Lecture Notes:
Real Analysis

Based on lectures by **So-Chin Chen**

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This is the lecture note taken in the course *Real Analysis* taught by **So-Chin Chen** at National Tsing Hua University, Taiwan in the semester I of 2021. The course is designed to provide a comprehensive understanding of the fundamental concepts and theories in Real Analysis.

Currently, these are just drafts of the lecture notes. There can be typos and mistakes anywhere. So, if you find anything that needs to be corrected or improved, please inform at `jingle.fu@graduateinstitute.ch`.

Contents

Lecture 1.

Preliminaries

Real analysis originates from calculus. When we were studying calculus, we deal with two sides: differentiation and integration, which are inverse operations. Real analysis is the study of these two sides in a more rigorous way: starting from generalizing the Riemann integral, but why?

Recall that when we were studying Riemann integral, we define $f : [a, b] \to \mathbb{R}$ as a bounded function on a closed interval $[a, b]$.

$$\int_a^b f(x)\,dx \text{ exists?} \tag{1.1}$$

The Riemann integral is then defined as the limit of Riemann sums as the partition of the interval becomes finer. We take the partition of the interval $[a, b]$ into n subintervals, $P = \{a < x_0 < x_1 < \cdots < x_n = b\}$, take $t_k \in [x_{k-1}, x_k]$ as a sample point in each subinterval, and then we form the Riemann sum:

$$R(f, P) = \sum_{k=1}^{n} f(t_k)\Delta x_k \tag{1.2}$$

where $\Delta x_k = x_k - x_{k-1}$ is the width of the k-th subinterval. The Riemann integral is then defined as:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R(f, P) \tag{1.3}$$

if the limit exists.

In Darboux integral, we define the upper and lower sums:

$$L(f, P) = \sum_{k=1}^{n} m_k\Delta x_k, \quad U(f, P) = \sum_{k=1}^{n} M_k\Delta x_k \tag{1.4}$$

where $m_k = \inf\limits_{x\in[x_{k-1},x_k]} f(x)$ and $M_k = \sup\limits_{x\in[x_{k-1},x_k]} f(x)$. The Darboux integral is then defined as:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} L(f, P) = \lim_{n\to\infty} U(f, P) \tag{1.5}$$

if the limit exists. The Riemann integral and the Darboux integral are equivalent, but the Darboux integral is more general.

Example 1.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

for $x \in [0, 1]$. The Riemann integral does not exist, but the Darboux integral exists and is equal to 0.

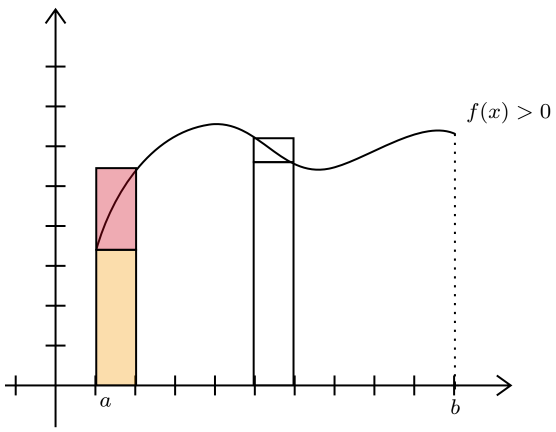
$$L(f, P) = 0, \quad U(f, P) = 1$$

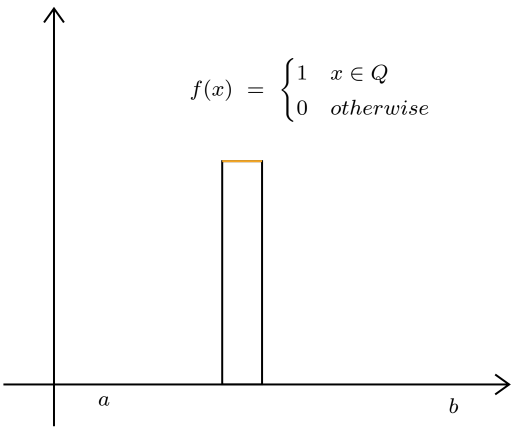
Lebesgue extends the idea of Riemann integral to more general functions.

Theorem 1.0.1 (Lebesgue's Criterion for Integrability).

Set $\mathcal{D} = \{x \in [a,b] | f \text{ is not continuous at } x\}$. Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then f is Lebesgue is integrable if and only if $\mathcal{D}$ is of measure zero.

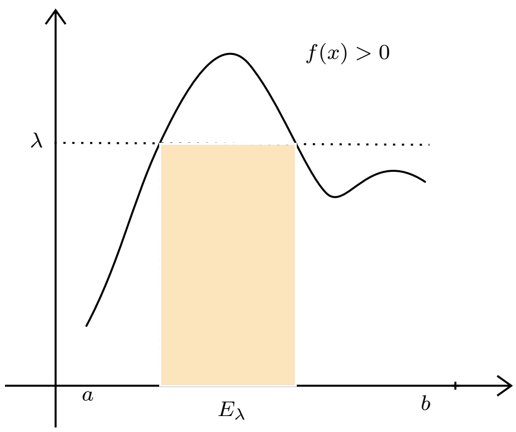
We first show the difference of Riemann integral and Darboux integral in the graph(left). Then let's look back at example 1 (right), we can see that the function is not continuous at any point in $[0,1]$. By the definition of Riemann integral, we can see that the Riemann integral does not exist.





What do we really want by taking integrals? Basically speaking, we want to find the area under the curve. In the case of Riemann and Darboux integrals, we partitioned the x-axis into subintervals, but we encountered problems.

It's quite common that we start to try partitioning the y-axis instead. We define the Lebesgue measure as $E_\lambda = \{x \in [a,b] | f(x) > \lambda\}$.



Assuming that E_λ is measurable, we know that the shaded area is $\lambda|E_\lambda|$.

Lecture 2.

Measure Theory

2.1 Outer Measure

Outer measure can be defined on every set.

Definition 2.1.1 (Outer Measure).

$E \subseteq \mathbb{R}^n$ is a set, I is a closed interval: $I = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n | a_i \leq x_i \leq b_i, i = 1, \cdots, n\}$, and $v(I)$ is the volume of the interval I,

$$v(I) = \begin{cases} \prod_{i=1}^{n}(b_i - a_i), & \text{if } a_i \leq b_i; \\ 0, & \text{otherwise.} \end{cases}$$

For set E, consider a *countable* collection of open, nounded intervals that cover E, $S = \{I_i\}_{i=1}^{\infty}$, in the sense that $E \subseteq \bigcup_{i=1}^{\infty} I_i$. For each such collection, consider the sum of the volumes of the intervals in the collection. We define

$$\sigma(S) = \sum_{i=1}^{\infty} v(I_i) \tag{2.1}$$

The outer measure of E, denoted by $m^*(E)$, is

$$m^*(E) = \inf \sigma(S) \tag{2.2}$$

the infimum is taken over all countable collections of closed intervals S.

Lemma 2.1.1.

If I is a closed interval, then $m^*(I) = v(I)$.

Proof.

By definition, I covers itself, so $m^*(I) \leq v(I)$. Given any $\varepsilon > 0$, $\exists S = \{I_i\}_{i=1}^{\infty}$, a closed interval cover, such that $\sigma(S) \leq m^*(I) + \varepsilon$. We need to show that $v(I) \leq \sum_{i=1}^{\infty} v(I_i) = \sigma(S)$. For each i, choose a bigger I_i^*, such that $I \subseteq int(I_i^*)$ and $v(I_i^*) \leq v(I_i)(1+\varepsilon)$. Then we have $I \subseteq \bigcup_{i=1}^{\infty} int(I_i^*)$. By compactness of I, (The Heine-Borel theorem), we can find an integer N such that $I \subseteq \bigcup_{i=1}^{N} int(I_i^*)$, hence

$$v(I) \leq \sum_{i=1}^{N} v(I_i^*) \leq (1+\varepsilon) \sum_{i=1}^{N} v(I_i) \leq (1+\varepsilon)\sigma(S).$$

So $v(I) \leq \sigma(S)$, if we take infimum over all S, we have $v(I) \leq m^*(I)$. □

From Lemma 2.1.1, we can see that the outer measure of the boundary of a closed interval is zero, i.e. $m^*(\partial I) = 0$.

Lemma 2.1.2.

Suppose we have two setws $E_1 \subseteq E_2$, then

$$m^*(E_1) \leq m^*(E_2).$$

Lemma 2.1.3.

Assume we have infinite sets: $E_1, \cdots, E_\infty$, $E_k \subseteq \mathbb{R}^n$, $k \in \mathbb{N}$. Let $E = \bigcup_{k=1}^{\infty} E_k$, then

$$m^*(E) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

Proof.

We may assume that $m^*(E_k) < \infty$ for all k. Given any $\varepsilon > 0$, for each k, we can find a countable collection of closed intervals $S_k = \{I_i^{(k)}\}_{i=1}^{\infty}$ such that $E_k \subseteq S_k$ and that

$$\sum_{i=1}^{\infty} v(I_i^{(k)}) \leq m^*(E_k) + \frac{\varepsilon}{2^k}.$$

Then we can take the union over all k to obtain a countable collection of closed intervals $S = \bigcup_{k=1}^{\infty} S_k$ such that

$$E = \sum_{k=1}^{\infty} E_k \subseteq \bigcup_{k=1}^{\infty} S_k = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{\infty} I_i^{(k)}.$$

So the outer measure $m^*(E)$ is bounded by the sum of the outer measures of the individual sets:

$$m^*(E) \leq \sigma(S) = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} v(I_i^{(k)}) = \sum_{k=1}^{\infty} \left(m^*(E_k) + \frac{\varepsilon}{2^k} \right) = \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon.$$

□

Example 2 (Singleton).

Singleton set $E = \{x\}$, where $x \in \mathbb{R}^n$. We can cover E with a single closed interval $I = [x, x]$. The volume of this interval is $v(I) = 0$. Therefore, the outer measure of a singleton set is $m^*(E) = 0$.

Example 3 (Countable Set).

For a countable set $E = \{x_1, x_2, \cdots, x_k, \cdots\}$, we can cover it with a finite collection of closed intervals. We can take $E = \bigcup_{k=1}^{\infty} x_k$. So the outer measure is

$$m^*(E) \leq \sum_{k=1}^{n} m^*(\{x_k\}) = 0.$$

As outer measure is non-negative, we have $m^*(E) = 0$.

2.1.1 Cantor Set

Definition 2.1.2 (Cantor Set).

The Cantor set C is defined as follows:

1. Start with the closed interval $[0, 1]$.

2. Remove the open middle third $(\frac{1}{3}, \frac{2}{3})$.

3. Repeat this process for each remaining closed interval.

The Cantor set is the intersection of all these sets after infinitely many steps.

$$C = \bigcap_{k=1}^{\infty} C_k, \quad \text{where } C_k \text{ is the set obtained after } k \text{ iterations.}$$

The Cantor set is uncountable, compact, and has Lebesgue measure zero.

$$m^*(C) \leq m^*(C_k) \leq 2^k \cdot \frac{1}{3^k} \to 0.$$

For each k, the set C_k consists of 2^k closed intervals (with $2^k - 1$ open intervals removed), each of length $\frac{1}{3^k}$.

2.1.2 Cantor Function (Devil's Staircase)

Definition 2.1.3 (Cantor Function)**.** The Cantor function $f : [0, 1] \to [0, 1]$ is defined as follows:

1. On the Cantor set C, f is defined by the ternary expansion without using digit 1.

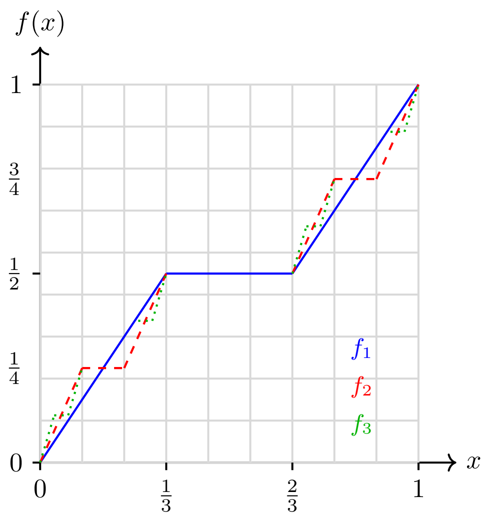
2. On each removed interval $\left(\frac{a}{3^k}, \frac{b}{3^k}\right)$, f is constant.

3. f is continuous, non-decreasing, and maps $[0, 1]$ onto $[0, 1]$.

For each iteration, we obtain a series of functions: $f_1, f_2, \cdots$, s.t. $f_k : [0, 1] \to [0, 1]$.

$$|f_k - f_m| \leq \frac{1}{2^k},$$

and that f_k is continuously monotone increasing. So $f = \lim_{k \to \infty} f_k$ exists and is a continuous function on $[0, 1]$, we call it the Cantor function. The Cantor function is also known as the "Devil's Staircase" due to its characteristic step-like appearance.



Remark. The Cantor function has several remarkable properties:

1. It is continuous and non-decreasing on $[0, 1]$.
2. It maps $[0, 1]$ onto $[0, 1]$ surjectively.
3. Its derivative is zero almost everywhere (on the complement of the Cantor set).
4. It increases only on the Cantor set, which has measure zero.
5. It is an example of a singular function: continuous but not absolutely continuous.

Lecture 3.

Appendix

Recommended Resources

Books

[1] Richard L. Wheeden Antoni Zygmund. *Measure and Integral An Introduction to Real Analysis*. 2nd ed. New York: CRC Press, 2015

[2] Walter Rudin. *Real and Complex Analysis*. 3rd ed. Singapore: McGraw-Hill, 1987

[3] Gerald B. Folland. *Real Analysis Modern Techniques and Their Applications*. 2nd ed. Toronto, Canada: Wiley, 1999

[4] P. M. Fitzpatrick Halsey L. Royden. *Real Analysis*. 5th ed. New Jersey: Pearson Education, Inc., 2023

[5] Elias M. Stein and Rami Shakarchi. *Real Analysis: Measure Theory, Integration, and Hilbert Spaces*. Princeton, New Jersey: Princeton University Press, 2005

[6] Sheldon Axler. *Measure, Integration & Real Analysis*. Springer Open, 2022