# Macroeconomics A; EI056

# Technical appendix: The New Keynesian model

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## 1 Baseline model with sticky prices

### 1.1 Building blocks

#### 1.1.1 Household

The representative household maximizes the expected intertemporal utility (we abstract from the utility of real balances, as we are not interested in deriving a money demand):

$$U_t = E_t \sum_{i=0}^{\infty} \beta^i \left[ \frac{(C_{t+i})^{1-\sigma}}{1-\sigma} - \chi \frac{(N_{t+i})^{1+\eta}}{1+\eta} \right] \tag{1}$$

The only asset is a risk-free one period bond, with a bond bought at time $t$ paying off $1+i_t$ at time $t+1$ (we abstract from cash as we are not interested in the money demand) The budget constraint is:

$$P_t C_t + B_t = (1 + i_{t-1}) B_{t-1} + W_t N_t + \Pi_t \tag{2}$$

where $P$ is the CPI, $B$ are bond holdings, $W$ is the wage, and $\Pi_t$ are the profits of the firms.

The consumption basket $C_t$ is an aggregate across the various available brands:

$$C_t = \left[ \int_0^1 [C_{j,t}]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}}$$

where $j$ is an index of brands. This is a CES (constant elasticity of scale) index, where $\theta > 1$ is the elasticity of substitution across brands.

We start with the allocation of consumption across brands. The agent chooses the consumption of brands to minimize the cost of a given basket $C$ (dropping the time subscript for brevity):

$$\min_{C_j\text{'s}} \int_0^1 P_j C_j dj \qquad \text{subject to: } \left[ \int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} = C$$

The Lagrangian is as follows:

$$\mathcal{L} = \int_0^1 P_j C_j dj + \lambda \left[ C - \left[ \int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} \right]$$

Take the optimality condition with respect to $C_j$:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial C_j} &= 0 \Rightarrow P_j = \frac{\theta}{\theta-1}\lambda \left[ \int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}-1} \frac{\theta}{\theta-1} [C_j]^{\frac{\theta-1}{\theta}-1} \\
&\Rightarrow \quad P_j = \lambda \left[ \int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}-1} [C_j]^{\frac{\theta-1}{\theta}-1} \qquad (3)
\end{aligned}$$

Multiply both sides by $C_j$ and add across all the available brands $i$:

$$\begin{aligned}
P_j &= \lambda \left[ \int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}-1} [C_j]^{\frac{\theta-1}{\theta}-1} \\
P_j C_j &= \lambda \left[ \int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}-1} [C_j]^{\frac{\theta-1}{\theta}} \\
\int_0^1 P_j C_j dj &= \int_0^1 \left\{ \lambda \left[ \int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}-1} [C_j]^{\frac{\theta-1}{\theta}} \right\} dj \\
\int_0^1 P_j C_j dj &= \lambda \left[ \int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}-1} \left[ \int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right] \\
\int_0^1 P_j C_j dj &= \lambda \left[ \int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} \\
PC &= \lambda C \\
P &= \lambda
\end{aligned}$$

where $P$ is an aggregate price index defined as:

$$\int_0^1 P_j C_j dj = PC$$

The multiplier is therefore the consumer price index.

Next, use this result in (3):

$$
\begin{aligned}
P_j &= \lambda \left[ \int_0^1 [C_j]^{\frac{\theta-1}{\theta}} \, dj \right]^{\frac{\theta}{\theta-1}-1} [C_j]^{\frac{\theta-1}{\theta}-1} \\
P_j &= P \left[ \int_0^1 [C_j]^{\frac{\theta-1}{\theta}} \, dj \right]^{\frac{\theta}{\theta-1}-1} [C_j]^{\frac{\theta-1}{\theta}-1} \\
P_j &= P \left[ \int_0^1 [C_j]^{\frac{\theta-1}{\theta}} \, dj \right]^{\frac{1}{\theta-1}} [C_j]^{-\frac{1}{\theta}} \\
P_j &= P \left[ \left[ \int_0^1 [C_j]^{\frac{\theta-1}{\theta}} \, dj \right]^{\frac{\theta}{\theta-1}} \right]^{\frac{1}{\theta}} [C_j]^{-\frac{1}{\theta}}
\end{aligned}
$$

$$
\begin{aligned}
P_j &= P \, [C]^{\frac{1}{\theta}} \, [C_j]^{-\frac{1}{\theta}} \\
[C_j]^{\frac{1}{\theta}} \, P_j &= P \, [C]^{\frac{1}{\theta}} \\
[C_j]^{\frac{1}{\theta}} &= \left[ \frac{P_j}{P} \right]^{-1} [C]^{\frac{1}{\theta}} \\
C_j &= \left[ \frac{P_j}{P} \right]^{-\theta} C \qquad (4)
\end{aligned}
$$

To derive the aggregate price index, use the demand for a particular brand (4) in the definition of the consumption basket:

$$
\begin{aligned}
C &= \left[ \int_0^1 [C_j]^{\frac{\theta-1}{\theta}} \, dj \right]^{\frac{\theta}{\theta-1}} \\
C &= \left[ \int_0^1 \left[ \left[ \frac{P_j}{P} \right]^{-\theta} C \right]^{\frac{\theta-1}{\theta}} \, dj \right]^{\frac{\theta}{\theta-1}} \\
C &= \left[ (C)^{\frac{\theta-1}{\theta}} \int_0^1 \left( \left[ \frac{P_j}{P} \right]^{-\theta} \right)^{\frac{\theta-1}{\theta}} \, dj \right]^{\frac{\theta}{\theta-1}} \\
C &= \left[ (C)^{\frac{\theta-1}{\theta}} \int_0^1 \left[ \frac{P_j}{P} \right]^{1-\theta} \, dj \right]^{\frac{\theta}{\theta-1}}
\end{aligned}
$$

Re-arrange further:

$$
\begin{aligned}
C &= C \left[ \int_0^1 \left[ \frac{P_j}{P} \right]^{1-\theta} \, dj \right]^{\frac{\theta}{\theta-1}} \\
1 &= \left[ \int_0^1 \left[ \frac{P_j}{P} \right]^{1-\theta} \, dj \right]^{\frac{\theta}{\theta-1}} \\
1 &= \int_0^1 \left[ \frac{P_j}{P} \right]^{1-\theta} \, dj
\end{aligned}
$$

$$
\begin{aligned}
1 &= \left[\frac{1}{P}\right]^{1-\theta} \int_0^1 [P_j]^{1-\theta} \, dj \\
(P)^{1-\theta} &= \int_0^1 [P_j]^{1-\theta} \, dj \\
P &= \left[\int_0^1 [P_j]^{1-\theta} \, dj\right]^{\frac{1}{1-\theta}} \qquad (5)
\end{aligned}
$$

Note that if all prices are identical ($P_j = \bar{P}$ for all $i$) then $P = \bar{P}$.

To sum up, the allocation of consumption across brands is:

$$
C_{j,t} = \left[\frac{P_{j,t}}{P_t}\right]^{-\theta} C_t \qquad ; \qquad P_t = \left[\int_0^1 [P_{j,t}]^{1-\theta} \, dj\right]^{\frac{1}{1-\theta}} \qquad (6)
$$

We now turn to the maximization of (1) with respect to hours worked and consumption, subject to (2), leads to the labor supply and Euler condition:

$$
\begin{aligned}
\frac{W_t}{P_t} (C_t)^{-\sigma} &= \chi (N_t)^{\eta} \Rightarrow \frac{W_t}{P_t} = \chi (N_t)^{\eta} (C_t)^{\sigma} \qquad (7) \\
\frac{1}{P_t} (C_t)^{-\sigma} &= \beta (1 + i_t) E_t \left[\frac{1}{P_{t+1}} (C_{t+1})^{-\sigma}\right] \qquad (8)
\end{aligned}
$$

### 1.1.2 Firms

A firm producing brand $i$ uses the following decreasing returns to scale technology in labor, with a stochastic productivity shifter (Walsh sets $a = 1$ for most of the chapter):

$$
C_{j,t} = Z_t (N_{j,t})^a \qquad (9)
$$

where we used the fact that as there is no investment, output of a brand is equal to consumption of that brand.

The demand faced by an individual firms is obtained by (6). Each period the firm has a probability $1 - \omega$ of being allowed to reset its price. Otherwise, it keeps the price unchanged. The price chosen by firms that have a chance to reset it is $P^*_{j,t}$ (I adopt a slightly different notation from Walsh). The firm chooses the price to maximize the following expected discounted profits:

$$
\begin{aligned}
&\sum_{i=0}^{\infty} E_t \left[\omega^i \Delta_{t,t+i} \left[P^*_{j,t} Y_{j,t+i|t} - W_{t+i} N_{j,t+i|t}\right]\right] \\
&= \sum_{i=0}^{\infty} E_t \left[\omega^i \Delta_{t,t+i} \left[P^*_{j,t} Y_{j,t+i|t} - W_{t+i} \left(\frac{Y_{j,t+i|t}}{Z_{t+i}}\right)^{\frac{1}{a}}\right]\right]
\end{aligned}
$$

where $Y_{j,t+i|t}$ is the output at time $t + i$ for a firm that last reset its price in period $t$, and $\Delta_{t,t+i}$ is the discount factor $\beta^i (C_{t+i}/C_t)^{-\sigma}$. From (6) we know that:

$$
Y_{j,t+i|t} = \left[\frac{P^*_{j,t}}{P_{t+i}}\right]^{-\theta} C_{t+i} \qquad (10)
$$

so we write the objective as:

$$\sum_{i=0}^{\infty} E_t \left[ \omega^i \Delta_{t,t+i} \left[ \left(P_{j,t}^*\right)^{1-\theta} \left(P_{t+i}\right)^{\theta} C_{t+i} - W_{t+i} \left(P_{j,t}^*\right)^{-\frac{\theta}{a}} \left(P_{t+i}\right)^{\frac{\theta}{a}} \left(\frac{C_{t+i}}{Z_{t+i}}\right)^{\frac{1}{a}} \right]\right]$$

The optimization with respect to $p_{j,t}$ proceeds as follows:

$$\begin{aligned}
0 &= \sum_{i=0}^{\infty} E_t \left[ \omega^i \Delta_{t,t+i} \left[ (1-\theta) \left(P_{j,t}^*\right)^{-\theta} \left(P_{t+i}\right)^{\theta} C_{t+i} + \frac{\theta}{a} W_{t+i} \left(P_{j,t}^*\right)^{-\frac{\theta}{a}-1} \left(P_{t+i}\right)^{\frac{\theta}{a}} \left(\frac{C_{t+i}}{Z_{t+i}}\right)^{\frac{1}{a}} \right]\right] \\
0 &= \sum_{i=0}^{\infty} E_t \left[ \omega^i \Delta_{t,t+i} \left[ (1-\theta) Y_{j,t+i|t} + \frac{\theta}{a} W_{t+i} \left(P_{j,t}^*\right)^{-1} \left(Y_{j,t+i|t}\right)^{\frac{1}{a}} \left(\frac{1}{Z_{t+i}}\right)^{\frac{1}{a}} \right]\right] \\
0 &= \sum_{i=0}^{\infty} E_t \left[ \omega^i \Delta_{t,t+i} Y_{j,t+i|t} \left[ 1 - \frac{\theta}{\theta-1} \frac{1}{a} W_{t+i} \left(P_{j,t}^*\right)^{-1} \left(Y_{j,t+i|t}\right)^{\frac{1-a}{a}} \left(\frac{1}{Z_{t+i}}\right)^{\frac{1}{a}} \right]\right] \\
0 &= \sum_{i=0}^{\infty} E_t \left[ \omega^i \Delta_{t,t+i} Y_{j,t+i|t} P_{t+i} \left[ \frac{P_{j,t}^*}{P_{t+i}} - \frac{\theta}{\theta-1} \frac{1}{a} \frac{W_{t+i}}{P_{t+i}} \left(Y_{j,t+i|t}\right)^{\frac{1-a}{a}} \left(\frac{1}{Z_{t+i}}\right)^{\frac{1}{a}} \right]\right]
\end{aligned}$$

It is useful to define the real marginal cost as follows:

$$\begin{aligned}
\Psi_{j,t+i|t} &= \frac{\partial \frac{W_{t+i}}{P_{t+i}} N_{j,t+i|t}}{\partial Y_{j,t+i|t}} \\
\Psi_{j,t+i|t} &= \frac{\partial}{\partial Y_{j,t+i|t}} \left( \frac{W_{t+i}}{P_{t+i}} \left(\frac{Y_{j,t+i|t}}{Z_{t+i}}\right)^{\frac{1}{a}} \right) \\
\Psi_{j,t+i|t} &= \frac{1}{a} \frac{W_{t+i}}{P_{t+i}} \left(Y_{j,t+i|t}\right)^{\frac{1-a}{a}} \left(\frac{1}{Z_{t+i}}\right)^{\frac{1}{a}} \qquad (11)
\end{aligned}$$

The optimality condition is then:

$$0 = \sum_{i=0}^{\infty} E_t \left[ \omega^i \Delta_{t,t+i} Y_{j,t+i|t} P_{t+i} \left[ \frac{P_{j,t}^*}{P_{t+i}} - \frac{\theta}{\theta-1} \Psi_{j,t+i|t} \right]\right] \qquad (12)$$

Note that the only source of heterogeneity across marginal costs for the firms resetting their prices at time $t$ is due to $Y_{j,t+i|t}$, which in turns only reflects heterogeneity across prices $p_{j,t}$. (12) then implies that the firms choose identical prices and: $P_{j,t}^* = P_t^*$.

#### 1.1.3 Equilibrium

In equilibrium bond holdings are zero, as bonds are in zero net supply $B = 0$. Note that the only role of bonds is to generate an equilibrium value for the nominal interest rate.

## 1.2 A linearized version

### 1.2.1 Steady state

In the steady state productivity is constant and normalized to unity ($Z_0 = 1$). (12) implies that all firms charge the same price. (9) then implies $N_0 = (C_0)^{1/a}$. Using (12) and (7) we get:

$$P_0 = \frac{\theta}{\theta - 1}\frac{W_0}{a}(C_0)^{\frac{1-a}{a}} \qquad ; \qquad \frac{W_0}{P_0} = \chi (C_0)^{\frac{\eta}{a}+\sigma}$$

which implies:

$$C_0 = \left(\frac{a}{\chi}\frac{\theta - 1}{\theta}\right)^{\frac{1}{\sigma + \frac{\eta+1-a}{a}}}$$

(8) implies:

$$\frac{1}{P_0}(C_0)^{-\sigma} = \beta(1 + i_0)\frac{1}{P_0}(C_0)^{-\sigma} \Rightarrow i_0 = \frac{1-\beta}{\beta}$$

#### 1.2.2 Log-linear expansions

We denotes log deviations from the steady state by lower case letters. (7) and (9) are exactly log linear, hence:

$$w_t - p_t = \eta n_t + \sigma c_t \tag{13}$$

$$y_{j,t} = z_t + a n_{j,t} \tag{14}$$

All firms resetting prices in a given period make identical choices: $p^*_{j,t} = p^*_t$ and $y_{j,t+i|t} = y_{t+i|t}$, hence $\varphi_{j,t+i|t} = \varphi_{t+i|t}$. The real marginal cost (11) and demands (10) are also exactly log linear:

$$\varphi_{t+i|t} = w_{t+i} - p_{t+i} - \frac{1}{a}z_{t+i} + \frac{1-a}{a}y_{t+i|t} \tag{15}$$

$$y_{t+i|t} = y_{t+i|t} = -\theta(p^*_t - p_{t+i}) + c_{t+i} \tag{16}$$

Combining (15) and (16) we get:

$$\varphi_{t+i|t} = w_{t+i} - p_{t+i} - \frac{1}{a}z_{t+i} - \frac{1-a}{a}\theta(p^*_t - p_{t+i}) + \frac{1-a}{a}c_{t+i} \tag{17}$$

Define an aggregate equivalent of (17):

$$\varphi_{t+i} = w_{t+i} - p_{t+i} - \frac{1}{a}z_{t+i} + \frac{1-a}{a}c_{t+i} = \varphi_{t+i|t} + \frac{1-a}{a}\theta(p^*_t - p_{t+i}) \tag{18}$$

The CPI (6) is approximated as:

$$p_t = \int_0^1 p_{j,t}dj$$

The inflation is then written as:

$$
\begin{aligned}
\pi_t &= p_t - p_{t-1} = \int_0^1 \left[p_{j,t} - p_{j,t-1}\right] dj \\
&= \int_{\text{do not reset at } t} \left[p_{j,t} - p_{j,t-1}\right] dj + \int_{\text{reset at } t} \left[p_{j,t} - p_{j,t-1}\right] dj \\
&= \int_{\text{reset at } t} \left[p_{j,t} - p_{j,t-1}\right] dj \\
&= \int_{\text{reset at } t} \left[p_t^* - p_{j,t-1}\right] dj = (1-\omega)\left(p_t^* - p_{t-1}\right) \qquad (19)
\end{aligned}
$$

We used the fact that the firms that adjust are evenly drawn around all firms, so the average of their lagged prices is the lagged CPI:

$$
\int_{\text{reset at } t} p_{j,t-1} dj = (1-\omega)\, p_{t-1}
$$

The Euler relation (8) is expanded as (recalling that consumption is equal to output):

$$
\begin{aligned}
& \frac{1}{P_0 (Y_0)^\sigma}\left[1 - \frac{P_t - P_0}{P_0} - \sigma \frac{Y_t - Y_0}{Y_0}\right] \\
&= \beta (1+i_0) \frac{1}{Q_0} \frac{1}{P_0 (Y_0)^\sigma}\left[1 + \frac{i_t - i_0}{1+i_0} + E_t\left[-\frac{P_{t+1} - P_0}{P_0} - \sigma \frac{Y_{t+1} - Y_0}{Y_0}\right]\right] \\
-p_t - \sigma y_t &= i_t + E_t\left[-p_{t+1} - \sigma y_{t+1}\right] \\
p_t + \sigma y_t &= -i_t + E_t\left[p_{t+1} + \sigma y_{t+1}\right] \\
\sigma y_t &= -i_t + E_t p_{t+1} - p_t + \sigma E_t y_{t+1} \\
\sigma y_t &= -\left(i_t - E_t\left[p_{t+1} - p_t\right]\right) + \sigma E_t y_{t+1} \\
\sigma y_t &= -\left(i_t - E_t \pi_{t+1}\right) + \sigma E_t y_{t+1} \\
y_t &= E_t y_{t+1} - \frac{1}{\sigma}\left(i_t - E_t \pi_{t+1}\right) \qquad (20)
\end{aligned}
$$

### 1.2.3 The new Keynesian Phillips curve

Next turn to the pricing (12). Recall that in the steady state $1 = \theta / (\theta - 1)\, \Psi_0$, therefore we can ignore the expansion of the terms $\Delta_{i,t+i} Y_{t+i|t} P_{t+i}$ as the multiply the steady state value of $P_t^* / P_{t+i} - \theta / (\theta - 1)\, \varphi_{t+i|t}$ which is zero. In addition, $\Delta_{t,t+i} = \beta^i$ in the steady state. (12) is then expanded as:

$$
\begin{aligned}
0 &= \sum_{i=0}^{\infty} E_t\left[\omega^i \Delta_{t,t+i} Y_{t+i|t} P_{t+i}\left[\frac{P_t^*}{P_{t+i}} - \frac{\theta}{\theta-1} \Psi_{t+i|t}\right]\right] \\
0 &= \sum_{i=0}^{\infty} E_t (\beta\omega)^i\left[\left(1 + \left(p_t^* - p_{t+i}\right)\right) - \frac{\theta}{\theta-1} \Psi_0\left(1 + \varphi_{t+i|t}\right)\right] \\
0 &= \sum_{i=0}^{\infty} E_t (\beta\omega)^i\left[p_t^* - p_{t+i} - \varphi_{t+i|t}\right]
\end{aligned}
$$

We re-arrange this as:

$$\begin{aligned}
\sum_{i=0}^{\infty} E_t (\beta\omega)^i p_t^* &= \sum_{i=0}^{\infty} E_t \left[ (\beta\omega)^i \left[ p_{t+i} + \varphi_{t+i|t} \right] \right] \\
p_t^* \sum_{i=0}^{\infty} E_t (\beta\omega)^i &= \sum_{i=0}^{\infty} (\beta\omega)^i E_t \left[ p_{t+i} + \varphi_{t+i|t} \right] \\
p_t^* \frac{1}{1-\omega\beta} &= \sum_{i=0}^{\infty} (\beta\omega)^i E_t \left[ p_{t+i} + \varphi_{t+i|t} \right] \\
p_t^* &= (1-\omega\beta) \sum_{i=0}^{\infty} (\beta\omega)^i E_t \left[ p_{t+i} + \varphi_{t+i|t} \right] \qquad (21)
\end{aligned}$$

Using (17) and (18), (21) becomes:

$$\begin{aligned}
p_t^* &= (1-\omega\beta) \sum_{i=0}^{\infty} (\beta\omega)^i E_t \left[ p_{t+i} + \varphi_{t+i} - \frac{1-a}{a}\theta \left( p_t^* - p_{t+i} \right) \right] \\
p_t^* \left[ 1 + (1-\omega\beta) \frac{1-a}{a}\theta \sum_{i=0}^{\infty} (\beta\omega)^i \right] &= (1-\omega\beta) \sum_{i=0}^{\infty} (\beta\omega)^i E_t \left[ \left( 1 + \frac{1-a}{a}\theta \right) p_{t+i} + \varphi_{t+i} \right] \\
p_t^* \left[ 1 + \frac{1-a}{a}\theta \right] &= (1-\omega\beta) \sum_{i=0}^{\infty} (\beta\omega)^i E_t \left[ \left( 1 + \frac{1-a}{a}\theta \right) p_{t+i} + \varphi_{t+i} \right] \\
p_t^* &= (1-\omega\beta) \sum_{i=0}^{\infty} (\beta\omega)^i E_t \left[ p_{t+i} + \frac{a}{a+(1-a)\theta} \varphi_{t+i} \right] \qquad (22)
\end{aligned}$$

The last step is to write (22) in terms of aggregate inflation. Using (19) notice that:

$$p_t^* = \frac{\pi_t}{1-\omega} + p_{t-1}$$

In addition:

$$\begin{aligned}
& (1-\omega\beta) \sum_{i=0}^{\infty} (\beta\omega)^i E_t \left[ p_{t+i} + \frac{a}{a+(1-a)\theta} \varphi_{t+i} \right] \\
= \; & (1-\omega\beta) \left( p_t + \frac{a}{a+(1-a)\theta} \varphi_t \right) \\
& + (1-\omega\beta) \sum_{i=1}^{\infty} (\beta\omega)^i E_t \left[ p_{t+i} + \frac{a}{a+(1-a)\theta} \varphi_{t+i} \right] \\
= \; & (1-\omega\beta) \left( p_t + \frac{a}{a+(1-a)\theta} \varphi_t \right) \\
& + \omega\beta (1-\omega\beta) \sum_{i=0}^{\infty} (\beta\omega)^i E_t \left[ p_{t+1+i} + \frac{a}{a+(1-a)\theta} \varphi_{t+1+i} \right] \\
= \; & (1-\omega\beta) \left( p_t + \frac{a}{a+(1-a)\theta} \varphi_t \right) + \omega\beta E_t p_{t+1}^* \\
= \; & (1-\omega\beta) \left( p_t + \frac{a}{a+(1-a)\theta} \varphi_t \right) + \omega\beta \left( \frac{E_t \pi_{t+1}}{1-\omega} + p_t \right)
\end{aligned}$$

Therefore we write (22) as:

$$
\begin{aligned}
p_t^* &= (1-\omega\beta)\sum_{i=0}^{\infty}(\beta\omega)^i E_t\left[p_{t+i}+\frac{a}{a+(1-a)\theta}\varphi_{t+i}\right] \\
\frac{\pi_t}{1-\omega}+p_{t-1} &= (1-\omega\beta)\left(p_t+\frac{a}{a+(1-a)\theta}\varphi_t\right)+\omega\beta\left(\frac{E_t\pi_{t+1}}{1-\omega}+p_t\right) \\
\frac{\pi_t}{1-\omega}+p_{t-1} &= p_t+(1-\omega\beta)\frac{a}{a+(1-a)\theta}\varphi_t+\frac{\omega\beta}{1-\omega}E_t\pi_{t+1} \\
\frac{\pi_t}{1-\omega} &= \pi_t+(1-\omega\beta)\frac{a}{a+(1-a)\theta}\varphi_t+\frac{\omega\beta}{1-\omega}E_t\pi_{t+1} \\
\frac{\omega}{1-\omega}\pi_t &= (1-\omega\beta)\frac{a}{a+(1-a)\theta}\varphi_t+\frac{\omega\beta}{1-\omega}E_t\pi_{t+1} \\
\pi_t &= \frac{(1-\omega\beta)(1-\omega)}{\omega}\frac{a}{a+(1-a)\theta}\varphi_t+\beta E_t\pi_{t+1} \\
\pi_t &= \frac{(1-\omega\beta)(1-\omega)}{\omega}\frac{a}{a+(1-a)\theta}\varphi_t+\beta E_t\pi_{t+1} \\
\pi_t &= \beta E_t\pi_{t+1}+\tilde{\kappa}\varphi_t \qquad (23)
\end{aligned}
$$

where:

$$\tilde{\kappa}=\frac{(1-\omega\beta)(1-\omega)}{\omega}\frac{a}{a+(1-a)\theta}$$

## 1.3 The flexible price allocation

If prices are fully flexible, the optimal pricing corresponds to the last bracket of (12):

$$
\begin{aligned}
\frac{P_{j,t}^*}{P_t} &= \frac{\theta}{\theta-1}\Psi_t \\
\frac{P_{j,t}^*}{P_t} &= \frac{\theta}{\theta-1}\frac{1}{a}\frac{W_t}{P_t}(Y_t)^{\frac{1-a}{a}}\left(\frac{1}{Z_t}\right)^{\frac{1}{a}}
\end{aligned}
$$

In equilibrium all firms set the same price ($P_{j,t}^*=P_t$), therefore the right-hand side is constant. In terms of log linear deviations we write:

$$0=(w_t-p_t)-\frac{1}{a}z_t+\frac{1-a}{a}y_t$$

Using (13) and (14) this becomes:

$$
\begin{aligned}
0 &= \eta n_t+\sigma y_t-\frac{1}{a}z_t+\frac{1-a}{a}y_t \\
0 &= \frac{\eta}{a}(y_t-z_t)+\sigma y_t-\frac{1}{a}z_t+\frac{1-a}{a}y_t \\
\left(\frac{1-a}{a}+\frac{\eta}{a}+\sigma\right)y_t &= \left(\frac{\eta}{a}+\frac{1}{a}\right)z_t \\
y_t^{\text{flex}} &= \frac{1+\eta}{1+\eta+(\sigma-1)a}z_t \qquad (24)
\end{aligned}
$$

where the "flex" superscript denotes the flexible price allocation.

The corresponding real interest rate is given by (20):

$$\begin{aligned} y_t^{\text{flex}} &= E_t y_{t+1}^{\text{flex}} - \frac{1}{\sigma} r_t^{\text{flex}} \\ r_t^{\text{flex}} &= \sigma \frac{1+\eta}{1+\eta+(\sigma-1)a} \left(E_t z_{t+1} - z_t\right) \end{aligned} \tag{25}$$

### 1.4 Deviations from the flexible price allocation

Consider the real marginal cost (18) under sticky prices:

$$\varphi_t = w_t - p_t - \frac{1}{a} z_t + \frac{1-a}{a} y_t$$

Using (13), (14) and (24) we write:

$$\begin{aligned} \varphi_t &= \eta n_t + \sigma y_t - \frac{1}{a} z_t + \frac{1-a}{a} y_t \\ \varphi_t &= \frac{\eta}{a} (y_t - z_t) + \sigma y_t - \frac{1}{a} z_t + \frac{1-a}{a} y_t \\ \varphi_t &= \left(\frac{\eta}{a} + \frac{1-a}{a} + \sigma\right) y_t - \frac{1+\eta}{a} z_t \\ \varphi_t &= \frac{1+\eta+(\sigma-1)a}{a} \left(y_t - \frac{1+\eta}{1+\eta+(\sigma-1)a} z_t\right) \\ \varphi_t &= \frac{1+\eta+(\sigma-1)a}{a} \left(y_t - y_t^{\text{flex}}\right) \\ \varphi_t &= \left(\sigma + \frac{1-a+\eta}{a}\right) x_t \end{aligned}$$

$x_t = y_t - y_t^{\text{flex}}$ is the output gap. The New Keynesian Phillips curve (23) is then:

$$\pi_t = \beta E_t \pi_{t+1} + \tilde{\kappa} \left(\sigma + \frac{1-a+\eta}{a}\right) x_t = \beta E_t \pi_{t+1} + \kappa x_t \tag{26}$$

The aggregate demand is given by the Euler (20). Using (25) we express it in terms of deviations from the flexible price allocation:

$$\begin{aligned} y_t &= E_t y_{t+1} - \frac{1}{\sigma} (i_t - E_t \pi_{t+1}) \\ y_t - y_t^{\text{flex}} &= E_t y_{t+1} - \frac{1}{\sigma} (i_t - E_t \pi_{t+1}) - y_t^{\text{flex}} \\ x_t &= E_t y_{t+1} - \frac{1}{\sigma} (i_t - E_t \pi_{t+1}) - E_t y_{t+1}^{\text{flex}} + \frac{1}{\sigma} r_t^{\text{flex}} \\ x_t &= E_t x_{t+1} - \frac{1}{\sigma} \left(i_t - E_t \pi_{t+1} - r_t^{\text{flex}}\right) \\ x_t &= E_t x_{t+1} - \frac{1}{\sigma} \tilde{r}_t \end{aligned} \tag{27}$$

where $\tilde{r}_t = i_t - E_t \pi_{t+1} - r_t^{\text{flex}}$ is the real interest rate gap.

### 1.5 Overall equilibrium

The overall equilibrium combines (26), (27) and an interest rate rule (for which we take a more general parametrization than Walsh):

$$
\begin{aligned}
\pi_t &= \beta E_t \pi_{t+1} + \kappa x_t \\
x_t &= E_t x_{t+1} - \frac{1}{\sigma} \tilde{r}_t \\
i_t &= \delta_\pi \pi_t + \delta_x x_t + v_t
\end{aligned}
$$

This leads to the following system:

$$
\begin{aligned}
\pi_t - \kappa x_t &= \beta E_t \pi_{t+1} \\
\left(1 + \frac{\delta_x}{\sigma}\right) x_t + \frac{\delta_\pi}{\sigma} \pi_t &= E_t x_{t+1} + \frac{1}{\sigma} E_t \pi_{t+1} - \frac{1}{\sigma} \left(v_t - r_t^{\text{flex}}\right)
\end{aligned}
$$

The system is stable if:

$$
\kappa \left(\delta_\pi - 1\right) + \left(1 - \beta\right) \delta_x > 0
$$

## 2 The gain from commitment

A trade-off between inflation and output is introduced through a cost push-shock:

$$
\begin{aligned}
\pi_t &= \beta E_t \pi_{t+1} + \kappa x_t + e_t \\
x_t &= E_t x_{t+1} - \frac{1}{\sigma} \tilde{r}_t
\end{aligned}
$$

The cost-push shocks are persistent:

$$
e_{t+1} = \rho e_t + \varepsilon_{t+1} \qquad ; \qquad E_t \varepsilon_{t+1} = 0 \tag{28}
$$

The central bank wants to minimize:

$$
\frac{1}{2} E_t \sum_{i=0}^{\infty} \beta^s \left[\lambda x_{t+i}^2 + \pi_{t+i}^2\right]
$$

### 2.1 Outcome under discretion

Under discretion, the central bank minimizes the one period loss function:

$$
\frac{1}{2} \left[\alpha_x x_t^2 + \pi_t^2\right] = \frac{1}{2} \left[\lambda x_t^2 + \left(\kappa x_t + \beta E_t \pi_{t+1} + e_t\right)^2\right]
$$

The first order condition with respect to $x_t$ taking $E_t\pi_{t+1}$ as given is:

$$\begin{aligned}
0 &= \lambda x_t + \kappa\left(\kappa x_t + \beta E_t\pi_{t+1} + e_t\right) \\
0 &= \lambda x_t + \kappa\pi_t \\
0 &= x_t + \frac{\kappa}{\lambda}\pi_t
\end{aligned}$$

$$\Rightarrow x_t = -\frac{\kappa}{\lambda}\pi_t \tag{29}$$

Using (29) in the NKPC we write:

$$\begin{aligned}
\pi_t &= \beta E_t\pi_{t+1} + \kappa x_t + e_t \\
\pi_t &= \beta E_t\pi_{t+1} - \frac{\kappa^2}{\lambda}\pi_t + e_t \\
\pi_t\left(1 + \frac{\kappa^2}{\lambda}\right) &= \beta E_t\pi_{t+1} + e_t \\
\pi_t &= \frac{\lambda\beta}{\lambda+\kappa^2}E_t\pi_{t+1} + \frac{\lambda}{\lambda+\kappa^2}e_t
\end{aligned}$$

Iterating forward and using (28) we get:

$$\begin{aligned}
\pi_t &= \frac{\lambda\beta}{\lambda+\kappa^2}E_t\pi_{t+1} + \frac{\lambda}{\lambda+\kappa^2}e_t \\
\pi_t &= \frac{\lambda\beta}{\lambda+\kappa^2}E_t\left(\frac{\lambda\beta}{\lambda+\kappa^2}\pi_{t+2} + \frac{\lambda}{\lambda+\kappa^2}e_{t+1}\right) + \frac{\lambda}{\lambda+\kappa^2}e_t \\
\pi_t &= \left(\frac{\lambda\beta}{\lambda+\kappa^2}\right)^2 E_t\pi_{t+2} + \frac{\lambda}{\lambda+\kappa^2}\left(e_t + \frac{\lambda\beta}{\lambda+\kappa^2}E_te_{t+1}\right) \\
\pi_t &= \frac{\lambda}{\lambda+\kappa^2}\sum_{i=0}^{\infty}E_t\left(\frac{\lambda\beta}{\lambda+\kappa^2}\right)^i E_te_{t+i} \\
\pi_t &= \frac{\lambda}{\lambda+\kappa^2}\sum_{i=0}^{\infty}\left(\frac{\lambda\beta\rho}{\lambda+\kappa^2}\right)^i e_t \\
\pi_t &= \frac{\lambda}{\lambda+\kappa^2}\frac{1}{1-\frac{\lambda\beta\rho}{\lambda+\kappa^2}}e_t \\
\pi_t &= \frac{\lambda}{\lambda+\kappa^2-\lambda\beta\rho}e_t \\
\pi_t &= \frac{\lambda}{\lambda\left(1-\beta\rho\right)+\kappa^2}e_t
\end{aligned}$$

Using this result into (29) the solution under discretion is then:

$$\pi_t^{\text{disc}} = \frac{\lambda}{\lambda\left(1-\beta\rho\right)+\kappa^2}e_t \tag{30}$$

$$x_t^{\text{disc}} = \frac{-\kappa}{\lambda\left(1-\beta\rho\right)+\kappa^2}e_t \tag{31}$$

## 2.2 Outcome under optimal commitment

The central bank commits to a sequence of inflation and the output gap over the entire future to minimize:

$$\frac{1}{2}E_t\sum_{i=0}^{\infty}\beta^i\left(\lambda x_{t+i}^2+\pi_{t+i}^2\right)$$

subject to two constraints, namely the New Keynesian Phillips curve and the Euler condition:

$$\begin{aligned}
E_t\pi_{t+i} &= E_t\left(\beta\pi_{t+i+1}+\kappa x_{t+i}+e_{t+i}\right)\\
E_t x_{t+i} &= E_t\left(x_{t+i+1}-\frac{1}{\sigma}\left(i_{t+i}-E_t\pi_{t+i+1}-r_{t+i}^{\text{flex}}\right)\right)
\end{aligned}$$

The problem is written as the following Lagrangian:

$$\mathcal{L}=E_t\sum_{i=0}^{\infty}\beta^i\left[\begin{array}{c}\frac{1}{2}\left(\lambda x_{t+i}^2+\pi_{t+i}^2\right)\\ +\psi_{t+i}\left(\pi_{t+i}-\beta\pi_{t+i+1}-\kappa x_{t+i}-e_{t+i}\right)\\ +\theta_{t+i}\left(x_{t+i}-x_{t+i+1}+\frac{1}{\sigma}\left(i_{t+i}-E_t\pi_{t+i+1}-r_{t+i}^{\text{flex}}\right)\right)\end{array}\right]$$

The first-order condition with respect to the interest rate is:

$$0=\frac{\partial\mathcal{L}}{\partial i_{t+i}}=E_t\beta^i\theta_{t+i}\frac{1}{\sigma}$$

provided the interest rate is positive, $i_{t+i}>0$. This shows that the Euler condition is not really a constraint as long as the interest rate can be set freely. Thus $E_t\theta_{t+i}=0$, and we can focus on the first two components of the Lagrangian, assuming that the central bank sets inflation and the output gap (it does not directly do so, but indirectly through the interest rate).

The first order condition with respect to inflation and the output gap are:

$$\begin{aligned}
0 &= \frac{\partial\mathcal{L}}{\partial x_{t+i}}=E_t\beta^i\left[\lambda x_{t+i}-\kappa\psi_{t+i}\right]\\
0 &= \frac{\partial\mathcal{L}}{\partial \pi_{t+i}}=E_t\beta^i\left[\pi_{t+i}+\psi_{t+i}\right]-E_t\beta^{i-1}\left[\psi_{t+i-1}\beta\right]
\end{aligned}$$

which can be re-written as:

$$\lambda x_{t+i}=\kappa\psi_{t+i}\qquad ; \qquad \pi_{t+i}+\psi_{t+i}=\psi_{t+i-1} \tag{32}$$

In addition, $\psi_{t-1}=0$ as $\pi_{t-1}$ is already set. The (32) at $i=0$ is then:

$$\begin{aligned}
\pi_t &= \psi_{t-1}-\psi_t\\
\pi_t &= 0-\psi_t\\
\pi_t &= -\frac{\lambda}{\kappa}x_t\\
x_t &= -\frac{\kappa}{\lambda}\pi_t
\end{aligned}$$

(32) at $i > 0$ implies:

$$\begin{aligned}
\pi_{t+i} &= \psi_{t+i-1} - \psi_{t+i} \\
\pi_{t+i} &= \frac{\lambda}{\kappa}\left(x_{t+i-1} - x_{t+i}\right) \\
x_{t+i} &= x_{t+i-1} - \frac{\kappa}{\lambda}\pi_{t+i}
\end{aligned}$$

In general terms we can write:

$$\begin{aligned}
x_t &= -\frac{\kappa}{\lambda}\pi_t = -\frac{\kappa}{\lambda}\left(p_t - p_{t-1}\right) \\
x_{t+1} &= x_t - \frac{\kappa}{\lambda}\pi_{t+1} = -\frac{\kappa}{\lambda}\left(\pi_{t+1} + \pi_t\right) = -\frac{\kappa}{\lambda}\left(p_{t+1} - p_{t-1}\right) \\
x_{t+2} &= x_{t+1} - \frac{\kappa}{\lambda}\pi_{t+2} = -\frac{\kappa}{\lambda}\left(\pi_{t+2} + p_{t+1} - p_{t-1}\right) = -\frac{\kappa}{\lambda}\left(p_{t+2} - p_{t-1}\right)
\end{aligned}$$

hence:

$$x_{t+i} = -\frac{\kappa}{\lambda}\left(p_{t+i} - p_{t-1}\right) \qquad ; \qquad i \geq 0 \tag{33}$$

Using (33) the NKPC for period $t+i$ is written as follows:

$$\begin{aligned}
\pi_{t+i} &= \beta E_{t+i}\pi_{t+i+1} + \kappa x_{t+i} + e_{t+i} \\
p_{t+i} - p_{t+i-1} &= \beta E_{t+i}\left(p_{t+i+1} - p_{t+i}\right) + \kappa x_{t+i} + e_{t+i} \\
\left(p_{t+i} - p_{t-1}\right) - \left(p_{t+i-1} - p_{t-1}\right) &= \beta E_{t+i}\left(\left(p_{t+i+1} - p_{t-1}\right) - \left(p_{t+i} - p_{t-1}\right)\right) \\
&\quad + \kappa x_{t+i} + e_{t+i} \\
\left(p_{t+i} - p_{t-1}\right) - \left(p_{t+i-1} - p_{t-1}\right) &= \beta E_{t+i}\left(p_{t+i+1} - p_{t-1}\right) - \beta\left(p_{t+i} - p_{t-1}\right) \\
&\quad - \frac{\kappa^2}{\lambda}\left(p_{t+i} - p_{t-1}\right) + e_{t+i} \\
\left(p_{t+i} - p_{t-1}\right)\left(1 + \beta + \frac{\kappa^2}{\lambda}\right) &= \beta E_{t+i}\left(p_{t+i+1} - p_{t-1}\right) + \left(p_{t+i-1} - p_{t-1}\right) + e_{t+i} \\
\left(p_{t+i} - p_{t-1}\right) &= \frac{\lambda\beta}{\lambda\left(1+\beta\right) + \kappa^2} E_{t+i}\left(p_{t+i+1} - p_{t-1}\right) \\
&\quad + \frac{\lambda}{\lambda\left(1+\beta\right) + \kappa^2}\left(p_{t+i-1} - p_{t-1}\right) + \frac{\lambda}{\lambda\left(1+\beta\right) + \kappa^2} e_{t+i}
\end{aligned}$$

which is a second order autoregressive system in $\left(p_{t+i} - p_{t-1}\right)$. The stationary solution is given by:

$$\left(p_{t+i} - p_{t-1}\right) = \delta\left(p_{t+i-1} - p_{t-1}\right) + \frac{\delta}{1 - \delta\beta\rho} e_{t+i} \qquad ; \qquad i \geq 0 \tag{34}$$

where:

$$\delta = \frac{1}{2\beta\frac{\lambda}{\lambda(1+\beta)+\kappa^2}}\left(1 - \sqrt{1 - 4\beta\left(\frac{\lambda}{\lambda\left(1+\beta\right) + \kappa^2}\right)^2}\right)$$

(34) provides a path for the price level, inflation follows immediately as $\pi_{t+si} = p_{t+i} - p_{t+i-1}$.

(33) gives the path for the output gap:

$$
\begin{aligned}
(p_{t+i} - p_{t-1}) &= \delta\left(p_{t+i-1} - p_{t-1}\right) + \frac{\delta}{1-\delta\beta\rho} e_{t+i} \\
-\frac{\lambda}{\kappa} x_{t+i} &= -\delta\frac{\lambda}{\kappa} x_{t+i-1} + \frac{\delta}{1-\delta\beta\rho} e_{t+i} \\
x_{t+i}^{\text{optimal commitment}} &= \delta x_{t+i-1} - \frac{\kappa}{\lambda}\frac{\delta}{1-\delta\beta\rho} e_{t+i}
\end{aligned}
\tag{35}
$$

## 2.3 Outcome under simpler commitment

The commitment rule above can be complex. Consider a simpler case of suboptimal commitment where the central bank commits to a rule linking the output gap to the shock:

$$x_t = -\xi e_t$$

where $\xi$ is to be determined.

Using (28), the NKPC becomes:

$$
\begin{aligned}
\pi_t &= \beta E_t \pi_{t+1} + \kappa x_t + e_t \\
\pi_t &= E_t \sum_{i=0}^{\infty} \beta^i \left[\kappa x_{t+i} + e_{t+i}\right] \\
\pi_t &= E_t \sum_{i=0}^{\infty} \beta^i \left(1-\kappa\xi\right) e_{t+i} \\
\pi_t &= \left(1-\kappa\xi\right) \sum_{i=0}^{\infty} \left(\beta\rho\right)^i e_t \\
\pi_t &= \frac{1-\kappa\xi}{1-\beta\rho} e_t
\end{aligned}
$$

We can now write the objective as a function of $\xi$:

$$
\begin{aligned}
& E_t \sum_{i=0}^{\infty} \beta^i \left[\lambda x_{t+i}^2 + \pi_{t+i}^2\right] \\
= \; & E_t \sum_{i=0}^{\infty} \beta^i \left[\lambda\xi^2 + \left(\frac{1-\kappa\xi}{1-\beta\rho}\right)^2\right] e_{t+i}^2 \\
= \; & \left[\lambda\xi^2 + \left(\frac{1-\kappa\xi}{1-\beta\rho}\right)^2\right] \sum_{i=0}^{\infty} \beta^i E_t e_{t+i}^2
\end{aligned}
$$

$\lambda\xi^2$ only enters in the first bracket. Setting the derivative to zero we obtain:

$$\begin{aligned}
0 &= \frac{\partial}{\partial\xi}\left[\lambda\xi^2+\left(\frac{1-\kappa\xi}{1-\beta\rho}\right)^2\right] \\
0 &= \lambda 2\xi + 2\left(\frac{1-\kappa\xi}{1-\beta\rho}\right)\frac{-\kappa}{1-\beta\rho} \\
0 &= \lambda\xi + (1-\kappa\xi)\frac{-\kappa}{(1-\beta\rho)^2} \\
\frac{\kappa}{(1-\beta\rho)^2} &= \left(\lambda+\frac{\kappa^2}{(1-\beta\rho)^2}\right)\xi \\
\xi &= \frac{\kappa}{\lambda(1-\beta\rho)^2+\kappa^2}
\end{aligned}$$

The output gap and inflation are then:

$$x_t^{\text{commitment}} = -\xi e_t = -\frac{\kappa}{\lambda(1-\beta\rho)^2+\kappa^2}e_t \tag{36}$$

$$\pi_t^{\text{commitment}} = \frac{1-\kappa\xi}{1-\beta\rho}e_t = \frac{\lambda(1-\beta\rho)}{\lambda(1-\beta\rho)^2+\kappa^2}e_t \tag{37}$$

## 2.4 Comparing discretion and simpler commitment

The simple commitment leads to a lower intertemporal loss:

$$E_t\sum_{i=0}^{\infty}\beta^i\left[\alpha_x\left(x_{t+s}^{\text{disc}}\right)^2+\left(\pi_{t+s}^{\text{disc}}\right)^2\right] > E_t\sum_{i=0}^{\infty}\beta^i\left[\alpha_x\left(x_{t+s}^{\text{commitment}}\right)^2+\left(\pi_{t+s}^{\text{commitment}}\right)^2\right]$$

We check whether this inequality hods.

$$\begin{aligned}
E_t\sum_{i=0}^{\infty}\beta^i\left[\lambda\left(x_{t+i}^{\text{disc}}\right)^2+\left(\pi_{t+i}^{\text{disc}}\right)^2\right] &> E_t\sum_{i=0}^{\infty}\beta^i\left[\lambda\left(x_{t+i}^{\text{commitment}}\right)^2+\left(\pi_{t+i}^{\text{commitment}}\right)^2\right] \\
\frac{\lambda\kappa^2+\lambda^2}{\left[\alpha_x(1-\beta\rho)+\kappa^2\right]^2}E_t\sum_{i=0}^{\infty}\beta^i\left(e_{t+i}\right)^2 &> \frac{\lambda\kappa^2+\lambda^2(1-\beta\rho)^2}{\left[\alpha_x(1-\beta\rho)^2+\kappa^2\right]^2}E_t\sum_{i=0}^{\infty}\beta^i\left(e_{t+i}\right)^2
\end{aligned}$$

This implies:

$$\begin{aligned}
0 &< \frac{\lambda\kappa^2+\lambda^2}{\left[\alpha_x(1-\beta\rho)+\kappa^2\right]^2}-\frac{\lambda\kappa^2+\lambda^2(1-\beta\rho)^2}{\left[\alpha_x(1-\beta\rho)^2+\kappa^2\right]^2} \\
0 &< \frac{\kappa^2+\lambda}{\left[\lambda(1-\beta\rho)+\kappa^2\right]^2}-\frac{\kappa^2+\lambda(1-\beta\rho)^2}{\left[\lambda(1-\beta\rho)^2+\kappa^2\right]^2} \\
0 &< \kappa^2\left[\left[\lambda(1-\beta\rho)^2+\kappa^2\right]^2-\left[\lambda(1-\beta\rho)+\kappa^2\right]^2\right] \\
&\quad -\lambda\left[(1-\beta\rho)^2\left[\lambda(1-\beta\rho)+\kappa^2\right]^2-\left[\lambda(1-\beta\rho)^2+\kappa^2\right]^2\right]
\end{aligned}$$

The term on the left-hand side is:

$$\begin{aligned}
& \kappa^2 \left[\left[\lambda (1-\beta\rho)^2 + \kappa^2\right]^2 - \left[\lambda (1-\beta\rho) + \kappa^2\right]^2\right] \\
= \quad & \kappa^2 \left[ \begin{array}{c} \left[\lambda (1-\beta\rho)^2\right]^2 + \left[\kappa^2\right]^2 + 2\lambda (1-\beta\rho)^2 \kappa^2 \\ -\left[\lambda (1-\beta\rho)\right]^2 - \left[\kappa^2\right]^2 - 2\lambda (1-\beta\rho) \kappa^2 \end{array} \right] \\
= \quad & \kappa^2 \left[2\lambda (1-\beta\rho)(-\beta\rho) \kappa^2 - \left[\lambda (1-\beta\rho)\right]^2 \left[1 - (1-\beta\rho)^2\right]\right] \\
= \quad & \lambda (1-\beta\rho) \kappa^2 \left[-2\beta\rho\kappa^2 - \lambda (1-\beta\rho) \left[1 - 1 - (\beta\rho)^2 + 2\beta\rho\right]\right] \\
= \quad & -\lambda\beta\rho (1-\beta\rho) \kappa^2 \left[2\kappa^2 + \lambda (1-\beta\rho)(2-\beta\rho)\right]
\end{aligned}$$

The term on the right-hand side is:

$$\begin{aligned}
& \lambda \left[(1-\beta\rho)^2 \left[\lambda (1-\beta\rho) + \kappa^2\right]^2 - \left[\lambda (1-\beta\rho)^2 + \kappa^2\right]^2\right] \\
= \quad & \lambda \left[ \begin{array}{c} (1-\beta\rho)^2 \left[\lambda (1-\beta\rho)\right]^2 + (1-\beta\rho)^2 \left[\kappa^2\right]^2 \\ +2 (1-\beta\rho)^2 \lambda (1-\beta\rho) \kappa^2 \\ -\left[\lambda (1-\beta\rho)^2\right]^2 - \left[\kappa^2\right]^2 - 2\lambda (1-\beta\rho)^2 \kappa^2 \end{array} \right] \\
= \quad & \lambda \left[ \begin{array}{c} (1-\beta\rho)^2 \left[\lambda (1-\beta\rho)\right]^2 - \left[\lambda (1-\beta\rho)^2\right]^2 \\ + \left[(1-\beta\rho)^2 - 1\right] \left[\kappa^2\right]^2 + 2\lambda (1-\beta\rho)^2 (-\beta\rho) \kappa^2 \end{array} \right] \\
= \quad & -\lambda\beta\rho\kappa^2 \left[(2-\beta\rho) \kappa^2 + 2\lambda (1-\beta\rho)^2\right]
\end{aligned}$$

Combining these results, the inequality becomes:

$$\begin{aligned}
0 \quad < \quad & -\lambda\beta\rho (1-\beta\rho) \kappa^2 \left[2\kappa^2 + \lambda (1-\beta\rho)(2-\beta\rho)\right] \\
& +\lambda\beta\rho\kappa^2 \left[(2-\beta\rho) \kappa^2 + 2\lambda (1-\beta\rho)^2\right] \\
0 \quad < \quad & -(1-\beta\rho) \left[2\kappa^2 + \lambda (1-\beta\rho)(2-\beta\rho)\right] \\
& + (2-\beta\rho) \kappa^2 + 2\lambda (1-\beta\rho)^2 \\
0 \quad < \quad & -(2-2\beta\rho) \kappa^2 - \lambda (1-\beta\rho)^2 (2-\beta\rho) \\
& + (2-\beta\rho) \kappa^2 + 2\lambda (1-\beta\rho)^2 \\
0 \quad < \quad & \beta\rho\kappa^2 + \lambda (1-\beta\rho)^2 \beta\rho \\
0 \quad < \quad & \kappa^2 + \lambda (1-\beta\rho)^2
\end{aligned}$$

which is the case.