Problem 1

Consider the following bivariate VAR(1):

$$y_t = \Phi_1 y_{t-1} + u_t \ , \quad u_t = \Phi_\varepsilon \varepsilon_t \ , \quad \varepsilon_t \overset{i.i.d.}{\sim} N(0, I).$$

where $y_t = (w_t, h_t)'$ is composed of log wages w_t and log hours h_t. The vector of structural shocks $\varepsilon_t = (\varepsilon_{a,t}, \varepsilon_{b,t})'$ is composed of a labor demand shock (technology shock) $\varepsilon_{a,t}$ and a labor supply shock (preference shock) $\varepsilon_{b,t}$.

(a) What condition does Φ_1 have to satisfy so that y_t is stationary?

Solution: All Eigenvalues of Φ_1 have to be less than 1 in absolute value.

(b) Suppose y_t is stationary, derive the autocovariances of order zero and one, denoted by $\Gamma_{yy}(0)$ and $\Gamma_{yy}(1)$.

Solution: Clearly, $\mathbb{E}[y_t] = 0$. Therefore, $\Gamma(h) = \mathbb{E}[y_t y'_{t-h}]$. There are two ways to obtain $\Gamma(h)$.

Inserting for y_{t-1} in the RHS of the VAR(1) equation, we get

$$\begin{aligned} y_t &= \Phi_1(\Phi_1 y_{t-2} + u_{t-1}) + u_t \\ &= \Phi_1^2 y_{t-2} + u_t + \Phi_1 u_{t-1} \ . \end{aligned}$$

Repeating this same process infinitely many times, we get

$$y_t = \sum_{l=0}^{\infty} \Phi_1^l u_{t-l} \ ,$$

since $\lim_{l\to\infty} \Phi_1^l y_{t-l} = 0$ under stationarity. Using this equation, we get

$$\begin{aligned} \Gamma(h) &= \mathbb{E}[y_t y'_{t-h}] \\ &= \mathbb{E}\left[\left(\sum_{l=0}^{\infty} \Phi_1^l u_{t-l}\right)\left(\sum_{k=0}^{\infty} \Phi_1^l u_{t-h-k}\right)'\right] \\ &= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} \Phi_1^l \mathbb{E}[u_{t-l} u'_{t-h-k}] \Phi_1^{l\prime} \\ &= \sum_{k=0}^{\infty} \Phi_1^{k+h} \Sigma \Phi_1^{k\prime} \ , \end{aligned}$$

since u_t is a WN processs and, therefore, $\mathbb{E}[u_{t-l} u'_{t-h-k}] = 0$ unless $l = h + k$. We can evaluate this expression for $h = 0$ and $h = 1$. Practically, we have to cut off this infinite sum after some finite number of terms, but this has little bearing on our results since, by stationarity,

Φ_1^k converges to zero as $k \to \infty$.

Using the second approach, we have

$$\begin{aligned}\Gamma(0) = \mathbb{E}[y_t y_t'] &= \mathbb{E}[(\Phi_1 y_{t-1} + \Phi_\varepsilon \varepsilon_t)(\Phi_1 y_{t-1} + \Phi_\varepsilon \varepsilon_t)'] \\ &= \Phi_1 \Gamma(0) \Phi_1' + \Phi_\varepsilon \Phi_\varepsilon' \ .\end{aligned}$$

Using $vec(ABC) = (A \otimes C')vec(B)$, we have

$$vec(\Gamma(0)) = (I - \Phi_1 \otimes \Phi_1)^{-1} vec(\Phi_\varepsilon \Phi_\varepsilon') \ .$$

The covariance matrices can be obtained using the Yule-Walker equations:

$$\Gamma(\tau) = \mathbb{E}[y_t y_{t-\tau}'] = \mathbb{E}[\Phi_1 y_{t-1} y_{t-\tau}'] + \mathbb{E}[\Phi_\varepsilon \varepsilon_t y_{t-\tau}'] = \Phi_1 \Gamma(\tau - 1) \ \forall \ \tau > 0 \ .$$

(c) Derive the impulse response function of y_{t+h}, $h = 0, 1, \ldots$ with respect to the vector of structural shocks ε_t. How do log wages react to a labor supply shock that occurred 3 periods before?

Solution: Using our derivations above, we have

$$y_{t+h} = \sum_{l=0}^{\infty} \Phi_1^l u_{t+h-l} = \sum_{l=0}^{\infty} \Phi_1^l \Phi_\varepsilon \varepsilon_{t+h-l} \ .$$

Therefore,

$$\frac{\partial y_{t+h}}{\partial \varepsilon_t} = \Phi_1^h \Phi_\varepsilon \ .$$

Since we are interested in the effect of the shock after 3 periods, we have $h = 3$. Also, since w_t is the first element of y_{t+h} and $\varepsilon_{b,t}$ is the second element in ε_t, we seek

$$\frac{\partial w_t}{\partial \varepsilon_{b,t}} = [\Phi_1^3 \Phi_\varepsilon]_{12} \ ,$$

i.e. the element in the first row and second column of the matrix $\Phi_1^3 \Phi_\varepsilon$

(d) Describe the identification problem in the context of this VAR.

Solution: It holds that $\Sigma = \Phi_\varepsilon \Phi_\varepsilon'$. When we estimate the reduced-form VAR, $y_t = \Phi_1 y_{t-1} + u_t$, we only obtain estimates of Φ_1 and $\mathbb{V}[u_t] = \Sigma$. This is not enough to pin down Φ_ε as Σ has $n(n+1)/2$ distinct elements, while Φ_ε has n^2. In our case, we have we 3 elements in Σ and 4 elements in Φ_ε. Therefore, we need (at least) one restriction so that we can identify all 4 elements in Φ_ε.

To see the identification problem more clearly, w.l.o.g. we can write

$$\Phi_\varepsilon = \Sigma_{tr}\Omega \ ,$$

where Σ_{tr} is the lower-triangular Cholesky factor of Σ and Ω is s.t. $\Omega\Omega' = I$. Since Σ is identified by the data, so is Σ_{tr}. However, any Ω that satisfies $\Omega\Omega' = I$ is consistent with the data (and there is a whole continuum of such matrices)!

(e) Supose we are willing to assume that, contemporaneously, hours worked are only affected by preferences, not technology. What restrictions does this assumption impose on Φ_ε? Is this enough to uniquely identify Φ_ε?

Solution: Yes. If h_t is contemporaneously only affected by $\varepsilon_{a,t}$, but not by $\varepsilon_{b,t}$, then this implies that $[\Phi_\varepsilon]_{22} = 0$. Therefore, the matrix Φ_ε has 3 non-zero elements. Since $n = 2$, we have $n(n+1)/2 = 3$ elements in Σ, which is exactly what we need to point-identify the remaining 3 elements in Φ_ε.

Coming back to our decomposition of Φ_ε into Σ_{tr} and Ω (which is a bit of an overkill when there is point-identification), it turns out that our assumption gives us a unique Ω that is consistent with the imposed restriction.

(f) Alternatively, suppose that we assume that a labor supply shock $\varepsilon_{b,t}$ moves wages and hours in opposite directions upon impact, whereas a demand shock $\varepsilon_{a,t}$ moves wages and hours in the same direction. What restrictions does this assumption impose on Φ_ε? Is this enough to uniquely identify Φ_ε?

Solution: Then we only know that Φ_ε has signs $\begin{bmatrix} + & + \\ + & - \end{bmatrix}$ (for example).

In this case we don't have point-identification but set-identification, as there is not a unique Φ_ε, i.e. unique values for each of its parameters, but a whole set of parameter-values in Φ_ε that satisfy the imposed restriction.

Now the decomposition of Φ_ε into Σ_{tr} and Ω is useful. As before (as always), we can perfectly tell Σ_{tr} from the data. In contrast to the identification assumption in the previous exercise, our assumptions here restrict the permissible set of Ωs, but do not perfectly tell us Ω: Ω now needs to satisfy $\Omega\Omega' = I$ as well as the imposed sign-restrictions on Φ_ε (which translate into sign-restrictions on Ω since $\Phi_\varepsilon = \Sigma_{tr}\Omega$).

We can think of wages and hours being determined by an interplay of labor supply and labor demand. Let $h_t = \varphi^D(w_t, y_{t-1}; \varepsilon_{a,t}, \varepsilon_{b,t})$ be the demand and $h_t = \varphi^S(w_t, y_{t-1}; \varepsilon_{a,t}, \varepsilon_{b,t})$ the supply function. They show the relationship between hours and wages (i.e. quantity and price in the labor market), whereby these functions (think of labor/supply curves) depend on current technology and preference shocks as well as past hours and wages.[1]

(h) Suppose we assume that the labor demand is only affected by the technology shock, not the preference shock, whereas labor supply is affected by both shocks:

$$h_t = \varphi^D(w_t, y_{t-1}; \varepsilon_{a,t}) \quad \text{and} \quad h_t = \varphi^S(w_t, y_{t-1}; \varepsilon_{a,t}, \varepsilon_{b,t}) \ .$$

Is this enough to uniquely identify Φ_ε?
Hint: remember that demand must equal supply at all times.

Solution: We don't get point-identification as Φ_ε still has 4 unrestricted elements, since w_t and h_t both depend on both shocks. For h_t, that's obvious when you look at the labour supply function. For w_t, you can see this by noting that if you set supply and demand equal and solve for w_t, the resulting function that determines w_t will also depend on both $\varepsilon_{a,t}$ and $\varepsilon_{b,t}$.

(i) Does your answer change if, on top of the above assumption, we assume that

$$\frac{\partial w_t}{\partial \varepsilon_{b,t}} = (\alpha - 1) \frac{\partial h_t}{\partial \varepsilon_{b,t}}$$

for some given α; conditional on α, is it possible to uniquely identify the elements of Φ_ε? If yes, show how you can solve for Φ_ε based on α and the reduced-form VAR parameters.

Solution: Under this additional assumption, we have that

$$\frac{\partial w_t}{\partial \varepsilon_{b,t}} = [\Phi_\varepsilon]_{12} = (1 - \alpha) \frac{\partial h_t}{\partial \varepsilon_{b,t}} = (1 - \alpha) \, [\Phi_\varepsilon]_{22} \ ,$$

where $[\Phi_\varepsilon]_{ij}$ is the (i,j)th element in Φ_ε. This is just a reparameterization of Φ_ε; it still has 4 parameters, which cannot be identified using only the 3 distinct elements/parameters in Σ. However, conditional on α, $\Sigma = \Phi_\varepsilon \Phi_\varepsilon'$ forms a system of 3 equations in 3 unknowns, which means that given Σ and α, we can solve for Φ_ε.

[1] In a structural economic model for the labor market, we would typically assume exogenous preference and technology processes (not i.i.d., but persistent, e.g. AR(1)s), which then, combined with the households' (workers') and firms' utility and profit maximization problems, lead to such demand and supply functions. Owing to the persistence preference and technology processes, the demand and supply functions will depend on past preference and technology shocks, which are summarized by past hours and wages in y_{t-1}.