

Demystifying DSGE Models

3. Case Study – The Canonical Smets-Wouters DSGE

- ***NK model*** introduces into basic RBC more realistic specifications:
 - ***monopolistic*** competition ✓
 - ***frictions*** in ***prices*** and in ***wages*** ✓
 - ***habit*** formation in consumption ✓
 - ***non-Ricardian*** agents ✓
 - ***adjustment costs*** in investment ✓
 - ***capacity*** (non-)utilisation ***costs*** ✓
 - ***government*** sector (***monetary*** and ***fiscal***) ✓
 - ***financial*** sector ⇒ Week 5
 - ***housing*** sector ⇒ Week 5
 - ***unemployment*** ⇒ Week 6
 - ***environment*** ⇒ Week 6
 - ***foreign*** trade ⇒ Week 7

- Smets-Wouters model is “workhorse” DSGE model used as basis for almost all central bank DSGEs
- Smets and Wouters both from Belgian (!) CB; Smets’ father was its Governor until retirement...
- Original version published two decades ago in 2003, but superseded in 2007 by version estimated using Bayesian techniques
- This model is ***so important*** – in terms both of historical development of DSGEs and their practical implementation outside purely academic framework – that it merits being studied in detail
- eg, NY Fed model = SW2007 + BGG (financial side)

- **SW2007** model uses most of the characteristics mentioned last week:
 - consumption with **habit** persistence
 - **monopolistic** competition
 - **sticky** prices and wages using **Calvo** fairy
 - investment **adjustment costs** + variable capital **utilisation**
- Major **new** feature of model: use of **seven structural shocks** to match behaviour of US economy
- Shocks are to: productivity (***a***), labour supply (***l***), investment-specific technology (***i***), risk (***b***), (wage) cost-push (***w***), fiscal policy (***g***) and monetary policy (***r***)

- Households maximize **[NEW] non-separable** utility function with two arguments (goods and labour effort) over infinite life horizon

$$E_t \sum_{s=0}^{\infty} \beta^s \left[\frac{1}{1 - \sigma_c} (C_{t+s}(j) - \lambda (C_{t+s-1}))^{1 - \sigma_c} \right] \exp \left(\frac{\sigma_c - 1}{1 + \sigma_l} L_{t+s}(j)^{1 + \sigma_l} \right)$$

- Habits are in form of time-varying **external habits**
- [NEW]** SW form of utility function:
- King-Plosser-Rebelo (KPR) preferences**

- General form of **KPR Preferences**:

$$u(C, L) = \frac{1}{1 - \sigma_c} C^{1 - \sigma_c} v(L)$$

σ_c is risk aversion parameter = inverse of intertemporal rate of substitution

- where $v(L)$ is increasing and concave if $0 < \sigma_c < 1$ or decreasing and convex if $\sigma_c > 1$
- In **limit case** of $\sigma_c = 1$, resulting preferences specification is **additively separable** and given by $u(C, L) = \ln C_t + v(L)$
- For SW, $v(L)$ is given by $\exp \left(\frac{\sigma_c - 1}{1 + \sigma_l} L_{t+s}(j)^{1 + \sigma_l} \right)$
- Labour L is **aggregated** by a **lazy** union \rightarrow **sticky nominal wages à la Calvo**

σ_l [last week φ !!] is inverse of “Frisch elasticity of labour supply” = percentage change in hours arising from a given percentage change in wages

- Households **save** through purchases of government **nominal bonds** (B_{t+1}), from which they earn an interest rate of R_t
- Return on these bonds is subject to a **risk shock** ϵ_t^b
- Households own all capital **stock** and **[NEW]** rent capital **services** to firms, deciding how much capital **stock** to accumulate given **capital adjustment costs**

$$K_t(j) = (1 - \delta)K_{t-1}(j) + \epsilon_t^i \left[1 - S \left(\frac{I_t(j)}{I_{t-1}(j)} \right) \right] I_t(j)$$

NB: SW use Dynare timing convention, so no “Predetermined_Variables K”

$$S(x_t) = \frac{\varphi}{2} (x_t - \gamma)^2$$

Quadratic form as seen last week

- As rental price of capital (R^k) changes, **utilisation** of capital **stock** can be adjusted at increasing **cost**

$$K_t^s(j) = Z_t(j) K_{t-1}(j) \leftarrow K^s = \text{capital services}; Z = \text{utilisation cost fn}$$

- Firms produce *differentiated* goods, decide on labour and capital inputs, and set *(sticky) prices*, again according to *Calvo model*
- Calvo Rule** here *[NEW]*: those prices/wage rates that are *not* re-optimised are *only partially indexed* to past inflation rates:

$$P_t(i) = (\pi_{t-1})^{\iota_p} (\pi_*)^{1-\iota_p} P_{t-1}(i)$$
 - π_* denotes steady-state *gross* inflation rate (eg, 1.02)
- $0 \leq \iota_p \leq 1$ is the (price) *indexation* parameter
- A positive value of ι_p introduces *structural inertia* into inflation process since P_{t-1} enters function
- Similarly* for *wages* (ι_w)

Some Detail: Intermediate Goods Sector

- **Intermediate Good Producer i** is assumed by SW to use standard **Cobb-Douglas** technology (with variable input costs) but subject also to a **fixed cost**

$$Y_t(i) = \epsilon_t^a K_t^s(i)^\alpha [\gamma^t L_t(i)]^{1-\alpha} - \gamma^t \Phi$$

Note that “fixed” cost grows with time !

- where
 - $K_t^s(i)$ is **capital services** used in production [flow]
 - $L_t(i)$ is a composite labour (services) input
 - **[NEW]** ϕ is a fixed cost
 - **[NEW]** γ^t represents a (labour-augmenting) deterministic **growth** rate of output

- ϵ_t^a is for ***total factor productivity*** and follows process

$$\ln \epsilon_t^a = (1 - \rho_z) \ln \epsilon^a + \rho_z \ln \epsilon_{t-1}^a + \eta_t^a, \eta_t^a \sim N(0, \sigma_a)$$

Steady-state value, usually assumed = 0

- Firm's ***profit*** is given by

$$P_t(i) Y_t(i) - W_t L_t(i) - R_t^k K_t^s(i)$$

- where
 - W_t is aggregate nominal wage rate
 - R_t^k is rental rate on capital ***services***
- Cost minimisation yields following first-order conditions:

$$(\partial L_t(i)) : \Theta_t(i) \gamma^{(1-\alpha)t} (1 - \alpha) \epsilon_t^a K_t^s(i)^\alpha L_t(i)^{-\alpha} = W_t$$

$$(\partial K_t^s(i)) : \Theta_t(i) \gamma^{(1-\alpha)t} \alpha \epsilon_t^a K_t^s(i)^{\alpha-1} L_t(i)^{1-\alpha} = R_t^k$$

- where $\Theta_t(i)$ is *Lagrange multiplier* (shadow value) associated with production function (and equals marginal cost MC_t)
- Combining these FOCs and noting that capital-labour ratio is equal across firms implies (*as usual*)

$$K_t^{\textcircled{s}} = \frac{\alpha}{1 - \alpha} \frac{W_t}{R_t^k} L_t$$

- **Marginal cost** MC_t is **same for all firms** and equals $MC_t = \alpha^{-\alpha} (1 - \alpha)^{-(1-\alpha)} W_t^{1-\alpha} R_t^{k\alpha} \gamma^{-(1-\alpha)t} (\epsilon_t^a)^{-1}$
- **Prices of intermediate goods** are determined by **Calvo** fairy/devil/lottery [$P \neq MC$ recall !!]
- As in previous Calvo models, in each period, each firm i faces a constant **probability** $1 - \xi_p$ of being able to **re-optimize** its price $P_t(i)$
- Probability that firm receives Calvo-fairy signal to re-optimize its price assumed **independent** of time that it last reset its price [“Markov Process”]

- Under Calvo pricing with SW *partial indexation* mentioned earlier,
- optimal price $\tilde{P}_t(i)$ set by firm that is allowed by Calvo fairy (with a probability $1-\xi_p$) to re-optimize
- results from solving following optimisation problem:

$$\max_{\tilde{P}_t(i)} E_t \sum_{s=0}^{\infty} \xi_p^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \left[\tilde{P}_t(i) \left(\prod_{l=1}^s \pi_{t+l-1}^{\ell_p} \pi_*^{1-\ell_p} \right) - MC_{t+s} \right] Y_{t+s}(i)$$

Net profits

“Stochastic discount factor”

- This looks formidable!
- But if we compare it to what we used in sticky-price model studied last week, we find clear similarities:

$$\max_{P_{j,t}^*} E_t \sum_{i=0}^{\infty} (\beta \theta)^i \left(P_{j,t}^* Y_{j,t+i} - TC_{j,t+i} \right)$$

$\xi_p \equiv \theta$

$$\max_{\tilde{P}_t(i)} E_t \sum_{s=0}^{\infty} \xi_p^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \left[\tilde{P}_t(i) \left(\prod_{l=1}^s \pi_{t+l-1}^{l_t} \pi_*^{1-l_p} \right) - MC_{t+s} \right] Y_{t+s}(i)$$

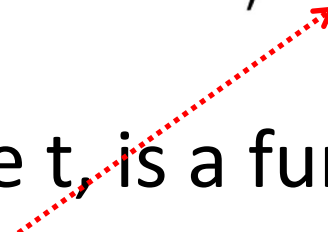
- Since $TC = MC * Y$, only really ***new element*** is ***indexation factor*** multiplying $P_t^{\sim} : \prod_{l=1}^s \pi_{t+l-1}^{l_t} \pi_*^{1-l_p}$

Product operator !!

- What is this ***indexation factor*** $\prod_{l=1}^s \pi_{t+l-1}^{l_t} \pi_*^{1-l_p}$
- Since π is a ***gross*** inflation rate, it has each period a value close to 1
- Thus, product $\prod_{l=1}^s \pi_{t+l-1}^{l_t}$ (which multiplies together inflation rates for all periods since previous price revision) also has a value near 1
- This is then scaled by factor $\pi_*^{1-l_p}$ which reflects ***weighted combination*** indexation
- Finally, as noted previously, curious expression $\frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}}$ is just the (nominal) discount factor

Ξ_t is here Lagrange Multiplier for consumption, so this is just β multiplied by (1) ratio of Lagrange multipliers [≈ 1] x (2) ratio of prices at t and $(t+s)$ [also ≈ 1]

- It is also clear from formulation of optimisation problem

$$\max_{\tilde{P}_t(i)} E_t \sum_{s=0}^{\infty} \xi_p^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \left[\tilde{P}_t(i) \left(\prod_{l=1}^s \pi_{t+l-1}^{l_t} \pi_*^{1-l_p} \right) - MC_{t+s} \right] Y_{t+s}(i)$$


- that price set by firm i , at time t , is a function of ***expected future*** marginal costs
- Price*** will therefore be a ***mark-up*** over these discount-weighted ***marginal costs***

- Given FOC for this optimisation problem, ***aggregate price index*** which results is

$$P_t = (1 - \xi_p) \tilde{P}_t(i) G'^{-1} \left[\frac{\tilde{P}_t(i) \tau_t}{P_t} \right] + \xi_p \pi_{t-1}^{l_p} \pi_*^{1-l_p} P_{t-1} G'^{-1} \left[\frac{\pi_{t-1}^{l_p} \pi_*^{1-l_p} P_{t-1} \tau_t}{P_t} \right]$$

- Compare this to what we used in first sticky-price model studied last week:

$$P_t = \left[\theta P_{t-1}^{1-\psi} + (1 - \theta) P_t^{*1-\psi} \right]^{\frac{1}{1-\psi}}$$

$\xi_p \equiv \theta$

- Function G (of which G' is used in definition above) is defined in Appendix; it comes from using **Kimball**

See later

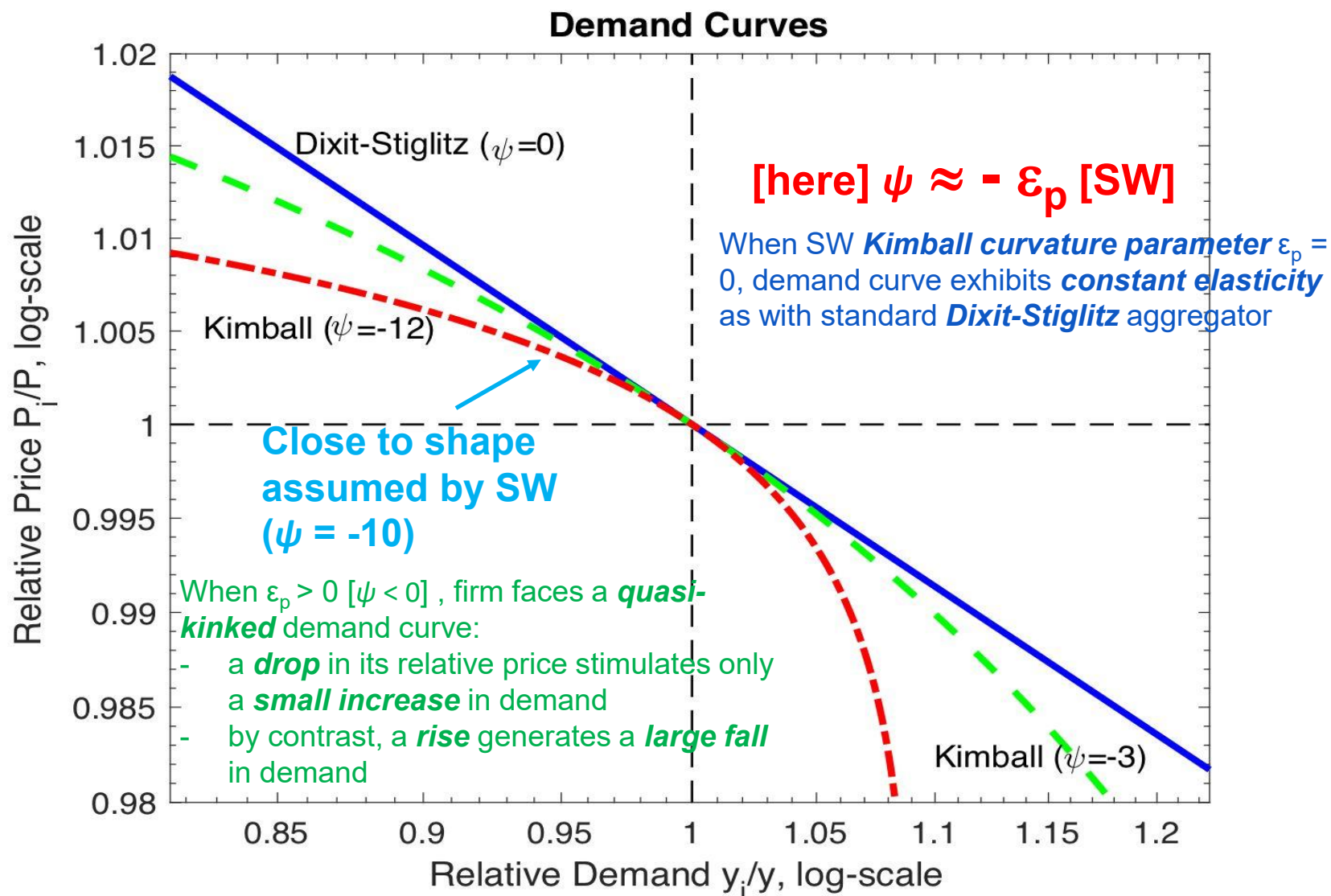
- Prices therefore determined (partially) by ***past inflation rate***
- ***Marginal costs*** are a function of **wages** and **rental rate** of capital

$$MC_t = \alpha^{-\alpha} (1 - \alpha)^{-(1-\alpha)} W_t^{1-\alpha} R_t^{k\alpha} \gamma^{-(1-\alpha)t} (\epsilon_t^a)^{-1}$$

- ***Same*** mechanism used by SW for **wages**

- Here endth such detail!
- Finally, a ***technical point***:
- In both goods and labour markets SW ***replace*** standard ***Dixit-Stiglitz*** aggregator
- with a ***Kimball (1995) aggregator***
- ➔ allows for ***time-varying demand elasticity***, which depends on ***relative price***
- Introduction of Kimball aggregator allows SW to estimate a ***more reasonable degree*** of price and wage ***stickiness***

Figure 1: Demand Curves -- Implications of Kimball vs. Dixit-Stiglitz Aggregators.



- For those who wish to see technical details of original non-linear SW2007 model, I have attached an Appendix which does so
- Here, we shall look instead at SW's ***published*** log-linearised equations

SW2007 Log-Linearised Model

- SW **detrended** their variables with **deterministic** trend γ and replaced **nominal** variables by their **real per-capita** counterparts – this is very **familiar!** [Some authors have later used a **time-varying** trend to do this for SW2007] *More on this next week*
- **Non-linear** system was then **linearised** around stationary **steady state** of **detrended** variables, in usual **log-linearisation** process which we have seen previously
- This was done primarily because system is highly **non-linear** and **multi-dimensional** (34 parameters plus 7 s.d. to be estimated **simultaneously**)

- **Log-linearised** model (equations taken directly from SW2007 in **AER**): start with **Supply Side**
- **Aggregate production function** is (eq.5)

$$y_t = \phi_p (\alpha k_t^s + (1 - \alpha) \ell_t) + \epsilon_t^a$$

- where

– y_t is (log-linearised) GDP

– ℓ_t is labour input

– ϵ_t^a is total factor productivity [shock **process**]

– **[NEW]** k_t^s is **capital services**, determined by **capital stock installed** in previous period

$$k_t^s = k_{t-1} + z_t$$

– and a **capacity utilisation** variable z (eq.6)

ϕ_p is (1 +) share of fixed costs in production, reflecting presence of *fixed costs* in production function

- **Aside:** here is how SW actually write these equations in their code

```
[name='FOC capacity utilization, SW Equation (7)']
    zcap = (1/(czcap/(1-czcap)))* rk ;|
[name='Definition capital services, SW Equation (6)']
    k = kp(-1)+zcap ;
[name='Law of motion for capital, SW Equation (8) ']
    kp = k1*kp(-1) + (1 - k1)*inve + k2*qs;
[name='Aggregate Production Function, SW Equation (5)']
    y = cfc*( calfa*k+(1-calfa)*lab +a );
```

- “kp” is physical capital [stock]
- “k” is capital services [flow]
- **Following Dynare convention**, “k” is determined by “kp(-1)”

- **Costs of adjusting** capital stock in use → rate of **capacity utilisation** linked to rental rate of capital (*eq.7*)

$$z_t = z_1 r_t^k$$

- **Rental rate of capital** is a function of capital-labour ratio and real wage (*eq.11*)

$$r_t^k = -(k_t - \ell_t) + w_t$$

Remember, these are logs:
log(ratio) = difference

- **Total factor productivity shock** evolves over time according to an **AR(1)** process

$$\epsilon_t^a = \rho_a \epsilon_{t-1}^a + \eta_t^a$$

This is a process; ρ_a is likely close to 1;
actual “shock” comes through η^a

- Now ***Demand Side***
- Expenditure formulation of aggregate resource constraint is familiar (*eq.1*)

$$y_t = c_y c_t + i_y i_t + z_y z_t + \epsilon_t^g$$

- where
 - y_t is GDP
 - c_t is consumption
 - i_t is investment
 - z_t is exogenous spending [= government + net exports]
 - ϵ_t^g is spending shock ***process***
 - c_y , i_y and z_y are constant ***parameters***, representing (as usual) ***steady-state shares*** of each variable in y

- **Consumption** is determined by usual **Euler Equation**

$$c_t = c_1 c_{t-1} + (1 - c_1) E_t c_{t+1} + c_2 (\ell_t - E_t \ell_{t+1}) - c_3 \left(r_t - E_t \pi_{t+1} + \epsilon_t^b \right)$$

- (eq.2) where

- c_1, c_2, c_3 are constant **coefficients**
- r_t is interest rate on a **one-period** risk-less **bond**
- ϵ_t^b evolves according to **AR(1)** process $\epsilon_t^b = \rho_b \epsilon_{t-1}^b + \eta_t^b$

- Note that this Euler Equation has a **backward-looking** element, deriving from “habit formation” [$C_t - \lambda C_{t-1}$ replaces C_t in utility function, recall]

- Term $c_2 (\ell_t - E_t \ell_{t+1})$ involving **labour** input allows for some **substitution** between consumption and labour input
- **Coefficients** c_1, c_2, c_3 are themselves functions of deeper **structural parameters**
- SW describe ϵ^b term as a “**risk premium**” shock determining willingness of households to hold one-period bond $c_3 \left(\overbrace{r_t - E_t \pi_{t+1}} + \epsilon_t^b \right)$
- It can also be seen as a type of **preference shock** that influences short-term consumption-saving decision (this is more **usual interpretation**)

- **Investment** is determined by (eq.3)

$$i_t = i_1 i_{t-1} + (1 - i_1) E_t i_{t+1} + i_2 q_t + \epsilon_t^i$$

- where **shadow price of capital stock** is (eq.4)

$$q_t = q_1 E_t q_{t+1} + (1 - q_1) r_{t+1}^k - (r_t - E_t \pi_{t+1} + \epsilon_t^b)$$

Same as in eq. 2

- **nominal value of installed capital** is given by (eq.8)

$$k_t = k_1 k_{t-1} + (1 - k_1) i_t + k_2 \epsilon_t^i$$

Same as in eq. 3

- Investment depends on **lagged** investment because \exists **adjustment cost** function that limits amount of new investment that can come “on line” immediately

Recall, SW use Dynare timing convention

- **Main driving force** behind investment is **shadow price of capital stock (Tobin's Q)** q_t (eq.4 above):

$$q_t = q_1 E_t q_{t+1} + (1 - q_1) (r_{t+1}^k - (r_t - E_t \pi_{t+1} + \epsilon_t^b))$$

- q_t depends **positively** on expected future marginal productivity of capital (r^k) and **negatively** on future real interest rate (and “risk premia”)
- Note: investment **shock** ϵ_t^i appears in **both** i_t **and** k_t equations → positive shock to investment also increases capital stock concurrently
- SW find that, **empirically, investment adjustment cost shocks are the most important in entire model**; by contrast (and curiously), **capacity utilisation costs** are estimated to be not very important

- Recall: $Y = C + I + [G + (X - M)]$ or here:

$$y_t = c_y c_t + i_y i_t + z_y z_t + \epsilon_t^g$$

- Exogenous spending** z_t has two components:
 - Government** spending G
 - an element related to **productivity** because “**net exports** ($X - M$) may be affected by domestic productivity developments” [SW’s words]
- exogenous spending shock ϵ_t^g changes over time according to **[NEW]** a **cross-equation** process:

$$\epsilon_t^g = \rho \epsilon_{t-1}^g + \eta_t^g + \rho_{ga} \eta_t^a$$

Productivity !

- **Mark-up** of **price** over marginal cost is determined in log-linearised model by difference between **marginal product of labour** and **real wage** (eq.9)

$$\mu_t^p = \text{mpl}_t - w_t = \alpha(k_t^s - l_t) + \epsilon_t^a - w_t$$

- where **marginal product of labour** is itself a (positive) function of **capital-labour ratio** and total factor **productivity (TFP)**
- → price **inflation** determined by “**New Keynesian Phillips Curve**” (eq.10)

$$\pi_t = \pi_1 \pi_{t-1} + \pi_2 E_t \pi_{t+1} - \pi_3 \mu_t^p + \epsilon_t^p$$

- **Lagged** inflation appears in NKPC as a result of **Calvo** pricing structure
- If **degree of indexation** to past inflation is **zero** ($\iota_p=0$), **eq.10** reverts to **purely forward-looking** Phillips curve (coefficient $\pi_1=0$) seen last week

$$\pi_t = \pi_1 \pi_{t-1} + \pi_2 E_t \pi_{t+1} - \pi_3 \mu_t^p + \epsilon_t^p$$

- **Speed of adjustment** to desired **mark-up** (coefficient π_3) depends on
 - **degree of price stickiness** (ξ_p)
 - **curvature** of **Kimball** goods market aggregator (ε_p)
 - steady-state price **mark-up** ($\varphi_p - 1$)

- In $\pi_t = \pi_1 \pi_{t-1} + \pi_2 E_t \pi_{t+1} - \pi_3 \mu_t^p + \epsilon_t^p$
- ϵ_t^p is **price mark-up** shock, which evolves as **[NEW] ARMA(1,1)**:

$$\epsilon_t^p = \rho^p \epsilon_{t-1}^p + \eta_t^p - \mu_p \eta_{t-1}^p$$

- **MA** component has **no memory**, recall
 → One-period-memory MA term captures **high-frequency** fluctuations in inflation
- **Price mark-up shock** ϵ_t^p is included because SW find it is **empirically very important in order** to capture **price dynamics**

- SW model treats **wages** similarly to prices:
- (Calvo) Sticky wages gradually adjust → real wages move only gradually over time
- to **equate real wages** with **marginal rate of substitution** between working and consuming
- Short-term **gap** between these is “**wage mark-up**” defined as (eq.12)

$$\begin{aligned}\mu_t^w &= w_t - mrs_t \\ &= w_t - \left(\sigma \ell_t - \frac{1}{1 - \lambda/\gamma} \left(c_t - \frac{\lambda}{\gamma} c_{t-1} \right) \right)\end{aligned}$$

- Wages are then given by ([eq.13](#))

$$w_t = w_1 w_{t-1} + (1 - w_1) (E_t w_{t+1} + E_t \pi_{t+1}) - w_2 \pi_t + w_3 \pi_{t-1} - w_4 \mu_t^w + \epsilon_t^w$$

- Parameter $w_4 \rightarrow$ **speed of adjustment** to desired **wage mark-up** depends on degree of wage **stickiness** (ξ_w) and demand elasticity for labour
- which itself is a function of **steady-state labour market mark-up** ($\varphi_w - 1$) and **curvature** of **Kimball** labour market aggregator (ε_w – do not **confuse** this with shock process ϵ_t^w !)
- Dynare** model therefore contains a complicated equation defining w_4

- The (again, **ARMA**) **wage mark-up shock** has form

$$\epsilon_t^w = \rho^w \epsilon_{t-1}^w + \eta_t^w - \mu_w \eta_{t-1}^w$$

- (As for prices) **wage mark-up shock** affects both current and lagged inflation and attempts also to capture **temporary** wage shocks with MA
- SW again find that wage mark-up shock is **empirically very important** to capture **wage dynamics**
- **Overall SW confirm, empirically, NK thesis** by finding that both price and wage **stickiness** are very **important**, with prices affecting **short-run** inflation and wages **longer-run** inflation

- Final element of model is rule for **monetary policy**
- **Central bank** sets short-term interest rates according to **modified Taylor Rule**, which in log-linearised form becomes (*eq.14*)

$$r_t = \rho r_{t-1} + (1 - \rho) (r_\pi \pi_t + r_y (y_t - y_t^p)) + r_{\Delta y} [(y_t - y_t^p) - (y_{t-1} - y_{t-1}^p)] + \epsilon_t^r$$

- where y^p is **potential output** (“flex-price” output)
- a monetary policy **shock** is included (of simple **AR(1)** form)

$$\epsilon_t^r = \rho^r \epsilon_{t-1}^r + \eta_t^r$$

- **Interest rate** depends on last period's interest rate while **gradually adjusting** towards a **target interest rate** $(r_\pi \pi_t + r_y(y_t - y_t^p))$ that depends on
 - inflation π_t
 - **output gap** $(y_t - \mathbf{y}_t^p)$ between **actual** and **potential** output (in SW, $\mathbf{y}^p \equiv \text{flex-price}$ level of output)
- Interest rate also depends on **growth rate** of this output gap

$$\left[(y_t - y_t^p) - (y_{t-1} - y_{t-1}^p) \right]$$
- (recall that for logs $\rightarrow \mathbf{dlog} = \%chg = \text{growth rate}$)

- **Potential output y_t^p** as used in Taylor Rule \equiv output obtained *if* prices and wages were **fully flexible**
- \rightarrow **[NEW]** model effectively needs to be **expanded** to add a **shadow** flexible-price economy
- \rightarrow in **Dynare** implementation \exists **repetitive** section defining variables in **flex-price** economy
- A different definition of Taylor Rule or of output gap would **avoid** need for this section
- In fact, **original** SW2003 **Euro area** model did **not** include “shadow” flexible-price economy
- SW2007 added it to help explain price dynamics

Dynare Model

- Implementation in **Dynare** was originally written by SW themselves, so we make use of it here
- Unfortunately, their ***code*** used a ***notation*** which ***differs considerably*** from that set out in their SW2007 AER paper, so I have used code re-written for our use in **Dynare**
- There are 14 equations in **Dynare** log-linearised model for sticky wage-price economy
- Look at just 4 of them

- **Consumption**

- **Eq.2** expressed consumption as

$$c_t = c_1 c_{t-1} + (1 - c_1) E_t c_{t+1} + c_2 (\ell_t - E_t \ell_{t+1}) - c_3 \left(r_t - E_t \pi_{t+1} + \epsilon_t^b \right)$$

- In **Dynare**, we have

#c1 = (lambda/gamma)/(1+lambda/gamma);

#c2 = ((sigma_c-1)*WL_C/(sigma_c*(1+lambda/gamma)));

#c3 = (1-lambda/gamma)/(sigma_c*(1+lambda/gamma));

c = c1 * c(-1) + (1 - c1) * c(+1) + c2 * (l - l(+1)) - c3 * (r - pinf(+1)) + eps_b;

Minor point: in SW's Dynare model, they separate out ϵ_b as here

$$c_1 = \frac{\lambda / \gamma}{1 + \lambda / \gamma} \quad c_2 = \frac{(\sigma_c - 1)(W_*^h L_* / C_*)}{\sigma_c (1 + \lambda / \gamma)} \quad c_3 = \frac{1 - \lambda / \gamma}{(1 + \lambda / \gamma) \sigma_c}$$

$$c_1 = \frac{\lambda / \gamma}{1 + \lambda / \gamma}$$

$$c_2 = \frac{(\sigma_c - 1)(W_*^h L_* / C_*)}{\sigma_c (1 + \lambda / \gamma)}$$

$$c_3 = \frac{1 - \lambda / \gamma}{(1 + \lambda / \gamma) \sigma_c}$$

- In c_1 , c_2 and c_3 , **deep parameters** are
 - **lambda** \equiv degree of (external) **habit persistence**
 - **sigma_c** \equiv **risk aversion** parameter
 - **gamma** \equiv gross **growth rate** [assuming **cointegration** of y, c, i]
- “WL_C” is a complicated combination of deep parameters and steady-state terms

- Next, **New Keynesian Phillips Curve** ([eq.10](#))

$$\pi_t = \pi_1 \pi_{t-1} + \pi_2 E_t \pi_{t+1} - \pi_3 \mu_t^p + \epsilon_t^p$$

- becomes

```
# pi1 = (1/(1+beta_bar*gamma*iota_p))* iota_p;
# pi2 = (1/(1+beta_bar*gamma*iota_p))* beta_bar*gamma;
# pi3 = (1/(1+beta_bar*gamma*iota_p))*
  ((1-xi_p)*(1-beta_bar*gamma*xi_p)/xi_p)/((phi_p-1)*curv_p+1) ;
pinf = pi1*pinf(-1) + pi2*pinf(1) + pi3*mc + eps_p;
```

$$\pi_1 = \frac{l_p}{1 + \beta \gamma^{1-\sigma_c} l_p}$$

$$\pi_2 = \frac{\beta \gamma^{1-\sigma_c}}{1 + \beta \gamma^{1-\sigma_c} l_p}$$

$$\pi_3 = \frac{1}{1 + \beta \gamma^{1-\sigma_c} l_p} \frac{(1 - \beta \gamma^{1-\sigma_c} \xi_p)(1 - \xi_p)}{\xi_p ((\phi_p - 1) \varepsilon_p + 1)}$$

- Repeating:

$$\pi_1 = \frac{l_p}{1 + \beta\gamma^{1-\sigma_c} l_p} \quad \pi_2 = \frac{\beta\gamma^{1-\sigma_c}}{1 + \beta\gamma^{1-\sigma_c} l_p} \quad \pi_3 = \frac{1}{1 + \beta\gamma^{1-\sigma_c} l_p} \frac{(1 - \beta\gamma^{1-\sigma_c} \xi_p)(1 - \xi_p)}{\xi_p ((\phi_p - 1)\varepsilon_p + 1)}$$

- Many new parameters here:

- ***beta_bar*** $\equiv \beta\gamma^{-\sigma_c}$ [ie, $\text{beta} * \text{gamma}^{(-\text{sigma}_c)}$] is growth-adjusted discount factor, beta being ***discount factor***

- ***iota_p*** = coefficient of ***indexation*** to past prices

- ***xi_p*** = ***Calvo parameter*** for prices

- ***curv_p*** = ***curvature of Kimball aggregator*** for prices (instead of ε_p to avoid confusion with shock ϵ^p)

- Hence ***beta_bar * gamma*** = $\beta\gamma^{1-\sigma_c}$

- **Wage Phillips Curve** (*eq.13*)

$$w_t = w_1 w_{t-1} + (1 - w_1) (E_t w_{t+1} + E_t \pi_{t+1}) \\ - w_2 \pi_t + w_3 \pi_{t-1} - w_4 \mu_t^w + \epsilon_t^w$$

- translates similarly into **Dynare** code below (once *eq.12* – next slide - is substituted in for μ_t^w)

```
#w1 = (1/(1+beta_bar*gamma));
#w2 = (1+beta_bar*gamma*iota_w)/(1+beta_bar*gamma);
#w3 = (iota_w/(1+beta_bar*gamma));
#w4 = - (1-xi_w)*(1-beta_bar*gamma*xi_w)/((1+beta_bar*gamma)*xi_w)*
      (1/((phi_w-1)*curv_w+1));
w = w1*w(-1) + (1 - w1)*w(1) + (1 - w1)*pinf(1) - w2*pinf + w3*pinf(-1)
    - w4*(sigma_l*l + (1/(1-lambda/gamma))*c -
      ((lambda/gamma)/(1-lambda/gamma))*c(-1) -w) + eps_w;
```

$$w_1 = \frac{1}{1 + \beta\gamma^{1-\sigma_c}} \quad w_2 = \frac{1 + \beta\gamma^{1-\sigma_c} \iota_w}{1 + \beta\gamma^{1-\sigma_c}} \quad w_3 = \frac{\iota_w}{1 + \beta\gamma^{1-\sigma_c}}$$

$$w_4 = \frac{1}{1 + \beta\gamma^{1-\sigma_c}} \frac{(1 - \beta\gamma^{1-\sigma_c} \xi_w)(1 - \xi_w)}{\xi_w ((\phi_w) - 1) \varepsilon_w + 1}$$

- New parameters here are
 - ι_w = coefficient of *indexation* to past *wages*
 - ξ_w = *Calvo parameter* for *wages*
 - curv_w = *curvature of Kimball aggregator* for *wages*
 - ϕ_w = gross steady-state *wage mark-up*
- *Wage mark-up* (eq. 12) is

$$\mu_t^w = w_t - mrs_t = w_t - \left(\sigma_l l_t + \frac{1}{1 - \lambda/\gamma} (c_t - \lambda/\gamma c_{t-1}) \right)$$

- Finally **Taylor Rule** ([eq.14](#))

$$r_t = \rho r_{t-1} + (1 - \rho) (r_\pi \pi_t + r_y (y_t - y_t^p)) + r_{\Delta y} [(y_t - y_t^p) - (y_{t-1} - y_{t-1}^p)] + \epsilon_t^r$$

- In **Dynare** implementation of this Taylor Rule, **potential output** variable y_t^p is defined as **yf, flex-price** (or “natural”) value of output
- ➔ introduce entire **flex-price economy** into model
- ➔ **11 additional equations**, in each case setting parameters which define “stickiness” of prices (and wages) to their “**zero stickiness**” values

- For example, ***Wage equation*** in ***flex-price*** economy becomes

$$w_f = \sigma_l * l_{abf} + (1/(1-\lambda/\gamma)) * c_f - (\lambda/\gamma)/(1-\lambda/\gamma) * c_f(-1) ;$$

- which is just ***last bit*** of Wage equation in ***sticky-price, sticky-wage*** economy because

$$- \textit{iota}_w = 0$$

$$- \textit{xi}_w = 1$$

in ***flex-price*** economy

Steady State

- One last step necessary before we can test out model: inserting information on model's ***steady state***
- These are derived as usual, for example

$$R_* = \bar{\beta}^{-1} \pi_*$$

$$k_* = \frac{\alpha}{1 - \alpha} \frac{w_*}{r_*^k} L_*$$

$$\dot{i}_* = (1 - (1 - \delta)/\gamma) \bar{k}_*$$

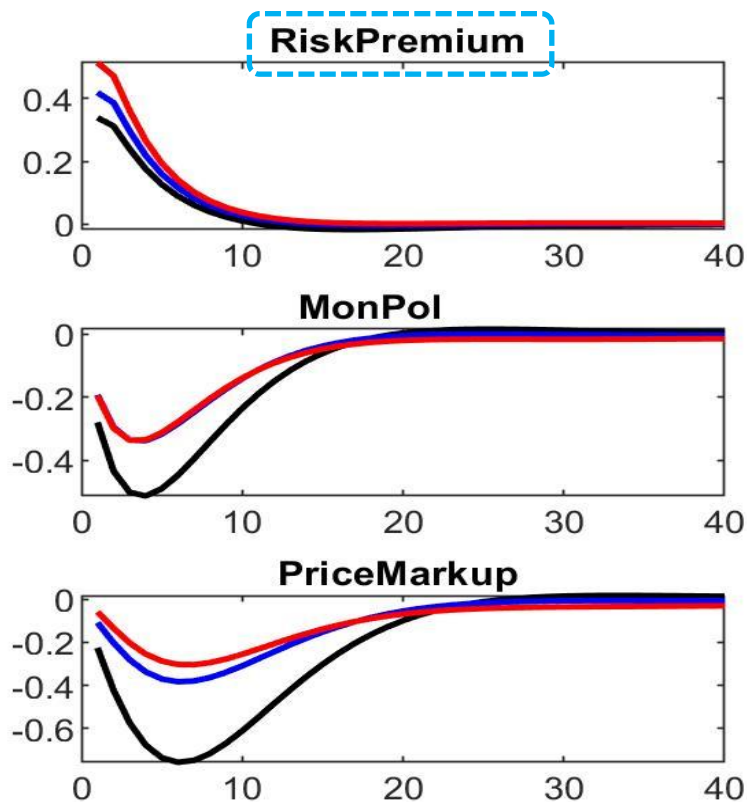
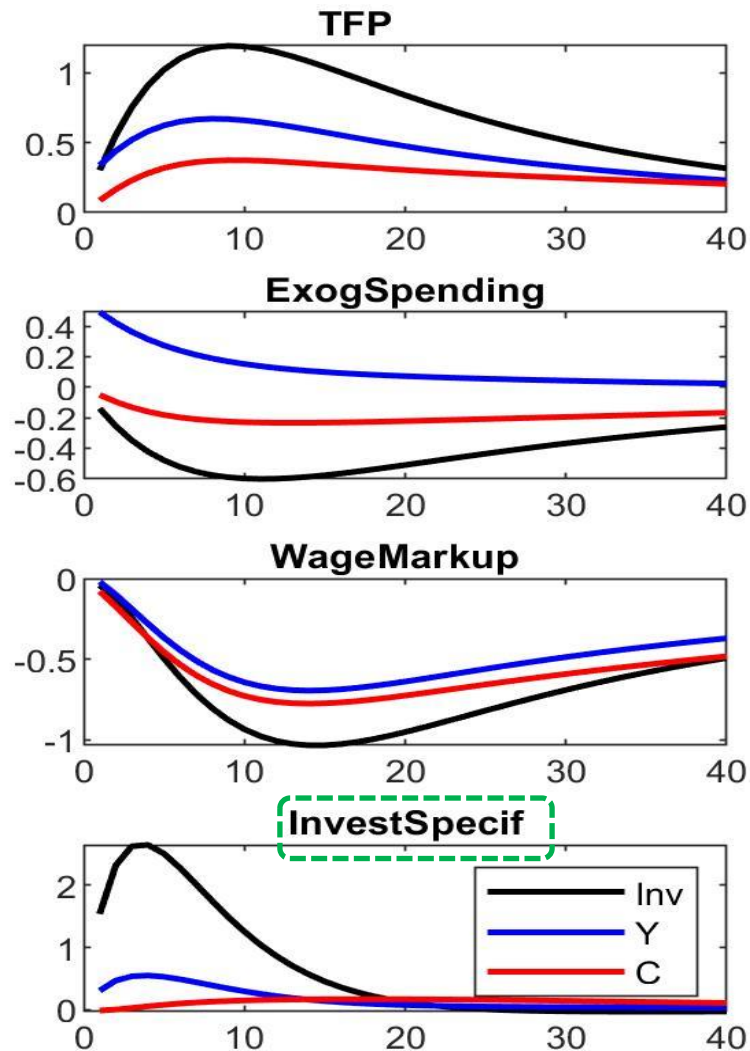
$$\frac{c_*}{y_*} + \frac{\dot{i}_*}{y_*} + g_* = 1$$

- These steady-state conditions are inserted into **Dynare** model as *restrictions on parameters*
- For example, steady-state rental rate r_*^k is defined in **Dynare** code as

$$\begin{aligned} r_*^k &= \bar{\beta}^{-1} - (1 - \delta) \\ &= \beta^{-1} \gamma^{\sigma_c} - (1 - \delta) \end{aligned}$$
- **#Rk = (beta^(-1)) * (gamma^sigma_c) - (1-delta);**
- where # sign informs **Dynare** that this is *not* a parameter to be *estimated directly*, but one *calculated* from various other “deep” parameters
- **# → model_local_variable**
- **Rk** is then used in value of capital *equation 4*, and also in definitions of other parameters (eg, steady state real wage and labour-to-capital ratio)

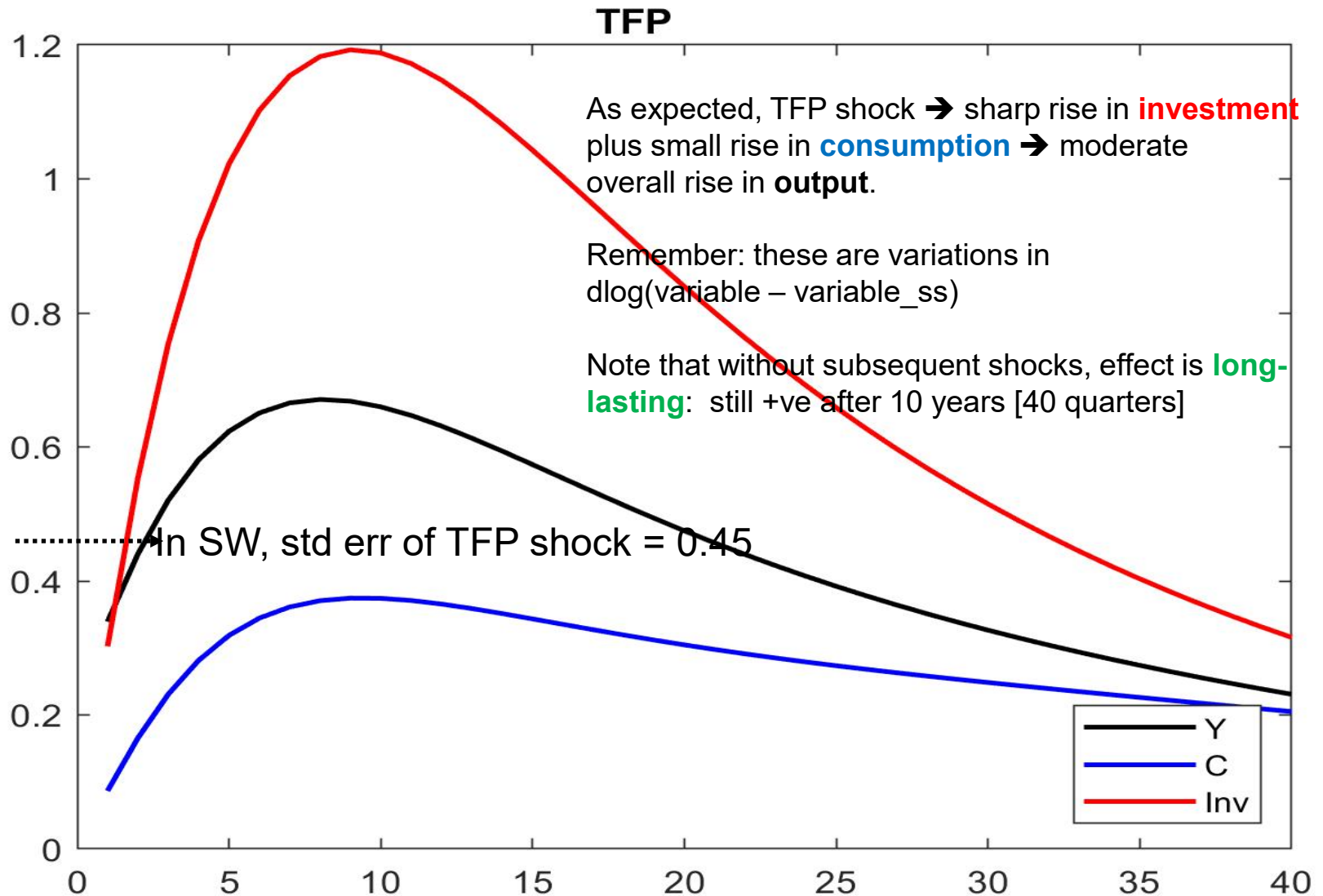
- Once this is done, it is possible to *simulate*
 - Slides below show impact of each shock on model's endogenous variables
 - Using *original SW2007* parameters
- As SW note, apart from *TFP shock* (ϵ^a), *risk premium* (ϵ^b) and *investment-specific* (ϵ^i) *shocks* are most powerful [observe *scale* of impact]
- Recall that SW describe ϵ^b term as a shock determining willingness of households to hold one-period bond
- But also seen as a type of *preference shock* that influences short-term consumption-saving decision

- Responses to all shocks – Y , C , Inv [see specifics next]

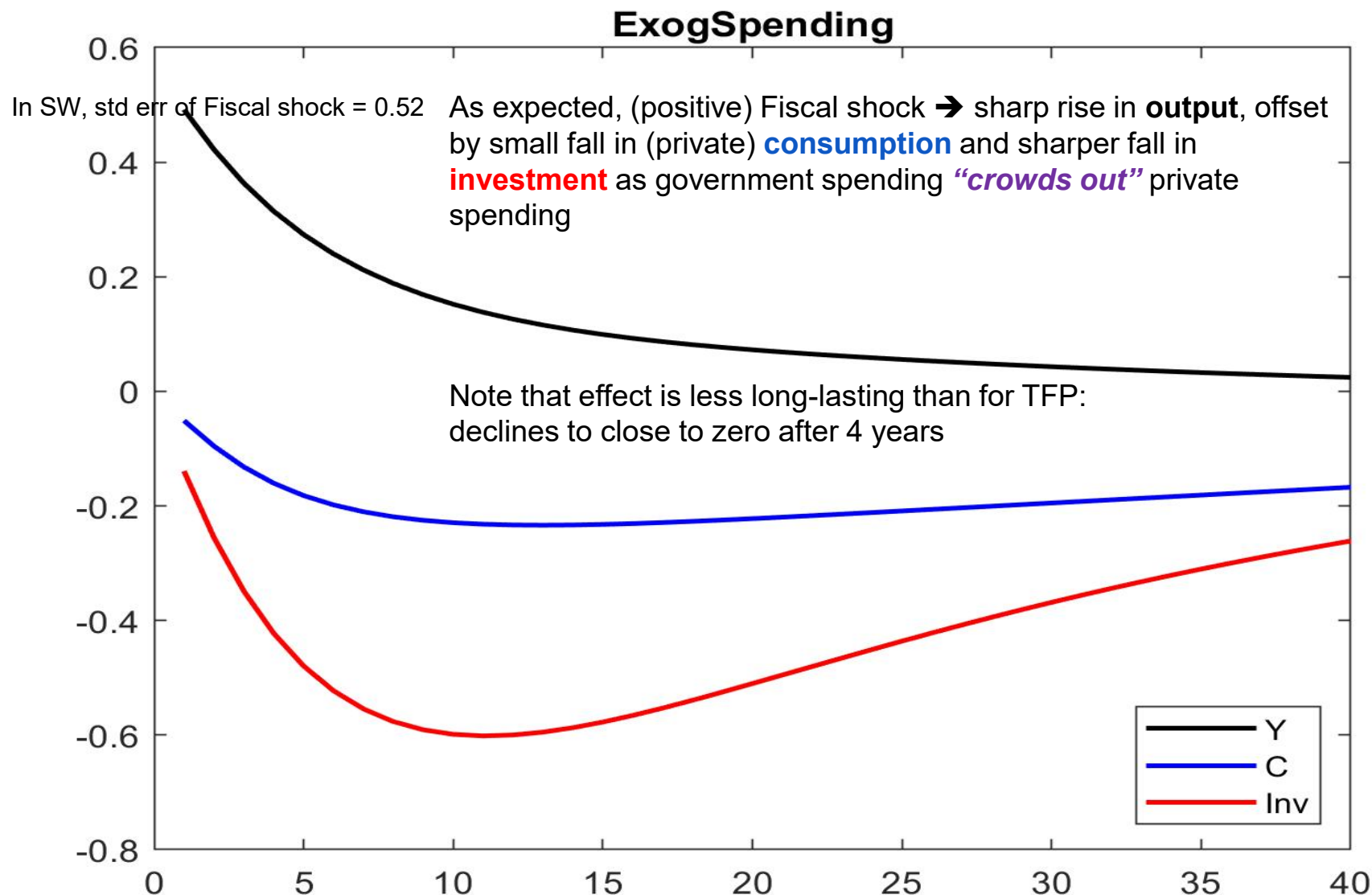


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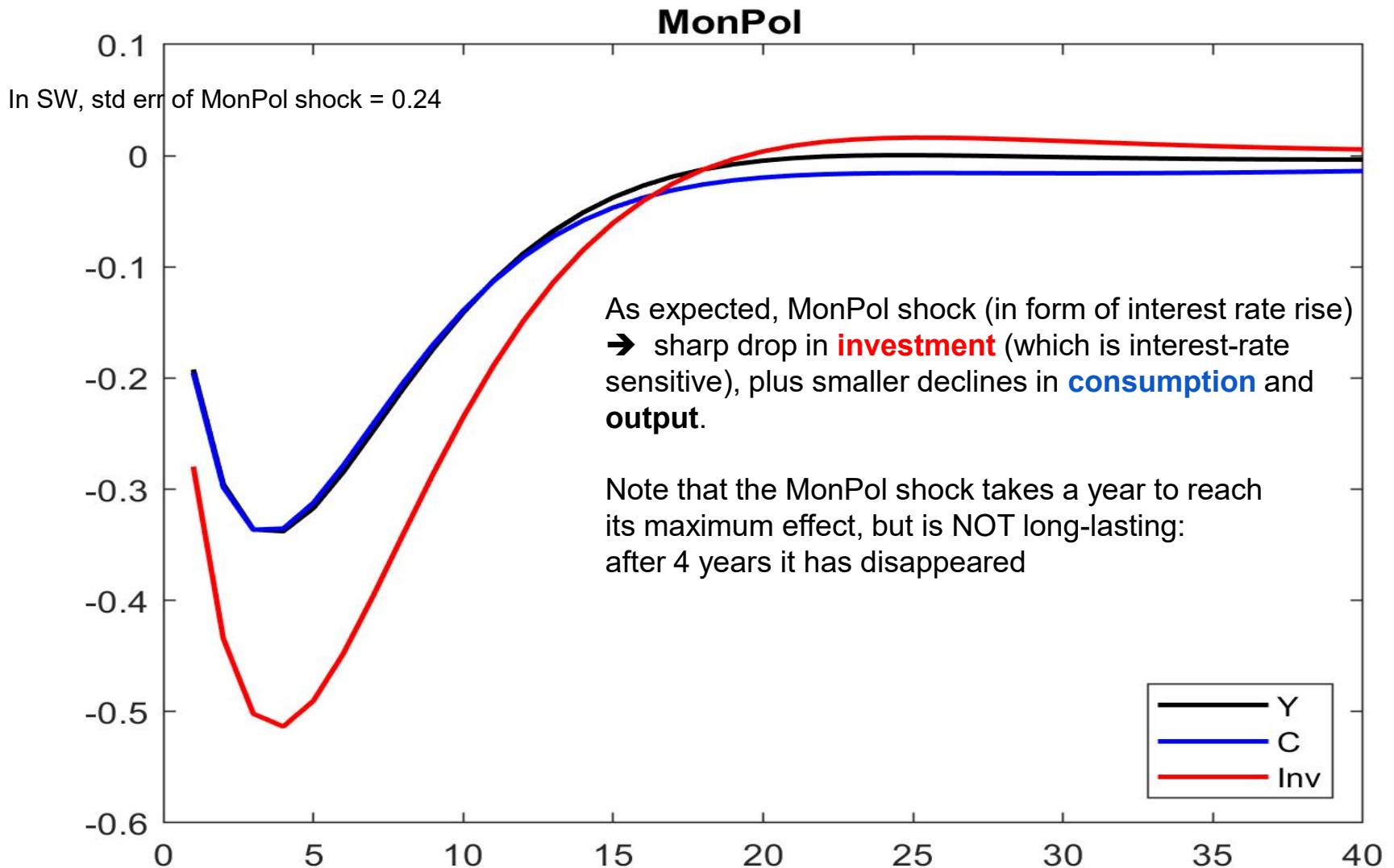
- Responses to TFP shock – Y, C, Inv



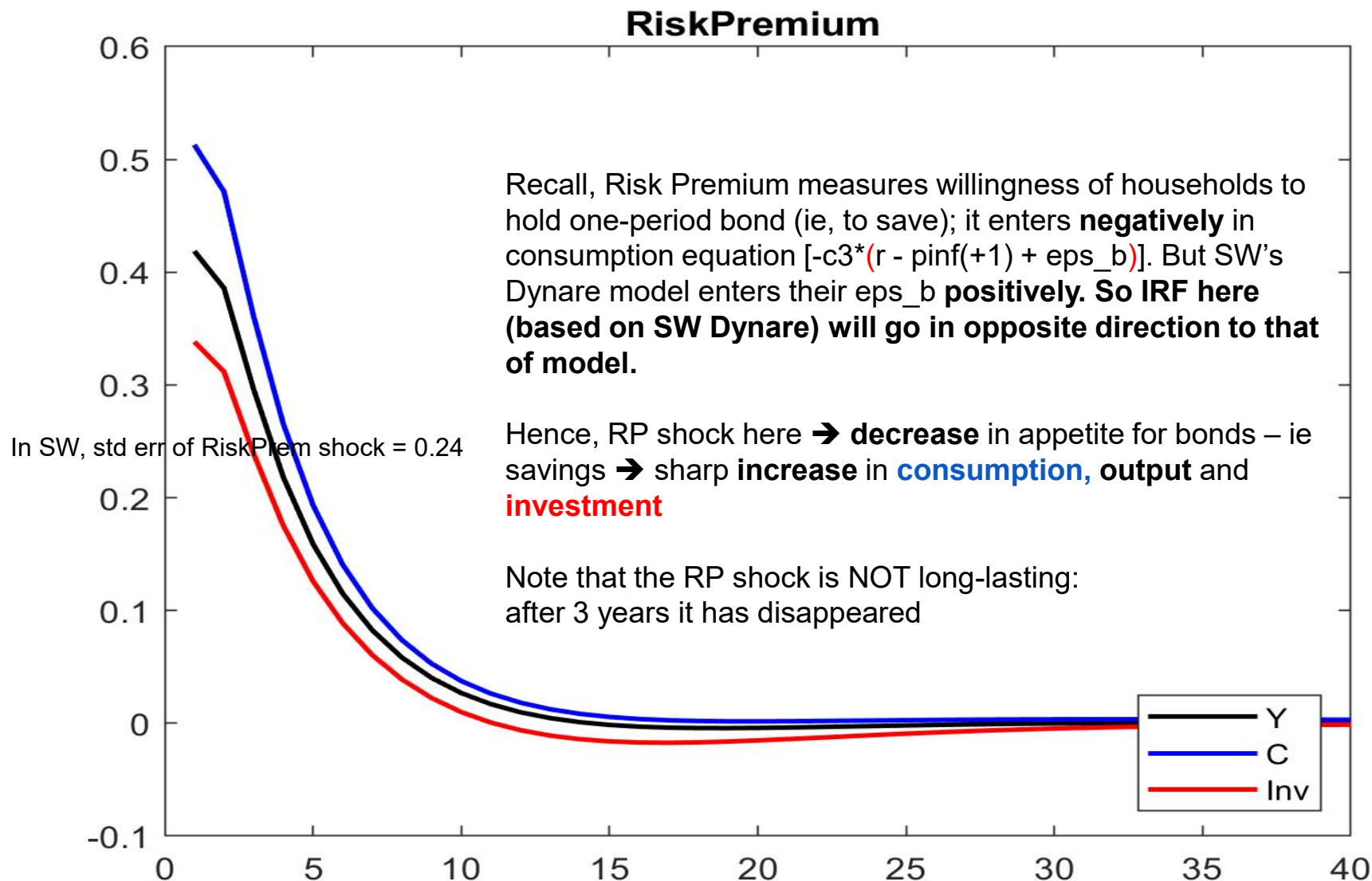
- Responses to Fiscal shock – Y, C, Inv



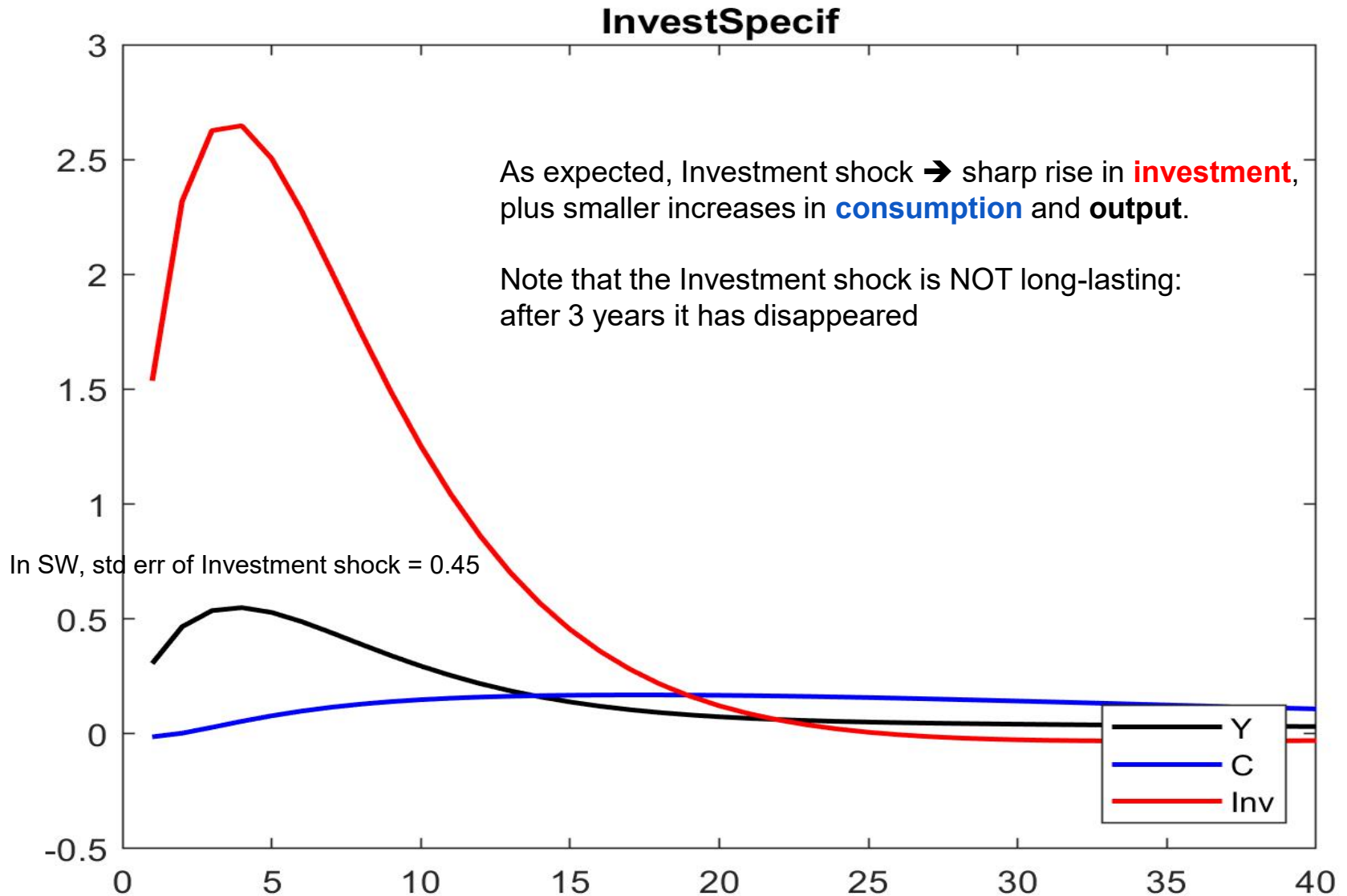
- Responses to MonPol shock – Y, C, Inv



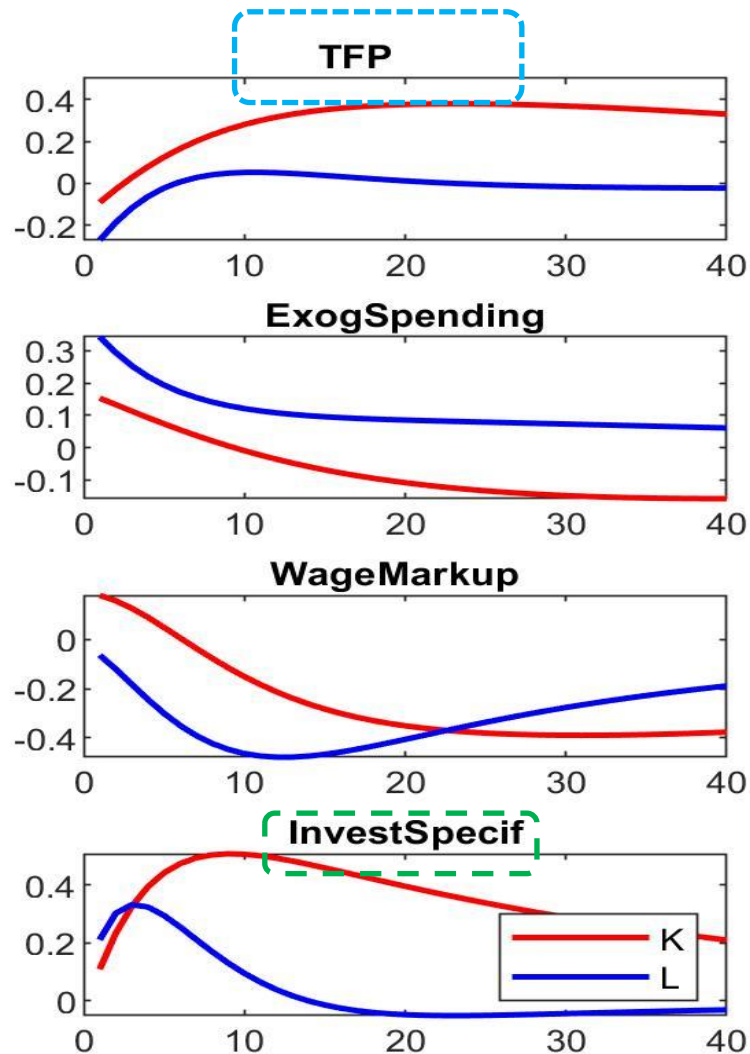
- Responses to Risk Premium shock – Y, C, Inv



- Responses to Investment shock – Y, C, Inv

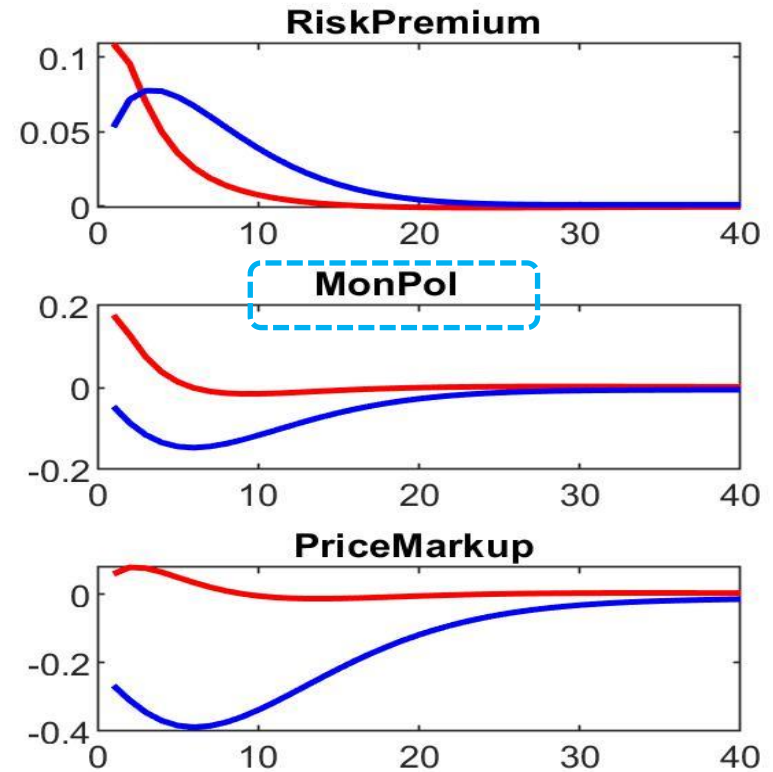
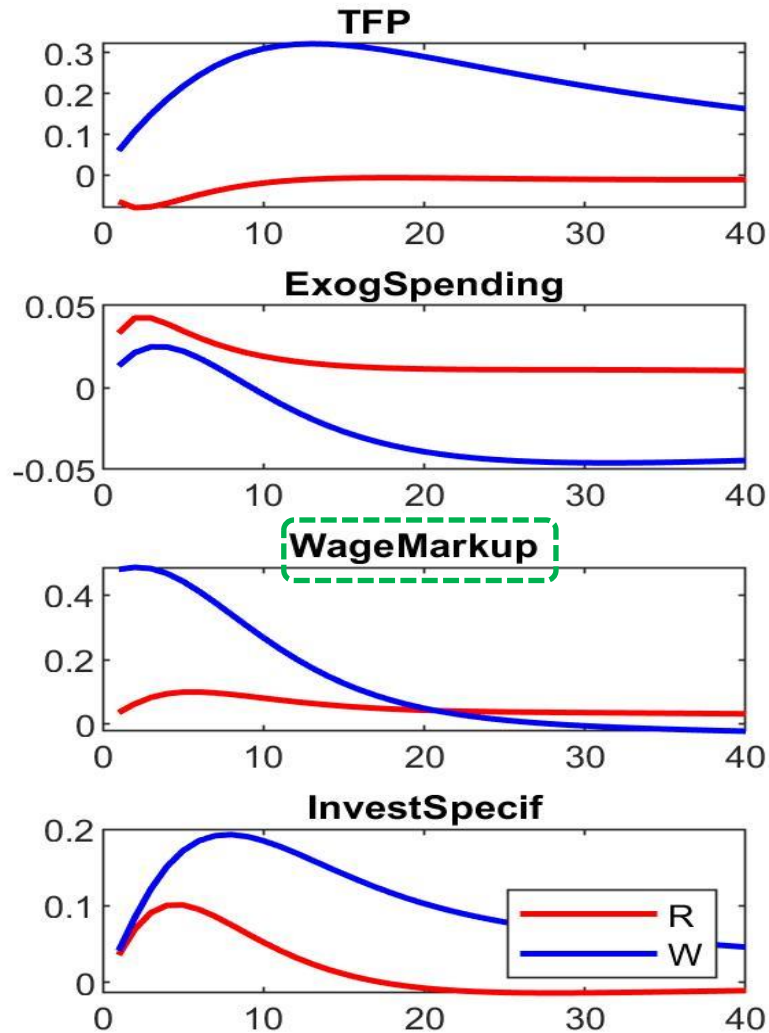


- Responses to all shocks – Capital and Labour



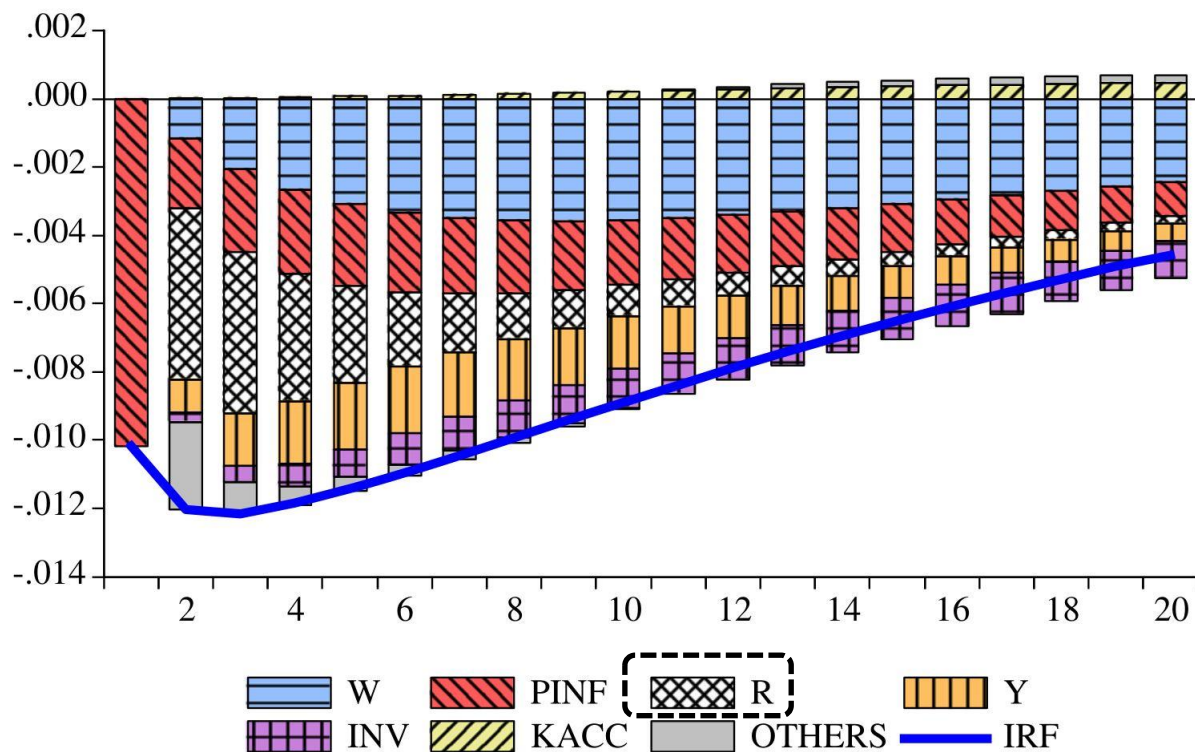
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- Responses to all shocks – Interest and Wage Rates



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- Labus and Labus (2019) construct a *decomposition* of SW *MonPol shock* effects on *inflation*:



- Note how lower *wage costs* and shrinking *demand* linked to decreasing *investments* additionally push down inflation rate beyond that caused by R

Final point

- Presently \exists growing interest in DSGE models that have ***more parameters, endogenous variables, exogenous shocks, and observables*** than SW2007 model
- and substantial ***additional complexities*** from
 - ***non-Gaussian*** distributions (“***fat tails***”)
 - incorporation of ***time-varying volatility***
- These higher-dimensional DSGE models are more ***realistic*** and potentially provide ***better statistical fit*** to observed data

- **Unfortunately**, Dynare is not currently capable of handling these more complex models
- **Fortunately**, staff at **Philadelphia Fed** have designed a user-friendly MATLAB software programme to reliably estimate high-dimensional DSGE models
- This is available freely to download, and is documented in:
- **Chib, Shin and Tan (2020)**, “High-Dimensional DSGE Models: Pointers on Prior, Estimation, Comparison, and Prediction”, **Federal Reserve Bank of Philadelphia** Working Paper 20-35 (September 2020)

- For those interested in model which Chib *et al* use to illustrate working of their software, it is:
- Leeper, Traum and Walker (2017), "Clearing Up the Fiscal Multiplier Morass", ***American Economic Review***, Vol. 107, No. 8, (2017), pp. 2409-2454
- To make this model more realistic, Chib *et al* introduce “fat-tailed” shocks and time-varying volatility
- Resulting model is ***high-dimensional***, consisting of
 - 51 parameters
 - 21 endogenous variables
 - 8 exogenous shocks
 - 8 observables
 - 1494 (!) non-Gaussian and nonlinear latent variables
- A simplified version of LTW model is available in ***Macroeconomic Model Data Base*** (version 3.1)

Appendix

- Slides below provide a more detailed exposition of the SW2007 model
- Do not be confused however by one notational difference with the main slides:
- Here, the ***shock processes*** are denoted by ε_t^x
- And the ***Kimball curvature parameters*** are denoted by $\epsilon_{w,p}$
- i.e., just the ***reverse*** of the notation in the main slides

Household Sector

- **As usual**, Household j chooses consumption $C_t(j)$, hours worked $L_t(j)$, bonds $B_t(j)$, investment $I_t(j)$ and capital utilisation $Z_t(j)$, so as to maximise an objective function, which SW define as

$$E_t \sum_{s=0}^{\infty} \beta^s \left[\frac{1}{1 - \sigma_c} (C_{t+s}(j) - \lambda C_{t+s-1})^{1 - \sigma_c} \right] \exp \left(\frac{\sigma_c - 1}{1 + \sigma_l} L_{t+s}(j)^{1 + \sigma_l} \right)$$

- Note that there is **external habit formation**, captured by the parameter λ
- σ_c is **coefficient of relative risk aversion** of households or inverse of the intertemporal elasticity of substitution
- σ_l represents inverse of the **elasticity of work effort** with respect to real wage

NB: No “leisure” shocks here !

- This particular form of non-separable utility function is chosen because ***standard, time-separable*** preferences ***cannot be made consistent*** with observation that a positive monetary policy shock should typically lead to a persistent decline in real interest rate and a hump-shaped rise in consumption
- It is a ***particular case*** of what are known as “***King-Plosser-Rebelo Preferences***” which are used in many DSGE models because they are compatible with balanced growth along the optimal steady state path

- **Footnote:**
- **Characteristic of most industrialised countries:** output per capita, consumption per capita, investment per capita and other variables exhibit **growth** over long periods of time
- These are known as the “Kaldor facts”
- Such long-run growth occurs at rates that are **roughly constant over time** within economies but differ across economies
- This pattern suggests steady state growth, which means that levels of certain key variables grow at a constant rate
- In that case, we say there is a **balanced growth path**

- General form of **KPR Preferences**:

$$u(C, L) = \frac{1}{1 - \sigma_c} C^{1 - \sigma_c} v(L)$$

- where $v(L)$ is increasing and concave if $0 < \sigma_c < 1$ or decreasing and convex if $\sigma_c > 1$
- In **limit** case of $\sigma_c = 1$, resulting preferences specification is **additively separable** and given by $u(C, L) = \ln C_t + v(L)$
- For SW, $v(L)$ is given by $\exp \left(\frac{\sigma_c - 1}{1 + \sigma_l} L_{t+s}(j)^{1 + \sigma_l} \right)$
- SW also include habit persistence in first part

- Reason why **KPR Preferences** work with a balanced growth path is that in a competitive equilibrium, marginal rate of substitution between consumption and leisure must equal (inverse of) marginal product of labour
- With KPR

$$u(c, l) = \frac{c^{1-\sigma}}{1-\sigma} v(l), \quad \sigma \neq 1, \sigma > 0$$

$$= \log c + v(l) \quad \text{for } \sigma = 1$$
- hence
- $$\frac{\partial u / \partial l}{\partial u / \partial c} = \frac{\frac{c^{1-\sigma}}{1-\sigma} v'(l)}{c^{-\sigma} v(l)} = \frac{c v'(l)}{(1-\sigma) v(l)} = MPL$$
- Along balanced growth path, MPL and c grow at same rate, so l (labour) can be constant

- Each period, household makes *sequence of decisions*
- **First**, consumption decision, capital accumulation decision, and decision on how many units of capital services to supply
- **Second**, purchases securities whose payoffs are contingent upon whether household can re-optimize its wage decision
- **Third**, sets wage rate at which it is prepared to work after finding out whether it can re-optimize or not
- **Finally**, receives lump-sum transfer from monetary authority

- Uncertainty faced by household over whether it can re-optimize its wage is *idiosyncratic*
- ➔ households work different amounts and earn *different* wage rates
- ➔ households are *heterogeneous* with respect to labour

- Household's ***budget constraint*** is

$$C_{t+s}(j) + I_{t+s}(j) + \frac{B_{t+s}(j)}{\varepsilon_t^b R_{t+s} P_{t+s}} - T_{t+s} \\ \leq \frac{B_{t+s-1}(j)}{P_{t+s}} + \frac{W_{t+s}^h(j) L_{t+s}(j)}{P_{t+s}} + \frac{R_{t+s}^k Z_{t+s}(j) K_{t+s-1}(j)}{P_{t+s}} - a(Z_{t+s}(j)) K_{t+s-1}(j) + \frac{Div_{t+s}}{P_{t+s}}$$

- One-period bond $B_{t+s}(j)$ is expressed on a discount basis, where R_{t+s} is interest rate
- T_{t+s} are lump sum taxes or subsidies
- Div_{t+s} are (per-capita) dividends distributed by intermediate goods producers and labour unions
- P_{t+s} is overall price level
- Penultimate term represents cost associated with variations in degree of ***capital utilisation***

- Households augment their financial assets through increasing their government nominal bond holdings (B_{t+1}), from which they earn an interest rate of R_t
- Return on these bonds is subject to a risk **shock** ε_t^b which may be considered as an exogenous **premium** in return to bonds, reflecting
 - inefficiencies in financial sector (leading to some premium on deposit rate versus risk-free rate set by central bank), or
 - a risk premium that households require to hold one period bond
- and
$$\ln \varepsilon_t^b = \rho_b \ln \varepsilon_{t-1}^b + \eta_t^b, \eta_t^b \sim N(0, \sigma_b)$$

- Investment is assumed by SW to augment household's physical capital stock according to

$$K_t(j) = (1 - \delta)_{t-1}(j) + \varepsilon_t^i \left[1 - S \left(\frac{I_t(j)}{I_{t-1}(j)} \right) \right] I_t(j)$$

- δ is depreciation rate
- $S(.)$ is **adjustment cost** function $S(x_t) = \frac{\varphi}{2} (x_t - \gamma)^2$
- ε_t^i is a stochastic **shock** to price of investment (relative to consumption goods) and follows an exogenous process

$$\ln \varepsilon_t^i = \rho_i \ln \varepsilon_{t-1}^i + \eta_t^i, \eta_t^i \sim N(0, \sigma_i)$$

- Households choose utilisation rate of capital
- Amount of ***effective capital*** that households can rent to firms is

$$K_t^s(j) = Z_t(j)K_{t-1}(j)$$

- Income from ***renting capital services*** is

$$R_t^k Z_t(j) K_{t-1}(j)$$

- Cost of changing ***capital utilisation*** is given by

$$Z_t = \kappa + R_t^k(1 - \psi)/\psi$$

- where ψ is normalized to be between zero and one
- κ is a constant

- When $\psi = 1$, it is extremely costly to change utilisation of capital and as a result capital utilisation remains constant
- In contrast, when $\psi = 0$, marginal cost of changing capital utilisation is constant \rightarrow in equilibrium rental rate on capital is constant

- FOCs for Household are

$$(\partial C_t) \quad \Xi_t = \exp \left(\frac{\sigma_c - 1}{1 + \sigma_l} L_t(j)^{1 + \sigma_l} \right) (C_t - \lambda C_{t-1})^{-\sigma_c}$$

$$(\partial L_t) \quad \left[\frac{1}{1 - \sigma_c} (C_t - h C_{t-1})^{1 - \sigma_c} \right] \exp \left(\frac{\sigma_c - 1}{1 + \sigma_l} L_t^{1 + \sigma_l} \right) (\sigma_c - 1) L_t^{\sigma_l} = -\Xi_t \frac{W_t^h}{P_t}$$

$$(\partial B_t) \quad \Xi_t = \beta \varepsilon_t^b R_t E_t \left[\frac{\Xi_{t+1}}{\pi_{t+1}} \right]$$

Lagrange
Multiplier

$$(\partial I_t) \quad \Xi_t = \Xi_t^k \varepsilon_t^i \left(1 - S\left(\frac{I_t}{I_{t-1}}\right) - S'\left(\frac{I_t}{I_{t-1}}\right) \frac{I_t}{I_{t-1}} \right) + \beta E_t \left[\Xi_{t+1}^k \varepsilon_{t+1}^i S'\left(\frac{I_{t+1}}{I_t}\right) \left(\frac{I_{t+1}}{I_t}\right)^2 \right]$$

$$(\partial \bar{K}_t) \quad \Xi_t^k = \beta E_t \left[\Xi_{t+1} \left(\frac{R_{t+1}^k}{P_{t+1}} Z_{t+1} - a(Z_{t+1}) \right) + \Xi_{t+1}^k (1 - \delta) \right]$$

$$(\partial u_t) \quad \frac{R_t^k}{P_t} = a'(Z_t) \quad \text{Tobin's } Q_t = \Xi_t^k / \Xi_t$$

$$\begin{aligned}\Xi_t = & \Xi_t^k \varepsilon_t^i \left(1 - S\left(\frac{I_t}{I_{t-1}}\right) - S'\left(\frac{I_t}{I_{t-1}}\right) \frac{I_t}{I_{t-1}} \right) \\ & + \beta E_t \left[\Xi_{t+1}^k \varepsilon_{t+1}^i S'\left(\frac{I_{t+1}}{I_t}\right) \left(\frac{I_{t+1}}{I_t}\right)^2 \right]\end{aligned}$$

- FOC for I is law of motion for shadow value of capital
- If adjustment cost were absent ($S=0$), FOC would simply say that $\Xi_t = \Xi_t^k \rightarrow$ marginal utility of consumption (Ξ_t) [shadow **cost** of taking resources away from consumption] = shadow **benefit** of putting these resources into investment (Ξ_t^k): Tobin's Q is one

$$\Xi_t^k = \beta E_t \left[\Xi_{t+1} \left(\frac{R_{t+1}^k}{P_{t+1}} Z_{t+1} - a(Z_{t+1}) \right) + \Xi_{t+1}^k (1 - \delta) \right]$$

- FOC for K says that if you buy a unit of capital today you have to pay its price in real terms (LHS: Ξ_t^k)
- but tomorrow (RHS)
- you will get proceeds from renting capital (first part)
- and you can sell back any capital that has not depreciated (second part)

Intermediate Goods Sector

- ***Intermediate Good Producer i*** is assumed by SW to use standard Cobb-Douglas technology (plus a fixed cost)

$$Y_t(i) = \epsilon_t^a K_t^s(i)^\alpha [\gamma^t L_t(i)]^{1-\alpha} - \gamma^t \Phi$$

- where
 - $K_t^s(i)$ is capital services used in production
 - $L_t(i)$ is a composite labour input
 - Φ is a fixed cost
 - γ^t represents a labour-augmenting deterministic growth rate in the economy

- ϵ_t^a is ***total factor productivity*** and follows process

$$\ln \epsilon_t^a = (1 - \rho_z) \ln \epsilon^a + \rho_z \ln \epsilon_{t-1}^a + \eta_t^a, \eta_t^a \sim N(0, \sigma_a)$$

- Firm's profit is given by

$$P_t(i) Y_t(i) - W_t L_t(i) - R_t^k K_t^s(i)$$

- where
 - W_t is aggregate nominal wage rate
 - R_t^k is rental rate on capital
- Cost minimization yields following first-order conditions:

$$(\partial L_t(i)) : \Theta_t(i) \gamma^{(1-\alpha)t} (1 - \alpha) \epsilon_t^a K_t^s(i)^\alpha L_t(i)^{-\alpha} = W_t$$

$$(\partial K_t^s(i)) : \Theta_t(i) \gamma^{(1-\alpha)t} \alpha \epsilon_t^a K_t^s(i)^{\alpha-1} L_t(i)^{1-\alpha} = R_t^k$$

- where $\Theta_t(i)$ is Lagrange multiplier associated with production function (and equals marginal cost MC_t)
- Combining these FOCs and noting that capital-labour ratio is equal across firms implies (as usual)

$$K_t^s = \frac{\alpha}{1 - \alpha} \frac{W_t}{R_t^k} L_t$$

- Marginal cost MC_t is ***same for all firms*** and equal to

$$MC_t = \alpha^{-\alpha} (1 - \alpha)^{-(1-\alpha)} W_t^{1-\alpha} R_t^{k\alpha} \gamma^{-(1-\alpha)t} (\epsilon_t^a)^{-1}$$

- Prices of ***intermediate goods*** are determined by Calvo fairy
- As in previous Calvo models, in each period, each firm i faces a constant probability $1-\xi_p$ of being able to re-optimize its price $P_t(i)$
- Probability that any firm receives a signal to re-optimize its price is assumed to be ***independent*** of time that it last reset its price

- Unlike in simpler earlier models, SW assume that if Calvo fairy does not allow a firm to optimise its price in a given period, it then adjusts its price by a ***weighted combination*** of lagged and steady-state inflation rates as:

$$P_t(i) = (\pi_{t-1})^{\iota_p} (\pi_*)^{1-\iota_p} P_{t-1}(i)$$

- where
 - $0 \leq \iota_p \leq 1$
 - π_{t-1} denotes gross inflation in period $t-1$
 - π_* denotes steady-state gross inflation rate
- A positive value of ι_p introduces ***structural inertia*** into inflation process

- Under Calvo pricing with this type of ***partial indexation***, optimal price $\tilde{P}_t(i)$ set by firm that is allowed by Calvo fairy (with a probability $1-\xi_p$) to re-optimize results from solving following optimisation problem:

$$\max_{\tilde{P}_t(i)} E_t \sum_{s=0}^{\infty} \xi_p^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \left[\tilde{P}_t(i) (\Pi_{l=1}^s \pi_{t+l-1}^{\ell_p} \pi_*^{1-\ell_p}) - MC_{t+s} \right] Y_{t+s}(i)$$

Nominal discount factor

$$s.t. Y_{t+s}(i) = Y_{t+s} G'^{-1} \left(\frac{P_t(i) X_{t,s}}{P_{t+s}} \tau_{t+s} \right)$$

- This looks formidable!
- But if we compare it to what we used in very first sticky-price model studied back at beginning of term, we find clear similarities:

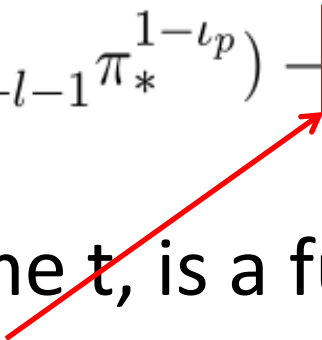
$$\max_{P_{j,t}^*} E_t \sum_{i=0}^{\infty} (\beta\theta)^i (P_{j,t}^* Y_{j,t+i} - TC_{j,t+i})$$

$$\max_{\tilde{P}_t(i)} E_t \sum_{s=0}^{\infty} \xi_p^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \left[\tilde{P}_t(i) (\Pi_{l=1}^s \pi_{t+l-1}^{\ell_p} \pi_*^{1-\ell_p}) - MC_{t+s} \right] Y_{t+s}(i)$$

- Since $TC = MC * Y$, only really ***new element*** is ***indexation factor*** multiplying P_t^{\sim} : $(\Pi_{l=1}^s \pi_{t+l-1}^{\ell_p} \pi_*^{1-\ell_p})$

- What is this indexation factor $(\prod_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p})$
- Since π is a **gross** inflation rate, it has each period a value close to 1
- Thus, product $\prod_{l=1}^s \pi_{t+l-1}^{\iota_p}$ (which multiplies together inflation rates for all periods since previous price revision) also has a value near 1
- This is then scaled by factor $\pi_*^{1-\iota_p}$ which reflects **weighted combination** indexation
- Finally, as noted previously, curious expression $\frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}}$ is just the (nominal) discount factor

- It is also clear from formulation of the optimisation problem

$$\max_{\tilde{P}_t(i)} E_t \sum_{s=0}^{\infty} \xi_p^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \left[\tilde{P}_t(i) (\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p}) - MC_{t+s} \right] Y_{t+s}(i)$$


- that price set by firm i , at time t , is a function of ***expected future*** marginal costs
- Price will be a ***mark-up*** over these weighted marginal costs
 - ***fixed*** over time if prices are perfectly flexible ($\iota_p = 0$), or
 - ***varying*** with sticky prices ($\iota_p \neq 0$)

- Given FOC for this optimisation problem, ***aggregate price index*** which results is

$$P_t = (1 - \xi_p) P_t(i) G'^{-1} \left[\frac{P_t(i) \tau_t}{P_t} \right] + \xi_p \pi_{t-1}^{\iota_p} \pi_*^{1-\iota_p} P_{t-1} G'^{-1} \left[\frac{\pi_{t-1}^{\iota_p} \pi_*^{1-\iota_p} P_{t-1} \tau_t}{P_t} \right]$$

- Compare this to what we used in very first sticky-price model studied back at beginning of term:

$$P_t = \left[\theta P_{t-1}^{1-\psi} + (1 - \theta) P_t^*{}^{1-\psi} \right]^{\frac{1}{1-\psi}}$$

- Function G (of which G' is used in definition above) is defined in Appendix

Final Good Sector

- As usual, ***final good*** Y_t is a composite made of a continuum of ***intermediate goods*** $Y_t(i)$
- ***Final Good Producers*** buy intermediate goods and package them into Y_t using not the Dixit-Stiglitz aggregator but instead the ***Kimball Aggregator***
- In contrast to Dixit–Stiglitz world of a constant elasticity and a ***constant*** desired mark-up of price over marginal cost, in Kimball's world desired mark-up is ***decreasing in a firm's relative price***
- Final Good Producers sell final good to consumers, investors and government in a perfectly competitive market

- Final Good Producers maximize profits

$$\max_{Y_t, Y_t(i)} P_t Y_t - \int_0^1 P_t(i) Y_t(i) di$$

- subject to Kimball Aggregator

$$\left[\int_0^1 G \left(\frac{Y_t(i)}{Y_t}; \varepsilon_t^p \right) di \right] = 1 \quad (\mu_{f,t})$$

- where P_t and $P_t(i)$ are price of final and intermediate goods respectively
- G is a ***strictly concave*** and increasing function (ie, ***decreasing slope G'***) characterised by $G(1) = 1$
- ε_t^p is an exogenous process that reflects ***shocks*** to aggregator function that result in changes in elasticity of demand and thus in mark-up

- ε_t^p follows an ARMA(1,1) process defined by

$$\varepsilon_t^p = \rho_p \varepsilon_t^p + \eta_t^p - \mu_p \eta_{t-1}^p$$

- Combining first-order conditions with respect to $Y_t(i)$ and Y_t results in

$$Y_t(i) = Y_t G'^{-1} \left[\frac{P_t(i)}{P_t} \int_0^1 G' \left(\frac{Y_t(i)}{Y_t} \right) \frac{Y_t(i)}{Y_t} di \right]$$

- Assumptions on G (specifically, G' decreasing) imply that demand for input $Y_t(i)$ is ***decreasing*** in its relative price
- while elasticity of demand is a positive function of relative price (or a negative function of relative output)

- For completeness, note that G is defined as

$$G_Y \left(\frac{Y_t(f)}{Y_t} \right) = \frac{\phi_t^p}{1 - (\phi_t^p - 1)\epsilon_p} \left[\left(\frac{\phi_t^p + (1 - \phi_t^p)\epsilon_p}{\phi_t^p} \right) \frac{Y_t(f)}{Y_t} + \frac{(\phi_t^p - 1)\epsilon_p}{\phi_t^p} \right] \frac{1 - (\phi_t^p - 1)\epsilon_p}{\phi_t^p - (\phi_t^p - 1)\epsilon_p} + \left[1 - \frac{\phi_t^p}{1 - (\phi_t^p - 1)\epsilon_p} \right]$$

- $\phi_t^p \geq 1$ is gross mark-up of the intermediate firms
- ϵ_p is degree of **curvature** of firm's demand (do not confuse this with ε_t^p , the price shock !!)

- When **curvature parameter** $\epsilon_p = 0$, demand curve exhibits **constant elasticity** as with standard Dixit-Stiglitz aggregator
- When ϵ_p is positive, firm instead faces a **quasi-kinked** demand curve, implying that a drop in its relative price only stimulates a **small increase** in demand
- On other hand, a rise in its relative price generates a **large fall** in demand
- Relative to standard Dixit-Stiglitz aggregator, this introduces more **strategic complementarity** in price setting which causes intermediate firms to **adjust** prices **less** to a given change in marginal cost (hence “stickier”)

- Let us now return to solution of the FOCs, given previously by

$$Y_t(i) = Y_t G'^{-1} \left[\frac{P_t(i)}{P_t} \int_0^1 G' \left(\frac{Y_t(i)}{Y_t} \right) \frac{Y_t(i)}{Y_t} di \right]$$

- Recall that $G'^{-1} \left[\frac{P_t(i)\tau_t}{P_t} \right]$ was used in defining P_t
- Definition of τ_t is $\tau_t = \int_0^1 G' \left(\frac{Y_t(i)}{Y_t} \right) \frac{Y_t(i)}{Y_t} di$
- So in fact we have simply that
- $Y_t(i)/Y_t = G'^{-1} \left[\frac{P_t(i)\tau_t}{P_t} \right]$
- which is why we earlier stated that demand for input $Y_t(i)$ is ***decreasing*** in its relative price

- When all intermediate firms produce the same amount, we have $\frac{Y_t(i)}{Y_t} = 1$
- But $G_Y(1) = 1$, implying **constant returns to scale** in this case

Labour Sector

- SW assume that Households supply their homogenous labour to an ***intermediate labour union*** which differentiates labour services, sets wages subject to a Calvo scheme and offers those labour services to intermediate ***labour packers***
- Labour used by intermediate goods producers L_t is a composite made of those differentiated labour services $L_t(i)$
- As with intermediate goods, ***Kimball aggregator*** is used, with a ***curvature parameter*** of ϵ_w

- Labour packers buy differentiated labour services, package L_t , and offer it to intermediate goods producers
- Labour packers maximize profits

$$\begin{aligned} \max_{L_t, L_t(i)} \quad & W_t L_t - \int_0^1 W_t(i) L_t(i) di \\ \text{s.t.} \quad & \left[\int_0^1 H \left(\frac{L_t(i)}{L_t}; \varepsilon_t^w \right) di \right] = 1 \quad (\mu_{l,t}) \end{aligned}$$

- where W_t and $W_t(i)$ are prices of composite and intermediate labour services respectively
- ***Like G previously***, function H is a strictly concave and increasing function characterised by $H(1) = 1$

- ε_t^w is an exogenous process that reflects **shocks** to aggregator function that result in changes in elasticity of demand and therefore in mark-up [do not confuse this with ϵ_w - the curvature !!]
- Similarly to ε_t^p , it is assumed that ε_t^w follows an ARMA(1,1) process

$$\varepsilon_t^w = \rho_w \varepsilon_{t-1}^w + \eta_t^w - \mu_w \eta_{t-1}^w$$

- Combining FOCs results (as for $Y_t(i)$ previously) in

$$L_t(i) = L_t H'^{-1} \left[\frac{W_t(i)}{W_t} \int_0^1 H' \left(\frac{L_t(i)}{L_t} \right) \frac{L_t(i)}{L_t} di \right]$$

- Labour unions are an intermediary between households and labour packers
- Under Calvo pricing with partial indexation, optimal wage set by union that is allowed to re-optimize its wage results from an optimisation problem similar to that for pricing

$$\max_{\widetilde{W}_t(i)} E_t \sum_{s=0}^{\infty} \xi_w^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \left[\widetilde{W}_t(i) (\Pi_{l=1}^s \gamma \pi_{t+l-1}^{\iota_w} \pi_*^{1-\iota_w}) - W_{t+s}^h \right] L_{t+s}(i)$$

$$s.t. \ L_{t+s}(i) = L_{t+s} H'^{-1} \left(\frac{W_t(i) X_{t,s}^w}{W_{t+s}} \tau_{t+s}^w \right)$$

- Again, similarly to case for pricing, ***aggregate wage index*** is in this case given by

$$W_t = (1 - \xi_w) \widetilde{W}_t H'^{-1} \left[\frac{\widetilde{W}_t \tau_t^w}{W_t} \right] + \xi_w \gamma \pi_{t-1}^{\ell_w} \pi_*^{1-\ell_w} W_{t-1} H'^{-1} \left[\frac{\gamma \pi_{t-1}^{\ell_w} \pi_*^{1-\ell_w} W_{t-1} \tau_t^w}{W_t} \right]$$

- ***Mark-up*** of aggregate wage over wage received by households is distributed to households in form of ***dividends*** (as already indicated in budget constraint)

Government Sector

- SW (both of whom work for central banks) assume that central bank follows a nominal interest rate rule by adjusting its instrument in response to deviations of inflation and output from their respective target levels

$$\frac{R_t}{R^*} = \left(\frac{R_{t-1}}{R^*} \right)^\rho \left[\left(\frac{\pi_t}{\pi_*} \right)^{r_\pi} \left(\frac{Y_t}{Y_t^*} \right)^{r_y} \right]^{1-\rho} \left(\frac{Y_t/Y_{t-1}}{Y_t^*/Y_{t-1}^*} \right)^{r_{\Delta y}} \epsilon_t^r$$

- where
 - R^* is steady state nominal rate (gross rate)
 - Y_t^* is “***natural output***”
 - ρ determines degree of interest rate smoothing
- Exogenous monetary policy ***shock*** ϵ_t^r is determined as

$$\ln \epsilon_t^r = \rho_r \ln \epsilon_{t-1}^r + \eta_t^r, \eta_t^r \sim N(0, \sigma_r)$$

- A new element is included in this version of Taylor Rule: “***natural output level***”
- SW define it as “the output in the flexible price and wage economy without mark-up shock in prices and wages”, ie ***NOT*** subject to sticky wages and prices
- In model eventually estimated below, there will be an ***entire section*** devoted to a “natural” or “***flex-price***” version of the economy, ***ONLY*** so as to be able to make use of the central bank reaction function defined by SW version of Taylor Rule – ***no need for this otherwise***

- Government budget constraint is of form

$$P_t G_t + B_{t-1} = T_t + \frac{B_t}{R_t}$$

- where T_t are nominal lump-sum taxes (or subsidies) that also appear in household's budget constraint
- SW assume that Government purchases G_t are exogenous
- They express ***government spending*** relative to steady state output path (“trend output”) as

$$\varepsilon_t^g = G_t / (Y \gamma^t)$$

- Path of ε_t^g is assumed to be given by

$$\ln \varepsilon_t^g = (1 - \rho_g) \ln \varepsilon^g + \rho_g \ln \varepsilon_{t-1}^g + \rho_{ga} \ln \varepsilon_t^a - \rho_{ga} \ln \varepsilon_{t-1}^a + \eta_t^g, \eta_t^g \sim N(0, \sigma_g)$$

- This allows for a reaction of government spending to productivity process γ^t
- Government purchases have no effect on marginal utility of private consumption, nor do they serve as an input into goods production

Market Equilibrium

- Finally, ***Market equilibrium***
- Final goods market is in equilibrium if production equals demand by households for consumption and investment and by government

$$C_t + I_t + G_t + a(Z_t)K_{t-1} = Y_t$$

- ***Capital rental market*** is in equilibrium when demand for capital by intermediate goods producers equals supply by households
- ***Labour market*** is in equilibrium if firms' demand for labour equals labour supply at wage level set by households
- In capital market, equilibrium means that ***government debt*** is held by domestic investors at market interest rate, R_t

Recall: General NK model introduced into basic RBC more realistic specifications:

- **imperfect** competition \Rightarrow *Last week*
- **frictions** in **prices** and in **wages** \Rightarrow *Last week*
- **habit** formation in consumption \Rightarrow *Last week*
- **non-Ricardian** agents \Rightarrow *Last week*
- **adjustment costs** in investment \Rightarrow *Last week*
- **capacity** (non-)utilisation **costs** \Rightarrow *Last week*
- **government** sector (**monetary** and **fiscal**) \Rightarrow *Last week*
- **financial** sector \Rightarrow *Week 6*
- **housing** sector \Rightarrow *Week 6*
- **unemployment** \Rightarrow *Week 7*
- **environment** \Rightarrow *Week 7*
- **foreign** trade \Rightarrow *Week 8*