PS1 Solutions

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1 First generation crisis model

1.1 Consumption under a peg

We begin with the Euler equation. Under fixed exchange rate, the exchange depreciation rate $\varepsilon_t = 0$. Thus the interest parity gives that $i = r$, so that $c_t^{-\frac{1}{\sigma}} = \lambda(1 + \theta + \alpha r)$, as we know that $r = \beta$ and θ is the tax rate on consumption, which is constant, consumption will be constant, we can write it as $\tilde{c}$.
We then identify the constant by the intertemporal budget constraint:

$$\begin{aligned}
\alpha_0 + \frac{y}{r} &= \int_0^\infty e^{-rt} \left(\tilde{c}(1+\theta) + r m_t\right) dt \\
&= \int_0^\infty e^{-rt} \left(\tilde{c}(1+\theta) + r\alpha c_t\right) dt \\
&= \int_0^\infty e^{-rt} \tilde{c}(1+\theta+r\alpha) dt \\
&= \tilde{c}(1+\theta+\alpha r) \int_0^\infty e^{-rt} dt \\
&= \tilde{c}(1+\theta+\alpha r) \frac{1}{r}
\end{aligned}$$

Thus we have:

$$\tilde{c} = \frac{\alpha_0 r + y}{1 + \theta + \alpha r}$$

1.2 Unsustainable peg

As we know that the government spending is over a threshold:

$$g > r h_0 + \frac{\alpha_0 r + y}{1 + \theta + \alpha r}$$

From 1.1, we know that the consumption is a constant $\tilde{c} = \frac{\alpha_0 r + y}{1+\theta+\alpha r}$, thus the government tax income would be $\theta\tilde{c} = \frac{\theta(\alpha_0 r + y)}{1+\theta+\alpha r}$.
The government tax revenue is:

$$s^p = \theta\tilde{c} - g = \frac{\theta(\alpha_0 r + y)}{1 + \theta + \alpha r} - g < -r h_0 < 0$$

Under a fixed exchange rate, $\varepsilon_t = 0$ and at the steady state, the real balance is constant, $\dot{m}_t = 0$, we know that the foreign reserves cahnge is:

$$\dot{h}_t = rh_t + s_t^p < rh_t - rh_0$$

at $t = 0$, $h_0 < 0$. As time passes, the foreign reserves will be negative and the government will default on its debt. Hence the government cannot continue to run a peg, and the peg is unsustainable.

1.3 Consumption and depreciation pre- and post-break

From the Euler equation, before the abandon of the peg, we have:

$$c_1^{-\frac{1}{\sigma}} = \lambda(1 + \theta + \alpha r)$$

and after the abandon, we have:

$$c_2^{-\frac{1}{\sigma}} = \lambda(1 + \theta + \alpha(r + \varepsilon)).$$

So,

$$\left(\frac{c_1}{c_2}\right) = \frac{1 + \theta + \alpha(r + \varepsilon)}{1 + \theta + \alpha r} > 1$$

as $\varepsilon > 0$, giving that after the abando of the peg, the consumption decrceases.
Now we use that budget constraint and the cash in advance constraint, we have:
With m constant and in steady state of reserves: $\dot{h}_t = 0$,

$$0 = rh_t + (\theta c_2 - g) + \varepsilon m_2 = \theta(c_2 - c_1) + \varepsilon \alpha c_2$$

since $\theta c_1 = g$, and we can solve:

$$\varepsilon = \frac{\theta}{\alpha}\left(\frac{c_1}{c_2} - 1\right)$$

At the equilibrium values, $\frac{c_1}{c_2} > 1$ and thus $\varepsilon > 0$.

1.4 Dynamics of reserves and assets

Before the break, $s_t^p = \theta c_1 - g = 0$, $\dot{m}_t = 0$ and $\varepsilon_t = 0$, so $\dot{h}_t = rh_t$.
For foreign assets, we have:

$$\dot{a}_t = ra_t + y - c_1(1 + \theta) - rm_1 = ra_t + y - c_1(1 + \theta + \alpha r)$$

and the uverall position is:

$$\dot{a}_t + \dot{h}_t = r(a_t + h_t) + y - c_1(1 + \theta + \alpha r) = r(a_t + h_t) + y - \frac{g(1 + \theta + \alpha r)}{\theta}$$

since $c_1 = \frac{g}{\theta}$.
Evaluated at $t = 0$, we have:

$$\dot{h}_0 = rh_0, \quad \dot{a}_0 + \dot{h}_0 = r(a_0 + h_0) + y - \frac{g(1+\theta+\alpha r)}{\theta}$$

which is exactly the current-account balance. Observing $\dot{a}+\dot{h} < 0$ ex-ante would signal an impending crisis.

1.5 Timing of break

Recall that the intertemporal budget constraint is:

$$\frac{g}{r} = h_0 + \int_0^\infty e^{-rt}\left[\theta c_t + \dot{m}_t + \varepsilon_t m_t\right] dt + e^{-rT}\left[m_T - m_{T-}\right].$$

Before the break fo the peg($0 \le t \le T$):

$$c_t = c_1, \quad \dot{m}_t = 0, \quad \varepsilon_t = 0$$

and after the break, ($t > T$):

$$c_t = c_2, \quad \dot{m}_t = 0, \quad \varepsilon_t = \varepsilon > 0$$

Bringing these conditions into the intertemporal budget constraint, we have:

$$\frac{g}{r} = h_0 + \int_0^T e^{-rt}\theta c_1 dt + \int_T^\infty e^{-rt}\left[\theta c_2 + \varepsilon m_2\right] dt + e^{-rT}\left[m_T - m_{T-}\right].$$

As $\theta c_1 = g$, we can write:

$$\int_0^T e^{-rt}\theta c_1 dt = g\int_0^T e^{-rt} dt = \frac{g\left(1-e^{-rT}\right)}{r}$$

and that the second term is:

$$\int_T^\infty e^{-rt}\left[\theta c_2 + \varepsilon m_2\right] dt = \left[\theta c_2 + \varepsilon m_2\right]\frac{e^{-rT}}{r}$$

Thus,

$$\frac{g}{r} = h_0 + \frac{g\left(1-e^{-rT}\right)}{r} + \frac{[\theta c_2 + \varepsilon m_2]e^{-rT}}{r} + e^{-rT}\left[m_T - m_{T-}\right].$$

As $m_2 = \alpha c_2$, $\varepsilon = \frac{\theta}{\alpha}\left(\frac{c_1}{c_2} - 1\right)$, we can further simplify that:

$$\theta c_2 + \varepsilon m_2 = \theta c_2 + \frac{\theta}{\alpha}\left(\frac{c_1}{c_2} - 1\right)\alpha c_2 = \theta c_2 + \theta(c_1 - c_2) = \theta c_1 = g$$

Thus the budget constraint is reduced to:

$$\begin{aligned}
\frac{g}{r} &= h_0 + \frac{g\left(1-e^{-rT}\right)}{r} + \frac{ge^{-rT}}{r} + e^{-rT}\left[m_T - m_{T-}\right] \\
&= h_0 + \frac{g}{r} + e^{-rT}[m_T - m_{T-}] \\
\Rightarrow h_0 &= -e^{-rT}[m_T - m_{T-}] \\
\Rightarrow T &= \frac{1}{r}\ln\left(\frac{m_{T-} - m_T}{h_0}\right)
\end{aligned}$$