Macroeconomics A: Review Session I

Constrained Optimization

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Outline

Table of contents

Advice for New Students

In general, as economists

- Intuition is important (diagnosis)
- So is technical ability (knowing what medicine to prescribe)
- So is presentation (bedside manner)

In the MIS

- Good teamwork matters
- Focus on improvement :)
- Take the initiative

Specifically, for this course

- Fully understand all the models and material
- Prof. Tille's office hours are very helpful!!
- It will not be easy, but there will be no surprises

Purpose

Developing a toolkit

- Today, we cover constrained optimaization in discrete time
- Many useful/important methods are not covered in class but are needed to solve models, review sessions will fill gaps

Deepening Intuition

- Some of the models we will see are not intuitive at first
- The only way to get comfortable is ‘learning-by-doing’
- We will spend some time looking at applications

Q&A

- Please send questions in 1-2 days in advance if you want detailed/correct answers

Structure of the Review Sessions

Format

- Time: Monday at 18h every week (but not the days before exams)
- Duration: As long as needed, maybe 2 hours but probably less
- Location: Room S7

Plan

- Main goal: get you ready for the midterm!
- Other goal: build confidence; clarify topics covered in class

Week	Topic
Sept. 25	constrained optimization in discrete time
Oct. 2	IS-TR model
Oct. 9	monetary policy (discretion vs. commitment)
Oct. 16	understanding phase diagrams
Oct. 23	exam prep
Nov...	TBD

Table of contents

Lagrange's Method

- A lot of economics is constrained optimization... we optimize utility subject to some budget constraint
- The basic statement of the problem becomes formulaic after a while, but it is useful to have an intuition for how constrained optimization works
- Let's start with the function $z = f(x, y)$ in $\mathbb{R}^3$
- The basis vectors $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ run 1 unit along the x and y axes
- The gradient $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$ gives the change in z as we move over the xy plane
- For a given point, we can find the change in z in any direction by multiplying $\nabla f(x, y)$ by a unit vector $\mathbf{u}$ in that direction

Note: this section follows Seth Leonard's guide

Finding the Gradient

- Let's define $f(x,y) = 8 - x^2 - y^2$ so that $\nabla f(x,y) = -2x\mathbf{i} - 2y\mathbf{j}$
- We want to find the slope of z in the direction $\mathbf{u} = \langle -\sqrt{0.5}, -\sqrt{0.5} \rangle$ at point $A = (0, -1)$
- Therefore we take $\langle -\sqrt{0.5}, -\sqrt{0.5} \rangle \cdot \langle 0, 2 \rangle = -2\sqrt{0.5}$

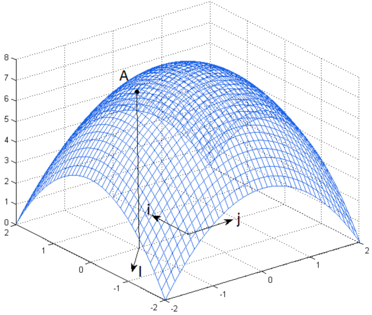

Figure: $f(x,y) = 8 - x^2 - y^2$

Finding Level Curves

- Let's set z equal to a constant k

$$f(x, y) = k$$

- If $\mathbf{u}$ is tangent to the level curve, then

$$\nabla f(x, y) \cdot \mathbf{u} = 0$$

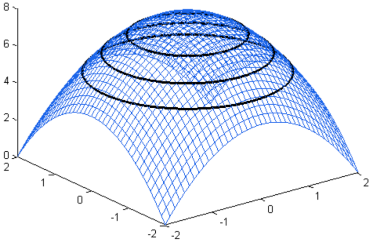

Figure: Level Curves $k = \{5, 6, 7\}$

Level Curve Tangent to the Constraint

- Let's say we have a constraint of the form $g(x, y) = 0$
- The constrained maximum (B) is the value of k tangent to $g(x, y)$

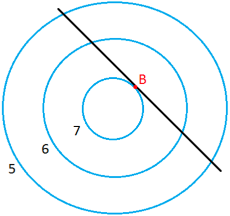

Figure: $g(x, y) = x + y - 2\sqrt{0.5}$

- If the two lines are tangent, their gradients have the same direction

$$\nabla f(x, y) = \lambda \nabla g(x, y) \tag{1}$$

- The term λ sets the magnitudes of the two gradients equal

Putting Things Together

- Recalling that $g(x, y) = 0$, we can write equation 1 as

$$\mathcal{L} = f(x, y) - \lambda g(x, y)$$

- The constrained optimum is at the point where

$$\mathcal{L}_x = \mathcal{L}_y = 0$$

Exercise: find the constrained optimum where

$$f(x, y) = 8 - x^2 - y^2 \quad \text{and} \quad g(x, y) = x + y - 2\sqrt{0.5}$$

Putting Things Together

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Exercise: find the constrained optimum (z) where

$$f(x, y) = 8 - x^2 - y^2 \quad \text{and} \quad g(x, y) = x + y - 2\sqrt{0.5}$$

(Answer here)

Applying to a One-Period Problem

Exercise: Let's solve a simple model with a constraint

$$\max_{c,\ell} \quad u(c,\ell) = \frac{c^{1-\sigma}}{1-\sigma} - \frac{\ell^{1+\eta}}{1+\eta}$$

$$\text{s.t.}$$

$$y = A\ell^{1-\alpha}$$

$$c = y$$

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(Answer here)

Table of contents

Discounting the Future

Most of the time, we make forward looking decisions and take into account both our present and future constraints

For a two-period problem, take that

$$\max_{c_t, a_t} \mathcal{L}_0 = \sum_{t=0}^{\infty} \beta^t \left[u(c_t) - \lambda g(c_t, a_{t-1})\right] \text{ where } a_t = y_t - c_t + (1 + r_{t-1}) a_{t-1}$$

Writing this out, we get

$$\max_{c_t, a_t} \quad \mathcal{L}_t = u(c_t) - \lambda g(c_t, a_{t-1}) + \beta \left[u(c_{t+1}) - \lambda g(c_{t+1}, a_t)\right] + ...$$

$$\beta^2 \left[u(c_{t+2}) - \lambda g(c_{t+2}, a_{t+1})\right] + \sum_{n=3}^{\infty} \beta^n \left[u(c_{t+n}) - \lambda g(c_{t+n}, a_{t+n-1})\right]$$

Note when $\beta < 1$, that $\lim_{t \to \infty} \beta^t = 0$

Euler Equation

Taking the previous example and setting $u(c_t) = \log(c_t)$

$$\mathcal{L}_t = \log(c_t) - \lambda_t \left(a_t - y_t + c_t - (1 + r_{t-1})a_{t-1}\right) + ...$$

$$\beta \left[\log(c_{t+1}) - \lambda_{t+1} \left(a_{t+1} - y_{t+1} + c_{t+1} - (1 + r_t)a_t\right)\right] + ...$$

Taking derivatives

$$\frac{\partial \mathcal{L}_t}{\partial c_t} = \frac{1}{c_t} - \lambda_t = 0$$
$$\frac{\partial \mathcal{L}_t}{\partial a_t} = -\lambda_t + \beta \lambda_{t+1}(1 + r_t) = 0$$

Combining terms, we get the celebrated Euler equation

$$\frac{c_{t+1}}{c_t} = \beta(1 + r_t)$$

Steady State

We can write out the Euler equation recursively so that

$$c_{t+2} = \beta c_{t+1}(1 + r_{t+1}) = \beta^2 c_t(1 + r_t)(1 + r_{t+1})$$

Setting $r = r_t = r_{t+1}$ for all t

$$c_{t+n} = \beta^n c_t(1 + r)^n$$

If $1 + r > 1/\beta$ then

$$\underset{n\to\infty}{c_{t+n}} = \infty \quad \text{(violates resource/borrowing constraint)}$$

Conversely if $1 + r < 1/\beta$

$$\underset{n\to\infty}{c_{t+n}} = 0 \quad \text{(marginal utility becomes infinite)}$$

Households would never choose this, so the only possible value for r is

$$r = \frac{1}{\beta} - 1$$