Lecture Notes:
Macroeconomics A

Based on lectures by **Johannes Boehm** in Autumn semester, 2024

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These lecture notes were taken in the course *Macroeconomics A* taught by professor **Johannes Boehm** at Graduate of International and Developoment Studies, Geneva as part of the International Economics program (Semester I, 2024).

Currently, these are just drafts of the lecture notes. There can be typos and mistakes anywhere. So, if you find anything that needs to be corrected or improved, please inform at `jingle.fu@graduateinstitute.ch`.

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Lecture 1.

Solow Model & Development Accounting

1.1 Solow model

Mechanism of Solow model

- Firms: producing output using capital
- Consumers: consume output and save fraction of income $\rightarrow$ future capital $\rightarrow$ produce $\rightarrow$ save

Basic Assumptions

- Discrete time
- Closed economy, single good: for consumption and investment
- Two actors in the economy: firms and households
  - Firms: produce output using capital and Labor
  - Households: receives labor and capital income, and profits from owning the firms(now $= 0$); consume and save
- Three markets: labor, capital and final goods

1.1.1 Households(HH)

Note.

Households do not optimize, which is the main difference to Ramsey model.

Assuming the received income as below:

$$Y_t = \underbrace{w(t)L^s(t)}_{\text{labor income}} + \underbrace{R(t)K^s(t)}_{\text{capital income}} + \underbrace{\Pi(t)}_{profits}$$

Keynesian consumption function

$$S_t = sY_t \quad C_t = (1-s)Y_t,$$

where s is the saving rate.

Technology

Firms produce according to the production function:

$$Y_t = F(K_t, L_t, A_t)$$

where L and K are labor and capital, A is the technology level.

Function F is the *neoclassical production function* with the following properties:

- Constant returns to scale:
$$F(\lambda K, \lambda L, A) = \lambda F(K, L, A)$$
- Diminishing marginal products for both K and L
$$\frac{\partial F}{\partial K} > 0, \quad \frac{\partial F^2}{\partial^2 K} < 0$$
- Inada conditions(likewise for both L and K)
$$\lim_{K\to 0} \frac{\partial F}{\partial K} = \infty, \quad \lim_{K\to\infty} \frac{\partial F}{\partial K} = 0$$

Assumption 1.1.1.

- All markets are perfectly competitive;
- Labor market clearing: $L^s(t) = L^d(t) \equiv L_t$;
- Capital market clearing: $K^s(t) = K^d(t) \equiv K_t$;
- Goods market clearing: $Y_t = C_t + I_t$: total supply equals to consumption plus investment.

Theorem 1.1.1 (Law of motion of capital).

$$K_{t+1} = (1-\delta)K_t + I_t$$

where δ is the depreciation rate, $K(0)$ is the initial capital endowment.

1.1.2 Firms

The firm's problem is to maximize profits:

$$\max_{K,L} \Pi(t) = F(K, L, A_t) - w(t)L_t - R(t)K_t.$$

Taking the FOCs:
$$\frac{\partial F}{\partial K} = R(t), \quad \frac{\partial F}{\partial L} = w(t).$$

Also, we know that
$$F(K, L, A) = \frac{\partial F}{\partial K} + \frac{\partial F}{\partial L} L,$$

so we can get:
$$\Pi = F(K, L, A) - R(t)K - w(t)L = 0.$$

The equilibrium of the Solow Model is:

- $K_{t+1} = (1-\delta)K_t + S_t$

- $C_t = (1-s)Y_t$
- $S_t = sY_t$

From the Fundamental Law of Motion 1.1.1, we can get:

$$\begin{aligned} K_{t+1} &= (1-\delta)K_t + S_t && (1.1)\\ &= (1-\delta)K_t + sY_t && (1.2)\\ &= (1-\delta)K_t + sF(K_t, L_t, A_t) && (1.3)\\ & && (1.4) \end{aligned}$$

1.1.3 The canonical Solow Model

Assumption 1.1.2.

- No populaiton growth $L_t = L$
- No technological progress $A_t = A$

Then, we transform the equation (3) into per-capita terms:

$$\begin{aligned} k_{t+1} = \frac{K_{t+1}}{L_{t+1}} &= (1-\delta)\frac{K_t}{L_t} + s\frac{F(K_t), L, A}{L_t} && (1.5)\\ &= (1-\delta)k_t + sF(k_t, 1, A) && (1.6)\\ &= (1-\delta)k_t + sf(k_t) && (1.7) \end{aligned}$$

where $k_t = \frac{K_t}{L_t}$ is the capital per capita, $f(k_t) = F(K_t/L_t, 1, A)$.

Similarly, we have the output per capita:

$$y(t) = \frac{Y_t}{L_t} = \frac{F(K_t, L, A)}{L} = f(k_t),$$

the rate of capital:

$$R(t) = F_K(K_t, L, A) = f'(k_t),$$

and the wage rate:

$$w(t) = F_L(K_t, L, A) = f(k_t) - k_t f'(k_t).$$

1.1.4 Dynamic evolution of the economy

Characterize everything in term sof uniqeu state variables k_t:

$$k_{t+1} = (1-\delta)k_t + sf(k_t)$$

with $k(0)$ given.

Definition 1.1.1 (Steady state).

A steady state is a state of the economy where all variables are constant over time.

The concept corresponding to the steady state in the basic model is the *balanced growth path* (some researchers still prefer to use the name “steady state” for the balanced growth path, because the normalized variables are “steady” also in this case).

Steady state of the Solow Model

There are actually two steady states in the Solow Model:

- The trivial steady state: $k^* = 0$
- The non-trivial steady state: $k^* > 0$

But we can actually ignore the one at $k_t = 0$.

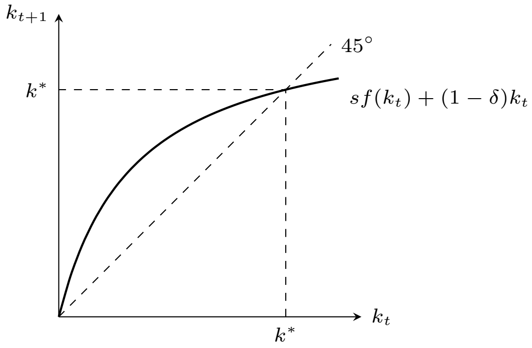


图 1.1: Steady state of Solow Model

Steady state anlaysis

At the steady-state, k^* satisfies:

$$\underbrace{\delta k^*}_{\text{Depreciation}} = \underbrace{sf(k^*)}_{\text{Investment}}$$

which means that the investment equals to the depreciation.

k^* defined by:

$$\frac{f(k^*)}{k^*} = \frac{\delta}{s}.$$

A different representation In the steady state, the amount of actual investment is exactly such that it compensates for depreciation.

Comparative statics

At the steady state, k^* satisfies:

$$\frac{f(k^*(s,\delta,A),A)}{k^*(s,\delta,A)} = \frac{\delta}{s}$$

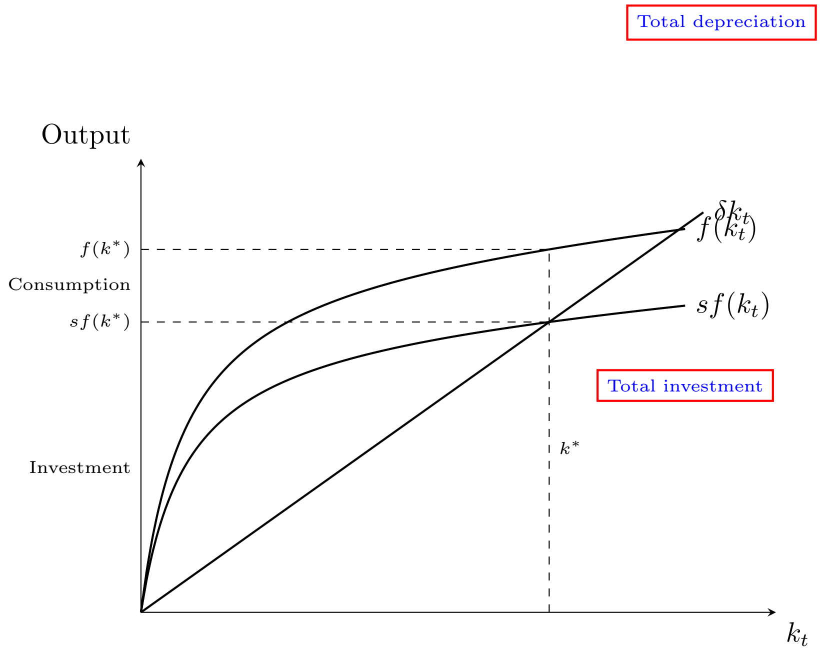
while the differentian yields the comparative statics:

$$\frac{\partial k^*(s,\delta,A)}{\partial \delta} < 0$$
$$\frac{\partial k^*(s,\delta,A)}{\partial s} > 0$$
$$\frac{\partial k^*(s,\delta,A)}{\partial A} > 0$$

The higher productivity A has the 'multiplier' effect on the output directly via k^*.

At the steady state, the consumption per capita is:

$$c^* = c(k^*(s,\delta,A)) = (1-s)f(k^*(s,\delta,A)).$$



s has two opposing effects:

- Higher s leads to higher k^*, which leads to higher y^*;
- Higher s leads to lower $1 - s$, which leads to lower fraction of output consumed.

Let $\hat{s}$ be the savings rate that maximizes consumption, i.e.

$$\hat{s} = \arg\max_s (1-s) f(k^*(s,\delta,A)) = \arg\max_s \{f(k^*(s,\delta,A)) - \delta k^*(s,\delta,A)\}.$$

Hence $\hat{s}$ is implicitly defined by:

$$f'(k^*(\hat{s},\delta)) = \delta.$$

Definition 1.1.2 (Golden Rule).

The Golden Rule is the savings rate that maximizes consumption in the steady state, which is $\hat{s}$ in our model. And, we have $k^{gold} = f'^{-1}(\delta)$ as the golden rule capital stock.

Note.

- Savings are excessive if depreciation is higher than the marginal product of capital;
- By saving less, consumption go up;
- The economy is dynamically inefficient if '$k > k^{gold}$'.

1.1.5 Growth in the Solow Model

Dynamics is fully determined by the evolution of capital:

$$g_k(t) = \frac{k_{t+1} - k_t}{k_t}$$

$$= \frac{(1-\delta)k_t + sf(k_t) - k_t}{k_t}$$
$$= s\frac{f(k_t)}{k_t} - \delta$$
$$= g_k(k_t)$$

So, in the long run, the growth rate of capital is constant and equal to $\frac{sf(k^*)}{k^*} - \delta$. As long as $k_t < k^*$, the growth rate of capital is positive, and vice versa.

1.1.6 The Growing Economy

Now we extend the model to the situation where A_t and L_t grow over time. In the previous section, the long-run outcome was the steady state where there is no growth. This extension is necessary for addressing the facts related to the growth issues described in later chapters.

We assume that the aggregate production function takes the form of

$$Y_t = F(K_t, A_t L_t).$$

There are two changes from the basic model: first, we allow the labor input (population times hours worked per person) L_t to grow over time. Second, and more importantly, we allow for technological progress. In this production function, the variable representing the technology level, A_t, is multiplied by labor input L_t. Technological progress thus takes a form of improving the labor input, and $A_t L_t$ is often referred to as the total number of efficiency units of labor (or effective labor).

This form of technological progress is labor-augmenting; it was introduced in the previous chapter. As was also asserted there, Uzawa (1961) proved that labor-augmenting technical change is the only form of technical progress that is consistent with exact balanced growth, that is, the growth path where aggregate variables such as output and capital grow at a constant rate.

We assume that the (net) growth rate of A_t is $\frac{\dot{A}_t}{A_t} = g$ and that the growth rate of L_t is $\frac{\dot{L}_t}{L_t} = n$. The same manipulations of equations as in the basic model yield

$$K_{t+1} = (1-\delta)K_t + sF(K_t, A_t L_t).$$

The dynamic equation for capital in continuous time is

$$\dot{K}_t = sF(K_t, A_t L_t) - \delta K_t.$$

Let us define the efficient unit of capital k_t by

$$k_t = \frac{K_t}{A_t L_t}.$$

and take the log derivative w.r.t. time,

$$\frac{\dot{k}_t}{k_t} = \frac{\dot{K}_t}{K_t} - \frac{\dot{A}_t}{A_t} - \frac{\dot{L}_t}{L_t} = \frac{\dot{k}_t}{k_t} - g - n.$$

Output per efficiency unit of labor is:

$$\hat{y}_t = \frac{Y_t}{A_t L_t} = F(k_t, 1) = f(k_t).$$

Here, $f(k_t) = F(k_t, 1)$ as in the basic model. Hence, the above equation can be written as:

$$\begin{aligned}\frac{\dot{k}_t}{k_t} &= \frac{sF(K_t, A_tL_t) - \delta K_t}{K_t} - g - n \\ &= \frac{sF(K_t, A_tL_t)}{K_t} - (\delta + g + n) \\ &= \frac{sf(k_t)}{k_t} - (\delta + g + n).\end{aligned}$$

Therefore, we can get the dynamic of efficient capital k_t as

$$\dot{k}_t = sf(k_t) - (\delta + g + n)k_t.$$

After the capital stock K_t is normalized by A_tL_t, we thus obtain a very similar difference equation as the fundamental equation in the basic model, with effective depreciation rate $\delta + g + n$.

In this case, upon entering a balanced growth path, the rate of per capita output growth depends uniquely on the rate of technological progress g.

1.1.7 Growth effects of changes in the savings rate

Consider the consumption path c^* per unit of effective labor, especially in the context of returning to equilibrium and whether the consumption level per unit of effective labor will be above, below, or equal to the level during the previous equilibrium.

This analysis depends on a proposition: an increase in the savings rate persistently raises the average capital stock per unit of effective labor, as well as savings and output levels. The question then arises whether it also implies an increase or decrease in the consumption level per unit of effective labor. Under what conditions does consumption per unit of effective labor reach its optimum?

Impact of Savings Rate Changes on Optimal Consumption per Unit of Effective Labor

The formula:

$$c^* = f(k^*) - sf(k^*) = f(k^*) - (n + g + \delta)k^*$$
$$\frac{\partial c^*}{\partial s} = [f'(k^*(s, n, g, \delta)) - (n + g + \delta)] \frac{\partial k^*}{\partial s}$$

reveals that whether an increase in the savings rate leads to a long-term rise in consumption per unit of effective labor depends on the comparison between the marginal product of capital at the steady-state per capita capital stock and the slope of the investment schedule.

This formula assists in addressing the query raised in the consumption path analysis diagram.

When the long-term steady-state capital stock per unit of effective labor satisfies the condition that $f'(k^*) = n + g + \delta$, *the capital stock per unit of effective labor* reaches the golden-rule(1.1.2) level, and *the consumption per unit of effective labor is optimal.*

Impact of Changes in the Savings Rate on the Output of Steady-State Efficient Labor, Capital Stocks

A rise in the savings rate can lead to a change in the amount of capital in steady state efficient labor:

$$sf(k^*(s, n, g, \delta)) = (n + g + \delta)k^*(s, n, g, \delta)$$

$$
\begin{aligned}
f(k^*(s,n,g,\delta)) + sf'(k^*)\frac{\partial k^*}{\partial s} &= (n+g+\delta)\frac{\partial k^*}{\partial s} \\
\frac{\partial k^*}{\partial s} &= \frac{f(k^*)}{(n+g+\delta) - sf'(k^*)} > 0
\end{aligned}
$$

The sensitivity of the steady state capital per effective labor to the savings rate is crucial. Under the condition that the derivative $s'f(k^*)$ is smaller than $(n+g+\delta)$, implying that the change in the steady state capital stock $dk^*/ds < 0$, an increase in the savings rate will initially reduce the consumption per effective worker until new equilibrium is achieved.

Further, the change in production per effective labor y^* with respect to the savings rate is given by:

$$
\frac{\partial y^*}{\partial s} = f'(k^*)\frac{\partial k^*(s,n,g,\delta)}{\partial s} = \frac{f'(k^*)f(k^*)}{(n+g+\delta) - sf'(k^*)} > 0
$$

Finally, the response of production per effective labor to changes in the savings rate can be expressed as:

$$
\begin{aligned}
\frac{s}{y^*}\frac{\partial y^*}{\partial s} &= \frac{s}{f(k^*)}\frac{f'(k^*)f(k^*)}{(n+g+\delta) - sf'(k^*)} \\
&= \frac{(n+g+\delta)k^*f'(k^*)}{f(k^*)\left[(n+g+\delta) - (n+g+\delta)k^*f'(k^*)/f(k^*)\right]} \\
&= \frac{k^*f'(k^*)/f(k^*)}{1 - k^*f'(k^*)/f(k^*)} \\
&= \frac{\alpha_k(k^*)}{1-\alpha_k(k^*)}
\end{aligned}
$$

where $\alpha_k(k^*)$ represents the output elasticity of capital at the steady state. This equation helps in determining whether the increase in the savings rate will eventually increase the consumption level per unit of effective labor in the long run.

Summary.

- Baseline Model:
  - No long-run growth, i.e. capital accumulation alone does not make the economy grow in the long run
  - Conditional convergence: if countries have the same parameters s, δ, A and the same production function F, they will converge to the same steady state. Countries further from the steady state will grow faster.
  - Crucial mechanism: decreasing marginal product to capital
- Extended Model:
  - Long-run growth poible if productivity grows: $g > 0$;
  - Consistant with Kaldor facts if $g > 0$;

But s and g are exogenous.

1.2 Development Accounting

Development accounting asks whether the factors of production can explain income levels(See Caselli 2005[8]).

Assuming the production function is Cobb-Douglas in physical and human capital(more general than labor):

$$
\begin{aligned}
Y_j &= A_j K_j^{\alpha} (L_j h_j)^{1-\alpha} \\
y_j &= F(k_j, h_j, A_j) = A_j k_j^{\alpha} h_j^{1-\alpha} = A_j y_j^{KH}
\end{aligned}
$$

Casselli(2005) use the following method to measure success:

$$\text{success} = \frac{\text{Var}\left[\log(y_j^{KH})\right]}{\text{Var}\left[\log(y_j)\right]}.$$

Then, if all countries had the same technology A_j, then the success would be 1.

How to measure human capital? Not directly observable. But: can see distribution of schooling in the population, and can see individual's wage and schooling.

From the production function

$$Y_j = K_j^{\alpha} (A_j H_j)^{1-\alpha}$$

we can calibrate the relative productivity differences relative to the US as

$$\frac{A_j}{A_{US}} = \left(\frac{Y_j}{Y_{US}}\right)^{\frac{1}{1-\alpha}} \left(\frac{K_{US}}{K_j}\right)^{\frac{\alpha}{1-\alpha}} \frac{H_{US}}{H_j}.$$

Lecture 2.

Neoclassical Growth Model

2.1 Neoclassical Growth Model

The same basic environment as Solow model without the assumption of the constant exogenous saving rate.

$$Y = F(K(t), L(t), Z(t))$$

As in basic general equilibrium theory, let us suppose that preference orderings can be represented by utility functions. In particular, suppose that there is a unique consumption good, and each household h has an *instantaneous utility function* given by: $u(c^h(t))$, where $c^h(t)$ is the consumption and $u : \mathbb{R}_+ \to \mathbb{R}$ is increasing and concave.

> **Note.** Negative levels of consumption are not allowed.

We shall make 2 major assumptions:

1. Household does not derive any utility from the consumption of other households, so consumption externalities are ruled out.

2. Impose the condition that overall utility is *time-separable and stationary*; that is, instantaneous utility at time t is independent of the consumption levels at past or future dates and is represented by the same utility function u^h at all dates.

Representative household with preferences:

$$U^h(T) = \sum_{t=0}^{T} (\beta^h)^t u^h(c^h(t))$$

where $\beta^h \in (0,1)$ is the time discount factor of household h.

A solution $\{x(t)\}_{t=1}^{T}$ to a dynamic optimization problem is *time-consistent* if the following is true: *When $\{x(t)\}_{t=1}^{T}$ is a solution to the continuation dynamic optimization problem starting from $t = 0$, $\{x(t)\}_{t=t'}^{T}$ is a solution to the continuation dynamic optimization starting from the time $t = t' > 0$.*

Let's consider the simplest case and suppose all households are infinitely-lived and identical. Then the demand side of the economy can be represented as the solution of the following maximization problem at time $t = 0$:

$$\max \sum_{t=0}^{\infty} \beta^t u(c(t))$$

where $\beta \in (0,1)$.

Budget constraint:

$$C(t) + K(t+1) = (1 + r(t))K(t) + w(t)L(t)$$

$r(t)$ is the rate of return on lending capital to firms.

Note. Optimal Growth

If the economy consists of a number of identical households, then this problem corresponds to the Pareto optimal allocation giving the same (Pareto) weight to all households. Therefore the optimal growth problem in discrete time with no uncertainty, no population growth, and no technological progress can be written as follows:

$$\max_{\{c(t),k(t)\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c(t))$$
$$\text{s.t.} \quad k(t+1) = f(k(t)) + (1-\delta)k(t) - c(t)$$

with $k(t) \geq 0$ and given $k(0) > 0$.

The constraint is straightforward to understand: *total output per capita produced with capital-labor ratio $k(t)$, $f(k(t))$, together with a fraction $(1-\delta)$ of the capital that is undepreciated make up the total resources of the economy at date t.*

Assuming that the representative household has $L(t)$ unit of labor supplied inelastically and denoting its assets at time t by $a(t)$, this problem can be written as:

$$\max_{\{c(t),k(t)\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c(t))$$
$$\text{s.t.} \quad a(t+1) = (1+r(t))a(t) - c(t) + w(t)L(t)$$

where $r(t)$ is the net rate of return on assets, so that $1+r(t)$ is the gross rate of return, and $w(t)$ is the equilibrium wage rate. Market clearing then requires $a(t) = k(t)$.

And, we have the non-Ponzi condition:

$$\lim_{t\to\infty} \left\{ K(t) \left[\prod_{s=1}^{t-1} \left(\frac{1}{1+r_s} \right) \right] \right\} \geq 0$$

which refers to: limit of the PDV of debt has to be nonnegative (you cannot die in debt)

Theorem 2.1.1 (Euler Equation).

$$\forall t: u'(c_t) = \beta(1+r_{t+1})u'(c_{t+1})$$

With an infinite horizon ($T \to \infty$), the Euler equations still determine the relative consumption levels. The terminal condition becomes transversality condition is the limit of the terminal condition:

$$\lim_{T\to\infty} K_{T+1}\beta^T u'(c_T) = \lim_{T\to\infty} K_{T+1}\beta^{T+1}u'(c_{T+1})(1+r_{T+1}) = 0$$

Hence

$$\lim_{t\to\infty} \beta^t u'(c_t)(1+r_t)K_t = 0$$

Then, we have two conditions involve the infinity:

$$\lim_{t\to\infty} \beta^t u'(c_t)(1+r_t)K_t = 0 \tag{2.1}$$

$$\lim_{t\to\infty}\left\{K(t)\left[\prod_{s=1}^{t-1}\left(\frac{1}{1+r_s}\right)\right]\right\}\geq 0 \tag{2.2}$$

The (2.1) is a condition for optimal behavior. The (2.2) is a condition to make sure that the flow budget constraints are consistent with a lifetime budget constraint and no perpetual debt.

Steady State (C^*, K^*)

From the Euler equation:

$$u'(C^*) = \beta(1 + F_K(K^*, L))u'(C^*)$$

hence

$$1 + r^* = 1 + F_K(k^*, 1) - \delta = \frac{1}{\beta}$$

which fully determines k^*.

> **Note.** This shows that k^* does not depend on $u(\cdot)$.

Level of per-capita consumption is then given by the economy's resource constraint:

$$C^* + K^* = (1-\delta)K^* + F(K^*, L)$$

hence,

$$c^* = F(k^*, 1) - \delta k^*$$
$$F_K(k^*, 1) = \delta + \left(\frac{1}{\beta} - 1\right)$$

In the Solow model the golden-rule level of capital (which maximizes steady-state consumption) satisfies

$$F_K(k_S^{\mathrm{GR}}, 1) = \delta$$

hence

$$k^*_{\mathrm{NGM}} < k_S^{\mathrm{GR}}$$

> **Note. Intuition:**
>
> In NGM model, people tend to consume more on the near future rather than in the far steady state.

2.2 NGM in continuous time

Assume labor supply L growth at rate n:

$$L(t) = e^{nt}$$

Households choose consumption/saving to maximize

$$U = \int_0^\infty e^{-\rho t} e^{nt} u(c(t))dt = \int_0^\infty e^{(n-\rho)t} u(c(t))dt$$

Intertemporal budget constraint: as in the discrete time case, but treat $\Delta K \to 0$:

$$\dot{K}(t) = w(t)L(t) + (1 + r(t))K(t) - C(t)$$

Write in per-capita units:

$$\dot{k}(t) = w(t) + (1 + r(t))k(t) - c(t) - nk(t)$$

Non-Ponzi condition:

$$\lim_{t\to\infty} \left\{ k(t) \exp \left[- \int_0^t (r(v) - n) dv \right] \right\} \geq 0$$

To solve this, we define the **Hamiltonian**:

$$H(c, k, \mu) = e^{-(\rho-n)t} u(c(t)) + \mu(t)[w(t) + (1 + r(t) - n)k(t) - c(t)]$$

optimality conditions are

$$\begin{aligned} H_c(c, k, \mu) &= 0 \\ H_k(c, k, \mu) &= -\dot{\mu}(t) \\ H_\mu(c, k, \mu) &= \dot{k}(t) \\ \lim_{t\to\infty} \mu(t)k(t) &= 0 \end{aligned}$$

Hence:

$$\begin{aligned} e^{-(\rho-n)t} u'(c(t)) - \mu(t) &= 0 \\ \mu(t)(1 + r(t) - n) &= -\dot{\mu}(t) \end{aligned}$$

and we get:

$$e^{-(\rho-n)t} u'(c(t))(1 + r(t) - n) = -e^{-(\rho-n)t} u''(c(t))\dot{c}(t) + (\rho - n)e^{-(\rho-n)t} u'(c(t))$$

Denote $\sigma(c) = -\frac{u'(c)}{u''(c)c}$, we have the Euler equation in continuous time:

$$\sigma(c)(1 + r(t) - \rho) = \frac{\dot{c}(t)}{c(t)}$$

$\sigma(c)$ is the intertemporal elasticity of substitution (IES).

Lecture 3.

Real Business Cycle Model(1)

Business Cycle Facts I

We'll study the **detrended macro times series:**

$$x_t = \log(X_t) - \log(X_t^*)$$

is the percentage deviation of variable X from its trend X^*.

How's the trend?

1. First linear,
2. more sophisticated filters: Baxter-King(Bankpass) filter, Hodrick-Prescott(HP) filter

Linear: just run OLS regression and use residuals.

Business Cycle Facts III:

Governmetn spending, not strongly correlated with level of output, main driver of business cycles.

Real wage is mildly pro-cyclical: average wage evolves differently than the wage of a continuously employed worker

$U_C(C, 1-L) > 0, U_{CC}(C, 1-L) < 0$ and $U_L(C, 1-L) > 0$, $U_{LL}(C, 1-L) < 0$

Lecture 4.

Review Session 2024.10.17

4.1 A simplified real-business cycle model with additive technology shocks

Problem 4.1.1 (1). Find the first-order condition (Euler equation) relating C_t and C_{t+1} in the following model:

Solution. The household's problem is to maximize

$$\max_{\{C_t\}} E_t \sum_{t=0}^{\infty} u(C_t)/(1+\rho)^t$$

subject to the budget constraint

$$K_{t+1} = K_t + Y_t - C_t$$

where $Y_t = AK_t + e_t$ and e_t is a random technology shock satisfying $e_t = \phi e_{t-1} + \varepsilon_t$, $-1 < \phi < 1$ and that ε_t is **i.i.d**.

Define the Lagrange function:

$$\mathcal{L} = E_t \sum_{t=0}^{\infty} \left[u(C_t)/(1+\rho)^t - \lambda_t (K_{t+1} - K_t - Y_t + C_t) \right] \tag{4.1}$$

The first-order condition with respect to C_t is:

$$\frac{\partial \mathcal{L}}{\partial C_t} = 0 \Rightarrow \frac{u'(C_t)}{(1+\rho)^t} - \lambda_t = 0 \tag{4.2}$$

The first-order condition with respect to K_{t+1} is:

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = 0 \Rightarrow -\lambda_t + E_t \left[\lambda_{t+1}\right](1+\rho) = 0 \tag{4.3}$$

Combining (2) and (3) we get:

$$\frac{u'(C_t)}{(1+\rho)^t} = E_t \left[\lambda_{t+1}\right](1+\rho) \tag{4.4}$$

We have the relationsjip between C_t and C_{t+1}:

$$u'(C_t) = E_t \left[u'(C_{t+1})\right] \tag{4.5}$$

Since $u(C_t) = C_t - \theta C_t^2$, we have $u'(C_t) = 1 - 2\theta C_t$. Thus, the Euler equation is:

$$1 - 2\theta C_t = E_t \left[1 - 2\theta C_{t+1}\right] \Rightarrow C_t = E_t \left[C_{t+1}\right] \tag{4.6}$$

Problem 4.1.2 (2). Guess that consumption takes the form$C_t = \alpha + \beta K_t + \gamma e_t$. What's K_t in this case?

Solution.

$$\begin{aligned} K_{t+1} &= K_t + Y_t - C_t \\ &= K_t + (AK_t + e_t) - \alpha - \beta K_t - \gamma e_t \\ &= (1 + A - \beta)K_t - \alpha + (1 - \gamma)e_t \end{aligned}$$

Problem 4.1.3 (3). What values must the parameters α, β , and γ have for the first-order condition in part 1 to be satisfied for all values of K_t and e_t?

Solution. From the Euler equation, we have:

$$C_t = E_t\left[C_{t+1}\right] \tag{4.7}$$

Substituting $C_t = \alpha + \beta K_t + \gamma e_t$ and $C_{t+1} = \alpha + \beta K_{t+1} + \gamma e_{t+1}$ into (7), we get:

$$\alpha + \beta K_t + \gamma e_t = E_t\left[\alpha + \beta K_{t+1} + \gamma e_{t+1}\right] \tag{4.8}$$

Rearranging using

$$E_t\left[e_{t+1}\right] = E_t\left[\phi e_t + \varepsilon\right] = \phi e_t.$$

Substituting $K_{t+1} = (1 + A - \beta)K_t - \alpha - (1 - \gamma)e_t$ into (8), we get:

$$\alpha + \beta K_t + \gamma e_t = E_t\left[\alpha + \beta((1 + A - \beta)K_t - \alpha + (1 - \gamma)e_t) + \phi\gamma e_t + \gamma\varepsilon_t\right] \tag{4.9}$$

Simplifying (9) we get:

$$\begin{aligned} \alpha + \beta K_t + \gamma e_t &= \alpha + \beta(1 + A - \beta)K_t - \alpha\beta + \beta(1 - \gamma)e_t + \phi\gamma e_t + \gamma E_t\left[\varepsilon_t\right] \\ \Rightarrow \beta K_t + \gamma e_t &= \beta(1 + A - \beta)K_t - \alpha\beta + \beta(1 - \gamma)e_t + \phi\gamma e_t \end{aligned}$$

Hence, we have:

$$\begin{aligned} \alpha\beta &= 0 \\ \beta &= \beta(1 + A - \beta) \\ \gamma &= \beta(1 - \gamma) + \phi\gamma \end{aligned}$$

SOlving the above equations, we get:

$$\begin{aligned} &\alpha = 0 \\ &\beta = A = \rho \\ &\gamma = \frac{\beta}{1 + \beta - \phi} = \frac{A}{1 + A - \phi} \end{aligned}$$

Problem 4.1.4 (4). What are the effects of a one-time shock to ε on the paths of Y, K, and C?

Solution.

$$K_{t+1} = (1+A-\beta)K_t - \alpha + (1-\gamma)e_t = K_t + \frac{1-\phi}{1+A-\phi}e_t \tag{4.10}$$

Thus, we can have:

$$\begin{aligned} K_{t+T} &= K_t + \sum_{j=0}^{T-1} \frac{1-\phi}{1+A-\phi}e_{t+j} \\ &= K_t + \frac{1-\phi}{1+A-\phi}\sum_{j=0}^{T-1}\phi^j e_t \\ \Rightarrow \lim_{T\to\infty} K_{t+T} &= K_t + \frac{1}{1+A-\phi}e_t \end{aligned}$$

For C_t:

$$\begin{aligned} C_t &= \alpha + \beta K_t + \gamma e_t \\ &= AK_t + \frac{A}{1+A-\phi}e_t \end{aligned}$$

$$\begin{aligned} C_{t+1} &= AK_{t+1} + \frac{A}{1+A-\phi}e_{t+1} \\ &= A\left[K_t + \frac{1-\phi}{1+A-\phi}e_t\right] + \frac{A}{1+A-\phi}e_{t+1} \\ &= AK_t + \frac{A(1-\phi)}{1+A-\phi}e_t + \frac{A}{1+A-\phi}e_{t+1} \end{aligned}$$

Similarly,

$$\begin{aligned} C_{t+T} &= AK_{t+T} + \frac{A}{1+A-\phi}e_{t+T} \\ &= AK_t + \frac{A}{1+A-\phi}\sum_{t=0}^{T-1}\phi^j e_t \\ \Rightarrow \lim_{T\to\infty} C_{t+T} &= AK_t + \frac{A}{1+A-\phi}\phi^T e_t + \frac{A}{1+A-\phi}e_t \end{aligned}$$

4.2 Solving the RBC model by finding the social optimum

Problem 4.2.1 (1). Explain the Bellman equation:

$$V(K_t, A_t) = \max_{C_t, L_t}\left\{\log C_t + b\log(1-L_t) + \beta E_t\left[V(K_{t+1}, A_{t+1})\right]\right\} \tag{4.11}$$

Solution.

Problem 4.2.2 (2). Given the log-linear structure of the model, let us guess that V takes the form:

$$V(K_t, A_t) = \gamma_0 + \gamma_K \log K_t + \gamma_A \log A_t$$

where the values of the β's are to be determined. Substituting this conjectured form and the facts that $K_{t+1} = Y_t - C_t$ and $E_t[\log A_{t+1}] = \rho \log A_t$ into the Bellman equation yields

$$V(K_t, A_t) = \max_{C_t, L_t} \{\log C_t + b \log(1 - L_t) + \beta[\gamma_0 + \gamma_K \log(Y_t - C_t) + \gamma_A \log A_t]\}.$$

Find the frst-order condition for C_t. Show that it implies that C_t/Y_t does not depend on K_t or A_t.

Solution. Based on our value function, the first-order condition for C_t is:

$$\frac{\partial V}{\partial C_t} = 0 \Rightarrow \frac{1}{C_t} = \beta\gamma_K \frac{1}{Y_t - C_t} \Rightarrow \frac{C_t}{Y_t} = \frac{1}{1 + \beta\gamma_K} \tag{4.12}$$

Hence, C_t/Y_t does not depend on K_t or A_t.

Problem 4.2.3 (3). Find the first-order condition for L_t. Use this condition and the result in part 2 to show that Lt does not depend on K_t or A_t.

Solution.

$$\frac{\partial V}{\partial L_t} = 0 \Rightarrow -\frac{b}{1 - L_t} + \beta\gamma_K \frac{1}{Y_t - C_t}(1 - \alpha)K_t^{\alpha} A_t^{1-\alpha} L_t^{-\alpha} = 0 \tag{4.13}$$

Simplifying (12) we get:

$$-\frac{b}{1 - L_t} + \frac{\beta\gamma_K}{Y_t - C_t}(1 - \alpha)\frac{Y_t}{L_t} = 0 \tag{4.14}$$

Replace $Y_t - C_t$ with $\beta\gamma_K C_t$:

$$\begin{aligned}
&-\frac{b}{1 - L_t} + \frac{\beta\gamma_K}{\beta\gamma_K C_t}(1 - \alpha)\frac{Y_t}{L_t} = 0 \\
\Rightarrow &\frac{1 - \alpha}{L_t}(1 + \beta\gamma_K) = \frac{b}{1 - L_t} \\
\Rightarrow &L_t = \frac{(1 - \alpha)(1 + \beta\gamma_K)}{(1 - \alpha)(1 + \beta\gamma_K) + b}
\end{aligned}$$

So, L_t does not depend on K_t or A_t.

Problem 4.2.4 (4). Substitute the production function and the results in questions 2 and 3 for the optimal C_t and L_t into the equation above for V, and show that the resulting expression has the form $V(K_t, A_t) = \gamma_0' + \gamma_K' \log K_t + \gamma_A' \log A_t$.

Solution. As we know that

$$\begin{aligned}
E_t[\log A_{t+1}] &= \rho \log A_t \\
E_t[K_{t+1}] &= \log(Y_t - C_t)
\end{aligned}$$

So, by substituting the optimal C_t and L_t into the equation for V, we get:

$$V(K_t, A_t) = \log \frac{Y_t}{1 + \beta\gamma_K} + b \log \frac{b}{(1 - \alpha)(1 + \beta\gamma_K) + b} + \beta\left\{\gamma_0 + \gamma_K \log \frac{\beta\gamma_K Y_t}{1 + \beta\gamma_K} + \gamma_A \rho \log A_t\right\}$$

Substitute $Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$, we have:

$$\begin{aligned} V(K_t, A_t) = &\alpha \log K_t + (1-\alpha)\log A_t + (1-\alpha)\log L_t - \log(1+\beta\gamma_K) \\ &+ b\log\frac{b}{(1-\alpha)(1+\beta\gamma_K)+b} + \beta\gamma_0 \\ &+ \beta\gamma_K\{\log(\beta\gamma_K) - \log(1+\beta\gamma_K) + \alpha\log K_t + (1-\alpha)\log A_t + (1-\alpha)\log L_t\} \\ &+ \beta\gamma_A\rho\log A_t \end{aligned}$$

Rearranging the terms, we get:

$$\begin{aligned} V(K_t, A_t) = &(\alpha + \alpha\beta\gamma_K)\log K_t + (1-\alpha+(1-\alpha)\beta\gamma_K + \beta\gamma_A\rho)\log A_t \\ &+ (1-\alpha+(1-\alpha)\beta\gamma_K)\log L_t \\ &- \log(1+\beta\gamma_K) + b\log\frac{b}{(1-\alpha)(1+\beta\gamma_K)+b} + \beta\gamma_0 \\ &+ \beta\gamma_K\log(\beta\gamma_K) - \beta\gamma_K\log(1+\beta\gamma_K) \end{aligned}$$

We define:

$$\begin{aligned} \gamma_K' &= \alpha(1+\beta\gamma_K) \\ \gamma_A' &= (1-\alpha)(\beta\gamma_K+1) + \beta\gamma_A\rho \\ \gamma_0' &= \beta\gamma_0 + b\log\frac{b}{(1-\alpha)(1+\beta\gamma_K)+b} - \log(1+\beta\gamma_K)(1+\beta\gamma_K) \\ &+ \beta\gamma_K\log(\beta\gamma_K) + (1-\alpha+(1-\alpha)\beta\gamma_K)\log L_t \end{aligned}$$

According to problem 3, we know that L_t does not depend on K_t or A_t, so, we could treat it as a constant in this expression about K_t and A_t. Thus, we have:

$$V(K_t, A_t) = \gamma_0' + \gamma_K'\log K_t + \gamma_A'\log A_t.$$

Problem 4.2.5 (5). What must γ_K and γ_A be so that $\gamma_K' = \gamma_K$ and $\gamma_A' = \gamma_A$?

Solution. Let $\gamma_K' = \gamma_K$ and $\gamma_A' = \gamma_A$, we have:

$$\begin{aligned} \gamma_K &= \alpha(1+\beta\gamma_K) \\ \gamma_A &= (1-\alpha)(\beta\gamma_K+1) + \beta\gamma_A\rho \\ \Rightarrow \gamma_K &= \frac{\alpha}{1-\alpha\beta} \\ \Rightarrow \gamma_A &= \frac{1-\alpha}{(1-\alpha\beta)(1-\beta\rho)} \end{aligned}$$

Problem 4.2.6 (6). What are the implied values of C_t/Y_t and L_t? Are they the same as those found in the lecture?

Solution. Substitute γ_K into the expression for C_t/Y_t and L_t, we have:

$$\begin{aligned} \frac{C_t}{Y_t} &= \frac{1}{1+\beta\gamma_K} = \frac{1}{1+\beta\frac{\alpha}{1-\alpha\beta}} = 1-\alpha\beta \\ L_t &= \frac{(1-\alpha)(1+\beta\gamma_K)}{(1-\alpha)(1+\beta\gamma_K)+b} = \frac{(1-\alpha)}{(1-\alpha)+b(1-\alpha\beta)} \end{aligned}$$

Lecture 5.

Problem Set 4

Question 1

Consider an RBC model with a representative household that maximizes utility

$$U_t = E_t \sum_{t=0}^{\infty} \beta^t \left(\log\left(C_t\right) + \nu\left(1 - L_t\right)\right)$$

where ν is an increasing and concave function. The production function is

$$Y_t = A_t L_t^{1-\alpha}\left(z_t K_t\right)^{\alpha}$$

with $0 < \alpha < 1$ and z_t denotes a capital utilization choice. For example, low capital utilization may mean running the machines only at half speed or half of the time. Capital accumulation is described by

$$K_{t+1} = \left(1 - \bar{\delta} z_t^{\phi}\right) K_t + Y_t - C_t.$$

Problem (1). Explain how variable utilitzation of capital is modeled here. Should we assume that $\phi < 1$ or $\phi > 1$?

Solution. In the given model, capital utilization z_t directly affects both production and capital depreciation: in the production function, $z_t K_t$ is the effective capital input at time period t, higher z_t increases the effective capital services, thus boosting output. The depreciation rate is $\delta_t = \bar{\delta} z_t^{\phi}$, and higher z_t leads to a higher depreciation rate.
Depreciation Function $\delta_t = \bar{\delta} z_t^{\phi}$ determines how sensitively depreciation responds to changes in utilization z_t.The marginal increase in depreciation due to higher z_t is given by:

$$\frac{\partial \delta_t}{\partial z_t} = \bar{\delta}\phi z_t^{\phi-1}.$$

Intuitively, if utilization increases, the marginal cost of utilizing capital rises, reflecting higher incremental damage, meaning that teh capital depreciates faster, which implies:

$$\frac{\partial^2 \delta_t}{\partial z_t^2} = \bar{\delta}\phi(\phi - 1) z_t^{\phi-2} > 0.$$

Hence, we should assume that $\phi > 1$.

Problem (2). Derive and explain the optimality conditions of the social planner's problem. The social planner's problem is to maximize total utility subject to the economy's physical resource constraints (i.e. production, and the law of motion for capital). The social planner can set the levels of consumption, investment, capacity utilization, and labor supply, without having to worry about prices or market clearing.

Solution. The social planner maximizes the expected discounted utility:

$$\max_{\{C_t, I_t, L_t, z_t\}} E_t \sum_{t=0}^{\infty} \beta^t \left(\log(C_t) + \nu(1 - L_t)\right)$$
$$\text{s.t.} \quad Y_t = A_t L_t^{1-\alpha}(z_t K_t)^\alpha$$
$$K_{t+1} = (1 - \bar{\delta} z_t^\phi) K_t + Y_t - C_t$$

Define the Lagrangian:

$$\mathcal{L} = E_t \sum_{t=0}^{\infty} \beta^t \left\{ \log(C_t) + \nu(1 - L_t) - \lambda_t \left[C_t + K_{t+1} - (1 - \bar{\delta} z_t^\phi) K_t - Y_t \right] \right\}$$

The first-order conditions are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = \beta^t \left(\frac{1}{C_t} - \lambda_t\right) = 0 \Rightarrow \frac{1}{C_t} = \lambda_t \tag{5.1}$$

$$\frac{\partial \mathcal{L}}{\partial L_t} = \beta^t \left[-\nu'(1 - L_t) + \lambda_t A_t (1 - \alpha) L_t^{-\alpha} (z_t K_t)^\alpha \right] = 0 \Rightarrow \nu'(1 - L_t) = \lambda_t A_t (1 - \alpha) L_t^{-\alpha} (z_t K_t)^\alpha \tag{5.2}$$

$$\frac{\partial \mathcal{L}}{\partial z_t} = \beta^t \lambda_t \left[A_t \alpha L_t^{1-\alpha} z_t^{\alpha-1} K_t^\alpha - \bar{\delta} \phi z_t^{\phi-1} K_t \right] = 0 \Rightarrow \bar{\delta} \phi z_t^{\phi-1} = A_t \alpha L_t^{1-\alpha} z_t^{\alpha-1} K_t^{\alpha-1} \tag{5.3}$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} \left[(1 - \bar{\delta} z_{t+1}^\phi) + \alpha \frac{Y_{t+1}}{K_{t+1}} \right] = 0 \Rightarrow \lambda_t = \beta \lambda_{t+1} \left(1 - \bar{\delta} z_{t+1}^\phi + \alpha \frac{Y_{t+1}}{K_{t+1}}\right) \tag{5.4}$$

From the FOCs, we could tell that:

1. The shadow price of consumption equals to the marginal utility of consumption.

2. For the derivative with respect to L_t, we can transform it into the following format:

$$\nu'(1 - L_t) = \frac{\partial Y_t}{C_t \partial L_t}$$

   The marginal utility of leisure equals the marginal utility gain from increased consumption made possible by supplying an extra unit of labor.

3. For the derivative with respect to z_t, we can transform it into the following format:

$$\frac{\partial \delta_t K_t}{\partial z_t} = \frac{\partial Y_t}{\partial z_t}$$

   The marginal cost(depreciation) of capital equals to the marginal gain of production of increasing capital utilization.

4. The standard inter-temporal condition balancing current and future utility.

Problem (3). Derive an expression for the steady-state capital utilization z^* in terms of the steady-state capital-output ratio. If we want to normalize $z^* = 1$, how should we calibrate ϕ?

Solution. At the steady state: $Y_t = Y$, $C_t = C$, $K_t = K$, $A_t = A$ and $L_t = L$. Since $C_t = C_{t+1}$, we know that $\lambda_t = \lambda_{t+1}$. From 5.3, we know that:

$$\alpha A L^{1-\alpha}(z^* K)^{\alpha-1} = \phi\bar{\delta}{z^*}^{\phi-1} = \frac{\alpha Y}{z^* K}$$
$$\Rightarrow \frac{Y}{K} = \frac{\phi\bar{\delta}{z^*}^{\phi}}{\alpha}$$

Denote $\kappa = \frac{K}{Y}$, which is the output ratio, we would have:

$$z^* = \left(\frac{\alpha}{\kappa\phi\bar{\delta}}\right)^{\frac{1}{\phi}}.$$

Then, we use 5.4 and $\lambda_t = \lambda_{t+1}$, we'll have:

$$\begin{aligned}
1 &= \beta(1 - \bar{\delta}{z^*}^{\phi} + \alpha\frac{Y}{K}) = \beta(1 - \bar{\delta}{z^*}^{\phi} + \alpha\frac{\phi\bar{\delta}{z^*}^{\phi}}{\alpha}) = \beta(1 - \bar{\delta}{z^*}^{\phi} + \phi\bar{\delta}{z^*}^{\phi}) \\
\Rightarrow& (\phi - 1)\bar{\delta}{z^*}^{\phi} = \frac{1-\beta}{\beta} \\
\Rightarrow& z^* = \left(\frac{1-\beta}{\beta(\phi-1)\bar{\delta}}\right)^{\frac{1}{\phi}}
\end{aligned}$$

Normalize $z^* = 1$, we simplify the equation into:

$$1 - \beta = \beta(\phi - 1)\bar{\delta},$$

which gives that

$$\phi = \frac{1-\beta}{\beta\bar{\delta}} + 1 = \frac{1-\beta+\beta\bar{\delta}}{\beta\bar{\delta}}$$

Or, we would simplify the expression of z^* with capital-output ratio κ:

$$\alpha = \kappa\phi\bar{\delta} \Rightarrow \phi = \frac{\alpha}{\kappa\bar{\delta}}$$

Problem (4). Compared to a standard RBC model (as in the lecture), do you expect this model to generate a stronger or weaker response of output to a TFP shock?

Solution. Compared to a standard RBC model, this model generates stronger response of output to a TFP shock.

In the Standard RBC Model:

Capital utilization is fixed ($z_t = 1$). Output response to TFP shocks relies solely on adjustments in labor and capital.

In This Model with Variable Utilization:

Firms can adjust z_t in response to TFP shocks. A positive TFP shock allows firms to temporarily increase z_t , boosting output beyond what is possible in the standard model, which gives higher fluctuations.

Problem (5). Assuming the model is correct, how is the standard empirical Solow residual $\log Y_t - (1-\alpha)\log L_t - \alpha \log K_t$ related to total factor productivity $\log A_t$?

Solution. If our model is correct, then we log-linearize the production function:

$$\log Y_t = \log A_t + (1-\alpha)\log L_t + \alpha \log K_t + \alpha \log z_t$$
$$\Rightarrow \text{Solow Residual} = \log Y_t - (1-\alpha)\log L_t - \alpha \log K_t = \log A_t + \alpha \log z_t$$

Question 2

We study a variation of the simplified RBC model from the lecture. Consider an economy populated by a mass of representative households and a mass of representative firms which take output and factor prices as given. The production function is Cobb-Douglas:

$$Y_t = e^{z_t} K_t^{\alpha} L_t^{1-\alpha}$$

where e_t^z is total factor productivity, K is capital, and L is labor. Capital fully depreciates in every periods so that

$$K_{t+1} = I_t$$

where I_t is investment.

The representative household supplies labor L_t and consumes C_t. It maximizes

$$U = E_0 \left[\sum_{t=0}^{\infty} \beta^t \left[\log C_t - e^{\chi_t} L_t\right] \right]$$

where χ_t is a preference shock. Capital is accumulated by the household and rented to the firm, as usual. Any profits of the firms are rebated to the households. As usual, we have $0 < \alpha < 1, 0 < \beta < 1$.

Problem (1). Write down the budget constraint of the household and the profit function of firms.

Solution. Assume the rental rate of capital is r_t and the wage rate is w_t. The budget constraint of the household is:

$$C_t + K_{t+1} = r_t K_t + w_t L_t + \Pi_t$$

The profit function of firms is:

$$\Pi_t = Y_t - r_t K_t - w_t L_t = e^{z_t} K_t^{\alpha} L_t^{1-\alpha} - r_t K_t - w_t L_t$$

Problem (2). Derive first-order conditions of firms and households.

Solution. From problem (1), we established:

$$\Pi_t = Y_t - r_t K_t - w_t L_t \Rightarrow r_t K_t + w_t L_t + \Pi_t = Y_t.$$

Thus, our households budget constraint is:

$$C_t + K_{t+1} = Y_t.$$

We define the Lagrangian of the household problem as:

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \beta^t \left[\log C_t - e^{\chi_t} L_t - \lambda_t (C_t + K_{t+1} - Y_t)\right].$$

The first-order conditions of households are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = \beta^t (\frac{1}{C_t} - \lambda_t) = 0 \Rightarrow \frac{1}{C_t} = \lambda_t$$
$$\frac{\partial \mathcal{L}}{\partial L_t} = \beta^t (-e^{\chi_t} + \lambda_t \frac{\partial Y_t}{\partial L_t}) = 0 \Rightarrow e^{\chi_t} = \lambda_t \frac{\partial Y_t}{\partial L_t}$$
$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = -\beta^t \lambda_t + \beta^{t+1} E_t \lambda_{t+1} \frac{\partial Y_{t+1}}{\partial K_{t+1}} = 0 \Rightarrow \lambda_t = \beta \lambda_{t+1} E_t \frac{\partial Y_{t+1}}{\partial K_{t+1}}$$

Let's compute $\frac{\partial Y_t}{\partial L_t}$ and $\frac{\partial Y_t}{\partial K_{t+1}}$:

$$\frac{\partial Y_t}{\partial L_t} = e^{z_t} K_t^{\alpha} (1-\alpha) L_t^{-\alpha} = (1-\alpha) \frac{Y_t}{L_t}$$
$$\frac{\partial Y_{t+1}}{\partial K_{t+1}} = \alpha e^{z_{t+1}} K_{t+1}^{\alpha-1} L_{t+1}^{1-\alpha} = \alpha \frac{Y_{t+1}}{K_{t+1}}$$

So, our households FOCs are:

$$\lambda_t = \frac{1}{C_t} \tag{5.5}$$
$$e^{\chi_t} = \lambda_t (1-\alpha) \frac{Y_t}{L_t} = \frac{(1-\alpha) Y_t}{C_t L_t} \tag{5.6}$$
$$\lambda_t = \alpha \beta \lambda_{t+1} E_t \frac{Y_{t+1}}{K_{t+1}} \Rightarrow \frac{1}{C_t} = \alpha \beta E_t \frac{Y_{t+1}}{C_{t+1} K_{t+1}} \tag{5.7}$$

The Lagrangian of the firm problem is:

$$\max_{K_t, L_t} \Pi_t = e^{z_t} K_t^{\alpha} L_t^{1-\alpha} - r_t K_t - w_t L_t$$

The first-order conditions of firms are:

$$\frac{\partial \Pi_t}{\partial K_t} = \alpha e^{z_t} K_t^{\alpha-1} L_t^{1-\alpha} - r_t = 0 \Rightarrow r_t = \alpha e^{z_t} K_t^{\alpha-1} L_t^{1-\alpha} = \alpha \frac{Y_t}{K_t} \tag{5.8}$$
$$\frac{\partial \Pi_t}{\partial L_t} = (1-\alpha) e^{z_t} K_t^{\alpha} L_t^{-\alpha} - w_t = 0 \Rightarrow w_t = (1-\alpha) e^{z_t} K_t^{\alpha} L_t^{-\alpha} = (1-\alpha) \frac{Y_t}{L_t} \tag{5.9}$$

Problem (3). Show that when factor markets and final good markets are competitive, firm profits are zero.

Solution. The competitive equilibrium requires that the rental rate of capital equals the marginal product of capital and the wage rate equals the marginal product of labor:

$$r_t = \alpha \frac{Y_t}{K_t}$$
$$w_t = (1-\alpha) \frac{Y_t}{L_t}$$

In this case, the profit function of firms becomes:

$$\Pi_t = Y_t - \alpha \frac{Y_t}{K_t} K_t - (1-\alpha) \frac{Y_t}{L_t} L_t = 0.$$

Problem (4). Solve the model and show that the process for output in a competitive equilibrium is

$$y_t = z_t + \alpha y_{t-1} - (1-\alpha)\chi_t$$

(where lowercase letters denote the log of the uppercase letter).

Solution. From the production function FOC 5.7, we have: $\frac{1}{C_t} = \alpha\beta E_t \frac{Y_{t+1}}{K_{t+1}C_{t+1}}$. Since $Y_{t+1} = C_{t+1} + I_{t+1} = C_{t+1} + K_{t+2}$, We have:

$$\frac{1}{C_t} = \alpha\beta E_t \frac{C_{t+1} + K_{t+2}}{K_{t+1}C_{t+1}}$$
$$\Rightarrow \frac{K_{t+1}}{C_t} = \alpha\beta E_t \left(1 + \frac{K_{t+2}}{C_{t+1}}\right)$$

According to the Banach's Fixed-point Theorem, $x = \alpha\beta(1+x)$, and we can solve

$$\frac{K_{t+1}}{C_t} = \frac{\alpha\beta}{1-\alpha\beta}.$$

As $K_{t+1} = Y_t - C_t$, we have $K_{t+1} = \alpha\beta Y_t$ and $C_t = (1-\alpha\beta)Y_t$. Then, by 5.6 and 5.9, we know that:

$$e^{\chi_t} = \frac{(1-\alpha)Y_t}{C_t L_t}$$
$$\Rightarrow e^{\chi_t} = \frac{(1-\alpha)Y_t}{(1-\alpha\beta)Y_t L_t} = \frac{1-\alpha}{1-\alpha\beta}\frac{1}{L_t}$$
$$\Rightarrow L_t = \frac{1-\alpha}{1-\alpha\beta}e^{-\chi_t}$$

$$Y_t = e^{z_t}K_t^{\alpha}L_t^{1-\alpha} = e^{z_t}(\alpha\beta Y_{t-1})^{\alpha}\left(\frac{1-\alpha}{1-\alpha\beta}e^{-\chi_t}\right)^{1-\alpha}$$

The the log of both sides, we have:

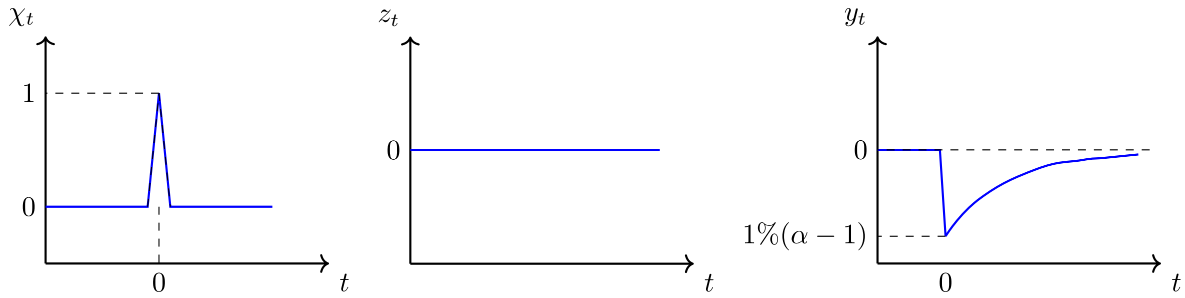
$$\log Y_t = z_t + \alpha\log(\alpha\beta) + \alpha\log Y_{t-1} + (1-\alpha)\log\left(\frac{1-\alpha}{1-\alpha\beta}\right) - (1-\alpha)\chi_t$$
$$= z_t + \alpha\log Y_{t-1} - (1-\alpha)\chi_t + \alpha\log(\alpha\beta) + (1-\alpha)\log\left(\frac{1-\alpha}{1-\alpha\beta}\right)$$
$$\Rightarrow y_t = z_t + \alpha y_{t-1} - (1-\alpha)\chi_t + \alpha\log(\alpha\beta) + (1-\alpha)\log\left(\frac{1-\alpha}{1-\alpha\beta}\right)$$

We drop the constant and gives: $y_t = z_t + \alpha y_{t-1} - (1-\alpha)\chi_t$.

Problem (5). Assume $y_{-1} = 0$ and $z_t = 0$ for all t, and $\chi_t = 0$ for all t except that $\chi_0 = 1\%$. Draw the time path for χ_t, z_t,and y_t. Explain why y_t is persistent.

Solution.

At time period 0, $y_0 = z_0 + \alpha y_{-1} - (1-\alpha)\chi_0 = 1\%(\alpha-1)$, and after time period 0, we have $z_t = 0$ and $\chi_t = 0$, thus $y_t = \alpha y_{t-1} = 1\%\alpha^t(\alpha-1)$. Then, we can draw the three graphs as below:



The initial preference shock χ_0 increases the disutility of labor, leading households to supply less labor at $t = 0$.
This reduces output at $t = 0$.
Since capital at $t + 1$ depends on savings from Y_t , the reduced output leads to lower capital K_{t+1}.
The lower capital stock in subsequent periods reduces output further, even though the preference shock is gone.
The capital accumulation process transmits the shock over time, causing persistence in output deviations from steady state.
But as the shock has ended, the output will finally return back to the steady state.

Lecture 6.

Real Business Cycle Model(2)

6.1 A more General Model

Assuming not full depreciation $0 < \delta < 1$. The household's problem is

$$U = E_t \sum_{t=0}^{\infty} \beta^i U(C_{t+i}, 1 - L_{t+i})$$

where the utility function is:

$$U(C_t, 1 - L_t) = \ln C_t + \theta \frac{(1 - L_t)^{1-\gamma_l}}{1 - \gamma_l}$$

log in consumption, CES is leisure. $\gamma_l = 1$ shows log in leisure as well.

6.1.1 The log-linearized version

Small letters denote the log-deviations from the non-stochastic BGP: $c_t = \ln C_t - \ln C_t^*$, $l_t = \ln L_t - \ln L_t^*$.

1. The production function The production function is:

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$$

rewrite as:

$$Y_t^* e^{y_t} = {K_t^*}^{\alpha} e^{\alpha k_t} (A_t^* L_t^*)^{1-\alpha} e^{(a_t + l_t)(1-\alpha)}$$

and use the production function to get:

$$y_t = \alpha k_t + (a_t + l_t)(1 - \alpha).$$

2. Capital accumulation

$$K_{t+1} = (1 - \delta)K_t + Y_t - C_t$$

re-write as

$$K_{t+1}^* e^{k_{t+1}} = (1 - \delta)K_t^* e^{k_t} + Y_t^* e^{y_t} - C_t^* e^{c_t}$$

Now use the fact that for x small enough $e^x \approx 1 + x$ to get:

$$K_{t+1}^* + K_{t+1}^* k_{t+1} \approx (1 - \delta)K_t^* + (1 - \delta)K_t^* k_t + Y_t^* + Y_t^* y_t - C_t^* - C_t^* c_t$$

which simplifies to

$$\frac{K_{t+1}^*}{K_t^*} k_{t+1} \approx (1 - \delta)k_t + \frac{Y_t^*}{K_t^*} y_t - \frac{C_t^*}{K_t^*} c_t.$$

3. The rental rate

$$R_t = \alpha \left(\frac{A_t L_t}{K_t} \right)^{1-\alpha} + (1 - \delta)$$

rewrite as

$$R_t^* e^{r_t} = \alpha \left(\frac{A_t^* L^*}{K_t^*} \right)^{1-\alpha} e^{(1-\alpha)(a_t + l_t - k_t)} + (1-\delta)$$

this is approximately

$$R_t^* r_t \approx \alpha(1-\alpha) \left(\frac{A_t^* L^*}{K_t^*} \right)^{1-\alpha} (a_t + I_t - k_t).$$

4. The wage rate

$$W_t = (1-\alpha) A_t^{1-\alpha} \left(\frac{K_t}{L_t} \right)^{\alpha}$$

re-write as

$$W_t^* e^{w_t} = (1-\alpha) A_t^{*^{1-\alpha}} e^{(1-\alpha)a_t} \left(\frac{K_t^*}{L_t^*} \right)^{\alpha} e^{\alpha(k_t - I_t)}$$

which simplifies to

$$w_t = (1-\alpha) a_t + \alpha(k_t - I_t).$$

5. The intra-temporal FOC

$$\frac{W_t}{C_t} = \theta(1 - L_t)^{-\gamma_l}$$

re-write and approximate:

$$\frac{W_t^* e^{W_t}}{C_t^* e^{C_t}} = \theta(1 - L^* e^{I_t})^{-\gamma_l} \Rightarrow \frac{W_t^* + W_t^* w_t}{C_t^* + C_t^* c_t} \approx \theta(1 - L^* - L^* I_t)^{-\gamma_l}$$

we do a first order taylor approximation around $c_t = w_t = I_t = 0$:

$$\frac{W_t^*}{C_t^*} + \frac{W_t^*}{C_t^*} w_t - \frac{W_t^*}{C_t^*} c_t \approx \theta(1-L^*)^{-\gamma_l} + \theta\gamma_l(1-L^*)^{-\gamma_l - 1} L^* I_t \frac{W_t^*}{C_t^*}(w_t - c_t) \approx \theta\gamma_l(1-L^*)^{-\gamma_l-1} L^* I_t$$

which simplifies to

$$w_t - c_t \approx \gamma_l \frac{L^*}{1-L^*} I_t.$$

6. The inter-temporal FOC

$$C_t = \beta E_t \left(\frac{R_{t+1}}{C_{t+1}} \right)$$

re-write as

$$C_t^* e^{C_t} = \beta E_t \left(\frac{R^* e^{r_{t+1}}}{C_{t+1}^* e^{c_{t+1}}} \right)$$

which is

$$e^{c_t} = E_t \left(\frac{e^{r_{t+1}}}{e^{c_{t+1}}} \right) \Rightarrow \frac{1}{1+c_t} \approx E_t \left(\frac{1 + r_{t+1}}{1 + c_{t+1}} \right)$$

finally, do a first order Taylor approximation

$$-c_t \approx E_t(r_{t+1} - c_{t+1}).$$

Blanchard-Kahn conditions

Lecture 7.

Money and the New Keynesian Model

7.1 About Monetary Economics

Does an unanticipated shock to money affect real side of the economy (quantities)?

- CLassical view: If prices are flexible, the shock will change wages and prices proportionally, but not change the quantities.

  If the gov prints money and spend it, the prices increase and nominal assets lose in value, which is negative wealth effect.

  Money supply increase acts like a tax on nominal asset holdings, some call it 'inflation tax'.

- Keynesian view: Prices are rigid, so the shock will give a positive wealth effect, this assumes that new money is distributed proportionally to existing money holdings.

How should we conduct monetary policy?

- Monetatists: steady money supply growth, in line with money demand growth.
- Keynesians: use expansionary MP to stimulate the economy in a recession.
- Inflation of 70's and subsequent successful tight MP(Volcker) led to Monetarists becoming very influential.

Problems of previous models

- Old models are subject to the Lucas critique(policy changes may affect people's response to the policies)
- Hard to take seriously from a quantitative perspective

Since RBC model has try to use dynamic method to explain the economy, people start putting money into the model.

- Outcome from this program is the "New Keynesian Synthesis": combine RBC models with price rigidities.
- Money is non-neutral in the short run (Keynesian element), and neutral in the long run (classical element).

7.2 The Monetary Transmission Mechanism

7.2.1 Households and nominal assets

Nominal assets are bonds B_t which pay interest at nominal rate i_t. Ignore the labor-leisure choice, the households maximize:

$$\begin{aligned} \max \quad & \mathbb{E}_t \sum_{i=0}^{\infty} U(C_{t+i}) \\ \text{s.t.} \quad & P_{t+i}C_{t+i} + P_{t+i}K_{t+i+1} + B_{t+i+1} = W_{t+i}L + P_{t+i}(1+r_{t+i})K_{t+i} + (1+i_{t+i})B_{t+i} \end{aligned}$$

where W is the nominal wage, P is the price of cunsumption goods, r_{t+i} is the rental rate of capital net of depreciation.

Define the Lagrangian:

$$\mathcal{L} = \mathbb{E}_t \sum_{i=0}^{\infty} \left[U(C_{t+i}) + \lambda_{t+i}\left(W_{t+i}L + P_{t+i}(1+r_{t+i})K_{t+i} + (1+i_{t+i})B_{t+i} - P_{t+i}C_{t+i} - P_{t+i}K_{t+i+1} - B_{t+i+1}\right)\right]$$

The FOCs are:

$$\begin{aligned} U'(C_{t+i}) &= \mathbb{E}_t P_{t+i}\lambda_{t+i} \\ P_{t+i}\lambda_{t+i} &= \mathbb{E}_t \lambda_{t+i+1}P_{t+i+1}(1+r_{t+i+1}) \\ \lambda_{t+i} &= \mathbb{E}_t \lambda_{t+i+1}(1+r_{t+i+1}) \end{aligned}$$

Combining the first two, we get the standard Euler equation:

$$U'(C_t) = \mathbb{E}_t\left(U'(C_{t+1})(1+r_{t+1})\right)$$

Conbine the first and the third:

$$U'(C_t) = \mathbb{E}_t\left(\frac{1+i_{t+1}}{1+\pi_{t+1}}U'(C_{t+1})\right)$$

where inflation $\pi = \frac{P_{t+1}-P_t}{P_t}$.

$$U'(C_t) = \mathbb{E}_t\left(\frac{1+i_{t+1}}{1+\pi_{t+1}}U'(C_{t+1})\right)$$

$$U'(C_t) = \mathbb{E}_t\left((1+r_{t+1})U'(C_{t+1})\right)$$

- What matters for the intertemporal consumption decision is the real interest rate, not the nominal interest rate.

- The FOC are necessary conditions for having both positive B and positive K. By the two equations above, we get:
$$1 + r_{t+1} = \frac{1 + i_{t+1}}{1 + \pi_{t+1}}.$$
So, the real return(interest rate) of a nominal asset with nominal interest rate $1 + i_{t+1}$ is: $\frac{1+i_{t+1}}{1+\pi_{t+1}}$.

Corollary 7.2.1 (Fisher equation)**.**

It's the relationship between asset returns that has to hold so that the household is willing to hold positive amounts of these assets in their portfolio (or: if you allow negative holdings, to clear asset markets).
When households are risk-neutral, the FOCs become:
$$\mathbb{E}_t(1 + r_{t+1}) = \mathbb{E}_t\left(\frac{1 + i_{t+1}}{1 + \pi_{t+1}}\right)$$
In a deterministic world, there's no E, which is often approxima

7.2.2 Firms

The firms would like to maximize profits, with perfect competition adn frictionless factor markets:
$$MPK = r$$
Again, real interest rate matters.

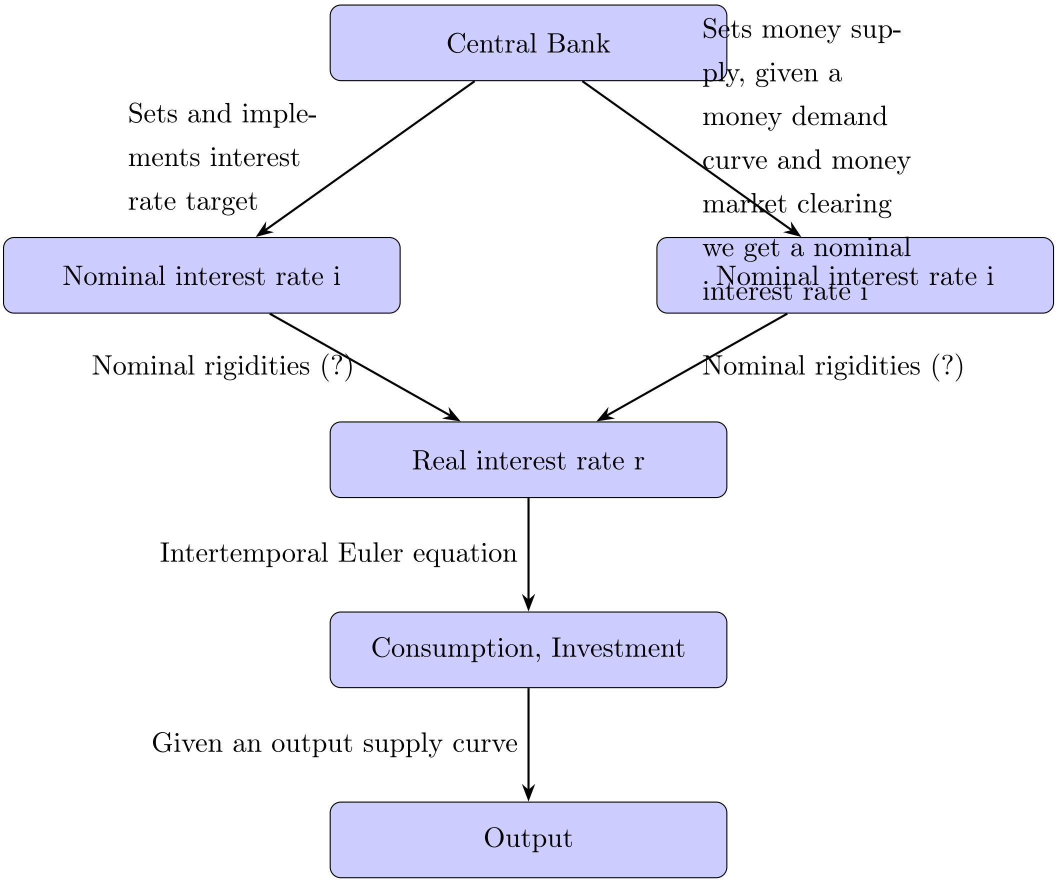
Conclusion: if monetary policy should affect demand, the real interest rate is the key price in the economy.

7.3 The tools of monetary policy

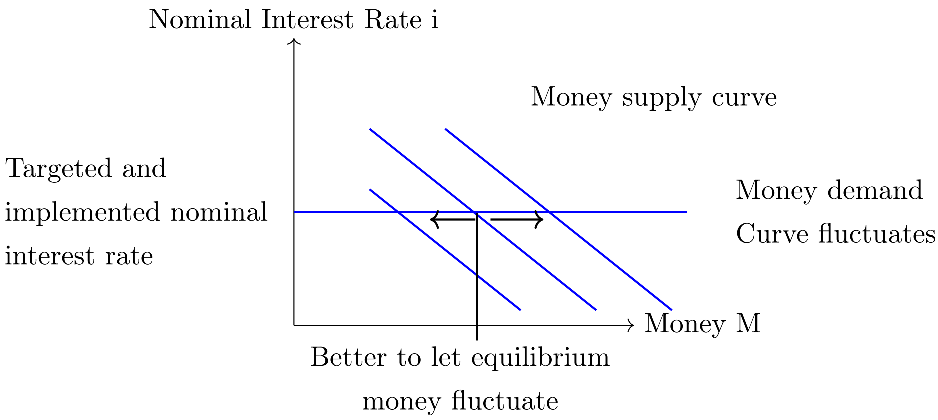
So, in order to stimulate demand, the central bank needs to change the real interest rate.

How?

- Change the money supply, with demand curve, this would implement a particular nominal interest rate.
- Change the nominal interest rate directly.



Money supply-demand equilibrium



7.4 A model of the interbank market

- Market participants: n banks.
- Banks hold reserves at the central bank.
- REserves are used for settling transfers between banks arising from the payment system.
- Banks are unsure of the transfers htey will need to make.

- Banks can borrow reserves from one another in the interbank market, but do not know how much reserve they'll require at the point they participate in the interbank market.
- Banks can also acquire reserves through the repo (repurchase agreement) market. The central bank can be a participant in this market.

Note.

B_j = reserves of bank j at beginning-of-period.(after previous debts settles)
I_j = net borrowing in the interbank market (negative for lending) by bank j.
P_j = net bond repos(sales and repurchases) by bank j.
T_j = net payment bank j make to other banks.
R_j = balance of bank j at end-of-period.
Accounting: $R_j = B_j + I_j + P_j - T_j$.

7.4.1 Interbank market and repo market

Interbank market:

- Uncollateralized loans between banks.(银行间无抵押贷款, potentially risky)
- Interest rate i.

Repo market:

- Sell bond now and agree to repurchase later.
- Effectively a collaboratized loan, so risk-free if collateral is good.

Note (Central bank facilities).

Deposit Facility: Centrla bank offers an interest rate of i_d on deposit balances in banks' account at the end of time period.

Borrowing facility: Central bank charges an interest rate i_b on negative balances in banks' accounts.

- Positive spread over deposit rate: $i_b > i_d$.
- Equivalent to an offer to make loans at the end of perios, with the oad credited to the bank's account.
- Collateral required – we should assune banks have enough collateral to use it.

7.4.2 Balance on account next period

Reserve balance of bank j at the end of period is:

$$B'_j = R_j - (1+i)(I_j + P_j) + \begin{cases} i_d R_j & \text{if } R_j \geq 0 \\ i_b R_j & \text{if } R_j < 0 \end{cases}$$

Since $R_j = B_j + I_j + P_j - T_j$, we have:

$$B'_j = B_j - T_j - i(I_j + P_j) + \begin{cases} i_d(B_j + I_j + P_j - T_j) & \text{if } B_j + I_j + P_j - T_j \geq 0 \\ i_b(B_j + I_j + P_j - T_j) & \text{if } B_j + I_j + P_j - T_j < 0 \end{cases}$$

Assume banks are risk-neutrala, and they aim to maximize their next period expected balance $\mathbb{E}(B'_j)$. Choice variabls are: interbank transaction I_j, repo market transaction P_j.

$$\begin{aligned} \mathbb{E}(B'_j) = B_j - i(I_j + P_j) + i_d \int_{-\infty}^{B_j+I_j+P_j} (B_j + I_j + P_j - T_j) dF(T_j) \\ + i_b \int_{B_j+I_j+P_j}^{\infty} (B_j + I_j + P_j - T_j) dF(T_j) \end{aligned}$$

The FOCs are:

$$\frac{\partial \mathbb{E}(B'_j)}{\partial I_j} = 0$$
$$\frac{\partial \mathbb{E}(B'_j)}{\partial P_j} = 0$$

Two equations lead to the same equation, so we only consider one of them.

Theorem 7.4.1 (Leibniz Rule)**.**

Let $f(x,t)$ be a function such that both $f(x,t)$ and its partial derivative $f_x(x,t)$ are continuous in t and x in some region of the xt-plane, including $a(x) \leq t \leq b(x)$, $x_0 \leq x \leq x_1$. Also suppose that the functions $a(x)$ and $b(x)$ are both continuous and both have continuous derivatives for $x_0 \leq x \leq x_1$. Then, for $x_0 \leq x \leq x_1$,

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = f(x,b(x)) \cdot \frac{d}{dx} b(x) - f(x,a(x)) \cdot \frac{d}{dx} a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt.$$

Use the Leibniz rule:

$$\begin{aligned} &\frac{\partial}{\partial I_j} \int_{-\infty}^{B_j+I_j+P_j} (B_j + I_j + P_j - T_j) dF(T_j) \\ &= \int_{-\infty}^{B_j+I_j+P_j} f(T_j) dT_j + ((B_j + I_j + P_j) - (B_j + I_j + P_j)) f(B_j + I_j + P_j) \\ &= F(B_j + I_j + P_j) \end{aligned}$$

Similarly,

$$\begin{aligned} &\frac{\partial}{\partial I_j} \int_{B_j+I_j+P_j}^{\infty} (B_j + I_j + P_j - T_j) dF(T_j) \\ &= 1 - F(B_j + I_j + P_j) \end{aligned}$$

Hence, the FOC reduces to:

$$-i + i_d F(B_j + I_j + P_j) + i_b(1 - F(B_j + I_j + P_j)) = 0$$

Equivalently,

$$(i - i_d) F(B_j + I_j + P_j) = (i_b - i)(1 - F(B_j + I_j + P_j))$$

The optimal demand for interbank and repo lending is characterized by the equation:

$$F(B_j + I_j + P_j) = \frac{i_b - i}{i_b - i_d}$$

7.4.3 Aggregation and market clearing

$$
\begin{aligned}
B &= \sum_{j=1}^{n} B_j \text{(Aggregate beginning-of-period balances)} \\
P &= \sum_{j=1}^{n} P_j \text{(Aggregate end-of-period balances)} \\
T &= \sum_{j=1}^{n} T_j = 0 \text{(payments between banks cancel out)} \\
R &= \sum_{j=1}^{n} R_j
\end{aligned}
$$

Since $R_j = B_j + I_j + P_j - T_j$, we have $R = B + P$ in the aggregate.

Since $B_j + I_j + P_j$ is the same for all banks: $B_j + I_j + P_j = \frac{R}{n}$.

7.4.4 Equilibrium interbank interest rate

$$
\begin{aligned}
F\left(\frac{R}{n}\right) &= \frac{i_b - i}{i_b - i_d} \\
\frac{R}{n} &= F^{-1}\left(\frac{i_b - i}{i_b - i_d}\right) \\
i &= i_d + \frac{i_b - i_d}{n} \int_{-\infty}^{F^{-1}\left(\frac{i_b - i}{i_b - i_d}\right)} x dF(x)
\end{aligned}
$$

THe central bank directly controls:

- i_d and i_b.
- Open market operations P.

Indirectly determinems

- Interbank and repo rate i.
- Total end-of-period reserves $R = B + P$.

7.4.5 Interbank market with channel system

- **Demand:** Equation $F\left(\frac{R}{n}\right) = \frac{i_b - i}{i_b - i_d}$, giving a negative relationship between i and R.
- **Supply:** Central bank sets $R = B + P$ by open market operations.

If we hold all other policy instruments constant,

- An increase in total reserves R will shift supply curve to the right, decreasing i.

- An increase in i_d will shift demand curve upward, increasing i.
- An increase in i_b will shift demand curve upward, increasing i.

Note.

A narrow channel has advantages:

- More precise control of i, less fluctuations.
- Less need for open market operations.

There'are also disadvantages, e.g. $i_b = i_b$:

- Trading in interbank market dries up.
- CB becomes the intermediary of all borrowing and lending between banks, hence it incurs the cost of monitoring credit-worthiness of banks. It's not the central bank's job.

7.4.6 Interest rate maturity

Question. How does this affect medium- and long-term interest rates? How are interest rates of different maturities related?

Example 1.

Imagine an investor invest for 2 years:

- Buy a 2-year bond, keep until it matures:

$$\text{return} = 1 + i_t^{(2y)}$$

- Buy a 1-year bond, then buy another 1-year bond:

$$\text{return} = (1 + i_t^{(1y)})(1 + i_{t+1}^{1y})$$

Expectations Hypothesis: Expectations of these two returns should be the same:

$$i_t^{(2y)} = \mathbb{E}_t(1 + i_t^{(1y)})(1 + i_{t+1}^{(1y)}).$$

This should hold exactly if agents are risk-neutral. WIth risk aversion, one would adjust the expected retuen for riskiness.

Lecture 8.

Problem Set 5

Question 1

Consider the model of the interbank market presented in the lectures. i is the interbank interest rate, id is the interest rate offered by the central bank on positive balances on banks' reserve accounts (deposit rate), ib is the interest rate charged by the central bank on negative reserve balances (borrowing rate). Banks face uncertainty about the payments they must make after the interbank market has closed. The probability that a bank's required payment is less than T is denoted by $F(T)$. Optimization by banks implies the following condition:

$$(i_b - i)(1 - F(R/n)) = (i - i_d)F(R/n)$$

or equivalently:

$$F(R/n) = \frac{i_b - i}{i_b - i_d}$$

where R is the total quantity of reserves and n is the number of banks.

Problem (1). Provide an intuitive explanation of the optimality condition for banks.

Solution.

Problem (2). Show how the equilibrium interbank interest rate is determined using a diagram.

Solution. The conditions gives that:

$$F(R/n) = \frac{i_b - i}{i_b - i_d} \in [0, 1] \Rightarrow i_d \leq i \leq i_b.$$

If the inter-bank rate

Problem (3). Suppose the central bank would like to raise the market interest rate. Explain how this can be done through:

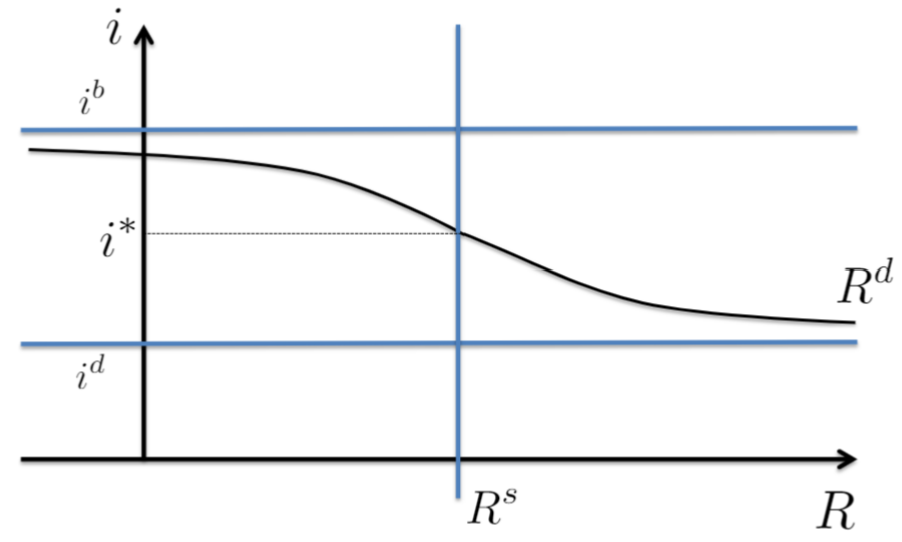
(a) Open market operations (only R is adjusted);

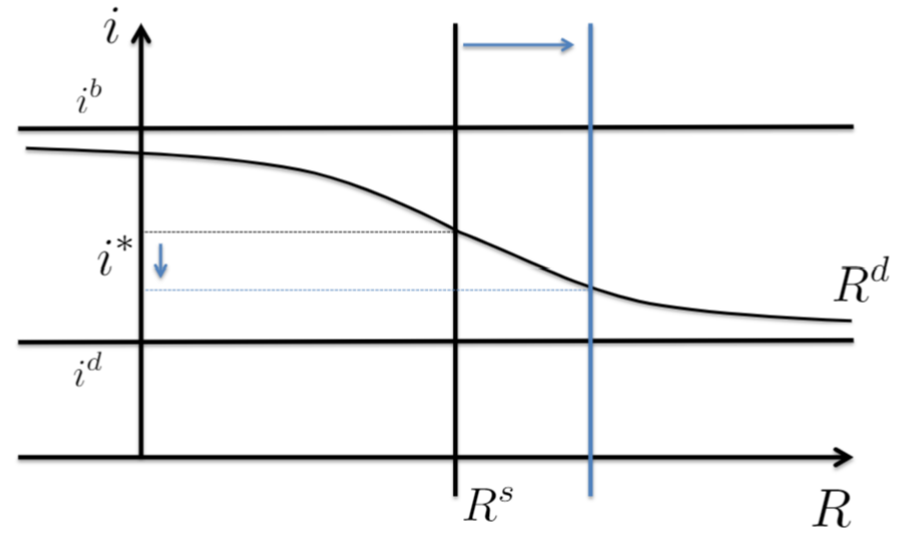
(b) Adjusting standing facility terms (only ib and id are adjusted)

Solution. Originally, the graph changes as the following graph:

Let's suppose the bank increase rate by x, say $i' = i + x$.

$$F(R/n) = \frac{(i_b + x) - (i + x)}{(i_b + x) - (i_d + x)} = \frac{i'_b - i'}{i'_b - i'_d}$$


i
i^b
i^*
R^d
i^d
R^s
R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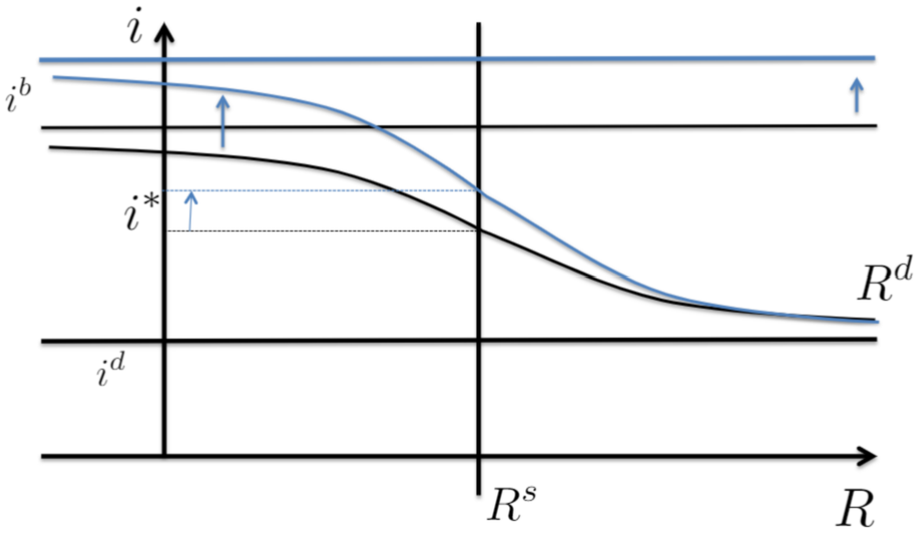
i
i^b
i^*
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R^s
R

Question 2

You are hired by the International Monetary Fund, and you're being assigned on a mission to provide technical assistance to Bank al-Maghrib, the central bank of Morocco. You're being asked to estimate the impact of monetary policy on unemployment. Knowing that your boss is not fond of DSGE models, but also distrusts the results from Christiano-Eichenbaum-Evans (because they were obtained with data from high-income countries), you decide to estimate a Vector Autoregression model (with interest rates and unemployment as the endogenous variables) yourself.

Problem (1). Write down the system of equations for a first-order Vector Autoregression where the nominal interest rate and unemployment depend on each other and a lagged variable for each. Both variables are also determined by a shock, η_t^i and η_t^u , which are orthogonal to past values of i and u.

Solution.

$$\begin{aligned}
i_t &= \phi_1 i_{t-1} + \phi_2 u_t + \phi_3 u_{t-1} + \eta_t^i \\
u_t &= \phi_4 i_t + \phi_5 u_{t-1} + \phi_6 i_{t-1} + \eta_t^u \\
&cov(x_{i_t}, \varepsilon_{i_t}) = 0
\end{aligned}$$

$$\Rightarrow i_t = \phi_1 i_t + \phi_2(\phi_4 i_{t-1} + \phi_5 u_{t-1} + \phi_6 i_{t-1} + \eta_t^u) + \phi_3 u_{t-1} + \eta_t^i$$

Solve the equation, we have:

$$\begin{aligned}
i_t &= \frac{1}{1-\phi_2\phi_4}\left[(\phi_1 + \phi_2\phi_6) i_{t-1} + (\phi_3 + \phi_2\phi_5) u_{t-1} + \phi_2 \eta_t^u + \eta_t^i\right] \\
u_t &= \frac{1}{1-\phi_2\phi_4}\left[(\phi_1\phi_4 + \phi_6) i_{t-1} + (\phi_3\phi_4 + \phi_5) u_{t-1} + \phi_4 \eta_t^i + \eta_t^u\right] \\
\varepsilon_{i_t} &= \frac{1}{1-\phi_2\phi_4}\left[\phi_2 \eta_t^u + \eta_t^i\right] \\
\varepsilon_{u_t} &= \frac{1}{1-\phi_2\phi_4}\left[\phi_4 \eta_t^i + \eta_t^u\right]
\end{aligned}$$

If we guess $\phi_4 = 0$, we have:

$$\begin{aligned}
\varepsilon_t^i &= \eta_t^i + \phi_2 \eta_t^u \\
\varepsilon_t^u &= \eta_t^u \\
\Rightarrow \varepsilon_t^i &= \phi_2 \varepsilon_t^u + \eta_t^i
\end{aligned}$$

Lecture 9.

New Keynesian Model

Start off with baseline RBC model, add a nominal asset(one-period bond)

Price stickiness only makes sense when firms can set prices; hence, need some degree of market power. Special case of Rotemberg & Woodford.

Nominal rigidities: Calvo pricing: firm-specific shock that tells you whether you're allowed to reset your price.

9.1 Households

$$\begin{aligned} \max \quad & \mathbb{E}_0 \sum_{i=0}^{\infty} \beta^i \left[u(c_{t+1}) - \nu(l_{t+i}) \right] \\ \text{s.t.} \quad & p_t c_t + b_t = (1 + i_{t-1}) b_{t-1} + p_t w_t l_t + p_t d_t \end{aligned}$$

where c_t is quantity of baskets of goods consumed, p_t is the prive of each basket in terms of money, d_t is the real wage, d_t are dividends(firms' profits) received from owning firms, i_t is the nominal interest rate, b_t is the quantity of nominal bonds held by the household.

The intertemporal optimality condition is as follows:

$$u'(c_t) = \beta \mathbb{E}_t (1 + r_t) u'(c_{t+1})$$

where r_t is the real interest rate given by: $1 + r_t = \frac{1+i_t}{1+\pi_{t+1}}$ and $\pi_{t+1} = \frac{p_{t+1} - p_t}{p_t}$ is the inflation rate between t and $t+1$.

Labor supply is given by:

$$MRS_{l,c} = \frac{\nu'(l_t)}{u'(c_t)} = w_t$$

Introduce the basket of imperfectly substitutable goods through a consumption aggregator, a description of consumer preferences over the whole set of goods (indexed by i).

The most common aggregator used is the CES:

$$c = \left(\int c(i)^{\frac{\varepsilon - 1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon - 1}}$$

where ε is the elasticity of substitution between goods.

9.1.1 Expenditure minimization

Suppose that the household faces prices p(i). The expenditure minimization problem is:

$$\min \quad \int p(i) c(i) di$$

$$\text{s.t.} \quad \left(\int c(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} = c$$

Define the Lagrangian:

$$\mathcal{L} = \int p(i)c(i)di + \lambda \left(c - \left(\int c(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} \right)$$

The first order conditions are:

$$\frac{\partial \mathcal{L}}{\partial c(i)} = p(i) - \lambda c(i)^{-\frac{1}{\varepsilon}} \left(\int c(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}-1} = 0$$

$$\Rightarrow p(i) = \lambda c(i)^{-\frac{1}{\varepsilon}} c^{\frac{1}{\varepsilon}}$$

From the first order conditions we get:

$$c(i) = c \left(\frac{p(i)}{\lambda} \right)^{-\varepsilon}$$

Substitute this into the FOC to get:

$$c = \left(\int \left(\frac{p(i)}{\lambda} \right)^{1-\varepsilon} c^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

$$\lambda = \left(\int p(i)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}}$$

Interpret λ as the shadow price:"If you tighten the constraint by one unit, the objective function increases by λ units." λ is the price of one basket, $p_t = \lambda$.

Hence, the demand for good i is:

$$c(i) = c \left(\frac{p(i)}{p} \right)^{-\varepsilon}$$

where

$$p = \left(\int p(i)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}}$$

Interpretation: demand depends on the relative price $p(i)/p$.

9.1.2 Production technology and market clearing

Output $y(i)$ is given by production function:

$$y_t(i) = a_t h_t(i)$$

where a_t is the total factor productivity and $h_t(i)$ is the hours of labor used in production of good i.

Assuming there's no investment of gov spending, the market clearing condition is:

$$y_t(i) = c_t(i), c_t = y_t$$

Interpret l_t as average labor supplied per good, the labor market clearing:

$$l_t = \int h_t(i)di$$

Firm i faces demand function:

$$y_t(i) = c_t(i) = y_t \left(\frac{p(i)}{p}\right)^{-\varepsilon}, p_t = \left(\int p(i)^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}$$

Firm i has the power to set price $p_t(i)$, but it is too small to affect the general price level p_t or aggregate demand y_t (monopolistic competition).

In a competitive market, firms can hire labor at the wage rate w_t, let z_t denote the real cost of production per unit of output. With linear production function for $y(i)$, we have:

$$z_t = \frac{w_t}{a_t}$$

with total revenue: $\frac{p_t(i)}{p_t} y_t(i)$, total cost: $z_t y_t(i)$.

The profits made by firm i at time t is:

$$d_t(i) = \frac{p_t(i)}{p_t} y_t(i) - z_t y_t(i) = \left(\frac{p_t(i)}{p_t} - z_t\right)\left(\frac{p_t(i)}{p_t}\right)^{-\varepsilon} y_t$$

Let $\mu_t(i) = \frac{p_t(i)}{p_t z_t}$ be the firm i's (gross) markup of price over marginal cost.

The expression for profits can be rewritten as:

$$d_t(i) = z_t^{1-\varepsilon} y_t (\mu_t(i)^{1-\varepsilon} - \mu_t(i)^{-\varepsilon})$$

Let's first consider a world with flexible prices. To maximize profits, firm i chooses $\mu_t(i)$ to solve:

$$\max_{\mu_t(i)} d_t(i) = z_t^{1-\varepsilon} y_t (\mu_t(i)^{1-\varepsilon} - \mu_t(i)^{-\varepsilon})$$

The first order condition is:

$$\frac{\partial d_t(i)}{\partial \mu_t(i)} = z_t^{1-\varepsilon} y_t ((1-\varepsilon)\mu_t(i)^{-\varepsilon} + \varepsilon \mu_t(i)^{-\varepsilon-1}) = 0 \Rightarrow \mu_t^* = \frac{\varepsilon}{\varepsilon - 1}$$

So, a higher price elasticity ε will cause a lower markup.

9.1.3 General Equilibrium with flexible prices

Assuming the following utility function:

$$u(c) = \frac{c^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}, \nu(l) = l$$

The labor supply condition is:

$$w_t = \frac{\nu'(l_t)}{u'(c_t)} = y_t^{\frac{1}{\sigma}}$$

Assuming the market clearing condition holds, $c_t = y_t$. The firm's profit maximization implies that

$$w_t = \frac{a_t}{\mu_t^*}.$$

Wage equals to the marginal revenue product of capital.

9.1.4 Flexible-price world: natural level of output

With flexible prices, all firms set the samse markup: $\mu_t(i) = \mu_t^*$. Hence the price would also be the same: $p_t(i) = p_t^*$ and the markups become:

$$\mu_t^* = \frac{1}{z_t}$$

The natural level of output is the level of output that would be produced if all prices were flexible. The natural level of output is given by:

$$y_t^{\frac{1}{\sigma}} = w_t = \frac{a_t}{\mu_t^*}$$

Hence the natural level of output is:

$$y_t^* = \left(\frac{a_t}{\mu_t^*}\right)^{\sigma}$$

- The natural output is higher when productivity a_t is higher.
- The natural output is lower when the markup(market power) μ_t^* is higher.

9.1.5 Natural real interest rate

The natural real interest rate is the real interest rate consistent with the natural level of output.

This requires $c_t = y_t = y_t^*$, which implies:

$$u'(y_t^*) = \beta(1 + r_t^*)u'(y_{t+1}^*)$$

with r_t^* being the natural real interest rate.

Using $u'(c) = c^{-\frac{1}{\sigma}}$, we get:

$$\begin{aligned}(y_t^*)^{-\frac{1}{\sigma}} &= \beta(1 + r_t^*)\left(y_{t+1}^*\right)^{-\frac{1}{\sigma}} \\ \frac{\mu_t^*}{a_t} &= \beta(1 + r_t^*)\frac{\mu_{t+1}^*}{a_{t+1}}\end{aligned}$$

The natural interest rate is given by:

$$1 + r_t^* = (1 + \bar{r})\frac{a_{t+1}}{a_t}\frac{\mu_t^*}{\mu_{t+1}^*}$$

where $\bar{r}$ is the average interest rate in the steady state with $\beta = \frac{1}{1+\bar{r}}$.

9.1.6 Efficiency of output

Imperfect competition makes market allocation of output not Pareto efficient.

Perfect competition is consistent with efficiency, so we can ask what level of output is efficient by finding the equilibrium in the special case of perfect competition:

- perfect competition: perfectly elastic demand curve because of perfect substitutability(no market power): $\varepsilon \to \infty$

- zero markup: $\mu_t^* \to 1$.

Efficient Output: $\hat{y}_t$ found using formula for natural output, and setting $\mu_t^* = 1$:

$$\hat{y}_t = a_t^{\sigma}$$

Efficient interest rate $\hat{r}_t$ is the real interest rate consistent with output as its efficient level:

$$1 + \hat{r}_t = (1 + \bar{r}) \frac{a_{t+1}}{a_t}.$$

9.1.7 Nominal Rigidities

Assumption 9.1.1 (Calvo and Yun).

Calvo pricing: Each period, a random selected fraction of $1 - \phi$ of firms has an opportunity to reset its price. All other firms must continue to use the same price as before.

This is a simple idea in a complicated approach:

- firms face a fixed physical cost of adjusting prices
- need to calculate the points in time when gains from adjusting prices exceed the costs
- hard to do in a dynamic model

9.1.8 Setting a new price

Suppose firm i set the new price s_t at time t. In each subsequent period, probability ϕ that the price won't be changed. Hence, the probability that the price is still used at time $T > t$ is ϕ^{T-t}.

The expected present value of profits from setting price s_t at time t is:

$$d_t(i) + \phi \frac{d_{t+1}(i)}{1 + r_t} + \phi^2 \frac{d_{t+2}(i)}{(1 + r_t)(1 + r_{t+1})} + \dots$$

where dividends(=profits) $d_t, d_{t+1}, \dots$ are calculated assuming the price remains 'sticky' at s_t: $p_t = p_{t+1} = \dots = s_t$.

From the firms' profits equation, we have:

$$d_t(i) = \left(\frac{p_t(i)}{p_t} - z_t \right) \left(\frac{p_t(i)}{p_t} \right)^{-\varepsilon} y_t$$

we get the objective function to maximize:

$$\max \mathbb{E}_t \sum_{i=0}^{\infty} \left(\prod_{j=1}^{i} \frac{1}{1 + r_{t+j-1}} \right) \phi^i \left(\frac{p_t(i)}{p_{t+i}} - z_{t+i} \right) \left(\frac{p_t(i)}{p_{t+i}} \right)^{-\varepsilon} y_{t+i}$$

the first order condition with respect to ε is:

$$\mathbb{E}_t \sum_{i=0}^{\infty} \left(\prod_{j=1}^{i} \frac{1}{1 + r_{t+j-1}} \right) \phi^i \left((1 - \varepsilon) \frac{p_t(i)}{p_{t+i}} + \varepsilon z_{t+i} \right) \frac{1}{p_t(i)} \left(\frac{p_t(i)}{p_{t+i}} \right)^{-\varepsilon} y_{t+i} = 0$$

plug in the Euler equation to get rid of the interest rates:

$$\mathbb{E}_t \sum_{i=0}^{\infty} y_{t+i}^{1+\frac{1}{\sigma}} \beta^i \phi^i \left((1-\varepsilon) \frac{p_t(i)}{p_{t+i}} + \varepsilon z_{t+i} \right) \frac{1}{p_t(i)} \left(\frac{p_t(i)}{p_{t+i}} \right)^{-\varepsilon} = 0$$

The optimal reset price $p_t(i)$ doesn't depend on the current price level p_t, denote by s_t for simplicity. We have:

$$\mathbb{E}_t \sum_{i=0}^{\infty} y_{t+i}^{1+\frac{1}{\sigma}} (\beta\phi)^i \left(\frac{s_t}{p_{t+i}} \right)^{1-\varepsilon} = \frac{\varepsilon}{1-\varepsilon} \mathbb{E}_t \sum_{i=0}^{\infty} y_{t+i}^{1+\frac{1}{\sigma}} (\beta\phi)^i z_{t+i} \left(\frac{s_t}{p_{t+i}} \right)^{-\varepsilon}$$

then log-linearize the above equation to get:

$$\tilde{s}_t = (1-\beta\phi)\mathbb{E}_t \left(\sum_{i=0}^{\infty} (\beta\phi)^i (\tilde{p}_{t+i} + \tilde{z}_{t+i}) \right)$$

This indicates that firms set price in accordance with current and expected future price level and cost deviations.

Write the equation for time $t+1$ and subtract from this equation to get:

$$\tilde{s}_t - \beta\phi\mathbb{E}_t \tilde{s}_{t+1} = (1-\beta\phi)(\tilde{p}_t + \tilde{z}_t)$$

9.1.9 Overall price level

Recall the overall price level:

$$p_t = \left(\int p_t(i)^{1-\varepsilon} di \right)^{\frac{1}{1-\varepsilon}}$$

The log-linearized version of the above equation is:

$$\tilde{p}_t = \int \tilde{p}_t(i) di$$

Lecture 10.

Problem Set 6

Question 1: The CES consumption aggregator

Consider the CES consumption aggregator $c = \left(\int c(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}}$.

Problem (1). Find the demand functions that minimize the expenditure $\int p(i)c(i)di$ needed to obtain a particular level of consumption c of the basket of goods.

Solution. The expenditure minimization problem is:

$$\begin{aligned} &\min_{c(i)} \int p(i)c(i)di \\ &\text{s.t.} \left(\int c(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} = c. \end{aligned}$$

The Lagrangian is:

$$\mathcal{L} = \int p(i)c(i)di + \lambda\left(c - \left(\int c(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}}\right).$$

The first-order condition is:

$$\begin{aligned} &\frac{\partial\mathcal{L}}{\partial c(i)} = p(i) - \lambda c(i)^{-\frac{1}{\varepsilon}} \left(\int c(i)^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}-1} = 0 \\ &\Rightarrow p(i) = \lambda c(i)^{-\frac{1}{\varepsilon}} c^{\frac{1}{\varepsilon}} \end{aligned}$$

From the first order conditions we get:

$$c(i) = c\left(\frac{p(i)}{\lambda}\right)^{-\varepsilon}$$

Substitute this into the FOC to get:

$$\begin{aligned} c &= \left(\int \left(\frac{p(i)}{\lambda}\right)^{1-\varepsilon} c^{\frac{\varepsilon-1}{\varepsilon}} di\right)^{\frac{\varepsilon}{\varepsilon-1}} \\ &= c\left(\frac{1}{\lambda}\right)^{-\varepsilon} \left(\int p(i)^{1-\varepsilon} di\right)^{\frac{\varepsilon}{\varepsilon-1}} \\ \lambda &= \left(\int p(i)^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}} \end{aligned}$$

Interpret λ as the shadow price:"If you tighten the constraint by one unit, the objective function increases by λ units." λ is the price of one basket, $p_t = \lambda$.

Hence, the demand for good i is:

$$c(i) = c\left(\frac{p(i)}{p}\right)^{-\varepsilon}$$

where

$$p = \left(\int p(i)^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}$$

Interpretation: demand depends on the relative price $p(i)/p$.

Problem (2). Show that the minimum expenditure required is $\int p(i)c(i)di = pc$ where $p = (\int p(i)^{1-\varepsilon} di)^{\frac{1}{1-\varepsilon}}$ is the price index.

Solution. Plug in the demand function into the expenditure function:

$$\begin{aligned}
\int p(i)c(i)di &= \int p(i)c\left(\frac{p(i)}{p}\right)^{-\varepsilon} di \\
&= c\int p(i)\left(\frac{p(i)}{p}\right)^{-\varepsilon} di \\
&= p^{\varepsilon}c\int p(i)^{1-\varepsilon} di \\
&= p^{\varepsilon}c\left[\left(\int p(i)^{1-\varepsilon} di\right)^{\frac{1}{1-\varepsilon}}\right]^{1-\varepsilon} \\
&= p^{\varepsilon}cp^{1-\varepsilon} \\
&= pc
\end{aligned}$$

Question 2: Deriving the IS curve

Consider the New Keynesian model. Definc the output gap x_t as

$$x_t = \frac{y_t}{\hat{y}_t}$$

where $\hat{y}_t$denotes the efficient level of output. Recall the Euler cquation for consumption (with $u'(c) = c_t^{-1/\sigma}$ and market clcaring, $y_t = c_t$)

$$y_t^{-\frac{1}{\sigma}} = \mathbb{E}_t\left(\frac{1+r_t}{1+\bar{r}} y_{t+1}^{-\frac{1}{\sigma}}\right)$$

where $\bar{r}$ is the real intercst rate in steady state, $\beta = 1/(1+\bar{r})$.

Problem (1). Log-lincarize the Euler cquation around the stcady state.

Problem (2). Noting that the approximatod Euler cquation should also hold in an cfficient world (i.c. for the efficient level of output and the cfficient real interest rate), usc approximations for the Fisher equation

$$r_t = i_t - \pi_{t+1}$$

and for the definition of the output gap (1) to show that

$$\widetilde{x}_t = \mathbb{E}_t\left(\widetilde{x}_{t+1} - \sigma(\widetilde{i}_t - \widetilde{\pi}_{t+1})\right) + \sigma\widetilde{\hat{r}}_t.$$

Solution. In the steady state, the Euler equation is:

$$y_t^{-\frac{1}{\sigma}} = \beta \mathbb{E}_t \left((1 + r_t) y_{t+1}^{-\frac{1}{\sigma}} \right)$$

where $\beta = \frac{1}{1+i}$. Log-linearize the Euler equation around the steady state and take 1st order approximation:

$$\begin{aligned}
\ln y_t^{-\frac{1}{\sigma}} &= \ln \beta + \mathbb{E}_t \left(\ln(1 + r_t) + \ln y_{t+1}^{-\frac{1}{\sigma}} \right) \\
-\frac{1}{\sigma} \ln y^* - \frac{1}{\sigma} \frac{1}{y^*}(y_t - y^*) &= \ln \beta + \ln(1 + r^*) + \frac{1}{1 + r^*}(1 + r_t - 1 - r^*) - \frac{1}{\sigma} \ln y^* \\
-\frac{1}{\sigma} \frac{1}{y^*}(y_t - y^*) &= \frac{r_t - r^*}{1 + r^*} \\
-\frac{1}{\sigma} \tilde{y}_t &= \mathbb{E}_t \left(\tilde{r}_t - \frac{1}{\sigma} \hat{y}_{t+1} \right) \\
\tilde{y}_t &= \mathbb{E}_t \left(\tilde{y}_{t+1} - \sigma \tilde{r}_t \right) \\
x_t &= \frac{y_t}{\hat{y}_t} \overset{approx}{\sim} \tilde{x}_t = \tilde{y}_t - \tilde{\hat{y}}_t \\
\Rightarrow \tilde{\hat{y}}_t &= \mathbb{E}_t \left(\tilde{\hat{y}}_{t+1} - \sigma \tilde{\hat{r}}_t \right) \\
\tilde{y}_t - \tilde{\hat{y}}_t &= \mathbb{E}_t \left(\tilde{y}_{t+1} - \sigma \tilde{r}_t - \tilde{\hat{y}}_{t+1} + \sigma \tilde{\hat{r}}_t \right) \\
\tilde{x}_t &= \mathbb{E}_t \left(\tilde{x}_{t+1} - \sigma(\tilde{i}_t - \tilde{\pi}_{t+1}) \right) + \sigma \tilde{\hat{r}}_t
\end{aligned}$$

where $\tilde{r}_t = \tilde{i}_t - \tilde{\pi}_{t+1}$.

Question 3: The three-equation New Keynesian model

Consider the New Keyncsian model, which consists of the New Keyncsian Phillips curve,

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + e_t,$$

the IS curve

$$x_t = \mathbb{E}_t \left(x_{t+1} - \sigma(i_t - \pi_{t+1}) \right) + v_t$$

where $v_t = \sigma \widetilde{\hat{r}}_t$ is a shock to the efficient real intercst rate, and a monctary policy rule of the form

$$i_t = \alpha_1 \pi_t + u_t$$

where u_t is a shock (with the usual properties: mean zero, uncorrelatcd with all other variables). All variables in the three equations, except the shocks, are in logdeviations from the stcady state (i.c. I dropped the $\sim$).

Note.

$$\begin{aligned}
r_t &= \gamma \mu_t^* \\
\mu_t^* &= \frac{\varepsilon}{\varepsilon - 1} \\
v_t &= \sigma \hat{r}_t
\end{aligned}$$

$$\hat{r}_t = (1+\bar{r})\frac{a_{t+1}}{a_t} - 1$$

Problem (1). Interpret the monctary policy rule. Which sign would you cxpect α to have?

Solution. The monetary policy rule is a Taylor rule where the interest rate is set as a function of inflation. The sign of α is positive since the central bank would want to raise interest rates when inflation is high.

Problem (2). State what effect the following havc on e_t, v_t, and u_t :

1. A temporary risc in productivity (TFP) a_t
2. A permancnt rise in productivity productivity at (i.c. the percentage incrcasc in a_t and $a_t + 1$ is the samc)
3. A temporary rise in the competitivencss of markets as measurcd by the price clasticity ε_t
4. A shift in monctary policy caused by new personncl on the monctary policy committec

Solution.
1. A temporary rise in a_t would cause a temporary decrease in $\hat{r}_t$, thus a decrease in v_t.
2. A permanent rise in a_t and a_{t+1} makes no change to v_t, thus the three equations stays the same, but since the TFP has increased permanently, the economy will rise to a new higher level equilibrium.
3. A temporary rise in the competitiveness of markets would cause a decrease in μ_t^*, thus a decrease in e_t.
4. A shift in monetary policy

Problem (3). Consider a temporary shock to e_t and assumc that this has no cffcct on cxpectations of the future, so that $\mathbb{E}_t\pi_{t+1} = 0$ and $\mathbb{E}_t x_{t+1} = 0$. No other shock occurs at the same time, thus $v_t = 0$ and $u_t = 0$. Find expressions for π_t and x_t in terms of e_t and interpret your results.

Solution. In this case, we have:

$$\begin{aligned}
\pi_t &= \kappa x_t + e_t \\
x_t &= \sigma i_t \\
i_t &= \alpha \pi_t
\end{aligned}$$

Thus,

$$\begin{aligned}
x_t &= -\sigma\alpha_1\pi_t \\
&= \frac{-\alpha_1\sigma}{1+\alpha\kappa\sigma}e_t \\
\pi_t &= \frac{e_t}{1+\kappa\alpha\sigma} \\
i_t &= \frac{\alpha_1}{1+\kappa\sigma\alpha}e_t
\end{aligned}$$

Problem (4). Suppose that the monctary policy rule is modified so that $i_t = \hat{r}_t + \alpha\pi_t + u_t$. If this interest rate rule is used then what are the effects of a temporary shock to the natural interest rate (and hence v_t) on inflation and the output gap?

Solution. We know that:

$$x_t = \mathbb{E}_t x_{t+1} - \sigma(i_t - \mathbb{E}_t\pi_{t+1}) + v_t,$$

$$\pi_t = \mathbb{E}_t\pi_{t+1} + \kappa x_t + e_t.$$

and

$$i_t = \alpha\pi_t + \hat{r}_t + u_t.$$

We have:

$$\mathbb{E}_t x_{t+1} = \mathbb{E}_t\pi_{t+1} = 0,$$

and $u_t = e_t = 0$, $v_t = \sigma\hat{r}_t$. So we have:

$$x_t = -\sigma(\alpha\pi_t + \hat{r}_t) + \sigma\hat{r}_t.$$

Problem (5). Is it possible to redcsign the monctary policy rule to achicve the same cffect found in part (d) but now in responsc to a shock to e_t? Explain why or why not.

Lecture 11.

Optimal Monetary Policy

11.1 Optimal MP Problem

Assume that CB cannot commit a policy,

Lecture 12.

Problem Set 7

The zero lower bound: discretion and commitment

Suppose the central bank has the following loss function:

$$L_t = \frac{1}{2}x_t^2,$$

where x_t is the output gap (defined relative to the efficient level of output). The output gap is determined by the IS equation

$$x_t = E_t x_{t+1} - \sigma(i_t - E_t\pi_{t+1} - \hat{r}_t),$$

where i_t is the nominal interest rate (controlled by the central bank), π_t is the inflation rate, and $\hat{r}_t$ is the efficient interest rate. Inflation is determined by the Phillips curve

$$\pi_t = E_t\pi_{t+1} + \kappa x_t$$

(assuming the discount factor $\beta = 1$, and ignoring cost-push shocks). Nominal interest rates are subject to a zero lower bound. Since all variables, including i_t, are given as percentage deviations from steady-state levels, and since the steady-state nominal interest rate is likely to be positive, the lower bound on i_t can be stated as $i_t \geq -b$ for some $b > 0$. (Note: i_t is the percentage deviation of the gross nominal interest rate from the steady-state level. The gross nominal interest rate is equal to one at the "zero lower bound"!)

Problem (1). What factors would determine the size of b?

Solution. As we know, $\tilde{i}_t = \frac{\hat{i}_t - i^*}{i^*}$, where $\tilde{i}_t$ is the percentage deviation of the nominal interest rate from the steady-state level, $\hat{i}_t$ is the nominal interest rate, and i^* is the steady-state nominal interest rate. Since the steady-state nominal interest rate is likely to be positive, the lower bound on $\tilde{i}_t$ can be stated as $\tilde{i}_t \geq -b$.

Problem (2). Now suppose at time t there is a temporary negative shock to the efficient interest rate $\hat{r}_t$, so that $\hat{r}_t = \hat{r} < 0$ and $\hat{r}_{t+1} = \hat{r}_{t+2} = \ldots = 0$. Find the optimal interest rate i_t when the central bank acts with discretion, and the resulting value of the output gap x_t. Distinguish between the cases $\hat{r} \geq -b$ and $\hat{r} < -b$ in your answer.

Solution. There are two cases to consider: $\hat{r} \geq -b$ and $\hat{r} < -b$. If $\hat{r} \geq -b$, then the optimal interest rate i_t is given by

$$i_t = E_t\pi_{t+1} + \kappa x_t + \hat{r}.$$

Substituting this into the IS equation, we get

$$x_t = E_t x_{t+1} - \sigma(E_t\pi_{t+1} + \kappa x_t + \hat{r} - E_t\pi_{t+1} - \hat{r}) = E_t x_{t+1} - \sigma\kappa x_t.$$

Rearranging, we get

$$x_t = \frac{1}{1+\sigma\kappa} E_t x_{t+1}.$$

Since $\hat{r}_{t+1} = \hat{r}_{t+2} = \ldots = 0$, we have $x_{t+1} = 0$, so $x_t = 0$.

Problem (3). Suppose that $\hat{r} < -b$, and that the shock now lasts for two periods, hence $\hat{r}_t = \hat{r}_{t+1} = \hat{r}$ and $\hat{r}_{t+2} = \hat{r}_{t+3} = \ldots = 0$. Find the level of the output gap x_t when the central bank follows the optimal policy with discretion. [Hint: work backwards, starting from period $t+2$.] Compare your answer to part 2 (assuming $\hat{r}$ is the same in both cases) and explain the intuition.

Solution. If $\hat{r} < -b$, the problem is:

$$\min_{i_t} L_t = \frac{1}{2}\left(-\sigma(i_t - r_t)\right)^2,$$

take the first-order condition:

$$\sigma(i_t - \hat{r}_t) = 0 \Rightarrow i_t = \hat{r}_t = \hat{r}.$$

Substituting this into the IS equation, we get

$$x_t = -\sigma(i_t - \hat{r}) = -\sigma(-b - \hat{r}) = \sigma(b + \hat{r}).$$

Problem (4). Suppose again that the shock is only temporary ($\hat{r}_{t+1} = \hat{r}_{t+2} = \ldots = 0$), and that $\hat{r}_t = \hat{r} < -b$. Assume the central bank is now able to commit to an interest-rate policy at time t for two periods, that is, it can choose both i_t and i_{t+1} at time t (but not influence outcomes further in the future). Find the optimal policy with commitment (minimizing the sum $L_t + L_{t+1}$, and assuming the shock is such that the zero lower bound will not bind at time $t+1$). Compare the total loss $L_t + L_{t+1}$ under discretion and commitment, and comment on the behaviour of long-term interest rates in the two cases.

Solution. In stage $t+2$, we have:

$$\mathbb{E}_t x_{t+2} = 0$$
$$\mathbb{E}_t \pi_{t+2} = 0$$
$$\mathbb{E}_t i_{t+2} = 0$$

We have:

$$x_{t+1} = \mathbb{E}_t x_{t+2} - \sigma(i_{t+1} - \mathbb{E}_t \pi_{t+2} - \hat{r}_{t+1}) = -\sigma(i_{t+1} - \hat{r}).$$

We aim to minimize the sum of losses $L_t + L_{t+1}$:

$$L_t + L_{t+1} = \frac{1}{2}x_t^2 + \frac{1}{2}x_{t+1}^2 = \frac{1}{2}\sigma^2(b+\hat{r})^2 + \frac{1}{2}\sigma^2(i_{t+1} - \hat{r})^2.$$

The first-order condition is:

$$i_{t+1} = -b$$

thus, $x_{t+1} = \sigma(b + \hat{r})$ and $\pi_{t+1} = \kappa\sigma(b + \hat{r})$. To solve:

$$\min \frac{1}{2}x_t^2$$

$$\text{s.t.} x_t = E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1} - \hat{r}_t)$$
$$\pi_{t+1} = \kappa\sigma x_{t+1}$$

So, we have:

$$x_t = \sigma(b+\hat{r}) - \sigma(i_t - \kappa\sigma(b+\hat{r}) - \hat{r}) = (1+\sigma\kappa)\sigma(b+\hat{r}) - \sigma(i_t - \hat{r}).$$

The problem is to solve:

$$\min_{i_t} \frac{1}{2}\left[(1+\sigma\kappa)\sigma(b+\hat{r}) - \sigma(i_{t-\hat{r}})\right]^2.$$

The first-order condition is:

$$-\sigma\left[(1+\sigma\kappa)\sigma(b+\hat{r}) - \sigma(i_t - \hat{r}_t)\right] = 0$$

which implies:

$$i_t = (1+\sigma\kappa)\sigma(b+\hat{r}) + \hat{r}.$$

The central bank could only set $i_t = -b$ in the commitment case, so that:

$$x_t = 2\sigma(b+\hat{r}) + \kappa\sigma^2(b+\hat{r}) < 0.$$

Then, we can find the total loss under discretion and commitment:

$$L_t + L_{t+1} = \frac{1}{2}\sigma^2(b+\hat{r})^2 + \frac{1}{2}\sigma^2(2\sigma(b+\hat{r}) + \kappa\sigma^2(b+\hat{r}))^2 = \frac{1}{2}\sigma^2(b+\hat{r})^2 + \frac{1}{2}\sigma^2(2\sigma(b+\hat{r}) + \kappa\sigma^2(b+\hat{r}))^2.$$

Problem (5). Describe the time-inconsistency problem* inherent in the optimal commitment found in part 4. Explain whether there is an analogy with the inflation bias problem*.

Solution. This time, we have: $\hat{r}_t = \hat{r} < -b$ and $\hat{r}_{t+1} = \hat{r}_{t+2} = \cdots = 0$. We still need to solve:

$$\begin{aligned}
\min_{i_t, i_{t+1}} \quad & L_t + L_{t+1} \\
\text{s.t.} \quad & x_t = E_t x_{t+1} - \sigma(i_t - E_t \pi_{t+1} - \hat{r}) \\
& x_{t+1} = \mathbb{E}_t x_{t+2} - \sigma(i_{t+1} - \mathbb{E}_t \pi_{t+2} - \hat{r}_{t+1}) = -\sigma i_{t+1} \\
& \pi_{t+1} = \mathbb{E}_t \pi_{t+2} = \kappa x_{t+1}
\end{aligned}$$

the problem can be then rewrite as:

$$\begin{aligned}
\min_{i_t, i_{t+1}} \quad & \frac{1}{2}x_t^2 + \frac{1}{2}\sigma^2 i_{t+1}^2 \\
\text{s.t.} \quad & x_t = -(1+\sigma\kappa)\sigma i_{t+1} - \sigma(i_t - \hat{r})
\end{aligned}$$

implement the condition into the objective function:

$$\min_{i_t} \frac{1}{2}\left[(1+\sigma\kappa)\sigma i_{t+1} + \sigma(i_t - \hat{r})\right]^2 + \frac{1}{2}\sigma^2 i_{t+1}^2.$$

the first-order condition with respect to i_t and i_{t+1} is:

$$\sigma\left[(1+\sigma\kappa)\sigma i_{t+1}+\sigma(i_t-\hat{r})\right]=0$$
$$\left[(1+\sigma\kappa)\sigma i_{t+1}+\sigma(i_t-\hat{r})\right](1+\kappa\sigma)\sigma+\sigma^2 i_{t+1}=0$$

the solution is:

$$i_t=\hat{r}$$
$$i_{t+1}=0$$

but as $\hat{r}<-b$, we cannot rech this policy, we can only set $i_t=-b$: Then, we need to solve:

$$\min_{i_t} \quad \frac{1}{2}\left[(1+\sigma\kappa)\sigma i_{t+1}+\sigma(i_t-\hat{r})\right]^2+\frac{1}{2}\sigma^2 i_{t+1}^2$$
$$\text{s.t.} \quad i_t=-b$$

the FOC with respect to i_{t+1} is:

$$\sigma(1+\kappa\sigma)\left[(1+\sigma\kappa)\sigma i_{t+1}+\sigma(i_t-\hat{r})\right]+\sigma^2 i_{t+1}=0$$

the solution is:

$$i_t=-b$$
$$i_{t+1}=\frac{1+\kappa\sigma}{1+(1+\sigma\kappa)^2}(b+\hat{r})<0$$

Lecture 13.

Labor Market

Lecture 14.

Problem Set 8

Monopolistically competitive firm

Consider a monopolistically competitive firm with real unit cost of z_t per unit sold at time t (marginal cost is constant at the firm level). The firm (firm i) sells output at price $p_t(i)$ and faces the following demand function for its output $y_t(i)$:

$$y_t(i) = \left(\frac{p_t(i)}{p_t}\right)^{-\varepsilon_t} y_t$$

where p_t is the general price level, y_t is aggregate output, and ε_t is the price elasticity of demand. The firm's price $p_t(i)$ implies a (gross) markup of $\mu_t(i) = \frac{p_t(i)}{p_t z_t}$ on marginal cost. Profits (in real terms) made by the firm at time t are therefore given by

$$\text{Profits}_{i,t} = z_t^{1-\varepsilon_t} y_t \left(\mu_t(i)^{1-\varepsilon_t} - \mu_t(i)^{-\varepsilon_t}\right)$$

Problem (a). Show that the profit-maximizing markup for a firm with flexible prices is $\mu_t^* = \frac{\varepsilon_t}{\varepsilon_t - 1}$.

Solution.

Take the first order derivative with respect to $\mu_t(i)$ and set it to zero, we have:

$$\begin{aligned}
&\frac{\partial \text{Profits}_{i,t}}{\partial \mu_t(i)} = \frac{\partial}{\partial \mu_t(i)} \left(z_t^{1-\varepsilon_t} y_t \left(\mu_t(i)^{1-\varepsilon_t} - \mu_t(i)^{-\varepsilon_t}\right)\right) = 0 \\
&\Rightarrow z_t^{1-\varepsilon_t} y_t \left((1-\varepsilon_t)\mu_t(i)^{-\varepsilon_t} + \varepsilon_t \mu_t(i)^{-\varepsilon_t - 1}\right) = 0 \\
&\Rightarrow (1-\varepsilon_t)\mu_t(i)^{-\varepsilon_t} + \varepsilon_t \mu_t(i)^{-\varepsilon_t - 1} = 0 \\
&\Rightarrow \mu_t^* = \frac{\varepsilon_t}{\varepsilon_t - 1}
\end{aligned}$$

The firm hires labor in a perfectly competitive market at wage w_t and each unit of labor has productivity a_t. Thus $z_t = w_t / a_t$. The wage is determined by the household labor supply condition

$$w_t = \frac{\nu'(l_t)}{u'(c_t)}$$

where l_t is labor supply and c_t is consumption.

Problem (b). Assume that $u(c_t) = 1 - c_t^{-1}$ and $\nu(l_t) = l_t$. Show that the natural level of output

y_t^* is

$$y_t^* = \sqrt{\frac{a_t}{\mu_t^*}}$$

and explain why this means that the efficient level of output is $\hat{y}_t = \sqrt{a_t}$.

Solution. From the household's labor supply condition:

$$w_t = \frac{\nu'(l_t)}{u'(c_t)} = \frac{1}{c_t^{-2}} = c_t^2.$$

In the flexible price equilibrium, firms set prices to maximize profits, leading to

$$\mu_t(i) = \frac{\varepsilon_t}{\varepsilon_t - 1} = \mu_t^*.$$

Hence the price would also be the same: $p_t(i) = p_t^*$ and the markups become:

$$\mu_t^* = \frac{1}{z_t}.$$

In a perfectly competitive market, the market clearing condition is: $y_t = c_t$. So, we have:

$$z_t = \frac{w_t}{a_t} = \frac{c_t^2}{a_t} = \frac{y_t^2}{a_t} = \frac{1}{\mu_t^*}$$
$$\Rightarrow y_t^* = \sqrt{\frac{a_t}{\mu_t^*}}$$

The efficient level of output occurs when there are no distortions, i.e. when the markup is eliminated: $\mu_t^* = 1$. Thus, $\hat{y}_t = \sqrt{a_t}$. The natural level of output y_t^* is lower than the efficient level $\hat{y}_t$due to the presence of monopolistic competition, which introduces a markup over marginal cost($\mu_t^* > 1$). This markup leads to higher prices and reduced output compared to the efficient (perfect competition) case. The efficient level of output is achieved when $\mu_t^* = 1$, eliminating the distortion caused by the markup, and thus maximizing output given the technology level a_t.

When firms can adjust their prices at random staggered intervals (Calvo pricing), inflation is determined by the New Keynesian Phillips curve:

$$\tilde{\pi}_t = \beta E_t \tilde{\pi}_{t+1} + \kappa \tilde{x}_t + \gamma \tilde{\mu}_t^*$$

where $\tilde{\pi}_t$ is inflation and $\tilde{x}_t$ is the output gap (defined relative to efficient output). The notation $\tilde{x}_t$ indicates the percentage deviation of x_t from its steady-state value.

Problem (c). In this environment, should monetary policy aim to stabilize the output gap $\tilde{x}_t$ completely following an unexpected temporary decrease in a_t? Why or why not?

Solution.

Yes. The monetary policy should aim to stabilize the output gap.

$\tilde{x}_t$ is defined as the difference between actual output y_t and efficient level of output $\hat{y}_t$:

$$\tilde{x}_t = \frac{y_t}{\hat{y}_t}.$$

Recall $z_t = \frac{w_t}{a_t}$, $w_t = y_t^2$ and $\hat{y}_t = \sqrt{a_t}$, we know that:

$$z_t = \tilde{x}_t^2$$

As a_t shock is exogenous, it won't affect the markup term $\tilde{\mu}_t^*$, so just cancel it out in this case, we have:

$$\tilde{\pi}_t = \beta \mathbb{E}_t \tilde{\pi}_{t+1} + \kappa \tilde{x}_t$$

If we stabilize the output gap completely in response to a temporary decrease in a_t, we have:

$$\tilde{x}_t = 0 \Rightarrow \tilde{\pi}_{t+1} = 0 \Rightarrow \tilde{\pi}_t = \beta \mathbb{E}_t \tilde{\pi}_{t+1} = 0$$

In this case, output is kept equal to the flexible-price equilibrium level of output. This also guarantees inflation equals zero, thereby eliminating the costly dispersion of relative prices that arises with inflation. When firms do not need to adjust their prices, the fact that prices are sticky is no longer relevant.

Now suppose that markets become more competitive when output is higher (perhaps because of more competition for new customers). In particular, assume that $\varepsilon_t = 1 + \frac{1}{2} y_t$. Price adjustment is staggered according to the Calvo model.

Problem (d). Derive an expression for the desired flexible-price markup μ_t^* in terms of the output gap $x_t \equiv y_t / \hat{y}_t$ and productivity a_t. (Hint: the efficient level of output is still $\hat{y}_t = \sqrt{a_t}$.) Find a log-linear approximation of this equation in terms of $\tilde{\mu}_t^*$, $\tilde{a}_t$, and $\tilde{x}_t$, and interpret it.

Solution. As given, $\varepsilon_t = 1 + \frac{1}{2} y_t$. The profit-maximizing markup is $\mu_t^* = \frac{\varepsilon_t}{\varepsilon_t - 1}$. So, we have:

$$\mu_t^* = \frac{1 + \frac{1}{2} y_t}{\frac{1}{2} y_t} = 1 + \frac{2}{y_t} = 1 + \frac{2}{x_t \sqrt{a_t}}.$$

We also know that $y_t = x_t \hat{y}_t = x_t \sqrt{a_t}$. Substitute this into the equation, we have:

$$\mu_t^* = 1 + \frac{2}{y_t} \Rightarrow y_t \mu_t^* = y_t + 2.$$

Take the log-linear approximation, we get:

$$\begin{aligned}
\tilde{y}_t &\approx \tilde{x}_t + \frac{1}{2} \tilde{a}_t \\
\tilde{y}_t + \tilde{\mu}_t^* &\approx \frac{\bar{y}}{2 + \bar{y}} \tilde{y}_t \\
\Rightarrow \tilde{\mu}_t^* &\approx -\frac{2}{2 + \bar{y}} \left(\tilde{x}_t + \frac{1}{2} \tilde{a}_t \right) \\
&= -\frac{2}{2 + \sqrt{\bar{a}}} \left(\tilde{x}_t + \frac{1}{2} \tilde{a}_t \right) \\
&\approx -\tilde{x}_t - \frac{1}{2} \tilde{a}_t
\end{aligned}$$

This indicates that the desired markup $\tilde{\mu}_t^*$ decreases when the output gap increases. This reflects the assumption that markets become more competitive when output is higher.

Higher productivity can enhance competition by enabling firms to produce more efficiently, which may encourage them to lower prices to gain market share.

Problem (e). In this environment, should monetary policy aim to stabilize the output gap $\tilde{x}_t$ completely following an unexpected temporary decrease in a_t? Why or why not?

Solution.

No, in this case, the monetary policy should not aim to stabilize the output gap completely following an unexpected temporary decrease in a_t.

As we have a negative coefficient, if we stabilize the output gap completely in response to a temporary decrease in a_t, we have:

$$\tilde{\mu}_t^* = -\frac{1}{2}\tilde{a}_t > 0.$$

Then, if we still expect the future inflation to be zero, we have:

$$\tilde{\pi}_t = \beta \mathbb{E}_t \tilde{\pi}_{t+1} + \kappa \tilde{x}_t + \gamma \tilde{\mu}_t^* = -\frac{\gamma}{2}\tilde{a}_t > 0.$$

This shows that we have a tradeoff between en stabilizing inflation and closing the output gap.

Problem (f). Compare the size of the response of inflation $\tilde{\pi}_t$ to the output gap $\tilde{x}_t$ in the two cases where the price elasticity ε_t is exogenous and where it depends positively on output y_t. Assuming the central bank is minimizing the same standard loss function in both cases, what implications does the dependence of ε_t on y_t have for the optimal balance between stabilizing inflation and stabilizing the output gap (when shocks shift the New Keynesian Phillips curve)? Why?

Solution.

With exonogenous ε_t, we have:

$$\tilde{\pi}_t = \beta \mathbb{E}_t \tilde{\pi}_{t+1} + \kappa \tilde{x}_t.$$

With $\varepsilon_t = 1 + \frac{1}{2}y_t$, we have:

$$\tilde{\pi}_t = \beta \mathbb{E}_t \tilde{\pi}_{t+1} + \kappa \tilde{x}_t + \gamma(-\tilde{x}_t - \frac{1}{2}\tilde{a}_t) = \beta \mathbb{E}_t \tilde{\pi}_{t+1} + (\kappa - \gamma)\tilde{x}_t - \frac{1}{2}\gamma\tilde{a}_t.$$

Thus, the response of inflation to the output gap is smaller when ε_t depends positively on output y_t.

Assuming the loss function is:

$$L_t = \frac{1}{2}\left(\tilde{\pi}_t^2 + a\tilde{x}_t^2\right).$$

The optimal balance problem is:

$$\begin{aligned}
\min_{\pi_t, x_t} \quad & L_t = \frac{1}{2}(\tilde{\pi}_t^2 + a\tilde{x}_t^2) \\
s.t. \quad & \tilde{\pi}_t = \beta \mathbb{E}_t \tilde{\pi}_{t+1} + \kappa \tilde{x}_t + \gamma(-\tilde{x}_t - \frac{1}{2}\tilde{a}_t) \\
& \tilde{\pi}_t = \beta \mathbb{E}_t \tilde{\pi}_{t+1} + \kappa \tilde{x}_t
\end{aligned}$$

Take the first order condition with respect to $\tilde{\pi}_t$ and $\tilde{x}_t$, we have:

$$\tilde{\pi}_t = -\frac{a}{\kappa}\tilde{x}_t \quad \text{under exogenous } \varepsilon_t;$$

$$\tilde{\pi}_t = -\frac{a}{\kappa - \gamma}\tilde{x}_t \quad \text{under } \varepsilon_t = 1 + \frac{1}{2}y_t.$$

So, the optimal balance between stabilizing inflation and stabilizing the output gap is affected by the dependence of ε_t on y_t. When ε_t depends positively on output y_t, the inflation rate is more volatile to the output gap. The monetary policy will prioritize inflation control over output gap stabilization.

Short Questions

Problem (2). Explain the channel system of conducting monetary policy, and illustrate its advantages and disadvantages.

Solution.

In a channel system, the central bank sets lower and upper bounds for the interest rate.

1. **Deposit Facility Rate (Floor):** The interest rate the central bank pays on excess reserves deposited by commercial banks.
2. **Lending Facility Rate (Ceiling):** The interest rate at which the central bank lends to commercial banks (typically through overnight loans).

The target policy rate is kept within this corridor, and the central bank uses open market operations to manage liquidity and guide the market rate towards the target.

Advantages:

- **Interest Rate Control:** The corridor provides clear boundaries for short-term interest rates, enhancing the central bank's ability to control market rates.
- **Liquidity Management:** By providing standing facilities, banks have predictable access to liquidity, reducing uncertainty in interbank markets.
- **Stability:** The system can mitigate volatility in interest rates by absorbing shocks through the corridor's floor and ceiling rates.

Disadvantages:

- **Interest Rate Volatility:** If the corridor is wide, market rates can fluctuate significantly within it, potentially leading to uncertainty.
- **Ineffective Transmission:** In times of excess reserves (e.g., during quantitative easing), the floor may become the de facto policy rate, weakening the central bank's control over other interest rates.
- **Dependency on Market Operations:** The system requires active management of liquidity through frequent open market operations, which can be resource-intensive.

Problem (3). Discuss whether (and in which way) optimal monetary policy with commitment is preferable over discretionary optimal monetary policy. Which assumption(s) of the New Keynesian model generate the dynamic externality that drives the differences between discretion and commitment?

Solution.

- **Commitment Policy:**
  - Under commitment, the central bank commits to future policy actions, influencing expectations and economic outcomes today.
  - It can implement time-inconsistent policies that are optimal in the long run but may not be optimal in the short run.
- **Discretionary Policy:**
  - The central bank optimizes policy period by period without committing to future actions.
  - This can lead to suboptimal outcomes due to the inability to influence expectations effectively.

If we suppose a timporary positive cost-push shock at time $t = 0$, then for the commitment policy, the central bank will increase the interest rate to reduce inflation, while for the discretionary policy, x_{t+j} would be negative for a while and the inflation expectations will fall $\Rightarrow \pi_0$ increases by less than under discretion without large decrease in x_0.

Hence, outcome leads to higher welfare.

Assumptions Generating the Dynamic Externality:

- **Forward-Looking Expectations:**
  - In the New Keynesian model, prices and wages are set based on expectations of future economic conditions.
  - This creates a dynamic externality where current policy affects future expectations and, therefore, current economic outcomes.
- **Sticky Prices and Wages:**
  - Price and wage rigidities cause adjustments to be gradual, making expectations about future policy more impactful.
  - The slow adjustment amplifies the importance of commitment in shaping expectations.
- **Monopolistic Competition and Price Setting:**
  - Firms set prices considering future marginal costs and the anticipated policy path.
  - Commitment allows the central bank to influence these expectations, improving allocation efficiency.

Lecture 15.

Labor Market(2)

Lecture 16.

Consumption

16.1 Job Creation

16.1.1 Wage determination

Recommended Resources

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