ECON 39: Undergraduate International Trade
Problem Set #3
Winter 2017, Professor Treb Allen
Due: Tuesday February 7 at the beginning of class

Preliminaries

This week, we are learning about the Specific Factor model of trade. This problem set is also pretty difficult, but after the last problem set, hopefully this will seem a little easier!

Questions

1. Consider a single country (call it country 1) which produces two products, product A and product B. The country is inhabited by L_1 identical workers and two specific factors, one for each product. Let the quantity of the specific factor in country 1 for product A be A_1 and the quantity for the specific factor in country 1 for product B be B_1. Suppose the production function for the production of product A can be written as:
$$Q_1^A = \left(L_1^A\right)^{\alpha} (A_1)^{1-\alpha},$$
where Q_1^A is the quantity produced of product A in country 1, L_1^A is the number of workers employed in the production of product A, and $\alpha \in (0,1)$. Similarly, suppose the production function for the production of product B can be written as:
$$Q_1^B = \left(L_1^B\right)^{\beta} (B_1)^{1-\beta},$$
where Q_1^B is the quantity produced of product B in country 1, L_1^B is the number of workers employed in the production of product B, and $\beta \in (0,1)$.

   (a) Suppose that a representative agent for country 1 has preferences $U_1\left(C_1^A, C_1^B\right)$, where C_1^A is the quantity consumed of product A and C_1^B is the quantity consumed of product B.

      i. List the exogenous model parameters.
         - *Production functions $Q_1^A(\cdot,\cdot)$ and $Q_1^B(\cdot,\cdot)$; factor endowments L_1, A_1; and B_1 and preferences $U_1(\cdot,\cdot)$.*

      ii. List the endogenous model outcomes.
         - *Quantities produced Q_1^A and Q_1^B; quantities consumed C_1^A and C_1^B; and the relative price $\frac{p^A}{p^B}$.*

      iii. Define the equilibrium.
         - *For any set of production functions $Q_1^A(\cdot,\cdot)$ and $Q_1^B(\cdot,\cdot)$; factor endowments L_1, A_1; and B_1 and preferences $U_1(\cdot,\cdot)$, equilibrium is defined as a set of quantities produced Q_1^A and Q_1^B; quantities consumed C_1^A and C_1^B; and the relative price $\frac{p^A}{p^B}$ such that (1) given prices and the behavior of all other workers, each worker chooses how much to produce of each good in order to maximize her revenue; (2) given prices and her income, the representative agent chooses how much of each good to consume in order to maximize her utility; and (3) markets clear, i.e. the total quantity of each good produced is equal to the total quantity of each good consumed.*

   (b) Using the labor market clearing condition (i.e. that $L_1 = L_1^A + L_1^B$), write down an equation for the production possibility frontier (i.e. a function for the maximum amount of Q_1^A that can be produced for a given quantity of Q_1^B produced).
      - *We start with the production function:*
$$Q_1^A = \left(L_1^A\right)^{\alpha} (A_1)^{1-\alpha} \tag{1}$$
Given the labor market clearing condition, we have $L_1^A = L_1 - L_1^B$, so that we can re-write the production function as:
$$Q_1^A = \left(L_1 - L_1^B\right)^{\alpha} (A_1)^{1-\alpha} \tag{2}$$

However, we need to express Q_1^A in terms of Q_1^B (not L_1^B). To do this, we use the production function of good B and solve for L_1^B in terms of Q_1^B:

$$\begin{aligned} Q_1^B &= \left(L_1^B\right)^{\beta}\left(B_1\right)^{1-\beta} \iff \\ L_1^B &= \left(Q_1^B\right)^{\frac{1}{\beta}}\left(B_1\right)^{\frac{\beta-1}{\beta}} \end{aligned} \tag{3}$$

Substituting equation (3) into equation (2) yields:

$$Q_1^A = \left(L_1 - \left(Q_1^B\right)^{\frac{1}{\beta}}\left(B_1\right)^{\frac{\beta-1}{\beta}}\right)^{\alpha}\left(A_1\right)^{1-\alpha},$$

as required.

(c) Find the marginal product of labor in the production of product A (i.e. $\frac{\partial Q_1^A}{\partial L_1^A}$) and the marginal product of labor in the production of product B (i.e. $\frac{\partial Q_1^B}{\partial L_1^B}$).

- *We have:*

$$\begin{aligned} \frac{\partial Q_1^A}{\partial L_1^A} &= \frac{\partial\left(L_1^A\right)^{\alpha}\left(A_1\right)^{1-\alpha}}{\partial L_1^A} = \alpha\left(L_1^A\right)^{\alpha-1}\left(A_1\right)^{1-\alpha} = \alpha\left(\frac{A_1}{L_1^A}\right)^{1-\alpha} \\ \frac{\partial Q_1^B}{\partial L_1^B} &= \frac{\partial\left(L_1^B\right)^{\beta}\left(B_1\right)^{1-\beta}}{\partial L_1^B} = \beta\left(L_1^B\right)^{\beta-1}\left(B_1\right)^{1-\beta} = \beta\left(\frac{B_1}{L_1^B}\right)^{1-\beta} \end{aligned}$$

(d) Calculate the equilibrium relationship between the relative price $\frac{p_1^A}{p_1^B}$ and the allocation of labor between the two products. (Hint: use the equilibrium condition that workers maximize their income).

- *In equilibrium, the wage in each sector is equalized:*

$$\begin{aligned} p_1^A MPL_1^A &= p_1^B MPL_1^B \iff \\ p_1^A\alpha\left(\frac{A_1}{L_1^A}\right)^{1-\alpha} &= p_1^B\beta\left(\frac{B_1}{L_1^B}\right)^{1-\beta} \iff \\ \frac{p_1^A}{p_1^B} &= \frac{\beta}{\alpha}\left(\frac{B_1}{L_1^B}\right)^{1-\beta}\left(\frac{L_1^A}{A_1}\right)^{1-\alpha} \end{aligned} \tag{4}$$

(e) Suppose that $U_1\left(C_1^A, C_1^B\right) = \gamma\log C_1^A + (1-\gamma)\log C_1^B$, where $\gamma \in (0,1)$. Calculate the equilibrium relationship between the relative price $\frac{p_1^A}{p_1^B}$ and the quantity consumed of the two products. (Hint: use the equilibrium condition that the representative agent maximizes her utility subject to the income earned in the country).

- *The Cobb-Douglas preferences imply:*

$$\begin{aligned} p_1^A C_1^A &= \gamma Y_1 \\ p_1^B C_1^B &= (1-\gamma)Y_1 \end{aligned}$$

Taking ratios yields a relationship between the relative prices and relative consumption:

$$\frac{p_1^A}{p_1^B} = \frac{\gamma}{1-\gamma}\frac{C_1^B}{C_1^A} \tag{5}$$

(f) Combining your previous two answers, write all the endogenous model outcomes as functions only of the exogenous model parameters.

- *Note that we can re-write the producer maximization equation (4) in terms of quantities produced:*

$$\frac{p_1^A}{p_1^B} = \frac{\beta}{\alpha}\frac{Q_1^B}{Q_1^A}\frac{L_1^A}{L_1^B}$$

Combining equations (4) and (5), we have:

$$\frac{\gamma}{1-\gamma}\frac{C_1^B}{C_1^A} = \frac{\beta}{\alpha}\frac{Q_1^B}{Q_1^A}\frac{L_1^A}{L_1^B} \tag{6}$$

Noting that in equilibrium consumption of each good is equal to production of each good (because we're in autarky), this equation implies:

$$\begin{aligned}\frac{\gamma}{1-\gamma} &= \frac{\beta}{\alpha}\frac{L_1^A}{L_1^B} \iff \\ L_1^A &= \frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}L_1^B\end{aligned}$$

Combining this equation with the labor market clearing condition $L_1 = L_1^A + L_1^B$ *gives us an equation for* L_1^B *that depends only on exogenous parameters:*

$$\begin{aligned}L_1 - L_1^B &= \frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}L_1^B \iff \\ L_1^B &= \frac{L_1}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}} \end{aligned}\tag{7}$$

Substituting this back into the labor market clearing condition then gives us:

$$\begin{aligned}L_1^A &= L_1 - L_1^B \iff \\ L_1^A &= L_1 - \left(\frac{L_1}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right) \iff \\ L_1^A &= L_1\left(1 - \frac{1}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right) \iff \\ L_1^A &= L_1\left(\frac{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}-1}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right) \iff \\ L_1^A &= L_1\left(\frac{\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)\end{aligned}\tag{8}$$

We can substitute equation (8) into the production functions to figure out the total quantity produced (and, from market clearing, consumed):

$$Q_1^A = C_1^A = \left(L_1\left(\frac{\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)\right)^{\alpha}(A_1)^{1-\alpha} \tag{9}$$

$$Q_1^B = C_1^B = \left(\frac{L_1}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)^{\beta}(B_1)^{1-\beta} \tag{10}$$

Finally, we can use equations (9) and (10) along with equation (5) to solve for the relative prices:

$$\begin{aligned}\frac{p_1^A}{p_1^B} &= \frac{\gamma}{1-\gamma}\frac{C_1^B}{C_1^A} \iff \\ \frac{p_1^A}{p_1^B} &= \frac{\gamma}{1-\gamma}\frac{\left(\left(\frac{L_1}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)^{\beta}(B_1)^{1-\beta}\right)}{\left(\left(L_1\left(\frac{\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)\right)^{\alpha}(A_1)^{1-\alpha}\right)}\end{aligned}\tag{11}$$

(g) Suppose that workers and owners of each specific factor all have identical preferences to the representative agent. Write down expressions for the *welfare* of workers and owners of each specific factor as a function only of the exogenous model parameters.

- *Let us normalize the price of good B to one. From the Cobb-Douglas preferences, for any income Y we have $C^A = \frac{\gamma Y}{p_1^A}$ and $C^B = (1-\gamma)Y$. As a result, given prices and income we can write the (indirect) utility function as:*

$$U = \gamma \log C^A + (1-\gamma)\log C^B \iff$$
$$U = \gamma \log\left(\frac{\gamma Y}{p_1^A}\right) + (1-\gamma)\log\left((1-\gamma)Y\right) \iff$$
$$U = \log Y - \gamma \log p_1^A + \gamma \log\gamma + (1-\gamma)\log(1-\gamma) \iff$$
$$U = \log Y - \gamma \log\left(\frac{\gamma}{1-\gamma}\frac{\left(\left(\frac{L_1}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)^\beta (B_1)^{1-\beta}\right)}{\left(\left(L_1\left(\frac{\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)\right)^\alpha (A_1)^{1-\alpha}\right)}\right) + \gamma\log\gamma + (1-\gamma)\log(1-\gamma), \tag{12}$$

where the last line used equation (11). Equation (12) says that to figure out the welfare of workers and each specific factor, we just need to substitute each factor's income into the "Y". To calculate the income of a worker, we note her wage is equal to her MPL in good B (since the p_1^B =1):

$$w_1 = \beta\left(\frac{B_1}{L_1^B}\right)^{1-\beta} \iff$$
$$w_1 = \beta\left(\frac{B_1\left(1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}\right)}{L_1}\right)^{1-\beta}$$

so that a worker's welfare is:

$$U_{worker} = \log\left(\beta\left(\frac{B_1\left(1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}\right)}{L_1}\right)^{1-\beta}\right) - \gamma\log\left(\frac{\gamma}{1-\gamma}\frac{\left(\left(\frac{L_1}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)^\beta (B_1)^{1-\beta}\right)}{\left(\left(L_1\left(\frac{\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)\right)^\alpha (A_1)^{1-\alpha}\right)}\right) + \log\gamma^\gamma(1-\gamma)^{1-\gamma}.$$

The specific factors A's income is equal to the value of revenue in sector A less the payments to labor:

$$Y_A = p_1^A Q_1^A - w_1 L_1^A \iff$$
$$Y_A = \left(\frac{\gamma}{1-\gamma}\frac{\left(\left(\frac{L_1}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)^\beta (B_1)^{1-\beta}\right)}{\left(\left(L_1\left(\frac{\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)\right)^\alpha (A_1)^{1-\alpha}\right)}\right)\left(\left(L_1\left(\frac{\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)\right)^\alpha (A_1)^{1-\alpha}\right)$$
$$-\left(\beta\left(\frac{B_1\left(1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}\right)}{L_1}\right)^{1-\beta}\right)\left(L_1\left(\frac{\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)\right) \iff$$
$$Y_A = \left(\frac{\gamma}{1-\gamma}\left(\left(\frac{L_1}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)^\beta (B_1)^{1-\beta}\right)\right) - \left(\beta\left(\frac{B_1\left(1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}\right)}{L_1}\right)^{1-\beta}\right)\left(L_1\left(\frac{\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)\right),$$

so the welfare of the specific factor A is:

$$U_A = \log\left(Y_A\right) - \gamma \log\left(\frac{\gamma}{1-\gamma}\frac{\left(\left(\frac{L_1}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)^{\beta}\left(B_1\right)^{1-\beta}\right)}{\left(\left(L_1\left(\frac{\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)\right)^{\alpha}\left(A_1\right)^{1-\alpha}\right)}\right) + \gamma\log\gamma + (1-\gamma)\log(1-\gamma).$$

The specific factor B's income is similar:

$$Y_B = p_1^B Q_1^B - w_1 L_1^B \iff$$

$$Y_B = p_1^B\left(\left(\frac{L_1}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)^{\beta}\left(B_1\right)^{1-\beta}\right) - \left(\beta\left(\frac{B_1\left(1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}\right)}{L_1}\right)^{1-\beta}\right)\left(\frac{L_1}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)$$

and hence the welfare is:

$$U_B = \log\left(Y_B\right) - \gamma \log\left(\frac{\gamma}{1-\gamma}\frac{\left(\left(\frac{L_1}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)^{\beta}\left(B_1\right)^{1-\beta}\right)}{\left(\left(L_1\left(\frac{\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}{1+\frac{\gamma}{1-\gamma}\frac{\alpha}{\beta}}\right)\right)^{\alpha}\left(A_1\right)^{1-\alpha}\right)}\right) + \gamma\log\gamma + (1-\gamma)\log(1-\gamma)$$

.

2. Now suppose that there exists another country (call it country 2). Suppose that country 2 is identical to country 1 in every way except that $A_2 < A_1$, i.e. country 2 is endowed with less of the specific factor for product A.

   (a) Using you answer to question 1(f), will the autarkic relative price of product A in country 2 (i.e. $\frac{p_2^A}{p_2^B}$) be higher or lower than the autarkic relative price of product A in country 1 (i.e. $\frac{p_1^A}{p_1^B}$)?

   - *If country 2 is endowed with less of the specific factor for product A, it will be able to produce less of product A, so to ensure that consumers in country 2 are willing to purchase less of product A, it's price will be higher. (See equation (11)).*

   (b) Suppose that country 1 and country 2 now open up to trade with each other.

      i. List the exogenous model parameters.

      - *Production functions $Q_i^A(\cdot,\cdot)$ and $Q_i^B(\cdot,\cdot)$; factor endowments L_i, A_i and B_i; and preferences $U_i(\cdot,\cdot)$ for all $i \in \{1,2\}$.*

      ii. List the endogenous model outcomes.

      - *Quantities produced Q_i^A and Q_i^B; quantities consumed C_i^A and C_i^B for all $i \in \{1,2\}$; and the relative price $\frac{p^A}{p^B}$.*

      iii. Define the trade equilibrium.

      - *For any set of production functions $Q_i^A(\cdot,\cdot)$ and $Q_i^B(\cdot,\cdot)$; factor endowments L_i, A_i; and B_i and preferences $U_i(\cdot,\cdot)$ for all $i \in \{1,2\}$, equilibrium is defined as a set of quantities produced Q_i^A and Q_i^B; quantities consumed C_i^A and C_i^B for all $i \in \{1,2\}$; and the relative price $\frac{p^A}{p^B}$ such that (1) in each country, given prices and the behavior of all other workers, each worker chooses how much to produce of each good in order to maximize her revenue; (2) in each country, given prices and her income, the representative agent chooses how much of each good to consume in order to maximize her utility; and (3) markets clear, i.e. the total quantity of each good produced ni the world is equal to the total quantity of each good consumed.*

   (c) Given your answer to question 2(a), which country do you expect will export product A and which country do you expect will export product B?

- *Since the autarkic relative price of good A is higher in country 2 than country 1, we expect that the world equilibrium relative price of good A will be higher than the autarkic price in country 1 and lower than the autarkic price in country 2. Hence, we expect that country 1 will specialize in the production of good A and hence export it, while country 2 will specialize in the production of good B and export it.*

(d) Write all the endogenous model outcomes as functions only of the exogenous model parameters. [Note: you should end up with a set of two equations and two unknowns. That's as far as you need to go!]

- *Unlike in problem 1(f), the quantity consumed in a country no longer is necessarily equal to the quantity produced in that country, which prevents me from using the clever trick in equation (6). Instead, what we will do is what we did in the in class example, which is solve for the equilibrium quantities produced and consumed solely as a function of the world price and then write down an expression (which we can't solve by hand) that could be solved to determine the world price on the computer. Let us define the relative price of good A to good B as p, i.e.* $p \equiv \frac{p^A}{p^B}$ *(i.e. normalize the price of good B to one). Using the worker indifference condition in equation (4), we can write the relationship between the number of workers employed in both sectors as:*

$$p = \frac{\beta}{\alpha}\left(\frac{B_i}{L_i^B}\right)^{1-\beta}\left(\frac{L_i^A}{A_i}\right)^{1-\alpha} \iff$$
$$L_i^A = p^{\frac{1}{1-\alpha}}\frac{\alpha}{\beta}B_i^{-\frac{1-\beta}{1-\alpha}}A_i\left(L_i^B\right)^{\frac{1-\beta}{1-\alpha}}$$

Substituting in the labor market clearing condition $L_i^A + L_i^B = L_i$, *we can write down an (implicit) function for the number of workers employed in each sector in each country solely as a function of the price. That is, we can say that* $L_i^B(p)$ *is the function that solves the following equation in country* $i \in \{1,2\}$ *for* L_i^B *as a function of the price:*

$$L_i - L_i^B = p^{\frac{1}{1-\alpha}}\frac{\alpha}{\beta}B_i^{-\frac{1-\beta}{1-\alpha}}A_i\left(L_i^B\right)^{\frac{1-\beta}{1-\alpha}}$$

Similarly, we can write the function for L_i^A *in country i as* $L_i^A(p) = L_i - L_i^B(p)$. *Note that there is no way (that I know of) to solve this problem for an arbitrary* α *and* β *by hand. Once we know* $L_i^A(p)$ *and* $L_i^B(p)$, *however, we can calculate the quantity produced of each good in each country i:*

$$Q_i^A(p) = \left(L_i^A(p)\right)^{\alpha}(A_i)^{1-\alpha}$$
$$Q_i^B(p) = \left(L_i^B(p)\right)^{\beta}(B_i)^{1-\beta}.$$

we can also calculate the income in each country:

$$Y_i = pQ_i^A(p) + Q_i^B(p).$$

Given preferences, we can then calculate the total consumption of each good in each country:

$$C_i^A(p) = \gamma\frac{pQ_i^A(p) + Q_i^B(p)}{p}$$
$$C_i^B(p) = (1-\gamma)\,pQ_i^A(p) + Q_i^B(p).$$

Finally, we can use the market clearing condition to solve for the equilibrium relative price. We can choose either sector:

$$Q_1^A(p) + Q_2^A(p) = C_1^A(p) + C_2^A(p) \iff$$
$$\left(L_1^A(p)\right)^{\alpha}(A_1)^{1-\alpha} + \left(L_2^A(p)\right)^{\alpha}(A_2)^{1-\alpha} = \frac{\gamma}{p}\left(pQ_1^A(p) + Q_1^B(p) + pQ_2^A(p) + Q_2^B(p)\right).$$

This is the second function that can be implicitly solved for p given the implicit function $L_i^A(p)$ *above.*

(e) How has the relative price changed in country 1 compared to its autarkic price? How has the equilibrium distribution of labor across the two sectors changed? Using your your answer to question 1(g), who gains from trade and who loses?

- *The relative price of good A has increased in country 1 with trade. This has caused labor to shift into the production of good A. As a result, the specific factor of good A has gained from trade, the specific factor of good B has lost from trade, and the effect of trade on workers is ambiguous (but the country as a whole is made better off).*

3. Suppose instead that $U_1\left(C_1^A, C_1^B\right) = \min\left\{C_1^A, C_1^B\right\}$. Show the overall gains from trade for country 1 and country 2 using pictures only.