Macroeconomics A: Review Session IV

Solow Growth Model

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Outline

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Solow Growth Model

- Solow growth model has a simple premise

$$\dot{k}(t) = s_k y(t) - (\delta + n + g)k(t)$$

- When $\dot{k} = 0$ (steady state), the share of output going to capital is equal to capital depreciation plus the growth rate of effective labor
- Given that $y(t) = (k(t))^\alpha$

$$k^* = \left(\frac{s_k}{\delta + n + g}\right)^{\frac{1}{1-\alpha}}$$

- This level of capital maximizes consumption
- As we will see, no need for utility function to pin equilibrium s_k down

Finding the Capital Law of Motion

- To find $\dot{k}(t)$ remember it is scaled capital

$$k(t) = \frac{K(t)}{A(t)L(t)}$$

- Taking the derivative with respect to time gives

$$\frac{dk(t)}{dt} = \frac{1}{A(t)L(t)}\frac{\partial K_t}{\partial t} - \underbrace{\frac{K(t)}{A(t)(L(t))^2}\frac{\partial L(t)}{\partial t}}_{k(t)\frac{1}{L(t)}\frac{\partial L(t)}{\partial t}} - \underbrace{\frac{K(t)}{(A(t))^2L(t)}\frac{\partial A_t}{\partial t}}_{k(t)\frac{1}{A(t)}\frac{\partial A(t)}{\partial t}}$$

- We assume that labor and productivity have constant growth rates

$$\frac{1}{L(t)}\frac{\partial L(t)}{\partial t} = \frac{\dot{L}(t)}{L(t)} = n \qquad \frac{1}{A(t)}\frac{\partial A(t)}{\partial t} = \frac{\dot{A}(t)}{A(t)} = g$$

Derivation

- Combining the result on the previous slide

$$\dot{k}(t) = \frac{\dot{K}(t)}{A(t)L(t)} - (n+g)k(t)$$

- By definition, the change in capital is equal to total saving less depreciation

$$\dot{K}(t) = s_K Y(t) - \delta K(t)$$

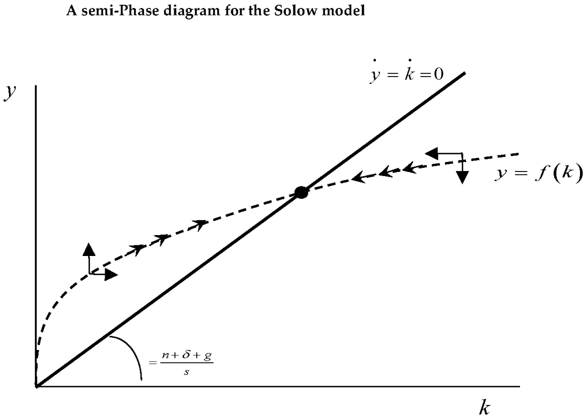
- Using this identity

$$\dot{k}(t) = \frac{s_K Y(t) - \delta K(t)}{A(t)L(t)} - (n+g)k(t)$$

$$\implies \dot{k}(t) = s_K y(t) - (\delta + n + g)k(t)$$

Saddle Path

Figure: Transition Dynamics in the Solow Growth Model



Off Equilibrium

- Almost by definition, a higher saving rate should increase k^* and y^*

$$\frac{\partial k^*}{\partial s_k} = \frac{1}{(1-\alpha)(\delta+n+g)}\left(\frac{s_k^*}{\delta+n+g}\right)^{\frac{1}{1-\alpha}-1} = \frac{1}{1-\alpha}\frac{k^*}{s_k} > 0$$

- However, this does not mean consumption increases

$$c^* = (k^*)^\alpha - (\delta+n+g)k^*$$

Exercise: find $\frac{\partial c^*}{\partial s_k}$. Under what condition is c^* maximized?

Hint: recall that $k^* = \left(\frac{s_k}{\delta+n+g}\right)^{\frac{1}{1-\alpha}}$

Off Equilibrium

Exercise: find $\frac{\partial c^*}{\partial s_k}$. Under what condition is c^* maximized?

$$\frac{\partial c^*}{\partial s_k} = \alpha (k^*)^{\alpha-1}\frac{\partial k^*}{\partial s_k} - (\delta + n + g)\frac{\partial k^*}{\partial s_k}$$
$$= \frac{\alpha}{1-\alpha}\frac{(k^*)^\alpha}{s_k} - \frac{1}{1-\alpha}\left(\frac{s_k}{\delta+n+g}\right)^{\frac{\alpha}{1-\alpha}}$$

Using the definition of k^*

$$\frac{\partial c^*}{\partial s_k} = \frac{\alpha}{1-\alpha}\frac{(k^*)^\alpha}{s_k} - \frac{1}{1-\alpha}(k^*)^\alpha$$
$$= \left(\frac{\alpha}{s_k} - 1\right)\frac{(k^*)^\alpha}{1-\alpha}$$

Finally, note that

$$s_k = \alpha \quad \implies \quad \frac{\partial c^*}{\partial s_k} = 0$$

Finding the Speed of Convergence

- We know the law of motion for capital and would like to make it an explicit function of time

$$\dot{k}(t) = s_k k(t)^{\alpha} - (\delta + n + g)k(t)$$

- Having the path of capital allows us to solve output at t

$$\frac{\dot{y}(t)}{y(t)} = \alpha \frac{\dot{k}(t)}{k(t)}$$

- One option is a 1st order Taylor approximation around $k(t) = k^*$

$$\begin{aligned}\dot{k}(t) &\approx \left[\alpha s_k (k^*)^{\alpha-1} - (\delta + n + g)\right](k(t) - k^*) \\ &\approx (\alpha - 1)(\delta + n + g)(k(t) - k^*)\end{aligned}$$

Nonhomogenous Differential Equation

- When $\dot{x} - ax(t) = b$, we can write

$$x(t) = -\frac{b}{a} + \left(x(0) + \frac{b}{a}\right) e^{at}$$

- The Taylor approximation on the previous slide gives

$$\dot{k}(t) - \underbrace{(\alpha - 1)(\delta + n + g)}_{a} k(t) = \underbrace{(1 - \alpha)(\delta + n + g)k^*}_{b}$$

- So we can write

$$k(t) = -\frac{b}{a} + \left(k(0) + \frac{b}{a}\right) e^{at}$$

The Effect of α on Convergence

- We can now write $k(t)$ as an explicit function of t

$$k(t) - k^* = (k(0) - k^*) \exp[(\alpha - 1)(\delta + n + g)t]$$

- Taking some interval $c = \frac{k(0)-k^*}{k(t)-k^*} > 1$ (assuming $k(t) \to k^*$)

$$\frac{1}{c} = \exp[(\alpha - 1)(\delta + n + g)t]$$

$$t = \frac{\log(c)}{(1-\alpha)(\delta + n + g)}$$

- Note that t is increasing in α and decreasing in the other parameters
- Is this intuitive? What does α represent?

Excel model is here

Lack of Convergence?

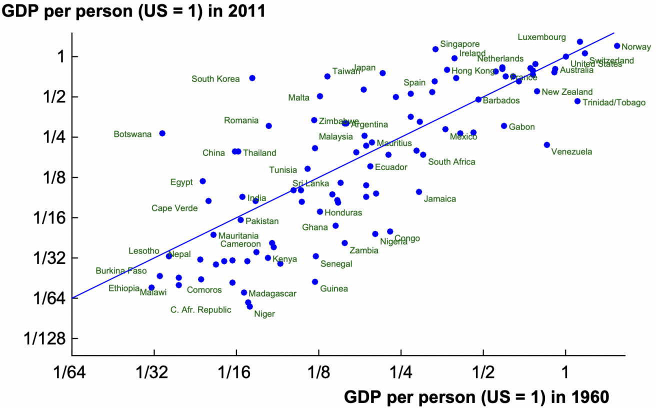


Fig. 24 GDP per person, 1960 and 2011. Source: *The Penn World Tables 8.0.*

From C.I. Jones excellent overview The Facts of Economic Growth

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Human Capital and Speed of Convergence

- We can add a human capital stock to the production function

$$y(t) = k(t)^{\alpha} h(t)^{\beta}$$

- The law of motion for $k(t)$ and $h(t)$ are similar

$$\dot{k}(t) = s_k y(t) - (\delta + n + g) k(t)$$

$$\dot{h}(t) = s_h y(t) - (\delta + n + g) h(t)$$

- The growth rate of y is given by $\partial y / \partial t$

$$\frac{\dot{y}(t)}{y(t)} = \alpha \frac{\dot{k}(t)}{k(t)} + \beta \frac{\dot{h}(t)}{h(t)} \tag{1}$$

- To find the speed of convergence, we need to find $\frac{\dot{k}(t)}{k(t)}$ and $\frac{\dot{h}(t)}{h(t)}$

Taylor Approximation Around Equilibrium

- Let's take a Taylor approximation around $k(t) = k^*$ and $h(t) = h^*$
- For $\dot{k}(t)/k(t)$

$$
\begin{aligned}
f(k(t), h(t)) &= f(k^*, h^*) + f_k(k^*, h^*)(k(t) - k^*) + f_h(k^*, h^*)(h(t) - h^*) \\
&= (\alpha - 1)s_k \frac{y^*}{(k^*)^2}(k(t) - k^*) + \beta s_k \frac{y^*}{k^* h^*}(h(t) - h^*) \\
&= (\delta + n + g)\left[(\alpha - 1)\frac{k(t) - k^*}{k^*} + \beta \frac{h(t) - h^*}{h^*}\right]
\end{aligned}
$$

- For $\dot{h}(t)/h(t)$

$$
\begin{aligned}
f(k(t), h(t)) &= f(k^*, h^*) + f_k(k^*, h^*)(k(t) - k^*) + f_h(k^*, h^*)(h(t) - h^*) \\
&= \alpha s_h \frac{y^*}{h^* k^*}(k(t) - k^*) + (1 - \beta)s_h \frac{y^*}{(h^*)^2}(h(t) - h^*) \\
&= (\delta + n + g)\left[\alpha \frac{k(t) - k^*}{k^*} + (\beta - 1)\frac{h(t) - h^*}{h^*}\right]
\end{aligned}
$$

Taylor Approximation of Output

- The notation is getting heavy, so let's specify a convenience variable
$$\theta(t) = \left[\alpha \frac{k(t) - k^*}{k^*} + \beta \frac{h(t) - h^*}{h^*}\right]$$
- Going back to equation 1, we can now write this as
$$\frac{\dot{y}(t)}{y^*} \approx (\alpha + \beta - 1)(\delta + n + g)\theta(t)$$
- A Taylor approximation of $y(t)$ at $y(t) = y^*$ gives
$$y(t) \approx y^* + \alpha y^* \frac{k(t) - k^*}{k^*} + \beta y^* \frac{h(t) - h^*}{h^*}$$
$$\implies \frac{y(t) - y^*}{y^*} = \theta(t)$$

Using a Differential Equation

- The law of motion for $y(t)$ is now given by

$$\dot{y}(t) \approx (\alpha + \beta - 1)(\delta + n + g)(y(t) - y^*)$$

- Using the same solution as before

$$y(t) - y^* = (y(0) - y^*) \exp[-(1 - \alpha - \beta)(\delta + n + g)t]$$

- Taking some interval $c = \frac{y(0) - y^*}{y(t) - y^*}$

$$\frac{1}{c} = \exp[-(1 - \alpha - \beta)(\delta + n + g)t]$$

$$t = \frac{\log(c)}{(1 - \alpha - \beta)(\delta + n + g)}$$

- Note that t is increasing in both α and β, decreasing in the other parameters