Macroeconomics A; EI060

Technical appendix: Corsetti-Pesenti stochastic "new open economy macroeconomics" model

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1 Closed economy

1.1 Household's optimization

1.1.1 Preferences

For simplicity we start with the optimal policy in a closed economy model, and then move to the open economy.

The representative household maximizes the following utility:

$$\mathcal{U} = E\left[\ln C - \kappa l + \ln \frac{M}{P}\right] \tag{1}$$

where the consumption index is:

$$C = \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}}$$

1.1.2 Consumption allocation

The optimal allocation of consumption across brands minimizes the cost of purchasing a given aggregate index. The Lagrangian is:

$$\mathcal{L}_C = \int_0^1 P_j C_j - \lambda \left[\left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}} - \bar{C}\right]$$

The optimization with respect to C_j is:

$$\frac{\partial \mathcal{L}_C}{\partial C_j} = 0 \Rightarrow P_j = \lambda \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}-1} [C_j]^{\frac{\theta-1}{\theta}-1}$$

Multiply both sides by C_j and integrate across the j's:

$$
\begin{aligned}
P_j &= \lambda \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} \, dj \right]^{\frac{\theta}{\theta-1}-1} [C_j]^{\frac{\theta-1}{\theta}-1} \\
P_j C_j &= \lambda \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} \, dj \right]^{\frac{\theta}{\theta-1}-1} [C_j]^{\frac{\theta-1}{\theta}} \\
\int_0^1 P_j C_j dj &= \lambda \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} \, dj \right]^{\frac{\theta}{\theta-1}-1} \int_0^1 [C_j]^{\frac{\theta-1}{\theta}} \, dh \\
\int_0^1 P_j C_j dj &= \lambda \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} \, dj \right]^{\frac{\theta}{\theta-1}} \\
PC &= \lambda C \\
P &= \lambda
\end{aligned}
$$

The optimality condition is then:

$$
\begin{aligned}
P_j &= P \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} \, dj \right]^{\frac{\theta}{\theta-1}-1} [C_j]^{\frac{\theta-1}{\theta}-1} \\
P_j &= P \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} \, dj \right]^{\frac{1}{\theta-1}} [C_j]^{-\frac{1}{\theta}} \\
P_j &= P \, [C_\tau]^{\frac{1}{\theta}} \, [C_j]^{-\frac{1}{\theta}} \\
C_j &= \left[\frac{P_j}{P} \right]^{-\theta} C \qquad (2)
\end{aligned}
$$

The consumer price index P is computed using (2) in the definition of the consumption index:

$$
\begin{aligned}
C &= \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} \, dh \right]^{\frac{\theta}{\theta-1}} \\
C &= \left[\int_0^1 \left[\left[\frac{P_j}{P} \right]^{-\theta} C \right]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}}
\end{aligned}
$$

$$
\begin{aligned}
C &= \left[\int_0^1 [C]^{\frac{\theta-1}{\theta}} \left[\frac{P_j}{P} \right]^{1-\theta} dj \right]^{\frac{\theta}{\theta-1}} \\
C &= C \left[\int_0^1 \left[\frac{P_j}{P} \right]^{1-\theta} dj \right]^{\frac{\theta}{\theta-1}} \\
1 &= \int_0^1 \left[\frac{P_j}{P} \right]^{1-\theta} dj \\
[P]^{1-\theta} &= \int_0^1 [P_j]^{1-\theta} \, dj
\end{aligned}
$$

$$P = \left[\int_0^1 [P_j]^{1-\theta}\, dj\right]^{\frac{1}{1-\theta}} \quad (3)$$

1.1.3 Intertemporal allocation

The representative household buys consumption and cash balances using her wage income, $W_\tau l_\tau$, the profits of the firms, $\mathcal{P}_\tau$ (all firms are owned by the household), minus any lump sum taxes $NETT_\tau$. The household can invest in a one-period bond paying a nominal interest rate i. The budget constraint is then:

$$M + PC + B = Wl + \mathcal{P} - NETT$$

The government does not purchase any goods, so the seigniorage revenue is repaid through a lump sum transfer. Using this relation to substitute for C into (1) we obtain:

$$\mathcal{U} = E\left[\ln\left[\frac{-M + Wl + \mathcal{P} - NETT}{P}\right] - \kappa l + \chi \ln \frac{M}{P}\right]$$

The derivatives with respect to bond holdings, nominal balances and labor lead to the following relations:

$$\begin{aligned} \frac{\partial \mathcal{U}}{\partial M} &= 0 \Rightarrow \frac{1}{PC} = \frac{1}{M} \\ &\Rightarrow \frac{M}{P} = C \quad (4) \\ \frac{\partial \mathcal{U}}{\partial l} &= 0 \Rightarrow W = \kappa PC \quad (5) \end{aligned}$$

Denote the value of nominal consumption by $\mu = PC$ (it reflects the stance of monetary policy). the AD relation is then mechanically:

$$C = \frac{\mu}{P} \quad (6)$$

The labor supply (5) then becomes:

$$W = \kappa \mu \quad (7)$$

1.2 Firms' optimization

There is a continuum of firm, each being the sole producer of a particular brand j. The firm takes the demands for its brand (2) into account. The technology is given by:

$$Y_j = Z l_j$$

The firms chooses P_j based on its ex-ante information in order to maximize a weighted average of ex-post profits. The firm does not simply maximize average profits, but the marginal utility of these profits. One additional dollar in profit leads to $1/P$ units of consumption. As we have a log utility, the marginal utility of consumption is $1/C$ The value of an additional dollar in profits is then $\frac{1}{PC}$, which we call the marginal utility of income. The reason profits are weighted is that the household is the firm's owner, hence the household's utility matters. The firm sets its price P_j to maximize:

$$\begin{aligned}
E\frac{1}{PC}\left[P_jY_j - Wl_j\right] &= E\frac{1}{PC}\left[P_jY_j - \frac{W}{Z}Y_j\right] \\
&= E\frac{1}{PC}\left[P_j\left[\frac{P_j}{P}\right]^{-\theta}C - \frac{W}{Z}\left[\frac{P_j}{P}\right]^{-\theta}C\right] \\
&= E\left[\left[\frac{P_j}{P}\right]^{1-\theta} - \frac{W}{PZ}\left[\frac{P_j}{P}\right]^{-\theta}\right] \\
&= E\left[\left[\frac{P_j}{P}\right]^{1-\theta} - \frac{W}{PZ}\left[\frac{P_j}{P}\right]^{-\theta}\right]
\end{aligned}$$

The first order condition with respect to P_j is:

$$(1-\theta)\,E\left[\frac{P_j}{P}\right]^{-\theta}\frac{1}{P} = \theta E\frac{W}{PZ}\left[\frac{P_j}{P}\right]^{-\theta-1}\frac{1}{P}$$

In equilibrium all firms are identical, hence $P = P_j$ and we get:

$$P = \frac{\theta}{\theta-1}E\frac{W}{Z} \tag{8}$$

If prices are fully flexible, then (8) holds in actual levels, and not just in expectations.

With all prices being identical, $Y_j = Y$ for all firms, hence the productivity gives an AS relation:

$$Y = Zl \tag{9}$$

1.3 Equilibrium

1.3.1 Flexible prices

If prices are flexible, (8) and (5) are combined as follows:

$$\begin{aligned}
P^{\text{flex}} &= \frac{\theta}{\theta-1}\frac{W}{Z} \\
P^{\text{flex}} &= \frac{\theta}{\theta-1}\frac{\kappa P^{\text{flex}}C^{\text{flex}}}{Z} \\
1 &= \frac{\theta\kappa}{\theta-1}\frac{C^{\text{flex}}}{Z}
\end{aligned}$$

$$Y^{\text{flex}} = C^{\text{flex}} = \frac{\theta-1}{\theta\kappa}Z \tag{10}$$

$$l^{\text{flex}} = \frac{Y^{\text{flex}}}{Z} = \frac{\theta-1}{\theta\kappa} = \bar{l} \tag{11}$$

The price level is inferred from (6):

$$P^{\text{flex}} = \frac{\mu}{C^{\text{flex}}} = \frac{\theta\kappa}{\theta-1}\frac{\mu}{Z} = \left(\bar{l}\right)^{-1}\frac{\mu}{Z} \tag{12}$$

Abstracting from the direct impact of real balances from (1) the ex-post utility is: $U = \ln C - \kappa l$. At the flexible price allocation, this is:

$$U^{\text{flex}} = \ln C^{\text{flex}} - \kappa l^{\text{flex}} = \ln Z + \ln \bar{l} - \kappa\bar{l}$$

Note that the slope of the indifference curve at the flexible price allocation is:

$$\begin{aligned} dU &= 0 \Rightarrow \frac{1}{C^{\text{flex}}}dC = \kappa dl \\ &\Rightarrow \frac{dC}{dl} = \kappa C^{\text{flex}} \\ &\Rightarrow \frac{dC}{dl} = \kappa\bar{l}Z \\ &\Rightarrow \frac{dC}{dl} = \frac{\theta-1}{\theta}Z < Z \end{aligned}$$

1.3.2 Sticky prices

If prices are sticky, we combine (8) and (5) to write:

$$\begin{aligned} P &= \frac{\theta}{\theta-1}E\frac{W}{Z} \\ P &= \frac{\theta}{\theta-1}E\frac{\kappa PC}{Z} \\ P &= \left(\bar{l}\right)^{-1}PE\frac{C}{Z} \end{aligned}$$

Now use the fact that $C = Y = Zl$ to write:

$$\begin{aligned} P &= \left(\bar{l}\right)^{-1}PE\frac{C}{Z} \\ 1 &= \left(\bar{l}\right)^{-1}E\frac{Zl}{Z} \\ 1 &= \left(\bar{l}\right)^{-1}El \\ El &= \bar{l} \end{aligned} \tag{13}$$

To solve for the actual price level, start from (13) and use (6):

$$\begin{aligned} \bar{l} &= El_t \\ \bar{l} &= E\frac{1}{Z}C \\ \bar{l} &= E\frac{1}{Z}\frac{\mu}{P} \\ \bar{l} &= \frac{1}{P}E\frac{\mu}{Z} \end{aligned}$$

$$P = \left(\bar{l}\right)^{-1} E\frac{\mu}{Z} \tag{14}$$

Note that the expected welfare one period ahead is then:

$$\begin{aligned}
EU &= E\ln C - \kappa E l \\
&= E\ln\frac{\mu}{P} - \kappa\bar{l} = E\ln\mu - \ln P - \kappa\bar{l} \\
&= E\ln\mu + \ln\bar{l} - \kappa\bar{l} - \ln E\frac{\mu}{Z}
\end{aligned}$$

1.3.3 Optimal monetary policy

Contrasting with the welfare level under flexible prices, we write:

$$\begin{aligned}
EU - EU^{\text{flex}} &= E\ln\mu - E\ln Z - \ln E\frac{\mu}{Z} \\
&= E\ln\frac{\mu}{Z} - \ln E\frac{\mu}{Z}
\end{aligned}$$

Recall that an expectation is the sum across state of natures k, weighted by the respective probabilities, π_k:

$$EU - EU^{\text{flex}} = \sum_k \pi_k \ln\frac{\mu_k}{Z_k} - \ln\sum_k \pi_k\frac{\mu_k}{Z_k}$$

The optimal monetary stance in state k maximizes this:

$$\frac{\partial}{\partial\mu_k}\left[EU - EU^{\text{flex}}\right] = 0$$

$$\begin{aligned}
0 &= \pi_k\frac{1}{\mu_k} - \frac{1}{\sum_k \pi_k\frac{\mu_k}{Z_k}}\pi_k\frac{1}{Z_k} \\
0 &= \pi_k - \frac{1}{\sum_k \pi_k\frac{\mu_k}{Z_k}}\pi_k\frac{\mu_k}{Z_k} \\
0 &= 1 - \frac{1}{\sum_k \pi_k\frac{\mu_k}{Z_k}}\frac{\mu_k}{Z_k}
\end{aligned}$$

$$\Rightarrow \frac{\mu_k}{Z_k} = E\frac{\mu}{Z} \Rightarrow \mu_k = \alpha Z_k$$

This rule implies that the welfare is the same under flexible or sticky prices:

$$\begin{aligned}
EU - EU^{\text{flex}} &= \sum_k \pi_k \ln\frac{\mu_k}{Z_k} - \ln\sum_k \pi_k\frac{\mu_k}{Z_k} \\
&= \sum_k \pi_k \ln\frac{\alpha Z_k}{Z_k} - \ln\sum_k \pi_k\frac{\alpha Z_k}{Z_k} \\
&= \sum_k \pi_k \ln\alpha - \ln\sum_k \pi_k\alpha \\
&= \ln\alpha\sum_k \pi_k - \ln\left(\alpha\sum_k \pi_k\right)
\end{aligned}$$

$$= \ln\alpha - \ln\alpha = 0$$

2 The open economy model

In many respect the model parallel the closed economy setup, so we focus on the aspects that are different.

The world consists of two countries, home and foreign, of equal size (set to 1). Foreign variables are indexed by an asterisk. In particular, prices with an asterisk are in foreign currency.

2.1 Household's optimization

2.1.1 Preferences

The preferences of the home representative agent are unchanged from (1). The foreign preferences are similar:

$$\mathcal{U}^* = E\left[\ln C^* - \kappa l^* + \ln \frac{M^*}{P^*}\right]$$

The consumption index is now different. In a first stage, consumption is allocated across two sub-baskets of home and foreign goods, with a unit elasticity of substitution:

$$C = [C_H]^{0.5}[C_F]^{0.5}$$

The subindexes are defined across the various relevant brands for home and foreign produced goods:

$$C_H = \left[\int_0^1 [C_{H,j}]^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}} \quad ; \quad C_F = \left[\int_0^1 [C_{F,j}]^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}}$$

2.1.2 Consumption allocation

We first allocate the aggregate consumption between the baskets of home and foreign goods. The agent minimizes the cost of purchasing a target consumption basket. The problem is described by the Lagrangian:

$$\mathcal{L}_{C1} = P_H C_H + P_F C_F - \varphi_1\left[[C_H]^{0.5}[C_F]^{0.5} - C\right]$$

The first order conditions are:

$$\begin{aligned}
\frac{\partial \mathcal{L}_{C1}}{\partial C_H} &= 0 \Rightarrow P_H = \frac{1}{2}\varphi_1 [C_H]^{-0.5}[C_F]^{0.5} \\
\frac{\partial \mathcal{L}_{C1}}{\partial C_F} &= 0 \Rightarrow P_F = \frac{1}{2}\varphi_1 [C_H]^{0.5}[C_F]^{-0.5}
\end{aligned}$$

Using these conditions, we write:

$$\begin{aligned}
P_H C_H + P_F C_F &= \frac{1}{2}\varphi_1 [C_H]^{0.5}[C_F]^{0.5} + \frac{1}{2}\varphi_1 [C_H]^{0.5}[C_F]^{0.5} \\
PC &= \frac{1}{2}\varphi_1 C + \frac{1}{2}\varphi_1 C \\
PC &= \varphi_1 C
\end{aligned}$$

$$P = \varphi_1$$

The optimality conditions are then:

$$C_H = \frac{1}{2}\frac{PC}{P_H} \qquad ; \qquad C_F = \frac{1}{2}\frac{PC}{P_F} \tag{15}$$

The consumer price index is computed from the definition of the consumption index:

$$\begin{aligned}
C &= [C_H]^{0.5}[C_F]^{0.5} \\
C &= \left[\frac{1}{2}\frac{PC}{P_H}\right]^{0.5}\left[\frac{1}{2}\frac{PC}{P_F}\right]^{0.5} \\
C &= PC\left[\frac{1}{P_H}\right]^{0.5}\left[\frac{1}{P_F}\right]^{0.5}\left[\frac{1}{2}\right]^{0.5}\left[\frac{1}{2}\right]^{0.5} \\
1 &= P\left[\frac{1}{P_H}\right]^{0.5}\left[\frac{1}{P_F}\right]^{0.5}\frac{1}{2} \\
P &= 2\,[P_H]^{0.5}[P_F]^{0.5} \qquad (16)
\end{aligned}$$

Next we proceed to the allocation of home brands. The Lagrangian is:

$$\mathcal{L}_{CH} = \int_0^1 P_{H,j}C_{H,j}dj - \lambda_H\left[\left[\int_0^1 [C_{H,j}]^{\frac{\theta-1}{\theta}}\,dj\right]^{\frac{\theta}{\theta-1}} - \bar{C}_H\right]$$

The optimization with respect to $C_{H,j}$ is:

$$\frac{\partial\mathcal{L}_{CH}}{\partial C_{H,j}} = 0 \Rightarrow P_{H,j} = \lambda_H\left[\int_0^1 [C_{H,j}]^{\frac{\theta-1}{\theta}}\,dj\right]^{\frac{\theta}{\theta-1}-1}[C_{H,j}]^{\frac{\theta-1}{\theta}-1}$$

Multiply both sides by $C_{H,j}$ and integrate across the j's:

$$\begin{aligned}
P_{H,j} &= \lambda_H\left[\int_0^1 [C_{H,j}]^{\frac{\theta-1}{\theta}}\,dj\right]^{\frac{\theta}{\theta-1}-1}[C_{H,j}]^{\frac{\theta-1}{\theta}-1} \\
P_{H,j}C_{H,j} &= \lambda_H\left[\int_0^1 [C_{H,j}]^{\frac{\theta-1}{\theta}}\,dj\right]^{\frac{\theta}{\theta-1}-1}[C_{H,j}]^{\frac{\theta-1}{\theta}} \\
\int_0^1 P_{H,j}C_{H,j}dj &= \lambda_H\left[\int_0^1 [C_{H,j}]^{\frac{\theta-1}{\theta}}\,dj\right]^{\frac{\theta}{\theta-1}-1}\int_0^1 [C_{H,j}]^{\frac{\theta-1}{\theta}}\,dj \\
\int_0^1 P_{H,j}C_{H,j}dj &= \lambda_H\left[\int_0^1 [C_{H,j}]^{\frac{\theta-1}{\theta}}\,dj\right]^{\frac{\theta}{\theta-1}} \\
P_HC_H &= \lambda_HC_H \\
P_H &= \lambda_H
\end{aligned}$$

The optimality condition is then:

$$
\begin{aligned}
P_{H,j} &= \lambda_H \left[\int_0^1 [C_{H,j}]^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}-1} [C_{H,j}]^{\frac{\theta-1}{\theta}-1} \\
P_{H,j} &= P_H [C_H]^{\frac{1}{\theta}} [C_{H,j}]^{\frac{-1}{\theta}} \\
C_{H,j} &= \left[\frac{P_{H,j}}{P_H}\right]^{-\theta} C_H
\end{aligned}
$$

The price index is derived as:

$$
\begin{aligned}
C_H &= \left[\int_0^1 [C_{H,j}]^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}} \\
C_H &= \left[[C_H]^{\frac{\theta-1}{\theta}} \int_0^1 \left[\frac{P_{H,j}}{P_H}\right]^{1-\theta} dj\right]^{\frac{\theta}{\theta-1}} \\
C_H &= C_H \left[\int_0^1 \left[\frac{P_{H,j}}{P_H}\right]^{1-\theta} dj\right]^{\frac{\theta}{\theta-1}} \\
1 &= \int_0^1 \left[\frac{P_{H,j}}{P_H}\right]^{1-\theta} dj \\
P_H &= \left[\int_0^1 [P_{H,j}]^{1-\theta} dj\right]^{\frac{1}{1-\theta}}
\end{aligned}
$$

We can follow similar steps for the allocation across foreign brands, and derive:

$$
\begin{aligned}
C_{F,j} &= \left[\frac{P_{F,j}}{P_F}\right]^{-\theta} C_F \\
P_{F,j} &= \left[\int_0^1 [P_{F,j}]^{1-\theta} dj\right]^{\frac{1}{1-\theta}}
\end{aligned}
$$

Using (15) this becomes:

$$
\begin{aligned}
C_{H,j} &= \frac{1}{2}\left[\frac{P_{H,j}}{P_H}\right]^{-\theta} \frac{PC}{P_H} \qquad (17) \\
C_{F,j} &= \frac{1}{2}\left[\frac{P_{F,j}}{P_F}\right]^{-\theta} \frac{PC}{P_F}
\end{aligned}
$$

We can compute the corresponding allocation for the foreign consumer, with all prices in foreign currency:

$$
\begin{aligned}
C^*_{H,j} &= \frac{1}{2}\left[\frac{P^*_{H,j}}{P^*_H}\right]^{-\theta} \frac{P^*C^*}{P^*_H} \qquad (18) \\
C^*_{F,j} &= \frac{1}{2}\left[\frac{P^*_{F,j}}{P^*_F}\right]^{-\theta} \frac{P^*C^*}{P^*_F} \\
P^* &= 2\,[P^*_H]^{0.5}\,[P^*_F]^{0.5} \qquad (19)
\end{aligned}
$$

2.1.3 Intertemporal allocation

The budget constraints of the home and foreign households are similar to the closed economy. We now assume that households can transact a complete set of state contingent securities denominated in home currency. The ex-ante price of a security paying off 1 unit of home currency ex-post if state k occurs is $q(k)$, in home currency. The budget constraint of the home agent is then:

$$M + PC + \sum_x q(x) B(x) = Wl + \mathcal{P} - NETT + B$$

where $B(k)$ is the number of securities paying of in state k that the home agent buys. The optimal portfolio is such that the cost of purchasing a particular security, adjusted by the marginal utility of consumption, is equal to its expected gain:

$$q(k) \sum_x \frac{1}{P(x) C(x)} = \pi(k) \frac{1}{P(k) C(k)}$$

where $\pi(k)$ is the probability that state k occurs. The budget constraint and optimality conditions for the home agent are (recalling that the securities are in home currency):

$$\begin{aligned} M^* + P^*C^* + \sum_x \frac{q(x)}{\mathcal{E}(x)} B^*(x) &= W^*l^* + \mathcal{P}^* - NETT^* + \frac{B^*}{\mathcal{E}} \\ q(k) \sum_x \frac{1}{\mathcal{E}(x) P^*(x) C^*(x)} &= \pi(k) \frac{1}{\mathcal{E}(k) P^*(k) C^*(k)} \end{aligned}$$

Combining the optimal portfolio conditions, we get:

$$\frac{\sum_x \frac{1}{\mathcal{E}(x)P^*(x)C^*(x)}}{\sum_x \frac{1}{P(x)C(x)}} = \frac{P(k) C(k)}{\mathcal{E}(k) P^*(k) C^*(k)} \qquad \forall k$$

$$\Rightarrow \frac{PC}{\mathcal{E}P^*C^*} = \Delta$$

As in the closed economy, we define the monetary stance as:

$$\mu = PC \qquad ; \qquad \mu^* = P^*C^*$$

which implies:

$$\frac{\mu}{\mathcal{E}\mu^*} = \Delta$$

for some initial state 0. This implies that the exchange rate is simply the ratio of monetary stances (without loss of generality set $\Delta = 1$):

$$\mathcal{E} = \frac{\mu}{\mu^*} \tag{20}$$

As in the closed economy, the labor supplies are:

$$W = \kappa\mu \qquad ; \qquad W^* = \kappa\mu^* \tag{21}$$

2.2 Firms' optimization

2.2.1 Demands

There is a continuum of firm, each being the sole producer of a particular brand j. The firm takes the demands for its brand (17) and (18) into account. The demands for a home and foreign firm are:

$$
\begin{aligned}
Y_j &= \frac{1}{2}\left[\frac{P_{H,j}}{P_H}\right]^{-\theta}\frac{PC}{P_H} + \frac{1}{2}\left[\frac{P^*_{H,j}}{P^*_H}\right]^{-\theta}\frac{P^*C^*}{P^*_H} \\
Y^*_j &= \frac{1}{2}\left[\frac{P_{F,j}}{P_F}\right]^{-\theta}\frac{PC}{P_F} + \frac{1}{2}\left[\frac{P^*_{F,j}}{P^*_F}\right]^{-\theta}\frac{P^*C^*}{P^*_F}
\end{aligned}
$$

The technologies is given by:

$$Y_j = Zl_j \qquad ; \qquad Y^*_j = Z^* l^*_j$$

The expected discounted profits, by the marginal utility of income, are then split across domestic and export sales:

$$
E\frac{1}{PC}\left[\begin{array}{c} P_{H,j}\frac{1}{2}\left[\frac{P_{H,j}}{P_H}\right]^{-\theta}\frac{PC}{P_H} - \frac{W}{Z}\frac{1}{2}\left[\frac{P_{H,j}}{P_H}\right]^{-\theta}\frac{PC}{P_H} \\ +\mathcal{E}P^*_{H,j}\frac{1}{2}\left[\frac{P^*_{H,j}}{P^*_H}\right]^{-\theta}\frac{P^*C^*}{P^*_H} - \frac{W}{Z}\frac{1}{2}\left[\frac{P^*_{H,j}}{P^*_H}\right]^{-\theta}\frac{P^*C^*}{P^*_H} \end{array}\right]
$$

and:

$$
E\frac{1}{P^*C^*}\left[\begin{array}{c} \frac{P_{F,j}}{\mathcal{E}}\frac{1}{2}\left[\frac{P_{F,j}}{P_F}\right]^{-\theta}\frac{PC}{P_F} - \frac{W^*}{Z^*}\frac{1}{2}\left[\frac{P_{F,j}}{P_F}\right]^{-\theta}\frac{PC}{P_F} \\ +P^*_{F,j}\frac{1}{2}\left[\frac{P^*_{F,j}}{P^*_F}\right]^{-\theta}\frac{P^*C^*}{P^*_F} - \frac{W^*}{Z^*}\frac{1}{2}\left[\frac{P^*_{F,j}}{P^*_F}\right]^{-\theta}\frac{P^*C^*}{P^*_F} \end{array}\right]
$$

2.2.2 Optimal sticky prices for domestic sales

The optimal choice for $P_{H,j}$ is such that:

$$
\begin{aligned}
0 &= E\frac{1-\theta}{PC}\frac{1}{2}\left[\frac{P_{H,j}}{P_H}\right]^{-\theta}\frac{PC}{P_H} + \theta E\frac{1}{PC}\frac{W}{Z}\frac{1}{2}\left[\frac{P_{H,j}}{P_H}\right]^{-\theta-1}\frac{1}{P_H}\frac{PC}{P_H} \\
0 &= E\left[\frac{P_{H,j}}{P_H}\right]^{-\theta}\frac{1}{P_H} - \frac{\theta}{\theta-1}E\frac{W}{Z}\left[\frac{P_{H,j}}{P_H}\right]^{-\theta-1}\frac{1}{P_H}\frac{1}{P_H} \\
0 &= P_{H,j}E\left[\frac{1}{P_H}\right]^{-\theta}\frac{1}{P_H} - \frac{\theta\kappa}{\theta-1}E\frac{\mu}{Z}\left[\frac{1}{P_H}\right]^{-\theta-1}\frac{1}{P_H}\frac{1}{P_H} \\
0 &= P_{H,j}E\left[P_H\right]^{\theta-1} - \frac{\theta\kappa}{\theta-1}E\frac{\mu}{Z}\left[P_H\right]^{\theta-1} \\
P_{H,j} &= \frac{\theta\kappa}{\theta-1}\frac{E\frac{\mu}{Z}\left[P_H\right]^{\theta-1}}{E\left[P_H\right]^{\theta-1}}
\end{aligned}
$$

Similarly, the optimal choice for $P^*_{F,j}$ is such that:

$$
0 = E\frac{1-\theta}{P^*C^*}\frac{1}{2}\left[\frac{P^*_{F,j}}{P^*_F}\right]^{-\theta}\frac{P^*C^*}{P^*_F} + \theta E\frac{1}{P^*C^*}\frac{W^*}{Z^*}\frac{1}{2}\left[\frac{P^*_{F,j}}{P^*_F}\right]^{-\theta-1}\frac{1}{P^*_F}\frac{P^*C^*}{P^*_F}
$$

$$
\begin{aligned}
0 &= E\left[\frac{P^*_{F,j}}{P^*_F}\right]^{-\theta}\frac{1}{P^*_F} - \frac{\theta}{\theta-1}E\frac{W^*}{Z^*}\left[\frac{P^*_{F,j}}{P^*_F}\right]^{-\theta-1}\frac{1}{P^*_F}\frac{1}{P^*_F} \\
0 &= P^*_{F,j}E\left[P^*_F\right]^{\theta-1} - \frac{\theta}{\theta-1}E\frac{W^*}{Z^*}\left[P^*_F\right]^{\theta-1} \\
P^*_{F,j} &= \frac{\theta\kappa}{\theta-1}\frac{E\frac{\mu^*}{Z^*}\left[P^*_F\right]^{\theta-1}}{E\left[P^*_F\right]^{\theta-1}}
\end{aligned}
$$

These relation clearly show that $P_{H,j}$ is the same for all j, hence $P_{H,j} = P_H$, Similarly: $P^*_{F,j} = P^*_F$, so the prices become:

$$
P_H = \frac{\theta\kappa}{\theta-1}E\frac{\mu}{Z} \qquad ; \qquad P^*_F = \frac{\theta\kappa}{\theta-1}E\frac{\mu^*}{Z^*} \tag{22}
$$

2.2.3 Optimal sticky prices for export sales

A home firm sets a price $\tilde{P}_{H,j}$ such that the price, in foreign currency, paid by the foreign consumer is:

$$
P^*_{H,j} = \tilde{P}_{H,j}\left(\mathcal{E}\right)^{-\gamma}
$$

where γ is the degree of exchange rate pass-through to foreign consumers. $\gamma = 1$ is the case of PCP and $\gamma = 0$ is the case of LCP. The revenue, in home currency, received by the home firm is:

$$
\tilde{P}_{H,j}\left(\mathcal{E}\right)^{1-\gamma}
$$

Using (20) the optimal price is then:

$$
\begin{aligned}
0 &= E\frac{1-\theta}{PC}\tilde{P}_{H,j}\left(\mathcal{E}\right)^{1-\gamma}\frac{1}{2}\left[\frac{\tilde{P}_{H,j}\left(\mathcal{E}\right)^{-\gamma}}{P^*_H}\right]^{-\theta}\frac{P^*C^*}{P^*_H} \\
&\quad +\theta E\frac{1}{PC}\frac{W}{Z}\frac{1}{2}\left[\frac{\tilde{P}_{H,j}\left(\mathcal{E}\right)^{-\gamma}}{P^*_H}\right]^{-\theta-1}\frac{\left(\mathcal{E}\right)^{-\gamma}}{P^*_H}\frac{P^*C^*}{P^*_H} \\
0 &= \tilde{P}_{H,j}E\frac{1-\theta}{PC}\left(\mathcal{E}\right)^{1-\gamma}\frac{1}{2}\left[\frac{\left(\mathcal{E}\right)^{-\gamma}}{P^*_H}\right]^{-\theta}\frac{P^*C^*}{P^*_H} + \theta E\frac{1}{PC}\frac{W}{Z}\frac{1}{2}\left[\frac{\left(\mathcal{E}\right)^{-\gamma}}{P^*_H}\right]^{-\theta-1}\frac{\left(\mathcal{E}\right)^{-\gamma}}{P^*_H}\frac{P^*C^*}{P^*_H} \\
0 &= \tilde{P}_{H,j}E\left(\frac{\mu}{\mu^*}\right)^{-\gamma}\frac{1}{2}\left[\frac{\left(\mathcal{E}\right)^{-\gamma}}{P^*_H}\right]^{-\theta}\frac{1}{P^*_H} - \frac{\theta}{\theta-1}E\frac{\mu}{\mu^*}\frac{W}{Z}\frac{1}{2}\left[\frac{\left(\mathcal{E}\right)^{-\gamma}}{P^*_H}\right]^{-\theta}\frac{1}{P^*_H} \\
0 &= \tilde{P}_{H,j}E\left[\left(\mathcal{E}\right)^{\gamma}P^*_H\right]^{\theta-1} - \frac{\theta}{\theta-1}E\left(\mathcal{E}\right)^{\gamma-1}\frac{W}{Z}\left[\left(\mathcal{E}\right)^{\gamma}P^*_H\right]^{\theta-1} \\
0 &= \tilde{P}_{H,j}E\left[\left(\mathcal{E}\right)^{\gamma}P^*_H\right]^{\theta-1} - \frac{\theta\kappa}{\theta-1}E\left(\mathcal{E}\right)^{\gamma-1}\frac{\mu}{Z}\left[\left(\mathcal{E}\right)^{\gamma}P^*_H\right]^{\theta-1} \\
\tilde{P}_{H,j} &= \frac{\theta\kappa}{\theta-1}\frac{E\frac{\mu}{Z}\left(\mathcal{E}\right)^{\gamma-1}\left[\left(\mathcal{E}\right)^{\gamma}P^*_H\right]^{\theta-1}}{E\left[\left(\mathcal{E}\right)^{\gamma}P^*_H\right]^{\theta-1}}
\end{aligned}
$$

Similarly a foreign firm sets a price $\tilde{P}_{F,j}$ such that the price, in home currency, paid by the foreign consumer is:

$$
P_{F,j} = \tilde{P}_{F,j}\left(\mathcal{E}\right)^{\gamma^*}
$$

The optimal price is then such that:

$$
\begin{aligned}
0 &= E\frac{1-\theta}{P^*C^*}\tilde{P}_{F,j}\left(\mathcal{E}\right)^{\gamma^*-1}\frac{1}{2}\left[\frac{\tilde{P}_{F,j}\left(\mathcal{E}\right)^{\gamma^*}}{P_F}\right]^{-\theta-1}\frac{\left(\mathcal{E}\right)^{\gamma^*}}{P_F}\frac{PC}{P_F} \\
&\quad +\theta E\frac{1}{P^*C^*}\frac{W^*}{Z^*}\frac{1}{2}\left[\frac{\tilde{P}_{F,j}\left(\mathcal{E}\right)^{\gamma^*}}{P_F}\right]^{-\theta-1}\frac{\left(\mathcal{E}\right)^{\gamma^*}}{P_F}\frac{PC}{P_F} \\
0 &= \tilde{P}_{F,j}E\frac{1-\theta}{P^*C^*}\left(\mathcal{E}\right)^{\gamma^*-1}\frac{1}{2}\left[\frac{\left(\mathcal{E}\right)^{\gamma^*}}{P_F}\right]^{-\theta-1}\frac{\left(\mathcal{E}\right)^{\gamma^*}}{P_F}\frac{PC}{P_F}+\theta E\frac{1}{P^*C^*}\frac{W^*}{Z^*}\frac{1}{2}\left[\frac{\left(\mathcal{E}\right)^{\gamma^*}}{P_F}\right]^{-\theta-1}\frac{\left(\mathcal{E}\right)^{\gamma^*}}{P_F}\frac{PC}{P_F} \\
0 &= \tilde{P}_{F,j}E\frac{\mu}{\mu^*}\left(\mathcal{E}\right)^{\gamma^*-1}\frac{1}{2}\left[\frac{\left(\mathcal{E}\right)^{\gamma^*}}{P_F}\right]^{-\theta}\frac{1}{P_t\left(F\right)}-\frac{\theta}{\theta-1}E\frac{\mu}{\mu^*}\frac{W^*}{Z^*}\frac{1}{2}\left[\frac{\left(\mathcal{E}\right)^{\gamma^*}}{P_F}\right]^{-\theta}\frac{1}{P_t\left(F\right)} \\
0 &= \tilde{P}_{F,j}E\left[\frac{\left(\mathcal{E}\right)^{\gamma^*}}{P_F}\right]^{1-\theta}-\frac{\theta\kappa}{\theta-1}E\frac{\mu^*}{Z^*}\left(\mathcal{E}\right)^{1-\gamma^*}\left[\frac{\left(\mathcal{E}\right)^{\gamma^*}}{P_F}\right]^{1-\theta} \\
\tilde{P}_{F,j} &= \frac{\theta\kappa}{\theta-1}\frac{E\frac{\mu^*}{Z^*}\left(\mathcal{E}\right)^{1-\gamma^*}\left[\left(\mathcal{E}\right)^{-\gamma^*}P_F\right]^{\theta-1}}{E\left[\left(\mathcal{E}\right)^{-\gamma^*}P_F\right]^{\theta-1}}
\end{aligned}
$$

These relations clearly show that $\tilde{P}_{H,j}$ and $\tilde{P}_{F,j}$ is the same for all j, hence $P_H^* = \tilde{P}_{H,j}\left(\mathcal{E}\right)^{-\gamma}$ and $P_F = \tilde{P}_{F,j}\left(\mathcal{E}\right)^{\gamma^*}$. We then write:

$$
\begin{aligned}
\tilde{P}_{H,j} &= \left(\mathcal{E}\right)^{\gamma}P_H^* = \frac{\theta\kappa}{\theta-1}E\frac{\mu}{Z}\left(\mathcal{E}\right)^{\gamma-1} \\
\tilde{P}_{F,j} &= \left(\mathcal{E}\right)^{-\gamma^*}P_F = \frac{\theta\kappa}{\theta-1}E\frac{\mu^*}{Z^*}\left(\mathcal{E}\right)^{1-\gamma^*}
\end{aligned}
\tag{23}
$$

2.3 Good markets clearing

As all firm in a country are identical, they set the same price. The output demands then become:

$$
Y=\frac{1}{2}\left(\frac{PC}{P_H}+\frac{P^*C^*}{P_H^*}\right) \qquad ; \qquad Y^*=\frac{1}{2}\left(\frac{PC}{P_F}+\frac{P^*C^*}{P_F^*}\right)
$$

Using (20) and the technologies this becomes:

$$
Y=\frac{1}{2}\left(\frac{PC}{P_H}+\frac{P^*C^*}{P_H^*}\right)=\frac{\mu}{2}\left(\frac{1}{P_H}+\frac{\mu^*}{\mu P_H^*}\right)=\frac{PC}{2}\left(\frac{1}{P_H}+\frac{1}{\mathcal{E}P_H^*}\right)
$$

$$
\Rightarrow C = Y\frac{1}{\frac{P}{2}\left(\frac{1}{P_H}+\frac{1}{\mathcal{E}P_H^*}\right)} = Zl\tau \tag{24}
$$

where:

$$
\tau = 2\left[\frac{P}{P_H}+\frac{P}{\mathcal{E}P_H^*}\right]^{-1} \tag{25}
$$

Similarly:

$$
Y^*=\frac{1}{2}\left(\frac{PC}{P_F}+\frac{P^*C^*}{P_F^*}\right)=\frac{\mu^*}{2}\left(\frac{\mathcal{E}_t}{P_F}+\frac{1}{P_F^*}\right)=\frac{P^*C^*}{2}\left(\frac{\mathcal{E}}{P_F}+\frac{1}{P_F^*}\right)
$$

$$\Rightarrow C^* = Y^* \frac{1}{\frac{P^*}{2}\left(\frac{\mathcal{E}}{P_F} + \frac{1}{P_F^*}\right)} = Z^* l^* \tau^* \tag{26}$$

where:

$$\tau^* = 2\left[\frac{\mathcal{E}P^*}{P_F} + \frac{P^*}{P_F^*}\right]^{-1} \tag{27}$$

2.4 Equilibrium

2.4.1 Flexible prices

If prices are flexible, the optimal prices are the same as (22)-(23), except that they hold in actual levels and not only in expectations:

$$P_H^{\text{flex}} = \mathcal{E}P_H^{*\text{flex}} = \frac{\theta\kappa}{\theta-1}\frac{\mu}{Z} \qquad ; \qquad P_F^{*\text{flex}} = \frac{P_F^{\text{flex}}}{\mathcal{E}} = \frac{\theta\kappa}{\theta-1}\frac{\mu^*}{Z^*} \tag{28}$$

which shows that the law of one price holds. From (16)-(19) we see that purchasing power parity holds:

$$\begin{aligned}
\mathcal{E}P^{*\text{flex}} &= 2\left[\mathcal{E}P_H^{*\text{flex}}\right]^{0.5}\left[\mathcal{E}P_F^{*\text{flex}}\right]^{0.5} \\
&= 2\left[P_H^{\text{flex}}\right]^{0.5}\left[P_F^{\text{flex}}\right]^{0.5} \\
&= P_t^{\text{flex}}
\end{aligned}$$

Furthermore:

$$\begin{aligned}
P_t^{\text{flex}} &= 2\left[P_H^{\text{flex}}\right]^{0.5}\left[P_F^{\text{flex}}\right]^{0.5} \\
&= 2\left[\frac{\theta\kappa}{\theta-1}\frac{\mu}{Z}\right]^{0.5}\left[\frac{\theta\kappa}{\theta-1}\frac{\mu^*}{Z^*}\mathcal{E}\right]^{0.5} \\
&= 2\frac{\theta\kappa}{\theta-1}\frac{\mu}{(Z)^{0.5}(Z^*)^{0.5}}
\end{aligned}$$

(24)-(27) then imply the following terms of trade:

$$\begin{aligned}
\tau^{\text{flex}} &= 2\left[2\frac{P^{\text{flex}}}{P_H^{\text{flex}}}\right]^{-1} = \frac{P_H^{\text{flex}}}{P^{\text{flex}}} = \frac{1}{2}\left(\frac{Z}{Z^*}\right)^{-0.5} \\
\tau^{*\text{flex}} &= 2\left[2\frac{P^{*\text{flex}}}{P_F^{*\text{flex}}}\right]^{-1} = \frac{P_F^{*\text{flex}}}{P^{*\text{flex}}} = \frac{1}{2}\left(\frac{Z}{Z^*}\right)^{0.5}
\end{aligned}$$

Employment in the home country are:

$$\begin{aligned}
C^{\text{flex}} &= Zl^{\text{flex}}\tau^{\text{flex}} \\
\frac{\mu}{P^{\text{flex}}} &= Zl^{\text{flex}}\frac{1}{2}\left(\frac{Z}{Z^*}\right)^{-0.5} \\
\frac{\mu}{2\frac{\theta\kappa}{\theta-1}\frac{\mu}{(Z)^{0.5}(Z^*)^{0.5}}} &= l^{\text{flex}}\frac{1}{2}(Z)^{0.5}(Z^*)^{0.5}
\end{aligned}$$

$$l^{\text{flex}} = \frac{\theta - 1}{\theta\kappa} = \bar{l}$$

Consumption and output are:

$$\begin{aligned} Y^{\text{flex}} &= Zl^{\text{flex}} = Z\bar{l} \\ C^{\text{flex}} &= Zl^{\text{flex}}\tau^{\text{flex}} = Z\frac{1}{2}\left(\frac{Z}{Z^*}\right)^{-0.5}\bar{l} = \frac{\bar{l}}{2}(Z)^{0.5}(Z^*)^{0.5} \end{aligned}$$

Following similar steps, the foreign variables are:

$$l^{*\text{flex}} = \bar{l} \qquad ; \qquad Y^{*\text{flex}} = Z^*\bar{l} \qquad ; \qquad C^{*\text{flex}} = C_t^{\text{flex}}$$

Abstracting from the direct impact of real balances, the ex-post utility (1) is:

$$U = \ln C - \kappa l$$

At the flexible price allocation, this is:

$$U^{\text{flex}} = \ln C^{\text{flex}} - \kappa l^{\text{flex}} = -\ln 2 + \frac{1}{2}\ln Z + \frac{1}{2}\ln Z^* + \ln \bar{l} - \kappa\bar{l}$$

The foreign utility is the same.

2.4.2 Sticky prices

If prices are sticky, they are given by (23). From (16) and (23) we write:

$$\begin{aligned} P &= 2\left[P_H\right]^{0.5}\left[P_F\right]^{0.5} = 2\left[P_H\right]^{0.5}\left[\tilde{P}_F(\mathcal{E})^{\gamma^*}\right]^{0.5} \\ &= 2\frac{\theta\kappa}{\theta-1}\left[E\frac{\mu}{Z}\right]^{0.5}\left[(\mathcal{E})^{\gamma^*}E\frac{\mu^*}{Z^*}(\mathcal{E})^{1-\gamma^*}\right]^{0.5} \end{aligned}$$

(25) then becomes:

$$\begin{aligned} \tau &= 2\left[\frac{P}{P_H} + \frac{P}{\mathcal{E}P_H^*}\right]^{-1} \\ &= \left[\left[\frac{P_H}{P_F}\right]^{-0.5} + \left[\frac{P_H}{P_F}\right]^{-0.5}\frac{P_H}{\mathcal{E}P_H^*}\right]^{-1} \\ &= \left[\frac{P_H}{P_F}\right]^{0.5}\left[1 + \frac{P_H}{\tilde{P}_H(\mathcal{E})^{1-\gamma}}\right]^{-1} \\ &= \left[\frac{E\frac{\mu}{Z}}{(\mathcal{E})^{\gamma^*}E\frac{\mu^*}{Z^*}(\mathcal{E})^{1-\gamma^*}}\right]^{0.5}\left[1 + \frac{E\frac{\mu}{Z}}{(\mathcal{E})^{1-\gamma}E\frac{\mu}{Z}(\mathcal{E})^{\gamma-1}}\right]^{-1} \end{aligned}$$

Similarly we write:

$$\tau^* = \left[\frac{E\frac{\mu^*}{Z^*}}{(\mathcal{E})^{-\gamma} E\frac{\mu}{Z}(\mathcal{E})^{\gamma-1}}\right]^{0.5}\left[\frac{\mathcal{E}E\frac{\mu^*}{Z^*}}{(\mathcal{E})^{\gamma^*} E\frac{\mu^*}{Z^*}(\mathcal{E})^{1-\gamma^*}}+1\right]^{-1}$$

Using (24), the effort in the home country is:

$$\begin{aligned}
l &= \frac{C}{\tau Z} = \frac{\mu}{Z}\frac{1}{\tau P} \\
&= \frac{\mu}{Z}\left[\frac{E\frac{\mu}{Z}}{(\mathcal{E})^{\gamma^*} E\frac{\mu^*}{Z^*}(\mathcal{E})^{1-\gamma^*}}\right]^{-0.5}\left[1+\frac{E\frac{\mu}{Z}}{(\mathcal{E})^{1-\gamma} E\frac{\mu}{Z}(\mathcal{E})^{\gamma-1}}\right] \\
&\quad E\frac{\mu}{Z}^{-0.5}\left[(\mathcal{E})^{\gamma^*} E\frac{\mu^*}{Z^*}(\mathcal{E})^{1-\gamma^*}\right]^{-0.5}\frac{\bar{l}}{2} \\
&= \frac{\mu}{Z}\frac{1}{E\frac{\mu}{Z}}\left[1+\frac{E\frac{\mu}{Z}}{(\mathcal{E})^{1-\gamma} E\frac{\mu}{Z}(\mathcal{E})^{\gamma-1}}\right]\frac{\bar{l}}{2} \\
&= \left[\frac{\frac{\mu}{Z}}{E\frac{\mu}{Z}}+\frac{\frac{\mu}{Z}(\mathcal{E})^{\gamma-1}}{E\frac{\mu}{Z}(\mathcal{E})^{\gamma-1}}\right]\frac{\bar{l}}{2}
\end{aligned}$$

Similarly the effort in the foreign country is:

$$l^* = \left[\frac{\frac{\mu^*}{Z^*}(\mathcal{E})^{1-\gamma^*}}{E\frac{\mu^*}{Z^*}(\mathcal{E})^{1-\gamma^*}}+\frac{\frac{\mu^*}{Z^*}}{E\frac{\mu^*}{Z^*}}\right]\frac{\bar{l}}{2}$$

These relations imply that expected effort is equal to its constant value under flexible prices:

$$El = El^* = \bar{l}$$

The home and foreign consumptions are then written as:

$$\begin{aligned}
C &= \frac{\mu}{P} = \frac{\mu}{2\left[E\frac{\mu}{Z}\right]^{0.5}\left[(\mathcal{E})^{\gamma^*} E\frac{\mu^*}{Z^*}(\mathcal{E})^{1-\gamma^*}\right]^{0.5}}\bar{l} \\
C^* &= \frac{\mu^*}{P^*} = \frac{\mu^*}{2\left[(\mathcal{E})^{-\gamma} E\frac{\mu}{Z}(\mathcal{E})^{\gamma-1}\right]^{0.5}\left[E\frac{\mu^*}{Z^*}\right]^{0.5}}\bar{l}
\end{aligned}$$

The expected welfare in the home country is:

$$\begin{aligned}
EU &= E\ln C - \kappa El \\
&= E\ln\mu - \frac{\gamma^*}{2}E\ln(\mathcal{E}) - \frac{1}{2}\ln\left[E\frac{\mu}{Z}\right] - \frac{1}{2}\ln\left[E\frac{\mu^*}{Z^*}(\mathcal{E})^{1-\gamma^*}\right] - \ln 2 + \ln\bar{l} - \kappa\bar{l} \\
&= \left(1-\frac{\gamma^*}{2}\right)E\ln\mu + \frac{\gamma^*}{2}E\ln\mu^* - \frac{1}{2}\ln\left[E\frac{\mu}{Z}\right] \\
&\quad -\frac{1}{2}\ln\left[E\frac{(\mu)^{1-\gamma^*}(\mu^*)^{\gamma^*}}{Z^*}\right] - \ln 2 + \ln\bar{l} - \kappa\bar{l}
\end{aligned}$$

The expected welfare in the foreign country is:

$$
\begin{aligned}
EU^{*} &= E\ln C^{*}-\kappa El^{*}\\
&= E\ln\mu^{*}+\frac{\gamma}{2}E\ln(\mathcal{E})-\frac{1}{2}\ln\left[E\frac{\mu}{Z}(\mathcal{E})^{\gamma-1}\right]-\frac{1}{2}\ln\left[E\frac{\mu^{*}}{Z^{*}}\right]-\ln 2+\ln\bar{l}-\kappa\bar{l}\\
&= \frac{\gamma}{2}E\ln\mu+\left(1-\frac{\gamma}{2}\right)E\ln\mu^{*}-\frac{1}{2}\ln\left[E\frac{(\mu)^{\gamma}(\mu^{*})^{1-\gamma}}{Z}\right]\\
&\quad -\frac{1}{2}\ln\left[E\frac{\mu^{*}}{Z^{*}}\right]-\ln 2+\ln\bar{l}-\kappa\bar{l}
\end{aligned}
$$

2.4.3 Optimal decentralized monetary policy

Contrasting with the welfare level under flexible prices, we write the home welfare as:

$$
\begin{aligned}
EU-EU^{\text{flex}} &= \left(1-\frac{\gamma^{*}}{2}\right)E\ln\mu+\frac{\gamma^{*}}{2}E\ln\mu^{*}-\frac{1}{2}E\ln Z-\frac{1}{2}E\ln Z^{*}\\
&\quad -\frac{1}{2}\ln\left[E\frac{\mu}{Z}\right]-\frac{1}{2}\ln\left[E\frac{(\mu)^{1-\gamma^{*}}(\mu^{*})^{\gamma^{*}}}{Z^{*}}\right]
\end{aligned}
$$

Recall that an expectation is the sum across state of natures k, weighted by the respective probabilities, π_k:

$$
\begin{aligned}
EU-EU^{\text{flex}} &= \left(1-\frac{\gamma^{*}}{2}\right)\sum_k\pi_k\ln\mu_k+\frac{\gamma^{*}}{2}\sum_k\pi_k\ln\mu_k^{*}\\
&\quad -\frac{1}{2}\sum_k\pi_k\ln Z_k-\frac{1}{2}\sum_k\pi_k\ln Z_k^{*}\\
&\quad -\frac{1}{2}\ln\left[\sum_k\pi_k\frac{\mu_k}{Z_k}\right]-\frac{1}{2}\ln\left[\sum_k\pi_k\frac{(\mu_k)^{1-\gamma^{*}}(\mu_k^{*})^{\gamma^{*}}}{Z_k^{*}}\right]
\end{aligned}
$$

The optimal home monetary stance in state k maximizes this:

$$
\begin{aligned}
0 &= \frac{\partial}{\partial\mu_k}\left[EU-EU^{\text{flex}}\right]\\
0 &= \left(1-\frac{\gamma^{*}}{2}\right)\pi_k\frac{1}{\mu_k}\\
&\quad -\frac{1}{2}\frac{1}{\sum_k\pi_k\frac{\mu_k}{Z_k}}\pi_k\frac{1}{Z_k}-\frac{1}{2}\frac{1-\gamma^{*}}{\sum_k\pi_k\frac{(\mu_k)^{1-\gamma^{*}}(\mu_k^{*})^{\gamma^{*}}}{Z_k^{*}}}\pi_k\frac{(\mu_k)^{-\gamma^{*}}(\mu_k^{*})^{\gamma^{*}}}{Z_k^{*}}\\
0 &= \left(1-\frac{1}{E\frac{\mu}{Z}}\frac{\mu_k}{Z_k}\right)+(1-\gamma^{*})\left(1-\frac{1}{E\frac{(\mu)^{1-\gamma^{*}}(\mu^{*})^{\gamma^{*}}}{Z^{*}}}\frac{(\mu_k)^{1-\gamma^{*}}(\mu_k^{*})^{\gamma^{*}}}{Z_k^{*}}\right)
\end{aligned}
$$

Taking a first-order log approximation of this relation, we get:

$$
\left[1+(1-\gamma^{*})^{2}\right]\ln(\mu_k)=\ln(Z_k)+(1-\gamma^{*})\ln(Z_k^{*})-\gamma^{*}(1-\gamma^{*})\ln(\mu_k^{*})
$$

Following similar steps for the foreign utility, we write the welfare as:

$$
\begin{aligned}
EU^{*}-EU^{\text{flex}} &= \frac{\gamma}{2}\sum_{k}\pi_{k}\ln\mu_{k}+\left(1-\frac{\gamma}{2}\right)\sum_{k}\pi_{k}\ln\mu_{k}^{*} \\
&-\frac{1}{2}\sum_{k}\pi_{k}\ln Z_{k}-\frac{1}{2}\sum_{k}\pi_{k}\ln Z_{k}^{*} \\
&-\frac{1}{2}\ln\left[\sum_{k}\pi_{k}\frac{(\mu_{k})^{\gamma}(\mu_{k}^{*})^{1-\gamma}}{Z_{k}}\right]-\frac{1}{2}\ln\left[\sum_{k}\pi_{k}\frac{\mu_{k}^{*}}{Z_{k}^{*}}\right]
\end{aligned}
$$

The optimal foreign monetary stance in state k maximizes this:

$$
\begin{aligned}
0 &= \frac{\partial}{\partial\mu_{k}^{*}}\left[EU^{*}-EU^{\text{flex}}\right] \\
0 &= \left(1-\frac{\gamma}{2}\right)\pi_{k}\frac{1}{\mu_{k}^{*}} \\
&-\frac{1}{2}\frac{1-\gamma}{E\frac{(\mu)^{\gamma}(\mu^{*})^{1-\gamma}}{Z}}\pi_{k}\frac{(\mu_{k})^{\gamma}(\mu_{k}^{*})^{-\gamma}}{Z_{k}}-\frac{1}{2}\frac{1}{E\frac{\mu^{*}}{Z^{*}}}\pi_{k}\frac{1}{Z_{k}^{*}} \\
0 &= (1-\gamma)\left(1-\frac{1}{E\frac{(\mu)^{\gamma}(\mu^{*})^{1-\gamma}}{Z}}\frac{(\mu_{k})^{\gamma}(\mu_{k}^{*})^{-\gamma}}{Z_{k}}\right)+\left(1-\frac{1}{E\frac{\mu^{*}}{Z^{*}}}\frac{\mu_{k}^{*}}{Z_{k}^{*}}\right)
\end{aligned}
$$

Taking a first-order log approximation of this relation, we get:

$$
\left[1+(1-\gamma)^{2}\right]\ln(\mu_{k}^{*})=(1-\gamma)\ln(Z_{k})+\ln(Z_{k}^{*})-\gamma(1-\gamma)\ln(\mu_{k})
$$

Combining the monetary policy rules we get the Nash solution as a function of shocks. If the degrees of pass-through are symmetric ($\gamma=\gamma^{*}$) we get:

$$
\begin{aligned}
\ln(\mu_{k}) &= \frac{1+(1-\gamma)^{3}}{1+(1-\gamma)^{2}(3-2\gamma)}\ln(Z_{k})+\frac{(2-\gamma)(1-\gamma)^{2}}{1+(1-\gamma)^{2}(3-2\gamma)}\ln(Z_{k}^{*}) \\
\ln(\mu_{k}^{*}) &= \frac{1+(1-\gamma)^{3}}{1+(1-\gamma)^{2}(3-2\gamma)}\ln(Z_{k}^{*})+\frac{(2-\gamma)(1-\gamma)^{2}}{1+(1-\gamma)^{2}(3-2\gamma)}\ln(Z_{k}) \\
\ln\left(\frac{\mu_{k}}{\mu_{k}^{*}}\right) &= \frac{1-(1-\gamma)^{2}}{1+(1-\gamma)^{2}(3-2\gamma)}\ln\left(\frac{Z_{k}}{Z_{k}^{*}}\right)
\end{aligned}
$$

Under PCP ($\gamma=\gamma^{*}=1$) we get:

$$
\ln(\mu_{k})=\ln(Z_{k})\qquad;\qquad\ln(\mu_{k}^{*})=\ln(Z_{k}^{*})
$$

Under LCP ($\gamma=\gamma^{*}=0$) we get:

$$
\ln(\mu_{k})=\ln(\mu_{k}^{*})=\frac{1}{2}\left[\ln(Z_{k})+\ln(Z_{k}^{*})\right]
$$

If there is asymmetric pass-through ($\gamma = 1$ and $\gamma^* = 0$) we get:

$$\ln(\mu_k) = \frac{1}{2}\left[\ln(Z_k) + \ln(Z_k^*)\right] \qquad ; \qquad \ln(\mu_k^*) = \ln(Z_k^*)$$

2.4.4 Optimal cooperative monetary policy

Under a cooperative setting the monetary authorities maximize the weighted average of utilities:

$$\begin{aligned}
& \frac{EU + EU^*}{2} - EU^{\text{flex}} \\
= & \frac{1}{2}\left(1 + \frac{\gamma - \gamma^*}{2}\right)\sum_k \pi_k \ln \mu_k + \frac{1}{2}\left(1 - \frac{\gamma - \gamma^*}{2}\right)\sum_k \pi_k \ln \mu_k^* \\
& - \frac{1}{2}\sum_k \pi_k \ln Z_k - \frac{1}{2}\sum_k \pi_k \ln Z_k^* \\
& - \frac{1}{4}\ln\left[\sum_k \pi_k \ln \frac{\mu_k}{Z_k}\right] - \frac{1}{4}\ln\left[\sum_k \pi_k \frac{(\mu_k)^{1-\gamma^*}(\mu_k^*)^{\gamma^*}}{Z_k^*}\right] \\
& - \frac{1}{4}\ln\left[\sum_k \pi_k \frac{(\mu_k)^{\gamma}(\mu_k^*)^{1-\gamma}}{Z_k}\right] - \frac{1}{4}\ln\left[\sum_k \pi_k \frac{\mu_k^*}{Z_k^*}\right]
\end{aligned}$$

The optimal monetary stance in either country in state k maximizes this:

$$0 = \frac{\partial}{\partial \mu_k}\left[\frac{EU + EU^*}{2} - EU^{\text{flex}}\right] = \frac{\partial}{\partial \mu_k^*}\left[\frac{EU + EU^*}{2} - EU^{\text{flex}}\right]$$

The resulting monetary stances are:

$$\begin{aligned}
& \left[1 + (1-\gamma^*)^2 + \gamma^2\right]\ln(\mu_k) + \left[\gamma^*(1-\gamma^*) + \gamma(1-\gamma)\right]\ln(\mu_k^*) \\
= & (1+\gamma)\ln(Z_k) + (1-\gamma^*)\ln(Z_k^*)
\end{aligned}$$

and:

$$\begin{aligned}
& \left[\gamma^*(1-\gamma^*) + \gamma(1-\gamma)\right]\ln(\mu_k) + \left[1 + (1-\gamma)^2 + (\gamma^*)^2\right]\ln(\mu_k^*) \\
= & (1-\gamma)\ln(Z_k) + (1+\gamma^*)\ln(Z_k^*)
\end{aligned}$$

Under symmetric PCP ($\gamma = \gamma^* = 1$) we get:

$$\ln(\mu_k) = \ln(Z_k) \qquad ; \qquad \ln(\mu_k^*) = \ln(Z_k^*)$$

Under symmetric LCP ($\gamma = \gamma^* = 0$) we get:

$$\ln(\mu_k) = \ln(\mu_k^*) = \frac{1}{2}\left[\ln(Z_k) + \ln(Z_k^*)\right]$$

If there is asymmetric pass-through ($\gamma = 1$ and $\gamma^* = 0$) we get:

$$\ln(\mu_k) = \frac{2}{3}\ln(Z_k) + \frac{1}{3}\ln(Z_k^*) \qquad ; \qquad \ln(\mu_k^*) = \ln(Z_k^*)$$