Intermediate Microeconomics

Examples: Substitution & Income Effects

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Cobb-Douglas Utility

- Utility: $U = x^{1/2}y^{1/2}$
- Budget line: let $p_x = p_y = 1$ and $I = 10$, hence $p_x x + p_y y = I \Rightarrow x + y = 10$
- Solving the utility-maximization problem

$$\max_{x,y} x^{1/2}y^{1/2}$$
$$s.t.\ x + y \leq 10$$

- The Lagrangian:

$$\mathcal{L} = x^{1/2}y^{1/2} + \lambda(10 - x - y)$$
$$\mathcal{L}'_x = (1/2)x^{-1/2}y^{1/2} - \lambda = 0$$
$$\mathcal{L}'_y = (1/2)x^{1/2}y^{-1/2} - \lambda = 0$$
$$\mathcal{L}'_\lambda = 10 - x - y = 0$$

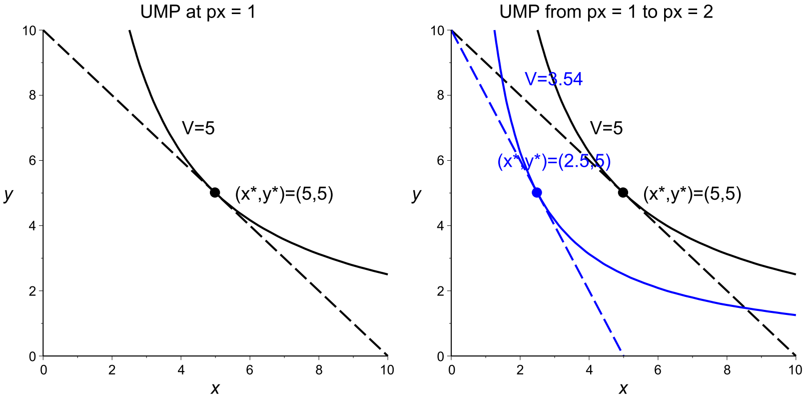
- The Marshallian demand (solutions): $x^* = y^* = 5$, $V = \sqrt{x^*y^*} = 5$.

- Consider an increase in p_x, from 1 to 2.
- The budget becomes $2x + y = 10$
- Solving the utility-maximization problem

$$\max_{x,y} x^{1/2} y^{1/2}$$
$$s.t.\ 2x + y \leq 10$$

- The solution (Mashallian demand after price change) is $x^* = 5/2, y^* = 5, V = \frac{5}{2}\sqrt{2} \approx 3.54$

Comparing the solutions before price change and after price change:



The horizontal distance of x between the two solutions is the **total effect** from a an increase of p_x on the consumption in x

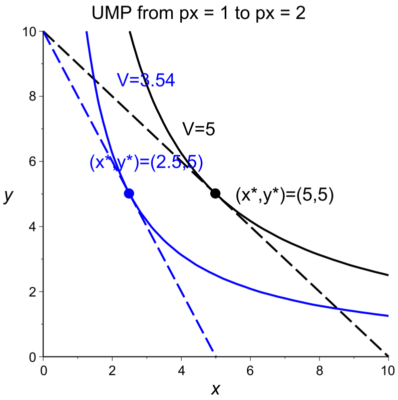
- Now let's decompose total effect into substitution and income effects.
- According to Slutsky equation: total effect $=$ substitution effect $+$ income effect.
  - The substitution effect (of x) is the difference between the optimal x evaluated at the original p_x, and the optimal x after an increase in p_x, fixing the original utility level.
  - The optimal x after an increase in p_x, is the Hicksian demand evaluated at the new price p_x and the original utility level.
- Solve the expenditure minimization problem (to obtain Hicksian demand h_x):

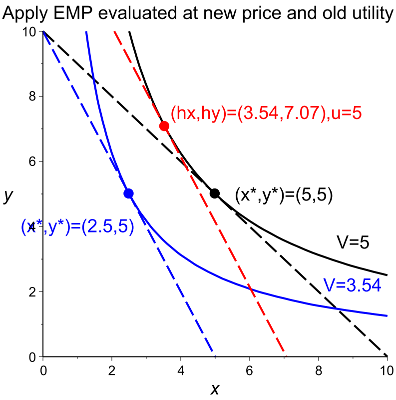
$$\min 2x + y$$

$$s.t.\ x^{1/2}y^{1/2} = 5$$

$$\Rightarrow h_x = \frac{5\sqrt{2}}{2} \approx 3.54, h_y = 5\sqrt{2}$$

Total effect $=$ substitution effect $+$ income effect

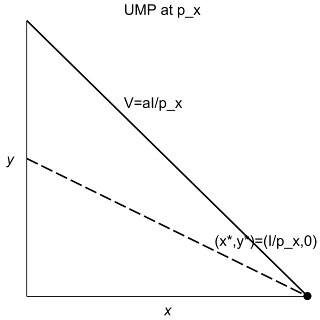




- Total effect: $x^* \to {\color{blue}x^*}$
- Substitution effect: $x^* \to {\color{red}h_x}$
- Income effect: ${\color{red}h_x} \to {\color{blue}x^*}$

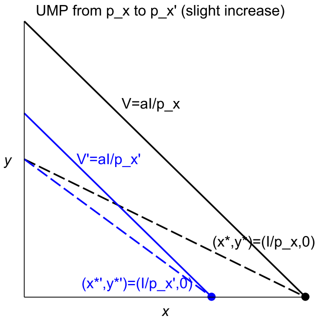
Perfect Substitutes

- Utility: $U = ax + by$, or $y = -\frac{a}{b}x + \frac{1}{b}u$
- Original budget line: $p_x x + p_y y = I$. Assume $\frac{p_x}{p_y} < \frac{a}{b}$.
- By comparing the relative slopes (the budget line is flatter; the indifference curve is steeper), the Mashallian demand is $y^* = 0 \Rightarrow x^* = \frac{I}{p_x}$. The indirect utility is $V = \frac{aI}{p_x}$



Perfect Substitutes: slight change in price

- Now consider a "slight" increase in p_x, from p_x to p'_x
- Assume that the price change is not large enough such that the relative size of slopes is unchanged: $\frac{p'_x}{p_y} < \frac{a}{b}$.
- The solution (Mashallian) after the price change is
$y^{*\prime} = 0 \Rightarrow x^{*\prime} = \frac{I}{p'_x}, V' = \frac{aI}{p'_x}$



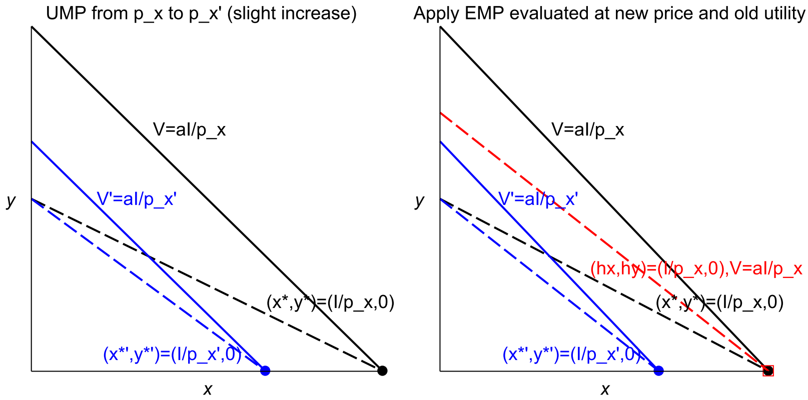
- Find the substitution effect: by solving the expenditure minimization problem, evaluated at
  - The original utility level $u = V = \frac{aI}{p_x}$
  - New price p'_x
- Solve EMP (Hicksian):

$$\min_{x,y} p'_x x + p_y y$$

$$s.t.\ ax + by = u = V = \frac{aI}{p_x}$$

$$\Rightarrow h_x = \frac{u}{a} = \frac{I}{p_x}, h_y = 0$$

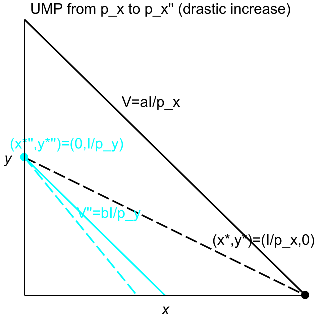
The solution is obtained by comparing the slopes of $\frac{a}{b}$ and $\frac{p'_x}{p_y}$



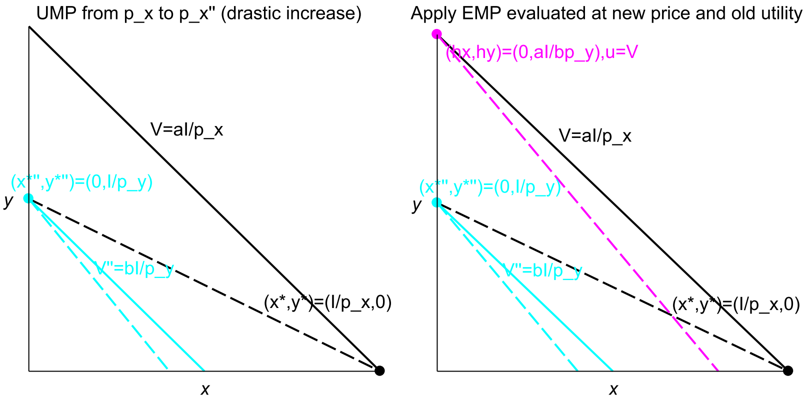
- Total effect: $x^* \to x^*$
- Substitution effect: $x^* \to h_x$, zero
- Income effect: $h_x \to x^*$, $=$ total effect

Perfect Substitutes: drastic change in price

- Now, consider a "drastic" price increase in p_x, from p_x to p_x'' such that the relative size of slopes is changed: $\frac{p_x''}{p_y} > \frac{a}{b}$
- Comparing the slopes, we obtain the Mashallian demand evaluated at p_x'': $x^{*''} = 0, y^{*''} = \frac{I}{p_y}, V'' = \frac{bI}{p_y}$



- To check the substitution & income effect for a drastic change of p_x to p_x'', compute Hicksian demand evaluated at:
  - Original utility $u = V = \frac{aI}{p_x}$
  - New price p_x''
- Solve EMP (Hicksian):
  - Minimize expenditure: $p_x''x + p_y y$
  - Subjected to utility at $u = V = \frac{aI}{p_x}$:
    $ax + by = \frac{aI}{p_y} \Leftrightarrow y = -\frac{a}{b}x + \frac{aI}{bp_y}$
- By comparing the slopes, the Hicksian demand is $h_x = 0, h_y = \frac{u}{b} = \frac{aI}{bp_y}$

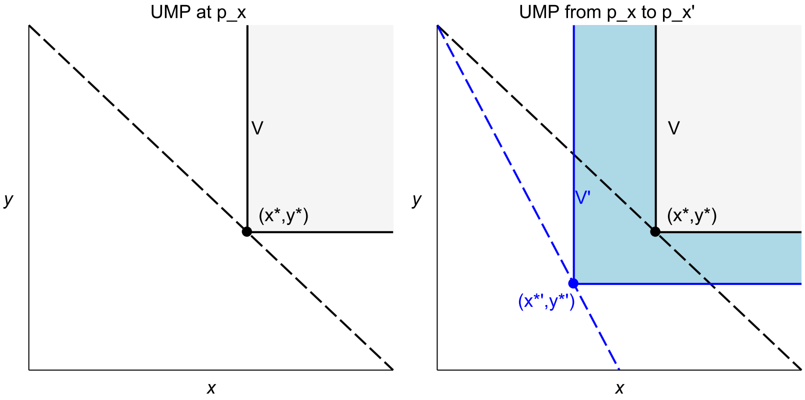


- Total effect: $x^* \to x^*$
- Substitution effect: $x^* \to h_x$, $=$ total effect
- Income effect: $h_x \to x^*$, zero

Perfect Complements

- Utility: $\min\{ax, by\}$
- Budget: $p_x x + p_y y = I$.
- Solve the UMP problem by checking the relative positions of ax and by:
  - If you buy $ax > by$, the utility is by, where the money spent on $ax - by$ is unnecessary (shaded region).
  - If you buy $ax < by$, the utility is ax, where the money spent on $by - ax$ is unnecessary (shaded region).
  - You should buy $ax = by$.
- Plug $ax = by$ into the budget line $p_x x + p_y y = I$, the Marshallian demand is $(x^*, y^*) = \left(\frac{bI}{bp_x + ap_y}, \frac{aI}{bp_x + ap_y}\right), V = \frac{abI}{bp_x + ap_y}$.

- Note that the shaded region denote for $ax > by^*$ and $ax^* < by$.
- Now consider an increase in p_x, from p_x to p'_x.
- Combing $ax = by$ and $p'_x x + p_y y = I$, the new Marshallian demand is $(x^{*\prime}, y^{*\prime}) = \left(\frac{bI}{bp'_x + ap_y}, \frac{aI}{bp'_x + ap_y}\right)$

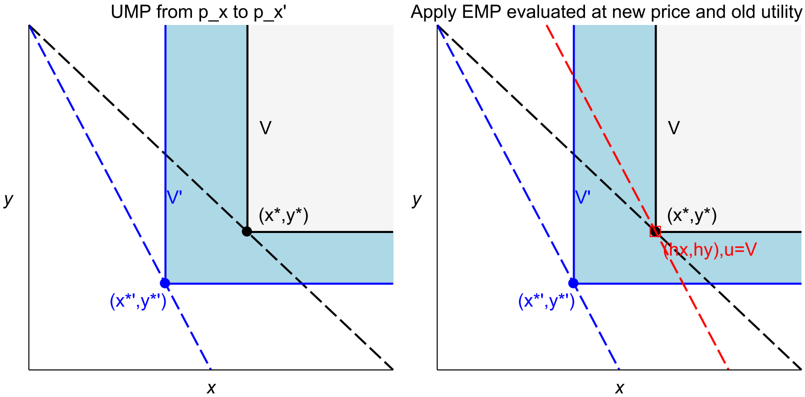


- To decompose the effect due to a change in p_x into substitution and income effects, we need to solve the Hicksian demand (EMP problem) evaluated at
  - The original utility level $u = V$
  - New price level p'_x
- The expenditure minimization problem is

$$\min_{x,y} p'_x x + p_y y$$

$$s.t. \ \min\{ax, by\} = u = V = \frac{abI}{bp_x + ap_y}$$

- $ax = u$ and $by = u$ imply $h_x = \frac{u}{a} = \frac{bI}{bp_x + ap_y} = x^*$ and $h_y = y^*$, and the minimized expenditure is $p'_x h_x + p_y h_y$.
- That is, we plot the budget line $p'_x x + p_y y = p'_x h_x + p_y h_y$ through the original indifference curve (here, is a point at (x^*, y^*)).



- Total effect: $x^* \to x^*$
- Substitution effect: $x^* \to h_x$, zero
- Income effect: $h_x \to x^* =$ total effect

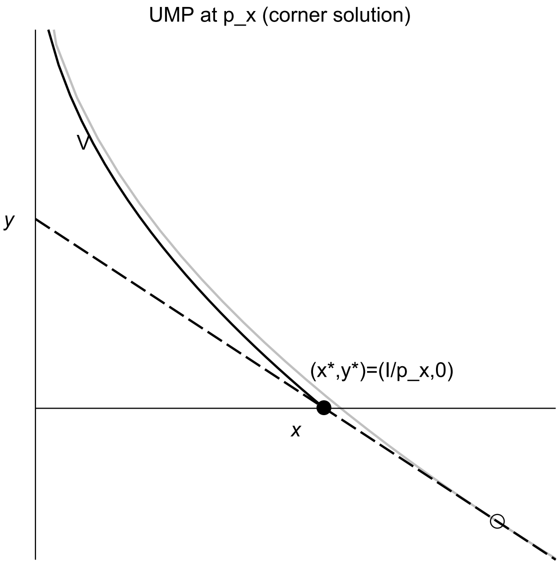
Quasi-linear Utility

- Utility: $U = u(x) + y$
- The initial price is p_x and p_y
- UMP is solved by:

$$\max_{x,y} u(x) + y$$

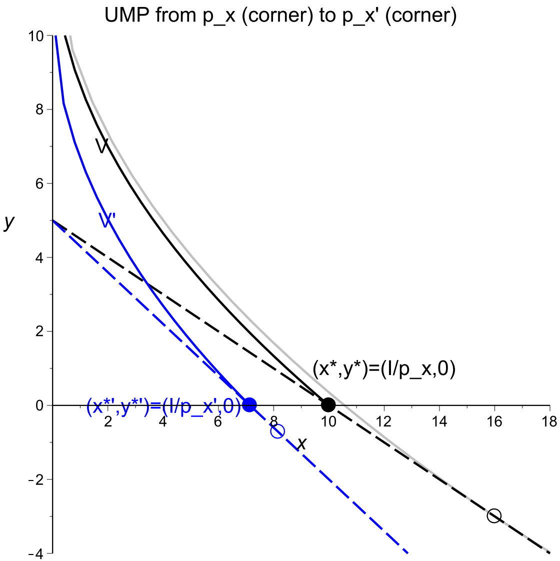
$$s.t.\ p_x x + p_y y = I$$

  - If we apply Lagrangian and solve the first-order condition and find $y^* < 0$, then the solution is not interior.
  - Corner solution is obtained by plugging $y = 0$ into the budget line: $x^* = \frac{I}{p_x}, y^* = 0$.


UMP at p_x (corner solution)
V
y
(x*,y*)=(I/p_x,0)
x

Quasi-linear: from corner to corner

- The corner solutions will exist as long as $MRS > \frac{p_x}{p_y}$
- Now consider a slight increase in p_x, from p_x to p'_x.
- The term "slight" means the p'_x is not so high that $MRS > \frac{p'_x}{p_y}$ and you still spend all your money on x.
- At p'_x, we still get corner solutions: $(x^{*\prime}, y^{*\prime}) = \left(\frac{I}{p'_x}\right), V'$


UMP from p_x (corner) to p_x' (corner)
y
x
V
V'
(x*,y*)=(I/p_x,0)
(x*',y*')=(I/p_x',0)
10
8
6
4
2
0
-2
-4
2
4
6
8
10
12
14
16
18

- To obtain substitution effect, solve the EMP (Hicksian) evaluated at
  - Original utility level $V = u(x^*) + y^* = u\left(\frac{I}{p_x}\right)$
  - New price level p_x'
- EMP:

$$\min_{x,y} p_x' x + p_y y$$

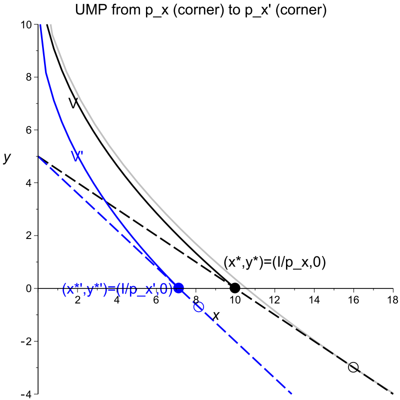
$$s.t.\ u(x) + y = V = u\left(\frac{I}{p_x}\right)$$

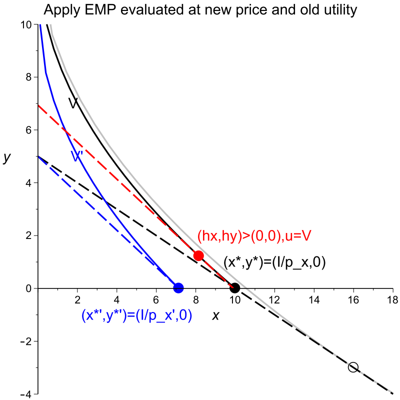
- Plug $y = V - u(x)$ into the objective, the first-order condition

$$u'(h_x) = \frac{p_x'}{p_y}$$

We can verify that $h_x < x^*$ and $h_y > 0$ (interior solution, why?).

Total effect = substitution effect + income effect





- Total effect: $x^* \rightarrow x^*$
- Substitution effect: $x^* \rightarrow h_x$
- Income effect: $h_x \rightarrow x^*$

Quasi-linear: from corner to interior

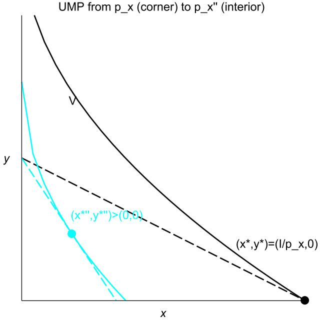
- Next, consider a further increase in p_x to p_x'' such that interior solution emerges (i.e., a positive amount of y will be purchased evaluated at the first-order condition).

At p_x'', solve the UMP:

$$\max_{x,y} u(x) + y$$

$$s.t.\ p_x'' x + p_y y = I$$

The interior solution is
$u'(x^{*''}) = \frac{p_x''}{p_y}$ and $(x^{*''}, y^{*''}) > (0,0)$



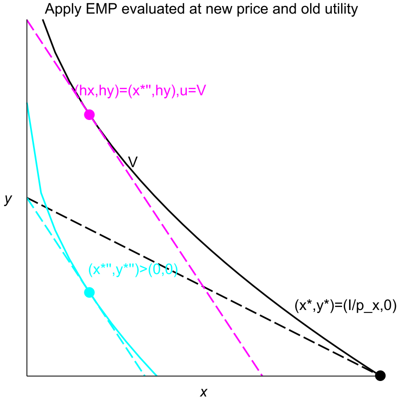
- To see the substitution & income effects, we should solve the Hicksian demand evaluated at the
  - Original utility level: $V = u\left(\frac{I}{p_x}\right)$
  - New price level: p_x''

Solve the EMP (Hicksian)

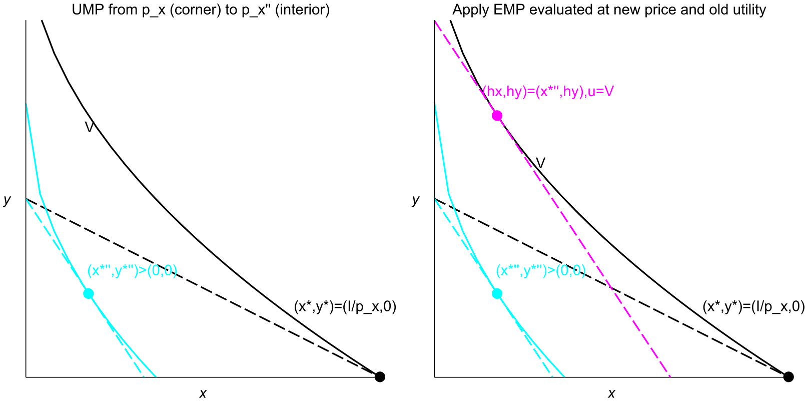
$$\min_{x,y} p_x'' x + p_y y$$

$$s.t.\ u(x) + y = V$$

The Hicksian demand is $u'(h_x) = \frac{p_x''}{p_y}$ and $(h_x, h_y) > (0,0)$. Therefore, $x^{*\prime\prime} = h_x$, i.e., no income effect.



Total effect = substitution effect + income effect



- Total effect: $x^* \to x^*$
- Substitution effect: $x^* \to h_x$ = total effect
- Income effect: $h_x \to x^*$, zero

Quasi-linear: price change (from interior to interior)

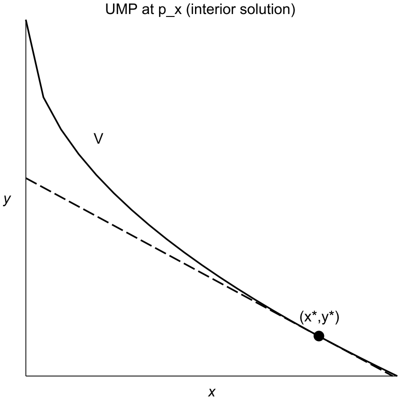
- Original utility: $U = u(x) + y$.
- Original budget line: $p_x x + p_y y = I$.
- Assume that evaluated at the initial price p_x, the interior solution is valid ($MRS = \frac{p_x}{p_y}$), and is equal to:

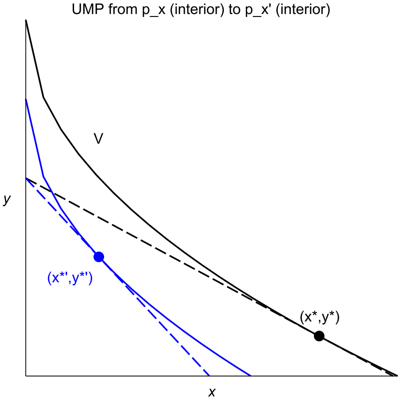
$$u'(x^*) = \frac{p_x}{p_y}$$

- Consider p_x increases from p_x to p'_x
- Evaluated at the new price level p'_x, we still have interior solutions

$$u'(x^{*\prime}) = \frac{p'_x}{p_y}$$

Before: p_x; After: p'_x



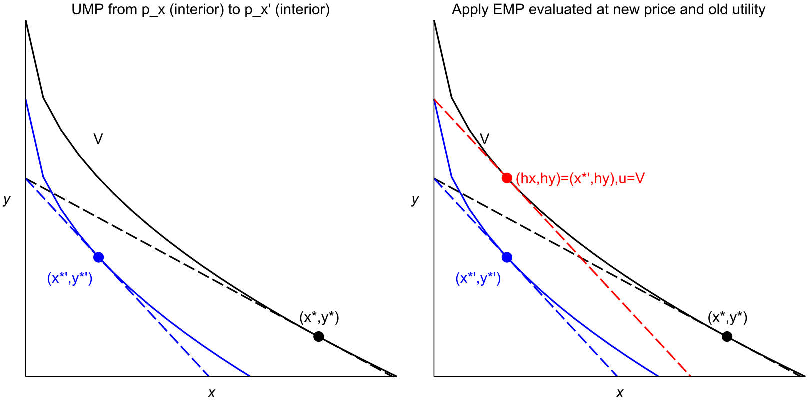


- To measure the income & substitution effect, solve the Hicksian demand evaluated at:
  - Original utility: V
  - New price: p'_x
- The Lagrangian of EMP:

$$\min_{x,y} p'_x x + p_y y$$
$$s.t.\ u(x) + y = V$$

The interior solution is $u'(h_x) = \frac{p'_x}{p_y} = u'(x^{*\prime}) \Rightarrow h_x = x^{*\prime}$

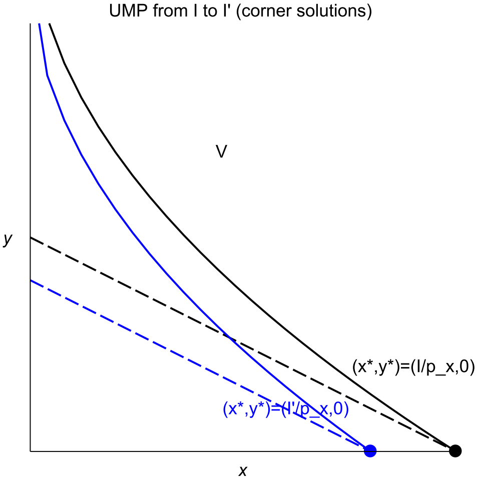
Total effect = substitution effect + income effect



- Total effect: $x^* \to x^*$
- Substitution effect: $x^* \to h_x$ = total effect
- Income effect: $h_x \to x^*$, zero

Quasi-linear: income change (from corner to corner)

- For quasi-linear utilities, another interesting issue is to consider the effect of income I on the optimal choice of x^*.
- Still, let's begin with the corner solution, i.e., the parametric values satisfy $MRS > \frac{p_x}{p_y}$.
- The original solution is: $(x^*, y^*) = \left(\frac{I}{p_x}, 0\right)$
- Now consider I decreases to I', then you should confirm that we still cannot apply the interior solution. (why?)
- The corner solution after change is $(x^*, y^*) = \left(\frac{I'}{p_x}, 0\right)$


UMP from I to I' (corner solutions)
V
y
x
(x*,y*)=(I/p_x,0)
(x*,y*)=(I'/p_x,0)

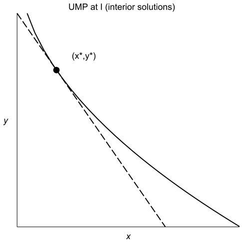
Quasi-linear: income change (from interior to interior)

- Now let's check the income effects within interior solutions.
- Assume initially, $MRS = \frac{p_x}{p_y}$.
- Solve the UMP:

$$\max_{x,y} u(x) + y$$

$$s.t.\ p_x x + p_y y = I$$

The solution is $u'(x^*) = \frac{p_x}{p_y}$

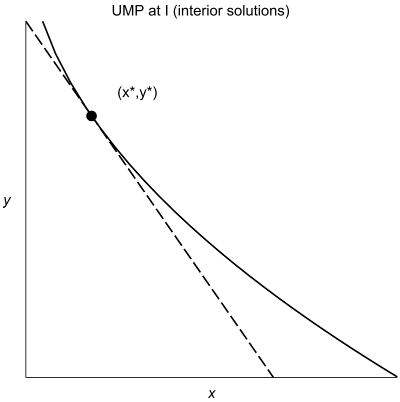


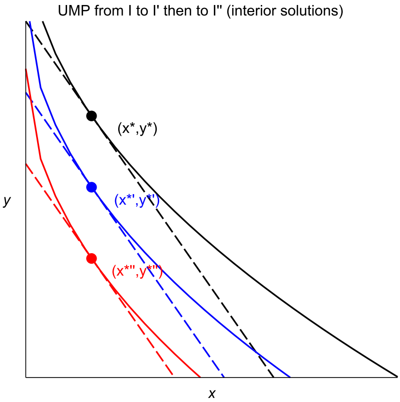
- Consider I decreases, from I to I' and I'', respectively.
- UMP:

$$u'(x^{*\prime}) = \frac{p_x}{p_y}, \ u'(x^{*\prime\prime}) = \frac{p_x}{p_y}$$

- The optimal x is orthogonal to income: $x^* = x^{*\prime} = x^{*\prime\prime}$
- From the budget line, $y = -\frac{p_x}{p_y}x + \frac{I}{p_y}$ is decreasing in I:
  - $y^* = -\frac{p_x}{p_y}x^* + \frac{I}{p_y}$
  - $y^{*\prime} = -\frac{p_x}{p_y}x^{*\prime} + \frac{I'}{p_y}$
  - $y^{*\prime\prime} = -\frac{p_x}{p_y}x^{*\prime\prime} + \frac{I''}{p_y}$

  Because $x^* = x^{*\prime} = x^{*\prime\prime}$ and $I > I' > I''$, hence $y^* > y^{*\prime} > y^{*\prime\prime}$


UMP at I (interior solutions)
(x*,y*)
y
x


UMP from I to I' then to I'' (interior solutions)
(x*,y*)
(x*',y*')
(x*'',y*'')
y
x