Macroeconomics A; EI060

Technical appendix: intertemporal appraoch to the current account

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1 Two period model with endowment

1.1 Consumption choice

A representative consumers picks consumption in period 1 and 2 to maximize:

$$u(C_1) + \beta u(C_2)$$

We consider a real economy (all prices are equal to 1) with only one good. The optimization is subject to the budget constraint:

$$C_1 + B_2 = Y_1 \qquad ; \qquad C_2 = (1+r) B_2 + Y_2$$

where B denotes the assets on the rest of the world, which are zero in a closed economy, and r is the real interest rate earned on them. Combining them gives the intertemporal constraint. The present value of consumption is equal to the present value of income Ω:

$$C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r} = \Omega$$

The Lagrangian is:

$$Z = u(C_1) + \beta u(C_2) + \lambda \left[\Omega - C_1 - \frac{C_2}{1+r} \right]$$

The first order conditions with respect to the two consumptions are:

$$0 = u'(C_1) - \lambda \qquad ; \qquad 0 = \beta u'(C_2) \frac{\lambda}{1+r}$$

Combining them gives the Euler condition. The marginal rate of substitution is equal to the interest rate:

$$u'(C_1) = \beta(1+r)\,u'(C_2) \qquad \Rightarrow \qquad \frac{\beta u'(C_2)}{u'(C_1)} = \frac{1}{1+r}$$

1.2 Autarky

If the country cannot lend and borrow with the rest of the world, $B_2 = 0$ and $C_1 = Y_1$, $C_2 = Y_2$. The endogenous variable is then the autarky interest rate:

$$1 + r^A = \frac{u'(Y_1)}{\beta u'(Y_2)}$$

It increases with Y_2 and decreases with Y_1.

1.3 Current account

The current account is the change in a country's net foreign claims, B, which corresponds to its savings (savings - investment more exactly):

$$\begin{aligned} CA_1 &= B_2 - B_1 = B_2 \\ CA_2 &= B_3 - B_2 = -B_2 \end{aligned}$$

as initial and final assets are $B_1 = B_3 = 0$.

1.4 Specific utility

Consider a constant relative risk aversion utility (the log utility is $\sigma = 1$):

$$u(C) = \frac{C^{1-\sigma}}{1-\sigma}$$

The Euler condition implies:

$$\begin{aligned} (C_1)^{-\sigma} &= \beta(1+r)(C_2)^{-\sigma} \\ C_2 &= [\beta(1+r)]^{\frac{1}{\sigma}} C_1 \end{aligned}$$

Using this in the intertemporal budget constraint, we get:

$$\begin{aligned} C_1 + \frac{C_2}{1+r} &= Y_1 + \frac{Y_2}{1+r} \\ C_1 + \frac{[\beta(1+r)]^{\frac{1}{\sigma}} C_1}{1+r} &= Y_1 + \frac{Y_2}{1+r} \\ C_1 &= \frac{1}{1+\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1}}\left[Y_1 + \frac{Y_2}{1+r}\right] \end{aligned}$$

The second period consumption and the current account are:

$$\begin{aligned}
C_2 &= [\beta(1+r)]^{\frac{1}{\sigma}} C_1 \\
&= \frac{\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}}}{1+\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1}}\left[Y_1+\frac{Y_2}{1+r}\right] \\
B_2 &= Y_1 - C_1 \\
&= \frac{1}{1+\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1}}\left[\beta^{\frac{1}{\sigma}}(1+r)^{\frac{1}{\sigma}-1}Y_1-\frac{Y_2}{1+r}\right]
\end{aligned}$$

2 Two period model with endogenous output

2.1 Technology and constraint

Consider that output is produced using capital (we set labor equal to 1 for simplicity) with decreasing returns:

$$Y_t = A_t F(K_t) \qquad ; \qquad F' > 0, \;\; F'' < 0$$

Initial capital K_1 is given by investment in the past, and we leave no capital at the end K_3.

Capital accumulates subject to depreciation δ and investment:

$$K_{t+1} = (1-\delta) K_t + I_t$$

The budget constraint takes account that the agent can now save in capital and the bond:

$$\begin{aligned}
B_{t+1} &= (1+r) B_t + Y_t - C_t - I_t \\
B_{t+1} &= (1+r) B_t + A_t F(K_t) + (1-\delta) K_t - C_t - K_{t+1}
\end{aligned}$$

The constraints for the two periods are:

$$\begin{aligned}
B_2 &= A_1 F(K_1) + (1-\delta) K_1 - C_1 - K_2 \\
0 &= (1+r) B_2 + A_2 F(K_2) + (1-\delta) K_2 - C_2
\end{aligned}$$

The intertemporal constraint is:

$$\begin{aligned}
A_1 F(K_1) + (1-\delta) K_1 - C_1 - K_2 &= -\frac{A_2 F(K_2) + (1-\delta) K_2}{1+r} + \frac{C_2}{1+r} \\
C_1 + \frac{C_2}{1+r} &= \frac{A_2 F(K_2) + (1-\delta) K_2}{1+r} \\
&\quad + A_1 F(K_1) + (1-\delta) K_1 - K_2
\end{aligned}$$

2.2 Production possibility frontier

In autarky, the budget constraint implies:

$$\begin{aligned}
C_2 &= A_2F(K_2) + (1-\delta)K_2 \\
C_2 &= A_2F(A_1F(K_1) + (1-\delta)K_1 - C_1) + (1-\delta)[A_1F(K_1) + (1-\delta)K_1 - C_1] \\
C_2 &= G(C_1)
\end{aligned}$$

This is a negatively-sloped concave relation between the two consumptions:

$$\begin{aligned}
\frac{\partial G(C_1)}{\partial C_1} &= -A_2F'(A_1F(K_1) + (1-\delta)K_1 - C_1) - (1-\delta) \\
&= -A_2F'(K_2) - (1-\delta) < 0 \\
\frac{\partial^2 G(C_1)}{(\partial C_1)^2} &= A_2F''(A_1F(K_1) + (1-\delta)K_1 - C_1) \\
&= A_2F''(K_2) < 0
\end{aligned}$$

2.3 Optimization

Using the constraints, the utility is:

$$u(A_1F(K_1) + (1-\delta)K_1 - B_2 - K_2) + \beta u((1+r)B_2 + A_2F(K_2) + (1-\delta)K_2)$$

The first-order condition with respect to B_2 and K_2 are:

$$\begin{aligned}
0 &= -u'(C_1) + \beta(1+r)u'(C_2) \\
0 &= -u'(C_1) + \beta(A_2F'(K_2) + (1-\delta))u'(C_2)
\end{aligned}$$

which gives the solution for capital:

$$A_2F'(K_2) = r + \delta$$

3 Infinite horizon model

3.1 Intertemporal constraint

The representative consumer now maximizes the intertemporal utility:

$$U_t = \sum_{s=t}^{\infty} \beta^{s-t} u(C_s)$$

The economy is a small open country, and the world interest rate is set at r. Output is produced using capital $Y = AF(K)$.

The flow budget constraint gives the current account ($CA_t = B_{t+1} - B_t$):

$$\begin{aligned} C_t + I_t + B_{t+1} &= (1+r) B_t + Y_t \\ CA_t &= rB_t + Y_t - C_t - I_t \end{aligned}$$

We iterate the constraint:

$$\begin{aligned} B_t &= \frac{C_t + I_t - Y_t}{1+r} + \frac{B_{t+1}}{1+r} \\ B_t &= \frac{C_t + I_t - Y_t}{1+r} + \frac{1}{1+r}\left(\frac{C_{t+1} + I_{t+1} - Y_{t+1}}{1+r} + \frac{B_{t+2}}{1+r}\right) \\ B_t &= \sum_{s=t}^{t+T} \frac{C_s + I_s - Y_s}{(1+r)^{s-t+1}} + \frac{B_{t+T+1}}{(1+r)^{T+1}} \end{aligned}$$

The transversality condition states that $\lim_{T\to\infty} B_{t+T+1}/(1+r)^{T+1} = 0$. The present value of consumption and investment is then equal to the present value of income plus the return on initial assets (including principal):

$$\sum_{s=t}^{\infty} \frac{C_s + I_s}{(1+r)^{s-t}} = (1+r) B_t + \sum_{s=t}^{\infty} \frac{Y_s}{(1+r)^{s-t}}$$

3.2 Consumption optimization

The Lagrangian is:

$$Z_t = \sum_{s=t}^{\infty} \beta^{s-t} u(C_s) + \lambda_t \left[(1+r) B_t + \sum_{s=t}^{\infty} \frac{Y_s}{(1+r)^{s-t}} - \sum_{s=t}^{\infty} \frac{C_s + I_s}{(1+r)^{s-t}}\right]$$

The first order conditions with respect toC_t and C_{t+1} are:

$$0 = u'(C_t) - \lambda_t \qquad ; \qquad 0 = \beta u'(C_{t+1}) \frac{\lambda_t}{1+r}$$

which again gives the Euler condition $u'(C_t) = \beta(1+r) u'(C_{t+1})$. Note that if $\beta(1+r) = 1$ consumption is constant and:

$$\begin{aligned} C_t \sum_{s=t}^{\infty} \frac{1}{(1+r)^{s-t}} &= (1+r) B_t + \sum_{s=t}^{\infty} \frac{Y_s - I_s}{(1+r)^{s-t}} \\ C_t \sum_{y=0}^{\infty} \frac{1}{(1+r)^{y}} &= (1+r) B_t + \sum_{s=t}^{\infty} \frac{Y_s - I_s}{(1+r)^{s-t}} \\ C_t \frac{1}{1 - \frac{1}{1+r}} &= (1+r) B_t + \sum_{s=t}^{\infty} \frac{Y_s - I_s}{(1+r)^{s-t}} \\ C_t &= rB_t + \frac{r}{1+r} \sum_{s=t}^{\infty} \frac{Y_s - I_s}{(1+r)^{s-t}} \end{aligned}$$

3.3 Permanent level and deviations

The permanent level of a variable X, denoted by $\tilde{X}$, is the constant value that gives the same net present value:

$$\sum_{s=t}^{\infty} \frac{\tilde{X}_t}{(1+r)^{s-t}} = \sum_{s=t}^{\infty} \frac{X_s}{(1+r)^{s-t}} \qquad \Rightarrow \qquad \tilde{X}_t = \frac{r}{1+r} \sum_{s=t}^{\infty} \frac{X_s}{(1+r)^{s-t}}$$

The current account is (assuming $\beta(1+r) = 1$ for simplicity):

$$\begin{aligned} CA_t &= rB_t + Y_t - C_t - I_t \\ CA_t &= rB_t + Y_t - I_t - rB_t - \frac{r}{1+r} \sum_{s=t}^{\infty} \frac{Y_s - I_s}{(1+r)^{s-t}} \\ CA_t &= \left(Y_t - \tilde{Y}_t\right) - \left(I_t - \tilde{I}_t\right) \end{aligned}$$

3.4 Uncertainty (from Obstfeld-Rogoff book)

The consumer maximizes the expected utility:

$$U_t = \sum_{s=t}^{\infty} \beta^{s-t} E_t \left[u\left(C_s\right)\right]$$

Denote a state of nature at time s by k, s, of probability $\pi_{k,s}$. The Lagrangian is now expressed with the flow budget constraints, by state (we consider for simplicity that the output is an endowment):

$$\begin{aligned} Z_t = & \sum_{s=t}^{\infty} \beta^{s-t} \sum_{k,s} \pi_{k,s} u\left(C_{k,s}\right) \\ & + \varphi_t \left[(1+r) B_t + Y_t - C_t - B_{t+1}\right] \\ & + \sum_{k,t+1} \varphi_{k,t+1} \left[(1+r) B_{t+1} + Y_{k,t+1} - C_{k,t+1} - B_{k,t+2}\right] \\ & + ... \end{aligned}$$

The first order conditions with respect toC_t, $C_{k,t+1}$ and B_{t+1} are:

$$\begin{aligned} 0 &= u'\left(C_t\right) - \varphi_t \\ 0 &= \beta \pi_{k,t+1} u'\left(C_{k,t+1}\right) - \varphi_{k,t+1} \\ 0 &= -\varphi_t + (1+r) \sum_{k,t+1} \varphi_{k,t+1} \end{aligned}$$

Combining these, we get the Euler condition:

$$\begin{aligned} u'\left(C_t\right) &= \beta(1+r) \sum_{k,t+1} \pi_{k,t+1} u'\left(C_{k,t+1}\right) \\ u'\left(C_t\right) &= \beta(1+r) E_t \left[u'\left(C_{t+1}\right)\right] \end{aligned}$$

For simplicity, we take a linear-quadratic utility, $(u(C) = C - (a/2) C^2$, and assume $\beta(1+r)$, as the Euler then condition implies that consumption follows a random walk: $E_t C_{s+1} = C_t$.

$$\begin{aligned}
u'(C_t) &= \beta(1+r) \sum_{k,t+1} \pi_{k,t+1} u'(C_{k,t+1}) \\
1 - aC_t &= \beta(1+r)\left[1 - aE_t C_{t+1}\right] \\
C_t &= E_t C_{t+1}
\end{aligned}$$

We can iterate forward the budget constraint, and use this random walk result to get:

$$C_t = \frac{1-\beta}{\beta} B_t + (1-\beta) \sum_{s=t}^{\infty} (\beta)^{s-t} E_t(Y_s)$$

Consider that output follows an AR1 process around a mean $\bar{Y}$:

$$Y_t - \bar{Y} = \rho\left(Y_{t-1} - \bar{Y}\right) + \epsilon_t$$

The consumption is then given by:

$$\begin{aligned}
C_t &= \frac{1-\beta}{\beta} B_t + (1-\beta) \sum_{s=t}^{\infty} (\beta)^{s-t} \left(\bar{Y} + \rho^{s-t}\left(Y_t - \bar{Y}\right)\right) \\
C_t &= \frac{1-\beta}{\beta} B_t + (1-\beta) \bar{Y} \sum_{s=t}^{\infty} (\beta)^{s-t} + (1-\beta)\left(Y_t - \bar{Y}\right) \sum_{s=t}^{\infty} (\beta\rho)^{s-t} \\
C_t &= \frac{1-\beta}{\beta} B_t + \bar{Y} + \frac{1-\beta}{1-\beta\rho}\left(Y_t - \bar{Y}\right)
\end{aligned}$$

The current account is written as:

$$\begin{aligned}
CA_t &= rB_t + Y_t - C_t \\
CA_t &= \frac{1-\beta}{\beta} B_t + Y_t - \frac{1-\beta}{\beta} B_t - \bar{Y} - \frac{1-\beta}{1-\beta\rho}\left(Y_t - \bar{Y}\right) \\
CA_t &= \left(Y_t - \bar{Y}\right) - \frac{1-\beta}{1-\beta\rho}\left(Y_t - \bar{Y}\right) \\
CA_t &= \frac{\beta(1-\rho)}{1-\beta\rho}\left(Y_t - \bar{Y}\right) \\
CA_t &= \frac{\beta(1-\rho)}{1-\beta\rho}\left(\rho\left(Y_{t-1} - \bar{Y}\right) + \epsilon_t\right)
\end{aligned}$$

4 Government spending, and solvency

4.1 Government spending

The government purchases an amount G_t of the goods, and the consumer is taxed an amount T_t. The consumer's budget constraints reflects the after tax income:

$$C_1 + B_2^{private} = Y_1 - T_1 \qquad ; \qquad C_2 = (1+r)\, B_2^{private} + Y_2 - T_2$$

The government budget constraints are:

$$G_1 + B_2^{public} = T_1 \qquad ; \qquad G_2 = (1+r)\, B_2^{public} + T_2$$

Adding up the two constraints, taxes cancel out and only government spending enters, as does private consumption:

$$C_1 + G_1 + \left(B_2^{private} + B_2^{public}\right) = Y_1 \qquad ; \qquad C_2 + G_2 = (1+r)\left(B_2^{private} + B_2^{public}\right) + Y_2$$

In terms of the intertenporal constraint:

$$C_1 + G_1 + \frac{C_2 + G_2}{1+r} = Y_1 + \frac{Y_2}{1+r} = \Omega$$

In an infinite horizon, we can follow the same steps as above to get the intertemporal constraint:

$$\sum_{s=t}^{\infty} \frac{C_s + G_s + I_s}{(1+r)^{s-t}} = (1+r)\, B_t^{total} + \sum_{s=t}^{\infty} \frac{Y_s}{(1+r)^{s-t}}$$

4.2 Solvency

4.2.1 Minimal assets

Using the intertemporal budget constraint, a positive net asset position $B_t^{total} > 0$ allows the country to fund a trade deficit ($NX_s < 0$) in net present value terms:

$$\begin{aligned}
\sum_{s=t}^{\infty} \frac{C_s + G_s + I_s}{(1+r)^{s-t}} &= (1+r)\, B_t^{total} + \sum_{s=t}^{\infty} \frac{Y_s}{(1+r)^{s-t}} \\
0 &= (1+r)\, B_t^{total} + \sum_{s=t}^{\infty} \frac{Y_s - (C_s + G_s + I_s)}{(1+r)^{s-t}} \\
0 &= (1+r)\, B_t^{total} + \sum_{s=t}^{\infty} \frac{NX_s}{(1+r)^{s-t}}
\end{aligned}$$

In other words, given the trade balance, the situation is sustainable as long as:

$$B_t^{total} > \quad B_t^{total,min} = \quad -\frac{1}{1+r} \sum_{s=t}^{\infty} \frac{NX_s}{(1+r)^{s-t}}$$

4.2.2 Solvency with floor consumption

Assume that investment and government spending are a constant share ϕ of output, so $(1-\phi)\, Y_t$ is left for consumption or exports. Output grows at a rate g, so $Y_{t+1} = (1+g)\, Y_t$.

Consider that their is a minimal value for consumption. At time t it is $C_t^{min} = \varphi^{min} Y_t$. This minimum grows at a rate $g^{Cmin} \in [0, g]$, hence:

$$
\begin{aligned}
C_{t+1}^{min} &= \left(1 + g^{Cmin}\right) \varphi^{min} Y_t \\
&= \left(\frac{1 + g^{Cmin}}{1 + g}\right) \varphi^{min} Y_{t+1} \\
C_{t+2}^{min} &= \left(1 + g^{Cmin}\right) C_{t+1}^{min} \\
&= \left(1 + g^{Cmin}\right) \left(\frac{1 + g^{Cmin}}{1 + g}\right) \varphi^{min} Y_{t+1} \\
&= \left(\frac{1 + g^{Cmin}}{1 + g}\right)^2 \varphi^{min} Y_{t+2} \\
C_{t+k}^{min} &= \left(\frac{1 + g^{Cmin}}{1 + g}\right)^k \varphi^{min} Y_{t+k}
\end{aligned}
$$

The lower limit on assets is the one when consumption is at the minimal level (hence net exports are as high as possible):

$$
\begin{aligned}
B_t^{total,min} &= -\frac{1}{1+r} \sum_{s=t}^{\infty} \frac{NX_s}{(1+r)^{s-t}} \\
B_t^{total,min} &= -\frac{1}{1+r} \sum_{s=t}^{\infty} \frac{(1-\phi) Y_s - C_s^{min}}{(1+r)^{s-t}} \\
B_t^{total,min} &= -(1-\phi) \frac{1}{1+r} \sum_{s=t}^{\infty} \frac{Y_s}{(1+r)^{s-t}} + \frac{1}{1+r} \sum_{s=t}^{\infty} \frac{C_s^{min}}{(1+r)^{s-t}} \\
B_t^{total,min} &= -(1-\phi) Y_t \frac{1}{1+r} \sum_{s=t}^{\infty} \left(\frac{1+g}{1+r}\right)^{s-t} + C_t^{min} \frac{1}{1+r} \sum_{s=t}^{\infty} \left(\frac{1+g^{Cmin}}{1+r}\right)^{s-t} \\
B_t^{total,min} &= -(1-\phi) Y_t \frac{1}{1+r} \frac{1}{1 - \frac{1+g}{1+r}} + C_t^{min} \frac{1}{1+r} \frac{1}{1 - \frac{1+g^{Cmin}}{1+r}} \\
B_t^{total,min} &= \frac{1}{r - g^{Cmin}} C_t^{min} - \frac{1-\phi}{r-g} Y_t
\end{aligned}
$$

Using the form for C_t^{min} we get

$$
\begin{aligned}
B_t^{total,min} &= \frac{1}{r - g^{Cmin}} \varphi^{min} Y_t - \frac{1-\phi}{r-g} Y_t \\
\frac{B_t^{total,min}}{Y_t} &= \frac{\varphi^{min}}{r - g^{Cmin}} - \frac{1-\phi}{r-g}
\end{aligned}
$$

In subsequent periods, we write:

$$
\begin{aligned}
B_{t+k}^{total,min} &= \frac{1}{r - g^{Cmin}} C_{t+k}^{min} - \frac{1-\phi}{r-g} Y_{t+k} \\
B_{t+k}^{total,min} &= \frac{1}{r - g^{Cmin}} \left(\frac{1 + g^{Cmin}}{1+g}\right)^k \varphi^{min} Y_{t+k} - \frac{1-\phi}{r-g} Y_{t+k}
\end{aligned}
$$

$$\frac{B_{t+k}^{total,min}}{Y_{t+k}} = \frac{\varphi^{min}}{r - g^{Cmin}} \left(\frac{1 + g^{Cmin}}{1 + g}\right)^k - \frac{1 - \phi}{r - g}$$

As k goes to infinity, if $g^{Cmin} < g$, the variables converge to the following values:

$$\begin{aligned}
\frac{B^{total,min}}{Y} &\to -\frac{1 - \phi}{r - g} \\
\frac{C^{min}}{Y} &\to 0 \\
\frac{NX}{Y} &\to 1 - \phi \\
\frac{CA}{Y} &= \frac{NX}{Y} + r\frac{B^{total,min}}{Y} \to -g\frac{1 - \phi}{r - g} = g\frac{B^{total,min}}{Y}
\end{aligned}$$

If $g^{Cmin} = g$, the ration of floor consumption to GDP is constant, and the situation is stable at the following values:

$$\begin{aligned}
\frac{B^{total,min}}{Y} &= -\frac{1 - \phi - \varphi^{min}}{r - g} \\
\frac{C^{min}}{Y} &= \varphi^{min} \\
\frac{NX}{Y} &= 1 - \phi - \varphi^{min} \\
\frac{CA}{Y} &= \frac{NX}{Y} + r\frac{B^{total,min}}{Y} = -g\frac{1 - \phi - \varphi^{min}}{r - g} = g\frac{B^{total,min}}{Y}
\end{aligned}$$