PS2 Solutions

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Solution 1. Production Theory

1.a. A production set represents all feasible combinations of inputs and outputs for a firm. The property of irreversibility implies that once a good is produced, it cannot be transformed back into its original inputs. Explaining in mathematical words: if a production set $y \in Y$ is feasible, then production set $-y$ cannot be feasible.

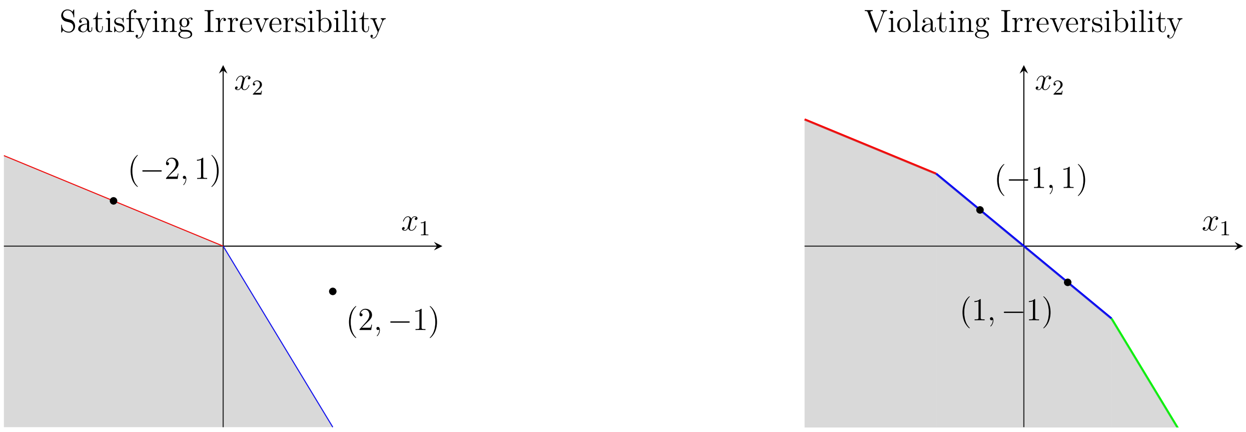


Figure 1: Production Sets Illustrating Irreversibility

1.b. For each production function, we'll derive the cost function $c(w, q)$ and the conditional factor demand functions $z(w, q)$, where $w = (w_1, w_2)$ denotes input prices and q denotes output.

(i) Perfect Substitutes: $f(z) = z_1 + z_2$ Inputs z_1 and z_2 are perfect substitutes. The firm will utilize the cheaper input to minimize costs.

* **Cost Function**:

$$c(w, q) = \begin{cases} w_1 q & \text{if } w_1 \leq w_2 \\ w_2 q & \text{if } w_1 > w_2 \end{cases}$$

* **Conditional Factor Demand Functions**:

$$z(w, q) = \begin{cases} (q, 0) & \text{if } w_1 < w_2 \\ \left\{(z_1, z_2) \in \mathbb{R}^2_+ : z_1 + z_2 = q\right\} & \text{if } w_1 = w_2 \\ (0, q) & \text{if } w_1 > w_2 \end{cases}$$

(ii) Leontief Technology: $f(z) = \min\{z_1, z_2\}$

Inputs are used in fixed proportions. To produce q units of output, the firm requires q units of both z_1 and z_2.

* **Cost Function**:

$$c(w, q) = (w_1 + w_2)q$$

* **Conditional Factor Demand Functions**:

$$z(w, q) = (q, q)$$

(iii) Constant Elasticity of Substitution (CES) Technology: $f(z) = (z_1^\rho + z_2^\rho)^{\frac{1}{\rho}}, \rho \leq 1$

The CES production function is given by:

$$q = (z_1^\rho + z_2^\rho)^{1/\rho},$$

where z_1, z_2 are input quantities and $\rho \leq 1$ is the substitution parameter.

The cost minimization problem is:

$$\min_{z_1, z_2} \; w_1 z_1 + w_2 z_2 \quad \text{subject to} \quad (z_1^\rho + z_2^\rho)^{1/\rho} \geq q.$$

The Lagrangian is:

$$\mathcal{L} = w_1 z_1 + w_2 z_2 + \lambda \left[(z_1^\rho + z_2^\rho)^{1/\rho} - q \right].$$

The FOCs are:

$$\frac{\partial \mathcal{L}}{\partial z_1} = w_1 - \lambda \cdot \frac{1}{\rho} \cdot z_1^{\rho-1} \cdot (z_1^\rho + z_2^\rho)^{\frac{1}{\rho}-1} = 0$$
$$\frac{\partial \mathcal{L}}{\partial z_2} = w_2 - \lambda \cdot \frac{1}{\rho} \cdot z_2^{\rho-1} \cdot (z_1^\rho + z_2^\rho)^{\frac{1}{\rho}-1} = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = (z_1^\rho + z_2^\rho)^{1/\rho} - q = 0$$

Dividing the two FOCs to eliminate λ, we get:

$$\frac{w_1}{w_2} = \frac{z_1^{\rho-1}}{z_2^{\rho-1}},$$

which simplifies to:

$$\frac{z_1}{z_2} = \left(\frac{w_1}{w_2} \right)^{1/(\rho-1)}.$$

With this ratio between z_1, z_2, substituting into $(z_1^\rho + z_2^\rho)^{1/\rho} = q$ and solving for z_2:

* **Conditional Factor Demand Functions**:

$$z_2 = q\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} \cdot w_2^{\frac{1}{\rho-1}}$$

$$z_1 = q\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} \cdot w_1^{\frac{1}{\rho-1}}$$

* **Cost Function**:

$$\begin{aligned} c(w,q) &= w_1 z_1 + w_2 z_2 \\ &= q\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{-\frac{1}{\rho}} \cdot \left(w_2^{\frac{\rho}{\rho-1}} + w_1^{\frac{\rho}{\rho-1}}\right) \\ &= q\left(w_1^{\frac{\rho}{\rho-1}} + w_2^{\frac{\rho}{\rho-1}}\right)^{\frac{\rho-1}{\rho}} \end{aligned}$$

1.c. For a firm with a constant returns to scale production function $f(x)$, paying each factor its marginal product implies:

$$w_i = p\frac{\partial f(x)}{\partial x_i}$$

Total revenue is $p \cdot f(x)$, and total cost is $\sum_i w_i x_i$. Substituting w_i gives:

$$\text{Total Cost} = \sum_i \left(p\frac{\partial f(x)}{\partial x_i}\right) x_i$$

By Euler's Theorem for homogeneous functions, since $f(x)$ is homogeneous of degree 1 (constant returns to scale):

$$f(x) = \sum_i \frac{\partial f(x)}{\partial x_i} x_i$$

Multiplying both sides by p:

$$p \cdot f(x) = \sum_i \left(p\frac{\partial f(x)}{\partial x_i}\right) x_i$$

Thus, total revenue equals total cost, leading to zero profit:

$$\text{Profit} = p \cdot f(x) - \sum_i w_i x_i = 0$$

1.d. Given a concave production function $f(z)$ with $L-1$ inputs, where $\frac{\partial f(z)}{\partial z_l} \geq 0$ for all l and $z \geq 0$, and the Hessian matrix $D^2 f(z)$ is negative definite for all z, we analyze the firm's profit-maximizing behavior in response to changes in output price and input prices.

The firm's profit maximization problem is:

$$\max_{z} \left[pf(z) - \sum_{l=1}^{L-1} w_l z_l \right]$$

where:

- p is the output price,
- w_l is the price of input z_l,
- $f(z)$ is the production function.

The FOC for maximization is:

$$p\nabla f(z) - w = 0$$

These conditions equate the value of the marginal product of each input to its price. Given that $D^2 f(z)$ is negative definite, the SOCs are satisfied, ensuring a unique maximum.

(i) Effect of an Increase in Output Price on Profit-Maximizing Output:

To analyze how an increase in p affects the optimal output level $q^* = f(z^*)$, we use the Implicit Function Theorem (IFT). As $z = z(p, w)$, we define the FOC as $g(\cdot)$ and apply the IFT, we have:

$$\begin{aligned} \frac{\partial z(p,w)}{\partial p} &= -\left(\frac{\partial g}{\partial z}\right)^{-1} \frac{\partial g}{\partial p} = -\left[pD^2 f(z)\right]^{-1} \nabla f(z) \\ \frac{\partial z(p,w)}{\partial w} &= -\left(\frac{\partial g}{\partial z}\right)^{-1} \frac{\partial g}{\partial w} = \left[pD^2 f(z)\right]^{-1} \end{aligned}$$

By the chain rule, we know that:

$$\begin{aligned} \frac{d}{dp}[f(z(p,w))] &= \nabla f(z(p,w)) \frac{\partial z(p,w)}{\partial p} \\ &= -\nabla f(z) \left[pD^2 f(z)\right]^{-1} \nabla f(z) \end{aligned}$$

Since $D^2 f(z)$ is negative definite, its inverse is negative definite, and $\frac{\partial f(z)}{\partial z_l} \geq 0$. Therefore, $\frac{d}{dp}[f(z(p,w))] \geq 0$, indicating that an increase in p leads to an increase in optimal output level.

(ii) Effect of an Increase in Output Price on Input Demand:

From the analysis above, $\frac{\partial f(z)}{\partial z_l} \geq 0$ for all l, and $\frac{d}{dp}[f(z(p,w))] \geq 0$ meaning that an increase in p increases the demand for some input z_l.

(iii) Effect of an Increase in Input Price on Input Demand:

As

$$\frac{\partial z(p,w)}{\partial w} = -\left(\frac{\partial g}{\partial z}\right)^{-1} \frac{\partial g}{\partial w} = \left[pD^2 f(z)\right]^{-1} < 0$$

is negative definite, and that $\frac{\partial z(p,w)}{\partial w_l}$ is the lth diagonal element of $\frac{\partial z(p,w)}{\partial w}$, we know that $\frac{\partial z(p,w)}{\partial w_l} < 0$. Thus, an increase in the price of an input w_l leads to a decrease in demand of output z_l.

Solution 2. Competitive Equilibrium and Welfare Theorems

2.a. (i) Suppose that a feasible allocation (x, y) is strong Pareto efficient, and take another allocation (x', y'), satisfying $u_i(x'_i) > u_i(x_i)$ for all i. By the definition of strong Pareto efficiency, (x', y') can not be feasible, then (x, y) must also be weak Pareto efficient.

(ii) Assume allocation (x, y) is not strong Pareto efficient, meaning that there exist another feasible allocation (x', y'), so that $u_i(x'_i) \geq u_i(x_i)$ for all $i \neq k$ and some k, $u_k(x'_k) > u_k(x_k)$.Since $X_i = \mathbb{R}^L_+$ and the preference is strongly monotone, we have: $u_k(x'_k) > u_k(x_k) \geq u_k(0)$. So, $x'_k \neq 0$, giving that there exist at least one commodity s, such that $x'_{ks} > 0$.

Take $0 < \varepsilon < x'_{ks} - x_{ks}$, define a new allocation (x'', y') as follows:

$$
\begin{aligned}
x''_{il} &= x'_{il} \text{ for all } i \neq k, l \neq s \\
x''_{is} &= x'_{is} + \frac{1}{I-1}\varepsilon \text{ for all } i \neq k \\
x''_{kl} &= x'_{kl} \text{ for all } l \neq s \\
x''_{ks} &= x'_{ks} - \varepsilon > x_{ks}
\end{aligned}
$$

As

$$
\begin{aligned}
\sum_{i,j} x''_{ij} &= \sum_{i\neq k, l\neq s} x'_{il} + \sum_{i\neq k}\left(x'_{is} + \frac{1}{I-1}\varepsilon\right) + \sum_{l\neq s} x'_{kl} + x'_{ks} - \varepsilon \\
&= \sum_{i,j} x'_{ij} + (I-1)\frac{1}{I-1}\varepsilon - \varepsilon \\
&= \sum_{i.j} x'_{ij}
\end{aligned}
$$

we could know that (x'', y') is also a feasible allocation.

Because utility is strongly monotone, we know that $u_k(x''_k) > u_k(x_k)$ and $u_i(x''_i) > u_i(x_i)$ for all $i \neq k$. This way, we find a feasible allocation (x'', y') so that it's strictly better than (x, y), giving that (x, y) is not weak Pareto efficient.

Combine with the result from (a), we know that strong Pareto efficient and weak Pareto efficient are equivalent.

(iii) Assume $X = 2$, and $L = 1$, $X_1 = X_2 = \mathbb{R}_+$, $Y = -\mathbb{R}_+$, take the following two utility function:

$$
\begin{aligned}
u_1(x_1) &= 0 \\
u_2(x_2) &= x_2
\end{aligned}
$$

In this case, the utility of x_1 won't change. Allocation $(x_1, x_2) = (1, 1)$ is weak Pareto efficient, but not strong Pareto efficient.

The reason is that the utility function of x_1 is not strongly monotone, so we could not change a weak Pareto efficient allocation to a strong one like we did in question (ii), by reallocating consumption from consumer 1 to consumer 2.

2.b. (i) The consumer's utility maximization problem is:

$$\max_{x,m} u(x, m) = \alpha + \beta \ln(x) + m, \quad \text{s.t. } m + px \leq w_m.$$

When the budget constraint binds, we have:

$$m = w_m - px.$$

Substituting m into the utility function:

$$u(x) = \alpha + \beta \ln(x) + w_m - px.$$

Taking the derivative with respect to x, the first-order condition (FOC) is:

$$\frac{\partial u}{\partial x} = \frac{\beta}{x} - p = 0 \implies x^* = \frac{\beta}{p}.$$

Thus, the consumer's demand function for good l is:

$$x_d(p) = \frac{\beta}{p}.$$

(ii) The firm's profit maximization problem is:

$$\max_{q} \pi = pq - c(q) = pq - \sigma q.$$

The first-order condition (FOC) for profit maximization is:

$$\frac{\partial \pi}{\partial q} = p - \sigma = 0 \implies p^* = \sigma.$$

Thus, the firm's supply function is:

$$q_s(p) = \begin{cases} 0 & \text{if } p < \sigma, \\ \infty & \text{if } p > \sigma, \\ q & \text{if } p = \sigma. \end{cases}$$

At equilibrium, supply equals demand:

$$q^* = x_d(p^*) \implies q^* = \frac{\beta}{p^*}.$$

From the firm's FOC, the equilibrium price is:

$$p^* = \sigma.$$

Substituting $p^* = \sigma$ into the demand function, the equilibrium quantity is:

$$q^* = \frac{\beta}{\sigma}.$$

We now analyze how the equilibrium price and quantity change with the parameters α, β, σ:

1. Effect of α: Since α only affects the constant term in the utility function, it does not affect the equilibrium price p^* or quantity q^*.
2. Effect of β:
$$\frac{\partial q^*}{\partial \beta} = \frac{1}{\sigma} > 0.$$
As β increases, the consumer's preference for good l increases, so the equilibrium quantity q^* increases.
3. Effect of σ:
$$\frac{\partial q^*}{\partial \sigma} = -\frac{\beta}{\sigma^2} < 0.$$
As σ increases, production becomes more costly, leading to a higher equilibrium price p^* and a lower equilibrium quantity q^*.

Solution 3. Strategic Interactions

3.a. The payoff function of this game could be written as:

$$u_i(h_i, h_{-i}) = \alpha \sum_i h_i + \beta \left(\prod_i h_i \right) - w_i (h_i)^2$$

If firm i has a strictly dominant strategy, then for all other strategies h_i' and unchanged h_{-i}, we have $u_i(h_i, h_{-i}) > u_i(h_i', h_{-i})$. Take the FOC, we have:

$$\begin{aligned} \alpha + \beta \left(\prod_{j \neq i} h_j \right) - 2 w_i h_i &= 0 \\ \Rightarrow h_i &= \frac{1}{2 w_i} \left[\alpha + \beta \left(\prod_{j \neq i} h_j \right) \right] \end{aligned}$$

Given that h_{-i} will not affect the payoff function, h_i should not be affected by any other $h_j (j \neq i)$. Thus, $\beta = 0$. Then, firm i's dominent strategy would be $h_i = \frac{\alpha}{2 w_i}$.

3.b. Assuming the strategy of player 2 is choosing U under probability α, and the strategy of player 3 is choosing l under probability β. Use the notation $\tilde{u}$ to represent the expectation payoff. We can have:

$$
\begin{aligned}
\tilde{u}_L &= (\pi + 4\varepsilon)\beta + (\pi - 4\varepsilon)(1-\beta) = \pi + 4\varepsilon(2\beta - 1) \\
\tilde{u}_M &= \pi + \left[-\alpha\beta - (1-\alpha)(1-\beta) + \frac{1}{2}\alpha(1-\beta) + \frac{1}{2}\beta(1-\alpha)\right]\eta \\
&= \pi + \left[\frac{3}{2}(\alpha+\beta) - 3\alpha\beta - 1\right]\eta \\
\tilde{u}_R &= \pi - 4\varepsilon\beta + 4\varepsilon(1-\beta) = \pi - 4\varepsilon(2\beta - 1)
\end{aligned}
$$

(i) To show that M is never a best response to any pair of strategies of players 2 and 3, (α, β), we have three cases:

[Case 1: $\beta > \frac{1}{2}$]

Note that in this case $\frac{\partial \tilde{u}_M}{\partial \alpha} = \eta\left[\frac{3}{2} - 3\beta\right] < 0$. Thus the highest payoff for player 1 if he plays M is obtained when $\alpha = 0$, because $\alpha \in [0,1]$. His payoff will be

$$
\tilde{u}_M(\alpha = 0) = \pi + \eta\left[\frac{3}{2}\beta - 1\right] < \pi + 4\varepsilon\left[\frac{3}{2}\beta - 1\right] < \pi + 4\varepsilon\left[2\beta - 1\right] = \tilde{u}_L.
$$

Further note that $\tilde{u}_L$ is independent of α, independent of α, so that these inequalities hold for all α. Therefore, M cannot be a best response.

[Case 2: $\beta < \frac{1}{2}$]

Now, $\frac{\partial \tilde{u}_M}{\partial \alpha} > 0$, the highest payoff for player 1 if he plays M is obtained when $\alpha = 1$, and his payoff is

$$
\begin{aligned}
\tilde{u}_M(\alpha = 1) &= \pi + \eta\left[\frac{3}{2} + \frac{3}{2}\beta - 3\beta - 1\right] = \pi + \eta\left[\frac{1}{2} - \frac{3}{2}\beta\right] \\
&< \pi + \eta\left[\frac{1}{2} - \frac{3}{2}\beta + \frac{1}{2} - \beta\right] < \pi + 4\varepsilon\left[1 - 2\beta\right] \\
&= \tilde{u}_R.
\end{aligned}
$$

Further note that $\tilde{u}_R$ is independent of α, so that these inequalities hold for all α. Therefore, M cannot be a best response in this case.

[Case 3: $\beta = \frac{1}{2}$]

In this case, $\tilde{u}_M = \pi - \frac{\eta}{4} < \pi = \tilde{u}_R = \tilde{u}_L$. This concludes that M can never be a best response.

(ii) Suppose in negation that there exists a mixed strategy, in which player 1 plays R with probability γ and L with probability $1-\gamma$, that strictly dominates M.

[Case 1: $\gamma \leq \frac{1}{2}$]

If $\beta = 0$ and $\alpha = 1$, then $\tilde{u}_M = \pi + \frac{\eta}{2} > \pi$. The mixed strategy will give a payoff of $\pi - 4\varepsilon(1 - 2\gamma) \leq \pi < \tilde{u}_M$. Therefore, M cannot be a strictly dominated by the mixed strategy in this case.

[Case 2: $\gamma > \frac{1}{2}$]

If $\beta = 1$ and $\alpha = 0$ then $\tilde{u}_M = \pi + \frac{\eta}{2} > \pi$. The mixed strategy will give a payoff of $\pi + 4\varepsilon(1 - 2\gamma) \le \pi < \tilde{u}_M$. Therefore, M cannot be a strictly dominated by the mixed strategy in this case. This implies a contradiction, so that M cannot be strictly dominated.

(iii) Suppose players correlate in the following way: Players 2 and 3 play (U, r) with probability $\frac{1}{2}$ and (D, l) with probability $\frac{1}{2}$.

Any mixed strategy for player 1 involving only L and R will give him a payoff of π.

$$\begin{aligned} EU_1(\gamma L + (1-\gamma)R) &= \gamma \left(\frac{1}{2}(\pi + 4\varepsilon) + \frac{1}{2}(\pi - 4\varepsilon) \right) \\ &\quad + (1-\gamma) \left(\frac{1}{2}(\pi - 4\varepsilon) + \frac{1}{2}(\pi + 4\varepsilon) \right) \\ &= \pi \end{aligned}$$

However, playing M will yield him a payoff of $\pi + \frac{\eta}{2}$.

$$EU_1(M) = \frac{1}{2}\left(\pi + \frac{\eta}{2}\right) + \frac{1}{2}\left(\pi + \frac{\eta}{2}\right) = \pi + \frac{\eta}{2}$$

Thus M is a best-response to the above correlated strategy of player 2 and 3.

3.c. Let's begin with the payoff matrix of this game:

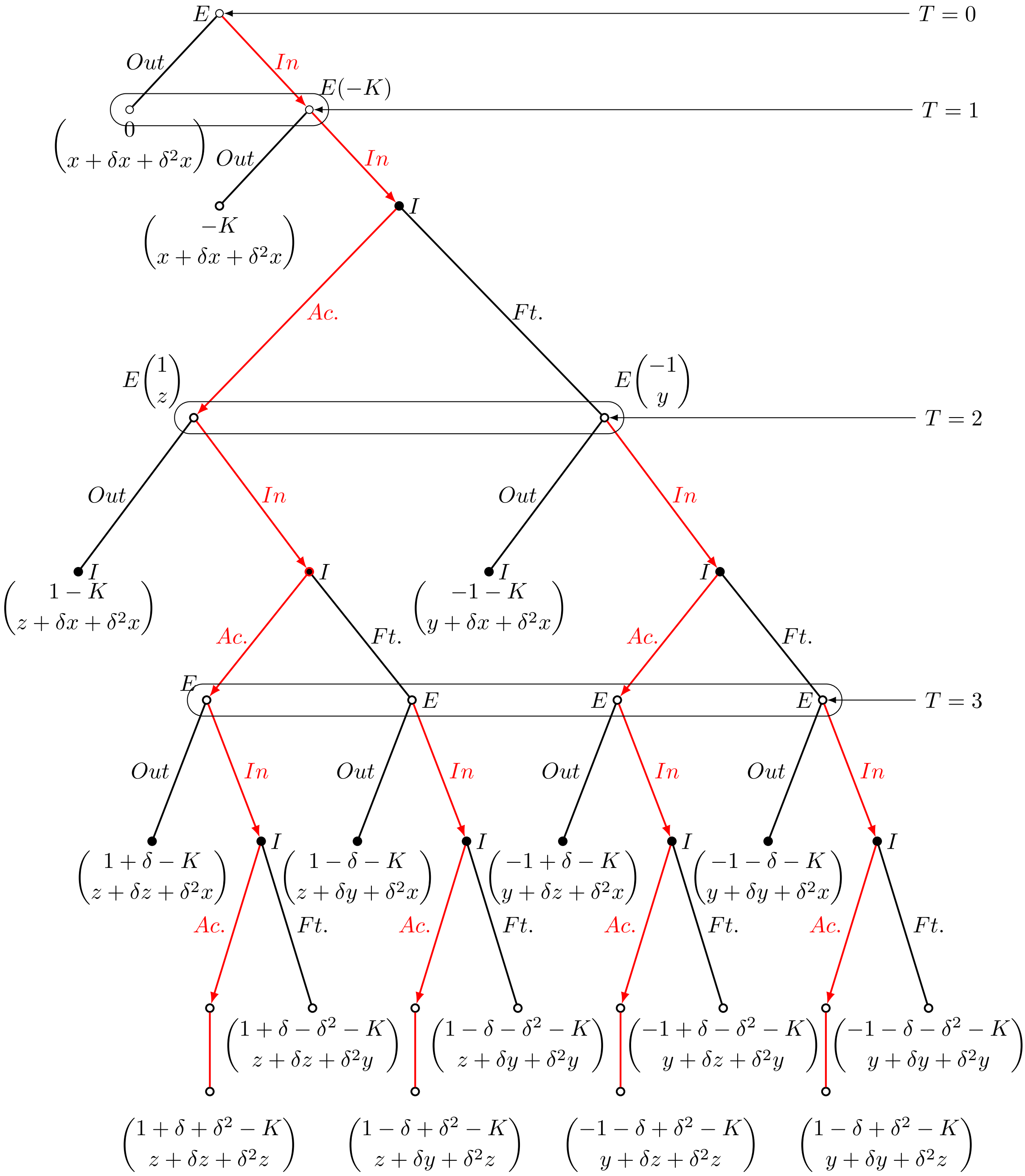
	$0 \le y \le 100 - x$	$y > 100 - x$
$0 \le x \le 100$	(x, y)	$(0, 0)$
$x > 100$	$(0, 0)$	$(0, 0)$

(i) We could tell from the payoff matrix that there's no strictly dominated strategy.

(ii) We could tell from the payoff matrix that any strategy giving total profit allocation over \$100 is weakly dominated.

(iii) If player 1's demand x is $0 \le x \le 100$, the best response of 2 is $y = 100 - x$, if player 2's demand $y = 100 - x$, similarly, the best response of player 1 is x. So, the Nash equilibrium of this problem is $(x, 100 - x)$ for all $0 \le x \le 100$.

3.d. (i) We can draw the extensive form of the game as below. Simple backward induction (using the assumptions) leads to the unique SPNE which is shown by arrows in the figure. Firm E enters at $t = 0$; and always plays 'In' thereafter. Firm I accommodates for all $t = 1, 2, 3$.



(ii) In this case, the game tree will be a bit simplified as below. Using backward induction, firm I will always accommodate in period $t = 3$, and therefore if $t = 3$ is reached, firm E will play In. This causes firm I to choose Fight in $t = 2$ since $y + \delta x > (1 + \delta)z$ by our second assumption. This causes firm E to exit the market forcibly at the beginning of period 3,which causes firm E to choose Out in $t = 2$.

Working backward we get that at $t = 1$, firm I chooses to accommodate and firm E choose In. However, the choice of firm E at $t = 0$ depends on the value of K.

1. If $K > 1$, then firm E will choose not to enter and firm I accommodates.
2. If $K < 1$, then firm E will enter, firm I will also choose to accommodate.
3. For $K = 1$, both are part of the (unique) continuation subgame perfect Nash Equilibrium, so there are up to two SPNEs in this case.