Demystifying DSGE Models

3. Case Study – The Canonical Smets-Wouters DSGE

Outline

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- Smets-Wouters model is “workhorse” DSGE model used as basis for almost all central bank DSGEs
- Smets and Wouters both from Belgian (!) CB; Smets’ father is its Governor ...
- Original version published two decades ago in 2003, but superseded in 2007 by version estimated using Bayesian techniques
- This model is ***so important*** – in terms both of historical development of DSGEs and their practical implementation outside purely academic framework – that it merits being studied in detail
- eg, NY Fed model = SW2007 + BGG (financial side)

- ***Recall:*** ***General NK model*** introduced into basic RBC more realistic specifications:
  - ***imperfect*** competition ⇨ *Last week*
  - ***frictions*** in ***prices*** and in ***wages*** ⇨ *Last week*
  - ***habit*** formation in consumption ⇨ *Last week*
  - ***non-Ricardian*** agents ⇨ *Last week*
  - ***adjustment costs*** in investment ⇨ *Last week*
  - ***capacity*** (non-)utilisation ***costs*** ⇨ *Last week*
  - ***government*** sector (***monetary*** and ***fiscal***) ⇨ *Last week - no!*
  - ***financial*** sector ⇨ *Week 6*
  - ***housing*** sector ⇨ *Week 6*
  - ***unemployment*** ⇨ *Week 7*
  - ***environment*** ⇨ *Week 7*
  - ***foreign*** trade ⇨ *Week 8*

- ***SW2007*** model uses most of the characteristics mentioned last week:
  - consumption with ***habit*** persistence
  - ***monopolistic*** competition
  - ***sticky*** prices and wages using ***Calvo*** fairy
  - investment ***adjustment costs*** + variable capital ***utilisation***
- Major new feature of model: use of ***seven structural shocks*** to match behaviour of US economy
- Shocks are to: productivity (***a***), labour supply (***l***), investment-specific technology (***i***), risk (***b***), (wage) cost-push (***w***), fiscal policy (***g***) and monetary policy (***r***)

- Households maximize *[NEW]* ***non-separable*** utility function with two arguments (goods and labour effort) over infinite life horizon

$$E_t \sum_{s=0}^{\infty} \beta^s \left[\frac{1}{1-\sigma_c} \left(C_{t+s}(j) - \lambda (C_{t+s-1}) \right)^{1-\sigma_c} \right] \exp \left(\frac{\sigma_c - 1}{1+\sigma_l} L_{t+s}(j)^{1+\sigma_l} \right)$$

- Habits are in form of time-varying ***external habits***
- *[NEW]* SW form of utility function:
- ***King-Plosser-Rebelo (KPR) preferences***

- General form of ***KPR Preferences***:

$$u(C,L) = \frac{1}{1-\sigma_c} C^{1-\sigma_c} v(L)$$

σ_c is risk aversion parameter = inverse of intertemporal rate of substitution

- where v(L) is increasing and concave if $0 < \sigma_c < 1$ or decreasing and convex if $\sigma_c > 1$
- In ***limit case*** of $\sigma_c = 1$, resulting preferences specification is ***additively separable*** and given by $u(C,L) = \ln C_t + v(L)$
- For SW, v(L) is given by $\exp\left(\frac{\sigma_c - 1}{1+\sigma_l} L_{t+s}(j)^{1+\sigma_l}\right)$
- Labour L is ***aggregated*** by a ***lazy*** union ➔ ***sticky nominal wages à la Calvo***

σ_l is inverse of "Frisch elasticity of labour supply", which is percentage change in hours arising from a given percentage change in wages

- Households ***save*** through purchases of government ***nominal bonds*** (B_{t+1}), from which they earn an interest rate of R_t
- Return on these bonds is subject to a ***risk shock*** ϵ_t^b
- Households own all capital ***stock*** and ***[NEW]*** rent capital ***services*** to firms, deciding how much capital ***stock*** to accumulate given ***capital adjustment costs***

$$K_t(j) = (1-\delta)K_{t-1}(j) + \epsilon_t^i \left[1 - S\left(\frac{I_t(j)}{I_{t-1}(j)}\right)\right] I_t(j)$$

NB: SW use Dynare timing convention, so no "Predetermined_ Variables K"

$$\boxed{S(x_t) = \frac{\varphi}{2}(x_t - \gamma)^2}$$

Quadratic form as seen last week

- As rental price of capital (R^k) changes, ***utilisation*** of capital ***stock*** can be adjusted at increasing ***cost***

$$K_t^s(j) = Z_t(j) K_{t-1}(j)$$

← K^s = capital ***services***; Z = ut. cost fn

- Firms produce ***differentiated*** goods, decide on labour and capital inputs, and set ***(sticky) prices***, again according to ***Calvo model***
- ***Calvo Rule*** here ***[NEW]***: those prices/wage rates that are ***not*** re-optimised are ***partially indexed*** to past inflation rates:

$$P_t(i) = (\pi_{\boxed{t-1}})^{\iota_p} (\pi_*)^{1-\iota_p} P_{t-1}(i)$$

  - π_* denotes steady-state gross inflation rate
- A positive value of ι_p introduces ***structural inertia*** into inflation process
- ***Similarly*** for ***wages*** (ι_w)

Some Detail: Intermediate Goods Sector

- ***Intermediate Good Producer i*** is assumed by SW to use standard ***Cobb-Douglas*** technology (with variable input costs) but subject also to a ***fixed cost***

$$Y_t(i) = \epsilon_t^a K_t^s(i)^\alpha \left[\boxed{\gamma^t} L_t(i)\right]^{1-\alpha} - \boxed{\gamma^t}\Phi$$

Odd that "fixed" cost varies with time !!

- where
  - $K_t^s(i)$ is ***capital services*** used in production
  - $L_t(i)$ is a composite labour (services) input
  - ***[NEW]*** Φ is a fixed cost
  - ***[NEW]*** $\boxed{\gamma^t}$ represents a (labour-augmenting) deterministic ***growth*** rate of output

- ϵ_t^a is ***total factor productivity*** and follows process

$$\ln \epsilon_t^a = (1-\rho_z)\boxed{ln \epsilon^a} + \rho_z \ln \epsilon_{t-1}^a + \eta_t^a, \eta_t^a \sim N(0, \sigma_a)$$

Steady-state value, usually assumed = 0

- Firm's ***profit*** is given by

$$P_t(i)Y_t(i) - W_t L_t(i) - R_t^k K_t^s(i)$$

- where
  - W_t is aggregate nominal wage rate
  - R_t^k is rental rate on capital services
- Cost minimisation yields following first-order conditions:

$$(\partial L_t(i)) : \Theta_t(i)\gamma^{(1-\alpha)t}(1-\alpha)\epsilon_t^a K_t^s(i)^\alpha L_t(i)^{-\alpha} = W_t$$

$$(\partial K_t^s(i)) : \Theta_t(i)\gamma^{(1-\alpha)t}\alpha\epsilon_t^a K_t^s(i)^{\alpha-1} L_t(i)^{1-\alpha} = R_t^k$$

- where $\Theta_t(i)$ is ***Lagrange multiplier*** (shadow value) associated with production function (and equals marginal cost MC_t)
- Combining these FOCs and noting that capital-labour ratio is equal across firms implies (***as usual***)

$$K_t^s = \frac{\alpha}{1-\alpha}\frac{W_t}{R_t^k}L_t$$

- ***Marginal cost*** MC_t is ***same for all firms*** and equals
$$MC_t = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} W_t^{1-\alpha} R_t^{k\alpha} \gamma^{-(1-\alpha)t} \left(\epsilon_t^a\right)^{-1}$$
- ***Prices*** of ***intermediate goods*** are determined by ***Calvo*** fairy/devil/lottery [P ≠ MC recall !!]
- As in previous Calvo models, in each period, each firm i faces a constant ***probability*** $1-\xi_p$ of being able to ***re-optimise*** its price $P_t(i)$
- Probability that firm receives Calvo-fairy signal to re-optimise its price assumed ***independent*** of time that it last reset its price ["Markov Process"]

- ***[NEW]*** ***Calvo Rule*** in SW: if Calvo fairy does ***not*** allow a firm to optimise its price in a given period, it then adjusts its price by a ***weighted combination*** of lagged and steady-state inflation rates as:

$$P_t(i) = (\pi_{\boxed{t-1}})^{\iota_p} (\pi_*)^{1-\iota_p} P_{t-1}(i)$$

- where
  - $0 \le \iota_p \le 1$ is the (price) ***indexation*** parameter
  - π_{t-1} denotes gross inflation in period t−1
  - π_* denotes steady-state gross inflation rate
- A positive value of ι_p introduces ***structural inertia*** into inflation process

- Under Calvo pricing with this type of ***partial indexation***,
- optimal price $\widetilde{P}_t(i)$ set by firm that is allowed by Calvo fairy (with a probability 1–ξ_p) to re-optimise
- results from solving following optimisation problem:

[Net profits]

$$\max_{\widetilde{P}_t(i)} E_t \sum_{s=0}^{\infty} \xi_p^s \boxed{\frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}}} \left[\widetilde{P}_t(i) (\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p}) - MC_{t+s} \right] Y_{t+s}(i)$$

Nominal discount factor

- This looks formidable!
- But if we compare it to what we used in sticky-price model studied last week, we find clear similarities:

$$\max_{P^*_{j,t}} E_t \sum_{i=0}^{\infty} \boxed{(\beta\theta)^i} \left(P^*_{j,t} Y_{j,t+i} - TC_{j,t+i} \right)$$

$$\max_{\widetilde{P}_t(i)} E_t \sum_{s=0}^{\infty} \boxed{\xi_p^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}}} \left[\widetilde{P}_t(i) \left(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_t} \pi_*^{1-\iota_p} \right) - MC_{t+s} \right] Y_{t+s}(i)$$

- Since TC = MC*Y, only really ***new element*** is ***indexation factor*** multiplying $P_t^{\sim}$: $\Pi_{l=1}^s \pi_{t+l-1}^{\iota_t} \pi_*^{1-\iota_p}$

Product operator !!

- What is this ***indexation factor*** $\boxed{\prod_{l=1}^{s}}\pi_{t+l-1}^{\iota_t}\pi_*^{1-\iota_p}$
- Since π is a ***gross*** inflation rate, it has each period a value close to 1
- Thus, product $\prod_{l=1}^{s}\pi_{t+l-1}^{\iota_t}$ (which multiplies together inflation rates for all periods since previous price revision) also has a value near 1
- This is then scaled by factor $\pi_*^{1-\iota_p}$ which reflects ***weighted combination*** indexation
- Finally, as noted previously, curious expression $\dfrac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}}$ is just the (nominal) discount factor

Ξ_t is Lagrange Multiplier for consumption, so this is just β multiplied by (1) ratio of Lagrange multipliers [≈ 1] x (2) ratio of prices at t and (t+s) [≈ 1]

- It is also clear from formulation of optimisation problem

$$\max_{\widetilde{P}_t(i)} E_t \sum_{s=0}^{\infty} \xi_p^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \left[\widetilde{P}_t(i) \left(\Pi_{l=1}^{s} \pi_{t+l-1}^{\iota_t} \pi_*^{1-\iota_p} \right) - \boxed{MC_{t+s}} \right] Y_{t+s}(i)$$

- that price set by firm i, at time t, is a function of ***expected future*** marginal costs
- Price will be a ***mark-up*** over these discount-weighted marginal costs

- Given FOC for this optimisation problem, ***aggregate price index*** which results is

$$P_t = (1-\xi_p)\boxed{\widetilde{P}_t}(i)G'^{-1}\left[\frac{\widetilde{P}_t(i)\tau_t}{P_t}\right] + \xi_p \pi_{t-1}^{\iota_p}\pi_*^{1-\iota_p}\boxed{P_{t-1}}G'^{-1}\left[\frac{\pi_{t-1}^{\iota_p}\pi_*^{1-\iota_p}P_{t-1}\tau_t}{P_t}\right]$$

- Compare this to what we used in first sticky-price model studied last week:

$$P_t = \left[\theta P_{t-1}^{1-\psi} + (1-\theta)P_t^{*1-\psi}\right]^{\frac{1}{1-\psi}}$$

- Function G (of which G' is used in definition above) is defined in Appendix

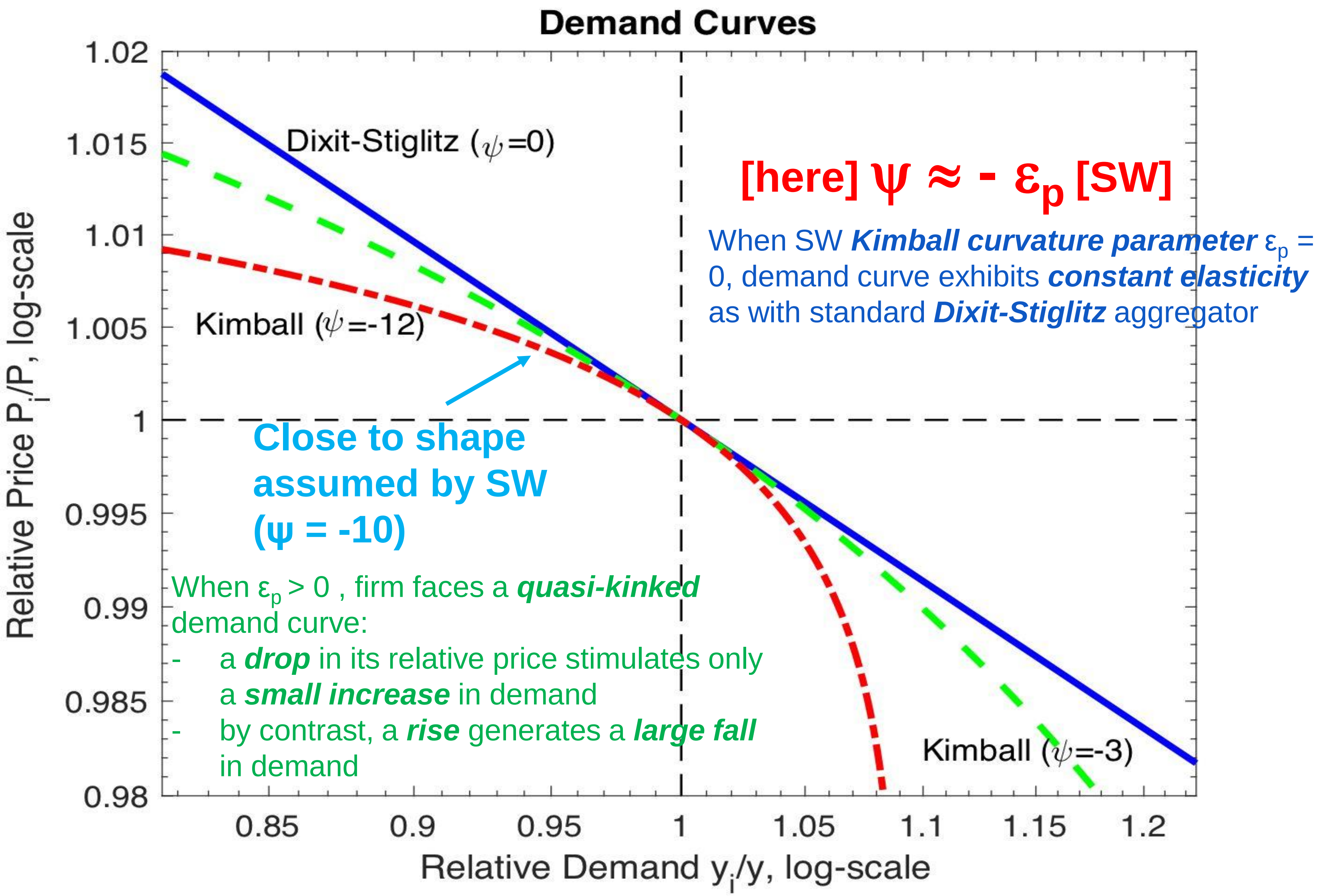
- Prices therefore determined (partially) by ***past inflation rate***
- ***Marginal costs*** are a function of ***wages*** and ***rental rate*** of capital

$$MC_t = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} \boxed{W_t}^{1-\alpha} \boxed{R_t^k}^{\alpha} \gamma^{-(1-\alpha)t} (\epsilon_t^a)^{-1}$$

- ***Same*** mechanism used by SW for ***wages***

- Here endth such detail!
- Finally, a ***technical point***:
- In both goods and labour markets SW ***replace*** standard ***Dixit-Stiglitz*** aggregator
- with a ***Kimball (1995) aggregator***
- ➔ allows for ***time-varying demand elasticity***, which depends on ***relative price***
- Introduction of Kimball aggregator allows SW to estimate a ***more reasonable degree*** of price and wage ***stickiness***

Figure 1: Demand Curves -- Implications of Kimball vs. Dixit-Stiglitz Aggregators.



- For those who wish to see technical details of original non-linear SW2007 model, I have attached an Appendix which does so
- Here, we shall look instead at SW's ***published*** log-linearised equations

SW2007 Log-Linearised Model

- SW ***detrended*** their variables with ***deterministic*** trend γ and replaced ***nominal*** variables by their ***real per-capita*** counterparts – this is very ***familiar***! [Some authors have later used a ***time-varying*** trend to do this for SW2007]
- ***Non-linear*** system was then ***linearised*** around stationary ***steady state*** of ***detrended*** variables, in usual ***log-linearisation*** process which we have seen previously
- This was done primarily because system is highly non-linear and ***multi-dimensional*** (34 parameters plus 7 s.d. to be estimated ***simultaneously***)

- ***Log-linearised*** model (equations taken directly from SW2007 in ***AER***): start with ***Supply Side***
- ***Aggregate production function*** is (*eq.5*)

$$y_t = \boxed{\phi_p} \left(\alpha k_t^s + (1-\alpha)\ell_t + \epsilon_t^a\right)$$

ϕ_p is (1 +) share of fixed costs in production, reflecting presence of *fixed costs* in production function

- where
  - y_t is (log-linearised) GDP
  - ℓ_t is labour input
  - ϵ_t^a is total factor productivity [shock ***process***]
  - *[NEW]* k^s is ***capital services***, determined by ***capital stock installed*** in previous period $k_t^s = k_{t-1} + z_t$
  - and a ***capacity utilisation*** variable ***z*** (*eq.6*)

- ***Costs of adjusting*** capital stock in use → rate of ***capacity utilisation*** linked to rental rate of capital (*eq.7*)

$$z_t = z_1 r_t^k$$

- ***Rental rate of capital*** is a function of capital-labour ratio and real wage (*eq.11*)

Remember, these are logs:
log(ratio) = difference

$$r_t^k = -(k_t - \ell_t) + w_t$$

- ***Total factor productivity*** evolves over time according to an ***AR(1)*** process

This is a process; ρ_a is likely close to 1; actual "shock" comes through η^a

$$\epsilon_t^a = \rho_a \epsilon_{t-1}^a + \boxed{\eta_t^a}$$

- Now ***Demand Side***
- Expenditure formulation of aggregate resource constraint is familiar (*eq.1*)

$$y_t = c_y c_t + i_y i_t + z_y z_t + \epsilon_t^g$$

- where
  - y_t is GDP
  - c_t is consumption
  - i_t is investment
  - z_t is exogenous spending [= government + net exports]
  - ϵ_t^g is spending shock ***process***
  - c_y, i_y and z_y are constant ***parameters***, representing (as usual) ***steady-state shares*** of each variable in y

- ***Consumption*** is determined by usual ***Euler Equation***

$$c_t = c_1 c_{t-1} + (1 - c_1)\, E_t c_{t+1} + c_2\,(\ell_t - E_t \ell_{t+1}) - c_3 \left(r_t - E_t \pi_{t+1} + \epsilon_t^b \right)$$

- (*eq.2*) where
  - c_1, c_2, c_3 are constant ***coefficients***
  - r_t is interest rate on a ***one-period*** risk-less ***bond***
  - ϵ_t^b evolves according to ***AR(1)*** process $\epsilon_t^b = \rho_b \epsilon_{t-1}^b + \eta_t^b$

- Note that this Euler Equation has a ***backward-looking*** element, deriving from “habit formation” [$C_t - \lambda C_{t-1}$ replaces C_t in utility function, recall]

- Term $c_2\left(\ell_t - E_t \ell_{t+1}\right)$ involving ***labour*** input allows for some ***substitution*** between consumption and labour input
- ***Coefficients*** c_1, c_2, c_3 are themselves functions of deeper ***structural parameters***
- SW describe ϵ^b term as a ***"risk premium" shock*** determining willingness of households to hold one-period bond $c_3\left(r_t - E_t \pi_{t+1} + \epsilon_t^b\right)$
- It can also be seen as a type of ***preference shock*** that influences short-term consumption-saving decision (this is more ***usual interpretation***)

- ***Investment*** is determined by (*eq.3*)

$$i_t = i_1 i_{t-1} + (1 - i_1) E_t i_{t+1} + i_2 q_t + \epsilon_t^i$$

- where ***shadow price of capital stock*** is (*eq.4*)

$$q_t = q_1 E_t q_{t+1} + (1 - q_1) r_{t+1}^k - (r_t - E_t \pi_{t+1} + \epsilon_t^b)$$

Same as in eq. 2

- ***nominal value of installed capital*** is given by (*eq.8*)

$$k_t = k_1 k_{t-1} + (1 - k_1) i_t + k_2 \epsilon_t^i$$

Same as in eq. 3

- Investment depends on ***lagged*** investment because ∃ ***adjustment cost*** function that limits amount of new investment that can come "on line" immediately

Recall, SW use Dynare timing convention

- ***Main driving force*** behind investment is ***shadow price of capital stock (Tobin's Q) q_t*** (*eq.4* above):

$$q_t = q_1 E_t q_{t+1} + (1 - q_1) r^k_{t+1} \boxed{- (r_t} - E_t \pi_{t+1} + \epsilon^b_t)$$

- q_t depends ***positively*** on expected future marginal productivities of capital (r^k) and ***negatively*** on future real interest rate (and "risk premia")
- Note: investment ***shock*** ϵ_t^i appears in ***both i_t and k_t*** equations ➔ positive shock to investment also increases capital stock
- SW find that, ***empirically, investment adjustment cost shocks are the most important in entire model***; by contrast (and curiously), ***capacity utilisation costs*** are estimated to be not very important

- Recall: Y = C + I + [G + (X – M)] or here:

$$y_t = c_y c_t + i_y i_t + z_y z_t + \epsilon_t^g$$

- ***Exogenous spending*** z_t has two components:
  - ***Government*** spending G
  - an element related to ***productivity*** because "***net exports*** (X – M) may be affected by domestic productivity developments"
- ➔ exogenous spending shock ϵ_t^g changes over time according to ***[NEW]*** a ***cross-equation*** process:

$$\boxed{\epsilon_t^g = \rho \epsilon_{t-1}^g} + \eta_t^g + \rho_{ga} \eta_t^a$$

Productivity !

- ***Mark-up*** of ***price*** over marginal cost is determined in log-linearised model by difference between ***marginal product of labour*** and ***real wage*** (*eq.9*)

$$\mu_t^p = mpl_t - w_t = \alpha\left(\boxed{k_t^s - l_t}\right) + \epsilon_t^a - w_t$$

- where ***marginal product of labour*** is itself a (positive) function of ***capital-labour ratio*** and total factor ***productivity (TFP)***

- ➔ price ***inflation*** determined by "***New Keynesian Phillips Curve***" (*eq.10*)

$$\pi_t = \pi_1 \pi_{t-1} + \pi_2 E_t \pi_{t+1} - \pi_3 \boxed{\mu_t^p} + \epsilon_t^p$$

- ***Lagged*** inflation appears in NKPC as a result of ***Calvo*** pricing structure
- If ***degree of indexation*** to past inflation is ***zero*** (ι_p=0), *eq.10* reverts to ***purely forward-looking*** Phillips curve (coefficient π_1=0) seen last week

$$\pi_t = \pi_1 \pi_{t-1} + \pi_2 E_t \pi_{t+1} - \pi_3 \mu_t^p + \epsilon_t^p$$

- ***Speed of adjustment*** to desired ***mark-up*** (coefficient π_3) depends on
  - ***degree of price stickiness*** (ξ_p)
  - ***curvature*** of ***Kimball*** goods market aggregator (ε_p)
  - steady-state price ***mark-up*** (φ_p - 1)

- In $$\pi_t = \pi_1 \pi_{t-1} + \pi_2 E_t \pi_{t+1} - \pi_3 \mu_t^p + \epsilon_t^p$$
- ϵ_t^p is ***price mark-up*** shock, which evolves as ***[NEW] ARMA(1,1)***:
$$\epsilon_t^p = \rho^p \epsilon_{t-1}^p + \boxed{\eta_t^p - \mu_p \eta_{t-1}^p}$$

- ***MA*** component has ***no memory***, recall

  ➔ ***One-period-memory*** MA term captures ***high-frequency*** fluctuations in inflation

- ***Price mark-up shock*** ϵ_t^p is included because SW find it is ***empirically very important in order*** to capture ***price dynamics***

- SW model treats ***wages*** similarly to prices:
- Sticky wages gradually adjust ➔real wages move only gradually over time
- to ***equate real wages*** with ***marginal rate of substitution*** between working and consuming
- Short-term ***gap*** between these is "***wage mark-up***" defined as (*eq.12*)

$$
\begin{aligned}
\mu_t^w &= w_t - mrs_t \\
&= w_t - \left(\sigma \ell_t - \frac{1}{1-\lambda/\gamma} \left(c_t - \frac{\lambda}{\gamma} c_{t-1} \right) \right)
\end{aligned}
$$

- Wages are then given by (*eq.13*)

$$w_t = w_1 w_{t-1} + (1 - w_1)(E_t w_{t+1} + E_t \pi_{t+1}) - w_2 \pi_t + w_3 \pi_{t-1} - w_4 \mu_t^w + \epsilon_t^w$$

- Parameter w_4 ➔ ***speed of adjustment*** to desired ***wage mark-up*** depends on degree of wage ***stickiness*** (ξ_w) and demand elasticity for labour
- which itself is a function of ***steady-state labour market mark-up*** (φ_w - 1) and ***curvature*** of ***Kimball*** labour market aggregator (ε_w – do not ***confuse*** this with shock process ϵ_t^w !)
- **Dynare** model therefore contains a complicated equation defining w_4

- The (again, ***ARMA***) ***wage mark-up shock*** has form

$$\epsilon_t^w = \rho^w \epsilon_{t-1}^w + \boxed{\eta_t^w - \mu_w \eta_{t-1}^w}$$

- (As for prices) ***wage mark-up shock*** affects both current and lagged inflation and attempts also to capture ***temporary*** wage shocks with MA
- SW again find that wage mark-up shock is ***empirically very important*** to capture ***wage dynamics***
- ***Overall SW confirm, empirically, NK thesis*** by finding that both price and wage ***stickiness*** are very ***important***, with prices affecting short-run inflation and wages longer-run inflation

- Final element of model is rule for ***monetary policy***
- ***Central bank*** sets short-term interest rates according to ***modified Taylor Rule***, which in log-linearised form becomes (*eq.14*)

$$\begin{aligned} r_t &= \rho r_{t-1} + (1-\rho)\left(r_\pi \pi_t + r_y \left(y_t - y_t^p\right)\right) \\ &\quad + r_{\Delta y}\left[\left(y_t - y_t^p\right) - \left(y_{t-1} - y_{t-1}^p\right)\right] + \epsilon_t^r \end{aligned}$$

- where **y^p** is ***potential output*** ("flex-price" output)
- a monetary policy ***shock*** is included (of simple ***AR(1)*** form)

$$\epsilon_t^r = \rho^r \epsilon_{t-1}^r + \eta_t^r$$

- ***Interest rate*** depends on last period's interest rate while ***gradually adjusting*** towards a ***target interest rate*** ($r_\pi \pi_t + r_y(y_t - y_t^p)$) that depends on
  - inflation π_t
  - ***output gap*** ($y_t - y_t^p$) between ***actual*** and ***potential*** output (in SW, $y^p \equiv$ ***flex-price*** level of output)
- Interest rate also depends on ***growth rate*** of this output gap $$\left[\left(y_t - y_t^p\right) - \left(y_{t-1} - y_{t-1}^p\right)\right]$$
- (recall these are logs ➔ ***dlog*** = %chg = growth rate)

- ***Potential output*** $\boldsymbol{y_t^p}$ as used in Taylor Rule ≡ output obtained ***if*** prices and wages were ***fully flexible***
- ➔ *[NEW]* model effectively needs to be ***expanded*** to add a ***shadow*** flexible-price economy
- ➔in **Dynare** implementation ∃ ***repetitive*** section defining variables in ***flex-price*** economy
- A different definition of Taylor Rule or of output gap would ***avoid*** need for this section
- In fact, ***original*** SW2003 ***Euro area*** model did ***not*** include "shadow" flexible-price economy
- SW2007 added it to help explain price dynamics

Dynare Model

- Implementation in **Dynare** was originally written by SW themselves, so we make use of it here
- Unfortunately, their ***code*** used a ***notation*** which ***differs considerably*** from that set out in their SW2007 AER paper, so I have re-written their code for our use in **Dynare**
- There are 14 equations in **Dynare** log-linearised model for sticky wage-price economy
- Look at just 4 of them

- ***Consumption***
- *Eq.2* expressed consumption as

$$c_t = c_1 c_{t-1} + (1 - c_1) E_t c_{t+1} + c_2 (\ell_t - E_t \ell_{t+1}) - c_3 \left(r_t - E_t \pi_{t+1} + \epsilon_t^b \right)$$

- In **Dynare**, we have

```
#c1 = (lambda/gamma)/(1+lambda/gamma);
#c2 =  ((sigma_c-1)*WL_C/(sigma_c*(1+lambda/gamma)));
#c3 = (1-lambda/gamma)/(sigma_c*(1+lambda/gamma));
c = c1 * c(-1) + (1 - c1) * c(+1) + c2 * (l - l(+1)) - c3 * (r - pinf(+1)) + eps_b;
```

Minor point: in SW's Dynare model, they separate out eps_b as here

$$c_1 = \frac{\lambda / \gamma}{1 + \lambda / \gamma} \qquad c_2 = \frac{(\sigma_c - 1)(W_*^h L_* / C_*)}{\sigma_c (1 + \lambda / \gamma)} \qquad c_3 = \frac{1 - \lambda / \gamma}{(1 + \lambda / \gamma)\sigma_c}$$

$$c_1 = \frac{\lambda / \gamma}{1 + \lambda / \gamma} \qquad c_2 = \frac{(\sigma_c - 1)\boxed{(W_*^h L_* / C_*)}}{\sigma_c (1 + \lambda / \gamma)} \qquad c_3 = \frac{1 - \lambda / \gamma}{(1 + \lambda / \gamma)\sigma_c}$$

- In c_1, c_2 and c_3, ***deep parameters*** are
  - ***lambda*** ≡ degree of (external) ***habit persistence***
  - ***sigma_c*** ≡ ***risk aversion*** parameter
  - ***gamma*** ≡ gross ***growth rate*** [assuming ***cointegration*** of y,c,i]
- "WL_C" is a complicated combination of deep parameters and steady-state terms

- Next, ***New Keynesian Phillips Curve*** (*eq.10*)

$$\pi_t = \pi_1 \pi_{t-1} + \pi_2 E_t \pi_{t+1} - \pi_3 \mu_t^p + \epsilon_t^p$$

- becomes

```
# pi1 = (1/(1+beta_bar*gamma*iota_p))* iota_p;
# pi2 = (1/(1+beta_bar*gamma*iota_p))* beta_bar*gamma;
# pi3 = (1/(1+beta_bar*gamma*iota_p))*
   ((1-xi_p)*(1-beta_bar*gamma*xi_p)/xi_p)/((phi_p-1)*curv_p+1) ;
pinf = pi1*pinf(-1) + pi2*pinf(1) + pi3*mc + eps_p;
```

$$\pi_1 = \frac{\iota_p}{1+\beta\gamma^{1-\sigma_c}\iota_p} \qquad \pi_2 = \frac{\beta\gamma^{1-\sigma_c}}{1+\beta\gamma^{1-\sigma_c}\iota_p} \qquad \pi_3 = \frac{1}{1+\beta\gamma^{1-\sigma_c}\iota_p}\frac{(1-\beta\gamma^{1-\sigma_c}\xi_p)(1-\xi_p)}{\xi_p((\phi_p-1)\varepsilon_p+1)}$$

- Repeating:

$$\pi_1 = \frac{\boxed{\iota_p}}{1+\beta\gamma^{1-\sigma_c}\iota_p} \qquad \pi_2 = \frac{\boxed{\beta\gamma^{1-\sigma_c}}}{1+\beta\gamma^{1-\sigma_c}\iota_p} \qquad \pi_3 = \frac{1}{1+\beta\gamma^{1-\sigma_c}\iota_p}\frac{(1-\beta\gamma^{1-\sigma_c}\xi_p)(1-\boxed{\xi_p})}{\xi_p((\phi_p-1)\boxed{\varepsilon_p}+1)}$$

- Many new parameters here:
  - ***beta_bar*** $\equiv \beta\gamma^{-\sigma_c}$ [ie, beta*gamma^(-sigma_c)] is growth-adjusted discount factor, beta being ***discount factor***
  - ***iota_p*** = coefficient of ***indexation*** to past prices
  - ***xi_p*** = ***Calvo parameter*** for prices
  - ***curv_p*** = ***curvature of Kimball aggregator*** for prices (instead of ε_p to avoid confusion with shock ϵ^p)
- Hence ***beta_bar*gamma*** = $\boxed{\beta\gamma^{1-\sigma_c}}$

- ***Wage Phillips Curve*** (*eq.13*)

$$w_t = w_1 w_{t-1} + (1 - w_1)(E_t w_{t+1} + E_t \pi_{t+1}) - w_2 \pi_t + w_3 \pi_{t-1} - w_4 \mu_t^w + \epsilon_t^w$$

- translates similarly into **Dynare** code below (once *eq.12* – next slide - is substituted in for μ_t^w)

```
#w1 = (1/(1+beta_bar*gamma));
#w2 = (1+beta_bar*gamma*iota_w)/(1+beta_bar*gamma);
#w3 = (iota_w/(1+beta_bar*gamma));
#w4 = - (1-xi_w)*(1-beta_bar*gamma*xi_w)/((1+beta_bar*gamma)*xi_w)*
        (1/((phi_w-1)*curv_w+1));
w = w1*w(-1) + (1 - w1)*w(1) + (1 - w1)*pinf(1) - w2*pinf + w3*pinf(-1)
    - w4*(sigma_l*l + (1/(1-lambda/gamma))*c -
    ((lambda/gamma)/(1-lambda/gamma))*c(-1) -w) + eps_w;
```

$$w_1 = \frac{1}{1+\beta\gamma^{1-\sigma_c}} \qquad w_2 = \frac{1+\beta\gamma^{1-\sigma_c}\boxed{\iota_w}}{1+\beta\gamma^{1-\sigma_c}} \qquad w_3 = \frac{\boxed{\iota_w}}{1+\beta\gamma^{1-\sigma_c}}$$

$$w_4 = \frac{1}{1+\beta\gamma^{1-\sigma_c}} \frac{(1-\beta\gamma^{1-\sigma_c}\xi_w)(1-\xi_w)}{\boxed{\xi_w}((\phi_w - 1)\varepsilon_w + 1)}$$

- New parameters here are
  - **_iota_w_** = coefficient of **_indexation_** to past **_wages_**
  - **_xi_w_** = **_Calvo parameter_** for **_wages_**
  - **_curv_w_** = **_curvature of Kimball aggregator_** for **_wages_**
  - **_phi_w_** = gross steady-state **_wage mark-up_**
- **_Wage mark-up_** (*eq. 12*) is

$$\mu_t^w = w_t - mrs_t = w_t - \left(\sigma_l l_t + \frac{1}{1-\lambda/\gamma}(c_t - \lambda/\gamma c_{t-1})\right)$$

- Finally ***Taylor Rule*** (*eq.14*)

$$
\begin{aligned}
r_t \quad = \quad & \rho r_{t-1} + (1-\rho)\left(r_\pi \pi_t + r_y \left(y_t - y_t^p\right)\right) \\
& + r_{\Delta y}\left[\left(y_t - y_t^p\right) - \left(y_{t-1} - y_{t-1}^p\right)\right] + \epsilon_t^r
\end{aligned}
$$

- In **Dynare** implementation of this Taylor Rule, ***potential output*** variable $\mathbf{y_t^p}$ is defined as **yf**, ***flex-price*** (or "natural") value of output
- ➔ introduce entire ***flex-price economy*** into model
- ➔***11 additional equations***, in each case setting parameters which define "stickiness" of prices (and wages) to their ***"zero stickiness"*** values

- For example, ***Wage equation*** in ***flex-price*** economy becomes

wf = sigma_l*labf +(1/(1-lambda/gamma))*cf -(lambda/gamma)/(1-lambda/gamma)*cf(-1) ;

- which is just ***last bit*** of Wage equation in ***sticky-price, sticky-wage*** economy because
  - ***iota_w= 0***
  - ***xi_w = 1***

  in ***flex-price*** economy

Steady State

- One last step necessary before we can test out model: inserting information on model's ***steady state***
- These are derived as usual, for example

$$R_* = \overline{\beta}^{-1} \pi_*$$

$$k_* = \frac{\alpha}{1-\alpha} \frac{w_*}{r_*^k} L_*$$

$$i_* \quad = \quad (1 - (1-\delta)/\gamma)\, \bar{k}_*$$

$$\frac{c_*}{y_*} + \frac{i_*}{y_*} + g_* = 1$$

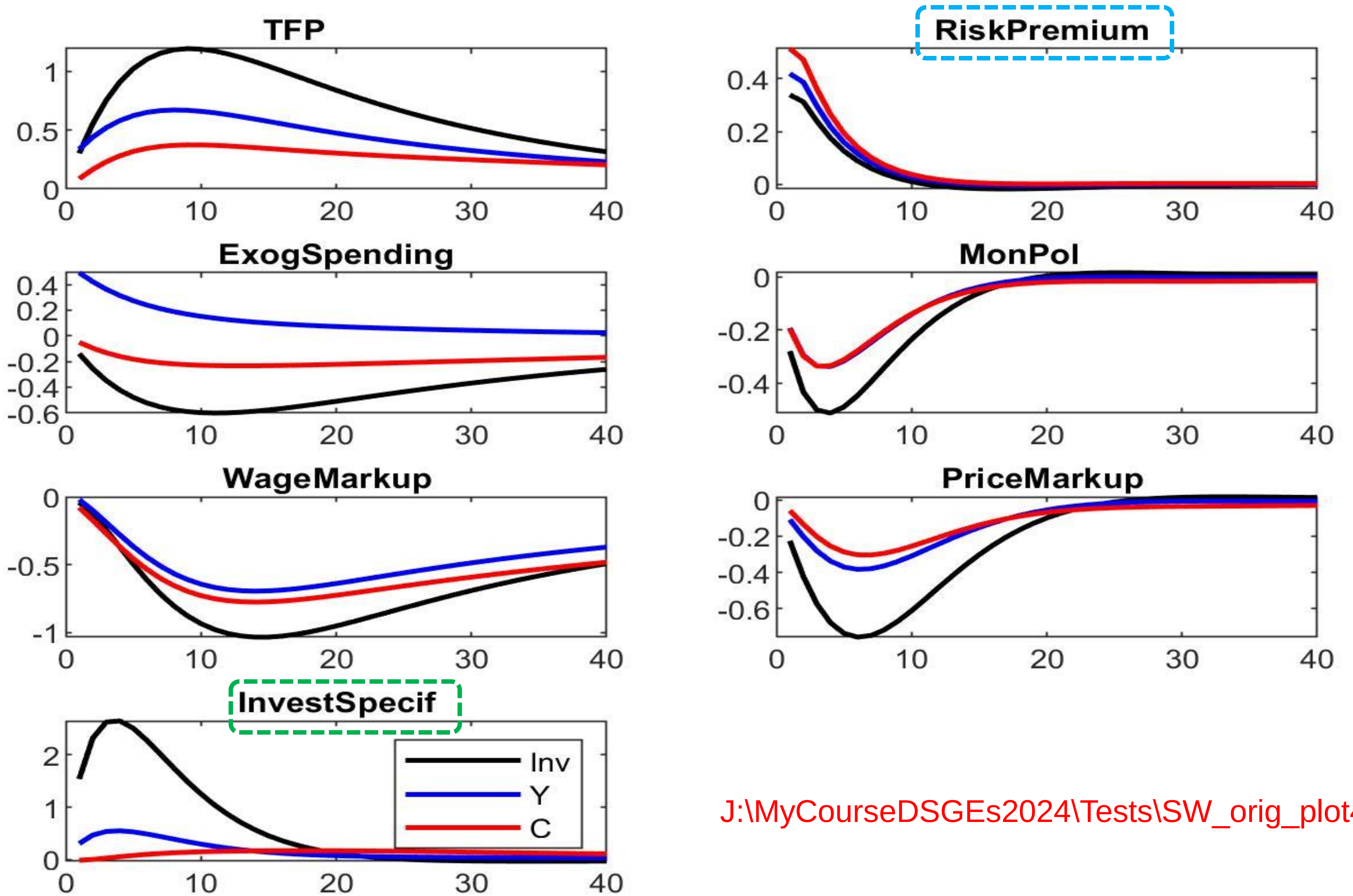
- These steady-state conditions are inserted into **Dynare** model as ***restrictions on parameters***
- For example, steady-state rental rate r_*^k is defined in **Dynare** code as

$$
\begin{aligned}
r_*^k &= \overline{\beta}^{-1} - (1-\delta) \\
&= \beta^{-1}\gamma^{\sigma_c} - (1-\delta)
\end{aligned}
$$

- **#Rk = (beta^(-1)) * (gamma^sigma_c) - (1-delta);**
- where # sign informs **Dynare** that this is ***not*** a parameter to be ***estimated directly***, but one ***calculated*** from various other "deep" parameters
- **#** ➔ ***model_local_variable***
- **Rk** is then used in value of capital *equation 4*, and also in definitions of other parameters (eg, steady state real wage and labour-to-capital ratio)

- Once this is done, it is possible to ***simulate***
  - Slides below show impact of each shock on model's endogenous variables
  - Using ***original SW2007*** parameters
- As SW note, apart from ***TFP shock (*** ϵ^a ***), risk premium (*** ϵ^b ***)*** and ***investment-specific (*** ϵ^i ***) shocks*** are most powerful [observe ***scale*** of impact]
- Recall that SW describe ϵ^b term as a shock determining willingness of households to hold one-period bond
- But also seen as a type of ***preference shock*** that influences short-term consumption-saving decision

- Responses to all shocks – Y, C, Inv



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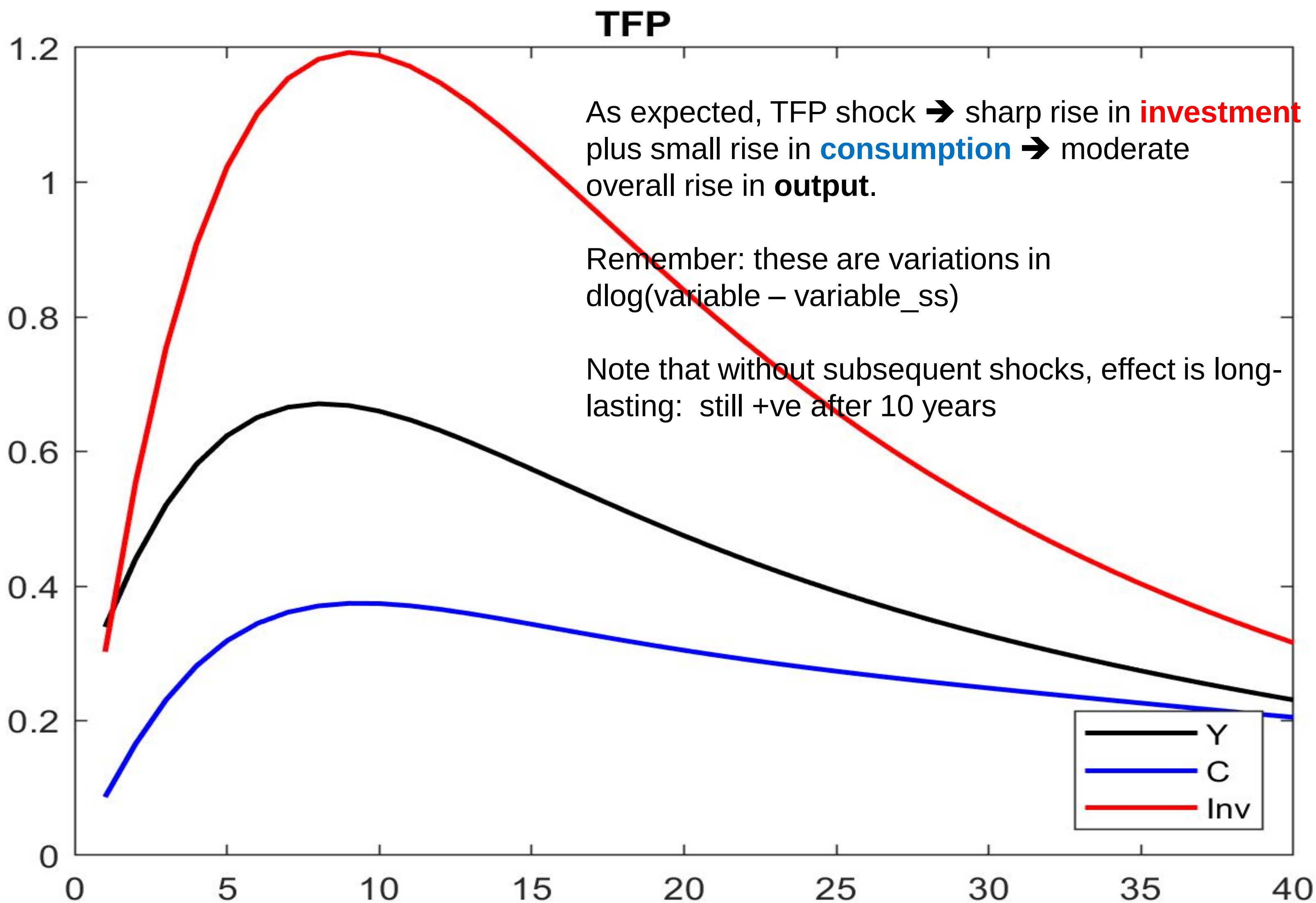
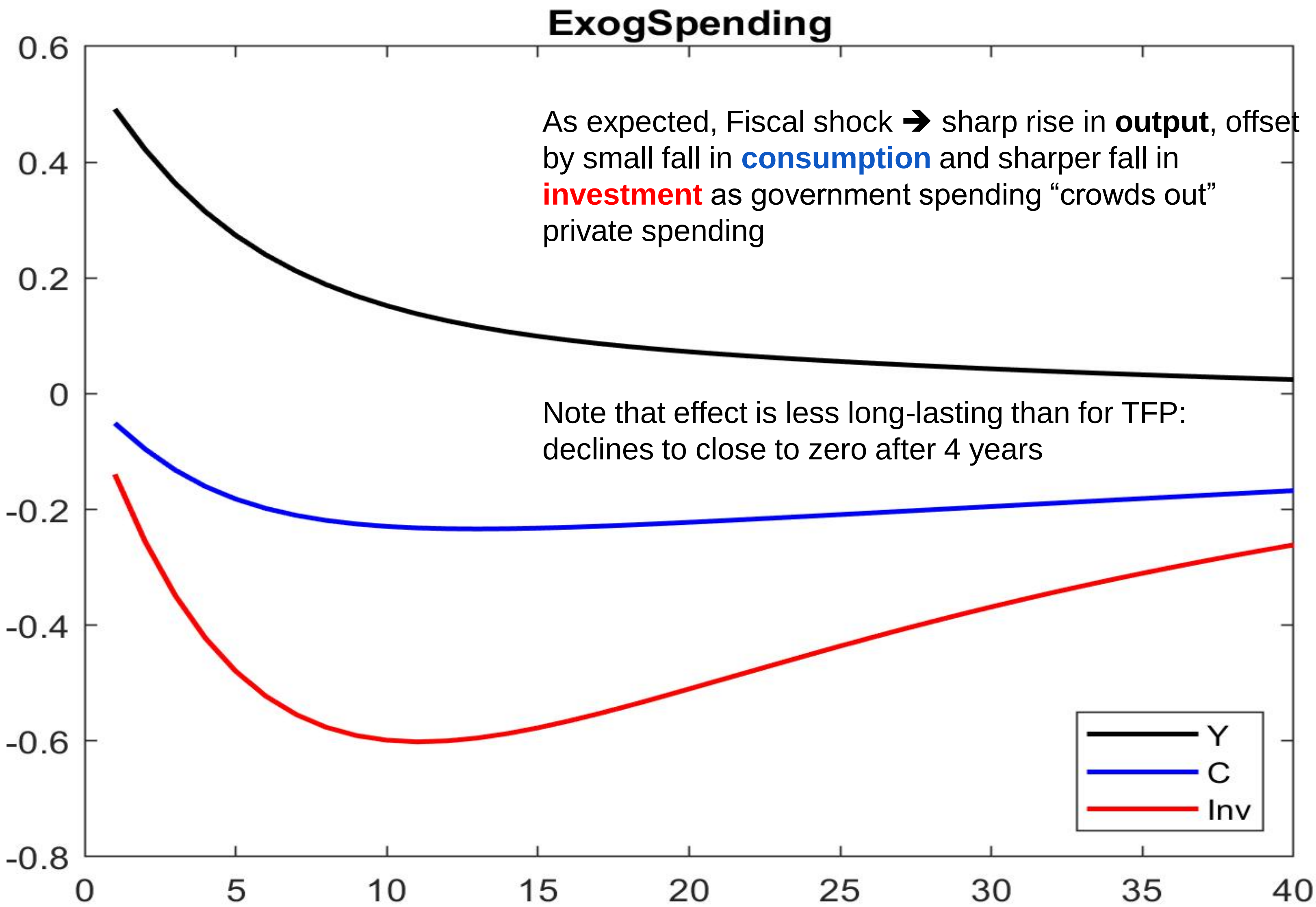
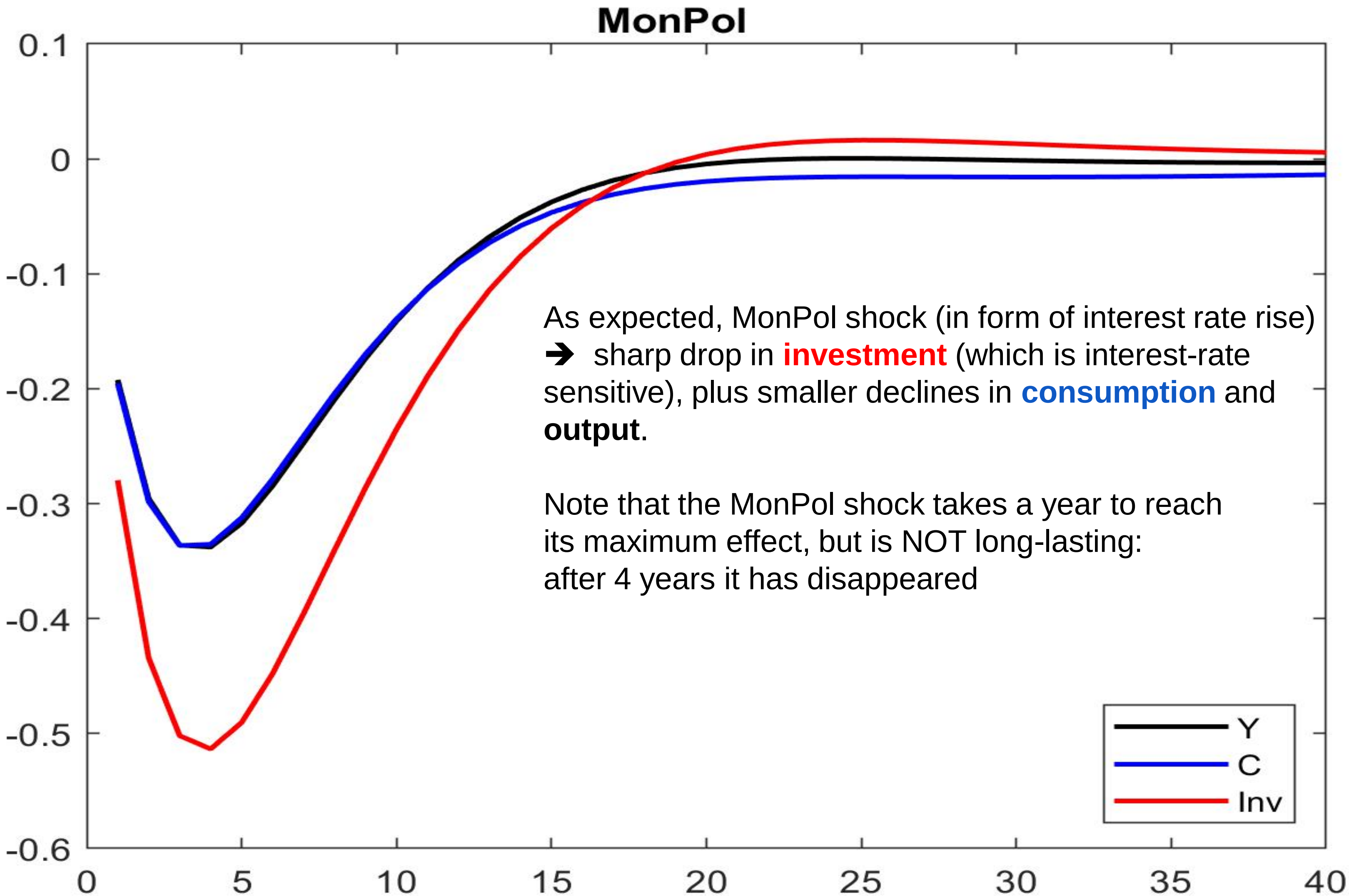
- Responses to TFP shock – Y, C, Inv



• Responses to Fiscal shock – Y, C, Inv



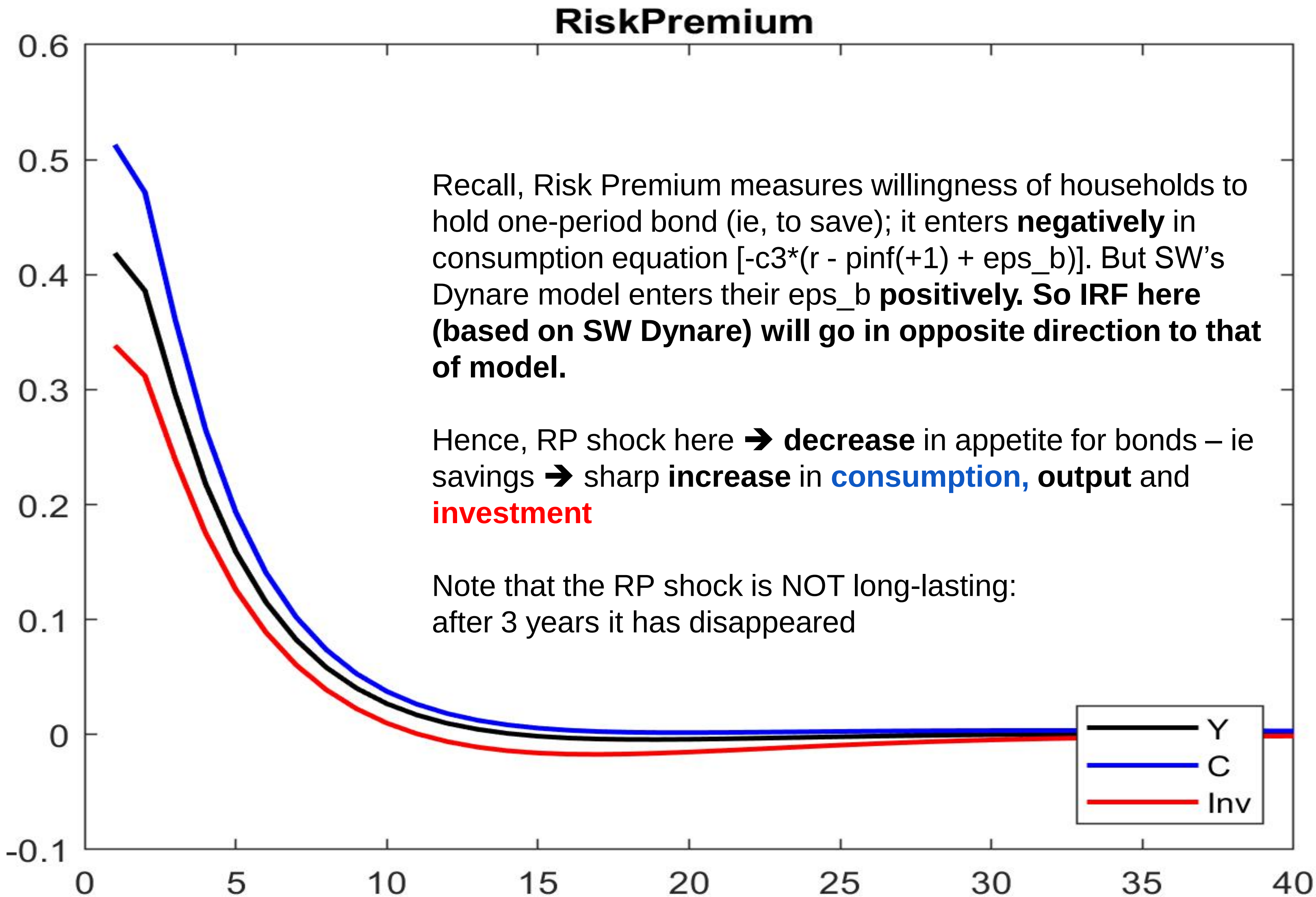
- Responses to MonPol shock – Y, C, Inv



As expected, MonPol shock (in form of interest rate rise) ➔ sharp drop in **investment** (which is interest-rate sensitive), plus smaller declines in **consumption** and **output**.

Note that the MonPol shock takes a year to reach its maximum effect, but is NOT long-lasting: after 4 years it has disappeared

- Responses to Risk Premium shock – Y, C, Inv

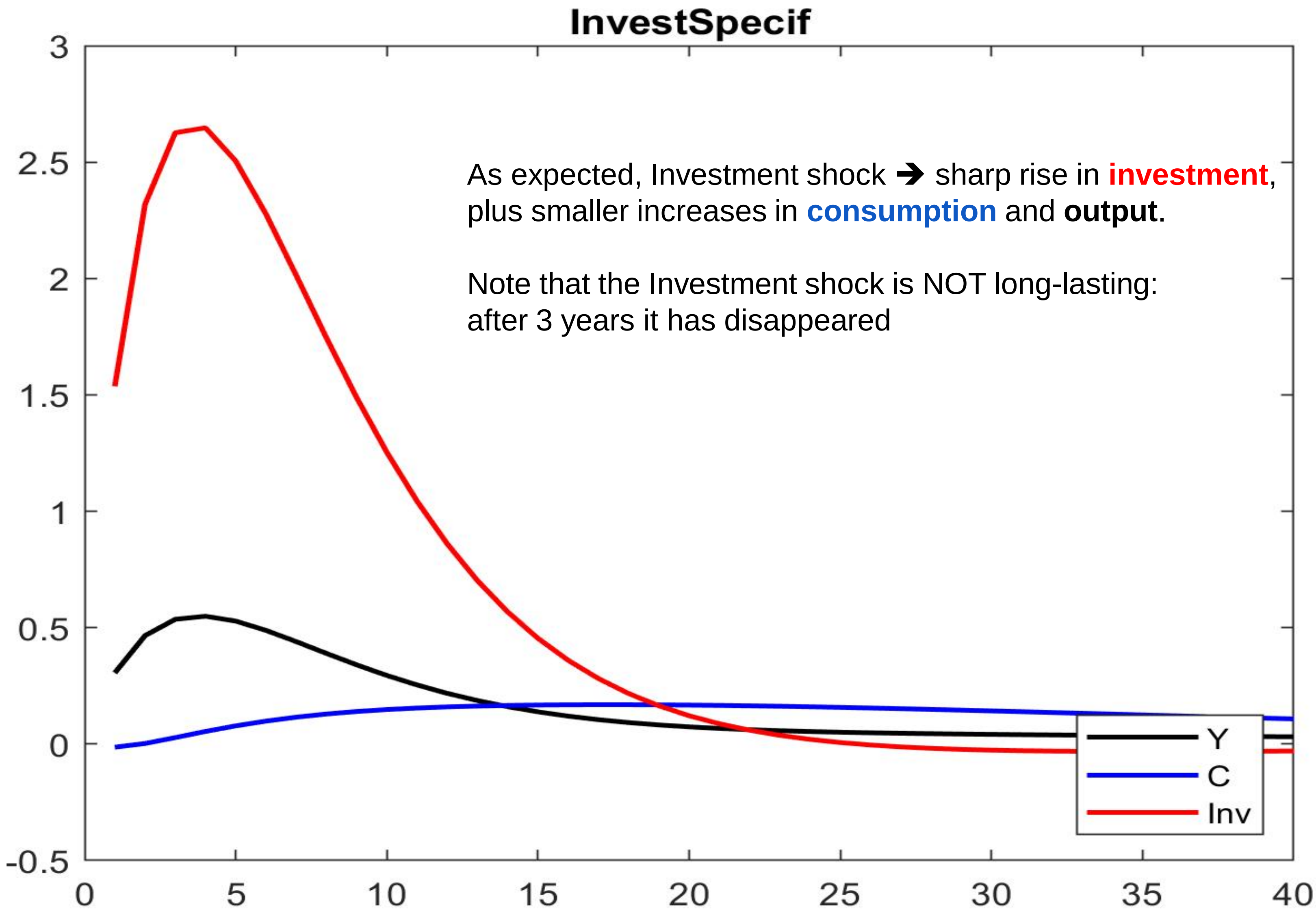


Recall, Risk Premium measures willingness of households to hold one-period bond (ie, to save); it enters **negatively** in consumption equation [-c3*(r - pinf(+1) + eps_b)]. But SW's Dynare model enters their eps_b **positively. So IRF here (based on SW Dynare) will go in opposite direction to that of model.**

Hence, RP shock here ➔ **decrease** in appetite for bonds – ie savings ➔ sharp **increase** in **consumption, output** and **investment**

Note that the RP shock is NOT long-lasting: after 3 years it has disappeared

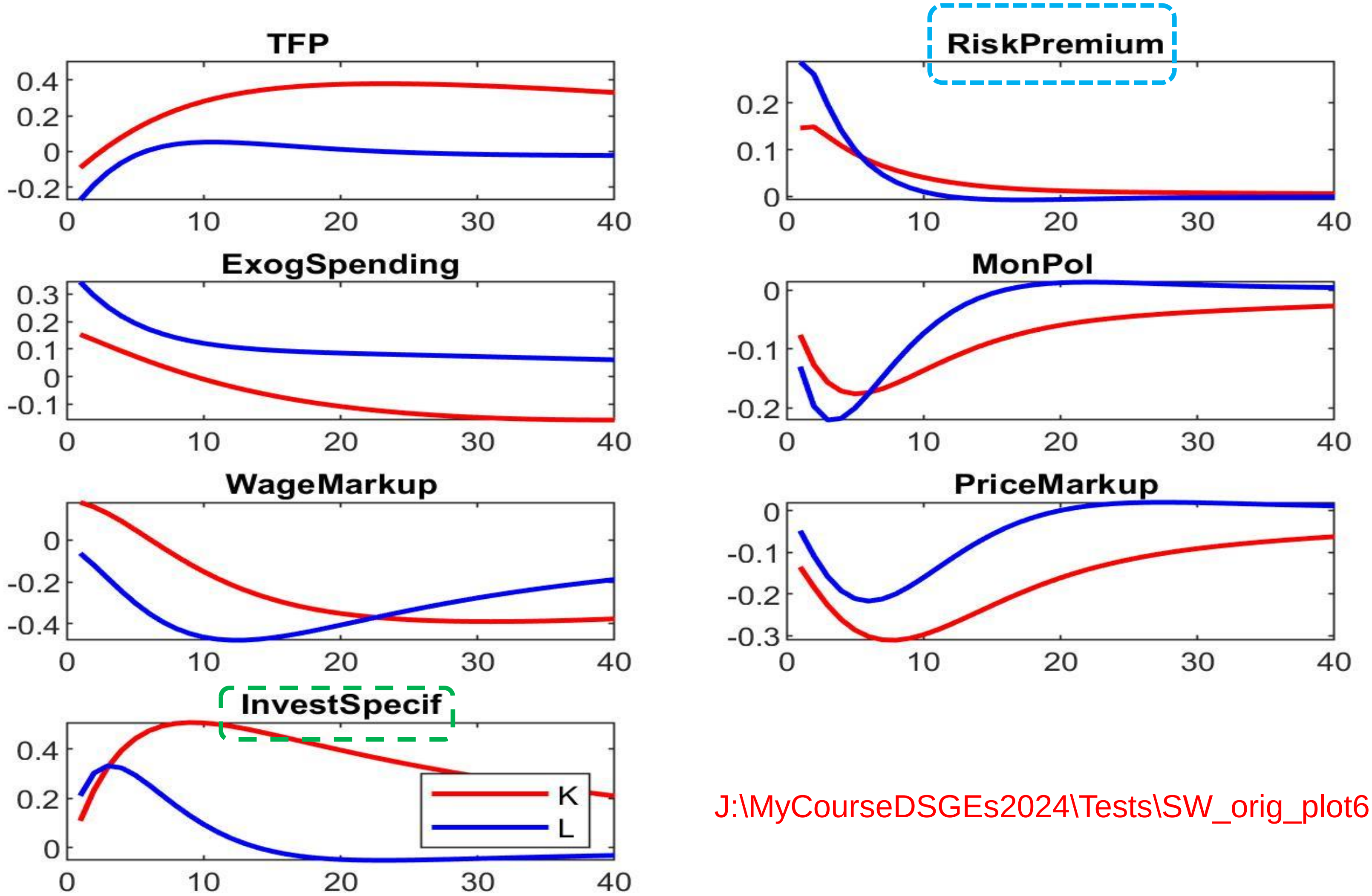
- Responses to Investment shock – Y, C, Inv



As expected, Investment shock ➔ sharp rise in **investment**, plus smaller increases in **consumption** and **output**.

Note that the Investment shock is NOT long-lasting: after 3 years it has disappeared

- Responses to all shocks – Capital and Labour



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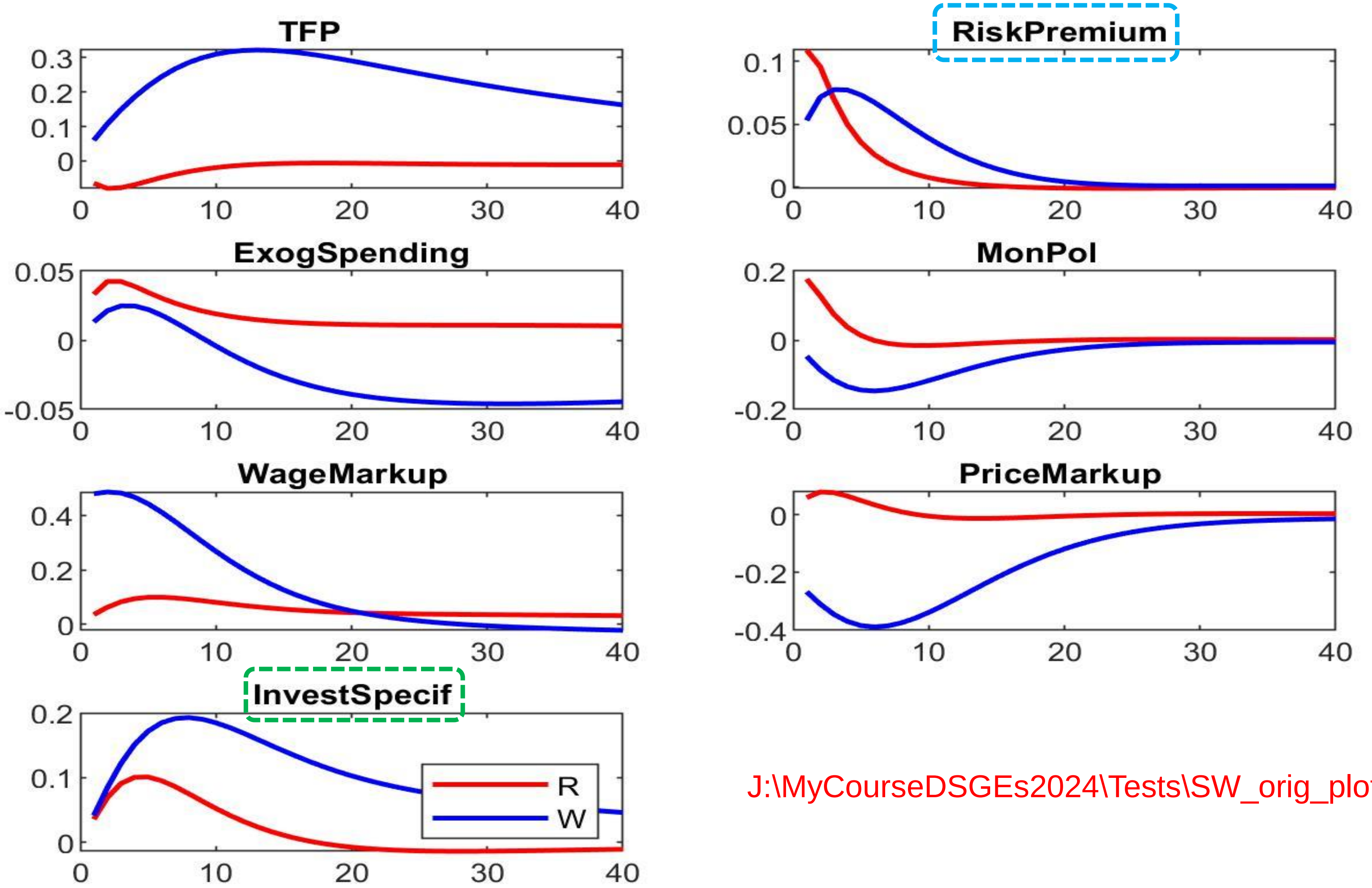
- Responses to all shocks – Interest and Wage Rates



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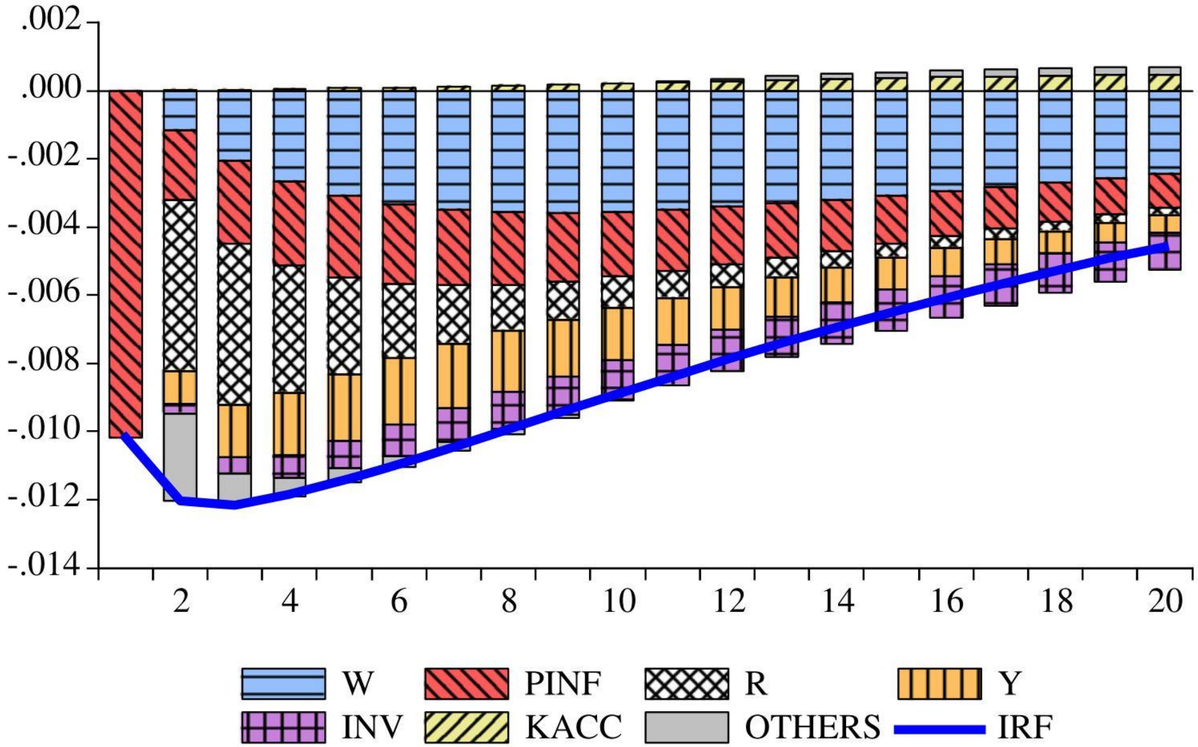
- Labus and Labus (2019) construct a ***decomposition*** of SW ***MonPol shock*** effects on ***inflation***:



- Note how lower ***wage costs*** and shrinking ***demand*** due to decreasing ***investments*** additionally push down inflation rate beyond that caused by R

Final point

- Presently ∃ growing interest in DSGE models that have ***more parameters, endogenous variables, exogenous shocks, and observables*** than SW2007 model
- and substantial ***additional complexities*** from
  - ***non-Gaussian*** distributions ("***fat tails***")
  - incorporation of ***time-varying volatility***
- These higher-dimensional DSGE models are more ***realistic*** and potentially provide ***better statistical fit*** to observed data

- ***Unfortunately***, **Dynare** is not currently capable of handling these more complex models
- ***Fortunately***, staff at ***Philadelphia Fed*** have designed a user-friendly MATLAB software programme to reliably estimate high-dimensional DSGE models
- This is available freely to download, and is documented in:
- ***Chib, Shin and Tan (2020)***, "High-Dimensional DSGE Models: Pointers on Prior, Estimation, Comparison, and Prediction", ***Federal Reserve Bank of Philadelphia*** Working Paper 20-35 (September 2020)

- For those interested in model which Chib *et al* use to illustrate working of their software, it is:
- Leeper, Traum and Walker (2017), "Clearing Up the Fiscal Multiplier Morass“, ***American Economic Review***, Vol. 107, No. 8, (2017), pp. 2409-2454
- To make this model more realistic, Chib *et al* introduce “fat-tailed” shocks and time-varying volatility
- Resulting model is ***high-dimensional***, consisting of
  – 51 parameters
  – 21 endogenous variables
  – 8 exogenous shocks
  – 8 observables
  – 1494 (!) non-Gaussian and nonlinear latent variables
- A simplified version of LTW model is available in ***Macroeconomic Model Data Base*** (version 3.1)

Appendix

- Slides below provide a more detailed exposition of the SW2007 model
- Do not be confused however by one notational difference with the main slides:
- Here, the ***shock processes*** are denoted by $\varepsilon^{x}{}_{t}$
- And the ***Kimball curvature parameters*** are denoted by $\epsilon_{w,p}$
- i.e., just the ***reverse*** of the notation in the main slides

Household Sector

- ***As usual***, Household j chooses consumption $C_t(j)$, hours worked $L_t(j)$, bonds $B_t(j)$, investment $I_t(j)$ and capital utilisation $Z_t(j)$, so as to maximise an objective function, which SW define as

$$E_t \sum_{s=0}^{\infty} \beta^s \left[\frac{1}{1-\sigma_c} \left(C_{t+s}(j) - \lambda C_{t+s-1} \right)^{1-\sigma_c} \right] \exp\left(\frac{\sigma_c - 1}{1+\sigma_l} L_{t+s}(j)^{1+\sigma_l} \right)$$

- Note that there is ***external habit formation***, captured by the parameter λ
- σ_c is ***coefficient of relative risk aversion*** of households or inverse of the intertemporal elasticity of substitution
- σ_l represents inverse of the ***elasticity of work effort*** with respect to real wage

NB: No “leisure” shocks here !

- This particular form of non-separable utility function is chosen because ***standard***, ***time-separable*** preferences ***cannot be made consistent*** with observation that a positive monetary policy shock should typically lead to a persistent decline in real interest rate and a hump-shaped rise in consumption
- It is a ***particular case*** of what are known as "***King-Plosser-Rebelo Preferences***" which are used in many DSGE models because they are compatible with balanced growth along the optimal steady state path

- ***Footnote***:
- ***Characteristic of most industrialised countries:*** output per capita, consumption per capita, investment per capita and other variables exhibit ***growth*** over long periods of time
- These are known as the “Kaldor facts”
- Such long-run growth occurs at rates that are ***roughly constant over time*** within economies but differ across economies
- This pattern suggests steady state growth, which means that levels of certain key variables grow at a constant rate
- In that case, we say there is a ***balanced growth path***

- General form of ***KPR Preferences***:

$$u(C,L) = \frac{1}{1-\sigma_c} C^{1-\sigma_c} v(L)$$

- where v(L) is increasing and concave if $0 < \sigma_c < 1$ or decreasing and convex if $\sigma_c > 1$
- In ***limit*** case of $\sigma_c = 1$, resulting preferences specification is ***additively separable*** and given by $u(C,L) = \ln C_t + v(L)$
- For SW, v(L) is given by $\exp\left(\frac{\sigma_c - 1}{1+\sigma_l} L_{t+s}(j)^{1+\sigma_l}\right)$
- SW also include habit persistence in first part

- Reason why ***KPR Preferences*** work with a balanced growth path is that in a competitive equilibrium, marginal rate of substitution between consumption and leisure must equal (inverse of) marginal product of labour
- With KPR

$$
\begin{aligned}
u(c,l) &= \frac{c^{1-\sigma}}{1-\sigma} v(l), \qquad \sigma \neq 1, \sigma > 0 \\
&= \log c + v(l) \qquad \text{for } \sigma = 1
\end{aligned}
$$

- hence
- $\dfrac{\partial u/\partial l}{\partial u/\partial c} = \dfrac{\frac{c^{1-\sigma}}{1-\sigma} v'(l)}{c^{-\sigma} v(l)} = \dfrac{c v'(l)}{(1-\sigma) v(l)} = MPL$
- Along balanced growth path, MPL and c grow at same rate, so l (labour) can be constant

- Each period, household makes ***sequence of decisions***
- ***First***, consumption decision, capital accumulation decision, and decision on how many units of capital services to supply
- ***Second***, purchases securities whose payoffs are contingent upon whether household can re-optimise its wage decision
- ***Third***, sets wage rate at which it is prepared to work after finding out whether it can re-optimise or not
- ***Finally***, receives lump-sum transfer from monetary authority

- Uncertainty faced by household over whether it can re-optimise its wage is ***idiosyncratic***
- ➔ households work different amounts and earn ***different*** wage rates
- ➔ households are ***heterogeneous*** with respect to labour

- Household's ***budget constraint*** is

$$C_{t+s}(j) + I_{t+s}(j) + \frac{B_{t+s}(j)}{\varepsilon_t^b R_{t+s} P_{t+s}} - T_{t+s}$$
$$\leq \frac{B_{t+s-1}(j)}{P_{t+s}} + \frac{W_{t+s}^h(j) L_{t+s}(j)}{P_{t+s}} + \frac{R_{t+s}^k Z_{t+s}(j) K_{t+s-1}(j)}{P_{t+s}} - a(Z_{t+s}(j)) K_{t+s-1}(j) + \frac{Div_{t+s}}{P_{t+s}}$$

- One-period bond $B_{t+s}(j)$ is expressed on a discount basis, where R_{t+s} is interest rate
- T_{t+s} are lump sum taxes or subsidies
- Div_{t+s} are (per-capita) dividends distributed by intermediate goods producers and labour unions
- P_{t+s} is overall price level
- Penultimate term represents cost associated with variations in degree of ***capital utilisation***

- Households augment their financial assets through increasing their government nominal bond holdings (B_{t+1}), from which they earn an interest rate of R_t
- Return on these bonds is subject to a risk ***shock*** ε_t^b which may be considered as an exogenous ***premium*** in return to bonds, reflecting
  - inefficiencies in financial sector (leading to some premium on deposit rate versus risk-free rate set by central bank), or
  - a risk premium that households require to hold one period bond
- and $$ln\varepsilon_t^b = \rho_b \ln \varepsilon_{t-1}^b + \eta_t^b, \eta_t^b \sim N(0, \sigma_b)$$

- Investment is assumed by SW to augment household's physical capital stock according to

$$K_t(j) = (1-\delta)_{t-1}(j) + \varepsilon_t^i \left[1 - S\left(\frac{I_t(j)}{I_{t-1}(j)}\right)\right] I_t(j)$$

- δ is depreciation rate
- S(.) is ***adjustment cost*** function $S(x_t) = \frac{\varphi}{2}(x_t - \gamma)^2$
- ε_t^i is a stochastic ***shock*** to price of investment (relative to consumption goods) and follows an exogenous process

$$ln\varepsilon_t^i \; = \rho_i \ln \varepsilon_{t-1}^i + \eta_t^i, \eta_t^i \sim N(0, \sigma_i)$$

- Households choose utilisation rate of capital
- Amount of ***effective capital*** that households can rent to firms is

$$K_t^s(j) = Z_t(j)K_{t-1}(j)$$

- Income from ***renting capital services*** is

$$R_t^k Z_t(j) K_{t-1}(j)$$

- Cost of changing ***capital utilisation*** is given by

$$Z_t = \kappa + R_t^k(1 - \psi)/\psi$$

- where ψ is normalized to be between zero and one
- κ is a constant

- When ψ =1, it is extremely costly to change utilisation of capital and as a result capital utilisation remains constant
- In contrast, when ψ =0, marginal cost of changing capital utilisation is constant ➔ in equilibrium rental rate on capital is constant

- FOCs for Household are

$$(\partial C_t) \qquad \Xi_t = \exp\left(\frac{\sigma_c-1}{1+\sigma_l} L_t(j)^{1+\sigma_l}\right)(C_t - \lambda C_{t-1})^{-\sigma_c}$$

$$(\partial L_t) \qquad \left[\frac{1}{1-\sigma_c}(C_t - hC_{t-1})^{1-\sigma_c}\right] \exp\left(\frac{\sigma_c-1}{1+\sigma_l} L_t^{1+\sigma_l}\right)(\sigma_c-1)L_t^{\sigma_l} = -\Xi_t \frac{W_t^h}{P_t}$$

$$(\partial B_t) \qquad \Xi_t = \beta \varepsilon_t^b R_t E_t\left[\frac{\Xi_{t+1}}{\pi_{t+1}}\right]$$

Lagrange Multiplier

$$(\partial I_t) \qquad \Xi_t = \Xi_t^k \varepsilon_t^i \left(1 - S\left(\frac{I_t}{I_{t-1}}\right) - S'\left(\frac{I_t}{I_{t-1}}\right)\frac{I_t}{I_{t-1}}\right) + \beta E_t\left[\Xi_{t+1}^k \varepsilon_{t+1}^i S'\left(\frac{I_{t+1}}{I_t}\right)\left(\frac{I_{t+1}}{I_t}\right)^2\right]$$

$$(\partial \bar{K}_t) \qquad \Xi_t^k = \beta E_t\left[\Xi_{t+1}\left(\frac{R_{t+1}^k}{P_{t+1}} Z_{t+1} - a(Z_{t+1})\right) + \Xi_{t+1}^k(1-\delta)\right]$$

$$(\partial u_t) \qquad \frac{R_t^k}{P_t} = a'(Z_t) \qquad\qquad \text{Tobin's } Q_t = \Xi_t^k / \Xi_t$$

$$\Xi_t = \Xi_t^k \varepsilon_t^i \left(1 - S(\frac{I_t}{I_{t-1}}) - S'(\frac{I_t}{I_{t-1}})\frac{I_t}{I_{t-1}}\right)$$
$$+\beta E_t \left[\Xi_{t+1}^k \varepsilon_{t+1}^i S'(\frac{I_{t+1}}{I_t})(\frac{I_{t+1}}{I_t})^2\right]$$

- FOC for I is law of motion for shadow value of capital
- If adjustment cost were absent (S=0), FOC would simply say that $\Xi_t = \Xi^k_t$ ➔ marginal utility of consumption (Ξ_t) [shadow ***cost*** of taking resources away from consumption] = shadow ***benefit*** of putting these resources into investment (Ξ^k_t): Tobin's Q is one

$$\Xi_t^k = \beta E_t \left[\Xi_{t+1} \left(\frac{R_{t+1}^k}{P_{t+1}} Z_{t+1} - a(Z_{t+1}) \right) + \Xi_{t+1}^k (1-\delta) \right]$$

- FOC for K says that if you buy a unit of capital today you have to pay its price in real terms (LHS: Ξ^k_t)
- but tomorrow (RHS)
- you will get proceeds from renting capital (first part)
- and you can sell back any capital that has not depreciated (second part)

Intermediate Goods Sector

- ***Intermediate Good Producer i*** is assumed by SW to use standard Cobb-Douglas technology (plus a fixed cost)

$$Y_t(i) = \epsilon_t^a K_t^s(i)^\alpha \left[\gamma^t L_t(i)\right]^{1-\alpha} - \gamma^t \Phi$$

- where
  - $K_t^s(i)$ is capital services used in production
  - $L_t(i)$ is a composite labour input
  - Φ is a fixed cost
  - γ^t represents a labour-augmenting deterministic growth rate in the economy

- ϵ_t^a is ***total factor productivity*** and follows process

$$\ln \epsilon_t^a = (1-\rho_z)\, ln\epsilon^a + \rho_z \ln \epsilon_{t-1}^a + \eta_t^a,\ \eta_t^a \sim N(0, \sigma_a)$$

- Firm's profit is given by

$$P_t(i)Y_t(i) - W_t L_t(i) - R_t^k K_t^s(i)$$

- where
  - W_t is aggregate nominal wage rate
  - R_t^k is rental rate on capital
- Cost minimization yields following first-order conditions:

$$(\partial L_t(i)) : \Theta_t(i)\gamma^{(1-\alpha)t}(1-\alpha)\epsilon_t^a K_t^s(i)^\alpha L_t(i)^{-\alpha} = W_t$$
$$(\partial K_t^s(i)) : \Theta_t(i)\gamma^{(1-\alpha)t}\alpha\epsilon_t^a K_t^s(i)^{\alpha-1} L_t(i)^{1-\alpha} = R_t^k$$

- where $\Theta_t(i)$ is Lagrange multiplier associated with production function (and equals marginal cost MC_t)
- Combining these FOCs and noting that capital-labour ratio is equal across firms implies (as usual)

$$K_t^s = \frac{\alpha}{1-\alpha}\frac{W_t}{R_t^k}L_t$$

- Marginal cost MC_t is ***same for all firms*** and equal to

$$MC_t = \alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)} W_t^{1-\alpha} R_t^{k\alpha} \gamma^{-(1-\alpha)t} (\epsilon_t^a)^{-1}$$

- Prices of ***intermediate goods*** are determined by Calvo fairy
- As in previous Calvo models, in each period, each firm i faces a constant probability $1-\xi_p$ of being able to re-optimise its price $P_t(i)$
- Probability that any firm receives a signal to re-optimise its price is assumed to be ***independent*** of time that it last reset its price

- Unlike in simpler earlier models, SW assume that if Calvo fairy does not allow a firm to optimise its price in a given period, it then adjusts its price by a ***weighted combination*** of lagged and steady-state inflation rates as:

$$P_t(i) = (\pi_{t-1})^{\iota_p} (\pi_*)^{1-\iota_p} P_{t-1}(i)$$

- where
  - $0 \leq \iota_p \leq 1$
  - π_{t-1} denotes gross inflation in period t−1
  - π_* denotes steady-state gross inflation rate
- A positive value of ι_p introduces ***structural inertia*** into inflation process

- Under Calvo pricing with this type of ***partial indexation***, optimal price $\widetilde{P}_t(i)$ set by firm that is allowed by Calvo fairy (with a probability $1-\xi_p$) to re-optimise results from solving following optimisation problem:

$$\max_{\widetilde{P}_t(i)} E_t \sum_{s=0}^{\infty} \xi_p^s \boxed{\frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}}} \left[\widetilde{P}_t(i) (\Pi_{l=1}^{s} \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p}) - MC_{t+s} \right] Y_{t+s}(i)$$

Nominal discount factor

$$s.t.\ Y_{t+s}(i) = Y_{t+s} G'^{-1} \left(\frac{P_t(i) X_{t,s}}{P_{t+s}} \tau_{t+s} \right)$$

- This looks formidable!
- But if we compare it to what we used in very first sticky-price model studied back at beginning of term, we find clear similarities:

$$\max_{P^*_{j,t}} E_t \sum_{i=0}^{\infty} (\beta\theta)^i (P^*_{j,t} Y_{j,t+i} - TC_{j,t+i})$$

$$\max_{\widetilde{P}_t(i)} E_t \sum_{s=0}^{\infty} \xi_p^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \left[\widetilde{P}_t(i) (\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p}) - MC_{t+s} \right] Y_{t+s}(i)$$

- Since TC = MC*Y, only really ***new element*** is ***indexation factor*** multiplying $P_t^{\sim}$: $(\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p})$

- What is this indexation factor $(\Pi_{l=1}^{s}\pi_{t+l-1}^{\iota_p}\pi_*^{1-\iota_p})$
- Since π is a ***gross*** inflation rate, it has each period a value close to 1
- Thus, product $\Pi_{l=1}^{s}\pi_{t+l-1}^{\iota_p}$ (which multiplies together inflation rates for all periods since previous price revision) also has a value near 1
- This is then scaled by factor $\pi_*^{1-\iota_p}$ which reflects ***weighted combination*** indexation
- Finally, as noted previously, curious expression $\frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}}$ is just the (nominal) discount factor

- It is also clear from formulation of the optimisation problem

$$\max_{\widetilde{P}_t(i)} E_t \sum_{s=0}^{\infty} \xi_p^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \left[\widetilde{P}_t(i) (\Pi_{l=1}^{s} \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p}) - \boxed{MC_{t+s}} \right] Y_{t+s}(i)$$

- that price set by firm i, at time t, is a function of ***expected future*** marginal costs
- Price will be a ***mark-up*** over these weighted marginal costs
  - ***fixed*** over time if prices are perfectly flexible ($\iota_p = 0$), or
  - ***varying*** with sticky prices ($\iota_p \neq 0$)

- Given FOC for this optimisation problem, ***aggregate price index*** which results is

$$P_t = (1-\xi_p)P_t(i)G'^{-1}\left[\frac{P_t(i)\tau_t}{P_t}\right] + \xi_p \pi_{t-1}^{\iota_p}\pi_*^{1-\iota_p}P_{t-1}G'^{-1}\left[\frac{\pi_{t-1}^{\iota_p}\pi_*^{1-\iota_p}P_{t-1}\tau_t}{P_t}\right]$$

- Compare this to what we used in very first sticky-price model studied back at beginning of term:

$$P_t = \left[\theta P_{t-1}^{1-\psi} + (1-\theta)P_t^{*\,1-\psi}\right]^{\frac{1}{1-\psi}}$$

- Function G (of which G' is used in definition above) is defined in Appendix

Final Good Sector

- As usual, ***final good*** Y_t is a composite made of a continuum of ***intermediate goods*** $Y_t(i)$
- ***Final Good Producers*** buy intermediate goods and package them into Y_t using not the Dixit-Stiglitz aggregator but instead the ***Kimball Aggregator***
- In contrast to Dixit–Stiglitz world of a constant elasticity and a ***constant*** desired mark-up of price over marginal cost, in Kimball's world desired mark-up is ***decreasing in a firm's relative price***
- Final Good Producers sell final good to consumers, investors and government in a perfectly competitive market

- Final Good Producers maximize profits

$$\max_{Y_t, Y_t(i)} P_t Y_t - \int_0^1 P_t(i) Y_t(i) di$$

- subject to Kimball Aggregator

$$\left[\int_0^1 G\left(\frac{Y_t(i)}{Y_t}; \varepsilon_t^p\right) di\right] = 1 \quad (\mu_{f,t})$$

- where P_t and $P_t(i)$ are price of final and intermediate goods respectively
- G is a ***strictly concave*** and increasing function (ie, ***decreasing slope G'***) characterised by G(1) = 1
- ε_t^p is an exogenous process that reflects ***shocks*** to aggregator function that result in changes in elasticity of demand and thus in mark-up

- ε_t^p follows an ARMA(1,1) process defined by

$$\varepsilon_t^p = \rho_p \varepsilon_t^p + \eta_t^p - \mu_p \eta_{t-1}^p$$

- Combining first-order conditions with respect to $Y_t(i)$ and Y_t results in

$$Y_t(i) = Y_t G'^{-1}\left[\frac{P_t(i)}{P_t}\int_0^1 G'\left(\frac{Y_t(i)}{Y_t}\right)\frac{Y_t(i)}{Y_t}di\right]$$

- Assumptions on G (specifically, G' decreasing) imply that demand for input $Y_t(i)$ is ***decreasing*** in its relative price
- while elasticity of demand is a positive function of relative price (or a negative function of relative output)

- For completeness, note that G is defined as

$$G_Y\left(\frac{Y_t(f)}{Y_t}\right) =$$

$$\frac{\phi_t^p}{1-\left(\phi_t^p-1\right)\epsilon_p}\left[\left(\frac{\phi_t^p+\left(1-\phi_t^p\right)\epsilon_p}{\phi_t^p}\right)\frac{Y_t(f)}{Y_t}+\frac{\left(\phi_t^p-1\right)\epsilon_p}{\phi_t^p}\right]^{\frac{1-\left(\phi_t^p-1\right)\epsilon_p}{\phi_t^p-\left(\phi_t^p-1\right)\epsilon_p}}$$

$$+\left[1-\frac{\phi_t^p}{1-\left(\phi_t^p-1\right)\epsilon_p}\right]$$

- $\phi_t^p \geq 1$ is gross mark-up of the intermediate firms
- ϵ_p is degree of ***curvature*** of firm's demand (do not confuse this with ε_t^p, the price shock !!)

- When ***curvature parameter*** $\epsilon_p = 0$, demand curve exhibits ***constant elasticity*** as with standard Dixit-Stiglitz aggregator
- When ϵ_p is positive, firm instead faces a ***quasi-kinked*** demand curve, implying that a drop in its relative price only stimulates a ***small increase*** in demand
- On other hand, a rise in its relative price generates a ***large fall*** in demand
- Relative to standard Dixit-Stiglitz aggregator, this introduces more ***strategic complementarity*** in price setting which causes intermediate firms to ***adjust*** prices ***less*** to a given change in marginal cost (hence "stickier")

- Let us now return to solution of the FOCs, given previously by

$$Y_t(i) = Y_t G'^{-1}\left[\frac{P_t(i)}{P_t}\int_0^1 G'\left(\frac{Y_t(i)}{Y_t}\right)\frac{Y_t(i)}{Y_t}di\right]$$

- Recall that $G'^{-1}\left[\frac{P_t(i)\tau_t}{P_t}\right]$ was used in defining P_t
- Definition of τ_t is $\tau_t = \int_0^1 G'\left(\frac{Y_t(i)}{Y_t}\right)\frac{Y_t(i)}{Y_t}di$
- So in fact we have simply that
- $Y_t(i)/Y_t$ = $G'^{-1}\left[\frac{P_t(i)\tau_t}{P_t}\right]$
- which is why we earlier stated that demand for input $Y_t(i)$ is ***decreasing*** in its relative price

- When all intermediate firms produce the same amount, we have $\frac{Y_t(i)}{Y_t}$ =1

- But $G_Y(1) = 1$, implying **constant returns to scale** in this case

Labour Sector

- SW assume that Households supply their homogenous labour to an ***intermediate labour union*** which differentiates labour services, sets wages subject to a Calvo scheme and offers those labour services to intermediate ***labour packers***
- Labour used by intermediate goods producers L_t is a composite made of those differentiated labour services $L_t(i)$
- As with intermediate goods, ***Kimball aggregator*** is used, with a ***curvature parameter*** of ϵ_w

- Labour packers buy differentiated labour services, package L_t, and offer it to intermediate goods producers
- Labour packers maximize profits

$$\max_{L_t, L_t(i)} W_t L_t - \int_0^1 W_t(i) L_t(i) di$$

$$s.t. \left[\int_0^1 H\left(\frac{L_t(i)}{L_t}; \varepsilon_t^w\right) di\right] = 1 \quad (\mu_{l,t})$$

- where W_t and $W_t(i)$ are prices of composite and intermediate labour services respectively
- ***Like G previously***, function H is a strictly concave and increasing function characterised by H(1) = 1

- ε_t^w is an exogenous process that reflects ***shocks*** to aggregator function that result in changes in elasticity of demand and therefore in mark-up [do not confuse this with ϵ_w - the curvature !!]
- Similarly to ε_t^p, it is assumed that ε_t^w follows an ARMA(1,1) process

$$\varepsilon_t^w = \rho_w \varepsilon_{t-1}^w + \eta_t^w - \mu_w \eta_{t-1}^w$$

- Combining FOCs results (as for $Y_t(i)$ previously) in

$$L_t(i) = L_t H'^{-1}\left[\frac{W_t(i)}{W_t}\int_0^1 H'\left(\frac{L_t(i)}{L_t}\right)\frac{L_t(i)}{L_t}di\right]$$

- Labour unions are an intermediary between households and labour packers
- Under Calvo pricing with partial indexation, optimal wage set by union that is allowed to re-optimise its wage results from an optimisation problem similar to that for pricing

$$\max_{\widetilde{W}_t(i)} E_t \sum_{s=0}^{\infty} \xi_w^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \left[\widetilde{W}_t(i) (\Pi_{l=1}^{s} \gamma \pi_{t+l-1}^{\iota_w} \pi_*{}^{1-\iota_w} - W_{t+s}^h \right] L_{t+s}(i)$$

$$s.t.\ L_{t+s}(i) = L_{t+s} H'^{-1} \left(\frac{W_t(i) X_{t,s}^w}{W_{t+s}} \tau_{t+s}^w \right)$$

- Again, similarly to case for pricing, ***aggregate wage index*** is in this case given by

$$W_t = (1-\xi_w)\widetilde{W}_t H'^{-1}\left[\frac{\widetilde{W}_t \tau_t^w}{W_t}\right] + \xi_w \gamma \pi_{t-1}^{\iota_w} \pi_*^{1-\iota_w} W_{t-1} H'^{-1}\left[\frac{\gamma \pi_{t-1}^{\iota_w} \pi_*^{1-\iota_w} W_{t-1} \tau_t^w}{W_t}\right]$$

- ***Mark-up*** of aggregate wage over wage received by households is distributed to households in form of ***dividends*** (as already indicated in budget constraint)

Government Sector

- SW (both of whom work for central banks) assume that central bank follows a nominal interest rate rule by adjusting its instrument in response to deviations of inflation and output from their respective target levels

$$\frac{R_t}{R^*} = \left(\frac{R_{t-1}}{R^*}\right)^{\rho} \left[\left(\frac{\pi_t}{\pi_*}\right)^{r_\pi} \left(\frac{Y_t}{Y_t^*}\right)^{r_y}\right]^{1-\rho} \left(\frac{Y_t/Y_{t-1}}{Y_t^*/Y_{t-1}^*}\right)^{r_{\Delta y}} \epsilon_t^r$$

- where
  - R* is steady state nominal rate (gross rate)
  - Y_t* is "***natural output***"
  - ρ determines degree of interest rate smoothing
- Exogenous monetary policy ***shock*** ϵ_t^r is determined as

$$\ln \varepsilon_t^r = \rho_r \ln \varepsilon_{t-1}^r + \eta_t^r, \eta_t^r \sim N\left(0, \sigma_r\right)$$

- A new element is included in this version of Taylor Rule: “***natural output level***”
- SW define it as “the output in the flexible price and wage economy without mark-up shock in prices and wages”, ie ***NOT*** subject to sticky wages and prices
- In model eventually estimated below, there will be an ***entire section*** devoted to a “natural” or ***“flex-price”*** version of the economy, ***ONLY*** so as to be able to make use of the central bank reaction function defined by SW version of Taylor Rule – ***no need for this otherwise***

- Government budget constraint is of form

$$P_t G_t + B_{t-1} = T_t + \frac{B_t}{R_t}$$

- where T_t are nominal lump-sum taxes (or subsidies) that also appear in household's budget constraint
- SW assume that Government purchases G_t are exogenous
- They express ***government spending*** relative to steady state output path ("trend output") as

$$\varepsilon_t^g = G_t / (Y\gamma^t)$$

- Path of ε_t^g is assumed to be given by

$$\ln \varepsilon_t^g = (1-\rho_g) ln\varepsilon^g + \rho_g ln\varepsilon_{t-1}^g + \rho_{ga} ln\varepsilon_t^a - \rho_{ga} ln\varepsilon_{t-1}^a + \eta_t^g, \eta_t^g \sim N(0, \sigma_g)$$

- This allows for a reaction of government spending to productivity process γ^t
- Government purchases have no effect on marginal utility of private consumption, nor do they serve as an input into goods production

Market Equilibrium

- Finally, ***Market equilibrium***
- Final goods market is in equilibrium if production equals demand by households for consumption and investment and by government

$$C_t + I_t + G_t + a(Z_t)K_{t-1} = Y_t$$

- ***Capital rental market*** is in equilibrium when demand for capital by intermediate goods producers equals supply by households
- ***Labour market*** is in equilibrium if firms' demand for labour equals labour supply at wage level set by households
- In capital market, equilibrium means that ***government debt*** is held by domestic investors at market interest rate, R_t