PS3

Exercise 2

1) $\sum_{t=0}^{\infty} \frac{u(C_t)}{(1-\rho)^t}$ w/ $u(C_t) = C_t - \theta C_t^2 ; \theta > 0$

$u'(C_t) = 1 - 2\theta C_t$

2) $Y_t = AK_t + e_t$

3) $K_{t+1} = K_t + Y_t - C_t$ => No depreciation

$e_t = \phi e_{t-1} + \varepsilon_t$ w/ $-1 < \phi < 1$ ε_t i.i.d. $A_t = \rho$

1) $$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} \left[u(C_t) - \lambda_t (K_{t+1} - K_t - Y_t + C_t) \right]$$

plug-in 2)

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} \left[u(C_t) - \lambda_t (K_{t+1} - K_t - AK_t - e_t + C_t) \right]$$

FOC

$$\frac{d\mathcal{L}}{dC_t} = \frac{1}{(1+\rho)^t} u'(C_t) = \frac{1}{(1+\rho)^t} \lambda_t \Rightarrow u'(C_t) = \lambda_t$$

$$\frac{d\mathcal{L}}{dK_{t+1}} = -\frac{1}{(1+\rho)^t} \lambda_t + \frac{1}{(1+\rho)^{t+1}} E_0 \left[\lambda_{t+1} - A\lambda_{t+1} \right]$$

$$\Rightarrow \lambda_t = \frac{1}{1+\rho} (1+A) E_0 \left[\lambda_{t+1} \right]$$

EULER EQUATION : $u'(C_t) = \frac{1+A}{1+\rho} E_0 u'(C_{t+1})$

$$u'(C_t) = 1 - 2\theta C_t$$

$$1 - 2\theta c_t = E_t\left[\frac{1}{1+\rho}(1+A)(1-2\theta c_{t+1})\right] \quad ; \text{ since } A=\rho$$

$$\Rightarrow c_t = E_t[c_{t+1}] \quad (2)$$

2) Guess consumption: $c_t = \alpha + \beta k_t + \gamma e_t$ (3)

What's k_{t+1}?

$$k_{t+1} = y_t + k_t - c_t = Ak_t + e_t + k_t - \alpha - \beta k_t - \gamma e_t$$

$$= (1+A-\beta)k_t + (1-\gamma)e_t - \alpha \quad (4)$$

3) For which α, β, γ the FOC holds $\forall k_t, e_t$?

Back to the FOC:

$$c_t = E_t[c_{t+1}] \Rightarrow \text{Plug-in (3) and (4)}$$

$$\alpha + \beta k_t + \gamma e_t = E_t[\alpha + \beta k_{t+1} + \gamma e_{t+1}] = \alpha + \beta\left[(1+A-\beta)k_t + (1-\gamma)e_t - \alpha\right] + \gamma e_{t+1}]$$

$$\alpha + \beta k_t + \gamma e_t = \alpha + \beta\left[(1+A-\beta)k_t + (1-\gamma)e_t - \alpha\right] + \gamma\phi e_t$$

This holds $\forall k_t, e_t$ iff:

$\alpha = 0$

$\cancel{\beta} = \cancel{\beta}(1+A-\beta) \Rightarrow 1 = 1 + A - \beta \Rightarrow A = \beta$

$\gamma = \beta(1-\gamma) + \gamma\phi \Rightarrow \gamma(1-\phi) = \beta(1-\gamma) \Rightarrow \dfrac{\gamma}{1-\gamma} = \dfrac{\beta}{1-\phi}$

$\gamma(1-\phi) = \beta - \gamma\beta \; ; \; \gamma(1-\phi+\beta) = \beta \Rightarrow \gamma = \dfrac{\beta}{1-\phi+\beta}$

If you plug these values for β, γ back in you can check that this is true.

4) One time shock e_t to Y_t, K, C ?

Start from K_{t+1} :

$K_{t+1} = (1+A-\beta)K_t + (1-\gamma)e_t - \alpha = K_t + (1-\gamma)e_t$ since $\alpha = 0$ & $A = \beta$

$$K_{t+1} = K_t + \left(1 - \frac{A}{1-\phi+A}\right)e_t = K_t + \frac{1-\phi}{1-\phi+A}e_t$$

$$K_{t+2} = K_{t+1} + \frac{1-\phi}{1-\phi+A}e_{t+1} = K_t + \frac{1-\phi}{1-\phi+A}e_t + \frac{1-\phi}{1-\phi+A}\phi e_t$$

$$K_{t+3} = K_{t+2} + \frac{(1-\phi)}{1+\phi+A}e_{t+2} = K_t + \frac{(1-\phi)}{1-\phi+A}e_t + \frac{1-\phi}{1-\phi+A}\phi e_t + \frac{1-\phi}{1-\phi+A}\phi^2 e_t$$

$\vdots$

$$K_{t+k} = K_t + \frac{1-\phi}{1-\phi+A}\sum_{n=0}^{k-1}\phi^n e_t = K_t + \frac{\cancel{1-\phi}}{1-\phi+A}\,\frac{1-\phi^k}{\cancel{1-\phi}}e_t$$

$$\boxed{\lim_{T\to\infty} K_T = K_t + \frac{e_t}{1-\phi+A}}$$

Now let's go to $C_t = \alpha + \beta K_t + \gamma e_t$

$$C_t = AK_t + \frac{A}{1-\phi+A}e_t$$

$$C_{t+1} = AK_{t+1} + \frac{A}{1-\phi+A}e_{t+1} = A\left[K_t + (1-\gamma)e_t\right] + \frac{A}{1-\phi+A}\phi e_t$$

$$C_{t+2} = AK_{t+2} + \frac{A}{1-\phi+A}e_{t+2} = A\left[K_{t+1} + (1-\gamma)e_{t+1}\right] + \frac{A}{1-\phi+A}\phi^2 e_t$$

$$= A\left[K_t + \frac{1-\phi}{1-\phi+A}e_t + \frac{1-\phi}{1-\phi+A}\phi e_t\right] + \frac{A}{1-\phi+A}\phi^2 e_t$$

$$C_{t+k} = Ak_{t+k} + \frac{A}{1-\phi+A} e_{t+k} = Ak_t + A\left(\frac{1-\phi}{1-\phi+A}\right)\sum_{n=0}^{k-1}\phi^n e_n + \frac{A}{1-\phi+A}\phi^k e_t$$

$$= C_t - \frac{A}{1-\phi+A} e_t + \frac{A}{1-\phi+A}(1-\phi^k)e_t + \frac{A\phi^k}{1-\phi+A} e_t = C_{t+k} \quad \checkmark$$

Now for Y_t:

$$Y_t = Ak_t + e_t$$

$$Y_{t+k} = Ak_{t+k} + e_{t+k} = A\left(k_t + \frac{1-\phi^k}{1+A-\phi} e_t\right) + \phi^k e_t$$

$$= Y_t - e_t + \frac{1-\phi^k}{1-\phi+A} A e_t + \phi^k e_t$$

$$= Y_t + A\frac{1-\phi^k}{1-\phi+A} e_t - (1-\phi^k) e_t = Y_t - (1-\phi^k) e_t \left[1 - \frac{A}{1-\phi+A}\right]$$

$$= Y_t - \frac{(1-\phi^k)(1-\phi)}{1-\phi+A} e_t \quad \checkmark$$

Exercise 2

$u(C_t, L_t) = \log C_t + b\log(1-L_t)$; BC : $C_t + K_{t+1} = W_t L_t + R_t K_t$

$Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$

$\delta = 1 \Rightarrow K_{t+1} = Y_t - C_t$

$\ln A_t = \rho \ln A_{t-1} + \varepsilon_{A,t}$ $\quad n = 0 \quad g = 0$

$$V(K_t, A_t) = \max_{\{C_t, L_t\}} E_t \sum_{\tau=0}^{\infty} \beta^\tau u(C_{t+\tau}, L_{t+\tau}) \}$$ VALUE FUNCTION!

1) $$V(K_t, A_t) = \max_{\{C_t, L_t\}} \left\{ u(C_t, L_t) + \beta E_t \left\{ \sum_{\tau=1}^{\infty} \beta^{\tau-1} u(C_{t+\tau}, L_{t+\tau}) \right\} \right\}$$

$$= \max_{\{C_t, L_t\}} \left\{ u(C_t, L_t) + \beta V(K_{t+1}, A_{t+1}) \right\}$$ BELLMAN EQUATION

2) $V(K_t, A_t) = \gamma_0 + \gamma_K \log K_t + \gamma_A \log A_t$ } GUESS

$$V(K_t, A_t) = \max_{\{C_t, L_t\}} \left\{ \log C_t + b\log(1-L_t) + \beta\left[\gamma_0 + \gamma_K \log K_{t+1} + \gamma_A \log A_{t+1}\right] \right\}$$

FOC remember $Y_t - C_t = K_{t+1}$

C_t: $\dfrac{1}{C_t} - \dfrac{\beta\gamma_K}{Y_t - C_t} = 0 \Rightarrow C_t(1 + \beta\gamma_K) = Y_t$

$$\frac{C_t}{Y_t} = \frac{1}{1+\beta\gamma_K}$$

L_t: Remember $Y_t = W_t L_t + R_t K_t$

FOC $\quad -\dfrac{b}{1-L_t} + \dfrac{\beta\gamma_K W_t}{Y_t - C_t} \Rightarrow W_t = MPL = (1-\alpha) K_t^\alpha L_t^{-\alpha} A_t^{1-\alpha}$

$$= (1-\alpha)\frac{Y_t}{L_t}$$

- $Y_t = C_t + \beta\gamma_K C_t$

$$\frac{b}{1-L_t} = \frac{\cancel{\beta\gamma_k} W_t}{\cancel{\beta\gamma_k} C_t} = (1-\alpha)\frac{Y_t}{L_t} \cdot \frac{1}{C_t} = (1-\alpha)(1+\beta\gamma_k)\frac{1}{L_t}$$

$$\frac{b}{1-L_t} = (1-\alpha)(1+\beta\gamma_k) \cdot \frac{1}{L_t}$$

$$bL_t + (1-\alpha)(1+\beta\gamma_k)L_t = (1-\alpha)(1+\beta\gamma_k)$$

$$L_t = \frac{(1-\alpha)(1+\beta\gamma_k)}{(1-\alpha)(1+\beta\gamma_k)+b}$$

$$1-L_t = \frac{b}{(1-\alpha)(1+\beta\gamma_k)+b}$$

3) $V(k_t, A_t) = \ln C_t + b\ln(1-L_t) + \beta\left[\gamma_0 + \gamma_K \ln(Y_t - C_t) + \gamma_A \rho_A \ln\right.$

$$= \ln\left[\frac{Y_t}{1+\beta\gamma_k}\right] + b\ln\frac{b}{(1-\alpha)(1+\beta\gamma_k)+b} + \beta\left[\gamma_0 + \gamma_k \ln\left(\frac{Y_t}{1+\beta\gamma_k}\right) + \gamma_A \rho_A \ln A_t\right.$$

- Substitute : $Y_t = K_t^\alpha (A_t L_t)^{1-\alpha}$

$$\alpha\ln k_t + (1-\alpha)\ln L_t + (1-\alpha)\ln A_t - \ln(1+\beta\gamma_k) + b\ln\frac{b}{(1-\alpha)(1+\beta\gamma_k)+b} +$$

$$+\beta\left[\gamma_0 + \gamma_k\alpha\ln K_t + \gamma_k(1-\alpha)\ln L_t + \gamma_k(1-\alpha)\ln A_t - \gamma_k \ln(1+\beta\gamma_k)\right.$$
$$\left. + \gamma_A\rho_A \ln A_t\right]$$

Rewrite :

$$\boxed{\alpha\ln K_t + \beta\gamma_k\alpha\ln K_t} \boxed{+(1-\alpha)\ln A_t + \beta\gamma_k(1-\alpha)\ln A_t + \gamma_A\rho_A\ln A_t} \boxed{+(1-\alpha)\ln L_t - \ln(1+\beta\gamma_k) + b\ln\frac{b}{(1-\alpha)(1+\beta\gamma_k)+b} + \beta\gamma_0 + \beta\gamma_k(1-\alpha)\ln L_t - \beta\gamma_k\ln(1+\beta\gamma_k)}$$

Define $\gamma_k' = \alpha(1+\beta\gamma_k)$

$\gamma_A' = (1-\alpha) + \beta\gamma_k(1-\alpha) + \gamma_A\rho_A = (1-\alpha)(1+\beta\gamma_k) + \gamma_A\rho_A$

$\gamma_0' \Rightarrow$ EVERYTHING ELSE SINCE IT DOES NOT DEPEND

ON THE STATES OF THE ECONOMY.

Then:

$$V(K_t, A_t) = \gamma_0' + \gamma_k' \ln K_t + \gamma_A' \ln A_t$$ || still linear in A_t & K_t

5) γ_K, γ_A to have $\gamma_K = \gamma_K'$ $\gamma_A = \gamma_A'$

Just equal the previous expressions

$$\gamma_K = \gamma_K' = \alpha(1 + \beta \gamma_K) \Rightarrow \gamma_K = \frac{\alpha}{1 - \beta\alpha}$$

$$\gamma_A = \gamma_A' = (1-\alpha)(1 + \beta\gamma_K) + \beta\gamma_A \rho_A$$ => rearrange to get

$$\gamma_A = \frac{(1-\alpha)}{(1-\beta\alpha)(1-\beta\rho_A)}$$

6) $$\frac{C_t}{Y_t} = \frac{1}{1+\beta\gamma_K} = \frac{1}{1 + \frac{\beta\alpha}{1-\alpha\beta}} = \frac{1}{\frac{1-\alpha\beta+\alpha\beta}{1-\alpha\beta}} = 1 - \alpha\beta$$

$$L_t = \frac{(1-\alpha)(1+\beta\gamma_K)}{(1-\alpha)(1+\beta\gamma_K) + b} = \frac{(1-\alpha)\left(1 + \frac{\beta\alpha}{1-\alpha\beta}\right)}{(1-\alpha)\left(1+\frac{\beta\alpha}{1-\alpha\beta}\right) + b}$$

$$= \frac{\frac{(1-\alpha)}{1-\alpha\beta}}{\frac{1-\alpha}{1-\alpha\beta} + b} = \frac{1-\alpha}{(1-\alpha) + b(1-\alpha\beta)}$$