Problem Set 2

Mathematics and Statistics for Economists

Due date: Thursday 8th September 2022

Optimisation

Exercise 1 Find the gradiant vector and the Hessian matrix of the following functions

$$\textbf{(a)} \quad f(x,y) = 3x + 4y^3 \quad \textbf{(b)} \quad f(x,y) = x^3 \ln y + 6x^2y^3 + e^{2x}y$$
$$\textbf{(c)} \quad f(x,y) = (x^2 + 4y^2)^{-1/2} \quad \textbf{(d)} \quad (x+4y)(e^{-2x} + e^{-3y})$$

Exercise 2 For the function

$$z = -2x^2 + 4xy - 3y^2 + 10x - 14y - 3$$

(a) Find the gradient vector and the Hessian matrix.

(b) Show that the function is concave, and find the global maximum value of z.

(c) Find the global maximum value subject to the constraints $x \geq 0$ and $y \geq 0$.

Exercise 3 Find the maximum and minimum values of $2x + 3y + z$ subject to

$$x^2 + y^2 + z^2 = 9$$

Exercise 4 Solve the problem

$$\min \quad (x-1)^2 + (y-1)^2 \quad \text{subject to } 3x + 4y \leq a$$

(a) if $a = 8$, (b) $a = 5$. In each case, verify that the Kuhn-Tucker conditions hold at the optimum.

Exercise 5 Consider an economy inhabited by a representative agent. She maximizes the utility of her consumption C over her infinite lifetime

$$U_t = \sum_{s=0}^{\infty} \beta^s \ln(C_{t+s})$$

where t is the time subscript. $U(C_t) = \ln C_t$. $\beta \in (0,1)$ is the discount factor.

The resource constraint indicates that the output Y at time t can be consumed or invested to increase the capital stock.

$$Y_t = C_t + I_t \quad \text{where } I_t = K_{t+1} - K_t$$

In this economy, output is produced using solely capital

$$Y_t = \frac{1}{\alpha} K_t^{\alpha}$$

(a) Write the Lagrangian for this constrained maximisation problem and indicate what are the control and state variables.

(b) Write the first-order conditions.

(c) Find a relation between C_t and C_{t+1}, the so-called Euler condition.

Probability and statistics

Exercise 6 Let the continuous random variable X have pdf $f_X(x) = c(x(x+1))$ for $1 \leq x \leq 2$. Find

(a) c.

(b) $E(X)$.

(c) $\text{var}(X)$.

Exercise 7 You roll a fair die twice. If $X = |$(roll on die 1) - (roll on die 2)$|$ (in absolute value), define the sample space S and give the probability distribution for X.

Exercise 8 Given $P(A) + P(B) = 1.2$, $P(B|A) = .6$ and $P(A \cup B) = .9$. Find $P(A)$.

Exercise 9 Find the moment generating function of the standard normal random variable $X = \mathcal{N}(0,1)$ and calculate the first three moments.

Exercise 10 The amount of time required to solve a certain Maths bootcamp exam is normally distributed with mean 40 minutes and standard deviation 8 minutes.

(a) Find the probability that a student solves the exam in under 30 minutes.

(b) One student finds that her solving time is greater than exactly 19% of the other students in the class. How long did she take to solve the exam (to the nearest minute)?

(c) If 60 students solve the exam independently, find the probability that their TOTAL solving time is greater that 2500 minutes.

(d) If 60 students solve the exam independently, find the probability that exactly 1 student solves the problem in under 20 minutes.