_t, L_t} \left[A_t K_t^{\alpha} L_{m,t}^{1-\alpha} - R_t K_t - W_t L_{m,t} \right]$$

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Q6. Show that the first order conditions associated with the firm's problem are:

$$\frac{\partial \Pi_t}{\partial K_t} : \quad R_t - \alpha A_t K_t^{\alpha-1} L_{m,t}^{1-\alpha} = 0$$
$$\frac{\partial \Pi_t}{\partial L_t} : W_t - (1-\alpha) A_t K_t^{\alpha} L_{m,t}^{-\alpha} = 0$$

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From the above equations, the following definitions obtain in terms of the remuneration to each of the productive factors:

$$R_t K_t = \alpha Y_t$$
$$W_t L_{m,t} = (1-\alpha) Y_t$$

Note that in this definition, the level of production of the economy takes into account only the activities of production of market goods and services, not including the goods produced by individuals in their homes.

The production function of household goods is given by:

$$C_{h,t} = B_t L_{h,t}^{\theta} \cdots (C2)$$

that is, the production of household goods is labour intensive. These domestic goods have to be consumed by the individual who produces them and cannot be sold in the market. B_t measures the productivity associated with the production of household goods.

Putting together equations A2, B2 and C2, the Lagrangian associated with the consumer problem is now defined by:

$$\underset{\{C_t, L_{m,t}, L_{h,t}, I_t\}_t^\infty}{Max} \mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ \begin{array}{c} \gamma \log \left[\omega C_{m,t}^{\eta} + (1-\omega) C_{h,t}^{\eta} \right]^{1/\eta} \\ +(1-\gamma) \log \left(1 - L_{m,t} - L_{h,t} \right) \\ -\lambda_t \left[\begin{array}{c} C_{m,t} + K_{t+1} - W_t L_{m,t} \\ -(R_t + 1 - \delta) K_t \end{array} \right] \\ -\zeta_t \left[C_{h,t} - B_t L_{h,t}^{\theta} \right] \end{array} \right\}.$$

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Q7. Show that the first order conditions of the above maximization problem are as follows:

$$\begin{array}{ll} \frac{\partial \mathcal{L}}{\partial C_{m,t}}: & \frac{\gamma \omega C_{m,t}^{\eta-1}}{\left[\omega C_{m,t}^{\eta} + (1-\omega) C_{h,t}^{\eta} \right]} - \lambda_t = 0 \cdots (H) \\ \frac{\partial \mathcal{L}}{\partial C_{h,t}}: & \frac{\gamma (1-\omega) C_{h,t}^{\eta-1}}{\left[\omega C_{m,t}^{\eta} + (1-\omega) C_{h,t}^{\eta} \right]} - \zeta_t = 0 \cdots (I) \\ \frac{\partial \mathcal{L}}{\partial L_{m,t}}: & \frac{(1-\gamma)}{1 - L_{m,t} - L_{h,t}} - \lambda_t W_t = 0 \cdots (J) \\ \frac{\partial \mathcal{L}}{\partial L_{h,t}}: & \frac{(1-\gamma)}{1 - L_{m,t} - L_{h,t}} - \zeta_t \theta B_t L_{h,t}^{\theta-1} = 0 \cdots (K) \\ \frac{\partial \mathcal{L}}{\partial K_t}: & \beta^t \left[\lambda_t \left(R_t + 1 - \delta \right) \right] - \lambda_{t-1} \beta^{t-1} = 0 \cdots (L) \end{array}$$

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From the first order condition (H), the shadow price of consumption of market goods is given by:

$$\lambda_t = \frac{\gamma \omega C_{m,t}^{\eta-1}}{\left[\omega C_{m,t}^{\eta} + (1-\omega) C_{h,t}^{\eta} \right]} \tag{1}$$

that is, it now also depends on the quantity of household goods consumed. For its part, the Lagrange parameter associated with the production of household goods, obtained from the first-order condition (I) is:

$$\zeta_t = \frac{\gamma(1-\omega)C_{h,t}^{\eta-1}}{\left[\omega C_{m,t}^{\eta} + (1-\omega)C_{h,t}^{\eta}\right]} \tag{2}$$

which indicates the shadow price associated with the consumption of domestic goods and which also depends on the consumption of market goods.

Combining the expression (H) with the expression (J), the equilibrium condition for the labour supply is obtained as:

$$\frac{(1-\gamma)}{1-L_{m,t}-L_{h,t}} = \frac{\gamma\omega C_{m,t}^{\eta-1}}{\left[\omega C_{m,t}^{\eta} + (1-\omega)C_{h,t}^{\eta}\right]} W_t \tag{3}$$

Combining the expression (H) with the expression (L), the Euler equation that determines the intertemporal consumption/investment decisions of the individual is given by:

$$\beta \frac{\gamma\omega C_{m,t}^{\eta-1}}{\left[\omega C_{m,t}^{\eta} + (1-\omega)C_{h,t}^{\eta}\right]} (R_t + 1 - \delta) = \frac{\gamma\omega C_{m,t-1}^{\eta-1}}{\left[\omega C_{m,t-1}^{\eta} + (1-\omega)C_{h,t-1}^{\eta}\right]} \tag{4}$$

Finally, combining the first order conditions (I) and (K), the expression that determines the amount of time that the individual will dedicate to the production of household chores is obtained as:

$$\frac{(1-\gamma)}{1-L_{m,t}-L_{h,t}} = \frac{\gamma(1-\omega)C_{h,t}^{\eta-1}}{\left[\omega C_{m,t}^{\eta} + (1-\omega)C_{h,t}^{\eta}\right]} \theta B_t L_{h,t}^{\theta-1} \tag{5}$$

The competitive equilibrium of the model is thus now given by a set of eleven equations that represent the behaviour of the nine endogenous variables, $Y_t, C_{m,t}, C_{h,t}, I_t, K_t, L_{m,t}, L_{h,t}, R_t, W_t$ and the two exogenous variables, A_t and B_t, (which in the model are endogenised through the definition of a (logarithmic) stochastic process for each one).

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Q8. Some of these eleven equations are given below; you must fill in the remainder:

$$\frac{(1-\gamma)}{1-L_{m,t}-L_{h,t}} = \frac{\gamma\omega C_{m,t}^{\eta-1}}{\left[\omega C_{m,t}^{\eta}+(1-\omega)C_{h,t}^{\eta}\right]} W_t$$

$$\beta \frac{\gamma\omega C_{m,t+1}^{\eta-1}}{\left[\omega C_{m,t+1}^{\eta}+(1-\omega)C_{h,t+1}^{\eta}\right]} (R_{t+1} + 1 - \delta) = \frac{\gamma\omega C_{m,t}^{\eta-1}}{\left[\omega C_{m,t}^{\eta}+(1-\omega)C_{h,t}^{\eta}\right]}$$

$$\frac{(1-\gamma)}{1-L_{m,t}-L_{h,t}} = \frac{\gamma(1-\omega) C_{h,t}^{\eta-1}}{\left[\omega C_{m,t}^{\eta}+(1-\omega)C_{h,t}^{\eta}\right]} \theta B_t L_{h,t}^{\theta-1}$$

$$Y_t = \cdots$$
$$C_{h,t} = \cdots$$
$$K_t = (1-\delta)K_{t-1} + I_t$$
$$I_t = \cdots$$
$$W_t = \cdots$$
$$R_t = \cdots$$
$$Ctot_t = \cdots$$
$$\ln A_t = (1-\rho_A)\ln \bar{A} + \rho_A \ln A_{t-1} + \varepsilon_t^A$$
$$\ln B_t = (1-\rho_B)\ln \bar{B} + \rho_B \ln B_{t-1} + \varepsilon_t^B$$
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To calibrate the model, it is necessary to determine the value of the following parameters:

$$\Omega = \{\alpha, \beta, \gamma, \delta, \eta, \omega, \theta, \rho_A, \sigma_A, \rho_B, \sigma_B\}$$

Table 3 shows the calibrated values of the parameters that you will use to simulate the model.

Table 3: Calibrated parameters

Parameter Value	Definition	
α	Capital share	0.350
β	Discount factor	0.970
γ	Preferences parameter	0.450
δ	Depreciation rate	0.060
η	Goods substitution parameter	0.800
ω	Relative weight of market vs home goods	0.400
θ	Home production parameter	0.800
ρ_A	TFP autoregressive parameter	0.950
σ_A	TFP standard deviation	0.001
ρ_B	Home production autoregressive parameter	0.950
σ_B	Home production standard deviation	0.001

As in Part 1, in order to implement this (non-linear) model, the steady-state needs to be found. The calculations can be done as before, the only difference being that now it is necessary to substitute L_m where previously only L was used. Thus,

$$Y_t = AK_t^{\alpha} L_{m,t}^{1-\alpha}$$

Hence in steady-state

$$\bar{Y} = \bar{A}\bar{K}^{\alpha}\bar{L_m}^{1-\alpha}$$

Thus, eventually we arrive at

$$\bar{Y} = \bar{A}^{\frac{1}{1-\alpha}} \left(\frac{\alpha\beta}{1-\beta+\beta\delta}\right)^{\frac{\alpha}{1-\alpha}} \bar{L_m}$$

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Q9. Calculate the steady-state values of the other model variables and check that your values are the same as those found in Table 4 below, bearing in mind that (as shown above) the production function is given by

$$Y_t = AK_t^{\alpha} L_{m,t}^{1-\alpha}$$

The calculations for $\bar{L}_h$ are difficult (highly non-linear); you may therefore just assume that the value is that shown in table 4. Compare your results to those found in Part 1.

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Table 4: Steady-state values

Variable	Value	Ratio w.r.t. $\bar{Y}$
$\bar{Y}$	0.58995	1.000
$\bar{C}_m$	0.45370	0.769
$\bar{C}_h$	0.34759	–
$\bar{I}$	0.13625	0.231
$\bar{K}$	2.27082	3.849
$\bar{L}_m$	0.28550	–
$\bar{L}_h$	0.26689	–
$\bar{R}$	0.09092	–
$\bar{W}$	1.34312	–
$\bar{A}$	1.00000	–
$\bar{B}$	1.00000	–

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Q10. Using the attached skeleton of a Dynare .mod file for this model, simulate it with the parameters indicated in Table 1 and comment on your results.

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