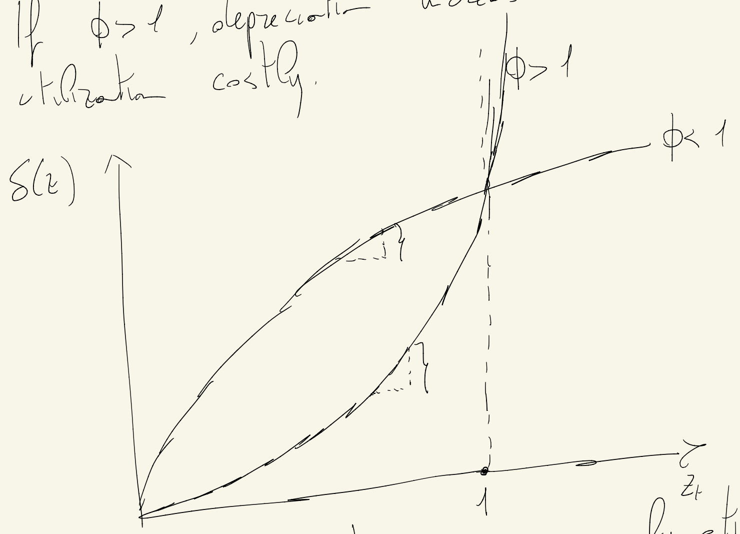
Exercise 1

$$U_t = E_t \sum_{t=0}^{\infty} \beta^t \left[\log(C_t) + v(1-L_t) \right]$$

where v is a concave and increasing function.

- $Y_t = A_t L_t^{1-\alpha} (z_t K_t)^{\alpha}$ => PRODUCTION FUNCTION
- $K_{t+1} = (1 - \bar{\delta} z_t^{\phi}) K_t + Y_t - C_t$

1) Capital utilization rate is governed by the ϕ parameter.
If $\phi < 1$ depreciation less than proportionally to capital utilization,
meaning it becomes relatively cheaper to increase utilization.
If $\phi > 1$, depreciation increases more than proportionally, making high utilization costly.



With $\phi > 1$ we have a convex function for the depreciation rate.
=> It is relatively more expensive to increase z_t at the margin and the agent takes this into account. So $\phi > 1$ makes more sense

2) Social Planner's Problem

Set-up the Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[\log(C_t) + v(1-L_t) + \lambda_t \left[(1-\bar{\delta} z_t^{\phi}) k_t + A_t L_t^{1-\alpha} (z_t k_t)^{\alpha} - C_t - k_{t+1} \right] \right]$$

FOC

1) $\dfrac{d\mathcal{L}}{dC_t} = 0 : \qquad \lambda_t = \dfrac{1}{C_t}$

2) $\dfrac{d\mathcal{L}}{dL_t} = 0 : \quad -\beta^t v'(1-L_t) + \beta^t \lambda_t (1-\alpha) A_t L_t^{1-\alpha-1} (z_t k_t)^{\alpha} = 0$

$$v'(1-L_t) = \lambda_t (1-\alpha) A_t \frac{Y_t}{L_t}$$

3) $\dfrac{d\mathcal{L}}{dk_{t+1}} = 0 : \quad -\beta^t \lambda_t + \beta^{t+1} \lambda_{t+1} (1 - \bar{\delta} z_{t+1}^{\phi}) + \lambda_{t+1} \beta^{t+1} \alpha A_{t+1} L_{t+1}^{1-\alpha} k_{t+1}^{\alpha-1} z_t^{\alpha}$

$$\lambda_t = \beta \lambda_{t+1} \left[\alpha A_{t+1} L_{t+1}^{1-\alpha} (z_{t+1})^{\alpha} k_{t+1}^{\alpha-1} + (1 - \bar{\delta} z_t^{\phi}) \right]$$

4) $\dfrac{d\mathcal{L}}{dz_t} = 0 : \quad \underbrace{\alpha A_t L_t^{1-\alpha} k_t^{\alpha} z_t^{\alpha-1}}_{\text{Marginal Benefit of capital utilization}} = \underbrace{\bar{\delta} \phi z_t^{\phi-1} k_t}_{\text{Marginal Cost of capital utilization}}$ } NOTICE THE ROLE OF ϕ

• Plug-in 1) in 2) and 3)

$$1)\quad V'(1-L_t) = \frac{(1-\alpha)A_t L_t^{-\alpha}(K_t Z_t)^{\alpha}}{C_t} = (1-\alpha)\frac{Y_t}{L_t C_t}$$ | INTRA-TEMP FOC

$$2)\quad \frac{C_{t+1}}{C_t} = \beta\left[\alpha A_{t+1} L_{t+1}^{1-\alpha} K_{t+1}^{\alpha-1} Z_{t+1}^{\alpha} + 1 - \bar{\delta} Z_t^{\phi}\right]$$ | EULER EQUATION

$$3)\quad \alpha A_t L_t^{1-\alpha} K_t^{\alpha} Z_t^{\alpha-1} = \bar{\delta}\phi Z_t^{\phi-1} K_t$$

EQUILIBRIUM: Social planner choose $\{C_t, L_t, Z_t, K_{t+1}, Y_t\}$ given $\{A_t\}$ and an initial condition for $K_t \Rightarrow \{K_0\}$ s.t.

- 1) + 2) + 3) holds and the production function and the capital accumulation equation.

3) Take eq. 3) remembering $Y_t = A_t L_t^{1-\alpha} K_t^{\alpha} Z_t^{\alpha}$

$$\alpha\frac{Y_t}{Z_t} = \bar{\delta}\phi Z_t^{\phi-1} K_t \Rightarrow \alpha\frac{Y_t}{K_t} = \bar{\delta}\phi Z_t^{\phi}$$

$$Z_t = \left(\frac{\alpha Y_t}{\bar{\delta}\phi K_t}\right)^{\frac{1}{\phi}} = \left(\frac{\alpha}{\bar{\delta}\phi}\right)^{\frac{1}{\phi}} \cdot \left(\frac{K_t}{Y_t}\right)^{-\frac{1}{\phi}}$$

$Z^* = 1$

$$\alpha\frac{Y^*}{K^*} = \bar{\delta}\phi \Rightarrow \boxed{\phi = \frac{Y^*}{K^*}\frac{\alpha}{\bar{\delta}}}$$

4) How does output behaves wrt to the standard RBC model when the economy it's hit by a TFP shock (A_t)?

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha} Z_t^{\alpha}$$

$$\frac{dY_t}{dA_t} = \frac{d}{dA_t}\left[A_t K_t^{\alpha} L_t^{1-\alpha} Z_t^{\alpha} \right] \Rightarrow \text{Looking at the FOC for } Z_t$$

$$Z_t = \left(\frac{\alpha Y_t}{\phi \delta K_t} \right)^{\frac{1}{\phi}}$$

$$\Downarrow$$

$$\frac{dZ_t}{dA_t} > 0 \quad \text{since } A_t \text{ is inside } Y_t$$

Notice also that ϕ governs the response of Z_t to a shock A_t $\Rightarrow \phi \uparrow \; Z_t \downarrow \Rightarrow$ IT IS MORE COSTLY TO INCREASE CAPITAL UTILIZATION

A_t affects directly Y_t in period t and also through Z_t decision. We have to wait 1-period to see the effect on K_{t+1} that then will affect output

$\Downarrow$

Higher effect of A_t relative to standard RBC

3) Solow Residual definition

$$\Rightarrow Y_t = A_t L_t^{1-\alpha} K_t^{\alpha}$$

Take logs

$$y_t = (1-\alpha) l_t + \alpha k_t + a_t \Rightarrow a_t = y_t - (1-\alpha) l_t - \alpha k_t$$

The estimate residual a_t is $\Rightarrow$ SOLOW RESIDUAL our case

$$Y_t = A_t (K_t Z_t)^{\alpha} L_t^{1-\alpha}$$

in logs

$$y_t = a_t + \alpha k_t + \alpha z_t + (1-\alpha) l_t$$

$$\underbrace{y_t - \alpha k_t - (1-\alpha) l_t}_{} = a_t + \alpha z_t$$

ADDITIONAL PART

If you just estimate this model you are overestimating the Solow Residual since you are not considering capital utilization $\Rightarrow \boxed{\alpha z_t}$

The correct residual for this model is:

$$\boxed{y_t - \alpha k_t - (1-\alpha) l_t - \alpha z_t = a_t}$$

Exercise 2: Technological shocks, Preference shocks, and the endogeneity of TFP

We consider here a variation of the simple analytical RBC model. We study a model economy populated with a representative household and a representative firm. The firm has a Cobb-Douglas technology:

$$Y_t = e^{z_t} K_t^{\gamma} N_t^{1-\gamma}$$

where K_t is capital, N_t is labor input, and e^{z_t} is the stochastic Total Factor Productivity (TFP). All profits of the firm are distributed to the household. Capital evolves according to

$$K_{t+1} = I_t$$

where I_t is investment in period t.

The representative household works N_t and consumes C_t. Preferences are given by

$$U = E_0 \sum_{t=0}^{\infty} \beta^t [\log C_t - e^{\chi_t} N_t]$$

where χ_t is a preference shock. Capital is accumulated by the household and rented to the firm. Let κ_t denote the real rental rate of capital, P_t the price of the final good, and W_t the nominal wage. It is assumed that γ and β are between 0 and 1.

1. Write down the budget constraint of the household and the profit function of the firm

The budget constraint of the household is:

$$P_t C_t + P_t K_{t+1} = W_t N_t + \kappa_t K_t$$

The profit function of the firm is:

$$\Pi_t = P_t Y_t - W_t N_t - \kappa_t K_t = e^{z_t} K_t^{\gamma} N_t^{1-\gamma} - W_t N_t - \kappa_t K_t$$

2. Derive the first-order conditions of the utility and profit maximization

In solving for the competitive equilibrium, individual firms and households make decisions based on prices, resulting in market-clearing outcomes. In contrast, the social planner aims to maximize total welfare by considering the overall utility of society without focusing on prices. While the planner allocates resources directly, firms and households respond to price signals. The two equilibria coincide under specific conditions, as stated in the 1st Fundamental Theorem of Welfare Economics.

Household Problem:

The household maximizes:

$$\max_{\{C_t, N_t, K_{t+1}\}} L = E_0 \sum_{t=0}^{\infty} \beta^t [\log C_t - e^{\chi_t} N_t - \lambda_t (P_t C_t + P_t K_{t+1} - W_t N_t - \kappa_t K_t)]$$

with K_0 given.

The first-order conditions are:

- w.r.t. C_t:

$$\frac{1}{C_t} = P_t \lambda_t$$

- w.r.t. N_t:

$$e^{\chi_t} = \lambda_t W_t \longrightarrow e^{\chi_t} = \frac{W_t}{P_t C_t}$$

- w.r.t. K_{t+1}:

$$\lambda_t P_t = \beta E_t [\lambda_{t+1} \kappa_{t+1}]$$

If we plug in the FOC for C_t it gives us the Euler Equation:

$$\frac{1}{P_t C_t} = \beta E_t \Big[\frac{\kappa_{t+1}}{P_{t+1} C_{t+1}} \Big]$$

Firm Problem:

The firm maximizes profits, i.e. total revenues ($P_t Y_t$) minus total costs ($W_t N_t + \kappa_t K_t$):

$$\max_{\{N_t, K_t\}} \Pi_t = P_t e^{z_t} K_t^{\gamma} N_t^{1-\gamma} - W_t N_t - \kappa_t K_t$$

The first-order conditions are: - w.r.t. N_t:

$$\frac{W_t}{P_t} = (1-\gamma) \frac{Y_t}{N_t}$$

- w.r.t. K_t:

$$\frac{\kappa_t}{P_t} = \gamma \frac{Y_t}{K_t}$$

Definition of Equilibrium A competitive equilibrium is a sequence of prices $\{P_t, W_t, \kappa_t\}$ and quantities $\{C_t, K_{t+1}, N_t\}$ such that:

- Quantities maximize profits and utility, given prices (i.e., the first-order conditions hold).
- Markets clear: $Y_t = C_t + K_{t+1}$.

Note that because of Walras' Law, one price can be normalized (for example, set $P_t = 1$).

3 Show that when factor markets and final good markets are competitive, firm profits are zero

If you take the profit function and substitute the expression for K_t and N_t derived from the FOC, profits will be zero.

4. Solve the model and show that the equilibrium process of output is:

$$y_t = z_t + \gamma y_{t-1} - (1-\gamma)\chi_t$$

Using the first-order conditions, we can derive the relationship between consumption and capital. I normalize $P_t = 1$ to get rid of this variable. Going back to the Euler Equation

$$\frac{1}{C_t} = \beta E_t \left(\frac{1}{C_{t+1}} \frac{\kappa_{t+1}}{P_{t+1}} \right)$$

From the firm's first-order conditions, $\frac{\kappa_t}{P_t} = \gamma \frac{Y_t}{K_t}$, we substitute this into the above equation to get:

$$\frac{1}{C_t} = \beta\gamma E_t \left(\frac{1}{C_{t+1}} \frac{Y_{t+1}}{K_{t+1}} \right)$$

Now we have to find a way to solve this equation. Remember from the market clearing condition that $Y_t = C_t + K_{t+1}$. We can rewrite the above equation as:

$$\frac{K_{t+1}}{C_t} = \beta\gamma E_t \left[\frac{C_{t+1} + K_{t+2}}{C_{t+1}} \right] = \beta\gamma E_t \left[1 + \frac{K_{t+2}}{C_{t+1}} \right]$$

and solve it forward

$$\frac{K_{t+1}}{C_t} = \beta\gamma E_{t+k} \sum_{k=0}^{\infty} (\beta\gamma)^k = \frac{\beta\gamma}{1-\beta\gamma}$$

If you substitute $C_t = Y_t - K_{t+1}$ you'll find that:

$$K_{t+1} = \beta\gamma Y_t \quad \text{and}$$

Substituting this into the market-clearing condition $Y_t = C_t + K_{t+1}$, we get:

$$C_t = (1-\beta\gamma) Y_t$$

Also notice that from the FOC for labor supply we can rearrange

$$e^{\chi_t} = (1-\gamma)\frac{Y_t}{N_t C_t} = (1-\gamma)\frac{Y_t}{N_t(1-\gamma\beta)C_t} \qquad \longrightarrow N_t = \frac{(1-\gamma)}{(1-\gamma\beta)} e^{-\chi_t}$$

Now go back to the production function and plug in the above expression for labor plus the expression for K_{t+1}

$$Y_t e^{z_t} K_t^{\gamma} N_t^{1-\gamma} = e^{z_t} \left(\beta\gamma Y_{t-1}\right)^{\gamma} \left(\frac{1-\gamma}{1-\beta\gamma} e^{-\chi_t} \right)^{1-\gamma}$$

Taking logs and dropping constants gives

$$y_t = z_t + \gamma y_{t-1} - (1-\gamma)\chi_t \tag{1}$$

5. Assume $y_{-1} = 0$, $z_t = 0$ for all t, $\chi_t = 0$ for all t except $\chi_0 = 1\%$. Draw the time path of χ_t, z_t, and y_t. Explain why y_t is persistent.

With $\chi_0 = 1\%$, we get an initial impact on output through the labor supply channel. However, due to the recursive nature of the output equation $y_t = \gamma y_{t-1} - (1-\gamma)\chi_t$, output will exhibit persistence over time even after the initial shock.

Figure 2: Impulse response to a one time shock χ, economy $\mathcal{A}$

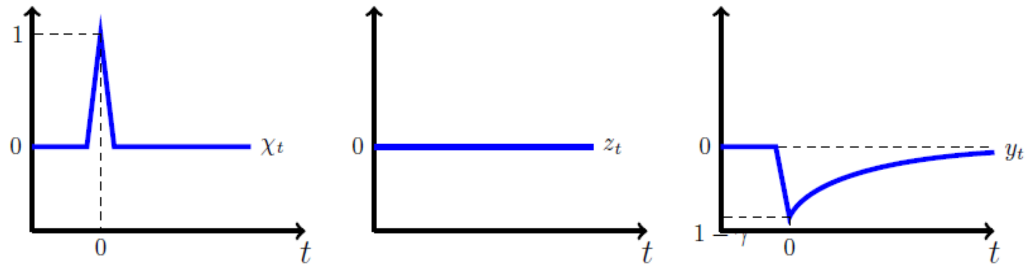


Figure 1: Impulse response of χ_t, z_t, and y_t to a one-time preference shock χ_0.

If you start from y_0 and you solve forward you'll see that there is an impact in at time 0 but then the effect dies out period by period until output goes back to its steady state value.