Exam Caleb Halverson-Fried

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Ex 1) $[M\ I] \to [I\ M^{-1}]$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 2 & 1 & 6 & 0 & 1 & 0 \\ 3 & 4 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3-3R_1]{R_2-2R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & -4 & -2 & 1 & 0 \\ 0 & 4 & -15 & -3 & 0 & 1 \end{array}\right) \xrightarrow{R_3-4R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & -4 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -4 & 1 \end{array}\right)$$

$$\xrightarrow{R_2+4R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 18 & -15 & 4 \\ 0 & 0 & 1 & 5 & -4 & 1 \end{array}\right) \xrightarrow{R_1-5R_3} \begin{pmatrix} 1 & 0 & 0 & -24 & 20 & -5 \\ 0 & 1 & 0 & 18 & -15 & 4 \\ 0 & 0 & 1 & 5 & -4 & 1 \end{pmatrix}$$

$$\boxed{M^{-1} = \begin{pmatrix} -24 & 20 & -5 \\ 18 & -15 & 4 \\ 5 & -4 & 1 \end{pmatrix}}$$ ok

Ex 2) $A-\lambda I = \cancel{\begin{vmatrix} 3-\lambda & 5 & -2 \\ 0 & 2-\lambda & 0 \\ 3 & 4 & -\lambda \end{vmatrix}}$ $\begin{vmatrix} 3-\lambda & 5 & -2 \\ 0 & 2-\lambda & 0 \\ 0 & 2 & 1-\lambda \end{vmatrix}$ $\begin{array}{l} -\lambda^3+3\lambda^2-2\lambda \\ +3\lambda^2-9\lambda+6 \end{array}$

$\det|A-\lambda I| = 0 \to 3-\lambda(\lambda^2-3\lambda+2)=0 \to (3-\lambda)(\lambda-2)(\lambda-1)=0$

a. characteristic polynomial: $(3-\lambda)(\lambda-2)(\lambda-1) = -\lambda^3+6\lambda^2-11\lambda+6$ ok

b. $-\lambda^3+6\lambda^2-11\lambda+6 = (3-\lambda)(\lambda-2)(\lambda-1)=0 \to \lambda = \{1, 2, 3\}$ ok

c. Since all eigenvalues > 0, A is positive definite ok

d. $\lambda_1 = 1 \to \begin{pmatrix} 2 & 5 & -2 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \to x_1 = k\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} k \\ 0 \\ k \end{pmatrix}$ where $k \in \mathbb{R}$ ok

$\lambda_2 = 2 \to \begin{pmatrix} 1 & 5 & -2 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{pmatrix} \to x_2 = k\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -k \\ k \\ 2k \end{pmatrix}$ where $k \in \mathbb{R}$ ok

$\lambda_3 = 3 \to \begin{pmatrix} 0 & 5 & -2 \\ 0 & -1 & 0 \\ 0 & 2 & -2 \end{pmatrix} \to x_3 = k\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} k \\ 0 \\ 0 \end{pmatrix}$ where $k \in \mathbb{R}$ ok

3) $z = x^2 - 2xy + y^2$, $x = r + \theta$, $y = r - \theta$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = (2x-2y)(1) + (-2x+2y)(1)$$
$$= 2x - 2y - 2x + 2y = 0$$

ok

$$\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} = (2x-2y)(1) + (-2x+2y)(-1)$$
$$= 2x - 2y + 2x - 2y = 4x - 4y$$
$$= 4(r+\theta) - 4(r-\theta)$$
$$= 4r + 4\theta - 4r + 4\theta = 8\theta$$

ok

4) $f(x,y) = \frac{1}{x} + xe^{-y}$, $\frac{\partial f}{\partial x} = -\frac{1}{x^2} + e^{-y}$, $\frac{\partial f}{\partial y} = -xe^{-y}$

$f_{xx} = \frac{2}{x^3}$, $f_{xy} = -e^{-y}$, $f_{yx} = -e^{-y}$, $f_{yy} = xe^{-y}$

gradient $= \begin{pmatrix} f_x \\ f_y \end{pmatrix} = \begin{pmatrix} -\frac{1}{x^2} + e^{-y} \\ -xe^{-y} \end{pmatrix}$ ok ; Hessian $= \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} \frac{2}{x^3} & -e^{-y} \\ -e^{-y} & xe^{-y} \end{pmatrix}$

ok

5) $f(x,y)$ is concave $\leftrightarrow D^2 f(x,y)$ is negative semidefinite

$f(x,y)$ is convex $\leftrightarrow D^2 f(x,y)$ is positive semidefinite

$f(x,y) = -6x^2 + (2a+4)xy - y^2 + 4ay$

$f_x = -12x + (2a+4)y$, $f_y = (2a+4)x - 2y + 4a$

$f_{xx} = -12$, $f_{xy} = 2a+4$, $f_{yx} = 2a+4$, $f_{yy} = -2$

Then $D^2 f(x,y) = \begin{pmatrix} -12 & 2a+4 \\ 2a+4 & -2 \end{pmatrix}$ $\quad \text{Det}(D^2 f(x,y)) = 24 - (2a+4)^2$

$(2a+4)^2 = 24 \rightarrow 2a+4 = \pm\sqrt{24}$

$2a = -4 \pm \sqrt{24}$

$a = -2 \pm \frac{1}{2}\sqrt{24}$

$a = -2 \pm \sqrt{6}$

Since the diagonals are negative, ~~[illegible]~~ $D^2 f(x,y)$ cannot be positive semidefinite anywhere so $f(x,y)$ is not convex anywhere. ok

$D^2 f(x,y)$ is negative semidefinite when $\det(D^2 f(x,y)) \geq 0$.

This happens when $a \in [-2-\sqrt{6}, -2+\sqrt{6}]$

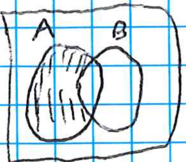
Therefore, $f(x,y)$ is concave when $a \in [-2-\sqrt{6}, -2+\sqrt{6}]$ ok and neither convex nor concave otherwise

8) $P(A) = .4$, $P(B) = .5$, $P(A \cap B) = .1$

a. A and B independent if $P(A \cap B) = P(A)P(B)$

$P(A)P(B) = (.4)(.5) = .2 \neq .1 = P(A \cap B) \rightarrow$ $\boxed{\text{not independent}}$ ok

b. $P(A \cup B) = P(A) + P(B) - P(A \cap B) = .4 + .5 - .1 = \boxed{.8}$ ok

c.  $P(A \cap B^c) = P(A) - P(A \cap B) = .4 - .1 = \boxed{.3}$ ok

d. $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{.1}{.5} = \boxed{.2}$ ok

e. $P(A|B^c) = \dfrac{P(A \cap B^c)}{P(B^c)} = \dfrac{.3}{1-.5} = \dfrac{.3}{.5} = \boxed{.6}$ ok

9) a. $P(X=Y) = P(X=1, Y=1) + P(X=2, Y=2) = .1 + .2 = \boxed{.3}$ ok

b. $P(X=2) = .6 = P(X=2, Y=1) + P(X=2, Y=2) = P(X=2, Y=1) + .2$

$\rightarrow P(X=2, Y=1) = \boxed{.4} = P(X=2 \cap Y=1)$ ok

c. $P(Y=2) = P(Y=2, X=1) + \cancel{P(Y=2)}\ P(Y=2, X=2) + P(Y=2, X=3)$

$= P(Y=2, X=1) + .3$

Since all probabilities sum to 1 and $P(X=2, Y=1) = .4$, $P(Y=2, X=1) = .2$

Then $P(Y=2) = \boxed{.5}$ ok

d. $P(X=2) = .6$, $P(Y=2) = .5$, $P(X=2)P(Y=2) = .3 \neq .2 = P(X=2 \cap Y=2)$

Therefore X and Y are not independent ok

e. $E(X) = E[E(X|Y)] = E(X|Y=1)\cdot(.5) + E(X|Y=2)\cdot .5$

$= \left(\frac{.1}{.5}\right)(.5) + \left(\frac{.4}{.5}\right)(.5)(2) + .5\left(\frac{.2}{.5} + \frac{.2}{.5}(2) + \frac{.1}{.5}(3)\right)$

$= (.2)(.5) + .8 + .5(.4 + .8 + .6)$

$= .1 + .8 + .5(1.8) = .9 + .9 = \boxed{1.8}$ ok

$E[X^2] = E(E(X^2|Y)) = E(X^2|Y=1)(.5) + E(X^2|Y=2)(.5)$

$= .5\left(\frac{.1}{.5}(1) + \frac{.4}{.5}(4) + 0\right) + .5\left(\frac{.2}{.5}(1) + \frac{.2}{.5}(4) + \frac{.1}{.5}(9)\right)$

$= .5(.2 + 3.2) + .5(.4 + 1.6 + 1.8) = .5(3.4) + .5(3.8)$

$= 1.7 + 1.9 = \boxed{3.6}$

$\boxed{E(X) = 1.8 \quad E(X^2) = 3.6}$

f. $\text{var}(X) = \sigma^2(X) = E(X^2) - E(X)^2 = 3.6 - 3.24 = .36$ ok

$\sigma(X) = \sqrt{\sigma^2(X)} = \sqrt{.36} = \boxed{.6}$ ok

6) $f(x,y) = x + y^2 \quad g(x,y) = x^2 + y^2 - 1 = 0$

$L(x,y,\lambda) = x + y^2 - \lambda(x^2 + y^2 - 1)$

$\frac{\partial L}{\partial x} = 1 - 2x\lambda = 0 \rightarrow 2x\lambda = 1 \rightarrow x = \frac{1}{2\lambda}$

$\frac{\partial L}{\partial y} = 2y - 2y\lambda = 0 \rightarrow 2y(1-\lambda) = 0 \rightarrow \lambda = 1$ or $y = 0$

$\frac{\partial L}{\partial \lambda} = 1 - x^2 - y^2 = 0$

If $y = 0 \rightarrow 1 - x^2 = 0 \rightarrow x = \pm 1$, critical points $(-1, 0, -1)$ and $(1, 0, 1)$

If $\lambda = 1$, $x = \frac{1}{2\lambda} = \frac{1}{2} \rightarrow 1 - (\frac{1}{2})^2 - y^2 = 0 \rightarrow y^2 = \frac{3}{4} \rightarrow y = \pm \frac{\sqrt{3}}{2}$

critical points $(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{5}{4})$ and $(\frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{5}{4})$

There are 4 critical points to analyze:

1) $(x^*, y^*) = (-1, 0)$, If $x = -1 \rightarrow \lambda = -\frac{1}{2} < 0 \rightarrow$ not maximum

2) $(x^*, y^*) = (1, 0)$, If $x = 1 \rightarrow \lambda = \frac{1}{2} > 0$, $\frac{\partial L}{\partial x}(1,0) = 0$, $\frac{\partial L}{\partial y}(1,0) = 0$, $g(1,0) = 0$
$\rightarrow$ maximum

3) $(x^*, y^*) = (\frac{1}{2}, \frac{\sqrt{3}}{2}) \rightarrow x = \frac{1}{2} \rightarrow \lambda = 1 > 0$, $g(\frac{1}{2}, \frac{\sqrt{3}}{2}) = 0 \rightarrow \lambda g(\frac{1}{2}, \frac{\sqrt{3}}{2}) = 0$
$\rightarrow$ maximum

4) $(x^*, y^*) = (\frac{1}{2}, -\frac{\sqrt{3}}{2}) \rightarrow x = \frac{1}{2}, \lambda = 1 > 0 \rightarrow$ maximum, $g(\frac{1}{2}, -\frac{\sqrt{3}}{2}) = 0$

ok

Thus the function is maximized at $(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{5}{4})$ and $(\frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{5}{4})$

7) $C_{H,t} = (1-\alpha)\left(\frac{P_{H,t}}{P_t}\right)^{-\eta} C_t \rightarrow \ln(C_{H,t}) = \ln(1-\alpha) - \eta \ln\left(\frac{P_{H,t}}{P_t}\right) + \ln(C_t)$

$\rightarrow \ln(C_{H,t}) = \ln(1-\alpha) - \eta(\ln(P_{H,t}) - \ln(P_t)) + \ln(C_t)$

ok

$\rightarrow \ln(C_{H,t}) = \ln(1-\alpha) - \eta(\ln(P_{H,t})) + \eta \ln(P_t) + \ln C_t$

$\rightarrow \ln(C_H^*) + \frac{1}{C_H^*}(C_{H,t} - C_H^*) = \ln(1-\alpha) - \eta \ln(P_H^*) - \frac{\eta}{P_H^*}(P_{H,t} - P_H^*) + \eta \ln(P^*) + \frac{\eta}{P^*}(P_t - P^*)$
$+ \ln(C^*) + \frac{1}{C^*}(C_t - C^*)$

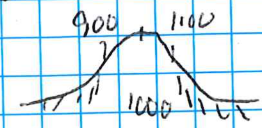
$\rightarrow \frac{C_{H,t} - C_H^*}{C_H^*} = -\frac{\eta}{P_H^*}(P_{H,t} - P_H^*) + \frac{\eta}{P^*}(P_t - P^*) + \frac{C_t - C^*}{C^*}$

$\hat{C}_{H,t} = -\eta \hat{P}_{H,t} + \eta \hat{P}_t + \hat{C}_t \rightarrow \boxed{\hat{C}_{H,t} = \eta(\hat{P}_t - \hat{P}_{H,t}) + \hat{C}_t}$ ok

Since $(1-\alpha)$ did not appear in the result for equation 1, by symmetry α won't appear in the log-linearization of equation 2

Then we have $\boxed{\hat{C}_F = \eta(\hat{P} - \hat{P}_F) + \hat{C}}$ ok

10) X = resistance, $\mu = 1000$, variance $= 2500$



2.23606

$$z_n = \frac{x-1000}{\sqrt{2500}} = \frac{x-1000}{50}$$

$P(\text{rejected}) = P(x \le 900) + P(x \ge 1100)$

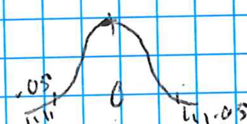
$= 2P(x \ge 1100) = 2(1 - P(x \le 1100))$

If $x = 1100 \rightarrow z = \frac{100}{50} = 2$

$2P(1 - P(z \le 2)) = 2(1 - .9772) = \boxed{.0456}$ ok

Bonus) μ unknown, $\sigma = 10$, $n = 5$, $H_0 : \mu = 0$, $H_a : \mu > 0$, $\alpha = .05$

a. $\bar{x} = \frac{1}{5}(-3+7+2+10+7) = \frac{23}{5} = 4.6$ ok



90% interval: $\left[\mu - \frac{1.65\sigma}{10/\sqrt{5}}, \mu + \frac{1.65\sigma}{10/\sqrt{5}}\right) = \left[-\frac{16.5}{\sqrt{5}}, \frac{16.5}{\sqrt{5}}\right] = [-7.38, 7.38]$

95% interval: $\left[\mu - \frac{1.96\sigma}{10/\sqrt{5}}, \mu + \frac{1.96\sigma}{10/\sqrt{5}}\right) = \left[-\frac{19.6}{\sqrt{5}}, \frac{19.6}{\sqrt{5}}\right] = [-8.77, 8.77]$ ok

b. $P\left(\frac{\bar{x} - 0}{10/\sqrt{5}}\right) = \frac{4.6}{10/\sqrt{5}} = \frac{4.6\sqrt{5}}{10} = \cancel{1.02859}$ 1.02859

p-value $= P(z \ge 1.02859) = \cancel{m}\, 1 - P(z \le 1.02859) = (1 - .8485)$

$.1515 \not< .05$ so we do not reject the null $= \boxed{.1515}$ ok

Then $\mu = 0$

c. rejected when $z \ge 1.65$ $(P \le .05)$

$1.65 = \frac{\bar{x}\sqrt{5}}{10} \rightarrow 16.5 = \bar{x}\sqrt{5} \rightarrow \bar{x} = 7.38$ ok

Thus the null hypothesis will be rejected when the sample mean is greater than 7.38.

This corresponds to 90% confidence interval