# Macroeconomics A; EI060

# Short problems

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## 1 Optimal exchange rate choice in a floating regime

**Question**: The output gap reflects unexpected exchange rate movements:

$$x_t = \theta \left( \Delta e_t - \Delta e_t^{\text{expected}} \right)$$

The loss function reflects the volatility of the output gap, around a reference point $\overline{x}$, the volatility of the exchange rate movements, and a fixed cost if the country deviates from the peg.

$$L = \frac{1}{2} \left\{ \phi \left( \theta \left( \Delta e_t - \Delta e_t^{\text{expected}} \right) - \overline{x} \right)^2 + (\Delta e_t)^2 \right\} + c_{\Delta e_t > 0}$$

Show that if the country deviates from the peg, it sets:

$$\Delta e_t = \frac{\phi \theta}{1 + \phi \theta^2} \left( \theta \Delta e_t^{\text{expected}} + \overline{x} \right)$$

Conditional on the country opting for a floating exchange rate, show that in equilibrium $\Delta e_t = \phi \theta \overline{x}$.

**Answer**: The loss function is:

$$L = \frac{1}{2} \left\{ \phi \left( x_t - \overline{x} \right)^2 + (\Delta e_t)^2 \right\} + c_{\Delta e_t > 0}$$

$$L = \frac{1}{2} \left\{ \phi \left( \theta \left( \Delta e_t - \Delta e_t^{\text{expected}} \right) - \overline{x} \right)^2 + (\Delta e_t)^2 \right\} + c_{\Delta e_t > 0}$$

The central chooses $\Delta e_t$ to minimize the loss, taken $\Delta e_t^{\text{expected}}$ as given. The optimal condition is:

$$
\begin{aligned}
0 &= \frac{\partial L}{\partial\left(\Delta e_t\right)} \\
0 &= \phi\theta\left(\theta\left(\Delta e_t-\Delta e_t^{\text{expected}}\right)-\overline{x}\right)+\left(\Delta e_t\right) \\
0 &= \phi\theta^2\left(\Delta e_t-\Delta e_t^{\text{expected}}\right)-\phi\theta\overline{x}+\left(\Delta e_t\right) \\
0 &= \phi\theta^2\left(\Delta e_t\right)-\phi\theta^2\left(\Delta e_t^{\text{expected}}\right)-\phi\theta\overline{x}+\left(\Delta e_t\right) \\
\left(1+\phi\theta^2\right)\left(\Delta e_t\right) &= \phi\theta^2\left(\Delta e_t^{\text{expected}}\right)+\phi\theta\overline{x} \\
\Delta e_t &= \frac{\phi\theta}{1+\phi\theta^2}\left(\theta\Delta e_t^{\text{expected}}+\overline{x}\right)
\end{aligned}
$$

In an equilibrium where the bank moves to a floating exchange rate, the actual exchange rate change is expected ($\Delta e_t=\Delta e_t^{\text{expected}}$):

$$
\begin{aligned}
\Delta e_t &= \frac{\phi\theta}{1+\phi\theta^2}\left(\theta\Delta e_t^{\text{expected}}+\overline{x}\right) \\
\Delta e_t &= \frac{\phi\theta}{1+\phi\theta^2}\left(\theta\Delta e_t+\overline{x}\right) \\
\left(1+\phi\theta^2\right)\Delta e_t &= \phi\theta^2\Delta e_t+\phi\theta\overline{x} \\
\Delta e_t &= \phi\theta\overline{x}
\end{aligned}
$$

## 2 Welfare

**Question**: Show that the loss function if the bank abandons the peg is:

$$
L^{\text{float}} = \frac{\phi}{2\left(1+\phi\theta^2\right)}\left(\overline{x}+\theta\Delta e_t^{\text{expected}}\right)^2+c
$$

Show that the loss function of the peg is maintained is:

$$
L^{\text{peg}} = \frac{\phi}{2}\left(\overline{x}+\theta\Delta e_t^{\text{expected}}\right)^2
$$

**Answer**: Take the loss function, and input the optimal exchange rate, conditional on the expected exchange rate, and conditional on being in a floating rate:

$$\begin{aligned}
L^{\text{float}} &= \frac{1}{2}\left\{\phi\left(\theta\left(\Delta e_t - \Delta e_t^{\text{expected}}\right) - \overline{x}\right)^2 + \left(\Delta e_t\right)^2\right\} + c \\
L^{\text{float}} &= \frac{1}{2}\left\{\phi\left(\theta\left(\frac{\phi\theta}{1+\phi\theta^2}\left(\theta\Delta e_t^{\text{expected}} + \overline{x}\right) - \Delta e_t^{\text{expected}}\right) - \overline{x}\right)^2 + \left(\frac{\phi\theta}{1+\phi\theta^2}\left(\theta\Delta e_t^{\text{expected}} + \overline{x}\right)\right)^2\right\} + c \\
L^{\text{float}} &= \frac{1}{2}\left\{\phi\left(\theta\left(\frac{\phi\theta}{1+\phi\theta^2}\overline{x} - \frac{1}{1+\phi\theta^2}\Delta e_t^{\text{expected}}\right) - \overline{x}\right)^2 + \left(\frac{\phi\theta}{1+\phi\theta^2}\left(\theta\Delta e_t^{\text{expected}} + \overline{x}\right)\right)^2\right\} + c \\
L^{\text{float}} &= \frac{1}{2}\left\{\phi\left(\frac{\phi\theta^2}{1+\phi\theta^2}\overline{x} - \frac{\theta}{1+\phi\theta^2}\Delta e_t^{\text{expected}} - \overline{x}\right)^2 + \left(\frac{\phi\theta}{1+\phi\theta^2}\left(\theta\Delta e_t^{\text{expected}} + \overline{x}\right)\right)^2\right\} + c \\
L^{\text{float}} &= \frac{1}{2}\left\{\phi\left(-\frac{1}{1+\phi\theta^2}\overline{x} - \frac{\theta}{1+\phi\theta^2}\Delta e_t^{\text{expected}}\right)^2 + \left(\frac{\phi\theta}{1+\phi\theta^2}\left(\theta\Delta e_t^{\text{expected}} + \overline{x}\right)\right)^2\right\} + c \\
L^{\text{float}} &= \frac{1}{2}\left\{\phi\left(\frac{1}{1+\phi\theta^2}\right)^2\left(\overline{x} + \theta\Delta e_t^{\text{expected}}\right)^2 + \left(\frac{\phi\theta}{1+\phi\theta^2}\left(\theta\Delta e_t^{\text{expected}} + \overline{x}\right)\right)^2\right\} + c \\
L^{\text{float}} &= \frac{1}{2}\left\{\phi + (\phi\theta)^2\right\}\left(\frac{1}{1+\phi\theta^2}\right)^2\left(\overline{x} + \theta\Delta e_t^{\text{expected}}\right)^2 + c \\
L^{\text{float}} &= \frac{1}{2}\phi\left(1+\phi\theta^2\right)\left(\frac{1}{1+\phi\theta^2}\right)^2\left(\overline{x} + \theta\Delta e_t^{\text{expected}}\right)^2 + c \\
L^{\text{float}} &= \frac{\phi}{2\left(1+\phi\theta^2\right)}\left(\overline{x} + \theta\Delta e_t^{\text{expected}}\right)^2 + c
\end{aligned}$$

If the peg is maintained, $\Delta e_t$, and the loss is (conditional on the expected exchange rate):

$$\begin{aligned}
L^{\text{peg}} &= \frac{1}{2}\left\{\phi\left(\theta\left(-\Delta e_t^{\text{expected}}\right) - \overline{x}\right)^2 + (0)^2\right\} \\
L^{\text{peg}} &= \frac{\phi}{2}\left(\overline{x} + \theta\Delta e_t^{\text{expected}}\right)^2
\end{aligned}$$

# 3 Threshold

**Question**: Show that the country chooses to abandon the peg if:

$$c < c^{\text{critical}} = \frac{(\phi\theta)^2}{2\left(1+\phi\theta^2\right)}\left(\overline{x} + \theta\Delta e_t^{\text{expected}}\right)^2$$

Show that $c = c^{\text{critical}}$ translates into a threshold for the exchange rate expectations:

$$\Delta\overline{e}_t^{\text{expected}} = \frac{\sqrt{2\left(1+\phi\theta^2\right)c}}{\phi\left(\theta\right)^2} - \frac{\overline{x}}{\theta}$$

Show that there is also another value of $\Delta\overline{e}_t^{\text{expected}}$ which is unambiguously negative

**Answer**: The central bank abandons the peg is the loss is higher under the peg:

$$\begin{aligned}
L^{\text{float}} &< L^{\text{peg}} \\
\frac{\phi}{2\left(1+\phi\theta^2\right)}\left(\overline{x}+\theta\Delta e_t^{\text{expected}}\right)^2 + c &< \frac{\phi}{2}\left(\overline{x}+\theta\Delta e_t^{\text{expected}}\right)^2 \\
c &< \left(1-\frac{1}{1+\phi\theta^2}\right)\frac{\phi}{2}\left(\overline{x}+\theta\Delta e_t^{\text{expected}}\right)^2 \\
c &< \frac{(\phi\theta)^2}{2\left(1+\phi\theta^2\right)}\left(\overline{x}+\theta\Delta e_t^{\text{expected}}\right)^2 \\
c &< c^{\text{critical}}
\end{aligned}$$

When the cost is at the critical value, we get:

$$\begin{aligned}
c &= \frac{(\phi\theta)^2}{2\left(1+\phi\theta^2\right)}\left(\overline{x}+\theta\Delta \overline{e}_t^{\text{expected}}\right)^2 \\
2\left(1+\phi\theta^2\right)c &= (\phi\theta)^2(\overline{x})^2 + 2(\phi\theta)^2\,\overline{x}\theta\Delta\overline{e}_t^{\text{expected}} + (\phi\theta)^2(\theta)^2\left(\Delta\overline{e}_t^{\text{expected}}\right)^2 \\
0 &= \left[(\phi\theta)^2(\overline{x})^2 - 2\left(1+\phi\theta^2\right)c\right] + 2(\phi\theta)^2\,\overline{x}\theta\Delta\overline{e}_t^{\text{expected}} + (\phi\theta)^2(\theta)^2\left(\Delta\overline{e}_t^{\text{expected}}\right)^2 \\
\Delta\overline{e}_t^{\text{expected}} &= \frac{-2(\phi\theta)^2\,\overline{x}\theta \pm \sqrt{\left[2(\phi\theta)^2\,\overline{x}\theta\right]^2 - 4(\phi\theta)^2(\theta)^2\left[(\phi\theta)^2(\overline{x})^2 - 2\left(1+\phi\theta^2\right)c\right]}}{2(\phi\theta)^2(\theta)^2} \\
\Delta\overline{e}_t^{\text{expected}} &= \frac{-2(\phi\theta)^2\,\overline{x}\theta \pm \sqrt{8(\phi\theta)^2(\theta)^2\left(1+\phi\theta^2\right)c}}{2(\phi\theta)^2(\theta)^2} \\
\Delta\overline{e}_t^{\text{expected}} &= \frac{-\phi\overline{x}\theta \pm \sqrt{2\left(1+\phi\theta^2\right)c}}{\phi(\theta)^2} \\
\Delta\overline{e}_t^{\text{expected}} &= \pm\frac{\sqrt{2\left(1+\phi\theta^2\right)c}}{\phi(\theta)^2} - \frac{\overline{x}}{\theta} \\
\Delta\overline{e}_t^{\text{expected}} &= \pm\frac{\sqrt{2\left(1+\phi\theta^2\right)c}}{\phi(\theta)^2} - \frac{\overline{x}}{\theta}
\end{aligned}$$

The two roots are:

$$\Delta\overline{e}_t^{\text{expected}} = -\frac{\sqrt{2\left(1+\phi\theta^2\right)c}}{\phi(\theta)^2} - \frac{\overline{x}}{\theta}$$

which is clearly negative, or:

$$\Delta\overline{e}_t^{\text{expected}} = \frac{\sqrt{2\left(1+\phi\theta^2\right)c}}{\phi(\theta)^2} - \frac{\overline{x}}{\theta}$$

# 4 Multiple equilibria

**Question**: Show that multiple equilibria are not feasible if $\Delta\overline{e}_t^{\text{expected}} < 0$ or $\Delta\overline{e}_t^{\text{expected}} > \phi\theta\overline{x}$. Show that multiple equilibria are possible if:

$$\overline{x} < \frac{\sqrt{2\left(1+\phi\theta^2\right)c}}{\phi\theta} < \left(1+\phi\theta^2\right)\overline{x}$$

**Answer**: If $\Delta\overline{e}_t^{\text{expected}} < 0$, then expected the peg to hold implies:

$$\Delta e_t^{\text{expected}} = 0 > \Delta\overline{e}_t^{\text{expected}}$$

As the expectation is above the threshold, the central bank abandons the peg. Expecting a peg is not an equilibrium, and only the floating exchange rate is an equilibrium.

If $\Delta\overline{e}_t^{\text{expected}} > \phi\theta\overline{x}$, the exchange rate that the central bank would choose if it abandoned the peg would be below the threshold, and thus the realized and expected exchange rate would not match. The only equilibrium is then the preservation of the peg.

Multiple equilibria can occur if:

$$\begin{aligned}
0 &< \Delta\overline{e}_t^{\text{expected}} < \phi\theta\overline{x} \\
0 &< \frac{\sqrt{2\left(1+\phi\theta^2\right)c}}{\phi\left(\theta\right)^2} - \frac{\overline{x}}{\theta} < \phi\theta\overline{x} \\
0 &< \frac{\sqrt{2\left(1+\phi\theta^2\right)c}}{\phi\theta} - \overline{x} < \phi\theta^2\overline{x} \\
\overline{x} &< \frac{\sqrt{2\left(1+\phi\theta^2\right)c}}{\phi\theta} < \left(1+\phi\theta^2\right)\overline{x}
\end{aligned}$$