Macroeconomics A: Review Session I

Useful Approximations

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Outline

1. **Taylor Expansion**
   - One variable
   - Two Variables
2. **Log-Linearization**
   - Sustitution Method
   - Via a Taylor Series Approximation

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One Variable

Taylor expansion for one variable around the point $x \approx a$

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + ... + \frac{1}{n!}f^{(n)}(a)(x-a)^n$$

where $f(x) = P_n(x) + R_n(x)$ and $f(x)$ is differentiable $n+1$ times

Here $P_n(x)$ is the approximation of $f(x)$, with a remainder $R_n(x)$

Exercise: find the 4^{th} order Taylor approximation of $f(x) = \log(x)$ near the point $a = 1$

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Exercise: find the 4^{th} order Taylor approximation of $f(x) = \log(x)$ near the point $a = 1$

$$P_4(x) = 0 + (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4$$

Taking $x = 1.1$ gives $P_4(x) \approx 0.09531 \approx \log(1.1)$

Two variables

For two variables around the points $x \approx x_0$ and $y \approx y_0$

$$P_n(x,y) = f(x_0,y_0) + f_x(x_0,y_0)\delta x + f_y(x_0,y_0)\delta y + ...$$
$$\frac{1}{2}\left[f_{xx}(...)(\delta x)^2 + 2f_{xy}(...)\delta x\delta y + f_{yy}(...)(\delta y)^2\right] + ...$$
$$\frac{1}{3!}\left[f_{xxx}\cdot(\delta x)^3 + 3f_{xxy}\cdot(\delta x)^2\delta y + 3f_{xyy}\cdot\delta x(\delta y)^2 + f_{yyy}\cdot(\delta y)^3\right] + ...$$

where $\delta x = (x - x_0)$ and $\delta y = (y - y_0)$

Exercise: find the 1^{st} order Taylor approximation of $Q = AK^{\alpha}L^{1-\alpha}$ around the points $K_0 = 3$ and $L_0 = 7$

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Exercise: find the 1^{st} order Taylor approximation of $Q = AK^{\alpha}L^{1-\alpha}$ around the points $K_0 = 3$ and $L_0 = 7$

$$P_1(K,L) = Q_0 + \frac{\alpha}{3}Q_0(K-3) + \frac{1-\alpha}{7}Q_0(L-7)$$

Uses and Intuition

Where we use Taylor expansions:

- Approximating complicated functions
- Root finding (Newton's method)
- Simplifying models

I can't give more intuition, but the following video is great:

3Blue1Brown

The materials here largely come from Alecos Papadopoulos' blog:

Economics for good... and econometrics forever

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Log-Linearization

Whenever we see a non-linear function, there is the temptation to linearize it by taking logs

$$Q_t = AK_t^{\alpha} L_t^{1-\alpha} \iff \log(Q) = \log(A) + \alpha \log(K_t) + (1-\alpha)\log(L_t)$$

From this, we could try to estimate α through linear regression, e.g.

$$\log(Q_t) = \beta_0 + \beta_1 \log(K_t) + \beta_2 \log(L_t)$$

This is one of many useful applications.

Furthermore, if we consider deviations from the steady state values, we don't have to worry about the constant A:

$$\widetilde{Q}_t = \alpha \widetilde{K}_t + (1-\alpha)\widetilde{L}_t \quad \text{where} \quad \widetilde{x}_t = \log(x_t) - \log(x_0)$$

Approximating Percent Change

Take that

$$\widetilde{x}_t = \log(x_t) - \log(x_0)$$

It is straightforward to show this close to a percent deviation, i.e.

$$\widetilde{x}_t \approx \frac{x_t - x_0}{x_0}$$

Take a 1^{st} order Taylor approximation of $\widetilde{x}_t$ around the point $x_t = x_0$

$$\log\left(\frac{x_t}{x_0}\right) \approx \log\left(\frac{x_0}{x_0}\right) + \frac{1}{x_0}(x_t - x_0)$$

That's it!

Log-Linearization and Addition

Also note that

$$\tilde{x}_t \approx \frac{x_t - x_0}{x_0} \iff x_t \approx x_0(1 + \tilde{x}_t)$$

or for exponents $\quad x_t^\alpha \approx x_0^\alpha(1 + \alpha\tilde{x}_t) \quad$ (hint: $x_t^\alpha = x_0^\alpha e^{\alpha\tilde{x}_t}$)

Therefore, we can write

$$Y_t = C_t + I_t \iff Y_0\left(1 + \tilde{Y}_t\right) = C_0\left(1 + \tilde{C}_t\right) + I_0\left(1 + \tilde{I}_t\right)$$

Since $Y_0 = C_0 + I_0$, we can cancel terms so that

$$Y_0\tilde{Y}_t = C_0\tilde{C}_t + I_0\tilde{I}_t \iff \tilde{Y}_t = \frac{C_0}{Y_0}\tilde{C}_t + \frac{I_0}{Y_0}\tilde{I}_t$$

Multiplication and Division

Take that

$$\frac{x_t}{y_t} = \frac{x_0 e^{\widetilde{x}_t}}{y_0 e^{\widetilde{y}_t}} = \frac{x_0}{y_0} e^{\widetilde{x}_t} e^{-\widetilde{y}_t}$$

Now let's find 1^{st} order Taylor approximation for two variables at the point where $\widetilde{x}_0 = 0$ and $\widetilde{y}_0 = 0$

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$$\begin{aligned} \frac{x_t}{y_t} &\approx \frac{x_0}{y_0} + \frac{x_0}{y_0} e^0 (\widetilde{x}_t - 0) - \frac{x_0}{y_0} e^0 (\widetilde{y}_t - 0) \\ &\approx \frac{x_0}{y_0} \left(1 + \widetilde{x}_t - \widetilde{y}_t\right) \end{aligned}$$

Dealing with Constants

If you want to log-linearize an equation with a constant

$$x_t + a = b y_t^{\alpha}$$

Just replace with a new variable

$$z_t = x_t + a \implies \tilde{z}_t = \alpha \tilde{y}_t$$

Where

$$\tilde{z}_t \approx \frac{(x_t + a) - (x_0 + a)}{x_0 + a}$$

Since

$$\tilde{x}_t \approx \frac{x_t - x_0}{x_0} \implies \tilde{x}_t = \tilde{z}_t \frac{x_0 + a}{x_0}$$

Wrapping Up Log-Linearization

Before we saw that $Q = AK^{\alpha}L^{1-\alpha}$ can be written

$$\widetilde{Q}_t = \alpha\widetilde{K}_t + (1-\alpha)\widetilde{L}_t$$

If we start with

$$Q_0(1+\widetilde{Q}_t) = AK_0^{\alpha}(1+\alpha\widetilde{K}_t)L_0^{1-\alpha}(1+(1-\alpha)\widetilde{L}_t)$$

We can divide through by the original expression, so that

$$\widetilde{Q}_t = \alpha\widetilde{K}_t + (1-\alpha)\widetilde{L}_t + \alpha(1-\alpha)\widetilde{K}_t\widetilde{L}_t$$

We are left with the term $\widetilde{K}_t\widetilde{L}_t$, but it is small and can be ignored

Simplifying Models Using Taylor Series

Often in economics, we have a law of motion

$$x_{t+1} = f(x_t)$$

Using a Taylor series approximation around the steady state $x_t = x_0$

$$x_{t+1} \approx f(x_0) + f'(x_0)(x_t - x_0)$$

In the steady state, $x_0 = f(x_0)$, so we can write

$$\frac{x_{t+1} - x_0}{x_0} \approx f'(x_0)\frac{x_t - x_0}{x_0}$$

In other words

$$\widetilde{x}_{t+1} \approx f'(x_0)\widetilde{x}_t$$

Finding Deviations in Steady State Capital

Let's assume capital has a law of motion

$$K_{t+1} = sK_t^{\alpha} + (1-\delta)K_t$$

where s is the saving rate and δ capital depreciation. Solving the steady state

$$K_0 = \left(\frac{\delta}{s}\right)^{\frac{1}{\alpha-1}}$$

Exercise: find the approximate law of motion if we perturb away from the steady state

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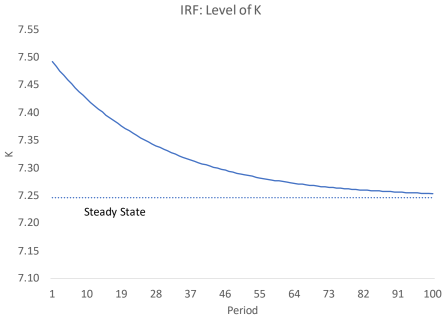
$$\widetilde{K}_{t+1} = \left[\alpha s K_0^{\alpha-1} + (1-\delta)\right]\widetilde{K}_t$$
$$\widetilde{K}_{t+1} = \left[\alpha\delta + (1-\delta)\right]\widetilde{K}_t$$

Modeling the Impulse Response Function

A simple Excel model used to calculate the IRF for K is here



Notes from this section are based on Joachim Zietz's guide