PS1 Solutions

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1 Consumption Allocation

Problem Setup

The Home agent's consumption basket is given by

$$C_t = \left(\frac{C_{T,t}}{\gamma}\right)^{\gamma}\left(\frac{C_{N,t}}{1-\gamma}\right)^{1-\gamma},$$

where:

- $C_{T,t}$ is the quantity of the traded good (its price is normalized to 1),
- $C_{N,t}$ is the quantity of the domestic non-traded good (price $P_{N,t}$),
- γ is the expenditure share on the traded good.

The consumer minimizes total expenditure subject to attaining a given consumption level C_t. The problem is

$$\begin{aligned}
\min_{C_{T,t},C_{N,t}} \; & P_t C_t = C_{T,t} + P_{N,t}C_{N,t} \\
\text{s.t.} \quad & C_t = \left(\frac{C_{T,t}}{\gamma}\right)^{\gamma}\left(\frac{C_{N,t}}{1-\gamma}\right)^{1-\gamma}.
\end{aligned}$$

Define the Lagrangian function:

$$\mathcal{L} = C_{T,t} + P_{N,t}C_{N,t} + \lambda\Big(C_t - \Big(\frac{C_{T,t}}{\gamma}\Big)^{\gamma}\Big(\frac{C_{N,t}}{1-\gamma}\Big)^{1-\gamma}\Big).$$

The FOCs with respect to $C_{T,t}$ and $C_{N,t}$ are:

$$\begin{aligned}
\mathcal{L}_{C_{T,t}} &= 1 - \lambda\gamma\Big(\frac{C_{T,t}}{\gamma}\Big)^{\gamma-1}\frac{1}{\gamma}\Big(\frac{C_{N,t}}{1-\gamma}\Big)^{1-\gamma} = 0, \\
\mathcal{L}_{C_{N,t}} &= P_{N,t} - \lambda(1-\gamma)\Big(\frac{C_{T,t}}{\gamma}\Big)^{\gamma}\frac{1}{1-\gamma}\Big(\frac{C_{N,t}}{1-\gamma}\Big)^{-\gamma} = 0 \\
\Rightarrow \quad \frac{1}{P_{N,t}} &= \frac{\gamma}{1-\gamma}\frac{C_{N,t}}{C_{T,t}}.
\end{aligned}$$

The dual (expenditure) minimization problem yields the unit cost function (composite price index) for the consumption bundle:

$$P_t C_t = \min \left\{ C_{T,t} + P_{N,t} C_{N,t} : \; C_t = \left(\frac{C_{T,t}}{\gamma}\right)^{\gamma} \left(\frac{C_{N,t}}{1-\gamma}\right)^{1-\gamma} \right\}.$$

So, we have:

$$\begin{aligned} P_t C_T &= C_{T,t} + P_{N,t} C_{N,t} \\ &= C_{T,t} + \frac{1-\gamma}{\gamma} C_{T,t} \\ \Rightarrow C_{T,t} &= \gamma P_t C_t \\ \Rightarrow C_{N,t} &= \frac{1-\gamma}{\gamma} \frac{C_{T,t}}{P_{N,t}} \\ &= (1-\gamma) P_t C_t. \end{aligned}$$

As P_t is the minimum expenditure required to attain the given consumption level $C_t = 1$, we have:

$$\begin{aligned} & \left(\frac{C_{T,t}}{\gamma}\right)^{\gamma} \left(\frac{C_{N,t}}{1-\gamma}\right)^{1-\gamma} = 1 \\ \Rightarrow \quad & (P_t C_t)^{\gamma} \left(\frac{P_t C_t}{P_N, t}\right)^{1-\gamma} = 1 \\ \Rightarrow \quad & P_t = (P_{N,t})^{1-\gamma}. \end{aligned}$$

Analogously, for the Foreign agent, we have

$$\begin{aligned} C^*_{T,t} &= \gamma P^*_t C^*_t \\ C^*_{N,t} &= (1-\gamma) P^*_t C^*_t \\ P^*_t &= (P^*_{N,t})^{1-\gamma}. \end{aligned}$$

Economic Intuition

- The parameter γ reflects the expenditure share on the traded good.
- Since the traded good is the numéraire (price normalized to 1), its cost enters directly, while the cost of the non-traded good is weighted by its price $P_{N,t}$.
- The composite price index P_t is a weighted geometric mean of the individual prices. With the traded good's price equal to 1, we have $P_t = (P_{N,t})^{1-\gamma}$.
- The optimal consumption choices allocate expenditure in a way that equates the marginal rate of substitution to the ratio of prices.

2 Market Clearing

Under market clearing, the quantities of traded and non-traded goods produced must equal the quantities consumed. We have:

Non-Traded Goods Market

For Home, market clearing in non-traded goods is:

$$nC_{N,t} = A_{N,t}(L_{N,t})^{1-\alpha}.$$

Substituting $C_{N,t} = (1-\gamma)\frac{P_t C_t}{P_{N,t}}$ with $P_t = (P_{N,t})^{1-\gamma}$, we obtain:

$$\boxed{n(1-\gamma)(P_{N,t})^{-\gamma}C_t = A_{N,t}(L_{N,t})^{1-\alpha}.}$$

For Foreign, the market clearing condition is: $(1-n)C^*_{N,t} = A^*_{N,t}(L^*_{N,t})^{(1-\alpha)}$

$$\boxed{(1-n)(1-\gamma)(P^*_{N,t})^{-\gamma}C^*_t = A^*_{N,t}(L^*_{N,t})^{1-\alpha}.}$$

Traded Goods Market

Global market clearing for traded goods is:

$$nC_{T,t} + (1-n)C^*_{T,t} = A_{T,t}(n - L_{N,t})^{1-\alpha} + A^*_{T,t}\big((1-n) - L^*_{N,t}\big)^{1-\alpha}.$$

Substituting $C_{T,t} = \gamma P_t C_t$ with $P_t = (P_{N,t})^{1-\gamma}$ (and similarly for Foreign), we have:

$$\boxed{n\gamma(P_{N,t})^{1-\gamma}C_t + (1-n)\gamma(P^*_{N,t})^{1-\gamma}C^*_t = A_{T,t}(n - L_{N,t})^{1-\alpha} + A^*_{T,t}\big((1-n) - L^*_{N,t}\big)^{1-\alpha}.}$$

Intuition: Non-traded goods are produced and consumed domestically, while traded goods are balanced across countries.

3 Intertemporal Allocation

Home Optimization

The Home agent maximizes:

$$U_t = \sum_{s=0}^{\infty}(\beta_{H,t+s})^s \ln C_{t+s},$$

subject to the period-t budget constraint:

$$nP_tC_t + nB_{t+1} = A_{T,t}(n - L_{N,t})^{1-\alpha} + P_{N,t}A_{N,t}(L_{N,t})^{1-\alpha} + n(1 + r_t)B_t.$$

Focusing on a single period (the infinite-horizon structure is standard), we write:

$$\mathcal{L} = \ln C_t + \beta_{H,t+1} \ln C_{t+1} - \lambda_t \Big[A_{T,t}(n - L_{N,t})^{1-\alpha} + P_{N,t}A_{N,t}(L_{N,t})^{1-\alpha} + n(1+r_t)B_t - nP_tC_t - nB_{t+1}\Big].$$

Take the FOCs:

$$\frac{\partial \mathcal{L}}{\partial C_t} = \frac{1}{C_t} - \lambda_t nP_t = 0 \quad \Rightarrow \quad \lambda_t = \frac{1}{nP_tC_t}$$
$$\frac{\partial \mathcal{L}}{\partial B_{t+1}} = -\lambda_t + \beta_{H,t+1}(1 + r_{t+1})\lambda_{t+1} = 0.$$

Substitute the expressions for λ_t and λ_{t+1}:

$$\frac{1}{nP_tC_t} = \beta_{H,t+1}(1 + r_{t+1})\frac{1}{nP_{t+1}C_{t+1}}.$$

Cancel n and rearrange:

$$\frac{1}{C_t} = \beta_{H,t+1}(1 + r_{t+1})\frac{P_t}{P_{t+1}}\frac{1}{C_{t+1}}.$$

Set:

$$1 + r^C_{t+1} \equiv (1 + r_{t+1})\frac{P_t}{P_{t+1}},$$

so the Euler equation becomes:

$$\boxed{C_{t+1} = \beta_{H,t+1}(1 + r^C_{t+1})C_t.}$$

From Question (1), we know that $P_t = (P_{N,t})^{(1-\gamma)}$, so, we have:

$$1 + r^C_{t+1} = (1 + r_{t+1})\frac{P_t}{P_{t+1}} = (1 + r_{t+1})\Big(\frac{P_{N,t}}{P_{N,t+1}}\Big)^{1-\gamma}.$$

Since

$$C_{T,t} = \gamma P_tC_t,$$

it follows that

$$C_{T,t+1} = \gamma P_{t+1}C_{t+1} = \gamma P_{t+1}\beta_{H,t+1}(1 + r^C_{t+1})C_t.$$

As we note that

$$\beta_{H,t+1}(1 + r^C_{t+1}) = \beta_{H,t+1}(1 + r_{t+1})\frac{P_t}{P_{t+1}},$$

so simplifying, we obtain:

$$\boxed{C_{T,t+1} = \beta_{H,t+1}(1 + r_{t+1})C_{T,t}.}$$

Remark. The Foreign agent's optimization yields analogous Euler equations:

$$C^*_{t+1} = \beta_{F,t+1}(1 + r^*_{C,t+1})C^*_t, \quad C^*_{T,t+1} = \beta_{F,t+1}(1 + r_{t+1})C^*_{T,t},$$

with

$$1 + r^{*C}_{t+1} = (1 + r_{t+1})\frac{P^*_t}{P^*_{t+1}}.$$

Foreign Optimization

Similarly, for the Foreign country:

$$\boxed{C^*_{t+1} = \beta_{F,t+1}(1 + r^*_{C,t+1})C^*_t, \qquad C^*_{T,t+1} = \beta_{F,t+1}(1 + r_{t+1})C^*_{T,t},}$$

with

$$1 + r^{*C}_{t+1} = (1 + r_{t+1})\left(\frac{P^*_{N,t}}{P^*_{N,t+1}}\right)^{1-\gamma}.$$

Intuition: Households equate the marginal utility cost of consuming today versus tomorrow, with intertemporal decisions affected by relative price changes.

4 Labor Allocation

The production functions are given by:

$$Y_{T,t} = A_{T,t}(n - L_{N,t})^{1-\alpha}, \quad Y_{N,t} = A_{N,t}(L_{N,t})^{1-\alpha}.$$

Compute the marginal product of labor in each sector:

$$\frac{\partial Y_{T,t}}{\partial(n - L_{N,t})} = (1 - \alpha)A_{T,t}(n - L_{N,t})^{-\alpha},$$

$$\frac{\partial Y_{N,t}}{\partial L_{N,t}} = (1 - \alpha)A_{N,t}(L_{N,t})^{-\alpha}.$$

The Home agent allocates labor so that the marginal value product in the traded sector equals the marginal value product (adjusted by the non-traded price) in the non-traded sector:

$$(1 - \alpha)A_{T,t}(n - L_{N,t})^{-\alpha} = P_{N,t}(1 - \alpha)A_{N,t}(L_{N,t})^{-\alpha}.$$

Cancel the common factor $1-\alpha$ and rearrange:

$$\boxed{A_{T,t}(n-L_{N,t})^{-\alpha} = P_{N,t}A_{N,t}(L_{N,t})^{-\alpha}.}$$

The analogous condition for the Foreign country is:

$$\boxed{A^*_{T,t}\big((1-n)-L^*_{N,t}\big)^{-\alpha} = P^*_{N,t}A^*_{N,t}(L^*_{N,t})^{-\alpha}.}$$

5 Resource Constraints and the Real Exchange Rate

Resource Constraints

Recall the Home budget constraint:

$$nP_tC_t + nB_{t+1} = A_{T,t}(n-L_{N,t})^{1-\alpha} + P_{N,t}A_{N,t}(L_{N,t})^{1-\alpha} + n(1+r_t)B_t.$$

From Question 1, we have:

$$C_{N,t} = (1-\gamma)\frac{P_t}{P_{N,t}}C_t.$$

Since $P_t = (P_{N,t})^{1-\gamma}$, then:

$$C_{N,t} = (1-\gamma)(P_{N,t})^{-\gamma}C_t.$$

Given that non-traded goods are produced solely for domestic consumption, we also have the production identity (from Question 2):

$$n(1-\gamma)(P_{N,t})^{-\gamma}C_t = A_{N,t}(L_{N,t})^{1-\alpha}.$$

Thus, the expenditure on traded goods (which uses the consumption price index) plus net asset accumulation must equal traded output plus bond returns:

$$\boxed{n\gamma(P_{N,t})^{1-\gamma}C_t + nB_{t+1} = A_{T,t}(n-L_{N,t})^{1-\alpha} + n(1+r_t)B_t.}$$

Similarly, for Foreign we obtain:

$$\boxed{(1-n)\gamma(P^*_{N,t})^{1-\gamma}C^*_t - nB_{t+1} = A^*_{T,t}\big((1-n)-L^*_{N,t}\big)^{1-\alpha} - n(1+r_t)B_t.}$$

Then, we define the real exchange rate as the ratio of Foreign to Home consumption price indices:

$$Q_t \equiv \frac{P^*_t}{P_t}.$$

Since

$$P_t = (P_{N,t})^{1-\gamma} \quad \text{and} \quad P^*_t = (P^*_{N,t})^{1-\gamma},$$

we have:

$$\boxed{Q_t = \left(\frac{P^*_{N,t}}{P_{N,t}}\right)^{1-\gamma}.}$$

6 Steady State

In steady state, consumption is constant so that $C_{t+1} = C_t$. The Euler equation for the Home agent is

$$C_{t+1} = \beta_0(1 + r^C_{t+1})C_t.$$

But by definition the real return in consumption units is

$$1 + r^C_{t+1} = (1 + r_{t+1})\left(\frac{P_t}{P_{t+1}}\right)^{1-\gamma}.$$

In steady state prices do not change ($P_t = P_{t+1}$) so that

$$1 + r^C_{t+1} = 1 + r_{t+1} \quad \Rightarrow \quad C_t = \beta_0(1 + r_0)C_t.$$

Dividing by $C_t > 0$ yields:

$$1 = \beta_0(1 + r_0).$$

Step 2. Price Normalization

By convention we normalize the steady state price of non-tradables to one:

$$P_{N,0} = 1, \quad \text{and similarly} \quad P^*_{N,0} = 1.$$

Then the consumption price indices become

$$P_0 = (P_{N,0})^{1-\gamma} = 1, \quad P^*_0 = 1.$$

Step 3. Labor Allocation in the Home Country

The Home intratemporal labor allocation condition is:

$$A_{T,0}(n - L_{N,0})^{-\alpha} = P_{N,0}A_{N,0}(L_{N,0})^{-\alpha}.$$

Since $P_{N,0} = 1$, this simplifies to:

$$A_{T,0}(n - L_{N,0})^{-\alpha} = A_{N,0}(L_{N,0})^{-\alpha}.$$

Rearrange by dividing both sides by $A_{T,0}$ and by $(L_{N,0})^{-\alpha}$:

$$\left(\frac{n - L_{N,0}}{L_{N,0}}\right)^{-\alpha} = \frac{A_{N,0}}{A_{T,0}}.$$

Taking the reciprocal and then the $1/\alpha$-th root,

$$\frac{n - L_{N,0}}{L_{N,0}} = \left(\frac{A_{T,0}}{A_{N,0}}\right)^{1/\alpha}.$$

Now, using the calibration

$$A_{N,0} = A_{T,0}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha},$$

we have

$$\frac{A_{T,0}}{A_{N,0}} = \left(\frac{\gamma}{1-\gamma}\right)^{\alpha}.$$

Then,

$$\frac{n - L_{N,0}}{L_{N,0}} = \left[\left(\frac{\gamma}{1-\gamma}\right)^{\alpha}\right]^{1/\alpha} = \frac{\gamma}{1-\gamma}.$$

Thus,

$$\boxed{L_{N,0} = n(1-\gamma).}$$

By symmetry, the corresponding condition for Foreign is:

$$A^*_{T,0}\Big((1-n) - L^*_{N,0}\Big)^{-\alpha} = P^*_{N,0}A^*_{N,0}(L^*_{N,0})^{-\alpha}.$$

Since $P^*_{N,0} = 1$, the same steps lead to:

$$\frac{(1-n) - L^*_{N,0}}{L^*_{N,0}} = \frac{\gamma}{1-\gamma},$$

so that

$$\boxed{L^*_{N,0} = (1-n)(1-\gamma).}$$

We derive the steady-state consumption from the market clearing conditions for non-traded and traded goods. The non-traded goods clearing condition is

$$nC_{N,0} = A_{N,0}(L_{N,0})^{1-\alpha}.$$

From Question 1 we have

$$C_{N,0} = (1-\gamma)\frac{P_0}{P_{N,0}}C_0.$$

Since $P_0 = 1$ and $P_{N,0} = 1$, it follows that

$$C_{N,0} = (1-\gamma)C_0.$$

Substitute into the clearing condition:

$$n(1-\gamma)C_0 = A_{N,0}(L_{N,0})^{1-\alpha}.$$

Recall that $L_{N,0} = n(1-\gamma)$, so

$$n(1-\gamma)C_0 = A_{N,0}\big[n(1-\gamma)\big]^{1-\alpha}.$$

Solve for C_0:

$$C_0^N = A_{N,0}\big[n(1-\gamma)\big]^{-\alpha}.$$

We then derive from the traded goods clearing condition:

$$\begin{aligned}
& n\gamma(P_{N,t})^{1-\gamma}C_t + (1-n)\gamma(P_{N,t}^*)^{1-\gamma}C_t^* = A_{T,t}(n-L_{N,t})^{1-\alpha} + A_{T,t}^*\big((1-n)-L_{N,t}^*\big)^{1-\alpha} \\
\Rightarrow \quad & n\gamma C_0 + \frac{\gamma}{1-\gamma}A_{N,0}^*(L_{N,0}^*)^{1-\alpha} = A_{T,0}(n-L_{N,0})^{1-\alpha} + A_{T,0}^*(1-n-L_{N,0}^*)^{1-\alpha} \\
\Rightarrow \quad & n\gamma C_0 + \frac{\gamma}{1-\gamma}A_{N,0}^*(1-n)^{1-\alpha}(1-\gamma)^{1-\alpha} = A_{T,0}(n-n(1-\gamma))^{1-\alpha} + \\
& A_{T,0}^*(1-n-(1-n)(1-\gamma))^{1-\alpha} \\
\Rightarrow \quad & n\gamma C_0 + \frac{\gamma}{1-\gamma}A_{N,0}^*(1-n)^{1-\alpha}(1-\gamma)^{1-\alpha} = A_{T,0}(n\gamma)^{1-\alpha} + A_{T,0}^*\Big[(1-n)\gamma\Big]^{1-\alpha} \\
\Rightarrow \quad & n\gamma C_0 + \frac{\gamma}{1-\gamma}A_{T,0}\Big(\frac{1-n}{n}\Big)^{\alpha}\Big(\frac{1-\gamma}{\gamma}\Big)^{\alpha}(1-n)^{1-\alpha}(1-\gamma)^{1-\alpha} = A_{T,0}(n\gamma)^{1-\alpha} + \\
& A_{T,0}\Big(\frac{1-n}{n}\Big)^{\alpha}\Big[(1-n)\gamma\Big]^{1-\alpha} \\
\Rightarrow \quad & n\gamma C_0 + A_{T,0}(1-n)n^{-\alpha}\gamma^{1-\alpha} = A_{T,0}(n\gamma)^{1-\alpha} + A_{T,0}(1-n)n^{-\alpha}\gamma^{1-\alpha} \\
\Rightarrow \quad & C_0^T = A_{T,0}(n\gamma)^{-\alpha}.
\end{aligned}$$

We take a weighted geometric mean with weights γ and $1-\gamma$. That is,

$$C_0 = (C_0^N)^{1-\gamma}\cdot(C_0^T)^{\gamma},$$

so that

$$C_0 = \big[A_{N,0}n^{-\alpha}(1-\gamma)^{-\alpha}\big]^{1-\gamma}\,\big[A_{T,0}n^{-\alpha}\gamma^{-\alpha}\big]^{\gamma}.$$

We obtain:

$$\boxed{C_0 = (A_{T,0})^{\gamma}(A_{N,0})^{1-\gamma}\,\big[n\gamma^{\gamma}(1-\gamma)^{1-\gamma}\big]^{-\alpha}.}$$

A similar derivation for Foreign (noting that the population is $1-n$) gives:

$$\boxed{C_0^* = (A_{T,0}^*)^{\gamma}(A_{N,0}^*)^{1-\gamma}\,\big[(1-n)\gamma^{\gamma}(1-\gamma)^{1-\gamma}\big]^{-\alpha}.}$$

Because of the calibration (and the fact that the relative productivities satisfy

$$\frac{A_{N,0}}{A_{T,0}} = \left(\frac{1-\gamma}{\gamma}\right)^{\alpha} \quad \text{and} \quad \frac{A^*_{N,0}}{A^*_{T,0}} = \left(\frac{1-n}{n}\right)^{\alpha}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha}),$$

we can check that indeed

$$\frac{C_0}{C_0^*} = 1.$$

7 Log-Linear Approximation

We linearize the equilibrium conditions around the steady state. Denote for any variable x_t its deviation from steady state by

$$\hat{x}_t = \frac{x_t - x_0}{x_0}.$$

We also define cross-country differences later but for now we linearize the Home equations.

A. Non-Traded Goods Market

The Home non-traded goods market clearing condition is:

$$n(1-\gamma)(P_{N,t})^{-\gamma}C_t = A_{N,t}(L_{N,t})^{1-\alpha}.$$

Taking logarithms, we have

$$\ln n + \ln(1-\gamma) - \gamma \ln P_{N,t} + \ln C_t = \ln A_{N,t} + (1-\alpha)\ln L_{N,t}.$$

Linearizing around the steady state (and noting that constants vanish in the difference), we obtain:

$$\boxed{-\gamma\widehat{P_{N,t}} + \widehat{C}_t = \widehat{A_{N,t}} + (1-\alpha)\widehat{L_{N,t}}.} \tag{7a}$$

B. Resource Constraint

The Home resource constraint is:

$$n\gamma(P_{N,t})^{1-\gamma}C_t + nB_{t+1} = A_{T,t}(n - L_{N,t})^{1-\alpha} + n(1+r_t)B_t.$$

Divide both sides by $n\gamma C_0$, we get:

$$\frac{(P_{N,t})^{1-\gamma}C_t}{C_0} + \widehat{B_{t+1}} = \frac{A_{T,t}(n-L_{N,t})^{1-\alpha}}{n\gamma C_0} + (1+r_t)\widehat{B_t}.$$

Taking logs and linearizing, we have:

$$(1-\gamma)\widehat{P_{N,t}} + \widehat{C}_t + \widehat{B_{t+1}} = \widehat{A_{T,t}} - \frac{(1-\alpha)(L_{N,t} - L_{N,0})}{n - L_{N,0}}\widehat{L_{N,t}} + \frac{1}{\beta_0}\widehat{B}_t.$$

Since in steady state $n - L_{N,0} = n\gamma$, and that $\widehat{L_{N,t}} = L_{N,t} - L_{N,0}$. Thus,

$$\boxed{(1-\gamma)\widehat{P_{N,t}} + \widehat{C}_t + \widehat{B_{t+1}} = \widehat{A_{T,t}} - (1-\alpha)\frac{1-\gamma}{\gamma}\widehat{L_{N,t}} + \frac{1}{\beta_0}\widehat{B}_t.} \tag{7c}$$

C. Euler Equation

Recall that:

$$C_{t+1} = C_t\beta_{H,t+1}(1 + r_{t+1})\left(\frac{P_{N,t}}{P_{N,t+1}}\right)^{1-\gamma}$$

Taking logs and linearizing, we have:

$$\widehat{C_{t+1}} = \widehat{C}_t + \widehat{\beta_{H,t+1}} + \frac{r_{t+1} - r_0}{1 + r_0} - (1-\gamma)(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}).$$

As $\beta_0(1 + r_0) = 1$,

$$\begin{aligned}
\widehat{r_t} &= r_t - \frac{1-\beta_0}{\beta_0} \\
&= r_t - \frac{1 - \frac{1}{1+r_0}}{\frac{1}{1+r_0}} \\
&= r_t - r_0
\end{aligned}$$

So, the Euler equation becomes:

$$\boxed{\widehat{C_{t+1}} = \widehat{C}_t + \widehat{\beta_{H,t+1}} + \beta_0\widehat{r_{t+1}} - (1-\gamma)(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}).} \tag{7e}$$

D. Labor Allocation

The intratemporal condition is:

$$A_{T,t}(n - L_{N,t})^{-\alpha} = P_{N,t}A_{N,t}(L_{N,t})^{-\alpha}.$$

Taking logarithms:

$$\ln A_{T,t} - \alpha\ln(n - L_{N,t}) = \ln P_{N,t} + \ln A_{N,t} - \alpha\ln L_{N,t}.$$

Linearize around steady state:

$$\widehat{A_{T,t}} - \alpha\frac{L_{N,t} - L_{N,0}}{n - L_{N,0}} = \widehat{P_{N,t}} + \widehat{A_{N,t}} - \alpha\widehat{L_{N,t}}.$$

Using the fact that in steady state $n - L_{N,0} = n\gamma$:

$$\widehat{A_{T,t}} - \alpha \frac{\widehat{L_{N,t}} n(1-\gamma)}{n\gamma} = \widehat{P_{N,t}} + \widehat{A_{N,t}} - \alpha \widehat{L_{N,t}}.$$

$$\boxed{\widehat{A_{T,t}} + \frac{\alpha}{\gamma}\widehat{L_{N,t}} = \widehat{P_{N,t}} + \widehat{A_{N,t}}.} \tag{7g}$$

E. Real Exchange Rate

As we know:

$$1 + r^C_{t+1} = (1 + r_{t+1})\Big(\frac{P_{N,t}}{P_{N,t+1}}\Big)^{1-\gamma},$$

take logs and linearize both sides, we have:

$$\frac{r^C_{t+1} - r^C_0}{1 + r^C_0} = \frac{r_{t+1} - r_0}{1 + r_0} + (1-\gamma)(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}).$$

$$\Rightarrow \beta_0 \widehat{r^C_{t+1}} = \beta_0 \widehat{r_{t+1}} + (1-\gamma)(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}).$$

$$\boxed{\widehat{r^C_{t+1}} = \widehat{r_{t+1}} + (1-\gamma)\frac{1}{\beta_0}(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}).} \tag{7i}$$

8 Worldwide Equilibrium

Define world aggregates as population-weighted averages (e.g., $\widehat{C^W_t} = n\widehat{C}_t + (1-n)\widehat{C^*_t}$). Then, from the above log-linearized equations one can show:

- **Non-Traded Goods Market:**

$$n \times (7a) + (1-n) \times (7b) \Rightarrow -\gamma \widehat{P^W_{N,t}} + \widehat{C^W_t} = \widehat{A^W_{N,t}} + (1-\alpha)\widehat{L^W_{N,t}}. \tag{8a}$$

- **Resource Constraint:**

$$n \times (7c) + (1-n) \times (7d) \Rightarrow (1-\gamma)\widehat{P^W_{N,t}} + \widehat{C^W_t} = \widehat{A^W_{T,t}} - (1-\alpha)\frac{1-\gamma}{\gamma}\widehat{L^W_{N,t}}. \tag{8b}$$

- **Euler Equation:**

$$n \times (7e) + (1-n) \times (7f) \Rightarrow \widehat{C^W_{t+1}} = \widehat{C^W_t} + (1-\gamma)(\widehat{P^W_{N,t}} - \widehat{P^W_{N,t+1}}) + \widehat{\beta^W_{H,t+1}} + \beta_0 \widehat{r_{t+1}}. \tag{8c}$$

- **Labor Allocation:**

$$n \times (7g) + (1-n) \times (7h) \Rightarrow \widehat{A^W_{T,t}} + \frac{\alpha}{\gamma}\widehat{L^W_{N,t}} = \widehat{P^W_{N,t}} + \widehat{A^W_{N,t}}. \tag{8d}$$

- **Real Exchange Rate:**

$$n \times (7i) + (1-n) \times (7j) \Rightarrow \beta_0 \widehat{r_{t+1}^{CW}} = (1-\gamma)(\widehat{P_{N,t}^W} - \widehat{P_{N,t+1}^W}) + \beta_0 \widehat{r_{t+1}}. \tag{8e}$$

Let (8a)-(8d), we get:

$$(1-\gamma)\widehat{P_{N,t}^W} + \widehat{C_t^W} = \widehat{A_{T,t}^W} - (1-\alpha+\frac{\alpha}{\gamma})\widehat{L_{N,t}^W}.$$

Combine with (8b), we have:

$$\begin{aligned}
(1-\alpha+\frac{\alpha}{\gamma})\widehat{L_{N,t}^W} &= -(1-\alpha)\frac{1-\gamma}{\gamma}\widehat{L_{N,t}^W} \\
&= (1-\alpha)\widehat{L_{N,t}^W} - \frac{1-\alpha}{\gamma}\widehat{L_{N,t}^W} \\
\Rightarrow \frac{1}{\gamma}\widehat{L_{N,t}^W} &= 0.
\end{aligned}$$

Let (8b)-(8a), we have:

$$\begin{aligned}
\widehat{P_{N,t}^W} &= \widehat{A_{T,t}^W} - \widehat{A_{N,t}^W} - (1-\alpha)\Big(\frac{1-\gamma}{\gamma} + 1\Big)\widehat{L_{N,t}^W} \\
&= \widehat{A_{T,t}^W} - \widehat{A_{N,t}^W}.
\end{aligned}$$

Take $(1-\gamma) \times$ (8a) $+ \gamma$(8b), we have:

$$(1-\gamma)\widehat{C_t^W} + \gamma\widehat{C_t^W} = \widehat{C_t^W} = (1-\gamma)\widehat{A_{N,t}^W} + \gamma\widehat{A_{T,t}^W}.$$

Finally, from (8c), we know that:

$$\begin{aligned}
\beta_0\widehat{r_{t+1}} + \widehat{\beta_{H,t+1}^W} &= \widehat{C_{t+1}^W} - \widehat{C_t^W} - (1-\gamma)\Big(\widehat{P_{N,t}^W} - \widehat{P_{N,t+1}^W}\Big) \\
&= \gamma\hat{A}_{t+1}^W + (1-\gamma)\widehat{A_{N,t+1}^W} - \gamma\widehat{A_{T,t+1}^W} - (1-\gamma)\widehat{A_{N,t+1}^W} \\
&- (1-\gamma)\Big(\widehat{A_{T,t}^W} - \widehat{A_{N,t}^W}\Big) + (1-\gamma)\Big(\widehat{A_{T,t+1}^W} - \widehat{A_{N,t+1}^W}\Big) \\
&= \widehat{A_{T,t+1}^W} - \widehat{A_{T,t}^W}.
\end{aligned}$$

Intuition: World aggregates respond only to symmetric shocks, with the real interest rate driven by global productivity changes and aggregate patience.

9 Cross-Country Differences

As we know that $Q_t = \left(\frac{P_{N,t}^*}{P_{N,t}}\right)^{1-\gamma}$, log-linearize the equation, we have:

$$\widehat{Q_t} = (1-\gamma)(\widehat{P_{N,t}^*}) - \widehat{P_{N,t}}.$$

Use (7a) - (7b), we get:

$$\begin{aligned}
&\widehat{C}_t - \widehat{C}^*_t - \gamma(\widehat{P_{N,t}} - \widehat{P^*_{N,t}}) = (\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) + (1-\alpha)(\widehat{L_{N,t}} - \widehat{L^*_{N,t}}) \\
&\quad \Rightarrow \widehat{C}_t - \widehat{C}^*_t + \frac{\gamma}{1-\gamma}\widehat{Q}_t = (\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) + (1-\alpha)(\widehat{L_{N,t}} - \widehat{L^*_{N,t}}). \qquad (9a)
\end{aligned}$$

Use (7c) - (7d), we get:

$$\begin{aligned}
&\text{LHS} = (1-\gamma)(\widehat{P_{N,t}} - \widehat{P^*_{N,t}}) + \widehat{C}_t - \widehat{C}^*_t + \frac{\widehat{B_{t+1}}}{1-n} \\
&\text{RHS} = (\widehat{A_{T,t}} - \widehat{A^*_{T,t}}) - (1-\alpha)\frac{1-\gamma}{\gamma}(\widehat{L_{N,t}} - \widehat{L^*_{N,t}}) + \frac{1}{\beta_0}\frac{\widehat{B_t}}{1-n} \\
&\text{As } \widehat{Q}_t = (1-\gamma)(\widehat{P^*_{N,t}} - \widehat{P_{N,t}}), \text{ we have} \\
&\text{LHS} = -\widehat{Q}_t + (\widehat{C}_t - \widehat{C}^*_t) + \frac{\widehat{B_{t+1}}}{1-n} = \text{RHS} \qquad (9b)
\end{aligned}$$

Use (7e) - (7f), we get:

$$\begin{aligned}
(\widehat{C_{t+1}} - \widehat{C^*_{t+1}}) &= (\widehat{C}_t - \widehat{C}^*_t) + (1-\gamma)(\widehat{P_{N,t}} - \widehat{P^*_{N,t}}) + (1-\gamma)(\widehat{P^*_{N,t+1}} - \widehat{P_{N,t+1}}) + (\widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}}) \\
&= (\widehat{C}_t - \widehat{C}^*_t) - \widehat{Q}_t + \widehat{Q_{t+1}} + \widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}}. \qquad (9c)
\end{aligned}$$

Use (7g) - (7h), we get:

$$\begin{aligned}
(\widehat{A_{T,t}} - \widehat{A^*_{T,t}}) + \frac{\alpha}{\gamma}(\widehat{L_{N,t}} - \widehat{L^*_{N,t}}) &= (\widehat{P_{N,t}} - \widehat{P^*_{N,t}}) + (\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) \\
&= -\frac{1}{1-\gamma}\widehat{Q}_t + (\widehat{A_{N,t}} - \widehat{A^*_{N,t}}). \qquad (9d)
\end{aligned}$$

Use (7i) - (7j), we get:

$$\begin{aligned}
\beta_0(\widehat{r^C_{t+1}} - \widehat{r^{C*}_{t+1}}) &= (1-\gamma)(\widehat{P_{N,t}} - \widehat{P^*_{N,t}}) + (1-\gamma)(\widehat{P^*_{N,t}} - \widehat{P_{N,t}}) \\
&= \widehat{Q_{t+1}} - \widehat{Q}_t. \qquad (9e)
\end{aligned}$$

Intuition: These equations link cross-country differences in consumption, labor, and the real exchange rate to differences in productivity and intertemporal preferences.

10 Long-Run Allocation (Period $t + 1$)

Assume that from $t + 1$ onward the economy reaches a new steady state with no further discount factor shocks ($\widehat{\beta_{H,t+2}} = \widehat{\beta_{F,t+2}} = 0$). In the steady state, the consumption growth rate is zero, the asset position is fixed and the real exchange rate is stable.

Using the labor allocation equation at $t+1$, we have:

$$\frac{\alpha}{\gamma}(\widehat{L_{N,t}} - \widehat{L^*_{N,t}}) = -\frac{1}{1-\gamma}\widehat{Q_{t+1}} + \Big[(\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) - (\widehat{A_{T,t}} - \widehat{A^*_{T,t}})\Big]. \tag{10.1}$$

Then, we use the market clearing condition for non-traded goods and the resource allocation constraints at $t+1$:

$$\frac{\gamma}{1-\gamma}\widehat{Q_{t+1}} + (\widehat{C_{t+1}} - \widehat{C^*_{t+1}}) = (\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) + (1-\alpha)(\widehat{L_{N,t+1}} - \widehat{L^*_{N,t+1}}) \tag{10.2}$$

$$-\widehat{Q_{t+1}} + (\widehat{C_{t+1}} - \widehat{C_{t+1}}^*) + \frac{\widehat{B_{t+2}}}{1-n} = (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) - (1-\alpha)\frac{1-\gamma}{\gamma}(\widehat{L_{N,t+1}} - \widehat{L^*_{N,t+1}}) + \frac{1}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}. \tag{10.3}$$

As $\widehat{B_{t+2}} = \widehat{B_{t+1}}$, using (10.2)-(10.3), we get:

$$\begin{aligned}
\Big(\frac{1-\gamma}{\gamma}+1\Big)\widehat{Q_{t+1}} &= \Big[(\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}})\Big] \\
&\quad + (1-\alpha)\Big(1+\frac{1-\gamma}{\gamma}\Big)(\widehat{L_{N,t+1}} - \widehat{L^*_{N,t+1}}) - \frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
\Rightarrow \quad \widehat{Q_{t+1}} &= (1-\gamma)\Big[(\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}})\Big] + (1-\alpha)\frac{1-\gamma}{\gamma}(\widehat{L_{N,t+1}} - \widehat{L^*_{N,t+1}}) \\
&\quad - (1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}.
\end{aligned} \tag{10.4}$$

Replacing $\widehat{L_{N,t+1}} - \widehat{L^*_{N,t+1}}$ using (10.1), we get:

$$\begin{aligned}
\widehat{Q_{t+1}} &= (1-\gamma)\Big[(\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}})\Big] - \frac{1-\alpha}{\alpha}\widehat{Q_{t+1}} \\
&\quad + \frac{1-\alpha}{\alpha}(1-\gamma)\Big[(\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}})\Big] - (1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
\Rightarrow \quad \Big(1+\frac{1-\alpha}{\alpha}\Big)\widehat{Q_{t+1}} &= (1-\gamma)\Big(1+\frac{1-\alpha}{\alpha}\Big)\Big[(\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}})\Big] \\
&\quad - (1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
\Rightarrow \quad \widehat{Q_{t+1}} &= -(1-\gamma)\Big[(\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) - (\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}})\Big] - \alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}.
\end{aligned} \tag{10a}$$

Comparing (10.4) and (10a), we have:

$$\begin{aligned}
&(1-\alpha)\frac{1-\gamma}{\gamma}(\widehat{L_{N,t+1}} - \widehat{L^*_{N,t+1}}) - (1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} = -\alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
\Rightarrow \quad &\widehat{L_{N,t+1}} - \widehat{L^*_{N,t+1}} = \gamma\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}.
\end{aligned} \tag{10b}$$

Implementing (10a) and (10b) back into (10.2), we get:

$$\begin{aligned}
\widehat{C_{t+1}} - \widehat{C^*_{t+1}} &= \widehat{Q_{t+1}} + (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) - \frac{(1-\alpha)(1-\gamma)}{\gamma}\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} + \frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
&= -(1-\gamma)(\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) + (1-\gamma)(\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) + (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) \\
&\quad - \alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} - (1-\alpha)(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} + \frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
&= \gamma(\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) + (1-\gamma)(\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) + \gamma\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \qquad \text{(10c)}
\end{aligned}$$

Interpretation:

- A positive $\widehat{B_{t+1}}$ (Home wealthier) implies higher relative consumption and a lower Q_{t+1} (Home's goods become relatively more expensive).
- Permanent productivity differences affect steady state consumption and prices directly.

11 Short-Run Allocation (Period t)

Assume initially $\widehat{B_t} = 0$. Solving the system (starting with the labor and non-traded market conditions, then using the resource constraint and Euler equations) yields:

$$\frac{\widehat{B_{t+1}}}{1-n} = \frac{\beta_0}{\gamma+\alpha(1-\gamma)}\Big[(\widehat{\beta_{H,t+1}} - \hat{\beta}_{F,t+1}) - (\hat{A}_{T,t+1} - \hat{A}^*_{T,t+1}) + (\widehat{A_{T,t}} - \hat{A}^*_{T,t})\Big], \qquad \text{(11a)}$$

$$\tilde{C}_t = \gamma(\widehat{A_{T,t}} - \hat{A}^*_{T,t}) + (1-\gamma)(\widehat{A_{N,t}} - \hat{A}^*_{N,t}) - \frac{\gamma\beta_0}{\gamma+\alpha(1-\gamma)}\Big[(\widehat{\beta_{H,t+1}} - \hat{\beta}_{F,t+1}) - (\hat{A}_{T,t+1} - \hat{A}^*_{T,t+1}) + (\widehat{A_{T,t}} - \hat{A}^*_{T,t})\Big], \qquad \text{(11b)}$$

$$\hat{Q}_t = -(1-\gamma)(\widehat{A_{T,t}} - \hat{A}^*_{T,t}) + (1-\gamma)(\widehat{A_{N,t}} - \hat{A}^*_{N,t}) + \frac{(1-\gamma)\alpha\beta_0}{\gamma+\alpha(1-\gamma)}\Big[(\widehat{\beta_{H,t+1}} - \hat{\beta}_{F,t+1}) - (\hat{A}_{T,t+1} - \hat{A}^*_{T,t+1}) + (\widehat{A_{T,t}} - \qquad \text{(11c)}$$

$$\tilde{L}_{N,t} = -\frac{\gamma\beta_0}{\gamma+\alpha(1-\gamma)}\Big[(\widehat{\beta_{H,t+1}} - \hat{\beta}_{F,t+1}) - (\hat{A}_{T,t+1} - \hat{A}^*_{T,t+1}) + (\widehat{A_{T,t}} - \hat{A}^*_{T,t})\Big]. \qquad \text{(11d)}$$

Interpretation:

- A temporary increase in Home patience (i.e. $\widehat{\beta_{H,t+1}} - \hat{\beta}_{F,t+1} > 0$) leads to $\widehat{B_{t+1}} > 0$ (Home runs a current account surplus), lower relative consumption, and a real exchange rate movement consistent with a trade surplus.
- Temporary traded or non-traded productivity shocks affect the current account and labor allocation differently.
- For permanent shocks ($\widehat{A_{T,t}} = \hat{A}_{T,t+1}$), intertemporal balance is restored with $\widehat{B_{t+1}} = 0$ and immediate adjustment to the new steady state.

12 Summary of Key Economic Insights

- **Consumption and Prices:** The structure of the consumption basket implies that a rise in the non-traded good price $P_{N,t}$ increases the overall consumption price P_t and shifts the consumption mix.
- **Market Clearing:** Non-traded goods are produced and consumed domestically, whereas traded goods are allocated internationally, linking domestic production choices to the real exchange rate.
- **Intertemporal Choices:** The Euler equations show that higher patience or higher real returns lead to deferred consumption. Cross-country differences in patience drive current account imbalances.
- **Labor Allocation:** Labor is reallocated across sectors until the marginal value products are equalized; productivity shifts affect both output composition and relative prices.
- **Steady State and Log-Linearization:** In a symmetric steady state, relative prices and allocations are balanced. Log-linearization permits analysis of small shocks and their propagation.
- **Short-Run vs. Long-Run Dynamics:** Temporary shocks generate current account imbalances and short-run reallocation, while permanent shocks adjust consumption and prices directly with no asset accumulation.
- **Wealth Effects:** A positive net asset position (Home wealthier) implies higher steady-state consumption and a relatively stronger (appreciated) currency.