PS1 Solutions

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1 Consumption Allocation

Problem Setup

The Home agent's consumption basket is given by

$$C_t = C_{T,t}^{\gamma}\, C_{N,t}^{1-\gamma},$$

where:

- $C_{T,t}$ is the quantity of the traded good (its price is normalized to 1),
- $C_{N,t}$ is the quantity of the domestic non-traded good (price $P_{N,t}$),
- γ is the expenditure share on the traded good.

We want to show that the cost–minimizing demands satisfy:

$$C_{T,t} = \gamma\, P_t\, C_t, \quad C_{N,t} = (1-\gamma)\, \frac{P_t}{P_{N,t}}\, C_t,$$

with the composite price index given by

$$P_t = (P_{N,t})^{\,1-\gamma}.$$

Step 1: Set Up the Expenditure Minimization Problem

The consumer minimizes total expenditure subject to attaining a given consumption level C_t. The problem is

$$\min_{C_{T,t},\, C_{N,t}} \; E = C_{T,t} + P_{N,t}\, C_{N,t} \quad \text{subject to} \quad C_{T,t}^{\gamma}\, C_{N,t}^{1-\gamma} = C_t.$$

Step 2: Form the Lagrangian

Define the Lagrangian function:

$$\mathcal{L} = C_{T,t} + P_{N,t}\, C_{N,t} + \lambda\Big(C_t - C_{T,t}^{\gamma}\, C_{N,t}^{1-\gamma}\Big),$$

where λ is the Lagrange multiplier.

Step 3: First-Order Conditions (FOCs)

Differentiate $\mathcal{L}$ with respect to $C_{T,t}$ and $C_{N,t}$.

With respect to $C_{T,t}$:

$$\frac{\partial \mathcal{L}}{\partial C_{T,t}} = 1 - \lambda\, \gamma\, C_{T,t}^{\gamma-1}\, C_{N,t}^{1-\gamma} = 0.$$

Thus, we have

$$\lambda = \frac{1}{\gamma\, C_{T,t}^{\gamma-1}\, C_{N,t}^{1-\gamma}}.$$

With respect to $C_{N,t}$:

$$\frac{\partial \mathcal{L}}{\partial C_{N,t}} = P_{N,t} - \lambda\, (1-\gamma)\, C_{T,t}^{\gamma}\, C_{N,t}^{-\gamma} = 0.$$

This implies

$$\lambda = \frac{P_{N,t}}{(1-\gamma)\, C_{T,t}^{\gamma}\, C_{N,t}^{-\gamma}}.$$

Step 4: Equate the Two Expressions for λ

Setting the two expressions equal:

$$\frac{1}{\gamma\, C_{T,t}^{\gamma-1}\, C_{N,t}^{1-\gamma}} = \frac{P_{N,t}}{(1-\gamma)\, C_{T,t}^{\gamma}\, C_{N,t}^{-\gamma}}.$$

Multiply both sides by $\gamma\, C_{T,t}^{\gamma-1}\, C_{N,t}^{1-\gamma}$ and $(1-\gamma)\, C_{T,t}^{\gamma}\, C_{N,t}^{-\gamma}$ to simplify:

$$(1-\gamma)\, C_{T,t} = \gamma\, P_{N,t}\, C_{N,t}.$$

Rearranging, the ratio of optimal consumptions is:

$$\frac{C_{T,t}}{C_{N,t}} = \frac{\gamma}{1-\gamma}\, P_{N,t}.$$

Step 5: Derive the Price Index and Conditional Demands

The dual (expenditure) minimization problem yields the unit cost function (composite price index) for the consumption bundle:

$$P_t\, C_t = \min \left\{ C_{T,t} + P_{N,t}\, C_{N,t} :\ C_{T,t}^{\gamma}\, C_{N,t}^{1-\gamma} = C_t \right\}.$$

For a Cobb-Douglas aggregator, the unit cost function is given by

$$P_t = 1^{\gamma} \, (P_{N,t})^{1-\gamma} = (P_{N,t})^{1-\gamma}.$$

Thus, the optimal (conditional) demands are:

$$C_{T,t} = \gamma \, \frac{P_t \, C_t}{1} = \gamma \, P_t \, C_t,$$

$$C_{N,t} = (1-\gamma) \, \frac{P_t \, C_t}{P_{N,t}}.$$

Step 6: Final Result

We have derived that:

$$\boxed{C_{T,t} = \gamma \, P_t \, C_t, \quad C_{N,t} = (1-\gamma) \, \frac{P_t}{P_{N,t}} \, C_t, \quad \text{with} \quad P_t = (P_{N,t})^{1-\gamma}.}$$

Economic Intuition

- The parameter γ reflects the expenditure share on the traded good.
- Since the traded good is the numéraire (price normalized to 1), its cost enters directly, while the cost of the non-traded good is weighted by its price $P_{N,t}$.
- The composite price index P_t is a weighted geometric mean of the individual prices. With the traded good's price equal to 1, we have $P_t = (P_{N,t})^{1-\gamma}$.
- The optimal consumption choices allocate expenditure in a way that equates the marginal rate of substitution to the ratio of prices.

2 Market Clearing

Non-Traded Goods Market

For Home, market clearing in non-traded goods is:

$$n \, C_{N,t} = A_{N,t} \, (L_{N,t})^{1-\alpha}.$$

Substituting $C_{N,t} = (1-\gamma) \, (P_t / P_{N,t}) \, C_t$ with $P_t = (P_{N,t})^{1-\gamma}$, we obtain:

$$\boxed{n(1-\gamma) \, (P_{N,t})^{-\gamma} \, C_t = A_{N,t} \, (L_{N,t})^{1-\alpha}.}$$

For Foreign:

$$\boxed{(1-n)(1-\gamma) \, (P^*_{N,t})^{-\gamma} \, C^*_t = A^*_{N,t} \, (L^*_{N,t})^{1-\alpha}.}$$

Traded Goods Market

Global market clearing for traded goods is:

$$n\, C_{T,t} + (1-n)\, C^*_{T,t} = A_{T,t}\, (n - L_{N,t})^{1-\alpha} + A^*_{T,t}\, \big((1-n) - L^*_{N,t}\big)^{1-\alpha}.$$

Substituting $C_{T,t} = \gamma\, P_t\, C_t$ with $P_t = (P_{N,t})^{1-\gamma}$ (and similarly for Foreign), we have:

$$\boxed{n\gamma\, (P_{N,t})^{1-\gamma}\, C_t + (1-n)\gamma\, (P^*_{N,t})^{1-\gamma}\, C^*_t = A_{T,t}\, (n - L_{N,t})^{1-\alpha} + A^*_{T,t}\, \big((1-n) - L^*_{N,t}\big)^{1-\alpha}.}$$

Intuition: Non-traded goods are produced and consumed domestically, while traded goods are balanced across countries.

3 Intertemporal Allocation

Home Optimization

The Home agent maximizes:

$$U_t = \sum_{s=0}^{\infty} (\beta_{H,t+s})^s \ln C_{t+s},$$

subject to the period-t budget constraint:

$$n\, P_t\, C_t + n\, B_{t+1} = A_{T,t}\, (n - L_{N,t})^{1-\alpha} + P_{N,t}\, A_{N,t}\, (L_{N,t})^{1-\alpha} + n(1+r_t)B_t.$$

The first-order condition gives the Euler equation:

$$\frac{1}{C_t} = \beta_{H,t+1}\, (1 + r_{C,t+1})\, \frac{1}{C_{t+1}},$$

with the consumption-based real return:

$$1 + r_{C,t+1} = (1 + r_{t+1}) \left(\frac{P_t}{P_{t+1}}\right) = (1 + r_{t+1}) \left(\frac{P_{N,t}}{P_{N,t+1}}\right)^{1-\gamma}.$$

Thus,

$$\boxed{C_{t+1} = \beta_{H,t+1}\, (1 + r_{C,t+1})\, C_t.}$$

Also, since $C_{T,t} = \gamma\, P_t\, C_t$:

$$\boxed{C_{T,t+1} = \beta_{H,t+1}\, (1 + r_{t+1})\, C_{T,t}.}$$

Foreign Optimization

Similarly, for the Foreign country:

$$\boxed{C^*_{t+1} = \beta_{F,t+1}\left(1 + r^*_{C,t+1}\right) C^*_t, \qquad C^*_{T,t+1} = \beta_{F,t+1}\left(1 + r_{t+1}\right) C^*_{T,t},}$$

with

$$1 + r^*_{C,t+1} = \left(1 + r_{t+1}\right)\left(\frac{P^*_{N,t}}{P^*_{N,t+1}}\right)^{1-\gamma}.$$

Intuition: Households equate the marginal utility cost of consuming today versus tomorrow, with intertemporal decisions affected by relative price changes.

4 Labor Allocation

Home Country

Firms allocate labor such that:

$$(1-\alpha)A_{T,t}\left(n - L_{N,t}\right)^{-\alpha} = P_{N,t}\,(1-\alpha)A_{N,t}\left(L_{N,t}\right)^{-\alpha}.$$

Canceling $(1-\alpha)$ and rearranging yields:

$$\boxed{A_{T,t}\left(n - L_{N,t}\right)^{-\alpha} = P_{N,t}\,A_{N,t}\left(L_{N,t}\right)^{-\alpha}.}$$

Foreign Country

Similarly, for Foreign:

$$\boxed{A^*_{T,t}\left((1-n) - L^*_{N,t}\right)^{-\alpha} = P^*_{N,t}\,A^*_{N,t}\left(L^*_{N,t}\right)^{-\alpha}.}$$

Intuition: Labor is allocated until the marginal value products in the traded and non-traded sectors are equalized (after accounting for relative prices).

5 Resource Constraints and the Real Exchange Rate

Resource Constraints

The Home resource constraint is given by:

$$n\gamma\left(P_{N,t}\right)^{1-\gamma} C_t + n\,B_{t+1} = A_{T,t}\left(n - L_{N,t}\right)^{1-\alpha} + n(1 + r_t)B_t.$$

Similarly, for Foreign:

$$(1-n)\gamma\,(P^*_{N,t})^{1-\gamma}\,C^*_t - n\,B_{t+1} = A^*_{T,t}\left((1-n) - L^*_{N,t}\right)^{1-\alpha} - n(1+r_t)B_t.$$

Real Exchange Rate

Define the real exchange rate as:

$$Q_t \equiv \frac{P^*_t}{P_t} = \left(\frac{P^*_{N,t}}{P_{N,t}}\right)^{1-\gamma}.$$

A higher Q_t implies that Foreign's consumption basket is relatively more expensive, equivalent to a real depreciation for Home.

6 Steady State

Assume a symmetric steady state with no cross-country assets ($B_0 = 0$) and constant discount factor β_0. Calibration is chosen as:

$$A_{N,0} = A_{T,0}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha}, \quad A^*_{T,0} = A_{T,0}\left(\frac{1-n}{n}\right)^{\alpha}, \quad A^*_{N,0} = A_{T,0}\left(\frac{1-n}{n}\right)^{\alpha}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha}.$$

Then:

- **Interest Rate:** $1 = \beta_0(1+r_0)$, so

$$r_0 = \frac{1-\beta_0}{\beta_0}.$$

- **Non-Traded Prices:** $P_{N,0} = P^*_{N,0} = 1$.
- **Labor Allocation:** From the intratemporal condition,

$$\frac{L_{N,0}}{n - L_{N,0}} = \frac{1-\gamma}{\gamma} \quad \Rightarrow \quad L_{N,0} = n(1-\gamma),$$

and similarly,

$$L^*_{N,0} = (1-n)(1-\gamma).$$

- **Consumption:** One can show that

$$C_0 = \left(A^{\gamma}_{T,0}A^{1-\gamma}_{N,0}\right)\left[n^{\gamma}(1-\gamma)^{1-\gamma}\right]^{-\alpha},$$

with a similar expression for C^*_0 so that $C_0 = C^*_0$.

Intuition: In the steady state, relative prices and allocations are balanced; country size differences aside, both countries have identical marginal conditions.

7 Log-Linear Approximation

Let the "hat" denote log-deviations from the steady state, e.g. $\hat{C}_t = \frac{C_t - C_0}{C_0}$.

Key Linearized Equations

Non-Traded Goods Market (Home):

$$-\gamma\, \hat{P}_{N,t} + \hat{C}_t = \hat{A}_{N,t} + (1-\alpha)\, \hat{L}_{N,t}. \tag{7a}$$

For Foreign:

$$-\gamma\, \hat{P}^*_{N,t} + \hat{C}^*_t = \hat{A}^*_{N,t} + (1-\alpha)\, \hat{L}^*_{N,t}.$$

Resource Constraint (Home):

$$(1-\gamma)\, \hat{P}_{N,t} + \hat{C}_t + \hat{B}_{t+1} = \hat{A}_{T,t} - \frac{(1-\alpha)(1-\gamma)}{\gamma}\, \hat{L}_{N,t} + \frac{1}{\beta_0}\, \hat{B}_t. \tag{7b}$$

Euler Equation (Home):

$$\hat{C}_{t+1} - \hat{C}_t = (1-\gamma)\,(\hat{P}_{N,t} - \hat{P}_{N,t+1}) + \hat{\beta}_{H,t+1} + \beta_0\, \hat{r}_{C,t+1}. \tag{7c}$$

Labor Allocation (Home):

$$\hat{A}_{T,t} + \frac{\alpha}{\gamma}\, \hat{L}_{N,t} = \hat{P}_{N,t} + \hat{A}_{N,t}. \tag{7d}$$

Real Exchange Rate:

$$\hat{Q}_t = (1-\gamma)(\hat{P}^*_{N,t} - \hat{P}_{N,t}).$$

Intuition: These equations capture the first-order responses of consumption, labor, and relative prices to shocks.

8 Worldwide Equilibrium

Define world aggregates as population-weighted averages (e.g., $\hat{C}^W_t = n\, \hat{C}_t + (1-n)\, \hat{C}^*_t$). Then, from the above log-linearized equations one can show:

- $\hat{L}^W_{N,t} = 0$, i.e., aggregate non-traded labor remains fixed.
- The world non-traded price satisfies

$$\hat{P}^W_{N,t} = \hat{A}^W_{T,t} - \hat{A}^W_{N,t}.$$

- World consumption is given by

$$\hat{C}_t^W = \gamma\, \hat{A}_{T,t}^W + (1-\gamma)\, \hat{A}_{N,t}^W.$$

- The world Euler equation becomes

$$\beta_0\, \hat{r}_{t+1} = -\hat{\beta}_{t+1}^W + \left(\hat{A}_{T,t+1}^W - \hat{A}_{T,t}^W\right).$$

Intuition: World aggregates respond only to symmetric shocks, with the real interest rate driven by global productivity changes and aggregate patience.

9 Cross-Country Differences

Define differences as (Home minus Foreign) for a variable x by $\tilde{x}_t = \hat{x}_t - \hat{x}_t^*$. Then:
Non-Traded Goods Market (Difference):

$$\frac{\gamma}{1-\gamma}\, \hat{Q}_t + \tilde{C}_t = \tilde{A}_{N,t} + (1-\alpha)\, \tilde{L}_{N,t}. \tag{9a}$$

Resource Constraints (Difference):

$$-\,\hat{Q}_t + \tilde{C}_t + \frac{\hat{B}_{t+1}}{1-n} = \tilde{A}_{T,t} - \frac{(1-\alpha)(1-\gamma)}{\gamma}\, \tilde{L}_{N,t}. \tag{9b}$$

Euler Equation (Difference):

$$\tilde{C}_{t+1} - \tilde{C}_t = (1-\gamma)\Big[(\hat{P}_{N,t} - \hat{P}_{N,t}^*) - (\hat{P}_{N,t+1} - \hat{P}_{N,t+1}^*)\Big] + (\hat{\beta}_{H,t+1} - \hat{\beta}_{F,t+1}) + \beta_0\Big(\hat{r}_{C,t+1} - \hat{r}_{C,t+1}^*\Big). \tag{9c}$$

Labor Allocation (Difference):

$$\frac{\alpha}{\gamma}\, \tilde{L}_{N,t} = -\frac{1}{1-\gamma}\, \hat{Q}_t - \tilde{A}_{N,t}. \tag{9d}$$

Intuition: These equations link cross-country differences in consumption, labor, and the real exchange rate to differences in productivity and intertemporal preferences.

10 Long-Run Allocation (Period $t+1$)

Assume that from $t+1$ onward the economy reaches a new steady state with no further discount factor shocks ($\hat{\beta}_{H,t+2} = \hat{\beta}_{F,t+2} = 0$). Taking the cross-country asset position $\hat{B}_{t+1}$ as given, one can show:

$$\hat{Q}_{t+1} = -(1-\gamma)\left[(\hat{A}_{T,t+1} - \hat{A}_{T,t+1}^*) - (\hat{A}_{N,t+1} - \hat{A}_{N,t+1}^*)\right] - \frac{\alpha(1-\gamma)}{1-\beta_0}\frac{1}{\beta_0}\frac{\hat{B}_{t+1}}{1-n},$$

$$\tilde{L}_{N,t+1} = \frac{\gamma}{1-\beta_0}\frac{1}{\beta_0}\frac{\hat{B}_{t+1}}{1-n},$$

$$\tilde{C}_{t+1} = \gamma\big(\hat{A}_{T,t+1} - \hat{A}^*_{T,t+1}\big) + (1-\gamma)\big(\hat{A}_{N,t+1} - \hat{A}^*_{N,t+1}\big) + \frac{\gamma}{1-\beta_0}\frac{1}{\beta_0}\frac{\hat{B}_{t+1}}{1-n}.$$

Interpretation:

- A positive $\hat{B}_{t+1}$ (Home wealthier) implies higher relative consumption and a lower Q_{t+1} (Home's goods become relatively more expensive).
- Permanent productivity differences affect steady state consumption and prices directly.

11 Short-Run Allocation (Period t)

Assume initially $\hat{B}_t = 0$. Solving the system (starting with the labor and non-traded market conditions, then using the resource constraint and Euler equations) yields:

$$\frac{\hat{B}_{t+1}}{1-n} = \frac{\beta_0}{\gamma + \alpha(1-\gamma)}\Big[(\hat{\beta}_{H,t+1} - \hat{\beta}_{F,t+1}) - (\hat{A}_{T,t+1} - \hat{A}^*_{T,t+1}) + (\hat{A}_{T,t} - \hat{A}^*_{T,t})\Big], \tag{11a}$$

$$\tilde{C}_t = \gamma(\hat{A}_{T,t} - \hat{A}^*_{T,t}) + (1-\gamma)(\hat{A}_{N,t} - \hat{A}^*_{N,t}) - \frac{\gamma\,\beta_0}{\gamma + \alpha(1-\gamma)}\Big[(\hat{\beta}_{H,t+1} - \hat{\beta}_{F,t+1}) - (\hat{A}_{T,t+1} - \hat{A}^*_{T,t+1}) + (\hat{A}_{T,t} - \hat{A}^*_{T,t})\Big], \tag{11b}$$

$$\hat{Q}_t = -(1-\gamma)(\hat{A}_{T,t} - \hat{A}^*_{T,t}) + (1-\gamma)(\hat{A}_{N,t} - \hat{A}^*_{N,t}) + \frac{(1-\gamma)\,\alpha\,\beta_0}{\gamma + \alpha(1-\gamma)}\Big[(\hat{\beta}_{H,t+1} - \hat{\beta}_{F,t+1}) - (\hat{A}_{T,t+1} - \hat{A}^*_{T,t+1}) + (\hat{A}_{T,t} - \tag{11c}$$

$$\tilde{L}_{N,t} = -\frac{\gamma\,\beta_0}{\gamma + \alpha(1-\gamma)}\Big[(\hat{\beta}_{H,t+1} - \hat{\beta}_{F,t+1}) - (\hat{A}_{T,t+1} - \hat{A}^*_{T,t+1}) + (\hat{A}_{T,t} - \hat{A}^*_{T,t})\Big]. \tag{11d}$$

Interpretation:

- A temporary increase in Home patience (i.e. $\hat{\beta}_{H,t+1} - \hat{\beta}_{F,t+1} > 0$) leads to $\hat{B}_{t+1} > 0$ (Home runs a current account surplus), lower relative consumption, and a real exchange rate movement consistent with a trade surplus.
- Temporary traded or non-traded productivity shocks affect the current account and labor allocation differently.
- For permanent shocks ($\hat{A}_{T,t} = \hat{A}_{T,t+1}$), intertemporal balance is restored with $\hat{B}_{t+1} = 0$ and immediate adjustment to the new steady state.

12 Summary of Key Economic Insights

- **Consumption and Prices:** The structure of the consumption basket implies that a rise in the non-traded good price $P_{N,t}$ increases the overall consumption price P_t and shifts the consumption mix.

- **Market Clearing:** Non-traded goods are produced and consumed domestically, whereas traded goods are allocated internationally, linking domestic production choices to the real exchange rate.
- **Intertemporal Choices:** The Euler equations show that higher patience or higher real returns lead to deferred consumption. Cross-country differences in patience drive current account imbalances.
- **Labor Allocation:** Labor is reallocated across sectors until the marginal value products are equalized; productivity shifts affect both output composition and relative prices.
- **Steady State and Log-Linearization:** In a symmetric steady state, relative prices and allocations are balanced. Log-linearization permits analysis of small shocks and their propagation.
- **Short-Run vs. Long-Run Dynamics:** Temporary shocks generate current account imbalances and short-run reallocation, while permanent shocks adjust consumption and prices directly with no asset accumulation.
- **Wealth Effects:** A positive net asset position (Home wealthier) implies higher steady-state consumption and a relatively stronger (appreciated) currency.