Intermediate Microeconomics Assignment 1

Due on October 24, 2021

Name Student ID

1. UMP: Consider the Cobb-Douglas utility function $U(x,y) = x^a y^b$ where $a > 0$ and $b > 0$. The price vector is $\boldsymbol{p} = (p_x, p_y)$, and the income is I.

   (1) Derive the Marshallian demand $x^*(p_x, p_y, I)$, $y^*(p_x, p_y, I)$, and the indirect utility (value function) $V(p_x, p_y, I)$.

   (2) If price p_x or p_y increases, is the consumer better-off or worse-off? If income increases, whether the consumer is better-off or worse-off.

2. EMP: Consider $U(x,y) = x^a y^{1-a}$.

   (1) Derive the Hicksian demand for $U(x,y) = x^a y^b$, and the expenditure function $E(p_x, p_y, u)$.

   (2) If price p_x or p_y increases, will the total expenditure increase or decrease?

   (3) Verify $x^*(p_x, p_y, I) = h_x(p_x, p_y, u)$ when $u = V$ where V is solved by 1 (1).

   (4) Verify $h_x(p_x, p_y, u) = x^*(p_x, p_y, I)$ when $I = E$ where E is solved by 2 (1).

3. Consider the quasi-linear utility $U(x,y) = u(x) + y$, where $u'(\cdot) > 0$, $u''(\cdot) < 0$ and $p_y = 1$. Assume that when the price of x increases from p_1 to p_2, the Mashallian demand changes from $x_1^*(p_1, p_y, I)$ to $x_2^*(p_2, p_y, I)$, where both x_1^* and x_2^* are interior solutions. Show that at p_2, the Hicksian demand $h_x(p_2, p_y, u_1)$ where u_1 is the original utility level before price increase must be an interior solution.

1. (1) The Lagrangian:

$$\begin{aligned}
\mathcal{L} &= x^a y^b + \lambda(I - p_x x - p_y y) \\
\frac{\partial \mathcal{L}}{\partial x} &= a x^{a-1} y^b - \lambda p_x = 0 \\
\frac{\partial \mathcal{L}}{\partial y} &= b x^a y^{b-1} - \lambda p_y = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= I - p_x x - p_y y = 0
\end{aligned}$$

The first two equations imply

$$\frac{a}{b}\frac{y}{x} = \frac{p_x}{p_y} \Rightarrow p_y y = \frac{b}{a} p_x x.$$

Combining the third equation:

$$I = p_x x + p_y y = p_x x + \frac{b}{a} p_x x = \frac{a+b}{a} p_x x \Rightarrow x^* = \frac{a}{a+b}\frac{I}{p_x}. \tag{1}$$

Similarly, $y^* = \frac{b}{a+b}\frac{I}{p_y}$.

The value function:

$$V = U(x^*, y^*) = \left(\frac{a}{a+b}\frac{I}{p_x}\right)^a \left(\frac{b}{a+b}\frac{I}{p_y}\right)^b = \left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b I^{a+b} p_x^{-a} p_y^{-b} \tag{2}$$

(2) Using equation (2),

$$\begin{aligned}
\frac{\partial V}{\partial p_x} &= -a \left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b I^{a+b} p_x^{-a-1} p_y^{-b} < 0 \\
\frac{\partial V}{\partial p_y} &= -b \left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b I^{a+b} p_x^{-a} p_y^{-b-1} < 0 \\
\frac{\partial V}{\partial I} &= (a+b) \left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b I^{a+b-1} p_x^{-a} p_y^{-b} > 0
\end{aligned}$$

That is, the consumer is worse-off due to price increase; and is better-off due to income increase.

2. (1) The Lagrangian:

$$\begin{aligned}
\mathcal{L} &= p_x x + p_y y + \lambda(u - x^a y^b) \\
\frac{\partial \mathcal{L}}{\partial x} &= p_x - \lambda a x^{a-1} y^b = 0 \\
\frac{\partial \mathcal{L}}{\partial y} &= p_y - \lambda b x^a y^{b-1} = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= u - x^a y^b = 0
\end{aligned}$$

The first two equations imply

$$\frac{p_x}{p_y} = \frac{a}{b}\frac{y}{x} \Rightarrow y = \frac{b}{a}\frac{p_x}{p_y} x$$

Combining the third equation:

$$u = x^a y^b = x^a \left(\frac{b}{a}\frac{p_x}{p_y}x\right)^b \Rightarrow h_x = \left(\frac{a}{b}\right)^{\frac{b}{a+b}} \left(\frac{p_y}{p_x}\right)^{\frac{b}{a+b}} u^{\frac{1}{a+b}} \tag{3}$$

Similarly, $h_y = \left(\frac{b}{a}\right)^{\frac{a}{a+b}} \left(\frac{p_x}{p_y}\right)^{\frac{a}{a+b}} u^{\frac{1}{a+b}}$. The expenditure function is given by

$$\begin{aligned} E(p_x, p_y, u) &= p_x h_x + p_y h_y = p_x \left(\frac{a}{b}\right)^{\frac{b}{a+b}} \left(\frac{p_y}{p_x}\right)^{\frac{b}{a+b}} u^{\frac{1}{a+b}} + p_y \left(\frac{b}{a}\right)^{\frac{a}{a+b}} \left(\frac{p_x}{p_y}\right)^{\frac{a}{a+b}} u^{\frac{1}{a+b}} \\ &= \left[\left(\frac{a}{b}\right)^{\frac{b}{a+b}} + \left(\frac{b}{a}\right)^{\frac{a}{a+b}}\right] p_x^{\frac{a}{a+b}} p_y^{\frac{b}{a+b}} u^{\frac{1}{a+b}} \end{aligned} \tag{4}$$

(2) You can directly use the Shepherd's Lemma to see $\frac{\partial E}{\partial p_x} = h_x > 0$. Alternatively, you can verify this:

$$\frac{\partial E}{\partial p_x} = \frac{a}{a+b}\left[\left(\frac{a}{b}\right)^{\frac{b}{a+b}} + \left(\frac{b}{a}\right)^{\frac{a}{a+b}}\right] p_x^{\frac{-b}{a+b}} p_y^{\frac{b}{a+b}} u^{\frac{1}{a+b}} > 0.$$

Similarly, $\frac{\partial E}{\partial p_y} = h_y > 0$. That is, total expenditure will increase if price increases.

(3) Verify $h_x = x^*$ evaluated at $u = V$. Plugging (2) into h_x:

$$\begin{aligned} h_x =& a^{\frac{b}{a+b}} b^{-\frac{b}{a+b}} p_x^{-\frac{b}{a+b}} p_y^{\frac{b}{a+b}} u^{\frac{1}{a+b}} \\ =& a^{\frac{b}{a+b}} b^{-\frac{b}{a+b}} p_x^{-\frac{b}{a+b}} p_y^{\frac{b}{a+b}} \left(\frac{a}{a+b}\right)^{\frac{a}{a+b}} \left(\frac{b}{a+b}\right)^{\frac{b}{a+b}} \frac{b}{a+b} I p_x^{-\frac{a}{a+b}} p_y^{-\frac{b}{a+b}} \\ =& a^{\frac{b}{a+b}+\frac{a}{a+b}} b^{-\frac{b}{a+b}+\frac{b}{a+b}} \left(\frac{1}{a+b}\right)^{\frac{a}{a+b}+\frac{b}{a+b}} I p_x^{-\frac{b}{a+b}-\frac{a}{a+b}} p_y^{\frac{b}{a+b}-\frac{b}{a+b}} \\ =& \frac{a}{a+b} I p_x^{-1} = x^*. \end{aligned}$$

(4) By plugging $I = E$ where E is given by (4) into x^*:

$$\begin{aligned} x^* = \frac{a}{a+b}\frac{I}{p_x} =& \frac{a}{a+b} p_x^{-1} E = \frac{a}{a+b} p_x^{-1} \left[\left(\frac{a}{b}\right)^{\frac{b}{a+b}} + \left(\frac{b}{a}\right)^{\frac{a}{a+b}}\right] p_x^{\frac{a}{a+b}} p_y^{\frac{b}{a+b}} u^{\frac{1}{a+b}} \\ =& \frac{a}{a+b}\left(a^{\frac{b}{a+b}} b^{-\frac{b}{a+b}} + b^{\frac{a}{a+b}} a^{-\frac{a}{a+b}}\right) p_x^{-\frac{b}{a+b}} p_y^{\frac{b}{a+b}} u^{\frac{1}{a+b}} \\ =& \left[\frac{a}{a+b} a^{\frac{b}{a+b}} b^{-\frac{b}{a+b}} + \frac{a^{\frac{b}{a+b}} b^{\frac{a}{a+b}}}{a+b}\right] p_x^{-\frac{b}{a+b}} p_y^{\frac{b}{a+b}} u^{\frac{1}{a+b}} \\ =& \frac{a^{\frac{b}{a+b}}}{a+b}\left[a b^{-\frac{b}{a+b}} + b^{\frac{a}{a+b}}\right] p_x^{-\frac{b}{a+b}} p_y^{\frac{b}{a+b}} u^{\frac{1}{a+b}} \\ =& \frac{a^{\frac{b}{a+b}}}{a+b} b^{-\frac{b}{a+b}} \left(a + b^{\frac{a}{a+b}} b^{\frac{b}{a+b}}\right) p_x^{-\frac{b}{a+b}} p_y^{\frac{b}{a+b}} u^{\frac{1}{a+b}} \\ =& \frac{\left(\frac{a}{b}\right)^{\frac{b}{a+b}}}{a+b} (a+b) \left(\frac{p_y}{p_x}\right)^{\frac{b}{a+b}} u^{\frac{1}{a+b}} = h_x. \end{aligned}$$

3. The UMP evaluated at p_1 and p_2 are:

$$\begin{aligned} &\max_{x,y} u(x) + y && \max_{x,y} u(x) + y \\ &s.t.\ p_1 x + y = I && s.t.\ p_2 x + y = I \\ &\Rightarrow u'(x_1^*) = p_1 && \Rightarrow u'(x_2^*) = p_2 \end{aligned}$$

Because $p_1 < p_2$ and $u''(\cdot) < 0$, then $u'(x_1^*) < u'(x_2^*) \Rightarrow x_1^* > x_2^*$.

The EMP after price increase is

$$\min_{x,y} p_2 x + y$$
$$s.t.\ u(x) + y = u_1$$

Plug $y = u_1 - u(x)$ into the objective, and the first-order derivative with respect to x is $p_2 - u'(x)$. If we want to show that $h_x(p_2, p_y, u_1)$ is an interior solution, it is equivalent to show that for h_x such that $p_2 - u'(h_x) = 0$, we have $h_y = u_1 - u(h_x) > 0$.

We will show that it is true indeed; otherwise, suppose that the interior solution is invalid, i.e., h_x solved by the first-order condition $p_2 - u'(h_x) = 0$ implies $h_y = u_1 - u(h_x) < 0 \Rightarrow u_1 < u(h_x)$. Combining the fact that x_2^* is an interior solution and $p_2 = u'(x_2^*)$ is true, we have $u'(h_x) = p_2 = u'(x_2^*) \Leftrightarrow h_x = x_2^*$. We have shown that $x_2^* < x_1^*$, and hence $h_x < x_1^*$. Along the original indifference curve $y = u_1 - u(x)$, because x_1^* is interior then $y^* = u_1 - u(x_1^*) > 0 \Rightarrow u_1 > u(x_1^*)$; along the same indifference curve, we supposed that the first-order condition gives a negative y, i.e., $h_y = u_1 - u(h_x) < 0 \Rightarrow u_1 < u(h_x)$. Therefore, $u(x_1^*) < u_1 < u(h_x)$, and due to $u'(\cdot) > 0$, then $x_1^* < h_x$, which contradicts with $h_x < x_1^*$.

Alternatively, you can use the condition $MRS = u'(x) \geq p_x/p_y$ to check whether h_x is interior or corner. Suppose h_x is corner, then $u'(h_x) > p_2 = u'(x_2^*) \Rightarrow h_x < x_2^* \Rightarrow h_x < x_1^*$. Because h_x and x_1^* locate at the same indifference curve $u(x) + y = u_1$, and since x_1^* is not located at the horizontal axis, and if $h_x < x_1^*$ then h_x must not locate at the horizontal axis, i.e., h_x cannot be corner.