Macroeconomics B: EI060

Problem set 1

Cédric Tille

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This assignment asks you to solve a two-country model of the real exchange rate and current account. There are traded and non-traded goods, and instead of endowments we consider that output is produced using labor which is mobile across sectors.

The assignment is structured in 11 questions. The questions build on each other, so if you find one challenging you can at first takes its results as given and move to the next question.

1 Consumption allocation

We consider a model with two countries, Home and Foreign. There are n Home households and $1-n$ Foreign households, each supplying one unit of labor. n is the Home country size and the small open economy cases corresponds to n going to zero.

The consumption basket of each agent consists of a traded good that is produced in both countries, and a domestic non-traded good. The elasticity of substitution between the traded and non-traded good is equal to 1, so the basket at time t of the Home agent is:

$$C_t = \frac{1}{\gamma^\gamma (1-\gamma)^{1-\gamma}} (C_{T,t})^\gamma (C_{N,t})^{1-\gamma}$$

The traded good is the numeraire, so we set its price to 1. $P_{N.t}$ is the price of the non-traded good in the Home country.

Show that:

$$\begin{aligned}
C_{T,t} &= \gamma P_t C_t \\
C_{N,t} &= (1-\gamma) \frac{P_t}{P_{N,t}} C_t \\
P_t &= (P_{N,t})^{1-\gamma}
\end{aligned}$$

The consumption basket in the Foreign country is:

$$C_t^* = \frac{1}{\gamma^\gamma (1-\gamma)^{1-\gamma}} \left(C_{T,t}^*\right)^\gamma \left(C_{N,T}^*\right)^{1-\gamma}$$

This implies (take this as given):

$$\begin{aligned} C_{T,t}^* &= \gamma P_t^* C_t^* \\ C_{N,t}^* &= (1-\gamma) \frac{P_t^*}{P_{N,t}^*} C_t^* \\ P_t^* &= \left(P_{N,t}^*\right)^{1-\gamma} \end{aligned}$$

where $P_{N,t}^*$ is the price of the non-traded good in the Foreign country.

2 Market clearing

Output is produced using labor with decreasing returns, subject to productivity A that is sector-country specific. $L_{N,t}$ and $L_{N,t}^*$ are the quantities of labor used in the production of non-traded goods.

The production functions are:

$$\begin{aligned} Y_{T,t} &= A_{T,t} \left(n - L_{N,t}\right)^{1-\alpha} \quad ; \quad Y_{N.t} = A_{N,t} \left(L_{N,t}\right)^{1-\alpha} \\ Y_{T,t}^* &= A_{T,t}^* \left(1 - n - L_N^*\right)^{1-\alpha} \quad ; \quad Y_{N,t}^* = A_{N,t}^* \left(L_{N,t}^*\right)^{1-\alpha} \end{aligned}$$

Show that the clearing of the two markets for non-traded goods are:

$$\begin{aligned} n (1-\gamma) \left(P_{N,t}\right)^{-\gamma} C_t &= A_{N,t} \left(L_{N,t}\right)^{1-\alpha} \\ (1-n)(1-\gamma) \left(P_{N,t}^*\right)^{-\gamma} C_t^* &= A_{N,t}^* \left(L_{N,t}^*\right)^{1-\alpha} \end{aligned}$$

Similarly, show that the clearing for the trade good market is:

$$n\gamma \left(P_{N,t}\right)^{1-\gamma} C_t + (1-n)\gamma \left(P_{N,t}^*\right)^{1-\gamma} C_t^* = A_{T,t} \left(n - L_{N,t}\right)^{1-\alpha} + A_{T,t}^* \left(1 - n - L_N^*\right)^{1-\alpha}$$

3 Intertemporal allocation

The Home agent maximizes an intertemporal log utility of consumption, with a discount factor that can vary through time:

$$U_t = \sum_{s=0}^{\infty} \left(\beta_{H,t+s}\right)^s \ln\left(C_{t+s}\right)$$

She can invest in a bond denominated in the traded good, with a return r. The income is the

GDP of the country, so that the budget constraint is:

$$nP_tC_t + nB_{t+1} = A_{T,t}\left(n - L_{N,t}\right)^{1-\alpha} + P_{N,t}A_{N,t}\left(L_{N,t}\right)^{1-\alpha} + n\left(1+r_t\right)B_t$$

where B_{t+1} is the quantity of bonds purchased, per household.

Show that the dynamics of the consumption basket are:

$$C_{t+1} = \beta_{H,t+1}\left(1+r^C_{t+1}\right)C_t$$

where the real interest rate, in terms of consumption basket, is:

$$1+r^C_{t+1} = \left(1+r_{t+1}\right)\left(\frac{P_{N,t}}{P_{N,t+1}}\right)^{1-\gamma}$$

Show that the dynamics of the traded good consumption are:

$$C_{T,t+1} = \beta_{H,t+1}\left(1+r_{t+1}\right)C_{T,t}$$

The utility, and the budget constraint for the Foreign country are, using the fact that assets of the Home country are liabilities of the Foreign country:

$$U^*_t = \sum_{s=0}^{\infty}\left(\beta_{F,t+s}\right)^s \ln\left(C^*_{t+s}\right)$$

$$\left(1-n\right)P^*_tC^*_t - nB_{t+1} = A^*_{T,t}\left(1-n-L^*_{N,t}\right)^{1-\alpha} + P^*_{N,t}A^*_{N,t}\left(L^*_{N,t}\right)^{1-\alpha} - n\left(1+r_t\right)B_t$$

This implies (take this as given):

$$\begin{aligned}
C^*_{t+1} &= \beta_{F,t+1}\left(1+r^{*C}_{t+1}\right)C^*_t \\
1+r^{*C}_{t+1} &= \left(1+r_{t+1}\right)\left(\frac{P^*_{N,t}}{P^*_{N,t+1}}\right)^{1-\gamma} \\
C^*_{T,t+1} &= \beta_{F,t+1}\left(1+r_{t+1}\right)C^*_{T,t}
\end{aligned}$$

4 Labor allocation

Show that the Home consumer allocates labor such that:

$$A_{T,t}\left(n-L_{N,t}\right)^{-\alpha} = P_{N,t}A_{N,t}\left(L_{N,t}\right)^{-\alpha}$$

The Foreign labor allocation is (take this as given):

$$A^*_{T,t}\left(1-n-L^*_{N,t}\right)^{-\alpha} = P^*_{N,t}A^*_{N,t}\left(L^*_{N,t}\right)^{-\alpha}$$

5 Resource constraints

Show that the resource constraints for the two countries are:

$$
\begin{aligned}
n\gamma \left(P_{N,t}\right)^{1-\gamma} C_t + nB_{t+1} &= A_{T,t}\left(n - L_{N,t}\right)^{1-\alpha} + n\left(1+r_t\right)B_t \\
(1-n)\gamma \left(P^*_{N,t}\right)^{1-\gamma} C^*_t - nB_{t+1} &= A^*_{T,t}\left(1 - n - L^*_{N,t}\right)^{1-\alpha} - n\left(1+r_t\right)B_t
\end{aligned}
$$

Show that the real exchange rate (the ratio of Foreign price index to Home one) is:

$$
Q_t = \left(\frac{P^*_{N,t}}{P_{N,t}}\right)^{1-\gamma}
$$

6 Steady state

The model is summarized by the 2 clearing of non-traded good markets, 2 resource constraints, 2 Euler conditions, 2 labor allocation conditions (note that the clearing of the goods market is redundant, as it corresponds to the sum of the resource constraints). We recall them for brevity:

$$
\begin{aligned}
n(1-\gamma)\left(P_{N,t}\right)^{-\gamma} C_t &= A_{N,t}\left(L_{N,t}\right)^{1-\alpha} \\
(1-n)(1-\gamma)\left(P^*_{N,t}\right)^{-\gamma} C^*_t &= A^*_{N,t}\left(L^*_{N,t}\right)^{1-\alpha} \\
n\gamma \left(P_{N,t}\right)^{1-\gamma} C_t + nB_{t+1} &= A_{T,t}\left(n - L_{N,t}\right)^{1-\alpha} + n\left(1+r_t\right)B_t \\
(1-n)\gamma \left(P^*_{N,t}\right)^{1-\gamma} C^*_t - nB_{t+1} &= A^*_{T,t}\left(1 - n - L^*_{N,t}\right)^{1-\alpha} - n\left(1+r_t\right)B_t \\
C_{t+1} &= C_t \beta_{H,t+1}\left(1+r_{t+1}\right)\left(\frac{P_{N,t}}{P_{N,t+1}}\right)^{1-\gamma} \\
C^*_{t+1} &= C^*_t \beta_{F,t+1}\left(1+r_{t+1}\right)\left(\frac{P^*_{N,t}}{P^*_{N,t+1}}\right)^{1-\gamma} \\
A_{T,t}\left(n - L_{N,t}\right)^{-\alpha} &= P_{N,t}A_{N,t}\left(L_{N,t}\right)^{-\alpha} \\
A^*_{T,t}\left(1 - n - L^*_{N,t}\right)^{-\alpha} &= P^*_{N,t}A^*_{N,t}\left(L^*_{N,t}\right)^{-\alpha}
\end{aligned}
$$

Consider a symmetric steady state where there are no cross-country assets ($B_0 = 0$) and the discount factor is β_0. The productivity levels are given by:

$$
\begin{aligned}
A_{N,0} &= A_{T,0}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha} \\
A^*_{T,0} &= A_{T,0}\left(\frac{1-n}{n}\right)^{\alpha} \\
A^*_{N,0} &= A_{T,0}\left(\frac{1-n}{n}\right)^{\alpha}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha}
\end{aligned}
$$

Show that:

$$
1 = \beta_0\left(1+r_0\right)
$$

$$
\begin{aligned}
P_{N,0} &= P^*_{N,0} = 1 \\
L_{N,0} &= n\,(1-\gamma) \\
L^*_{N,0} &= (1-n)\,(1-\gamma) \\
C_0 &= (A_{T,0})^{\gamma}\,(A_{N,0})^{1-\gamma}\left[n\,(\gamma)^{\gamma}\,(1-\gamma)^{1-\gamma}\right]^{-\alpha} \\
C^*_0 &= \left(A^*_{T,0}\right)^{\gamma}\left(A^*_{N,0}\right)^{1-\gamma}\left[(1-n)\,(\gamma)^{\gamma}\,(1-\gamma)^{1-\gamma}\right]^{-\alpha}
\end{aligned}
$$

Note that

$$
\frac{C_0}{C^*_0} = \frac{(A_{T,0})^{\gamma}\left(A_{T,0}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha}\right)^{1-\gamma}(n)^{-\alpha}}{\left(A_{T,0}\left(\frac{1-n}{n}\right)^{\alpha}\right)^{\gamma}\left(A_{T,0}\left(\frac{1-n}{n}\right)^{\alpha}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha}\right)^{1-\gamma}(1-n)^{-\alpha}} = 1
$$

7 Log linear approximation

We take log linear approximations around the steady state. We denote log deviations by hatted variables. For instance: $\widehat{L_{N,t}} = (L_{N,t} - L_{N,0})/L_{N,0}$. We also define $\widehat{B_t} = B_t/(\gamma C_0)$ and $\widehat{r_t} = r_t - \frac{1-\beta_0}{\beta_0}$.

Show that the clearing of the non-traded good market, in the two countries, are approximated as:

$$
\begin{aligned}
-\gamma\widehat{P_{N,t}} + \widehat{C_t} &= \widehat{A_{N,t}} + (1-\alpha)\,\widehat{L_{N,t}} \\
-\gamma\widehat{P^*_{N,t}} + \widehat{C^*_t} &= \widehat{A^*_{N,t}} + (1-\alpha)\,\widehat{L^*_{N,t}}
\end{aligned}
$$

Show that the resource constraints are approximated as:

$$
\begin{aligned}
(1-\gamma)\,\widehat{P_{N,t}} + \widehat{C_t} + \widehat{B_{t+1}} &= \widehat{A_{T,t}} - (1-\alpha)\,\frac{1-\gamma}{\gamma}\widehat{L_{N,t}} + \frac{1}{\beta_0}\widehat{B_t} \\
(1-\gamma)\,\widehat{P^*_{N,t}} + \widehat{C^*_t} - \frac{n}{1-n}\widehat{B_{t+1}} &= \widehat{A^*_{T,t}} - (1-\alpha)\,\frac{1-\gamma}{\gamma}\widehat{L^*_{N,t}} - \frac{1}{\beta_0}\frac{n}{1-n}\widehat{B_t}
\end{aligned}
$$

Show that the Euler conditions are approximated as:

$$
\begin{aligned}
\widehat{C_{t+1}} &= \widehat{C_t} + (1-\gamma)\left(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}\right) + \widehat{\beta_{H,t+1}} + \beta_0\widehat{r_{t+1}} \\
\widehat{C^*_{t+1}} &= \widehat{C^*_t} + (1-\gamma)\left(\widehat{P^*_{N,t}} - \widehat{P^*_{N,t+1}}\right) + \widehat{\beta_{F,t+1}} + \beta_0\widehat{r_{t+1}}
\end{aligned}
$$

Show that the labor allocations are approximated as:

$$
\begin{aligned}
\widehat{A_{T,t}} + \frac{\alpha}{\gamma}\widehat{L_{N,t}} &= \widehat{P_{N,t}} + \widehat{A_{N,t}} \\
\widehat{A^*_{T,t}} + \frac{\alpha}{\gamma}\widehat{L^*_{N,t}} &= \widehat{P^*_{N,t}} + \widehat{A^*_{N,t}}
\end{aligned}
$$

Show that the real interest rates, in terms of consumption baskets, are:

$$\begin{aligned}
\widehat{r_{t+1}^{C}} &= (1-\gamma)\frac{1}{\beta_0}\left(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}\right) + \widehat{r_{t+1}} \\
\widehat{r_{t+1}^{C*}} &= (1-\gamma)\frac{1}{\beta_0}\left(\widehat{P_{N,t}^{*}} - \widehat{P_{N,t+1}^{*}}\right) + \widehat{r_{t+1}}
\end{aligned}$$

8 Worldwide solution

We define worldwide variables as weighted averages across the two countries. For instance $\widehat{C_t^W} = n\widehat{C_t} + (1-n)\widehat{C_t^*}$. Show that the market clearing conditions for the non-traded good, the resource constraints, the Euler conditions, the labor allocations, and the real consumption interest rates imply:

$$\begin{aligned}
-\gamma\widehat{P_{N,t}^W} + \widehat{C_t^W} &= \widehat{A_{N,t}^W} + (1-\alpha)\widehat{L_{N,t}^W} \\
(1-\gamma)\widehat{P_{N,t}^W} + \widehat{C_t^W} &= \widehat{A_{T,t}^W} - (1-\alpha)\frac{1-\gamma}{\gamma}\widehat{L_{N,t}^W} \\
\widehat{C_{t+1}^W} &= \widehat{C_t^W} + (1-\gamma)\left(\widehat{P_{N,t}^W} - \widehat{P_{N,t+1}^W}\right) + \widehat{\beta_{t+1}^W} + \beta_0\widehat{r_{t+1}} \\
\widehat{A_{T,t}^W} + \frac{\alpha}{\gamma}\widehat{L_{N,t}^W} &= \widehat{P_{N,t}^W} + \widehat{A_{N,t}^W} \\
\beta_0\widehat{r_{t+1}^{CW}} &= (1-\gamma)\left(\widehat{P_{N,t}^W} - \widehat{P_{N,t+1}^W}\right) + \beta_0\widehat{r_{t+1}}
\end{aligned}$$

Show that:

$$\begin{aligned}
\widehat{L_{N,t}^W} &= 0 \\
\widehat{P_{N,t}^W} &= \widehat{A_{T,t}^W} - \widehat{A_{N,t}^W} \\
\widehat{C_t^W} &= \gamma\widehat{A_{T,t}^W} + (1-\gamma)\widehat{A_{N,t}^W} \\
\beta_0\widehat{r_{t+1}} &= -\widehat{\beta_{t+1}^W} + \left(\widehat{A_{T,t+1}^W} - \widehat{A_{T,t}^W}\right)
\end{aligned}$$

Explains what drives the relative price of goods, consumption, and the real interest rate.

9 Cross country differences

We now express the relations in terms of difference between the Home and Foreign countries.

Show that the market clearing conditions for the non-traded good, the resource constraints, the Euler conditions, the labor allocations, and the real consumption interest rates imply:

$$\begin{aligned}
\frac{\gamma}{1-\gamma}\widehat{Q_t} + \left(\widehat{C_t} - \widehat{C_t^*}\right) &= \left(\widehat{A_{N,t}} - \widehat{A_{N,t}^*}\right) + (1-\alpha)\left(\widehat{L_{N,t}} - \widehat{L_{N,t}^*}\right) \\
-\widehat{Q_t} + \left(\widehat{C_t} - \widehat{C_t^*}\right) + \frac{\widehat{B_{t+1}}}{1-n} &= \left(\widehat{A_{T,t}} - \widehat{A_{T,t}^*}\right) - (1-\alpha)\frac{1-\gamma}{\gamma}\left(\widehat{L_{N,t}} - \widehat{L_{N,t}^*}\right) + \frac{1}{\beta_0}\frac{\widehat{B_t}}{1-n}
\end{aligned}$$

$$
\begin{aligned}
\left(\widehat{C_{t+1}}-\widehat{C_{t+1}^*}\right) &= \left(\widehat{C_t}-\widehat{C_t^*}\right)+\left(\widehat{Q_{t+1}}-\widehat{Q_t}\right)+\left(\widehat{\beta_{H,t+1}}-\widehat{\beta_{F,t+1}}\right)\\
\left(\widehat{A_{T,t}}-\widehat{A_{T,t}^*}\right)+\frac{\alpha}{\gamma}\left(\widehat{L_{N,t}}-\widehat{L_{N,t}^*}\right) &= -\frac{1}{1-\gamma}\widehat{Q_t}+\left(\widehat{A_{N,t}}-\widehat{A_{N,t}^*}\right)\\
\beta_0\left(\widehat{r_{t+1}^C}-\widehat{r_{t+1}^{C*}}\right) &= \left(\widehat{Q_{t+1}}-\widehat{Q_t}\right)
\end{aligned}
$$

where $\widehat{Q_t}=-(1-\gamma)\left(\widehat{P_{N,t}}-\widehat{P_{N,t}^*}\right)$ is the real exchange rate.

10 Long run allocation

Consider that from period $t+1$ onward we are in a new steady state, where discount factors have come back to the steady state ($\widehat{\beta_{H,t+2}}=\widehat{\beta_{F,t+2}}=0$).

Take the initial cross-country asset for that long run, $\widehat{B_{t+1}}$, as given. Show that:

$$
\begin{aligned}
\widehat{Q_{t+1}} &= -(1-\gamma)\left[\left(\widehat{A_{T,t+1}}-\widehat{A_{T,t+1}^*}\right)-\left(\widehat{A_{N,t+1}}-\widehat{A_{N,t+1}^*}\right)\right]-\alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}\\
\widehat{L_{N,t+1}}-\widehat{L_{N,t+1}^*} &= \gamma\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}\\
\widehat{C_{t+1}}-\widehat{C_{t+1}^*} &= \gamma\left(\widehat{A_{T,t+1}}-\widehat{A_{T,t+1}^*}\right)+(1-\gamma)\left(\widehat{A_{N,t+1}}-\widehat{A_{N,t+1}^*}\right)+\gamma\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}
\end{aligned}
$$

What is the impact of the Home country being wealthier than the Foreign one ($\widehat{B_{t+1}}>0$)?

What is the long run impact of productivity (aside from any indirect impact through $\widehat{B_{t+1}}$)?

11 Short run allocation

We now turn to the allocation in period t. Consider that we start in a situation with no cross-country assets ($\widehat{B_t}=0$).

Show that (hint: start solving for $\widehat{L_{N,t}}-\widehat{L_{N,t}^*}$ and $\widehat{Q_t}$ as functions of $\widehat{C_t}-\widehat{C_t^*}$ using the labor allocation and the market clearing for the non-traded good, then use this and the resource constraint to solve for $\widehat{C_t}-\widehat{C_t^*}$ as function of $\widehat{B_{t+1}}$, and finally use the Euler condition and the results above for the long-run variables):

$$
\frac{\widehat{B_{t+1}}}{1-n} = \frac{\beta_0}{\gamma+\alpha(1-\gamma)}\left[\left(\widehat{\beta_{H,t+1}}-\widehat{\beta_{F,t+1}}\right)-\left(\widehat{A_{T,t+1}}-\widehat{A_{T,t+1}^*}\right)+\left(\widehat{A_{T,t}}-\widehat{A_{T,t}^*}\right)\right]
$$

Then, show that:

$$
\begin{aligned}
\widehat{C_t}-\widehat{C_t^*} &= \gamma\left(\widehat{A_{T,t}}-\widehat{A_{T,t}^*}\right)+(1-\gamma)\left(\widehat{A_{N,t}}-\widehat{A_{N,t}^*}\right)\\
&\quad -\frac{\gamma\beta_0}{\gamma+\alpha(1-\gamma)}\left[\left(\widehat{\beta_{H,t+1}}-\widehat{\beta_{F,t+1}}\right)-\left(\widehat{A_{T,t+1}}-\widehat{A_{T,t+1}^*}\right)+\left(\widehat{A_{T,t}}-\widehat{A_{T,t}^*}\right)\right]\\
\widehat{Q_t} &= -(1-\gamma)\left(\widehat{A_{T,t}}-\widehat{A_{T,t}^*}\right)+(1-\gamma)\left(\widehat{A_{N,t}}-\widehat{A_{N,t}^*}\right)
\end{aligned}
$$

$$
\begin{aligned}
& & & +\frac{(1-\gamma)\,\alpha\beta_0}{\gamma+\alpha(1-\gamma)}\left[\left(\widehat{\beta_{H,t+1}}-\widehat{\beta_{F,t+1}}\right)-\left(\widehat{A_{T,t+1}}-\widehat{A^*_{T,t+1}}\right)+\left(\widehat{A_{T,t}}-\widehat{A^*_{T,t}}\right)\right] \\
& \widehat{L_{N,t}}-\widehat{L^*_{N,t}} &=& -\frac{\gamma\beta_0}{\gamma+\alpha(1-\gamma)}\left[\left(\widehat{\beta_{H,t+1}}-\widehat{\beta_{F,t+1}}\right)-\left(\widehat{A_{T,t+1}}-\widehat{A^*_{T,t+1}}\right)+\left(\widehat{A_{T,t}}-\widehat{A^*_{T,t}}\right)\right] \\
& \beta_0\left(\widehat{r^C_{t+1}}-\widehat{r^{C*}_{t+1}}\right) &=& -(1-\gamma)\left[\left(\widehat{A_{T,t+1}}-\widehat{A^*_{T,t+1}}\right)-\left(\widehat{A_{N,t+1}}-\widehat{A^*_{N,t+1}}\right)\right] \\
& & & +(1-\gamma)\left[\left(\widehat{A_{T,t}}-\widehat{A^*_{T,t}}\right)-\left(\widehat{A_{N,t}}-\widehat{A^*_{N,t}}\right)\right] \\
& & & -\frac{(1-\gamma)\,\alpha}{\gamma+\alpha(1-\gamma)}\left[\left(\widehat{\beta_{H,t+1}}-\widehat{\beta_{F,t+1}}\right)-\left(\widehat{A_{T,t+1}}-\widehat{A^*_{T,t+1}}\right)+\left(\widehat{A_{T,t}}-\widehat{A^*_{T,t}}\right)\right]
\end{aligned}
$$

What is the impact of a temporary increase in Home patience ($\widehat{\beta_{H,t+1}}-\widehat{\beta_{F,t+1}}>0$)?

What is the impact of a temporary shock in Home traded productivity ($\widehat{A_{T,t}}>0$)? What if the shock is in non-traded productivity ($\widehat{A_{N,t}}>0$)?

What is the impact of permanent productivity shocks?