Macroeconomics A; EI056

Technical appendix: Financial frictions, banking panics, and the financial accelerator

Cédric Tille

Class of December 5, 2023

1 Adverse selection

1.1 General idea

A model of adverse selection explains why lenders will not always increase the interest rate to clear the loan supply and demand, but may instead choose to limit the amount they lend (credit rationing).

Consider that there are two types of borrowers: borrowers of type G repay with probability q_g and borrowers of type B repay with probability $q_b < q_g$. The lender as the outside option to put his funds into a bond paying a return r. If the lender can observes the type of borrower, he charges a return R_g or R_b that gives an expected payoff of r:

$$r = q_g R_g \Rightarrow R_g = r/q_g$$

and similarly $R_b = r/q_b$.

If the lender cannot distinguish between borrowers, he charges all a rate r_l. The expected return $\bar{r}$ if the fraction of borrowers G is equal to g is:

$$\bar{r} = r_l \left(g q_g + (1-g) q_b \right)$$

As the lender wants to equalize this expected return $\bar{r}$ to r, we can thus determine the return r_l as a function of g. If g is a decreasing function of r_l (bad borrowers always stay, bit good ones may decide not to borrow), the expected return is not necessarily increasing in r_l:

$$\frac{\partial \bar{r}}{\partial r_l} = \left[g q_g + (1-g) q_b \right] + r_l \left(q_g - q_b \right) \frac{\partial g}{\partial r_l}$$

The first bracket is positive, but the second term is negative as $q_g - q_b > 0$ and $\partial g / \partial r_l < 0$.

1.2 A specific model

There are three components of a lending contract: the interest rate r_l, the amount of the loan L, and the amount of collateral posted by the lender C. Considers that the borrower uses the money to invest in a project with ex-post payoff R. The borrower can decide to repay the loan with interest (in which case the lender gets $(1+r_l)L$), or default. In case of default the borrower looses its collateral, and the lender gets the collateral plus the return on the project (which he can seize). Repaying is a better choice for the borrower if:

$$\begin{aligned} R-(1+r_l)L &> -C \\ R+C &> (1+r_l)L \end{aligned}$$

a higher R thus makes default less likely.

The payoff is random, equal to $R'-x$ with probability 0.5 and $R'+x$ with probability 0.5. The expected payoff is thus R' and he variance x^2. We assume that the borrower defaults when the return is bad (otherwise he would never default), that is:

$$\begin{aligned} R'-x-(1+r_l)L &< -C \\ R'-x &< (1+r_l)L-C \end{aligned}$$

In case of success the borrower gets $R'+x-(1+r_l)L$. The expected payoff of the borrower is thus:

$$\begin{aligned} E\pi^B(x) &= \frac{1}{2}(R'+x-(1+r_l)L)+\frac{1}{2}(-C) \\ &= \frac{1}{2}(R'+x-(1+r_l)L-C) \end{aligned}$$

This is positive if x exceeds a threshold value x^* which is increasing in the interest rate, the amount of the loan, the amount of collateral, and decreasing in the expected payoff:

$$\begin{aligned} 0 &= R'+x^*-(1+r_l)L-C \\ x^*(r_l,L,C,R') &= (1+r_l)L+C-R' \end{aligned}$$

A borrower accepts to enter the contract only when he has a value of x above x^*, i.e. only risky borrowers find it optimal to borrow. Intuitively, the borrower has a limited loss in case of failure (i.e. $-C$), so more risk raises his payoff in case of success without raising his cost of failure (in other words there is limited liability). Increasing the interest rate r_l raises the threshold, and thus leads to safe borrowers opting out.

The expected payoff to the lender is:

$$E\pi^L(x)=\frac{1}{2}(1+r_l)L+\frac{1}{2}(R'-x+C)-(1+r)L$$

where the last term represents the foregone revenue as the lender did not invest in the safe bond.

A higher x clearly reduces the lender's expected payoff.

We now consider that there are two types of borrowers: safe borrowers of type G with x_g and risky borrowers of type B with $x_b > x_g$. Each group represents half of the population.

Consider that $x_b > x_g > x^*$. In that case borrowers of both types borrow. The expected payoff to the lender is:

$$\begin{aligned}
E\pi^L_{B,G} &= \frac{1}{2}E\pi^B(x_g) + \frac{1}{2}E\pi^B(x_b) \\
&= \frac{1}{4}(1+r_l)L + \frac{1}{4}(R' - x_g + C) + \frac{1}{4}(1+r_l)L + \frac{1}{4}(R' - x_b + C) - (1+r)L \\
&= \frac{1}{2}(1+r_l)L + \frac{1}{2}\left(R' - \frac{x_g + x_b}{2} + C\right) - (1+r)L
\end{aligned}$$

The expected payoff thus reflects the average riskiness. This expected profit is increasing in r_l, as long as both types of borrowers remains present.

If the interest rate r_l keeps increasing, there is a threshold r_l^* at which the safe borrower is indifferent between participating or not:

$$x_g = x^*(r_l^*, L, C, R') = (1+r_l^*)L + C - R'$$

If r_l increases further, the safe borrowers opt out. The lender is then left with only risky borrowers, and the expected payoff is:

$$\begin{aligned}
E\pi^L_B &= E\pi^B(x_b) \\
&= \frac{1}{2}(1+r_l)L + \frac{1}{2}(R' - x_b + C) - (1+r)L
\end{aligned}$$

We can thus see that there is a drop at r_l^*:

$$\begin{aligned}
E\pi^L_{B,G} - E\pi^L_B &= \frac{1}{2}(1+r_l^*)L + \frac{1}{2}\left(R' - \frac{x_g + x_b}{2} + C\right) - (1+r)L \\
&\quad -\frac{1}{2}(1+r_l^*)L - \frac{1}{2}(R' - x_b + C) + (1+r)L \\
&= \frac{1}{2}\left(R' - \frac{x_g + x_b}{2} + C\right) - \frac{1}{2}(R' - x_b + C) \\
&= -\frac{x_g + x_b}{4} + \frac{1}{2}x_b \\
&= \frac{-x_g - x_b}{4} + \frac{2}{4}x_b \\
&= \frac{x_b - x_g}{4} \\
&> 0
\end{aligned}$$

Therefore the lender will not increase the interest rate beyond r_l^*. If at that rate credit demand exceeds credit supply, the lender will instead opt to reduce the amount of the loan L. This reduces $x^*(r_l, L, C, R')$ and thus can allow for a bit of higher r_l.

2 Moral hazard

Consider a model where the borrower can invest in a safe project A which pays R^a if successful (with probability p^a) and zero otherwise, and a risky project B which pays R^b if successful (with probability p^b) and zero otherwise. We assume that project B pays more when successful, $R^a < R^b$, but is less likely to succeed, $p^a > p^b$, and has a lower expected payoff, $p^a R^a > p^b R^b$ (project B is akin to gambling with a small probability of a very high payoff).

The borrower takes a loan L with interest r_l. If he makes enough money, he repays the loan with interest. Otherwise he defaults, in which case the lender takes the payoff of the project as well as collateral C. If the project fails, the borrower has nothing to repay and thus has no choice but to default. If the project is successful the borrower repays the loan.

The expected payoffs for the lender when the borrower invests in projects A and B are:

$$\begin{aligned} E\pi^A_{Lend} &= p^a (1+r_l) L + (1-p^a) C \\ E\pi^B_{Lend} &= p^b (1+r_l) L + \left(1-p^b\right) C \end{aligned}$$

The lender clearly prefers project A:

$$\begin{aligned} E\pi^A_{Lend} - E\pi^B_{Lend} &= p^a (1+r_l) L - p^b (1+r_l) L + (1-p^a) C - \left(1-p^b\right) C \\ &= \left(p^a - p^b\right)(1+r_l) L - \left(p^a - p^b\right) C \\ &= \left(p^a - p^b\right)((1+r_l) L - C) \end{aligned}$$

The expected payoffs for the borrower of investing in projects A and B are:

$$\begin{aligned} E\pi^A_{Bor} &= p^a (R^a - (1+r_l) L) - (1-p^a) C \\ E\pi^B_{Bor} &= p^b \left(R^b - (1+r_l) L\right) - \left(1-p^b\right) C \end{aligned}$$

Investing in project A is more appealing if:

$$\begin{aligned} E\pi^A_{Bor} &> E\pi^B_{Bor} \\ p^a (R^a - (1+r_l) L) - (1-p^a) C &> p^b \left(R^b - (1+r_l) L\right) - \left(1-p^b\right) C \\ p^a R^a &> p^b R^b + \left(p^a - p^b\right)(1+r_l) L - \left(p^a - p^b\right) C \\ \frac{p^a R^a - p^b R^b}{p^a - p^b} &> (1+r_l) L - C \end{aligned}$$

The right hand side is increasing in the interest rate on the loan. Therefore, increasing the interest rate can induce borrowers to choose the risky (and inefficient) project. This is the case if r_l exceeds a thresholds r_l^*:

$$\frac{p^a R^a - p^b R^b}{p^a - p^b} = (1+r_l^*) L - C$$

When the borrower is indifferent between the two projects, the lender clearly prefers project A:

$$E\pi^A_{Lend} - E\pi^B_{Lend} = \left(p^a - p^b\right)((1+r_l^*) L - C)$$

$$
\begin{aligned}
&= p^a R^a - p^b R^b \\
&> 0
\end{aligned}
$$

Therefore, the lender will not increase the interest rate above r_l^*.

3 The Diamond-Dybvig model of bank runs

3.1 Setting and autarky allocation

A 3 periods model: at time 0 agents get one unit of good which can be invested. If the investment is held until period 2 it yields $R > 1$. If it is liquidated at period 1 it yields 1.

At time 1 a share t of agents learn that they want to consume right away (liquidity shock), with utility $u(c_1)$. The other agents do not need to consume until tomorrow , with utility $u(c_2)$.

Consider an autarky allocation for each agent. If the agent is impatient, she consumes 1, if not she consumes R. The expected utility is:

$$
U^{\text{autarky}} = t\frac{1}{1-\sigma}(1)^{1-\sigma} + (1-t)\frac{1}{1-\sigma}(R)^{1-\sigma}
$$

3.2 Insurance

Consider now that a planner can pool agents' resources. She maximizes the average utility across patient and impatient agents:

$$
U = t\frac{1}{1-\sigma}(c_1)^{1-\sigma} + (1-t)\frac{1}{1-\sigma}(c_2)^{1-\sigma}
$$

subject to the resource constraint:

$$
(1-t)\,c_2 = R(1-tc_1)
$$

The first order condition is:

$$
\begin{aligned}
0 &= \frac{\partial U}{\partial c_1} \\
0 &= \frac{1}{\partial c_1}\left[t\frac{1}{1-\sigma}(c_1)^{1-\sigma} + (1-t)\frac{1}{1-\sigma}\left(\frac{R(1-tc_1)}{1-t}\right)^{1-\sigma}\right] \\
0 &= t(c_1)^{-\sigma} - (1-t)\left(\frac{R(1-tc_1)}{1-t}\right)^{-\sigma}\frac{Rt}{1-t} \\
0 &= (c_1)^{-\sigma} - R(c_2)^{-\sigma} \\
\frac{c_2}{c_1} &= (R)^{\frac{1}{\sigma}} > 1
\end{aligned}
$$

The budget constraint then implies:

$$
(1-t)\,c_2 = R(1-tc_1)
$$

$$
\begin{aligned}
(1-t)(R)^{\frac{1}{\sigma}}c_1 &= R(1-tc_1) \\
(1-t)(R)^{\frac{1-\sigma}{\sigma}}c_1 &= 1-tc_1 \\
c_1 &= \frac{1}{1-(1-t)\left[1-(R)^{\frac{1-\sigma}{\sigma}}\right]}
\end{aligned}
$$

which implies:

$$
\begin{aligned}
c_2 &= R\frac{(R)^{\frac{1-\sigma}{\sigma}}}{1-(1-t)\left[1-(R)^{\frac{1-\sigma}{\sigma}}\right]} \\
c_2 &= R\frac{(R)^{\frac{1-\sigma}{\sigma}}}{t+(1-t)(R)^{\frac{1-\sigma}{\sigma}}} \\
c_2 &= R\frac{1}{t(R)^{\frac{\sigma-1}{\sigma}}+(1-t)} \\
c_2 &= R\frac{1}{1+t\left[(R)^{\frac{\sigma-1}{\sigma}}-1\right]}
\end{aligned}
$$

If $\sigma > 1$ we get $(R)^{\frac{1-\sigma}{\sigma}} < 1$ and $(R)^{\frac{\sigma-1}{\sigma}} > 1$, hence consumption is smoother than under autarky:

$$1 < c_1 < c_2 < R$$

3.3 Bank deposits

Bank provides insurance. Agents deposit their endowment in period 0. The bank invests all the deposits. At period 1 any agent can go to the bank and withdraw $c_1 > 1$. The bank pays for it by liquidating some investment. Agents waiting until period 2 get $c_2 > c_1$. Picking consumption levels that match the insurance outcome is feasible.

If more than s agents ask for their money back in period 1, the bank cannot pay c_2 in period 2 as it liquidates too much investment. Self-fulfilling liquidity runs: if patient agents expects that many agents withdraw in period 1, she runs to the bank to get something out.

4 The Bernanke Gertler financial accelerator model

This section outlines the solution of the model of Bernanke and Gertler (1989) “Agency costs, net worth, and business fluctuations”, American Economic Review 79, pp. 14-31.

4.1 The key idea

Before considering the general equilibrium model, we illustrate the key aspect.

4.1.1 Agents

The economy consists of a lender and borrowers (entrepreneurs). Entrepreneurs invest in a risky project. While entrepreneurs can observe the return on the project, the lender can only do so at a cost.

Entrepreneur are indexed by a measure of inability $\omega \in [0,1]$. An entrepreneur with ability ω has a cost of completing a project that is $x(\omega)$, with $x' > 0$. A higher ω thus denotes an entrepreneur that is not efficient. ω is uniformly distributed between 0 and 1. Each entrepreneur has a wealth of S. We assume that $S < x(0)$, so even the best entrepreneur cannot fully fund his project.

A project is either successful, producing κ_2, or unsuccessful producing $\kappa_1 < \kappa_2$. Success occurs with probability π_2. The expected output of a project is thus $\kappa = \pi_2\kappa_2 + (1-\pi_2)\kappa_1$.

Entrepreneurs decide whether or not to invest in the project, depending on their ω. Entrepreneurs who do not invest in a project instead purchase a bond with a return r. The lender can also purchase the bond, so r is the outside option for lenders.

4.1.2 The frictionless allocation

Consider that the lender can observe the outcome of the project. The interest rate repaid in case of success is r_2, and the gross interest rate repaid in case of failure is r_1. The lender requires that the expected return is equal to the return on the bond:

$$(1-\pi_2)\, r_1 + \pi_2 r_2 = r$$

If the entrepreneur undertakes the project, he borrows $x(\omega) - S$ and gets an expected return:

$$\begin{aligned} & (1-\pi_2)\,\kappa_1 + \pi_2\kappa_2 - [(1-\pi_2)\, r_1 + \pi_2 r_2]\,(x(\omega) - S) \\ = \quad & \kappa - r\,(x(\omega) - S) \end{aligned}$$

Alternatively, the entrepreneur could have invested his wealth S in the bond and gotten $(1+r)\,S$. The entrepreneur thus prefers to invest if:

$$\begin{aligned} \kappa - r\,(x(\omega) - S) &> rS \\ \kappa &> rx(\omega) \end{aligned}$$

Therefore, all entrepreneurs with $\omega < \omega^*$, where $\kappa = rx(\omega^*)$ undertake the project.

4.1.3 Introducing frictions

Now consider that each entrepreneur is the only one who knows whether the project has been successful or not. The lender can learn the outcome by paying a cost $\bar{c}$.

The entrepreneur announces the return on the project and then a repayment takes place. The entrepreneur has an incentive to always announce κ_1, as repayment is lower under failure than under success. The lender will thus double-check the announcement.

Specifically, if the entrepreneur announces κ_2, the lender believes him (there is no incentive to announce success when instead failure occurred) and the lender receives an amount $\kappa_2 - P_2$, so the entrepreneur keeps P_2. If the entrepreneur truthfully announces κ_1, the lenders audits him with probability p to learn the true return, in which case the borrowers keeps P_1^a (and the lender gets $\kappa_1 - P_1^a - \bar{c}$), or does not audit in which case the borrowers keeps P_1 (and the lender gets $\kappa_1 - P_1$). If the entrepreneur lies and announces κ_1 and does not get audited, he pays $\kappa_1 - P_1$ to the lender and gets to keep $\kappa_2 - \kappa_1 + P_1$. If he gets audited, the lie is discovered and the lender takes everything.

The contract thus requires setting the probability of audit and the payoffs. The goal is to maximize the entrepreneur's expected payoff:

$$(1-\pi_2)\left(pP_1^a + (1-p)\,P_1\right) + \pi_2 P_2$$

subject to the constraints that P_1^a and P_1 are not negative, that the lender gets an expected return r on the loan:

$$(1-\pi_2)\left(p\left(\kappa_1 - P_1^a - \bar{c}\right) + (1-p)\left(\kappa_1 - P_1\right)\right) + \pi_2\left(\kappa_2 - P_2\right) = r\left(x\left(\omega\right) - S\right)$$

Finally, the borrower must be induced to tell the truth. If the project is truly successful and he tells the truth, he gets P_2. If he lies and gets audited he gets nothing. If he lies and does not get audited he keeps $\kappa_2 - \kappa_1 + P_1$. Telling the truth is thus optimal if $P_2 > (1-p)\left(\kappa_2 - \kappa_1 + P_1\right)$.

The Lagrangian is thus:

$$\begin{aligned}
\mathcal{L} \;=\;& (1-\pi_2)\left(pP_1^a + (1-p)\,P_1\right) + \pi_2 P_2 \\
& +\mu_1\left[(1-\pi_2)\left(p\left(\kappa_1 - P_1^a - \bar{c}\right) + (1-p)\left(\kappa_1 - P_1\right)\right) + \pi_2\left(\kappa_2 - P_2\right) - r\left(x\left(\omega\right) - S\right)\right] \\
& +\mu_2\left[P_2 - (1-p)\left(\kappa_2 - \kappa_1 + P_1\right)\right] \\
& +\mu_3 P_1^a + \mu_4 P_1
\end{aligned}$$

The optimality condition with respect to the audit probability is:

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial p} = (1-\pi_2)\left(P_1^a - P_1\right) + \mu_1(1-\pi_2)\left(-P_1^a - \bar{c} + P_1\right) + \mu_2\left(\kappa_2 - \kappa_1 + P_1\right) \\
0 &= (1-\pi_2)\left[\left(P_1^a - P_1\right) + \mu_1\left(P_1 - P_1^a - \bar{c}\right)\right] + \mu_2\left(\kappa_2 - \kappa_1 + P_1\right)
\end{aligned}$$

and the conditions with respect to the payoffs are:

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial P_1^a} = (1-\pi_2)\,p\,(1-\mu_1) + \mu_3 \\
0 &= \frac{\partial \mathcal{L}}{\partial P_1} = (1-\pi_2)(1-p)(1-\mu_1) - \mu_2(1-p) + \mu_4 \\
0 &= \frac{\partial \mathcal{L}}{\partial P_2} = \pi_2(1-\mu_1) + \mu_2
\end{aligned}$$

As $\mu_3 \geq 0$ the optimality condition with respect to P_1^a implies that $\mu_1 \geq 1$ and thus $\mu_1 > 0$ so the lender gets exactly an expected return r (there is not point in giving him more). Therefore we have:

$$r\left(x\left(\omega\right) - S\right) = (1-\pi_2)\left(p\left(\kappa_1 - P_1^a - \bar{c}\right) + (1-p)\left(\kappa_1 - P_1\right)\right) + \pi_2\left(\kappa_2 - P_2\right)$$

$$\begin{aligned}
r\left(x\left(\omega\right)-S\right) &= -\left[\left(1-\pi_2\right)\left(pP_1^a+\left(1-p\right)P_1\right)+\pi_2P_2\right] \\
&\quad +\left(1-\pi_2\right)\left(p\left(\kappa_1-\bar{c}\right)+\left(1-p\right)\kappa_1\right)+\pi_2\kappa_2 \\
r\left(x\left(\omega\right)-S\right) &= -\left[\left(1-\pi_2\right)\left(pP_1^a+\left(1-p\right)P_1\right)+\pi_2P_2\right]+\left(1-\pi_2\right)\left(\kappa_1-p\bar{c}\right)+\pi_2\kappa_2 \\
r\left(x\left(\omega\right)-S\right) &= -\left[\left(1-\pi_2\right)\left(pP_1^a+\left(1-p\right)P_1\right)+\pi_2P_2\right]+\kappa-\left(1-\pi_2\right)p\bar{c}
\end{aligned}$$

This implies that the expected payoff to the entrepreneur is:

$$\left(1-\pi_2\right)\left(pP_1^a+\left(1-p\right)P_1\right)+\pi_2P_2=\kappa-r\left(x\left(\omega\right)-S\right)-\left(1-\pi_2\right)p\bar{c}$$

Thus maximizing this payoff is equivalent to minimizing the expected auditing cost $\left(1-\pi_2\right)p\bar{c}$, i.e minimizing p.

Note that no auditing is needed for entrepreneurs who borrow so little that even in the case of failure they are able to repay the loan with an interest rate r:

$$\kappa_1>r\left(x\left(\omega\right)-S\right)$$

This can hold if the entrepreneur is very efficient, or as enough own resources S.

If auditing is needed, it is costly and thus it is optimal to minimize its probability p. The probability is set such that the borrower is just induced not to lie in case of success. To make truth telling more appealing in case of success, it makes sense to give the entrepreneur as much as possible then (P_2 is high), and pay for this by giving it little in case of failure ($P_1^a=P_1=0$). The entrepreneur is willing to tell the truth when:

$$P_2=\left(1-p\right)\left(\kappa_2-\kappa_1\right)$$

We substitute this in the condition that the lender gets an expected return equal to r:

$$\begin{aligned}
r\left(x\left(\omega\right)-S\right) &= \left(1-\pi_2\right)\left(p\left(\kappa_1-P_1^a-\bar{c}\right)+\left(1-p\right)\left(\kappa_1-P_1\right)\right)+\pi_2\left(\kappa_2-P_2\right) \\
r\left(x\left(\omega\right)-S\right) &= \left(1-\pi_2\right)\left(\kappa_1-p\bar{c}\right)+\pi_2\left(\kappa_2-P_2\right) \\
r\left(x\left(\omega\right)-S\right) &= \left(1-\pi_2\right)\left(\kappa_1-p\bar{c}\right)+\pi_2\left(\kappa_2-\left(1-p\right)\left(\kappa_2-\kappa_1\right)\right) \\
r\left(x\left(\omega\right)-S\right) &= \left(1-\pi_2\right)\kappa_1+\pi_2\left(\kappa_2-\left(\kappa_2-\kappa_1\right)\right)+\left[-\left(1-\pi_2\right)\bar{c}+\pi_2\left(\kappa_2-\kappa_1\right)\right]p \\
r\left(x\left(\omega\right)-S\right) &= \kappa_1+\left[\pi_2\left(\kappa_2-\kappa_1\right)-\left(1-\pi_2\right)\bar{c}\right]p \\
p &= \frac{r\left(x\left(\omega\right)-S\right)-\kappa_1}{\pi_2\left(\kappa_2-\kappa_1\right)-\left(1-\pi_2\right)\bar{c}}
\end{aligned}$$

The probability of audit is higher the higher the borrowed amount $x\left(\omega\right)-S$.

4.2 A general equilibrium model

We now show how the setting affects the dynamics of a general equilibrium model.

4.2.1 Agents

Consider an overlapping generation model, where agents live for two periods. They supply labor when young to firms producing the consumption good. The technology of the firms producing the consumption good is:

$$y_t = \theta_t f(k_t)$$

where θ_t is a stochastic productivity. The total labor supply is normalized to one. There are two types of agents within each generation. $1-\eta$ agents are "lenders". Lenders born at time t maximize the following utility of consumption when young and old:

$$U\left(c_t^y\right) + \beta E_t c_{t+1}^o$$

where c_t^y is consumption when young, c_{t+1}^o is consumption when old, β is the time discount. Notice that agents are risk-neutral with respect to consumption when old, which simplifies the analysis. Lenders can access a storage technology with fixed return r. The budget constraints are then:

$$S_t = w_t L - c_t^y \qquad ; \qquad c_{t+1}^o = r S_t$$

where S denotes saving, w is the waged earned when young. The optimization leads to a demand for consumption when young which is a function of the interest rate: $c_t^y = c_y^*(r)$.

The other η agents are entrepreneurs. They can undertake a project that produces capital. As above, entrepreneur differ by a measure of inability ω. An entrepreneur with ability ω has a cost of completing a project that is $x(\omega)$, with $x' > 0$. A higher ω thus denotes an entrepreneur that is not efficient. ω is uniformly distributed between 0 and 1. A project is either successful, producing κ_2 units of capital, or unsuccessful, producing $\kappa_1 < \kappa_2$. Success occurs with probability π_2. The expected output of the project is thus $\kappa = \pi_2\kappa_2 + (1-\pi_2)\kappa_1$. The price of capital, in terms of the consumption good, is denoted by q.

Entrepreneurs consume only when old (abstracting from intertemporal choice simplifies the problem). At the end of their youth they have savings $S_t = w_t L$ that they can store, lend (at interest rate r) or invest in their own project.

4.2.2 The frictionless equilibrium

In the frictionless case the outcome of projects (success or failure) is observed by everybody. Consider entrepreneur ω. She can use resources $x(\omega)$ to undertake her project. The expected payoff of the project undertaken at time t is:

$$\kappa E_t q_{t+1}$$

Instead of investing in the project, the entrepreneur could store her resources and get a certain return $rx(\omega)$. Entrepreneur ω^* is indifferent between the two options:

$$rx(\omega^*) = \kappa E_t q_{t+1} \tag{1}$$

As x is an increasing function of ω, all the entrepreneurs with $\omega > \omega^*$ prefer to store their savings instead of undertaking their project (the difference with the baseline setting above is that we now have the price of capital q). As ω is uniformly distributed, the number of undertaken projects is $\bar{\omega}\eta$.

The aggregate capital stock at $t+1$, k_{t+1}, is equal to the number of projects undertaken, $i_t = \omega^* \eta$, times the expected payoff on each:

$$\begin{aligned} k_{t+1} &= \left[\pi \kappa_2 + (1-\pi)\,\kappa_1\right] i_t \\ &= \kappa \omega^* \eta \end{aligned}$$

Note that this is not uncertain from the point of view of period t. We can then write:

$$\omega^* = \frac{k_{t+1}}{\kappa \eta}$$

(1) then becomes:

$$\begin{aligned} rx\left(\omega^*\right) &= \kappa E_t q_{t+1} \\ rx\left(\frac{k_{t+1}}{\kappa\eta}\right) &= \kappa E_t q_{t+1} \\ E_t q_{t+1} &= \frac{r}{\kappa} x \left(\frac{k_{t+1}}{\kappa\eta}\right) \qquad (2) \end{aligned}$$

As $x' > 0$ this gives a positive capital supply relation between $E_t q_{t+1}$ and k_{t+1}. The higher the future price of capital, the more entrepreneurs for whom the project is worth undertaking, hence the more future capital.

The capital demand comes from the production of consumption goods, where the marginal product is equal to the price, both in actual and expected terms:

$$E_t q_{t+1} = f'\left(k_{t+1}\right) E_t \theta_{t+1} \qquad (3)$$

With decreasing returns to scale this gives a downward sloping capital demand. Assuming that θ_{t+1} is not persistent (i.e. it is identically independently distributed across periods), the expected price of capital and capital are constant.

4.2.3 Introducing frictions

Individual entrepreneur problem Now consider that each entrepreneur is the only one who knows whether the project has been successful or not. Other agents can learn the outcome by paying a cost $\bar{c}$. There is then a tension between an entrepreneur who borrows and the lender. If the entrepreneur does not need to borrow, i.e. $S > x(\omega)$ then there is no problem as he is self-financed.

The steps parallel the ones in the presentation of the baseline mechanism above If the entrepreneur borrows, he needs to provide the lender with an expected return:

$$r\left(x\left(\omega\right) - S\right)$$

The problem is that the entrepreneur has an incentive to always announce that the project failed, to get away with a low repayment (there is not point in announcing success if the project truly failed).

The optimal contract takes the following form. The entrepreneur announces whether the project was successful or not. If he claims success, the lender does not double check. If he claims failure, the lender checks with probability p, in which case he incurs the cost $\bar{c}$ in terms of unit of capital (i.e. $\hat{q}\bar{c}$ in terms of consumption good). The entrepreneur maximizes expected consumption:

$$(1-\pi)\left(pc^a + (1-p)\,c_1\right) + \pi c_2$$

where c_2 is consumption in case of success, c_1 is consumption in case of failure and no audit, c_1^a is consumption in case of failure and audit. In case of success the lender gets $\hat{q}\kappa_2 - c_2$, where $\hat{q}$ is the expected price of capital. In case of failure and no audit he gets $\hat{q}\kappa_1 - c_1$. In case of failure and audit he gets $\hat{q}\kappa_1 - c_1^a - \bar{c}$. The lender must get an expected rate of return r:

$$\begin{aligned} & (1-\pi)\left(\hat{q}\kappa_1 - p\left(c_1^a + \hat{q}\bar{c}\right) - (1-p)\,c_1\right) + \pi\left[\hat{q}\kappa_2 - c_2\right] \\ = \; & r\,(x-S) \end{aligned}$$

In addition, an entrepreneur never finds it optimal to claim failure when he in fact was successful (if caught lying, he looses everything):

$$c_2 > (1-p)\left[\hat{q}\left(\kappa_2 - \kappa_1\right) + c_1\right]$$

Finally consumption cannot be negative.

The optimal outcome is such that under failure the lender gets everything: $c_1^a = c_1 = 0$. This implies:

$$c_2 = (1-p)\,\hat{q}\left(\kappa_2 - \kappa_1\right)$$

The constraint on the lender's expected return is then:

$$\begin{aligned} r\,(x-S) &= (1-\pi)\left(\hat{q}\kappa_1 - p\hat{q}\bar{c}\right) + \pi\left[\hat{q}\kappa_2 - c_2\right] \\ r\,(x-S) &= (1-\pi)\left(\hat{q}\kappa_1 - p\hat{q}\bar{c}\right) + \pi\left[\hat{q}\kappa_2 - (1-p)\,\hat{q}\left(\kappa_2-\kappa_1\right)\right] \\ r\,(x-S) &= \left[-(1-\pi)\,\hat{q}\bar{c} + \pi\hat{q}\left(\kappa_2-\kappa_1\right)\right]p + \hat{q}\kappa_1 \\ p &= \frac{r\,(x-S) - \hat{q}\kappa_1}{\pi\hat{q}\left(\kappa_2-\kappa_1\right) - (1-\pi)\,\hat{q}\bar{c}} \end{aligned}$$

which is of the same form of what we had above.

4.2.4 Aggregating

Consider two efficiency threshold, $\underline{\omega}$ and $\overline{\omega}$ with $\underline{\omega} < \overline{\omega}$. Entrepreneur with $\omega < \underline{\omega}$ have a positive expected consumption even if they are audited with certainty in case of failure:

$$\hat{q}\kappa - r\left(\underline{\omega}\right) - (1-\pi)\,\hat{q}\bar{c} = 0$$

Entrepreneurs with $\omega > \overline{\omega}$ have a positive expected consumption only if they are never audited in case of failure:

$$\hat{q}\kappa - r\left(\overline{\omega}\right) = 0$$

The "bad" entrepreneurs ($\omega > \overline{\omega}$) don't invest in their project. The "good" ones ($\omega < \underline{\omega}$) invest and borrow, thus facing the cost of audit. The "medium" ones ($\underline{\omega} < \omega < \overline{\omega}$) invest but without borrowing from lenders. That is, they self finance as a group randomly drawing some members to lend to the other.

The capital accumulation can be shown to be:

$$k_{t+1} = \eta \left[\kappa\bar{\omega} - (1-\pi)\int_{0}^{\underline{\omega}} p\left(\omega\right) d\omega + \int_{\underline{\omega}}^{\overline{\omega}} \kappa\left(1 - g\left(\omega\right)\right) d\omega\right]$$

where the probability of audit for an entrepreneur ω is:

$$p\left(\omega\right) = \frac{r\left(x\left(\omega\right) - S\right) - \hat{q}\kappa_1}{\pi\hat{q}\left(\kappa_2 - \kappa_1\right) - (1-\pi)\,\hat{q}\bar{c}}$$

$g\left(\omega\right)$ is the fraction of "medium" entrepreneurs of ability ω who get to invest:

$$g\left(\omega\right) = \frac{rS}{rx\left(\omega\right) - \hat{q}\kappa_1}$$

Working out the dynamics of the model is complex, but the intuition is the one presented through the diagram in class.