PS1 Solutions

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1 Consumption Allocation

Problem Setup

The Home agent's consumption basket is given by

$$C_t = \left(\frac{C_{T,t}}{\gamma}\right)^{\gamma}\left(\frac{C_{N,t}}{1-\gamma}\right)^{1-\gamma},$$

where:

- $C_{T,t}$ is the quantity of the traded good (its price is normalized to 1),
- $C_{N,t}$ is the quantity of the domestic non-traded good (price $P_{N,t}$),
- γ is the expenditure share on the traded good.

The consumer minimizes total expenditure subject to attaining a given consumption level C_t. The problem is

$$\begin{aligned}
\min_{C_{T,t},C_{N,t}} \; & P_t C_t = C_{T,t} + P_{N,t}C_{N,t} \\
\text{s.t.} \quad & C_t = \left(\frac{C_{T,t}}{\gamma}\right)^{\gamma}\left(\frac{C_{N,t}}{1-\gamma}\right)^{1-\gamma}.
\end{aligned}$$

Define the Lagrangian function:

$$\mathcal{L} = C_{T,t} + P_{N,t}C_{N,t} + \lambda\Big(C_t - \Big(\frac{C_{T,t}}{\gamma}\Big)^{\gamma}\Big(\frac{C_{N,t}}{1-\gamma}\Big)^{1-\gamma}\Big).$$

The FOCs with respect to $C_{T,t}$ and $C_{N,t}$ are:

$$\begin{aligned}
\mathcal{L}_{C_{T,t}} &= 1 - \lambda\gamma\Big(\frac{C_{T,t}}{\gamma}\Big)^{\gamma-1}\frac{1}{\gamma}\Big(\frac{C_{N,t}}{1-\gamma}\Big)^{1-\gamma} = 0, \\
\mathcal{L}_{C_{N,t}} &= P_{N,t} - \lambda(1-\gamma)\Big(\frac{C_{T,t}}{\gamma}\Big)^{\gamma}\frac{1}{1-\gamma}\Big(\frac{C_{N,t}}{1-\gamma}\Big)^{-\gamma} = 0 \\
\Rightarrow \quad \frac{1}{P_{N,t}} &= \frac{\gamma}{1-\gamma}\frac{C_{N,t}}{C_{T,t}}.
\end{aligned}$$

The dual (expenditure) minimization problem yields the unit cost function (composite price index) for the consumption bundle:

$$P_t C_t = \min \left\{ C_{T,t} + P_{N,t} C_{N,t} : \ C_t = \left(\frac{C_{T,t}}{\gamma} \right)^{\gamma} \left(\frac{C_{N,t}}{1-\gamma} \right)^{1-\gamma} \right\}.$$

So, we have:

$$\begin{aligned} P_t C_t &= C_{T,t} + P_{N,t} C_{N,t} \\ &= C_{T,t} + \frac{1-\gamma}{\gamma} C_{T,t} \\ \Rightarrow C_{T,t} &= \gamma P_t C_t && \text{(1a)} \\ \Rightarrow C_{N,t} &= \frac{1-\gamma}{\gamma} \frac{C_{T,t}}{P_{N,t}} \\ &= (1-\gamma) P_t C_t. && \text{(1b)} \end{aligned}$$

$$\begin{aligned} & \left(\frac{C_{T,t}}{\gamma} \right)^{\gamma} \left(\frac{C_{N,t}}{1-\gamma} \right)^{1-\gamma} = C_t \\ \Rightarrow \quad & (P_t C_t)^{\gamma} \left(\frac{P_t C_t}{P_N, t} \right)^{1-\gamma} = C_t \\ \Rightarrow \quad & P_t = (P_{N,t})^{1-\gamma}. && \text{(1c)} \end{aligned}$$

Analogously, for the Foreign agent, we have

$$\begin{aligned} C^*_{T,t} &= \gamma P^*_t C^*_t && \text{(1d)} \\ C^*_{N,t} &= (1-\gamma) P^*_t C^*_t && \text{(1e)} \\ P^*_t &= (P^*_{N,t})^{1-\gamma}. && \text{(1f)} \end{aligned}$$

2 Market Clearing

Under market clearing, the quantities of traded and non-traded goods produced must equal the quantities consumed. We have:

Non-Traded Goods Market

For Home, market clearing in non-traded goods is:

$$n C_{N,t} = A_{N,t} (L_{N,t})^{1-\alpha}.$$

Substituting $C_{N,t} = (1-\gamma) \frac{P_t C_t}{P_{N,t}}$ with $P_t = (P_{N,t})^{1-\gamma}$, we obtain:

$$n(1-\gamma)(P_{N,t})^{-\gamma} C_t = A_{N,t} (L_{N,t})^{1-\alpha}. \tag{2a}$$

For Foreign, the market clearing condition is: $(1-n)C^*_{N,t} = A^*_{N,t}(L^*_{N,t})^{1-\alpha}$. Following the same method, we have:

$$(1-n)(1-\gamma)(P^*_{N,t})^{-\gamma}C^*_t = A^*_{N,t}(L^*_{N,t})^{1-\alpha}. \tag{2b}$$

Traded Goods Market

Global market clearing for traded goods is:

$$nC_{T,t} + (1-n)C^*_{T,t} = A_{T,t}(n-L_{N,t})^{1-\alpha} + A^*_{T,t}\big((1-n) - L^*_{N,t}\big)^{1-\alpha}.$$

Substituting $C_{T,t} = \gamma P_t C_t$ with $P_t = (P_{N,t})^{1-\gamma}$ (and similarly for Foreign), we have:

$$n\gamma(P_{N,t})^{1-\gamma}C_t + (1-n)\gamma(P^*_{N,t})^{1-\gamma}C^*_t = A_{T,t}(n-L_{N,t})^{1-\alpha} + A^*_{T,t}\big((1-n) - L^*_{N,t}\big)^{1-\alpha}. \tag{2c}$$

3 Intertemporal Allocation

Home Optimization

The Home agent maximizes:

$$U_t = \sum_{s=0}^{\infty}(\beta_{H,t+s})^s \ln C_{t+s},$$

subject to the period-t budget constraint:

$$nP_tC_t + nB_{t+1} = A_{T,t}(n-L_{N,t})^{1-\alpha} + P_{N,t}A_{N,t}(L_{N,t})^{1-\alpha} + n(1+r_t)B_t.$$

Focusing on a single period (the infinite-horizon structure is standard), we write:

$$\mathcal{L}_t = \sum_{s=0}^{\infty}(\beta_{H,t+s})^s \ln C_{t+s} + \lambda_t\Big[A_{T,t}(n-L_{N,t})^{1-\alpha} + P_{N,t}A_{N,t}(L_{N,t})^{1-\alpha} + n(1+r_t)B_t - nP_tC_t - nB_{t+1}\Big].$$

Take the FOCs:

$$\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial C_t} &= \frac{1}{C_t} - \lambda_t n P_t = 0 \quad \Rightarrow \quad \lambda_t = \frac{1}{nP_tC_t} \\
\frac{\partial \mathcal{L}_t}{C_{t+1}} &= \frac{\beta_{H,t+1}}{C_{t+1}} - nP_{t+1}\lambda_{t+1} = 0 \\
\frac{\partial \mathcal{L}_t}{\partial B_{t+1}} &= -n\lambda_t + n(1+r_{t+1})\lambda_{t+1} = 0.
\end{aligned}$$

Substitute the expressions for λ_t and λ_{t+1}:

$$\frac{1}{nP_tC_t} = \beta_{H,t+1}(1+r_{t+1})\frac{1}{nP_{t+1}C_{t+1}}.$$

Cancel n and rearrange:

$$\frac{1}{C_t} = \beta_{H,t+1}(1+r_{t+1})\frac{P_t}{P_{t+1}}\frac{1}{C_{t+1}}.$$

Set:

$$1 + r^C_{t+1} \equiv (1 + r_{t+1})\frac{P_t}{P_{t+1}},$$

so the Euler equation becomes:

$$C_{t+1} = \beta_{H,t+1}(1 + r^C_{t+1})C_t. \tag{3a}$$

From Question (1), we know that $P_t = (P_{N,t})^{(1-\gamma)}$, so, we have:

$$1 + r^C_{t+1} = (1 + r_{t+1})\frac{P_t}{P_{t+1}} = (1 + r_{t+1})\Big(\frac{P_{N,t}}{P_{N,t+1}}\Big)^{1-\gamma}. \tag{3b}$$

Since

$$C_{T,t} = \gamma P_t C_t,$$

it follows that

$$C_{T,t+1} = \gamma P_{t+1}C_{t+1} = \gamma P_{t+1}\beta_{H,t+1}(1 + r^C_{t+1})C_t.$$

As we note that

$$\beta_{H,t+1}(1 + r^C_{t+1}) = \beta_{H,t+1}(1 + r_{t+1})\frac{P_t}{P_{t+1}},$$

so simplifying, we obtain:

$$C_{T,t+1} = \beta_{H,t+1}(1 + r_{t+1})C_{T,t}. \tag{3c}$$

Remark. The Foreign agent's optimization yields analogous Euler equations:

$$C^*_{t+1} = \beta_{F,t+1}(1 + r^*_{C,t+1})C^*_t, \quad C^*_{T,t+1} = \beta_{F,t+1}(1 + r_{t+1})C^*_{T,t},$$

with

$$1 + r^{*C}_{t+1} = (1 + r_{t+1})\frac{P^*_t}{P^*_{t+1}}.$$

Foreign Optimization

Similarly, for the Foreign country:

$$C^*_{t+1} = \beta_{F,t+1}(1 + r^*_{C,t+1})C^*_t \tag{3d}$$

$$C^*_{T,t+1} = \beta_{F,t+1}(1 + r_{t+1})C^*_{T,t} \tag{3e}$$

with

$$1 + r^{*C}_{t+1} = (1 + r_{t+1})\left(\frac{P^*_{N,t}}{P^*_{N,t+1}}\right)^{1-\gamma}. \tag{3f}$$

4 Labor Allocation

The production functions are given by:

$$Y_{T,t} = A_{T,t}(n - L_{N,t})^{1-\alpha}, \quad Y_{N,t} = A_{N,t}(L_{N,t})^{1-\alpha}.$$

Compute the marginal product of labor in each sector:

$$\frac{\partial Y_{T,t}}{\partial(n-L_{N,t})} = (1-\alpha)A_{T,t}(n-L_{N,t})^{-\alpha},$$

$$\frac{\partial Y_{N,t}}{\partial L_{N,t}} = (1-\alpha)A_{N,t}(L_{N,t})^{-\alpha}.$$

The Home agent allocates labor so that the marginal value product in the traded sector equals the marginal value product (adjusted by the non-traded price) in the non-traded sector:

$$(1-\alpha)A_{T,t}(n-L_{N,t})^{-\alpha} = P_{N,t}(1-\alpha)A_{N,t}(L_{N,t})^{-\alpha}.$$

Cancel the common factor $1-\alpha$ and rearrange:

$$A_{T,t}(n-L_{N,t})^{-\alpha} = P_{N,t}A_{N,t}(L_{N,t})^{-\alpha}. \tag{4a}$$

As $Y^*_{T,t} = A^*_{T,t}\Big(1-n-L^*_{N,t}\Big)^{-\alpha}$, the analogous condition for the Foreign country is:

$$A^*_{T,t}\big((1-n)-L^*_{N,t}\big)^{-\alpha} = P^*_{N,t}A^*_{N,t}(L^*_{N,t})^{-\alpha}. \tag{4b}$$

5 Resource Constraints and the Real Exchange Rate

Resource Constraints

Recall the Home budget constraint:

$$nP_tC_t + nB_{t+1} = A_{T,t}(n-L_{N,t})^{1-\alpha} + P_{N,t}A_{N,t}(L_{N,t})^{1-\alpha} + n(1+r_t)B_t.$$

Given (2a),

$$P_{N,t}A_{N,t}\big(L_{N,t}\big)^{1-\alpha} = n(1-\gamma)\big(P_{N,t}\big)^{1-\gamma}C_t \tag{5.1}$$

By (1a) and (5.1), we have:

$$nP_tC_t - P_{N,t}A_{N,t}\big(L_{N,t}\big)^{1-\alpha} = n\big(P_{N,t}\big)^{1-\gamma}C_t\big(1-(1-\gamma)\big) = n\gamma(P_{N,t})^{1-\gamma}C_t \tag{5.2}$$

Bring (5.2) back to the budget constraint, we have:

$$n\gamma\big(P_{N,t}\big)^{1-\gamma}C_t + nB_{t+1} = A_{T,t}(n-L_{N,t})^{1-\alpha} + n(1+r_t)B_t. \tag{5a}$$

Similarly, for Foreign we obtain:

$$(1-n)\gamma(P^*_{N,t})^{1-\gamma}C^*_t - nB_{t+1} = A^*_{T,t}\big((1-n)-L^*_{N,t}\big)^{1-\alpha} - n(1+r_t)B_t. \tag{5b}$$

Then, we define the real exchange rate as the ratio of Foreign to Home consumption price indices:

$$Q_t \equiv \frac{P^*_t}{P_t}.$$

Since

$$P_t = (P_{N,t})^{1-\gamma} \quad \text{and} \quad P_t^* = (P_{N,t}^*)^{1-\gamma},$$

we have:

$$Q_t = \left(\frac{P_{N,t}^*}{P_{N,t}}\right)^{1-\gamma}. \tag{5c}$$

6 Steady State

In steady state, consumption is constant so that $C_{t+1} = C_t$. The Euler equation for the Home agent is

$$C_{t+1} = \beta_{H,t+1}(1 + r_{t+1}^C)C_t. \tag{6.1}$$

But by definition the real return in consumption units is

$$1 + r_{t+1}^C = (1 + r_{t+1})\left(\frac{P_t}{P_{t+1}}\right)^{1-\gamma}. \tag{6.2}$$

In steady state prices do not change ($P_t = P_{t+1}$) so that

$$1 + r_{t+1}^C = 1 + r_{t+1} \quad \Rightarrow \quad C_t = \beta_t(1 + r_t)C_t.$$

Dividing by $C_t > 0$ and take $t = 0$, yields:

$$1 = \beta_0(1 + r_0). \tag{6a}$$

Step 2. Price Normalization

By convention we normalize the steady state price of non-tradables to one:

$$P_{N,0} = 1, \quad \text{and similarly} \quad P_{N,0}^* = 1.$$

Then the consumption price indices become

$$P_0 = (P_{N,0})^{1-\gamma} = 1, \quad {}_0P^* = 1. \tag{6b}$$

Step 3. Labor Allocation in the Home Country

The Home intratemporal labor allocation condition is:

$$A_{T,0}(n - L_{N,0})^{-\alpha} = P_{N,0}A_{N,0}(L_{N,0})^{-\alpha}.$$

Since $P_{N,0} = 1$, this simplifies to:

$$A_{T,0}(n - L_{N,0})^{-\alpha} = A_{N,0}(L_{N,0})^{-\alpha}.$$

Rearrange by dividing both sides by $A_{T,0}$ and by $(L_{N,0})^{-\alpha}$:

$$\left(\frac{n - L_{N,0}}{L_{N,0}}\right)^{-\alpha} = \frac{A_{N,0}}{A_{T,0}}.$$

Taking the reciprocal and then the $1/\alpha$-th root,

$$\frac{n - L_{N,0}}{L_{N,0}} = \left(\frac{A_{T,0}}{A_{N,0}}\right)^{1/\alpha}.$$

Now, using the calibration

$$A_{N,0} = A_{T,0}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha},$$

we have

$$\frac{A_{T,0}}{A_{N,0}} = \left(\frac{\gamma}{1-\gamma}\right)^{\alpha}.$$

Then,

$$\frac{n - L_{N,0}}{L_{N,0}} = \left[\left(\frac{\gamma}{1-\gamma}\right)^{\alpha}\right]^{1/\alpha} = \frac{\gamma}{1-\gamma}.$$

Thus,

$$L_{N,0} = n(1-\gamma). \tag{6c}$$

By symmetry, the corresponding condition for Foreign is:

$$A^*_{T,0}\Big((1-n) - L^*_{N,0}\Big)^{-\alpha} = P^*_{N,0}A^*_{N,0}(L^*_{N,0})^{-\alpha}.$$

Since $P^*_{N,0} = 1$, the same steps lead to:

$$\frac{(1-n) - L^*_{N,0}}{L^*_{N,0}} = \frac{\gamma}{1-\gamma},$$

so that

$$L^*_{N,0} = (1-n)(1-\gamma). \tag{6d}$$

We derive the steady-state consumption from the market clearing conditions for non-traded and traded goods. The non-traded goods clearing condition is

$$nC_{N,0} = A_{N,0}(L_{N,0})^{1-\alpha}.$$

From Question 1 we have

$$C_{N,0} = (1-\gamma)\frac{P_0}{P_{N,0}}C_0.$$

Since $P_0 = 1$ and $P_{N,0} = 1$, it follows that

$$C_{N,0} = (1-\gamma)C_0.$$

Substitute into the clearing condition:

$$n(1-\gamma)C_0 = A_{N,0}(L_{N,0})^{1-\alpha}.$$

Recall that $L_{N,0} = n(1-\gamma)$, so

$$n(1-\gamma)C_0 = A_{N,0}\big[n(1-\gamma)\big]^{1-\alpha}.$$

Solve for C_0(as it is an expression of $A_{N,0}$, we denote by C_0^N):

$$C_0^N = A_{N,0}\big[n(1-\gamma)\big]^{-\alpha}.$$

We then derive from the traded goods clearing condition:

$$\begin{aligned}
&n\gamma(P_{N,t})^{1-\gamma}C_t + (1-n)\gamma(P^*_{N,t})^{1-\gamma}C^*_t = A_{T,t}(n-L_{N,t})^{1-\alpha} + A^*_{T,t}\big((1-n)-L^*_{N,t}\big)^{1-\alpha}\\
\Rightarrow& n\gamma C_0 + \frac{\gamma}{1-\gamma}A^*_{N,0}(L^*_{N,0})^{1-\alpha} = A_{T,0}(n-L_{N,0})^{1-\alpha} + A^*_{T,0}(1-n-L^*_{N,0})^{1-\alpha}\\
\Rightarrow& n\gamma C_0 + \frac{\gamma}{1-\gamma}A^*_{N,0}(1-n)^{1-\alpha}(1-\gamma)^{1-\alpha} = A_{T,0}(n-n(1-\gamma))^{1-\alpha}\\
&\qquad + A^*_{T,0}(1-n-(1-n)(1-\gamma))^{1-\alpha}\\
\Rightarrow& n\gamma C_0 + \frac{\gamma}{1-\gamma}A^*_{N,0}(1-n)^{1-\alpha}(1-\gamma)^{1-\alpha} = A_{T,0}(n\gamma)^{1-\alpha} + A^*_{T,0}\Big[(1-n)\gamma\Big]^{1-\alpha}\\
\Rightarrow& n\gamma C_0 + \frac{\gamma}{1-\gamma}A_{T,0}\Big(\frac{1-n}{n}\Big)^{\alpha}\Big(\frac{1-\gamma}{\gamma}\Big)^{\alpha}(1-n)^{1-\alpha}(1-\gamma)^{1-\alpha} = A_{T,0}(n\gamma)^{1-\alpha}+\\
&A_{T,0}\Big(\frac{1-n}{n}\Big)^{\alpha}\Big[(1-n)\gamma\Big]^{1-\alpha}\\
\Rightarrow& n\gamma C_0 + A_{T,0}(1-n)n^{-\alpha}\gamma^{1-\alpha} = A_{T,0}(n\gamma)^{1-\alpha} + A_{T,0}(1-n)n^{-\alpha}\gamma^{1-\alpha}\\
\Rightarrow& C_0^T = A_{T,0}(n\gamma)^{-\alpha}.
\end{aligned}$$

We take a weighted geometric mean with weights γ and $1-\gamma$. That is,

$$C_0 = (C_0^N)^{1-\gamma}\cdot(C_0^T)^{\gamma},$$

so that

$$C_0 = \big[A_{N,0}n^{-\alpha}(1-\gamma)^{-\alpha}\big]^{1-\gamma}\big[A_{T,0}n^{-\alpha}\gamma^{-\alpha}\big]^{\gamma}.$$

We obtain:

$$C_0 = (A_{T,0})^{\gamma}(A_{N,0})^{1-\gamma}\left[n\gamma^{\gamma}(1-\gamma)^{1-\gamma}\right]^{-\alpha}. \tag{6e}$$

A similar derivation for Foreign (noting that the population is $1-n$) gives:

$$C_0^* = (A^*_{T,0})^{\gamma}(A^*_{N,0})^{1-\gamma}\left[(1-n)\gamma^{\gamma}(1-\gamma)^{1-\gamma}\right]^{-\alpha}. \tag{6f}$$

Because of the calibration and the fact that the relative productivities satisfy

$$\frac{A_{N,0}}{A_{T,0}} = \left(\frac{1-\gamma}{\gamma}\right)^{\alpha} \quad\text{and}\quad \frac{A^*_{N,0}}{A^*_{T,0}} = \left(\frac{1-n}{n}\right)^{\alpha}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha},$$

we can check that indeed

$$\frac{C_0}{C_0^*} = 1. \tag{6g}$$

7 Log-Linear Approximation

A. Non-Traded Goods Market

The Home non-traded goods market clearing condition is:

$$n(1-\gamma)(P_{N,t})^{-\gamma}C_t = A_{N,t}(L_{N,t})^{1-\alpha}.$$

Taking logarithms, we have

$$\ln n + \ln(1-\gamma) - \gamma \ln P_{N,t} + \ln C_t = \ln A_{N,t} + (1-\alpha)\ln L_{N,t}.$$

Linearizing around the steady state (and noting that constants vanish in the difference), we obtain:

$$-\gamma \widehat{P_{N,t}} + \widehat{C}_t = \widehat{A_{N,t}} + (1-\alpha)\widehat{L_{N,t}}. \tag{7a}$$

For foreign non-traded goods, we have:

$$\ln(1-n) + \ln(1-\gamma) - \gamma \ln P^*_{N,t} + \ln C^*_t = \ln A^*_{N,t} + (1-\alpha)\ln L^*_{N,t}.$$

Linearizing around the steady state, we obtain:

$$-\gamma \widehat{P^*_{N,t}} + \widehat{C^*_t} = \widehat{A^*_{N,t}} + (1-\alpha)\widehat{L^*_{N,t}}. \tag{7b}$$

B. Resource Constraint

The Home resource constraint is:

$$n\gamma(P_{N,t})^{1-\gamma}C_t + nB_{t+1} = A_{T,t}(n - L_{N,t})^{1-\alpha} + n(1+r_t)B_t.$$

Divide both sides by $n\gamma C_0$, we get:

$$\frac{(P_{N,t})^{1-\gamma}C_t}{C_0} + \widehat{B_{t+1}} = \frac{A_{T,t}(n-L_{N,t})^{1-\alpha}}{n\gamma C_0} + (1+r_t)\widehat{B_t}.$$

Taking logs and linearizing, we have:

$$(1-\gamma)\widehat{P_{N,t}} + \widehat{C}_t + \widehat{B_{t+1}} = \widehat{A_{T,t}} - \frac{(1-\alpha)(L_{N,t}-L_{N,0})}{n-L_{N,0}}\widehat{L_{N,t}} + \frac{1}{\beta_0}\widehat{B_t}.$$

Since in steady state $n - L_{N,0} = n\gamma$, and that $\widehat{L_{N,t}} = \frac{L_{N,t}-L_{N,0}}{L_{N,0}}$. Thus,

$$(1-\gamma)\widehat{P_{N,t}} + \widehat{C}_t + \widehat{B_{t+1}} = \widehat{A_{T,t}} - (1-\alpha)\frac{1-\gamma}{\gamma}\widehat{L_{N,t}} + \frac{1}{\beta_0}\widehat{B_t}. \tag{7c}$$

Recall foreign budget constraint (5b):

$$(1-n)\gamma(P_{N,t}^*)^{1-\gamma}C_t^* - nB_{t+1} = A_{T,t}^*\big((1-n)-L_{N,t}^*\big)^{1-\alpha} - n(1+r_t)B_t.$$

Following similar steps, we have:

$$(1-n)(1-\gamma)\big(\widehat{P_{N,t}^*}+\widehat{C_t^*}\big) - n\widehat{B_{t+1}} = \widehat{A_{T,t}^*} - (1-\alpha)\frac{1-\gamma}{\gamma}\widehat{L_{N,t}^*} - \frac{n}{\beta_0}\widehat{B_t}.$$

Divide both sides by $1-n$, we get:

$$(1-\gamma)\widehat{P_{N,t}^*} + \widehat{C_t^*} - \frac{n}{1-n}\widehat{B_{t+1}} = \widehat{A_{T,t}^*} - (1-\alpha)\frac{1-\gamma}{\gamma}\widehat{L_{N,t}^*} - \frac{1}{\beta_0}\frac{n}{1-n}\widehat{B_t}. \tag{7d}$$

C. Euler Equation

Recall that:

$$C_{t+1} = C_t\beta_{H,t+1}(1+r_{t+1})\Big(\frac{P_{N,t}}{P_{N,t+1}}\Big)^{1-\gamma}$$

Taking logs and linearizing, we have:

$$\widehat{C_{t+1}} = \widehat{C}_t + \widehat{\beta_{H,t+1}} + \frac{r_{t+1}-r_0}{1+r_0} + (1-\gamma)(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}).$$

As $\beta_0(1+r_0)=1$,

$$\begin{aligned}\widehat{r_t} &= r_t - \frac{1-\beta_0}{\beta_0}\\ &= r_t - \frac{1-\frac{1}{1+r_0}}{\frac{1}{1+r_0}}\\ &= r_t - r_0\end{aligned}$$

So, the Euler equation becomes:

$$\widehat{C_{t+1}} = \widehat{C}_t + \widehat{\beta_{H,t+1}} + \beta_0\widehat{r_{t+1}} + (1-\gamma)(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}). \tag{7e}$$

Similarly for Foreign:

$$\widehat{C_{t+1}^*} = \widehat{C_t^*} + \widehat{\beta_{F,t+1}} + \beta_0\widehat{r_{t+1}}^* + (1-\gamma)(\widehat{P_{N,t}^*} - \widehat{P_{N,t+1}}^*). \tag{7f}$$

D. Labor Allocation

The intratemporal condition is:

$$A_{T,t}(n-L_{N,t})^{-\alpha} = P_{N,t}A_{N,t}(L_{N,t})^{-\alpha}.$$

Taking logarithms:

$$\ln A_{T,t} - \alpha\ln(n-L_{N,t}) = \ln P_{N,t} + \ln A_{N,t} - \alpha\ln L_{N,t}.$$

Linearize around steady state:

$$\widehat{A_{T,t}} - \alpha \frac{L_{N,t} - L_{N,0}}{n - L_{N,0}} = \widehat{P_{N,t}} + \widehat{A_{N,t}} - \alpha \widehat{L_{N,t}}.$$

Using the fact that in steady state $n - L_{N,0} = n\gamma$:

$$\widehat{A_{T,t}} - \alpha \frac{\widehat{L_{N,t}} n(1-\gamma)}{n\gamma} = \widehat{P_{N,t}} + \widehat{A_{N,t}} - \alpha \widehat{L_{N,t}}.$$

$$\widehat{A_{T,t}} + \frac{\alpha}{\gamma} \widehat{L_{N,t}} = \widehat{P_{N,t}} + \widehat{A_{N,t}}. \tag{7g}$$

Similarly for Foreign:

$$\widehat{A^*_{T,t}} + \frac{\alpha}{1-\gamma} \widehat{L^*_{N,t}} = \widehat{P^*_{N,t}} + \widehat{A^*_{N,t}}. \tag{7h}$$

E. Real Exchange Rate

As we know:

$$1 + r^C_{t+1} = (1 + r_{t+1}) \left(\frac{P_{N,t}}{P_{N,t+1}} \right)^{1-\gamma},$$

take logs and linearize both sides, we have:

$$\frac{r^C_{t+1} - r^C_0}{1 + r^C_0} = \frac{r_{t+1} - r_0}{1 + r_0} + (1-\gamma)(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}).$$

$$\Rightarrow \beta_0 \widehat{r^C_{t+1}} = \beta_0 \widehat{r_{t+1}} + (1-\gamma)(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}).$$

$$\widehat{r^C_{t+1}} = \widehat{r_{t+1}} + (1-\gamma)\frac{1}{\beta_0}(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}). \tag{7i}$$

Similarly for Foreign:

$$\widehat{r^{*C}_{t+1}} = \widehat{r^*_{t+1}} + (1-\gamma)\frac{1}{\beta_0}(\widehat{P^*_{N,t}} - \widehat{P^*_{N,t+1}}). \tag{7j}$$

8 Worldwide Equilibrium

Define world aggregates as population-weighted averages (e.g., $\widehat{C^W_t} = n\widehat{C_t} + (1-n)\widehat{C^*_t}$). Then, from the above log-linearized equations one can show:

- **Non-Traded Goods Market:**

$$n \times (7a) + (1-n) \times (7b) \Rightarrow -\gamma \widehat{P^W_{N,t}} + \widehat{C^W_t} = \widehat{A^W_{N,t}} + (1-\alpha)\widehat{L^W_{N,t}}. \tag{8a}$$

- **Resource Constraint:**

$$n \times (7c) + (1-n) \times (7d) \Rightarrow (1-\gamma)\widehat{P^W_{N,t}} + \widehat{C^W_t} = \widehat{A^W_{T,t}} - (1-\alpha)\frac{1-\gamma}{\gamma}\widehat{L^W_{N,t}}. \tag{8b}$$

- **Euler Equation:**

$$n \times (7e) + (1-n) \times (7f) \Rightarrow \widehat{C^W_{t+1}} = \widehat{C^W_t} + (1-\gamma)(\widehat{P^W_{N,t}} - \widehat{P^W_{N,t+1}}) + \widehat{\beta^W_{H,t+1}} + \beta_0 \widehat{r_{t+1}}. \tag{8c}$$

- **Labor Allocation:**

$$n \times (7g) + (1-n) \times (7h) \Rightarrow \widehat{A^W_{T,t}} + \frac{\alpha}{\gamma}\widehat{L^W_{N,t}} = \widehat{P^W_{N,t}} + \widehat{A^W_{N,t}}. \tag{8d}$$

- **Real Exchange Rate:**

$$n \times (7i) + (1-n) \times (7j) \Rightarrow \beta_0 \widehat{r^{CW}_{t+1}} = (1-\gamma)(\widehat{P^W_{N,t}} - \widehat{P^W_{N,t+1}}) + \beta_0 \widehat{r_{t+1}}. \tag{8e}$$

Let (8a)-(8d), we get:

$$(1-\gamma)\widehat{P^W_{N,t}} + \widehat{C^W_t} = \widehat{A^W_{T,t}} + \Big(1 - \alpha + \frac{\alpha}{\gamma}\Big)\widehat{L^W_{N,t}}.$$

Combine with (8b), we have:

$$\begin{aligned}
\Big(1 - \alpha + \frac{\alpha}{\gamma}\Big)\widehat{L^W_{N,t}} &= -(1-\alpha)\frac{1-\gamma}{\gamma}\widehat{L^W_{N,t}} \\
&= (1-\alpha)\widehat{L^W_{N,t}} - \frac{1-\alpha}{\gamma}\widehat{L^W_{N,t}} \\
\Rightarrow \frac{1}{\gamma}\widehat{L^W_{N,t}} &= 0.
\end{aligned} \tag{8f}$$

Let (8b)-(8a), we have:

$$\begin{aligned}
\widehat{P^W_{N,t}} &= \widehat{A^W_{T,t}} - \widehat{A^W_{N,t}} - (1-\alpha)\Big(\frac{1-\gamma}{\gamma} + 1\Big)\widehat{L^W_{N,t}} \\
&= \widehat{A^W_{T,t}} - \widehat{A^W_{N,t}}.
\end{aligned} \tag{8g}$$

Take $(1-\gamma) \times (8a) + \gamma(8b)$, we have:

$$(1-\gamma)\widehat{C^W_t} + \gamma\widehat{C^W_t} = \widehat{C^W_t} = (1-\gamma)\widehat{A^W_{N,t}} + \gamma\widehat{A^W_{T,t}}. \tag{8h}$$

Finally, from (8c), we know that:

$$\begin{aligned}
\beta_0\widehat{r_{t+1}} + \widehat{\beta^W_{H,t+1}} &= \widehat{C^W_{t+1}} - \widehat{C^W_t} - (1-\gamma)\Big(\widehat{P^W_{N,t}} - \widehat{P^W_{N,t+1}}\Big) \\
&= \gamma\hat{A}^W_{t+1} + (1-\gamma)\widehat{A^W_{N,t+1}} - \gamma\widehat{A^W_{T,t+1}} - (1-\gamma)\widehat{A^W_{N,t+1}} \\
&\quad - (1-\gamma)\Big(\widehat{A^W_{T,t}} - \widehat{A^W_{N,t}}\Big) + (1-\gamma)\Big(\widehat{A^W_{T,t+1}} - \widehat{A^W_{N,t+1}}\Big) \\
&= \widehat{A^W_{T,t+1}} - \widehat{A^W_{T,t}}.
\end{aligned} \tag{8i}$$

Relative Price of Goods:
Equation (8g) shows that the relative price of non-traded goods is driven by sectoral productivity differentials:

- **Productivity Differences:** An increase in traded-sector productivity ($\widehat{A^W_{T,t}} > 0$) raises the economy-wide efficiency in producing traded goods. This reduces the cost of tradables relative to non-traded goods, increasing the price of non-tradables in relative terms.

- **Relative Demand Effects:** Shocks that affect consumption (e.g. through changes in discount factors) alter demand for non-traded goods, hence influencing $P_{N,t}$.

Consumption:
Aggregate consumption in each country is determined by both traded and non-traded output. In log-linear terms, world consumption is given by

$$\widehat{C^W_t} = \gamma\,\widehat{A^W_{T,t}} + (1-\gamma)\,\widehat{A^W_{N,t}},$$

where $\widehat{A^W_{T,t}}$ and $\widehat{A^W_{N,t}}$ are the population-weighted productivity shocks in the traded and non-traded sectors. This shows that consumption is driven by a weighted average of productivity growth across traded and non-traded sectors, where γ represents the share of traded goods in consumption.
Real Interest Rate:
Equation (8i) shows that the real interest rate is primarily driven by expected future productivity growth in the traded sector:

- Higher future traded productivity growth ($\widehat{A^W_{T,t+1}} > \widehat{A^W_{T,t}}$) leads to a higher real interest rate. Since future income is expected to be higher, households are willing to borrow more today, driving up interest rates.

- Conversely, if traded productivity growth slows, the real interest rate declines. This happens because expectations of lower future income reduce borrowing and investment demand.

Additionally, changes in household patience also influences interest rates:

- If households become more patient, they increase saving, which lowers the real interest rate.

- If they become less patient, saving decreases, and the real interest rate rises.

9 Cross-Country Differences

As we know that $Q_t = \left(\frac{P^*_{N,t}}{P_{N,t}}\right)^{1-\gamma}$, log-linearize the equation, we have:

$$\widehat{Q_t} = (1-\gamma)(\widehat{P^*_{N,t}}) - \widehat{P_{N,t}}.$$

Use (7a) - (7b), we get:

$$\begin{aligned}
&\widehat{C_t} - \widehat{C^*_t} - \gamma(\widehat{P_{N,t}} - \widehat{P^*_{N,t}}) = (\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) + (1-\alpha)(\widehat{L_{N,t}} - \widehat{L^*_{N,t}}) \\
&\Rightarrow \widehat{C_t} - \widehat{C^*_t} + \frac{\gamma}{1-\gamma}\widehat{Q_t} = (\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) + (1-\alpha)(\widehat{L_{N,t}} - \widehat{L^*_{N,t}}).
\end{aligned} \tag{9a}$$

Use (7c) - (7d), we get:

$$\begin{aligned}
&\text{LHS} = (1-\gamma)(\widehat{P_{N,t}} - \widehat{P^*_{N,t}}) + \widehat{C}_t - \widehat{C^*_t} + \frac{\widehat{B_{t+1}}}{1-n} \\
&\text{RHS} = (\widehat{A_{T,t}} - \widehat{A^*_{T,t}}) - (1-\alpha)\frac{1-\gamma}{\gamma}(\widehat{L_{N,t}} - \widehat{L^*_{N,t}}) + \frac{1}{\beta_0}\frac{\widehat{B_t}}{1-n} \\
&\text{As } \widehat{Q_t} = (1-\gamma)(\widehat{P^*_{N,t}} - \widehat{P_{N,t}}), \text{ we have} \\
&\text{LHS} = -\widehat{Q}_t + (\widehat{C}_t - \widehat{C^*_t}) + \frac{\widehat{B_{t+1}}}{1-n} = \text{RHS}
\end{aligned} \tag{9b}$$

Use (7e) - (7f), we get:

$$\begin{aligned}
(\widehat{C_{t+1}} - \widehat{C^*_{t+1}}) &= (\widehat{C}_t - \widehat{C^*_t}) + (1-\gamma)(\widehat{P_{N,t}} - \widehat{P^*_{N,t}}) + (1-\gamma)(\widehat{P^*_{N,t+1}} - \widehat{P_{N,t+1}}) + (\widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}}) \\
&= (\widehat{C}_t - \widehat{C^*_t}) - \widehat{Q}_t + \widehat{Q_{t+1}} + \widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}}.
\end{aligned} \tag{9c}$$

Use (7g) - (7h), we get:

$$\begin{aligned}
(\widehat{A_{T,t}} - \widehat{A^*_{T,t}}) + \frac{\alpha}{\gamma}(\widehat{L_{N,t}} - \widehat{L^*_{N,t}}) &= (\widehat{P_{N,t}} - \widehat{P^*_{N,t}}) + (\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) \\
&= -\frac{1}{1-\gamma}\widehat{Q}_t + (\widehat{A_{N,t}} - \widehat{A^*_{N,t}}).
\end{aligned} \tag{9d}$$

Use (7i) - (7j), we get:

$$\begin{aligned}
\beta_0(\widehat{r^C_{t+1}} - \widehat{r^{C*}_{t+1}}) &= (1-\gamma)(\widehat{P_{N,t}} - \widehat{P^*_{N,t}}) + (1-\gamma)(\widehat{P^*_{N,t}} - \widehat{P_{N,t}}) \\
&= \widehat{Q_{t+1}} - \widehat{Q}_t.
\end{aligned} \tag{9e}$$

10 Long-Run Allocation (Period $t+1$)

Assume that from $t+1$ onward the economy reaches a new steady state with no further discount factor shocks ($\widehat{\beta_{H,t+2}} = \widehat{\beta_{F,t+2}} = 0$). In the steady state, the consumption growth rate is zero, the asset position is fixed and the real exchange rate is stable.
Using the labor allocation equation at $t+1$, we have:

$$\frac{\alpha}{\gamma}(\widehat{L_{N,t}} - \widehat{L^*_{N,t}}) = -\frac{1}{1-\gamma}\widehat{Q_{t+1}} + \Big[(\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) - (\widehat{A_{T,t}} - \widehat{A^*_{T,t}})\Big]. \tag{10.1}$$

Then, we use the market clearing condition for non-traded goods and the resource allocation constraints at $t+1$:

$$\frac{\gamma}{1-\gamma}\widehat{Q_{t+1}} + (\widehat{C_{t+1}} - \widehat{C^*_{t+1}}) = (\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) + (1-\alpha)(\widehat{L_{N,t+1}} - \widehat{L^*_{N,t+1}}) \tag{10.2}$$

$$-\widehat{Q_{t+1}} + (\widehat{C_{t+1}} - \widehat{C_{t+1}}^*) + \frac{\widehat{B_{t+2}}}{1-n} = (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) - (1-\alpha)\frac{1-\gamma}{\gamma}(\widehat{L_{N,t+1}} - \widehat{L^*_{N,t+1}}) + \frac{1}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}. \tag{10.3}$$

As $\widehat{B_{t+2}} = \widehat{B_{t+1}}$, using (10.2)-(10.3), we get:

$$\begin{aligned}
\Big(\frac{1-\gamma}{\gamma}+1\Big)\widehat{Q_{t+1}} &= \Big[(\widehat{A_{N,t+1}}-\widehat{A^*_{N,t+1}})-(\widehat{A_{T,t+1}}-\widehat{A^*_{T,t+1}})\Big] \\
&\quad + (1-\alpha)\Big(1+\frac{1-\gamma}{\gamma}\Big)(\widehat{L_{N,t+1}}-\widehat{L^*_{N,t+1}}) - \frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
\Rightarrow \quad \widehat{Q_{t+1}} &= (1-\gamma)\Big[(\widehat{A_{N,t+1}}-\widehat{A^*_{N,t+1}})-(\widehat{A_{T,t+1}}-\widehat{A^*_{T,t+1}})\Big] + (1-\alpha)\frac{1-\gamma}{\gamma}(\widehat{L_{N,t+1}}-\widehat{L^*_{N,t+1}}) \\
&\quad - (1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}. \qquad (10.4)
\end{aligned}$$

Replacing $\widehat{L_{N,t+1}}-\widehat{L^*_{N,t+1}}$ using (10.1), we get:

$$\begin{aligned}
\widehat{Q_{t+1}} &= (1-\gamma)\Big[(\widehat{A_{N,t+1}}-\widehat{A^*_{N,t+1}})-(\widehat{A_{T,t+1}}-\widehat{A^*_{T,t+1}})\Big] - \frac{1-\alpha}{\alpha}\widehat{Q_{t+1}} \\
&\quad + \frac{1-\alpha}{\alpha}(1-\gamma)\Big[(\widehat{A_{N,t+1}}-\widehat{A^*_{N,t+1}})-(\widehat{A_{T,t+1}}-\widehat{A^*_{T,t+1}})\Big] - (1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
\Rightarrow \quad \Big(1+\frac{1-\alpha}{\alpha}\Big)\widehat{Q_{t+1}} &= (1-\gamma)\Big(1+\frac{1-\alpha}{\alpha}\Big)\Big[(\widehat{A_{N,t+1}}-\widehat{A^*_{N,t+1}})-(\widehat{A_{T,t+1}}-\widehat{A^*_{T,t+1}})\Big] \\
&\quad - (1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
\Rightarrow \quad \widehat{Q_{t+1}} &= -(1-\gamma)\Big[(\widehat{A_{T,t+1}}-\widehat{A^*_{T,t+1}})-(\widehat{A_{N,t+1}}-\widehat{A^*_{N,t+1}})\Big] - \alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}. \qquad (10a)
\end{aligned}$$

Comparing (10.4) and (10a), we have:

$$\begin{aligned}
(1-\alpha)\frac{1-\gamma}{\gamma}(\widehat{L_{N,t+1}}-\widehat{L^*_{N,t+1}}) - (1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} &= -\alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
\Rightarrow \quad \widehat{L_{N,t+1}}-\widehat{L^*_{N,t+1}} &= \gamma\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}. \qquad (10b)
\end{aligned}$$

Implementing (10a) and (10b) back into (10.2), we get:

$$\begin{aligned}
\widehat{C_{t+1}}-\widehat{C^*_{t+1}} &= \widehat{Q_{t+1}} + (\widehat{A_{T,t+1}}-\widehat{A^*_{T,t+1}}) - \frac{(1-\alpha)(1-\gamma)}{\gamma}\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} + \frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
&= -(1-\gamma)(\widehat{A_{T,t+1}}-\widehat{A^*_{T,t+1}}) + (1-\gamma)(\widehat{A_{N,t+1}}-\widehat{A^*_{N,t+1}}) + (\widehat{A_{T,t+1}}-\widehat{A^*_{T,t+1}}) \\
&\quad - \alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} - (1-\alpha)(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} + \frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
&= \gamma(\widehat{A_{T,t+1}}-\widehat{A^*_{T,t+1}}) + (1-\gamma)(\widehat{A_{N,t+1}}-\widehat{A^*_{N,t+1}}) + \gamma\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \qquad (10c)
\end{aligned}$$

(1) Impact of Home Being Wealthier ($\widehat{B_{t+1}} > 0$)
When $\widehat{B_{t+1}} > 0$, Home has accumulated net foreign assets. In the long run, this wealth effect leads to:

- Since $\widehat{B_{t+1}} > 0$, the term $\gamma\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}$ is positive, indicating that Homes consumption is permanently higher than Foreigns. This reflects the fact that Homes past savings allow it to sustain a higher consumption path without reducing its wealth, as the income from foreign assets supports increased spending.

- From equation (10b), the labor allocation in Home is higher than in Foreign, reflecting the wealth effect. The higher labor allocation in Homes non-tradable sector is consistent with the intuition that a wealthier Home can afford more non-tradable goods.

- From equation (10a), we know that $\widehat{Q_{t+1}}$ decreases. This means that Homes real exchange rate appreciates in the long run. The appreciation occurs because higher wealth increases demand for non-traded goods, raising their price relative to Foreigns, which increases the overall price level in Home compared to Foreign.

- The long-run real interest rate is determined by the intertemporal equilibrium condition. Since we assume that no further discount factor shocks occur beyond $t+1$, the real interest rate stabilizes at a steady-state level that balances global savings and investment. $\widehat{r^C_{t+1}} - \widehat{r^{*C}_{t+1}}$ remains the same.

(2) Long-Run Impact of Productivity (Excluding the Asset Channel)

- From equation (10c), long-run consumption differences are entirely determined by productivity differences in the traded and non-traded sectors. A permanent increase in Homes productivity raises the output in the traded / non-traded sector. This increases Home's income and consumption by the factor γ and, via the Balassa–Samuelson mechanism, raises wages and non-traded prices, leading to a real appreciation.

- From equation (10b), as we ignore the asset channel, labor allocation does not change across countries due to productivity differences alone. The reason is that long-run equilibrium adjustments in wages and prices ensure that sectoral labor distributions reach a stable proportion across Home and Foreign.

- From equation (10a), If Homes traded productivity, the first term dominates, and $\widehat{Q_{t+1}}$ decreases, implying a real appreciation of Homes exchange rate. If Homes non-traded productivity, the second term dominates, and $\widehat{Q_{t+1}}$ increases, implying a real depreciation of Homes exchange rate.

11 Short-Run Allocation (Period t)

To simplify notation, we denote $\widehat{A_{N,t}} - \widehat{A^*_{N,t}} = \widetilde{A_{N,t}}$, $\widehat{A_{T,t}} - \widehat{A^*_{T,t}} = \widetilde{A_{T,t}}$, $\widehat{L_{N,t}} - \widehat{L^*_{N,t}} = \widetilde{L_{N,t}}$. Combining (9a) and (9d), we can get:

$$\widetilde{C}_t + \frac{\gamma}{1-\gamma}\widehat{Q_t} - \gamma\widetilde{A_{T,t}} - \alpha\widetilde{L_{N,t}} = \widetilde{A_{N,t}}(1-\gamma) + (1-\alpha)\widetilde{L_{N,t}} + \frac{\gamma}{1-\gamma}\widehat{Q_t}$$
$$\Rightarrow \widetilde{L_{N,t}} = \widetilde{C}_t - \gamma\widetilde{A_{T,t}} - (1-\gamma)\widetilde{A_{N,t}} \tag{11.1}$$

and that

$$(1-\alpha)\gamma\widetilde{A_{T,t}} + \alpha(1-\alpha)\widetilde{L_{N,t}} + \frac{\alpha\gamma}{1-\gamma}\widehat{Q_t} + \alpha\widetilde{C_t} = -\frac{\gamma(1-\alpha)}{1-\gamma}\widehat{Q_t} + (1-\alpha)\gamma\widetilde{A_{N,t}} + \alpha\widetilde{A_{N,t}} + \alpha(1-\alpha)\widetilde{L_{N,t}}$$

$$\Rightarrow \widehat{Q_t} = \Big(\frac{1-\gamma}{\gamma}\Big)[(1-\alpha)\gamma+\alpha]\widetilde{A_{N,t}} - (1-\alpha)(1-\gamma)\widetilde{A_{T,t}} - \frac{\alpha(1-\gamma)}{\gamma}\widetilde{C_t} \tag{11.2}$$

Implementing (11.1) and (11.2) back into (9b), we get:

$$\begin{aligned}
&-\Big(\frac{1-\gamma}{\gamma}\Big)[(1-\alpha)\gamma+\alpha]\widetilde{A_{N,t}} + (1-\alpha)(1-\gamma)\widetilde{A_{T,t}} + \frac{\alpha(1-\gamma)}{\gamma}\widetilde{C_t} + \widetilde{C_t} + \frac{\widehat{B_{t+1}}}{1-n} \\
&= \widehat{A_{T,t}} - \frac{(1-\alpha)(1-\gamma)}{\gamma}\widetilde{C_t} + (1-\alpha)(1-\gamma)\widetilde{A_{T,t}} + \frac{(1-\alpha)(1-\gamma)^2}{\gamma}\widetilde{A_{N,t}} \\
&\Rightarrow \frac{\widehat{B_{t+1}}}{1-n} = \widetilde{A_{T,t}} + \frac{1-\gamma}{\gamma}\widetilde{A_{N,t}} - \frac{1}{\gamma}\widetilde{C_t} \\
&\Rightarrow \widetilde{C_t} = \gamma\widetilde{A_{T,t}} + (1-\gamma)\widetilde{A_{N,t}} - \gamma\frac{\widehat{B_{t+1}}}{1-n}
\end{aligned} \tag{11.3}$$

Bring (11.3) back to (11.2), we get:

$$\widehat{Q_t} = (1-\gamma)(\widetilde{A_{N,t}} - \widetilde{A_{T,t}}) + \gamma\frac{\widehat{B_{t+1}}}{1-n} \tag{11.4}$$

Subtracting (11.4) from (10a), we get:

$$\widehat{Q_{t+1}} - \widehat{Q_t} = (1-\gamma)(\widetilde{A_{N,t+1}} - \widetilde{A_{N,t}}) - (1-\gamma)(\widetilde{A_{T,t+1}} - \widetilde{A_{T,t}}) - \alpha(1-\gamma)\frac{1}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \tag{11.5}$$

Subtracting (11.3) from (10c), we get:

$$\widetilde{C_{t+1}} - \widetilde{C_t} = \gamma(\widetilde{A_{T,t+1}} - \widetilde{A_{T,t}}) + (1-\gamma)(\widetilde{A_{N,t+1}} - \widetilde{A_{N,t}}) + \frac{\gamma}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \tag{11.6}$$

Bring (11.5) and (11.6) back to (9c), we get:

$$\begin{aligned}
&\gamma(\widetilde{A_{T,t+1}} - \widetilde{A_{T,t}}) + (1-\gamma)(\widetilde{A_{N,t+1}} - \widetilde{A_{N,t}}) + \frac{\gamma}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
&= (1-\gamma)(\widetilde{A_{N,t+1}} - \widetilde{A_{N,t}}) - (1-\gamma)(\widetilde{A_{T,t+1}} - \widetilde{A_{T,t}}) - \alpha(1-\gamma)\frac{1}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} + \widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}} \\
&\Rightarrow \Big[\gamma + \alpha(1-\gamma)\Big]\frac{1}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} = \widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}} - (\widetilde{A_{T,t+1}} - \widetilde{A_{T,t}}) \\
&\Rightarrow \frac{\widehat{B_{t+1}}}{1-n} = \frac{\beta_0}{\gamma+\alpha(1-\gamma)}\Big[\widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}} - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) + (\widehat{A_{T,t}} - \widehat{A^*_{T,t}})\Big]
\end{aligned} \tag{11a}$$

Bring (11a) back to (11.3), we get:

$$\begin{aligned}\widehat{C_{t+1}} - \widehat{C}_t = \gamma(\widehat{A_{T,t}} - \widehat{A^*_{T,t}}) + (1-\gamma)(\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) \\ - \frac{\gamma\beta_0}{\gamma + \alpha(1-\gamma)}\Big[\widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}} - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) + (\widehat{A_{T,t}} - \widehat{A^*_{T,t}})\Big]\end{aligned} \tag{11b}$$

Bring (11a) back to (11.4), we get:

$$\begin{aligned}\widehat{Q_t} = -(1-\gamma)(\widehat{A_{T,t}} - \widehat{A^*_{T,t}}) + (1-\gamma)(\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) \\ + \frac{\alpha(1-\gamma)\beta_0}{\gamma + \alpha(1-\gamma)}\Big[\widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}} - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) + (\widehat{A_{T,t}} - \widehat{A^*_{T,t}})\Big]\end{aligned} \tag{11c}$$

Bring (11b) and (11c) back to (11.1), we get:

$$\widehat{L_{N,t}} - \widehat{L^*_{N,t}} = -\frac{\gamma\beta_0}{\gamma + \alpha(1-\gamma)}\Big[\widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}} - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) + (\widehat{A_{T,t}} - \widehat{A^*_{T,t}})\Big] \tag{11d}$$

By (9c), we know that:

$$\beta_0(\widehat{r^C_{t+1}} - \widehat{r^{C*}_{t+1}}) = \widehat{Q_{t+1}} - \widehat{Q_t}$$

Implementing (10.4) and (11c) back to the equation, we get:

$$\begin{aligned}\beta_0(\widehat{r^C_{t+1}} - \widehat{r^{C*}_{t+1}}) = -(1-\gamma)\Big[(\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) - (\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}})\Big] - \alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\ - \Big\{-(1-\gamma)(\widehat{A_{T,t}} - \widehat{A^*_{T,t}}) + (1-\gamma)(\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) + \alpha(1-\gamma)\frac{\widehat{B_{t+1}}}{1-n}\Big\}\end{aligned}$$

Bring (11a) back to the equation, we get:

$$\begin{aligned}\beta_0(\widehat{r^C_{t+1}} - \widehat{r^{C*}_{t+1}}) = -(1-\gamma)\Big[(\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) - (\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}})\Big] - \alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\ + (1-\gamma)\Big[(\widehat{A_{T,t}} - \widehat{A^*_{T,t}}) - (\widehat{A_{N,t}} - \widehat{A^*_{N,t}})\Big] \\ - \frac{\alpha(1-\gamma)}{\gamma + \alpha(1-\gamma)}\Big[\widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}} - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) + (\widehat{A_{T,t}} - \widehat{A^*_{T,t}})\Big]\end{aligned} \tag{11e}$$

(1) Impact of a Temporary Increase in Home Patience ($\widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}} > 0$)

- Home households postpone current consumption and save more, leading to a positive net asset accumulation, i.e., $\widehat{B_{t+1}} > 0$.
- The current account surplus (positive $\widehat{B_{t+1}}$) reduces Home's current consumption relative to Foreign, so that the cross-country consumption gap $\widetilde{C}_t$ decreases.
- As Home saves more, there is a reallocation of labor away from the non-traded sector (since non-tradables are largely consumed domestically) and the real exchange rate adjusts accordingly. Typically, increased saving puts downward pressure on domestic non-tradable prices, leading

to a real depreciation, which helps balance the external sector.

- From (11c), we see that $\widehat{Q_t}$ is higher. This means that Homes goods become cheaper relative to Foreigns, which facilitates the necessary trade surplus by making Homes exports more competitive and discouraging imports.
- From (11e), we can tell that if $\widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}} > 0$, $\widehat{r^C_{t+1}} - \widehat{r^{C*}_{t+1}}$ decreases,implying that the real interest rate in Home falls relative to Foreign. This lower interest rate reflects Homes increased supply of savings, which pushes down borrowing costs.

(2) Impact of a Temporary Shock in Home Traded Productivity ($\widehat{A_{T,t}} > 0$) versus Non-Traded Productivity ($\widehat{A_{N,t}} > 0$)

Temporary Traded Productivity Shock ($\widehat{A_{T,t}} > 0$):

- A positive temporary shock in $\widehat{A_{T,t}}$ increases Home's output of tradable goods.
- Because tradables are internationally traded, Home can export the surplus. However, households do not instantaneously increase their consumption by the full amount of the productivity gain; instead, they smooth consumption by saving some of the extra income.
- As tradable goods supply increases, Home real exchange rate depreciates and reduces their relative prices.
- As a result, Home runs a current account surplus ($\widehat{B_{t+1}} > 0$) and, by the mechanism of reallocation, its non-traded goods sector experiences higher demandleading to higher domestic prices and a real appreciation of Home's currency.

Temporary Non-Traded Productivity Shock ($\widehat{A_{N,t}} > 0$):

- A temporary shock in $\widehat{A_{N,t}}$ boosts the production of non-tradable goods, which are consumed domestically.
- Since non-tradables are not exported, the entire output effect is absorbed by domestic consumption, raising Home's real consumption without significantly altering external trade.
- The increased supply of non-tradables tends to lower their relative price, so the domestic consumption price index falls. This produces a real depreciation of Home's currency.
- Additionally, with most of the income effect confined to domestic consumption, there is little pressure for a current account surplus or deficit.

(3) Impact of Permanent Productivity Shocks

- Home's consumption jumps to a higher level that reflects the permanent increase in productivity.
- From equation (11d), labor shifts in the long run toward the sector with higher relative productivity, but no transitory reallocation occurs.
- As the real exchange rate is about cross-country differences, a permanent productivity shock doesnt impact the real excahnge rate.
- Since permanent shocks do not create a temporary savings imbalance, the real interest rate remains unchanged in the long run, determined by steady-state preferences.