The gains from trade

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1 Introduction

In the previous few lectures we have discussed how trade maximizes global welfare, but we have been largely silent on how trade affects the welfare of each individual country. In Arkolakis, Costinot, and Rodríguez-Clare (2012), the authors show that all of the gravity trade models we have considered in class thus far have the same implications for the relationship between trade flows and the welfare of each country. This paper has become highly-influential for two reasons: first, it provides a very simple equation for evaluating the welfare gains of any trade shock; second, its somewhat surprising implication that a large set of gravity trade models – regardless of the microeconomic foundations of the trade model – yield the same welfare gains undermined the contributions of the new margins through which gains from trade occur in the heterogeneous firm trade models such as Melitz (2003). Furthermore, this characterization has an important implication for how changes in exogenous model parameters affect world welfare for all gravity models satisfying the condition.

2 The Basic Idea

The basic idea of the "ACR" paper is that the gains from trade can be inferred from the change in the openness of a country and the elasticity of trade flows to the variable trade cost. Let us first present the basic idea before getting into the heart of the Arkolakis, Costinot, and Rodríguez-Clare (2012) paper.

Consider any gravity trade model that yield the following expression for trade flows:

$$X_{ij} = \frac{F_{ij} w_i^{\alpha}}{\sum_{k \in S} F_{kj} w_k^{\alpha}} Y_j, \tag{1}$$

where $F_{ij} > 0$ is exogenous, $\alpha < 0$ is the (partial) trade elasticity, and the consumer price index can be written as:

$$P_j = \left(\sum_{k \in S} F_{kj} w_k^{\alpha} \right)^{\frac{1}{\alpha}}. \tag{2}$$

As we have seen, this set-up includes the Armington model of Anderson (1979), the Krugman (1980) model, the Eaton and Kortum (2002) model, and the Melitz (2003) model (with the Pareto distribution of productivities).

Define $\lambda_{ij} \equiv \frac{X_{ij}}{Y_j}$ to be the fraction of expenditure in $j \in S$ on goods from $i \in S$. Then note that λ_{jj} reflects the fraction of goods that a country consumes that are produced locally, i.e. $1 - \lambda_{ii}$ is a measure of the openness of a country. Then from equation (2) we can write:

$$\lambda_{jj} = \frac{F_{jj} w_j^\alpha}{\sum_{k \in S} F_{kj} w_k^\alpha} = F_{jj} \left(\frac{w_j}{P_j} \right)^\alpha,$$

where the second equality substituted in equation (2). Define $W_j \equiv \frac{w_j}{P_j}$ to be the real wage (i.e. the welfare). Then we have:

$$\begin{aligned} \lambda_{jj} &= F_{jj} W_j^\alpha \iff \\ W_j &= \left(\frac{\lambda_{jj}}{F_{jj}} \right)^{\frac{1}{\alpha}}. \end{aligned} \tag{3}$$

Equation (3) says that the welfare in a location depends only on its trade with itself, the elasticity of trade α and exogenous domestic parameters F_{jj}. Hence, for any foreign shock to model parameters (i.e. a shock that changes $\mathbf{F} \equiv \{F_{ij}\}$ but leaves F_{jj} unchanged), the change in welfare in $j \in S$ can be written as:

$$\hat{W}_j = \left(\hat{\lambda}_{jj} \right)^{\frac{1}{\alpha}}. \tag{4}$$

Equation (4) has become famous enough to be known as the "ACR" equation. It says that the change in welfare for country $j \in S$ from any foreign trade shock can be inferred from how the shock affected the openness of country j; since $\alpha < 0$, the more a country increased its openness (i.e. decreased its λ_{jj}), the greater the welfare gain. [Class question: why is the change in welfare smaller if α is of smaller magnitude?]

Note that there are two things one can use the "ACR" equation for. First, since in autarky $\lambda_{jj} = 1$, if α is known, the "ACR" equation immediately tells us the gains of moving from autarky to the current equilibrium. Second, *after* any trade shock, the ACR equation allows us to evaluate the change in welfare based on the changes in the observed openness. However, the "ACR" equation does *not* allow us to predict what the welfare gains from a counterfactual trade shock would be, as to know this, one would have to know how the trade shock would affect λ_{jj}.[1] Hence, the ACR equation (4) is an "ex-post" equatoin for the welfare gains from trade.

3 The Armington model

Prior to seeing how the "ACR" equation holds more generally (which is a fairly involved proof), it is helpful to go through the methodology of Arkolakis, Costinot, and Rodríguez-

[1] In Arkolakis, Costinot, and Rodríguez-Clare (2012), the authors do provide a general formula for calculating how λ_{jj} changes from an arbitrary trade shock based on the methodology of Dekle, Eaton, and Kortum (2008) (see their Proposition 2). Since we will be seeing that paper in a few lectures, here we focus only of the "ex-post" ACR equation.

Clare (2012) for the simplest case: the Armington model. Recall that in the Armington model, trade flows can be written as:

$$X_{ij} = \tau_{ij}^{1-\sigma} A_i^{\sigma-1} w_i^{1-\sigma} P_j^{\sigma-1} Y_j, \tag{5}$$

where $\sigma > 1$ is the elasticity of substitution and

$$P_j \equiv \left(\sum_{i \in S} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} w_i^{1-\sigma} \right)^{\frac{1}{1-\sigma}} \tag{6}$$

is the Dixit-Stiglitz price index. Suppose also that the labor market clearing condition holds, i.e. for all $i \in S$ we have:

$$Y_i = w_i L_i \tag{7}$$

Now consider country $j \in S$ and suppose there is any foreign shock that affects trade costs $\tau \equiv \{\tau_{ij}\}_{i,j \in S}$, labor supply $\mathbf{L} \equiv \{L_i\}_{i \in S}$, or productivities $\mathbf{A} \equiv \{A_i\}_{i \in S}$ but does not affect the local trade costs τ_{jj}, labor supply L_j, or productivity A_j. How does this affect the welfare $W_j \equiv \frac{Y_j}{P_j}$ in country j?

To answer this question, first note that without loss of generality we can normalize the wage in country j to one. Since the labor supply in country j remains fixed, from the labor market clearing condition (7) we have:

$$d \ln Y_j = d \ln w_j + d \ln L_j = 0.$$

Hence we have:

$$d \ln W_j = d \ln Y_j - d \ln P_j = -d \ln P_j,$$

i.e. the change in log welfare can be solely determined by the change in the log price index.

From the price index equation (6) we have:

$$\begin{aligned}
\ln P_j &= \left(\frac{1}{1-\sigma} \right) \ln \left(\sum_{i \in S} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} w_i^{1-\sigma} \right) \implies \\
d \ln P_j &= \left(\frac{1}{1-\sigma} \right) \frac{\sum_{i \in S} (1-\sigma) \times \tau_{ij}^{1-\sigma} A_i^{\sigma-1} w_i^{1-\sigma} \times (d \ln \tau_{ij} + d \ln w_i - d \ln A_i)}{\left(\sum_{i \in S} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} w_i^{1-\sigma} \right)} \iff \\
d \ln P_j &= \sum_{i \in S} \frac{\tau_{ij}^{1-\sigma} A_i^{\sigma-1} w_i^{1-\sigma}}{\left(\sum_{i \in S} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} w_i^{1-\sigma} \right)} (d \ln \tau_{ij} + d \ln w_i - d \ln A_i) \qquad (8)
\end{aligned}$$

Define $\lambda_{ij} \equiv \frac{X_{ij}}{Y_j}$ to be the fraction of j's expenditure on goods from i. From the gravity equation (5) we have:

$$\begin{aligned}
\lambda_{ij} &= \tau_{ij}^{1-\sigma} A_i^{\sigma-1} w_i^{1-\sigma} P_j^{\sigma-1} \iff \\
\lambda_{ij} &= \frac{\tau_{ij}^{1-\sigma} A_i^{\sigma-1} w_i^{1-\sigma}}{\sum_{i \in S} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} w_i^{1-\sigma}}, \qquad (9)
\end{aligned}$$

where the second line substituted in the price index (6). Substituting equation (9) into equation (8) then yields an expression for the change in log welfare:

$$\begin{aligned} d\ln P_j &= \sum_{i\in S}\lambda_{ij}\left(d\ln\tau_{ij}+d\ln w_i-d\ln A_i\right) \implies \\ d\ln W_j &= -\sum_{i\in S}\lambda_{ij}\left(d\ln\tau_{ij}+d\ln w_i-d\ln A_i\right). \end{aligned} \tag{10}$$

Finally, note that from the gravity equation (5) we can write the change in the relative

$$\begin{aligned} \frac{\lambda_{ij}}{\lambda_{jj}} &= \frac{\tau_{ij}^{1-\sigma}A_i^{\sigma-1}w_i^{1-\sigma}}{\tau_{jj}^{1-\sigma}A_j^{\sigma-1}w_j^{1-\sigma}} \implies \\ d\ln\lambda_{ij}-d\ln\lambda_{jj} &= (1-\sigma)\left(d\ln\tau_{ij}+d\ln w_i-d\ln A_i\right) \end{aligned} \tag{11}$$

Substituting equation (11) into (10) yields:

$$\begin{aligned} d\ln W_j &= -\sum_{i\in S}\lambda_{ij}\left(d\ln\tau_{ij}+d\ln w_i-d\ln A_i\right) \iff \\ &= -\frac{1}{1-\sigma}\sum_{i\in S}\lambda_{ij}\left(d\ln\lambda_{ij}-d\ln\lambda_{jj}\right) \iff \\ &= -\frac{1}{1-\sigma}\left(\sum_{i\in S}\lambda_{ij}d\ln\lambda_{ij}\right)+\frac{1}{1-\sigma}d\ln\lambda_{jj}\sum_{i\in S}\lambda_{ij} \iff \\ &= -\frac{1}{1-\sigma}\left(\sum_{i\in S}d\lambda_{ij}\right)+\frac{1}{1-\sigma}d\ln\lambda_{jj} \iff \\ &= \frac{1}{1-\sigma}d\ln\lambda_{jj}, \end{aligned} \tag{12}$$

where I used the fact that $\sum_{i\in S}\lambda_{ij}=1$, which implies that $\sum_{i\in S}d\lambda_{ij}=0$. Equation (12) implies that the change in (log) welfare from any foreign shock can be expressed as a change in the (log) openness of the location. Equation (12) also allows us to also express the change in welfare from any foreign shock. Let a "1" superscript denote post shock variable and a "0" superscript denote a pre shock variable. Then applying the second fundamental theorem of calculus yields:

$$\begin{aligned} \frac{d\ln W_j}{d\ln\lambda_{jj}} &= \frac{1}{1-\sigma} \implies \\ \ln W_j^1-\ln W_j^0 &= \int_{\lambda_{jj}^0}^{\lambda_{jj}^1}\frac{d\ln W_j}{d\lambda}d\lambda \iff \end{aligned} \tag{13}$$

$$\begin{aligned} \ln W_j^1-\ln W_j^0 &= \int_{\lambda_{jj}^0}^{\lambda_{jj}^1}\frac{d\ln W_j}{d\ln\lambda}d\ln\lambda \iff \\ \ln W_j^1-\ln W_j^0 &= \frac{1}{1-\sigma}\left(\ln\lambda_{jj}^1-\ln\lambda_{jj}^0\right) \iff \\ \hat{W}_j &= \left(\hat{\lambda}_{jj}\right)^{\frac{1}{1-\sigma}}, \end{aligned} \tag{14}$$

Hence, in a slightly more complicated way, we have rederived the "ACR" equation.

4 The general setup

We now consider a more general gravity structure to show that the "ACR" equation (14) holds more generally. The setup of the model is similar to the gravity models we have considered previously. Suppose there is a set of countries $i \in S$ and goods $\omega \in \Omega$. Labor is the only factor of production and is supplied inelastically; let L_i denote the (exogenous) number of workers in country $i \in S$. Let w_i denote the (endogenous) wage of a worker in country $i \in S$.

We now discuss the assumptions of the model. These assumptions are of two types: "micro-level" assumptions about preferences and market structure and "macro-level" restrictions on trade flows and aggregate variables.

4.1 Micro Assumptions

The first micro assumption is that there is a **representative agent in each country with Dixit-Stiglitz preferences**, so that the price index in country $i \in S$ is:

$$P_i = \left(\int_{\omega \in \Omega} p_i (\omega)^{1-\sigma} \, d\omega \right)^{\frac{1}{1-\sigma}}, \tag{15}$$

where $p_i (\omega)$ is the price of good $\omega \in \Omega$ in country i. Since not all goods will necessarily be available in country i, we let the price of unavailable goods be infinitely high.

The second micro assumption is that each good $\omega \in \Omega$ is **produced with a constant marginal cost $c_{ij} (\omega)$ and a fixed cost of production $f_{ij} (\omega)$**, where:[2]

$$c_{ij} (\omega) = \frac{\tau_{ij} \times w_i}{\alpha_i (\omega)}$$

$$f_{ij} (\omega) = \xi_{ij} \times w_i,$$

and τ_{ij} / ξ_{ij} are exogenous parameters common to all goods, w_i are the origin wages, and $\alpha_i (\omega)$ are exogenous parameters that may vary across goods.

The third micro assumption is that the market structure is either **perfectly competitive** or **monopolistically competitive**. Under perfect competition, it is assumed that $\phi_{ij} (\omega) = 0$ for all $\omega \in \Omega$ (i.e. there are no fixed costs of exporting) and all firms maximize profits taking prices and wages as given. Under monopolistic competition, we consider both the case where the number of firms producing, N_i is exogenous and where the number of firms N_i is pinned down by a free entry condition such that the total profits in a country Π_i are equal to the entry cost $w_i F_i$.

4.2 Macro assumptions

The first macro assumption is that **trade is balanced**, i.e. for all $j \in S$ we have:

$$\sum_{i \in S} X_{ij} = \sum_{i \in S} X_{ji}. \tag{16}$$

[2] These functional forms are slightly less general than the original paper, but are more consistent with what we have seen so far.

The second macro assumption is that **aggregate profits are a constant share of revenues**. Let Π_j denote the aggregate profits in $j \in S$ and let R_j denote the aggregate revenue in $j \in S$ (which is equal to aggregate income). Then this restriction requires that:

$$\frac{\Pi_j}{R_j} = c \geq 0. \tag{17}$$

The third macro assumption is that the **import demand system is "CES."** Formally, for any importer $j \in S$ and any two exporters $i \in S$ and $i' \in S$ such that $i \neq j$ and $i' \neq j$, we have:

$$\frac{\partial \ln \left(\frac{X_{ij}}{X_{jj}}\right)}{\partial \ln \tau_{i'j}} = \begin{cases} 0 & \text{if } i \neq i' \\ \varepsilon < 0 & \text{if } i = i' \end{cases}$$

Intuitively, changing the trade costs affects only the demand for the good for which the trade costs are being changed and that elasticity is constant for all countries.

5 The welfare effect of a foreign shock

Suppose that there is a **foreign shock** to some country $j \in S$. That is, suppose the set of exogenous model parameters $(\mathbf{L}, \mathbf{F}, \tau, \xi)$ changes to $(\mathbf{L}', \mathbf{F}', \tau', \xi')$, holding fixed the domestic model parameters L_j (the labor supply), F_j (the fixed cost of entry), τ_{jj} (the domestic trade costs) and ξ_{ii} (the domestic fixed "export" cost).

Theorem 1. *For any foreign shock to country j, the change in overall welfare $W_j \equiv \frac{Y_j}{P_j}$ can be written as a function of the change in the country's openness and the partial elasticity of trade flows to variable trade costs:*

$$\hat{W}_j = \hat{\lambda}_{jj}^{\frac{1}{\varepsilon}}. \tag{18}$$

Proof. We prove the (more complicated) case where there is monopolistic competition and refer you to the paper for the perfect competition case.

Without loss of generality, normalize the wage in country j to one. Then as with the Armington model, we can write the change in (log) welfare as the negative of the change in the (log) price index:

$$\begin{aligned} \ln W_j &\equiv \ln Y_j - \ln P_j \iff \\ \ln W_j &\equiv \ln w_j + \ln L_j - \ln P_j \implies \\ d \ln W_j &= -d \ln P_j, \end{aligned} \tag{19}$$

since the local labor supply does not change and the local wages are normalized to be one. We now consider the price index. From equation (15) the price index can be written as:

$$\begin{aligned} P_j &= \left(\int_{\omega \in \Omega} p_i (\omega)^{1-\sigma} d\omega \right)^{\frac{1}{1-\sigma}} \iff \\ P_j &= \left(\sum_{i \in S} \int_{\Omega_{ij}} p_{ij} (\omega)^{1-\sigma} d\omega \right)^{\frac{1}{1-\sigma}}, \end{aligned}$$

where Ω_{ij} is the set of goods exported from $i \in S$ to $j \in S$.

From Shepherd's lemma (!), trade shares can be written as:

$$
\begin{aligned}
q_{ij}(\omega) &= W_j \frac{\partial P_j}{\partial p_{ij}(\omega)} \implies \\
q_{ij}(\omega) &= W_j p_{ij}(\omega)^{-\sigma} \left(\sum_{i \in S} \int_{\Omega_{ij}} p_{ij}(\omega)^{1-\sigma} d\omega \right)^{\frac{\sigma}{1-\sigma}} \iff \\
x_{ij}(\omega) &= \frac{Y_j}{P_j} p_i(\omega)^{1-\sigma} \left(\sum_{i \in S} \int_{\Omega_{ij}} p_{ij}(\omega)^{1-\sigma} d\omega \right)^{\frac{\sigma}{1-\sigma}} \iff \\
x_{ij}(\omega) &= \frac{p_{ij}(\omega)^{1-\sigma}}{\sum_{i \in S} \int_{\Omega_{ij}} p_{ij}(\omega)^{1-\sigma} d\omega} Y_j \implies \\
\int_{\Omega_{ij}} x_{ij}(\omega) d\omega &= \frac{\int_{\Omega_{ij}} p_{ij}(\omega)^{1-\sigma} d\omega}{\sum_{i \in S} \int_{\Omega_{ij}} p_{ij}(\omega)^{1-\sigma} d\omega} Y_j \iff \\
X_{ij} &= \frac{\int_{\Omega_{ij}} p_{ij}(\omega)^{1-\sigma} d\omega}{\sum_{i \in S} \int_{\Omega_{ij}} p_{ij}(\omega)^{1-\sigma} d\omega} Y_j. \qquad (20)
\end{aligned}
$$

From monopolistic competition, we know that the price will be a constant mark-up over marginal cost so that the price of good $\omega \in \Omega$ that is shipped from i to j will be:

$$
p_{ij}(\omega) = \frac{\sigma}{\sigma - 1} \frac{\tau_{ij} w_i}{\alpha_i(\omega)}
$$

However, only firms that are sufficiently productive will choose to export from i to j. Recall that the profits of a firm can be written as:

$$
\pi_{ij}(\omega) = \frac{1}{\sigma} \left(\frac{\sigma}{\sigma - 1} \frac{\tau_{ij} w_i}{\alpha_i(\omega)} \right)^{1-\sigma} Y_j P_j^{\sigma-1}
$$

so that a firm will choose to export if and only if:

$$
\begin{aligned}
\pi_{ij}(\omega) &\geq f_{ij} \iff \\
\frac{1}{\sigma} \left(\frac{\sigma}{\sigma - 1} \frac{\tau_{ij} w_i}{\alpha_i(\omega)} \right)^{1-\sigma} Y_j P_j^{\sigma-1} &\geq \xi_{ij} \times w_i
\end{aligned}
$$

Let α_{ij}^* denote the minimum "productivity" of a firm and let $G_i(\alpha)$ denote the distribution of α. Note that:

$$
\begin{aligned}
\frac{1}{\sigma} \left(\frac{\sigma}{\sigma - 1} \frac{\tau_{ij} w_i}{\alpha_{ij}^*} \right)^{1-\sigma} Y_j P_j^{\sigma-1} &= \xi_{ij} \times w_i \iff \\
\alpha_{ij}^* &= \left(\frac{1}{\sigma} \right)^{\frac{1}{1-\sigma}} \left(\frac{\sigma}{\sigma - 1} \tau_{ij} w_i \right) \left(\frac{Y_j}{\xi_{ij} \times w_i} \right)^{\frac{1}{1-\sigma}} \frac{1}{P_j}. \qquad (21)
\end{aligned}
$$

Then we have that the distribution of prices of goods being sold from i to j is:

$$\int_{\Omega_{ij}} p_{ij}(\omega)^{1-\sigma} d\omega = N_i \int_{\alpha_{ij}^*}^{\infty} \left(\frac{\sigma}{\sigma-1}\frac{\tau_{ij}w_i}{\alpha}\right)^{1-\sigma} dG_i(\alpha) \iff$$
$$\int_{\Omega_{ij}} p_{ij}(\omega)^{1-\sigma} d\omega = N_i \left(\frac{\sigma}{\sigma-1}\tau_{ij}w_i\right)^{1-\sigma} \int_{\alpha_{ij}^*}^{\infty} \alpha^{\sigma-1} dG_i(\alpha) \tag{22}$$

so that the price index becomes:

$$P_j = \left(\sum_{i\in S}\int_{\Omega_{ij}} p_{ij}(\omega)^{1-\sigma} d\omega\right)^{\frac{1}{1-\sigma}} \iff$$
$$P_j = \left(\sum_{i\in S} N_i \left(\frac{\sigma}{\sigma-1}\tau_{ij}w_i\right)^{1-\sigma} \int_{\alpha_{ij}^*}^{\infty} \alpha^{\sigma-1} dG_i(\alpha)\right)^{\frac{1}{1-\sigma}} \tag{23}$$

Substituting (22) into the gravity equation (20) yields:

$$X_{ij} = \frac{N_i\left(\frac{\sigma}{\sigma-1}\tau_{ij}w_i\right)^{1-\sigma}\int_{\alpha_{ij}^*}^{\infty}\alpha^{\sigma-1}dG_i(\alpha)}{\sum_{i\in S}N_i\left(\frac{\sigma}{\sigma-1}\tau_{ij}w_i\right)^{1-\sigma}\int_{\alpha_{ij}^*}^{\infty}\alpha^{\sigma-1}dG_i(\alpha)}Y_j. \tag{24}$$

Differentiating the price index using equation (23) yields:

$$d\ln P_j = \frac{1}{1-\sigma}\sum_{i\in S}\frac{N_i\left(\frac{\sigma}{\sigma-1}\tau_{ij}w_i\right)^{1-\sigma}\int_{\alpha_{ij}^*}^{\infty}\alpha^{\sigma-1}dG_i(\alpha)}{\sum_{i\in S}N_i\left(\frac{\sigma}{\sigma-1}\tau_{ij}w_i\right)^{1-\sigma}\int_{\alpha_{ij}^*}^{\infty}\alpha^{\sigma-1}dG_i(\alpha)}\left(d\ln\left(N_i\left(\frac{\sigma}{\sigma-1}\tau_{ij}w_i\right)^{1-\sigma}\int_{\alpha_{ij}^*}^{\infty}\alpha^{\sigma-1}dG_i(\alpha)\right)\right)$$
$$d\ln P_j = \sum_{i\in S}\lambda_{ij}\left(d\ln(\tau_{ij}w_i) + \frac{d\ln N_i + \gamma_{ij}d\ln\alpha_{ij}^*}{1-\sigma}\right), \tag{25}$$

where the second line used the gravity equation (24) and $\gamma_{ij} \equiv \frac{d\ln\int_{\alpha_{ij}^*}^{\infty}\alpha^{\sigma-1}dG_i(\alpha)}{d\ln\alpha_{ij}^*}$. Substituting in expression (21) for α_{ij}^* yields:

$$d\ln P_j = \sum_{i\in S}\lambda_{ij}\left(d\ln(\tau_{ij}w_i) + \frac{d\ln N_i + \gamma_{ij}d\ln\alpha_{ij}^*}{1-\sigma}\right) \iff$$
$$d\ln P_j = \sum_{i\in S}\lambda_{ij}\left(d\ln(\tau_{ij}w_i) + \frac{d\ln N_i + \gamma_{ij}d\ln\left(\left(\frac{1}{\sigma}\right)^{\frac{1}{1-\sigma}}\left(\frac{\sigma}{\sigma-1}\tau_{ij}w_i\right)\left(\frac{Y_j}{\xi_{ij}\times w_i}\right)^{\frac{1}{1-\sigma}}\frac{1}{P_j}\right)}{1-\sigma}\right) \iff$$
$$d\ln P_j = \sum_{i\in S}\left(\frac{1}{1-\sigma}\right)\lambda_{ij}\left((1-\sigma+\gamma_{ij})\,d\ln(\tau_{ij}w_i) + d\ln N_i - \frac{\gamma_{ij}}{1-\sigma}\left(d\ln(\xi_{ij}w_i)\right) - \gamma_{ij}d\ln P_j\right) \iff$$
$$d\ln P_j = \sum_{i\in S}\left(\frac{\lambda_{ij}}{1-\sigma+\sum_{i\in S}\lambda_{ij}\gamma_{ij}}\right)\left((1-\sigma+\gamma_{ij})\,d\ln(\tau_{ij}w_i) + d\ln N_i - \frac{\gamma_{ij}}{1-\sigma}\left(d\ln(\xi_{ij}w_i)\right)\right). \tag{26}$$

Let us now differentiate the gravity equation (24):

$$
\begin{aligned}
\lambda_{ij} &= \frac{N_i\left(\frac{\sigma}{\sigma-1}\tau_{ij}w_i\right)^{1-\sigma}\int_{\alpha_{ij}^*}^{\infty}\alpha^{\sigma-1}dG_i\left(\alpha\right)}{\sum_{i\in S}N_i\left(\frac{\sigma}{\sigma-1}\tau_{ij}w_i\right)^{1-\sigma}\int_{\alpha_{ij}^*}^{\infty}\alpha^{\sigma-1}dG_i\left(\alpha\right)} \implies \\
\frac{\lambda_{ij}}{\lambda_{jj}} &= \frac{N_i\left(\tau_{ij}w_i\right)^{1-\sigma}\int_{\alpha_{ij}^*}^{\infty}\alpha^{\sigma-1}dG_i\left(\alpha\right)}{N_j\left(\tau_{jj}w_j\right)^{1-\sigma}\int_{\alpha_{jj}^*}^{\infty}\alpha^{\sigma-1}dG_i\left(\alpha\right)} \implies \\
d\ln\lambda_{ij} - d\ln\lambda_{jj} &= d\ln N_i - d\ln N_j + (1-\sigma)\left(d\ln\tau_{ij}w_i\right) + \gamma_{ij}d\ln\alpha_{ij}^* - \gamma_{jj}d\ln\alpha_{jj}^*.
\end{aligned}
$$

Substituting in expression (21) for α_{ij}^* yields:

$$
\begin{aligned}
d\ln\frac{\lambda_{ij}}{\lambda_{jj}} &= d\ln N_i - d\ln N_j + (1-\sigma)\left(d\ln\tau_{ij}w_i\right) + \gamma_{ij}d\ln\alpha_{ij}^* - \gamma_{jj}d\ln\alpha_{jj}^* \iff \\
d\ln\frac{\lambda_{ij}}{\lambda_{jj}} &= d\ln\frac{N_i}{N_j} + (1-\sigma)\left(d\ln\tau_{ij}w_i\right) + \gamma_{ij}d\ln\left(\left(\tau_{ij}w_i\right)\left(\frac{Y_j}{\xi_{ij}w_i}\right)^{\frac{1}{1-\sigma}}\frac{1}{P_j}\right) - \gamma_{jj}d\ln\left(\left(\tau_{jj}w_j\right)\left(\frac{Y_j}{\xi_{jj}w_j}\right)^{\frac{1}{1-\sigma}}\frac{1}{P_j}\right. \\
d\ln\frac{\lambda_{ij}}{\lambda_{jj}} &= d\ln N_i - d\ln N_j + \left(1-\sigma+\gamma_{ij}\right)\left(d\ln\tau_{ij}w_i\right) - \frac{\gamma_{ij}}{1-\sigma}\left(d\ln\xi_{ij} + d\ln w_i\right) \qquad (27)
\end{aligned}
$$

Finally (!!) substituting equation (27) into equation (26) yields:

$$
\begin{aligned}
d\ln P_j &= \sum_{i\in S}\left(\frac{\lambda_{ij}}{1-\sigma+\sum_{i\in S}\lambda_{ij}\gamma_{ij}}\right)\left(\left(1-\sigma+\gamma_{ij}\right)d\ln\left(\tau_{ij}w_i\right) + d\ln N_i - \frac{\gamma_{ij}}{1-\sigma}\left(d\ln\xi_{ij} + d\ln w_i\right)\right) \iff \\
d\ln P_j &= \sum_{i\in S}\left(\frac{\lambda_{ij}}{1-\sigma+\sum_{i\in S}\lambda_{ij}\gamma_{ij}}\right)\left(d\ln\lambda_{ij} - d\ln\lambda_{jj} + d\ln N_j\right) \iff \\
d\ln P_j &= \frac{1}{1-\sigma+\sum_{i\in S}\lambda_{ij}\gamma_{ij}}\left(\sum_{i\in S}\lambda_{ij}d\ln\lambda_{ij} + d\ln\lambda_{jj}\sum_{i\in S}\lambda_{ij} + \sum_{i\in S}\lambda_{ij}d\ln N_j\right) \iff \\
d\ln P_j &= \frac{1}{1-\sigma+\sum_{i\in S}\lambda_{ij}\gamma_{ij}}\left(d\ln\lambda_{jj} + \sum_{i\in S}\lambda_{ij}d\ln N_j\right) \iff \\
d\ln W_j &= -\frac{1}{1-\sigma+\sum_{i\in S}\lambda_{ij}\gamma_{ij}}\left(d\ln\lambda_{jj} + \sum_{i\in S}\lambda_{ij}d\ln N_j\right). \qquad (28)
\end{aligned}
$$

We have two more things to do. The first is to show that $d\ln N_j = 0$. Note that if there is fixed entry, then $d\ln N_j = 0$ trivially. If there is free entry, then the free entry condition implies that $\Pi_j = N_jF_j$. From the assumption that aggregate profits are a constant share of revenues we then have that:

$$\begin{aligned}
\Pi_j &= N_j w_j F_j \iff \\
\ln N_j &= \ln \Pi_j - \ln F_j - \ln w_j \iff \\
\ln N_j &= \ln c + \ln R_j - \ln F_j - \ln w_j \iff \\
\ln N_j &= \ln c + \ln Y_j - \ln F_j - \ln w_j \iff \\
\ln N_j &= \ln c + \ln w_j + \ln L_j - \ln F_j - \ln w_j \iff \\
\ln N_j &= \ln c + \ln L_j - \ln F_j \implies \\
d \ln N_j &= 0.
\end{aligned}$$

Hence, we can write equation (28) as:

$$d \ln W_j = -\frac{1}{1 - \sigma + \sum_{i \in S} \gamma_{ij}} d \ln \lambda_{jj}. \tag{29}$$

The second thing to do is to show that $1 - \sigma + \sum_{i \in S} \lambda_{ij} \gamma_{ij} = \varepsilon$. This comes directly from the "CES" import demand system to equation (24). Hence, equation (29) becomes:

$$d \ln W_j = -\frac{1}{\varepsilon} d \ln \lambda_{jj} \tag{30}$$

As in the Armington model, we can integrate (30) to get the "ACR" condition:

$$\begin{aligned}
\frac{d \ln W_j}{d \ln \lambda_{jj}} &= \frac{1}{1 - \sigma} \implies \\
\ln W_j^1 - \ln W_j^0 &= \int_{\lambda_{jj}^0}^{\lambda_{jj}^1} \frac{d \ln W_j}{d \ln \lambda} d\lambda \iff \\
\ln W_j^1 - \ln W_j^0 &= \frac{1}{1 - \sigma} \left(\ln \lambda_{jj}^1 - \ln \lambda_{jj}^0 \right) \iff \\
\hat{W}_j &= \left(\hat{\lambda}_{jj} \right)^{\frac{1}{1 - \sigma}},
\end{aligned}$$

as required. □

6 World Welfare

The previous results provide a relationship between the change in welfare of a country and the change in its openness. The limitation of the ACR equation, however, is that it does not tell us what the effect of a change in exogenous parameters would be on welfare. It turns out that such a characterization is (reasonably) straightforward if we consider world welfare (instead of country welfare).

consider the problem of maximizing a weighted average of world welfare subject to only the aggregate feasibility constraint. Of course, in the absence of a micro-foundation of the gravity trade model nothing can be directly said about the welfare of the equilibrium (as we have not specified preferences). However, Arkolakis, Costinot, and Rodríguez-Clare (2012) show that for a large class of trade models, the welfare of a location can be written solely as an increasing function of its openness to trade and an exogenous parameter, i.e. for all $i \in S$, welfare in location i, can be written as:[3]

$$W_i = C_i^W \lambda_{ii}^{-1/\rho} = C_i^W \left(B_i \gamma_i^{\alpha-1} \delta_i^{\beta-1}\right)^{1/\rho}, \tag{31}$$

where $C_i^W > 0$ is an (exogenous) parameter and $\rho > 0$ is an exogenous scalar. If welfare can be written as in equation (31), we can define world welfare as a weighted average of the welfare in each location:

$$W \equiv \sum_{i \in S} \omega_i W_i = \sum_{i \in S} \omega_i C_i^W \left(B_i \gamma_i^{\alpha-1} \delta_i^{\beta-1}\right)^{1/\rho},$$

where $\omega_i > 0$ are the weights placed on the welfare in each location. Then the following world welfare maximization problem is well defined:

$$\max_{\{\gamma\},\{\delta\}} W$$
$$\text{s.t.} \sum_{i \in S} \sum_{j \in S} K_{ij} \gamma_i \delta_j = \sum_{i \in S} B_i \gamma_i^{\alpha} \delta_i^{\beta}. \tag{32}$$

It turns out that the solution tothe world welfare maximization problem (32) is the solution to the general equilibrium gravity model, which we prove in the following proposition:

Proposition 1. *Consider any general equilibrium gravity model. If $\alpha + \beta > 2$ or $\alpha + \beta < 0$ (which guarantees uniqueness), then:*

If welfare can be expressed as in equation (31), then there exists a set of weights $\{\omega_i\}$ such that the solution of the general equilibrium trade model is equivalent to the solution of the world welfare maximization problem (32).

Proof. With the assumption we made that the utility for country i is expressed in the following form

$$u_i = \left(B_i \left(\gamma_i\right)^{\alpha-1} \left(\delta_i\right)^{\beta-1}\right)^{1/\rho},$$

the welfare maximization problem is to maximize the weighted sum of $\{u_i\}_i$ subject to the same constraints. To show that the competitive allocation is Pareto efficient, we show that under a particular choice of (θ_i), the competitive allocation $\left(\gamma_i^{CE}, \delta_i^{CE}\right)_i$ solves the planning problem.

Set the Pareto weights $(\omega_i)_i$ as follows.

$$(\omega_i) = \sum_k \frac{\left(B_k\right)^{1/\rho} \left(\gamma_k^{CE}\right)^{(\alpha-1)/\rho} \left(\delta_k^{CE}\right)^{(\beta-1)/\rho}}{\left(B_i\right)^{1/\rho} \left(\gamma_i^{CE}\right)^{(\alpha-1)/\rho} \left(\delta_i^{CE}\right)^{(\beta-1)/\rho} \frac{\sum_i (B_i) \left(\gamma_i^{CE}\right)^{\alpha} \left(\delta_i^{CE}\right)^{\beta}}{\sum_j K_{j,i} \gamma_j^{CE} \delta_i^{CE}}} (\omega_k).$$

[3] In addition to CES preferences, this includes a larger class of homothetic demand functions including the symmetric translog demand function (see also Feenstra (2003)) and the Kimball demand function (see **?**); see Arkolakis, Costinot, Donaldson, and Rodríguez-Clare (2012).

From Karlin and Nirenberg (1967), we know there is a solution to the system.

The associated Lagrangian is

$$\mathcal{L} = \sum_i \omega_i B_i^{1/\rho} \gamma_i^{(\alpha-1)/\rho} \delta_i^{(\beta-1)/\rho} - \lambda \left(\sum_i \sum_j K_{ij} \gamma_i \delta_j - \sum_i B_i \gamma_i^\alpha \delta_i^\beta \right).$$

Taking the FONCs w.r.t. γ_i and δ_i, we get

$$\rho^{-1} (\alpha - 1) \omega_i B_i^{1/\rho} \gamma_i^{(\alpha-1)/\rho} \delta_i^{(\beta-1)/\rho} = \lambda \sum_j K_{ij} \gamma_i \delta_j - \alpha \lambda B_i \gamma_i^\alpha \delta_i^\beta$$

$$\rho^{-1} (\beta - 1) \omega_i B_i^{1/\rho} \gamma_i^{(\alpha-1)/\rho} \delta_i^{(\beta-1)/\rho} = \lambda \sum_j K_{ji} \gamma_j \delta_i - \lambda \beta B_i \gamma_i^\alpha \delta_i^\beta.$$

Adding the two equations, and solving for λ, we have

$$\lambda = \frac{1}{\rho} \frac{W}{Y}.$$

Substitute this expression into the FONCs.

$$\left(\frac{\alpha - 1}{\alpha} \right) \left(\frac{\omega_i B_i^{1/\rho} \gamma_i^{(\alpha-1)/\rho} \delta_i^{(\beta-1)/\rho}}{W} Y^W - \sum_j K_{ij} \gamma_i \delta_j \right) + \sum_j K_{ij} \gamma_i \delta_j = B_i \gamma_i^\alpha \delta_i^\beta$$

$$\left(\frac{\beta - 1}{\beta} \right) \left(\frac{\omega_i B_i^{1/\rho} \gamma_i^{(\alpha-1)/\rho} \delta_i^{(\beta-1)/\rho}}{W} Y^W - \sum_j K_{ji} \gamma_j \delta_i \right) + \sum_j K_{ji} \gamma_j \delta_i = B_i \gamma_i^\alpha \delta_i^\beta.$$

From the construction of ω_i, the bracket term is zero if we evaluate the system at $\left(\gamma_i^{CE}, \delta_i^{CE}\right)_i$.

$$\left(\omega_i B_i^{1/\rho} \left(\gamma_i^{CE}\right)^{(\alpha-1)/\rho} \left(\delta_i^{CE}\right)^{(\beta-1)/\rho} \frac{\sum_j B_j \left(\gamma_j^{CE}\right)^\alpha \left(\delta_j^{CE}\right)^\beta}{\sum_j \omega_i B_i^{1/\rho} \left(\gamma_i^{CE}\right)^{(\alpha-1)/\rho} \left(\delta_i^{CE}\right)^{(\beta-1)/\rho}} - \sum_j K_{ij} \gamma_i^{CE} \delta_j^{CE} \right) = 0.$$

Then the second equation is solved at $\left(\gamma_i^{CE}, \delta_i^{CE}\right)_i$ since

$$\sum_j K_{ji} \gamma_j^{CE} \delta_i^{CE} = B_i \left(\gamma_i^{CE}\right)_i^\alpha \left(\delta_i^{CE}\right)^\beta.$$

. □

An advantage of this dual approach is that it allows us to apply the envelope theorem to derive an expression for how any change in bilateral trade frictions affects world welfare. Applying the envelope theorem to the world welfare maximization interpretation, the elasticity of world welfare to K_{ij} is even simpler:

$$\frac{\partial \ln W}{\partial \ln K_{ij}} = \frac{1}{\rho} \frac{X_{ij}}{Y^W}. \tag{33}$$

This expression has been derived for gravity models with CES demand by Atkeson and Burstein (2010), **?**, and **?**; our derivation extends this result to any gravity trade model where welfare can be expressed as in equation (31).

7 Conclusion and next steps

The publication of Arkolakis, Costinot, and Rodríguez-Clare (2012) led to quite a controversy, as it undermined the contributions of the new margin for the gains from trade in the heterogeneous firm theory based on the reallocation of resources to more productive firms. In very recent work, Melitz and Redding (2014) develop a heterogenous firm trade model where the degree of heterogeneity is one of the structural parameters. They show that for any set of trade costs, if the productivity of the homogeneous firm version of the model is calibrated so that it delivers the same welfare as the heterogeneous firm version of the model with those same trade costs, then for any change in the trade costs, the welfare will be greater with the heterogenous firm trade model. Note that this is fundamentally a different experiment than in Arkolakis, Costinot, and Rodríguez-Clare (2012), as equating the welfare in the "before" world between the two models does not restrict the own-country trade shares or the elasticity of trade to variable trade costs to be the same in both models.

We were going to spend a class going through Melitz and Redding (2014), but since we need one more class to go through the class presentations than originally allocated, we will move to the empirical part of the course. Next class we will discuss about how to estimate the gravity equation.

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