

Demystifying DSGE Models

3. Case Study – The Canonical Smets-Wouters DSGE

- **NK model** introduces into basic RBC more realistic specifications:
 - *monopolistic* competition
 - *frictions* in *prices* and in *wages*
 - *habit* formation in consumption
 - *non-Ricardian* agents
 - *adjustment costs* in investment
 - *capacity* (non-)utilisation *costs*
 - *government* sector (*monetary* and *fiscal*)
 - *financial* sector ⇒ Week 5
 - *housing* sector ⇒ Week 5
 - *unemployment* ⇒ Week 6
 - *environment* ⇒ Week 6
 - *foreign* trade ⇒ Week 7

- Smets-Wouters model is “workhorse” DSGE model used as basis for almost all central bank DSGEs
- Smets and Wouters both from Belgian (!) CB; Smets’ father was its Governor until retirement...
- Original version published two decades ago in 2003, but superseded in 2007 by version estimated using Bayesian techniques
- This model is ***so important*** – in terms both of historical development of DSGEs and their practical implementation outside purely academic framework – that it merits being studied in detail
- eg, NY Fed model = SW2007 + BGG (financial side)

- **SW2007** model uses most of the characteristics mentioned last week:
 - consumption with *habit* persistence
 - *monopolistic* competition
 - *sticky* prices and wages using *Calvo* fairy
 - investment *adjustment costs* + variable capital *utilisation*
- Major **new** feature of model: use of **seven structural shocks** to match behaviour of US economy
- Shocks are to: productivity (*a*), labour supply (*l*), investment-specific technology (*i*), risk (*b*), (wage) cost-push (*w*), fiscal policy (*g*) and monetary policy (*r*)

- Households maximize **[NEW] non-separable** utility function with two arguments (goods and labour effort) over infinite life horizon

$$E_t \sum_{s=0}^{\infty} \beta^s \left[\frac{1}{1 - \sigma_c} (C_{t+s}(j) - \lambda(C_{t+s-1}))^{1-\sigma_c} \right] \exp \left(\frac{\sigma_c - 1}{1 + \sigma_l} L_{t+s}(j)^{1+\sigma_l} \right)$$

- Habits are in form of time-varying ***external habits***
- **[NEW]** SW form of utility function:
- ***King-Plosser-Rebelo (KPR) preferences***

- General form of **KPR Preferences**:

$$u(C, L) = \frac{1}{1 - \sigma_c} C^{1-\sigma_c} v(L)$$

σ_c is risk aversion parameter = inverse of intertemporal rate of substitution

- where $v(L)$ is increasing and concave if $0 < \sigma_c < 1$ or decreasing and convex if $\sigma_c > 1$
- In **limit case** of $\sigma_c = 1$, resulting preferences specification is **additively separable** and given by $u(C, L) = \ln C_t + v(L)$
- For SW, $v(L)$ is given by $\exp\left(\frac{\sigma_c - 1}{1 + \sigma_l} L_{t+s}(j)^{1+\sigma_l}\right)$
- Labour L is **aggregated** by a **lazy union** → **sticky nominal wages à la Calvo**

σ_l [last week φ !!] is inverse of “Frisch elasticity of labour supply” = percentage change in hours arising from a given percentage change in wages

- Households **save** through purchases of government ***nominal bonds*** (B_{t+1}), from which they earn an interest rate of R_t
- Return on these bonds is subject to a **risk shock** ϵ_t^b
- Households own all capital **stock** and **[NEW]** rent capital **services** to firms, deciding how much capital **stock** to accumulate given **capital adjustment costs**

$$K_t(j) = (1 - \delta)K_{t-1}(j) + \epsilon_t^i \left[1 - S \left(\frac{I_t(j)}{I_{t-1}(j)} \right) \right] I_t(j)$$

NB: SW use Dynare timing convention, so no "Predetermined Variables K"

$$S(x_t) = \frac{\varphi}{2} (x_t - \gamma)^2$$

Quadratic form as seen last week

- As rental price of capital (R^k) changes, **utilisation** of capital **stock** can be adjusted at increasing **cost**

$$K_t^s(j) = Z_t(j) K_{t-1}(j) \leftarrow K^s = \text{capital services}; Z = \text{utilisation cost fn}$$

- Firms produce ***differentiated*** goods, decide on labour and capital inputs, and set (***sticky***) ***prices***, again according to ***Calvo model***
- ***Calvo Rule*** here [NEW]: those prices/wage rates that are ***not*** re-optimised are ***only partially indexed*** to past inflation rates:

$$P_t(i) = (\pi_{\boxed{t-1}})^{\iota_p} (\pi_*)^{1-\iota_p} P_{t-1}(i)$$

- π_* denotes steady-state gross inflation rate (eg, 1.02)
- $0 \leq \iota_p \leq 1$ is the (price) ***indexation*** parameter
- A positive value of ι_p introduces ***structural inertia*** into inflation process since P_{t-1} enters function
- ***Similarly for wages*** (ι_w)

Some Detail: Intermediate Goods Sector

- *Intermediate Good Producer* i is assumed by SW to use standard **Cobb-Douglas** technology (with variable input costs) but subject also to a ***fixed cost***

$$Y_t(i) = \epsilon_t^a K_t^s(i)^\alpha [\gamma^t L_t(i)]^{1-\alpha} - \gamma^t \Phi$$

Note that “fixed” cost grows with time !

- where

- $K_t^s(i)$ is **capital services** used in production [flow]
- $L_t(i)$ is a composite labour (services) input
- **[NEW]** ϕ is a fixed cost
- **[NEW]** γ^t represents a (labour-augmenting) deterministic ***growth*** rate of output

- ϵ_t^a is for ***total factor productivity*** and follows process

$$\ln \epsilon_t^a = (1 - \rho_z) \ln \epsilon^a + \rho_z \ln \epsilon_{t-1}^a + \eta_t^a, \eta_t^a \sim N(0, \sigma_a)$$

Steady-state value, usually assumed = 0

- Firm's ***profit*** is given by

$$P_t(i) Y_t(i) - W_t L_t(i) - R_t^k K_t^s(i)$$

- where
 - W_t is aggregate nominal wage rate
 - R_t^k is rental rate on capital ***services***
- Cost minimisation yields following first-order conditions:

$$(\partial L_t(i)) : \Theta_t(i) \gamma^{(1-\alpha)t} (1 - \alpha) \epsilon_t^a K_t^s(i)^\alpha L_t(i)^{-\alpha} = W_t$$

$$(\partial K_t^s(i)) : \Theta_t(i) \gamma^{(1-\alpha)t} \alpha \epsilon_t^a K_t^s(i)^{\alpha-1} L_t(i)^{1-\alpha} = R_t^k$$

- where $\Theta_t(i)$ is *Lagrange multiplier* (shadow value) associated with production function (and equals marginal cost MC_t)
- Combining these FOCs and noting that capital-labour ratio is equal across firms implies (*as usual*)

$$K_t^s = \frac{\alpha}{1 - \alpha} \frac{W_t}{R_t^k} L_t$$

- **Marginal cost** MC_t is *same for all firms* and equals

$$MC_t = \alpha^{-\alpha} (1 - \alpha)^{-(1-\alpha)} W_t^{1-\alpha} R_t^{k\alpha} \gamma^{-(1-\alpha)t} (\epsilon_t^a)^{-1}$$
- **Prices of intermediate goods** are determined by **Calvo** fairy/devil/lottery [$P \neq MC$ recall !!]
- As in previous Calvo models, in each period, each firm i faces a constant **probability** $1 - \xi_p$ of being able to **re-optimise** its price $P_t(i)$
- Probability that firm receives Calvo-fairy signal to re-optimise its price assumed **independent** of time that it last reset its price [“Markov Process”]

- Under Calvo pricing with SW *partial indexation* mentioned earlier,
- optimal price $\tilde{P}_t(i)$ set by firm that is allowed by Calvo fairy (with a probability $1-\xi_p$) to re-optimize
- results from solving following optimisation problem:

Net profits

$$\max_{\tilde{P}_t(i)} E_t \sum_{s=0}^{\infty} \xi_p^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \left[\tilde{P}_t(i) (\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p}) - MC_{t+s} \right] Y_{t+s}(i)$$

“Stochastic discount factor”

- This looks formidable!
- But if we compare it to what we used in sticky-price model studied last week, we find clear similarities:

$$\max_{P_{j,t}^*} E_t \sum_{i=0}^{\infty} (\beta\theta)^i \left(P_{j,t}^* Y_{j,t+i} - TC_{j,t+i} \right)$$

$\xi_p \equiv \theta$

$$\max_{\tilde{P}_t(i)} E_t \sum_{s=0}^{\infty} \xi_p^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \left[\tilde{P}_t(i) \left(\prod_{l=1}^s \pi_{t+l-1}^{\iota_l} \pi_*^{1-\iota_p} \right) - MC_{t+s} \right] Y_{t+s}(i)$$

- Since $TC = MC^*Y$, only really ***new element*** is ***indexation factor*** multiplying \tilde{P}_t : $\prod_{l=1}^s \pi_{t+l-1}^{\iota_l} \pi_*^{1-\iota_p}$

- What is this ***indexation factor*** $\prod_{l=1}^s \pi_{t+l-1}^{l_t} \pi_*^{1-l_p}$
 - Since π is a ***gross*** inflation rate, it has each period a value close to 1
 - Thus, product $\prod_{l=1}^s \pi_{t+l-1}^{l_t}$ (which multiplies together inflation rates for all periods since previous price revision) also has a value near 1
 - This is then scaled by factor $\pi_*^{1-l_p}$ which reflects ***weighted combination*** indexation
 - Finally, as noted previously, curious expression

$$\frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}}$$
 is just the (nominal) discount factor
- Ξ_t is here Lagrange Multiplier for consumption, so this is just β multiplied by (1) ratio of Lagrange multipliers [≈ 1] x (2) ratio of prices at t and $(t+s)$ [also ≈ 1]

- It is also clear from formulation of optimisation problem

$$\max_{\tilde{P}_t(i)} E_t \sum_{s=0}^{\infty} \xi_p^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \left[\tilde{P}_t(i) \left(\prod_{l=1}^s \pi_{t+l-1}^{\iota_l} \pi_*^{1-\iota_p} \right) - \boxed{MC_{t+s}} \right] Y_{t+s}(i)$$

- that price set by firm i , at time t , is a function of ***expected future*** marginal costs
- ***Price*** will therefore be a ***mark-up*** over these discount-weighted ***marginal costs***

- Given FOC for this optimisation problem, ***aggregate price index*** which results is

$$P_t = (1 - \xi_p) \tilde{P}_t(i) G'^{-1} \left[\frac{\tilde{P}_t(i) \tau_t}{P_t} \right] + \xi_p \pi_{t-1}^{l_p} \pi_*^{1-l_p} P_{t-1} G'^{-1} \left[\frac{\pi_{t-1}^{l_p} \pi_*^{1-l_p} P_{t-1} \tau_t}{P_t} \right]$$

- Compare this to what we used in first sticky-price model studied last week:

$$P_t = \left[\theta P_{t-1}^{1-\psi} + (1 - \theta) P_t^{*1-\psi} \right]^{\frac{1}{1-\psi}}$$

$\xi_p \equiv \theta$

- Function G (of which G' is used in definition above) is defined in Appendix; it comes from using **Kimball**
See later

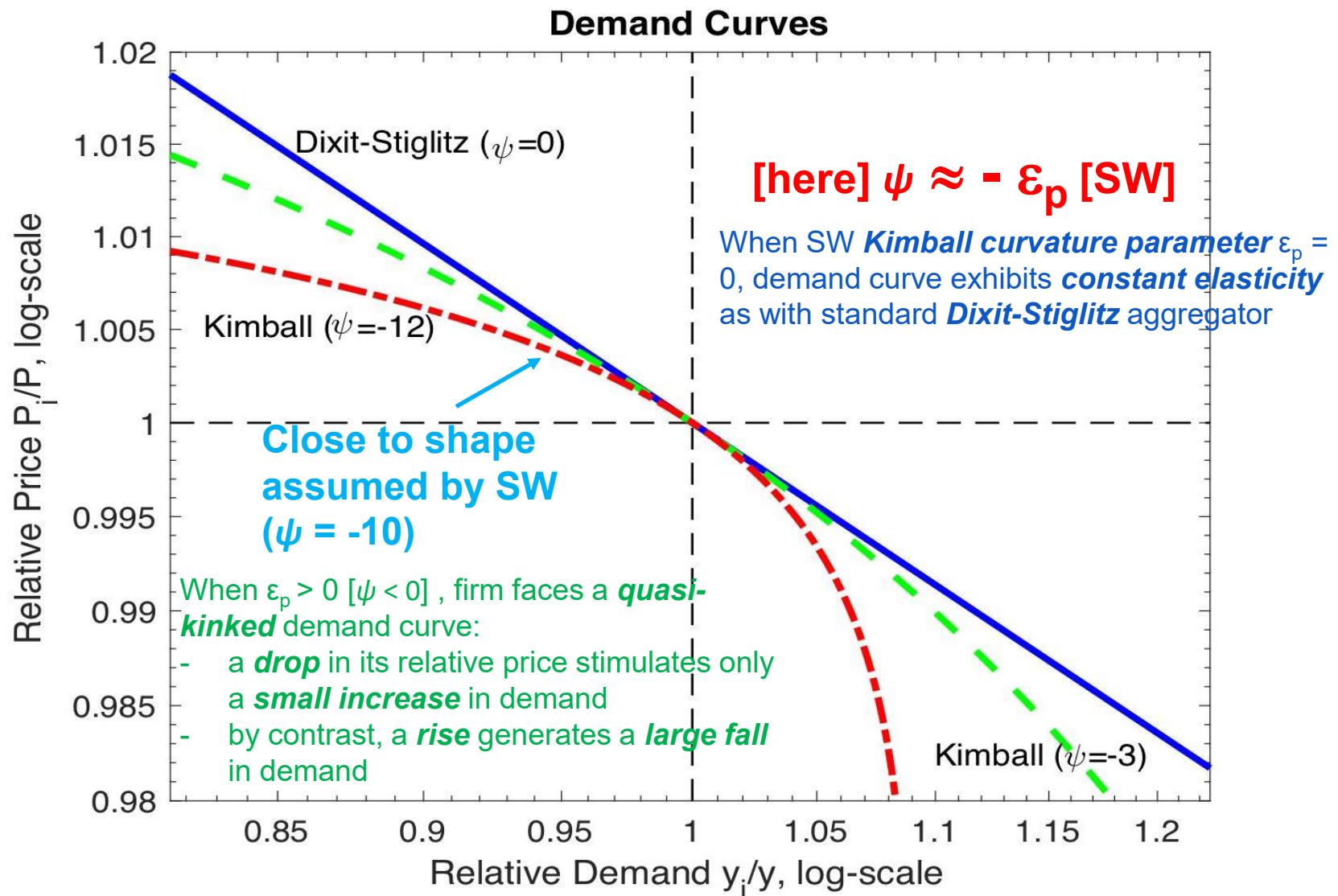
- Prices therefore determined (partially) by *past inflation rate*
- *Marginal costs* are a function of *wages* and *rental rate* of capital

$$MC_t = \alpha^{-\alpha} (1 - \alpha)^{-(1-\alpha)} W_t^{1-\alpha} R_t^{k\alpha} \gamma^{-(1-\alpha)t} (\epsilon_t^\alpha)^{-1}$$

- *Same* mechanism used by SW for *wages*

- Here endth such detail!
- Finally, a *technical point*:
- In both goods and labour markets SW *replace* standard *Dixit-Stiglitz* aggregator
- with a *Kimball (1995) aggregator*
- → allows for *time-varying demand elasticity*, which depends on *relative price*
- Introduction of Kimball aggregator allows SW to estimate a *more reasonable degree* of price and wage *stickiness*

Figure 1: Demand Curves -- Implications of Kimball vs. Dixit-Stiglitz Aggregators.



- For those who wish to see technical details of original non-linear SW2007 model, I have attached an Appendix which does so
- Here, we shall look instead at SW's *published* log-linearised equations

SW2007 Log-Linearised Model

- SW *detrended* their variables with *deterministic* trend γ and replaced *nominal* variables by their *real per-capita* counterparts – this is very *familiar!*
[Some authors have later used a *time-varying* trend to do this for SW2007] *More on this next week*
- *Non-linear* system was then *linearised* around stationary *steady state* of *detrended* variables, in usual *log-linearisation* process which we have seen previously
- This was done primarily because system is highly *non-linear* and *multi-dimensional* (34 parameters plus 7 s.d. to be estimated *simultaneously*)

- ***Log-linearised*** model (equations taken directly from SW2007 in *AER*): start with *Supply Side*
- ***Aggregate production function*** is (*eq.5*)

$$y_t = \phi_p (\alpha k_t^s + (1 - \alpha) \ell_t + \epsilon_t^a)$$

- where

- y_t is (log-linearised) GDP
- ℓ_t is labour input
- ϵ_t^a is total factor productivity [shock *process*]
- **[NEW]** k^s is *capital services*, determined by *capital stock installed* in previous period $k_t^s = k_{t-1} + z_t$
- and a *capacity utilisation* variable z (*eq.6*)

- *Aside*: here is how SW actually write these equations in their code

```
[name='FOC capacity utilization, SW Equation (7)']
    zcap = (1/(czcap/(1-czcap)))* rk ;
[name='Definition capital services, SW Equation (6)']
    k = kp(-1)+zcap ;
[name='Law of motion for capital, SW Equation (8)']
    kp = k1*kp(-1) + (1 - k1)*inve + k2*qs;
[name='Aggregate Production Function, SW Equation (5)']
    y = cfc*( calfa*k+(1-calfa)*lab +a );
```

- “kp” is physical capital [stock]
- “k” is capital services [flow]
- *Following Dynare convention*, “k” is determined by “kp(-1)”

- ***Costs of adjusting*** capital stock in use → rate of ***capacity utilisation*** linked to rental rate of capital (eq.7)
$$z_t = z_1 r_t^k$$
- ***Rental rate of capital*** is a function of capital-labour ratio and real wage (eq.11)

Remember, these are logs:
 $\log(\text{ratio}) = \text{difference}$

$$r_t^k = -(k_t - \ell_t) + w_t$$
- ***Total factor productivity shock*** evolves over time according to an ***AR(1)*** process

$$\epsilon_t^a = \rho_a \epsilon_{t-1}^a + \eta_t^a$$

This is a process; ρ_a is likely close to 1;
 actual “shock” comes through η^a

- Now *Demand Side*
- Expenditure formulation of aggregate resource constraint is familiar (*eq.1*)

$$y_t = c_y c_t + i_y i_t + z_y z_t + \epsilon_t^g$$

- where
 - y_t is GDP
 - c_t is consumption
 - i_t is investment
 - z_t is exogenous spending [= government + net exports]
 - ϵ_t^g is spending shock ***process***
 - c_y , i_y and z_y are constant ***parameters***, representing (as usual) ***steady-state shares*** of each variable in y

- **Consumption** is determined by usual **Euler Equation**

$$c_t = c_1 c_{t-1} + (1 - c_1) E_t c_{t+1} + \\ c_2 (\ell_t - E_t \ell_{t+1}) - c_3 \left(r_t - E_t \pi_{t+1} + \epsilon_t^b \right)$$

- (eq.2) where
 - c_1, c_2, c_3 are constant **coefficients**
 - r_t is interest rate on a **one-period** risk-less **bond**
 - ϵ_t^b evolves according to **AR(1)** process $\epsilon_t^b = \rho_b \epsilon_{t-1}^b + \eta_t^b$
- Note that this Euler Equation has a **backward-looking** element, deriving from “habit formation” [$C_t - \lambda C_{t-1}$ replaces C_t in utility function, recall]

- Term $c_2(\ell_t - E_t \ell_{t+1})$ involving ***labour*** input allows for some ***substitution*** between consumption and labour input
- ***Coefficients*** c_1, c_2, c_3 are themselves functions of deeper ***structural parameters***
- SW describe ϵ^b term as a “***risk premium***” ***shock*** determining willingness of households to hold one-period bond $c_3 \left(\overbrace{r_t - E_t \pi_{t+1}} + \epsilon_t^b \right)$
- It can also be seen as a type of ***preference shock*** that influences short-term consumption-saving decision (this is more ***usual interpretation***)

- **Investment** is determined by (eq.3)

$$i_t = i_1 i_{t-1} + (1 - i_1) E_t i_{t+1} + i_2 q_t + \epsilon_t^i$$

- where **shadow price of capital stock** is (eq.4)

$$q_t = q_1 E_t q_{t+1} + (1 - q_1) r_{t+1}^k - (r_t - E_t \pi_{t+1} + \epsilon_t^b)$$

Same as
in eq. 2

- **nominal value of installed capital** is given by (eq.8)

$$k_t = k_1 k_{t-1} + (1 - k_1) i_t + k_2 \epsilon_t^i$$

Same as
in eq. 3

- Investment depends on **lagged** investment because \exists **adjustment cost** function that limits amount of new investment that can come “on line” immediately

Recall, SW use Dynare timing convention

- **Main driving force** behind investment is ***shadow price of capital stock (Tobin's Q)*** q_t ([eq.4](#) above):

$$q_t = q_1 E_t q_{t+1} + (1 - q_1) r_{t+1}^k - (r_t - E_t \pi_{t+1} + \epsilon_t^b)$$

- q_t depends **positively** on expected future marginal productivity of capital (r^k) and **negatively** on future real interest rate (and “risk premia”)
- Note: investment **shock** ϵ_t^i appears in **both i_t and k_t** equations → positive shock to investment also increases capital stock concurrently
- SW find that, ***empirically, investment adjustment cost shocks are the most important in entire model***; by contrast (and curiously), ***capacity utilisation costs*** are estimated to be not very important

- Recall: $Y = C + I + [G + (X - M)]$ or here:

$$y_t = c_y c_t + i_y i_t + z_y z_t + \epsilon_t^g$$
- ***Exogenous spending*** z_t has two components:
 - ***Government*** spending G
 - an element related to ***productivity*** because “***net exports*** ($X - M$) may be affected by domestic productivity developments” [SW’s words]
- → exogenous spending shock ϵ_t^g changes over time according to **[NEW]** a ***cross-equation*** process:

$$\epsilon_t^g = \rho \epsilon_{t-1}^g + \eta_t^g + \rho_{ga} \eta_t^a$$

Productivity!

- ***Mark-up of price*** over marginal cost is determined in log-linearised model by difference between ***marginal product of labour*** and ***real wage*** (*eq.9*)

$$\mu_t^p = \textcolor{red}{mpl}_t - w_t = \alpha(\textcolor{blue}{k_t^s - l_t}) + \textcolor{green}{\epsilon_t^a} - w_t$$

- where ***marginal product of labour*** is itself a (positive) function of ***capital-labour ratio*** and total factor ***productivity (TFP)***
- → price ***inflation*** determined by “***New Keynesian Phillips Curve***” (*eq.10*)

$$\pi_t = \pi_1 \pi_{t-1} + \pi_2 E_t \pi_{t+1} - \pi_3 \textcolor{blue}{\mu_t^p} + \textcolor{green}{\epsilon_t^p}$$

- *Lagged* inflation appears in NKPC as a result of *Calvo* pricing structure
- If *degree of indexation* to past inflation is *zero* ($\iota_p=0$), eq.10 reverts to *purely forward-looking* Phillips curve (coefficient $\pi_1=0$) seen last week

$$\pi_t = \pi_1 \pi_{t-1} + \pi_2 E_t \pi_{t+1} - \pi_3 \mu_t^p + \epsilon_t^p$$

- *Speed of adjustment* to desired *mark-up* (coefficient π_3) depends on
 - *degree of price stickiness* (ξ_p)
 - *curvature* of *Kimball* goods market aggregator (ε_p)
 - steady-state price *mark-up* ($\phi_p - 1$)

- $\ln \pi_t = \pi_1 \pi_{t-1} + \pi_2 E_t \pi_{t+1} - \pi_3 \mu_t^p + \epsilon_t^p$
- ϵ_t^p is ***price mark-up*** shock, which evolves as **[NEW]**
ARMA(1,1):

$$\epsilon_t^p = \rho^p \epsilon_{t-1}^p + \eta_t^p - \mu_p \eta_{t-1}^p$$

- MA component has ***no memory***, recall
 → ***One-period-memory*** MA term captures
high-frequency fluctuations in inflation
- ***Price mark-up shock*** ϵ_t^p is included because SW
 find it is ***empirically very important in order*** to
 capture ***price dynamics***

- SW model treats **wages** similarly to prices:
- (Calvo) Sticky wages gradually adjust → real wages move only gradually over time
- to ***equate real wages*** with ***marginal rate of substitution*** between working and consuming
- Short-term **gap** between these is “**wage mark-up**” defined as (**eq.12**)

$$\begin{aligned}\mu_t^w &= w_t - mrs_t \\ &= w_t - \left(\sigma \ell_t - \frac{1}{1 - \lambda/\gamma} \left(c_t - \frac{\lambda}{\gamma} c_{t-1} \right) \right)\end{aligned}$$

- Wages are then given by (*eq.13*)

$$w_t = w_1 w_{t-1} + (1 - w_1) (E_t w_{t+1} + E_t \pi_{t+1}) \\ - w_2 \pi_t + w_3 \pi_{t-1} - w_4 \mu_t^w + \epsilon_t^w$$

- Parameter w_4 → ***speed of adjustment*** to desired ***wage mark-up*** depends on degree of wage ***stickiness*** (ξ_w) and demand elasticity for labour
- which itself is a function of ***steady-state labour market mark-up*** ($\varphi_w - 1$) and ***curvature of Kimball*** labour market aggregator (ε_w – do not ***confuse*** this with shock process ϵ_t^w !)
- **Dynare** model therefore contains a complicated equation defining w_4

- The (again, *ARMA*) **wage mark-up shock** has form

$$\epsilon_t^w = \rho^w \epsilon_{t-1}^w + \eta_t^w - \mu_w \eta_{t-1}^w$$

- (As for prices) **wage mark-up shock** affects both current and lagged inflation and attempts also to capture ***temporary*** wage shocks with MA
- SW again find that wage mark-up shock is ***empirically very important*** to capture **wage dynamics**
- ***Overall SW confirm, empirically, NK thesis*** by finding that both price and wage ***stickiness*** are very ***important***, with prices affecting ***short-run*** inflation and wages ***longer-run*** inflation

- Final element of model is rule for *monetary policy*
- *Central bank* sets short-term interest rates according to *modified Taylor Rule*, which in log-linearised form becomes (eq.14)

$$r_t = \rho r_{t-1} + (1 - \rho) (r_\pi \pi_t + r_y (y_t - y_t^p)) + r_{\Delta y} [(y_t - y_t^p) - (y_{t-1} - y_{t-1}^p)] + \epsilon_t^r$$

- where y^p is *potential output* (“flex-price” output)
- a monetary policy *shock* is included (of simple $AR(1)$ form)

$$\epsilon_t^r = \rho^r \epsilon_{t-1}^r + \eta_t^r$$

- ***Interest rate*** depends on last period's interest rate while ***gradually adjusting*** towards a ***target interest rate*** ($r_\pi \pi_t + r_y (y_t - y_t^p)$) that depends on
 - inflation π_t
 - ***output gap*** ($y_t - y_t^p$) between ***actual*** and ***potential*** output (in SW, $y^p \equiv \text{flex-price}$ level of output)
- Interest rate also depends on ***growth rate*** of this output gap

$$[(y_t - y_t^p) - (y_{t-1} - y_{t-1}^p)]$$
- (recall that for logs $\rightarrow d\log = \% \text{chg} = \text{growth rate}$)

- **Potential output** y_t^p as used in Taylor Rule \equiv output obtained *if* prices and wages were **fully flexible**
- → [NEW] model effectively needs to be *expanded* to add a **shadow** flexible-price economy
- → in Dynare implementation ∃ *repetitive* section defining variables in **flex-price** economy
- A different definition of Taylor Rule or of output gap would *avoid* need for this section
- In fact, *original* SW2003 **Euro area** model did **not** include “shadow” flexible-price economy
- SW2007 added it to help explain price dynamics

Dynare Model

- Implementation in **Dynare** was originally written by SW themselves, so we make use of it here
- Unfortunately, their *code* used a *notation* which *differs considerably* from that set out in their SW2007 AER paper, so I have used code re-written for our use in **Dynare**
- There are 14 equations in **Dynare** log-linearised model for sticky wage-price economy
- Look at just 4 of them

- *Consumption*
- *Eq.2* expressed consumption as

$$c_t = c_1 c_{t-1} + (1 - c_1) E_t c_{t+1} + \\ c_2 (\ell_t - E_t \ell_{t+1}) - c_3 (r_t - E_t \pi_{t+1} + \epsilon_t^b)$$

- In Dynare, we have

#c1 = (lambda/gamma)/(1+lambda/gamma);

#c2 = ((sigma_c-1)*WL_C/(sigma_c*(1+lambda/gamma)));

#c3 = (1-lambda/gamma)/(sigma_c*(1+lambda/gamma));

c = c1 * c(-1) + (1 - c1) * c(+1) + c2 * (l - l(+1)) - c3 * (r - pinf(+1)) + eps_b;

Minor point: in SW's Dynare model, they separate out `eps_b` as here

$$c_1 = \frac{\lambda / \gamma}{1 + \lambda / \gamma} \quad c_2 = \frac{(\sigma_c - 1)(W_*^h L_* / C_*)}{\sigma_c (1 + \lambda / \gamma)} \quad c_3 = \frac{1 - \lambda / \gamma}{(1 + \lambda / \gamma) \sigma_c}$$

$$c_1 = \frac{\lambda/\gamma}{1 + \lambda/\gamma}$$

$$c_2 = \frac{(\sigma_c - 1)(W_*^h L_* / C_*)}{\sigma_c(1 + \lambda/\gamma)}$$

$$c_3 = \frac{1 - \lambda/\gamma}{(1 + \lambda/\gamma)\sigma_c}$$

- In c_1 , c_2 and c_3 , *deep parameters* are
 - *lambda* ≡ degree of (external) *habit persistence*
 - *sigma_c* ≡ *risk aversion* parameter
 - *gamma* ≡ gross *growth rate* [assuming *cointegration* of y, c, i]
- “WL_C” is a complicated combination of deep parameters and steady-state terms

- Next, **New Keynesian Phillips Curve (eq.10)**

$$\pi_t = \pi_1 \pi_{t-1} + \pi_2 E_t \pi_{t+1} - \pi_3 \mu_t^p + \epsilon_t^p$$

- becomes

```
# pi1 = (1/(1+beta_bar*gamma*iota_p))* iota_p;
# pi2 = (1/(1+beta_bar*gamma*iota_p))* beta_bar*gamma;
# pi3 = (1/(1+beta_bar*gamma*iota_p))*  

|   ((1-xi_p)*(1-beta_bar*gamma*xi_p)/xi_p)/((phi_p-1)*curv_p+1) ;
pinf = pi1*pinf(-1) + pi2*pinf(1) + pi3*mc + eps_p;
```

$$\pi_1 = \frac{\iota_p}{1 + \beta\gamma^{1-\sigma_c}\iota_p}$$

$$\pi_2 = \frac{\beta\gamma^{1-\sigma_c}}{1 + \beta\gamma^{1-\sigma_c}\iota_p}$$

$$\pi_3 = \frac{1}{1 + \beta\gamma^{1-\sigma_c}\iota_p} \frac{(1 - \beta\gamma^{1-\sigma_c}\xi_p)(1 - \xi_p)}{\xi_p((\phi_p - 1)\varepsilon_p + 1)}$$

- Repeating:

$$\pi_1 = \frac{\iota_p}{1 + \beta\gamma^{1-\sigma_c} \iota_p} \quad \pi_2 = \frac{\beta\gamma^{1-\sigma_c}}{1 + \beta\gamma^{1-\sigma_c} \iota_p} \quad \pi_3 = \frac{1}{1 + \beta\gamma^{1-\sigma_c} \iota_p} \frac{(1 - \beta\gamma^{1-\sigma_c} \xi_p)(1 - \xi_p)}{\xi_p ((\phi_p - 1)\varepsilon_p + 1)}$$

- Many new parameters here:

- ***beta_bar*** $\equiv \beta\gamma^{-\sigma_c}$ [ie, beta*gamma^(-sigma_c)] is growth-adjusted discount factor, beta being ***discount factor***
- ***iota_p*** = coefficient of ***indexation*** to past prices
- ***xi_p*** = ***Calvo parameter*** for prices
- ***curv_p*** = ***curvature of Kimball aggregator*** for prices (instead of ε_p to avoid confusion with shock ϵ^p)
- Hence ***beta_bar*gamma*** = $\beta\gamma^{1-\sigma_c}$

- **Wage Phillips Curve (eq.13)**

$$w_t = w_1 w_{t-1} + (1 - w_1) (E_t w_{t+1} + E_t \pi_{t+1}) \\ - w_2 \pi_t + w_3 \pi_{t-1} - w_4 \mu_t^w + \epsilon_t^w$$

- translates similarly into **Dynare** code below (once *eq.12* – next slide - is substituted in for μ_t^w)

```
#w1 = (1/(1+beta_bar*gamma));
#w2 = (1+beta_bar*gamma*iota_w)/(1+beta_bar*gamma);
#w3 = (iota_w/(1+beta_bar*gamma));
#w4 = - (1-xi_w)*(1-beta_bar*gamma*xi_w)/((1+beta_bar*gamma)*xi_w)*
        (1/((phi_w-1)*curv_w+1));
w = w1*w(-1) + (1 - w1)*w(1) + (1 - w1)*pinf(1) - w2*pinf + w3*pinf(-1)
    - w4*(sigma_l*l + (1/(1-lambda/gamma))*c -
        ((lambda/gamma)/(1-lambda/gamma))*c(-1) -w) + eps_w;
```

$$w_1 = \frac{1}{1 + \beta\gamma^{1-\sigma_c}} \quad w_2 = \frac{1 + \beta\gamma^{1-\sigma_c} l_w}{1 + \beta\gamma^{1-\sigma_c}} \quad w_3 = \frac{l_w}{1 + \beta\gamma^{1-\sigma_c}}$$

$$w_4 = \frac{1}{1 + \beta\gamma^{1-\sigma_c}} \frac{(1 - \beta\gamma^{1-\sigma_c} \xi_w)(1 - \xi_w)}{\xi_w ((\phi_w - 1)\varepsilon_w + 1)}$$

- New parameters here are
 - ***iota_w*** = coefficient of *indexation* to past *wages*
 - ***xi_w*** = *Calvo parameter* for *wages*
 - ***curv_w*** = *curvature of Kimball aggregator* for *wages*
 - ***phi_w*** = gross steady-state *wage mark-up*
- ***Wage mark-up*** (*eq. 12*) is

$$\mu_t^w = w_t - mrs_t = w_t - \left(\sigma_l l_t + \frac{1}{1 - \lambda/\gamma} (c_t - \lambda/\gamma c_{t-1}) \right)$$

- Finally **Taylor Rule** (eq.14)

$$r_t = \rho r_{t-1} + (1 - \rho) (r_\pi \pi_t + r_y (y_t - y_t^p)) \\ + r_{\Delta y} [(y_t - y_t^p) - (y_{t-1} - y_{t-1}^p)] + \epsilon_t^r$$

- In Dynare implementation of this Taylor Rule, ***potential output*** variable y_t^p is defined as ***yf, flex-price*** (or “natural”) value of output
- → introduce entire ***flex-price economy*** into model
- → ***11 additional equations***, in each case setting parameters which define “stickiness” of prices (and wages) to their “***zero stickiness***” values

- For example, *Wage equation* in *flex-price* economy becomes

$wf = \sigma_l * labf + (1/(1-\lambda/\gamma)) * cf - (\lambda/\gamma)/(1-\lambda/\gamma) * cf(-1)$;

- which is just *last bit* of Wage equation in *sticky-price, sticky-wage* economy because
 - $\iota_w = 0$
 - $\xi_w = 1$
 in *flex-price* economy

Steady State

- One last step necessary before we can test out model: inserting information on model's *steady state*
- These are derived as usual, for example

$$R_* = \bar{\beta}^{-1} \pi_*$$

$$k_* = \frac{\alpha}{1 - \alpha} \frac{w_*}{r_*^k} L_*$$

$$i_* = (1 - (1 - \delta)/\gamma) \bar{k}_*$$

$$\frac{c_*}{y_*} + \frac{i_*}{y_*} + g_* = 1$$

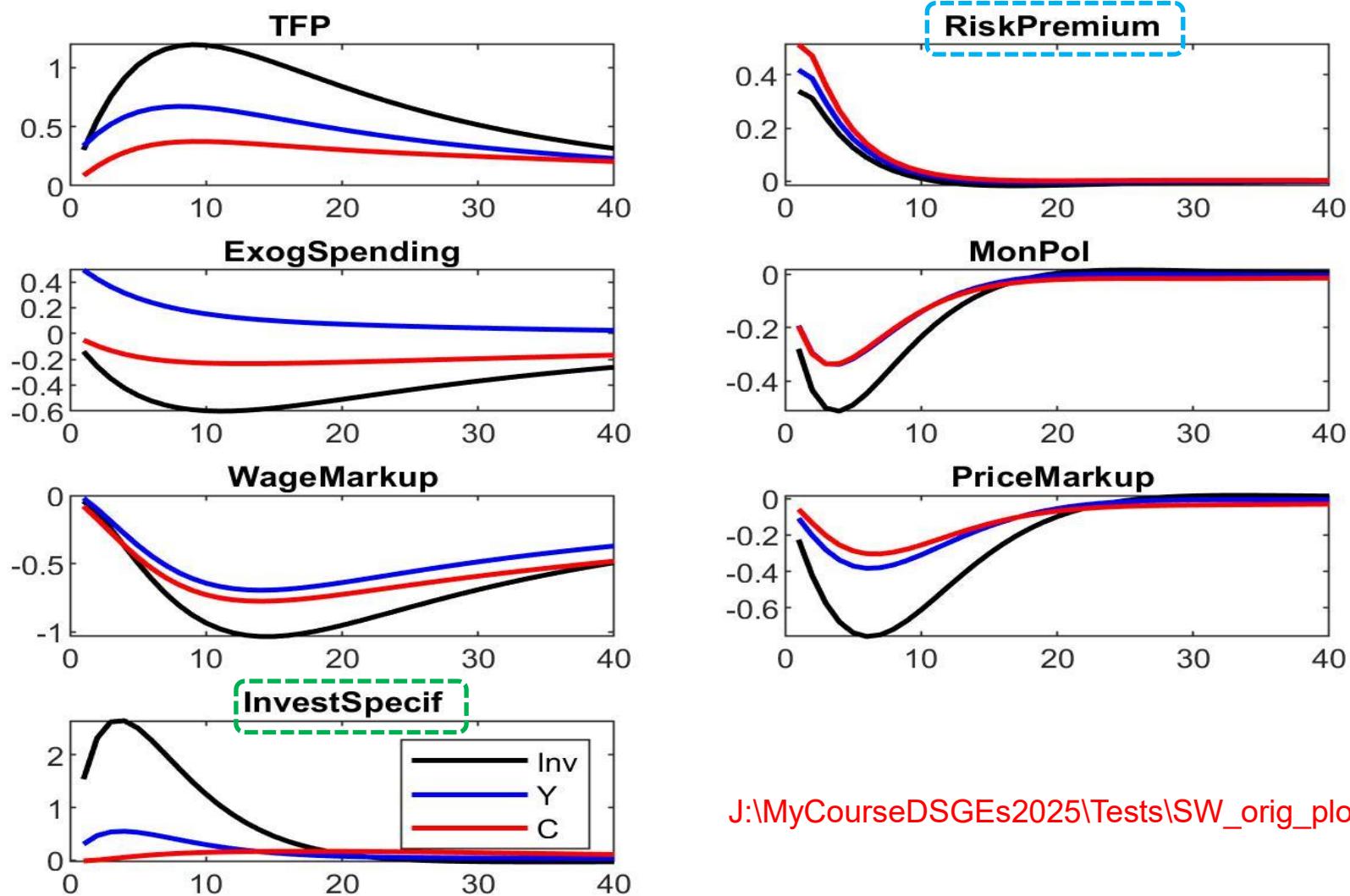
- These steady-state conditions are inserted into **Dynare** model as ***restrictions on parameters***
- For example, steady-state rental rate r_*^k is defined in **Dynare** code as

$$\begin{aligned} r_*^k &= \bar{\beta}^{-1} - (1 - \delta) \\ &= \beta^{-1} \gamma^{\sigma_c} - (1 - \delta) \end{aligned}$$

- **#Rk = (beta^-1) * (gamma^sigma_c) - (1-delta);**
- where # sign informs **Dynare** that this is ***not*** a parameter to be ***estimated directly***, but one ***calculated*** from various other “deep” parameters
- **# → model_local_variable**
- **Rk** is then used in value of capital **equation 4**, and also in definitions of other parameters (eg, steady state real wage and labour-to-capital ratio)

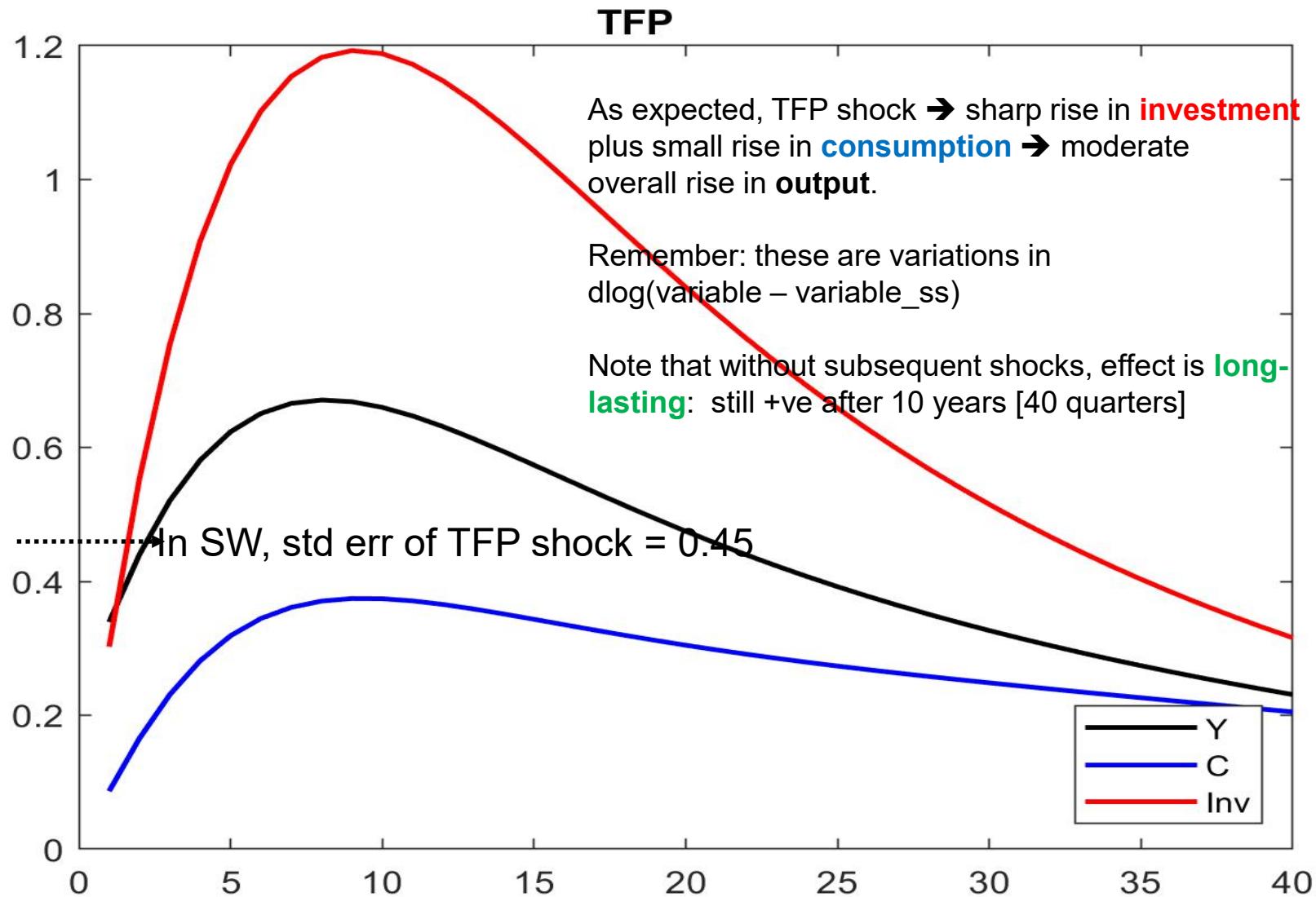
- Once this is done, it is possible to *simulate*
 - Slides below show impact of each shock on model's endogenous variables
 - Using *original SW2007* parameters
- As SW note, apart from *TFP shock (ϵ^a)*, *risk premium (ϵ^b)* and *investment-specific (ϵ^i) shocks* are most powerful [observe *scale* of impact]
- Recall that SW describe ϵ^b term as a shock determining willingness of households to hold one-period bond
- But also seen as a type of *preference shock* that influences short-term consumption-saving decision

- Responses to all shocks – Y , C , Inv [see specifics next]

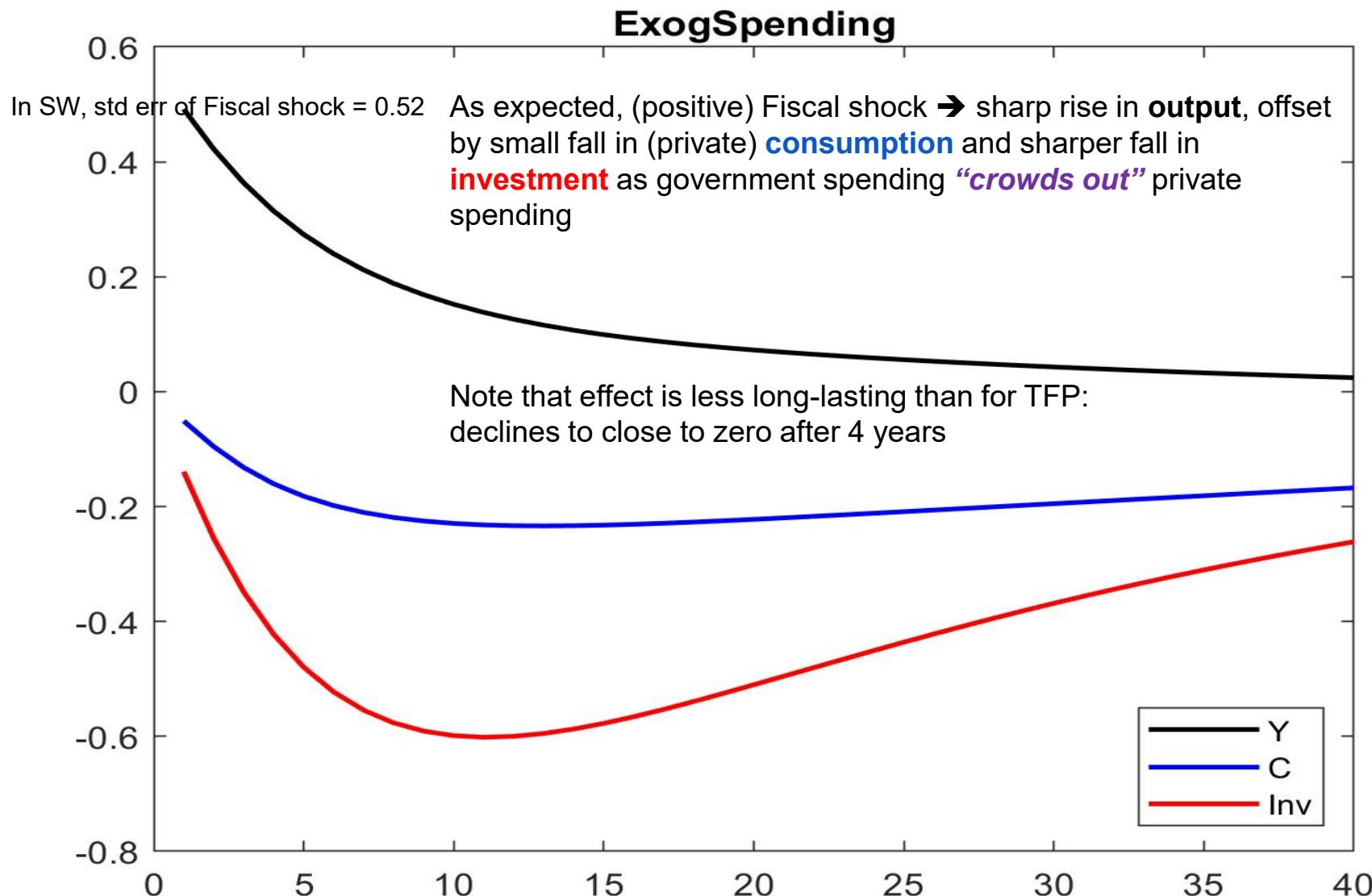


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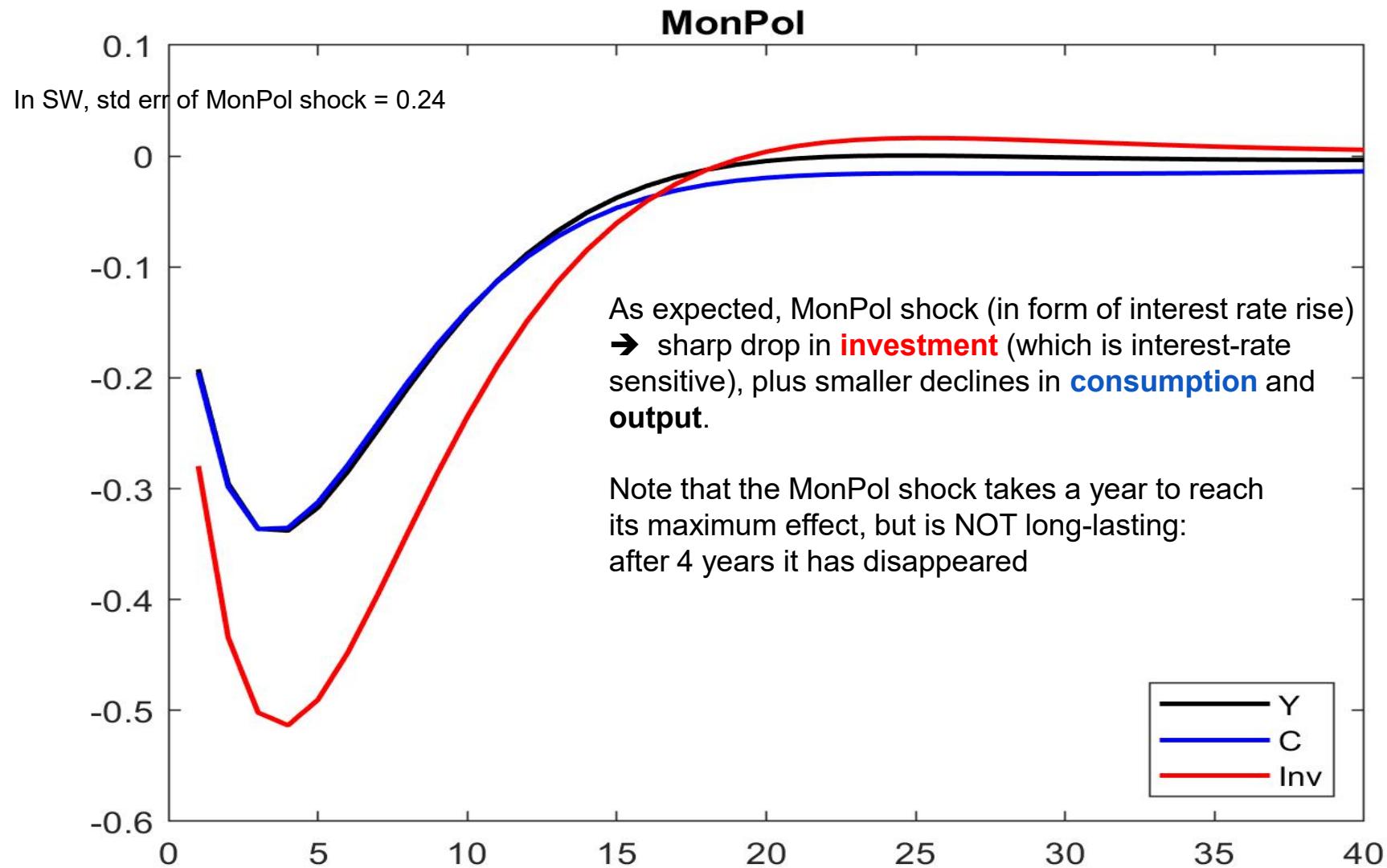
- Responses to TFP shock – Y, C, Inv



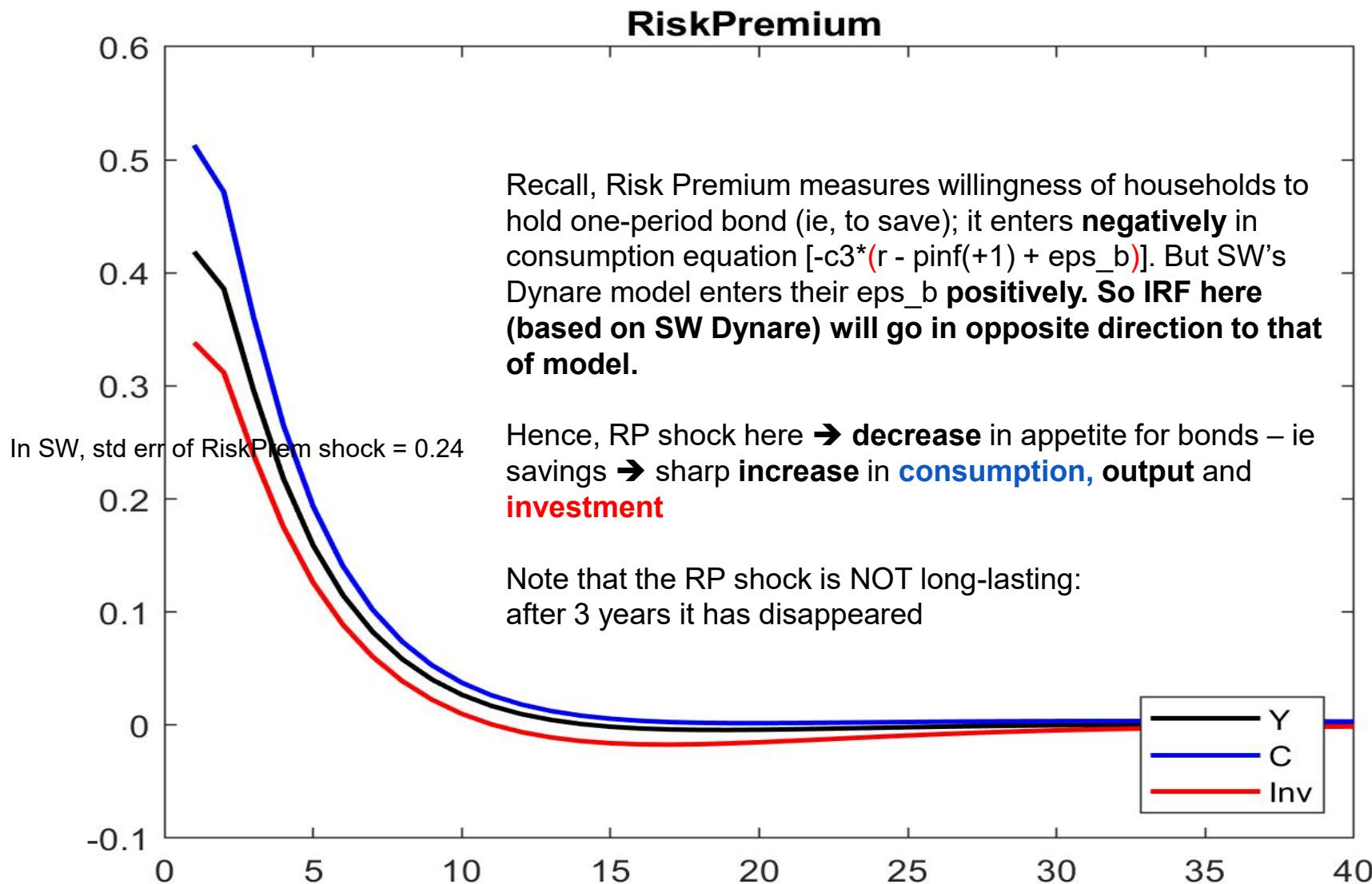
- Responses to Fiscal shock – Y , C , Inv



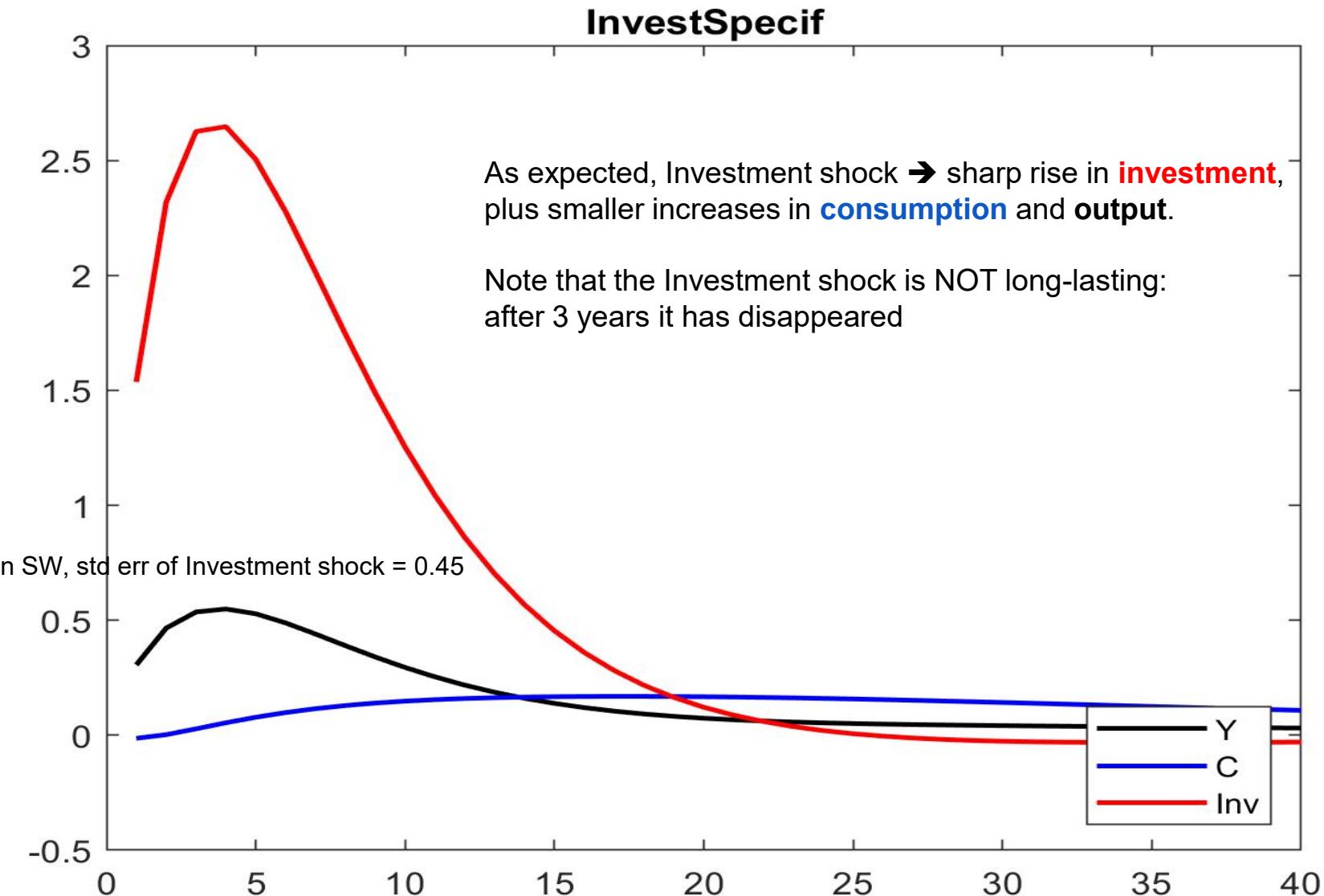
- Responses to MonPol shock – Y , C , Inv



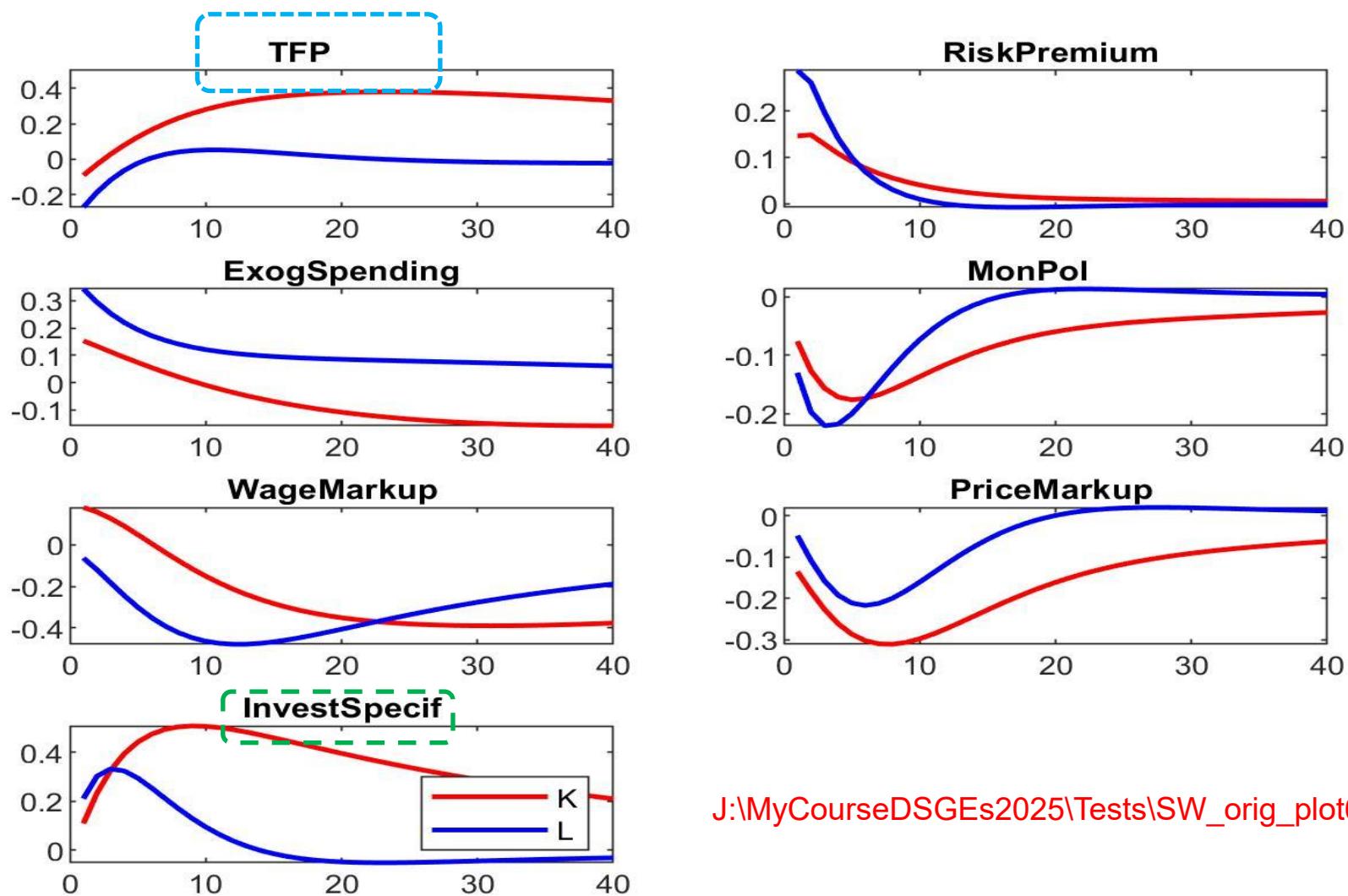
- Responses to Risk Premium shock – Y, C, Inv



- Responses to Investment shock – Y, C, Inv

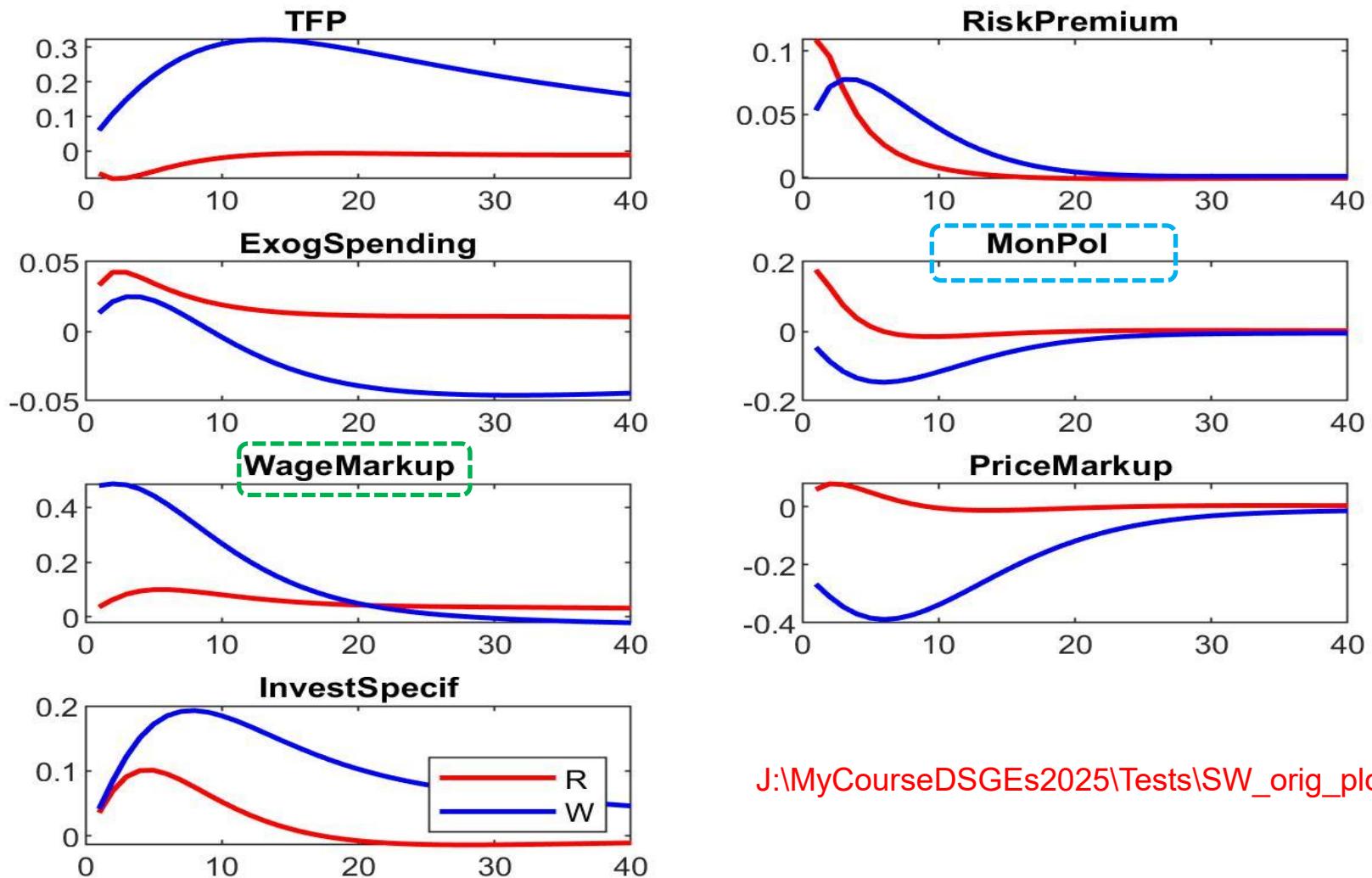


- Responses to all shocks – Capital and Labour



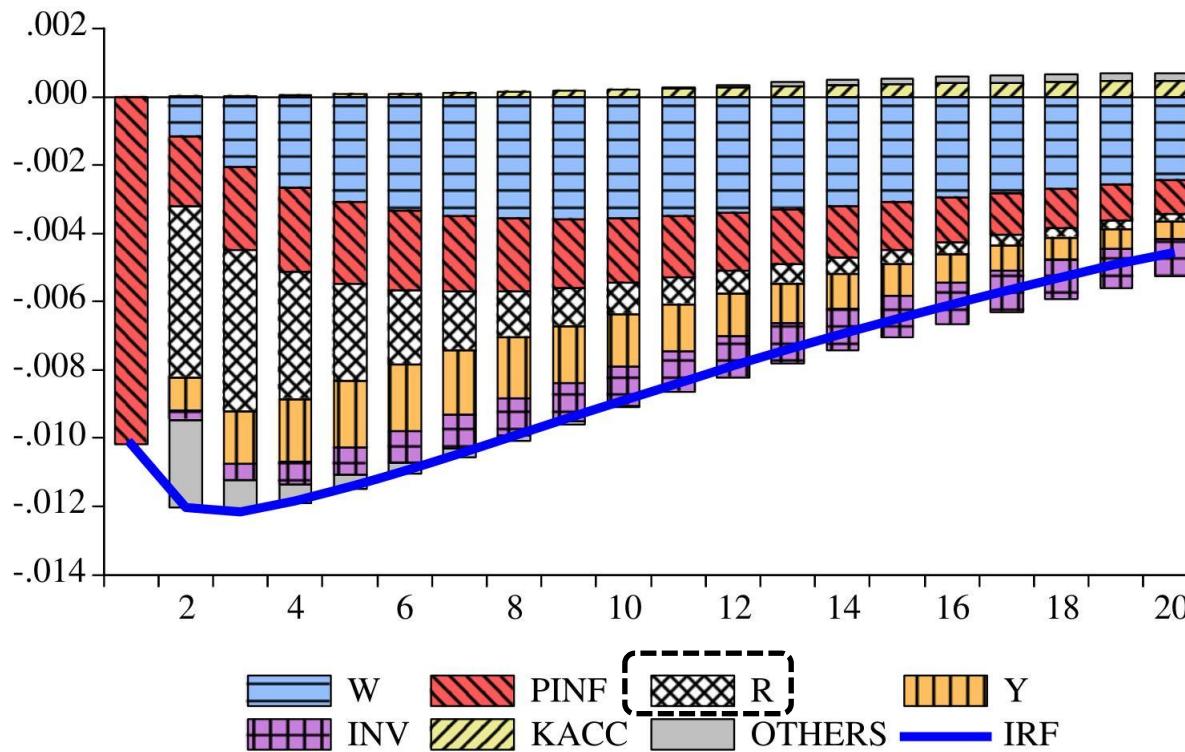
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- Responses to all shocks – Interest and Wage Rates



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- Labus and Labus (2019) construct a *decomposition* of SW *MonPol shock* effects on *inflation*:



- Note how lower *wage costs* and shrinking *demand* linked to decreasing *investments* additionally push down inflation rate beyond that caused by R

Final point

- Presently there is growing interest in DSGE models that have *more parameters, endogenous variables, exogenous shocks, and observables* than SW2007 model
- and substantial *additional complexities* from
 - *non-Gaussian* distributions (“*fat tails*”)
 - incorporation of *time-varying volatility*
- These higher-dimensional DSGE models are more *realistic* and potentially provide *better statistical fit* to observed data

- **Unfortunately**, Dynare is not currently capable of handling these more complex models
- **Fortunately**, staff at *Philadelphia Fed* have designed a user-friendly MATLAB software programme to reliably estimate high-dimensional DSGE models
- This is available freely to download, and is documented in:
- ***Chib, Shin and Tan (2020)***, “High-Dimensional DSGE Models: Pointers on Prior, Estimation, Comparison, and Prediction”, *Federal Reserve Bank of Philadelphia* Working Paper 20-35 (September 2020)

- For those interested in model which Chib *et al* use to illustrate working of their software, it is:
- Leeper, Traum and Walker (2017), "Clearing Up the Fiscal Multiplier Morass", *American Economic Review*, Vol. 107, No. 8, (2017), pp. 2409-2454
- To make this model more realistic, Chib *et al* introduce “fat-tailed” shocks and time-varying volatility
- Resulting model is ***high-dimensional***, consisting of
 - 51 parameters
 - 21 endogenous variables
 - 8 exogenous shocks
 - 8 observables
 - 1494 (!) non-Gaussian and nonlinear latent variables
- A simplified version of LTW model is available in ***Macroeconomic Model Data Base*** (version 3.1)

Appendix

- Slides below provide a more detailed exposition of the SW2007 model
- Do not be confused however by one notational difference with the main slides:
- Here, the ***shock processes*** are denoted by ε_t^x
- And the ***Kimball curvature parameters*** are denoted by $\epsilon_{w,p}$
- i.e., just the ***reverse*** of the notation in the main slides

Household Sector

- **As usual**, Household j chooses consumption $C_t(j)$, hours worked $L_t(j)$, bonds $B_t(j)$, investment $I_t(j)$ and capital utilisation $Z_t(j)$, so as to maximise an objective function, which SW define as

$$E_t \sum_{s=0}^{\infty} \beta^s \left[\frac{1}{1 - \sigma_c} (C_{t+s}(j) - \lambda C_{t+s-1})^{1-\sigma_c} \right] \exp \left(\frac{\sigma_c - 1}{1 + \sigma_l} L_{t+s}(j)^{1+\sigma_l} \right)$$

- Note that there is ***external habit formation***, captured by the parameter λ
- σ_c is ***coefficient of relative risk aversion*** of households or inverse of the intertemporal elasticity of substitution
- σ_l represents inverse of the ***elasticity of work effort*** with respect to real wage

NB: No “leisure” shocks here !

- This particular form of non-separable utility function is chosen because *standard, time-separable* preferences *cannot be made consistent* with observation that a positive monetary policy shock should typically lead to a persistent decline in real interest rate and a hump-shaped rise in consumption
- It is a *particular case* of what are known as “*King-Plosser-Rebelo Preferences*” which are used in many DSGE models because they are compatible with balanced growth along the optimal steady state path

- ***Footnote:***
- ***Characteristic of most industrialised countries:*** output per capita, consumption per capita, investment per capita and other variables exhibit ***growth*** over long periods of time
- These are known as the “Kaldor facts”
- Such long-run growth occurs at rates that are ***roughly constant over time*** within economies but differ across economies
- This pattern suggests steady state growth, which means that levels of certain key variables grow at a constant rate
- In that case, we say there is a ***balanced growth path***

- General form of **KPR Preferences**:

$$u(C, L) = \frac{1}{1 - \sigma_c} C^{1 - \sigma_c} v(L)$$

- where $v(L)$ is increasing and concave if $0 < \sigma_c < 1$ or decreasing and convex if $\sigma_c > 1$
- In ***limit*** case of $\sigma_c = 1$, resulting preferences specification is ***additively separable*** and given by
 $u(C, L) = \ln C_t + v(L)$
- For SW, $v(L)$ is given by $\exp\left(\frac{\sigma_c - 1}{1 + \sigma_l} L_{t+s}(j)^{1 + \sigma_l}\right)$
- SW also include habit persistence in first part

- Reason why **KPR Preferences** work with a balanced growth path is that in a competitive equilibrium, marginal rate of substitution between consumption and leisure must equal (inverse of) marginal product of labour

- With KPR

$$\begin{aligned} u(c, l) &= \frac{c^{1-\sigma}}{1-\sigma} v(l), \quad \sigma \neq 1, \sigma > 0 \\ &= \log c + v(l) \quad \text{for } \sigma = 1 \end{aligned}$$

- hence

- $\frac{\partial u / \partial l}{\partial u / \partial c} = \frac{\frac{c^{1-\sigma}}{1-\sigma} v'(l)}{c^{-\sigma} v(l)} = \frac{cv'(l)}{(1-\sigma)v(l)} = MPL$

- Along balanced growth path, MPL and c grow at same rate, so l (labour) can be constant

- Each period, household makes *sequence of decisions*
- *First*, consumption decision, capital accumulation decision, and decision on how many units of capital services to supply
- *Second*, purchases securities whose payoffs are contingent upon whether household can re-optimize its wage decision
- *Third*, sets wage rate at which it is prepared to work after finding out whether it can re-optimize or not
- *Finally*, receives lump-sum transfer from monetary authority

- Uncertainty faced by household over whether it can re-optimize its wage is *idiosyncratic*
- → households work different amounts and earn *different* wage rates
- → households are *heterogeneous* with respect to labour

- Household's ***budget constraint*** is

$$C_{t+s}(j) + I_{t+s}(j) + \frac{B_{t+s}(j)}{\varepsilon_t^b R_{t+s} P_{t+s}} - T_{t+s}$$

$$\leq \frac{B_{t+s-1}(j)}{P_{t+s}} + \frac{W_{t+s}^h(j)L_{t+s}(j)}{P_{t+s}} + \frac{R_{t+s}^k Z_{t+s}(j)K_{t+s-1}(j)}{P_{t+s}} - a(Z_{t+s}(j))K_{t+s-1}(j) + \frac{Div_{t+s}}{P_{t+s}}$$

- One-period bond $B_{t+s}(j)$ is expressed on a discount basis, where R_{t+s} is interest rate
- T_{t+s} are lump sum taxes or subsidies
- Div_{t+s} are (per-capita) dividends distributed by intermediate goods producers and labour unions
- P_{t+s} is overall price level
- Penultimate term represents cost associated with variations in degree of ***capital utilisation***

- Households augment their financial assets through increasing their government nominal bond holdings (B_{t+1}), from which they earn an interest rate of R_t
- Return on these bonds is subject to a risk ***shock*** ε_t^b which may be considered as an exogenous ***premium*** in return to bonds, reflecting
 - inefficiencies in financial sector (leading to some premium on deposit rate versus risk-free rate set by central bank), or
 - a risk premium that households require to hold one period bond
- and $\ln \varepsilon_t^b = \rho_b \ln \varepsilon_{t-1}^b + \eta_t^b, \eta_t^b \sim N(0, \sigma_b)$

- Investment is assumed by SW to augment household's physical capital stock according to

$$K_t(j) = (1 - \delta)_{t-1}(j) + \varepsilon_t^i \left[1 - S \left(\frac{I_t(j)}{I_{t-1}(j)} \right) \right] I_t(j)$$

- δ is depreciation rate
- $S(\cdot)$ is ***adjustment cost*** function $S(x_t) = \frac{\varphi}{2} (x_t - \gamma)^2$
- ε_t^i is a stochastic ***shock*** to price of investment (relative to consumption goods) and follows an exogenous process

$$\ln \varepsilon_t^i = \rho_i \ln \varepsilon_{t-1}^i + \eta_t^i, \eta_t^i \sim N(0, \sigma_i)$$

- Households choose utilisation rate of capital
- Amount of ***effective capital*** that households can rent to firms is

$$K_t^s(j) = Z_t(j)K_{t-1}(j)$$

- Income from ***renting capital services*** is

$$R_t^k Z_t(j) K_{t-1}(j)$$

- Cost of changing ***capital utilisation*** is given by
 $Z_t = \kappa + R_t^k (1 - \psi) / \psi$
- where ψ is normalized to be between zero and one
- κ is a constant

- When $\psi = 1$, it is extremely costly to change utilisation of capital and as a result capital utilisation remains constant
- In contrast, when $\psi = 0$, marginal cost of changing capital utilisation is constant → in equilibrium rental rate on capital is constant

- FOCs for Household are

$$(\partial C_t) \quad \Xi_t = \exp\left(\frac{\sigma_c - 1}{1 + \sigma_l} L_t(j)^{1 + \sigma_l}\right) (C_t - \lambda C_{t-1})^{-\sigma_c}$$

$$(\partial L_t) \quad \left[\frac{1}{1 - \sigma_c} (C_t - h C_{t-1})^{1 - \sigma_c} \right] \exp\left(\frac{\sigma_c - 1}{1 + \sigma_l} L_t^{1 + \sigma_l}\right) (\sigma_c - 1) L_t^{\sigma_l} = -\Xi_t \frac{W_t^h}{P_t}$$

Lagrange Multiplier

$$(\partial B_t) \quad \Xi_t = \beta \varepsilon_t^b R_t E_t \left[\frac{\Xi_{t+1}}{\pi_{t+1}} \right]$$

$$(\partial I_t) \quad \Xi_t = \Xi_t^k \varepsilon_t^i \left(1 - S\left(\frac{I_t}{I_{t-1}}\right) - S'\left(\frac{I_t}{I_{t-1}}\right) \frac{I_t}{I_{t-1}} \right) \\ + \beta E_t \left[\Xi_{t+1}^k \varepsilon_{t+1}^i S'\left(\frac{I_{t+1}}{I_t}\right) \left(\frac{I_{t+1}}{I_t}\right)^2 \right]$$

$$(\partial \bar{K}_t) \quad \Xi_t^k = \beta E_t \left[\Xi_{t+1} \left(\frac{R_{t+1}^k}{P_{t+1}} Z_{t+1} - a(Z_{t+1}) \right) + \Xi_{t+1}^k (1 - \delta) \right]$$

$$(\partial u_t) \quad \frac{R_t^k}{P_t} = a'(Z_t) \quad \text{Tobin's } Q_t = \Xi_t^k / \Xi_t$$

$$\begin{aligned}\Xi_t &= \Xi_t^k \varepsilon_t^i \left(1 - S\left(\frac{I_t}{I_{t-1}}\right) - S'\left(\frac{I_t}{I_{t-1}}\right) \frac{I_t}{I_{t-1}} \right) \\ &\quad + \beta E_t \left[\Xi_{t+1}^k \varepsilon_{t+1}^i S'\left(\frac{I_{t+1}}{I_t}\right) \left(\frac{I_{t+1}}{I_t}\right)^2 \right]\end{aligned}$$

- FOC for I is law of motion for shadow value of capital
- If adjustment cost were absent ($S=0$), FOC would simply say that $\Xi_t = \Xi_t^k \rightarrow$ marginal utility of consumption (Ξ_t) [shadow **cost** of taking resources away from consumption] = shadow **benefit** of putting these resources into investment (Ξ_t^k): Tobin's Q is one

$$\Xi_t^k = \beta E_t \left[\Xi_{t+1} \left(\frac{R_{t+1}^k}{P_{t+1}} Z_{t+1} - a(Z_{t+1}) \right) + \Xi_{t+1}^k (1 - \delta) \right]$$

- FOC for K says that if you buy a unit of capital today you have to pay its price in real terms (LHS: Ξ_t^k)
- but tomorrow (RHS)
- you will get proceeds from renting capital (first part)
- and you can sell back any capital that has not depreciated (second part)

Intermediate Goods Sector

- *Intermediate Good Producer i* is assumed by SW to use standard Cobb-Douglas technology (plus a fixed cost)

$$Y_t(i) = \epsilon_t^a K_t^s(i)^\alpha [\gamma^t L_t(i)]^{1-\alpha} - \gamma^t \Phi$$

- where
 - $K_t^s(i)$ is capital services used in production
 - $L_t(i)$ is a composite labour input
 - Φ is a fixed cost
 - γ^t represents a labour-augmenting deterministic growth rate in the economy

- ϵ_t^a is ***total factor productivity*** and follows process

$$\ln \epsilon_t^a = (1 - \rho_z) \ln \epsilon^a + \rho_z \ln \epsilon_{t-1}^a + \eta_t^a, \eta_t^a \sim N(0, \sigma_a)$$

- Firm's profit is given by

$$P_t(i) Y_t(i) - W_t L_t(i) - R_t^k K_t^s(i)$$

- where
 - W_t is aggregate nominal wage rate
 - R_t^k is rental rate on capital
- Cost minimization yields following first-order conditions:

$$(\partial L_t(i)) : \Theta_t(i) \gamma^{(1-\alpha)t} (1 - \alpha) \epsilon_t^a K_t^s(i)^\alpha L_t(i)^{-\alpha} = W_t$$

$$(\partial K_t^s(i)) : \Theta_t(i) \gamma^{(1-\alpha)t} \alpha \epsilon_t^a K_t^s(i)^{\alpha-1} L_t(i)^{1-\alpha} = R_t^k$$

- where $\Theta_t(i)$ is Lagrange multiplier associated with production function (and equals marginal cost MC_t)
- Combining these FOCs and noting that capital-labour ratio is equal across firms implies (as usual)

$$K_t^s = \frac{\alpha}{1 - \alpha} \frac{W_t}{R_t^k} L_t$$

- Marginal cost MC_t is *same for all firms* and equal to

$$MC_t = \alpha^{-\alpha} (1 - \alpha)^{-(1-\alpha)} W_t^{1-\alpha} R_t^{k\alpha} \gamma^{-(1-\alpha)t} (\epsilon_t^a)^{-1}$$

- Prices of *intermediate goods* are determined by Calvo fairy
- As in previous Calvo models, in each period, each firm i faces a constant probability $1 - \xi_p$ of being able to re-optimize its price $P_t(i)$
- Probability that any firm receives a signal to re-optimize its price is assumed to be *independent* of time that it last reset its price

- Unlike in simpler earlier models, SW assume that if Calvo fairy does not allow a firm to optimise its price in a given period, it then adjusts its price by a ***weighted combination*** of lagged and steady-state inflation rates as:

$$P_t(i) = (\pi_{t-1})^{\iota_p} (\pi_*)^{1-\iota_p} P_{t-1}(i)$$

- where
 - $0 \leq \iota_p \leq 1$
 - π_{t-1} denotes gross inflation in period $t-1$
 - π_* denotes steady-state gross inflation rate
- A positive value of ι_p introduces ***structural inertia*** into inflation process

- Under Calvo pricing with this type of ***partial indexation***, optimal price $\tilde{P}_t(i)$ set by firm that is allowed by Calvo fairy (with a probability $1-\xi_p$) to re-optimize results from solving following optimisation problem:

$$\max_{\tilde{P}_t(i)} E_t \sum_{s=0}^{\infty} \xi_p^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \left[\tilde{P}_t(i) (\prod_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p}) - MC_{t+s} \right] Y_{t+s}(i)$$

s.t. $Y_{t+s}(i) = Y_{t+s} G'^{-1} \left(\frac{P_t(i) X_{t,s}}{P_{t+s}} \tau_{t+s} \right)$

Nominal discount factor

- This looks formidable!
- But if we compare it to what we used in very first sticky-price model studied back at beginning of term, we find clear similarities:

$$\max_{P_{j,t}^*} E_t \sum_{i=0}^{\infty} (\beta\theta)^i (P_{j,t}^* Y_{j,t+i} - TC_{j,t+i})$$

$$\max_{\tilde{P}_t(i)} E_t \sum_{s=0}^{\infty} \xi_p^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \left[\tilde{P}_t(i) \left(\prod_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p} \right) - MC_{t+s} \right] Y_{t+s}(i)$$

- Since $TC = MC^*Y$, only really ***new element*** is ***indexation factor*** multiplying \tilde{P}_t : $\left(\prod_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p} \right)$

- What is this indexation factor $(\prod_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p})$
- Since π is a ***gross*** inflation rate, it has each period a value close to 1
- Thus, product $\prod_{l=1}^s \pi_{t+l-1}^{\iota_p}$ (which multiplies together inflation rates for all periods since previous price revision) also has a value near 1
- This is then scaled by factor $\pi_*^{1-\iota_p}$ which reflects ***weighted combination*** indexation
- Finally, as noted previously, curious expression $\frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}}$ is just the (nominal) discount factor

- It is also clear from formulation of the optimisation problem

$$\max_{\tilde{P}_t(i)} E_t \sum_{s=0}^{\infty} \xi_p^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \left[\tilde{P}_t(i) (\Pi_{l=1}^s \pi_{t+l-1}^{\iota_p} \pi_*^{1-\iota_p}) - \boxed{MC_{t+s}} \right] Y_{t+s}(i)$$

- that price set by firm i , at time t , is a function of ***expected future*** marginal costs
- Price will be a ***mark-up*** over these weighted marginal costs
 - ***fixed*** over time if prices are perfectly flexible ($\iota_p = 0$), or
 - ***varying*** with sticky prices ($\iota_p \neq 0$)

- Given FOC for this optimisation problem, ***aggregate price index*** which results is

$$P_t = (1 - \xi_p) P_t(i) G'^{-1} \left[\frac{P_t(i) \tau_t}{P_t} \right] + \xi_p \pi_{t-1}^{\iota_p} \pi_*^{1-\iota_p} P_{t-1} G'^{-1} \left[\frac{\pi_{t-1}^{\iota_p} \pi_*^{1-\iota_p} P_{t-1} \tau_t}{P_t} \right]$$

- Compare this to what we used in very first sticky-price model studied back at beginning of term:

$$P_t = \left[\theta P_{t-1}^{1-\psi} + (1 - \theta) P_t^* {}^{1-\psi} \right]^{\frac{1}{1-\psi}}$$

- Function G (of which G' is used in definition above) is defined in Appendix

Final Good Sector

- As usual, *final good* Y_t is a composite made of a continuum of *intermediate goods* $Y_t(i)$
- *Final Good Producers* buy intermediate goods and package them into Y_t using not the Dixit-Stiglitz aggregator but instead the *Kimball Aggregator*
- In contrast to Dixit–Stiglitz world of a constant elasticity and a *constant* desired mark-up of price over marginal cost, in Kimball’s world desired mark-up is *decreasing in a firm’s relative price*
- Final Good Producers sell final good to consumers, investors and government in a perfectly competitive market

- Final Good Producers maximize profits

$$\max_{Y_t, Y_t(i)} P_t Y_t - \int_0^1 P_t(i) Y_t(i) di$$

- subject to Kimball Aggregator

$$\left[\int_0^1 G\left(\frac{Y_t(i)}{Y_t}; \varepsilon_t^p\right) di \right] = 1 \quad (\mu_{f,t})$$

- where P_t and $P_t(i)$ are price of final and intermediate goods respectively

- G is a ***strictly concave*** and increasing function (ie, ***decreasing slope G'***) characterised by $G(1) = 1$

- ε_t^p is an exogenous process that reflects ***shocks*** to aggregator function that result in changes in elasticity of demand and thus in mark-up

- ε_t^p follows an ARMA(1,1) process defined by

$$\varepsilon_t^p = \rho_p \varepsilon_t^p + \eta_t^p - \mu_p \eta_{t-1}^p$$

- Combining first-order conditions with respect to $Y_t(i)$ and Y_t results in

$$Y_t(i) = Y_t G'^{-1} \left[\frac{P_t(i)}{P_t} \int_0^1 G' \left(\frac{Y_t(i)}{Y_t} \right) \frac{Y_t(i)}{Y_t} di \right]$$

- Assumptions on G (specifically, G' decreasing) imply that demand for input $Y_t(i)$ is ***decreasing*** in its relative price
- while elasticity of demand is a positive function of relative price (or a negative function of relative output)

- For completeness, note that G is defined as

$$G_Y \left(\frac{Y_t(f)}{Y_t} \right) = \frac{\phi_t^p}{1 - (\phi_t^p - 1)\epsilon_p} \left[\left(\frac{\phi_t^p + (1 - \phi_t^p)\epsilon_p}{\phi_t^p} \right) \frac{Y_t(f)}{Y_t} + \frac{(\phi_t^p - 1)\epsilon_p}{\phi_t^p} \right] \frac{1 - (\phi_t^p - 1)\epsilon_p}{\phi_t^p - (\phi_t^p - 1)\epsilon_p}$$

$$+ \left[1 - \frac{\phi_t^p}{1 - (\phi_t^p - 1)\epsilon_p} \right]$$

- $\phi_t^p \geq 1$ is gross mark-up of the intermediate firms
- ϵ_p is degree of ***curvature*** of firm's demand (do not confuse this with ε_t^p , the price shock !!)

- When *curvature parameter* $\epsilon_p = 0$, demand curve exhibits *constant elasticity* as with standard Dixit-Stiglitz aggregator
- When ϵ_p is positive, firm instead faces a *quasi-kinked* demand curve, implying that a drop in its relative price only stimulates a *small increase* in demand
- On other hand, a rise in its relative price generates a *large fall* in demand
- Relative to standard Dixit-Stiglitz aggregator, this introduces more *strategic complementarity* in price setting which causes intermediate firms to *adjust* prices *less* to a given change in marginal cost (hence “stickier”)

- Let us now return to solution of the FOCs, given previously by

$$Y_t(i) = Y_t G'^{-1} \left[\frac{P_t(i)}{P_t} \int_0^1 G' \left(\frac{Y_t(i)}{Y_t} \right) \frac{Y_t(i)}{Y_t} di \right]$$

- Recall that $G'^{-1} \left[\frac{P_t(i)\tau_t}{P_t} \right]$ was used in defining P_t
- Definition of τ_t is $\tau_t = \int_0^1 G' \left(\frac{Y_t(i)}{Y_t} \right) \frac{Y_t(i)}{Y_t} di$
- So in fact we have simply that
- $Y_t(i)/Y_t = G'^{-1} \left[\frac{P_t(i)\tau_t}{P_t} \right]$
- which is why we earlier stated that demand for input $Y_t(i)$ is ***decreasing*** in its relative price

- When all intermediate firms produce the same amount, we have $\frac{Y_t(i)}{Y_t} = 1$
- But $G_Y(1) = 1$, implying **constant returns to scale** in this case

Labour Sector

- SW assume that Households supply their homogenous labour to an *intermediate labour union* which differentiates labour services, sets wages subject to a Calvo scheme and offers those labour services to intermediate *labour packers*
- Labour used by intermediate goods producers L_t is a composite made of those differentiated labour services $L_t(i)$
- As with intermediate goods, *Kimball aggregator* is used, with a *curvature parameter* of ϵ_w

- Labour packers buy differentiated labour services, package L_t , and offer it to intermediate goods producers
- Labour packers maximize profits

$$\begin{aligned} & \max_{L_t, L_t(i)} W_t L_t - \int_0^1 W_t(i) L_t(i) di \\ & s.t. \left[\int_0^1 H\left(\frac{L_t(i)}{L_t}; \varepsilon_t^w\right) di \right] = 1 \quad (\mu_{l,t}) \end{aligned}$$

- where W_t and $W_t(i)$ are prices of composite and intermediate labour services respectively
- ***Like G previously,*** function H is a strictly concave and increasing function characterised by $H(1) = 1$

- ε_t^w is an exogenous process that reflects *shocks* to aggregator function that result in changes in elasticity of demand and therefore in mark-up [do not confuse this with ϵ_w - the curvature !!]
- Similarly to ε_t^p , it is assumed that ε_t^w follows an ARMA(1,1) process

$$\varepsilon_t^w = \rho_w \varepsilon_{t-1}^w + \eta_t^w - \mu_w \eta_{t-1}^w$$

- Combining FOCs results (as for $Y_t(i)$ previously) in

$$L_t(i) = L_t H'^{-1} \left[\frac{W_t(i)}{W_t} \int_0^1 H' \left(\frac{L_t(i)}{L_t} \right) \frac{L_t(i)}{L_t} di \right]$$

- Labour unions are an intermediary between households and labour packers
- Under Calvo pricing with partial indexation, optimal wage set by union that is allowed to re-optimize its wage results from an optimisation problem similar to that for pricing

$$\max_{\widetilde{W}_t(i)} E_t \sum_{s=0}^{\infty} \xi_w^s \frac{\beta^s \Xi_{t+s} P_t}{\Xi_t P_{t+s}} \left[\widetilde{W}_t(i) (\prod_{l=1}^s \gamma \pi_{t+l-1}^{\iota_w} \pi_*^{1-\iota_w} - W_{t+s}^h) \right] L_{t+s}(i)$$

$$s.t. L_{t+s}(i) = L_{t+s} H'^{-1} \left(\frac{W_t(i) X_{t,s}^w}{W_{t+s}} \tau_{t+s}^w \right)$$

- Again, similarly to case for pricing, ***aggregate wage index*** is in this case given by

$$W_t = (1 - \xi_w) \widetilde{W}_t H'^{-1} \left[\frac{\widetilde{W}_t \tau_t^w}{W_t} \right] + \xi_w \gamma \pi_{t-1}^{\iota_w} \pi_*^{1-\iota_w} W_{t-1} H'^{-1} \left[\frac{\gamma \pi_{t-1}^{\iota_w} \pi_*^{1-\iota_w} W_{t-1} \tau_t^w}{W_t} \right]$$

- Mark-up*** of aggregate wage over wage received by households is distributed to households in form of ***dividends*** (as already indicated in budget constraint)

Government Sector

- SW (both of whom work for central banks) assume that central bank follows a nominal interest rate rule by adjusting its instrument in response to deviations of inflation and output from their respective target levels

$$\frac{R_t}{R^*} = \left(\frac{R_{t-1}}{R^*} \right)^\rho \left[\left(\frac{\pi_t}{\pi_*} \right)^{r_\pi} \left(\frac{Y_t}{Y_t^*} \right)^{r_y} \right]^{1-\rho} \left(\frac{Y_t/Y_{t-1}}{Y_t^*/Y_{t-1}^*} \right)^{r_{\Delta y}} \epsilon_t^r$$

- where
 - R^* is steady state nominal rate (gross rate)
 - Y_t^* is “***natural output***”
 - ρ determines degree of interest rate smoothing
- Exogenous monetary policy **shock** ϵ_t^r is determined as

$$\ln \epsilon_t^r = \rho_r \ln \epsilon_{t-1}^r + \eta_t^r, \eta_t^r \sim N(0, \sigma_r)$$

- A new element is included in this version of Taylor Rule: “*natural output level*”
- SW define it as “the output in the flexible price and wage economy without mark-up shock in prices and wages”, ie *NOT* subject to sticky wages and prices
- In model eventually estimated below, there will be an *entire section* devoted to a “natural” or “*flex-price*” version of the economy, *ONLY* so as to be able to make use of the central bank reaction function defined by SW version of Taylor Rule – *no need for this otherwise*

- Government budget constraint is of form

$$P_t G_t + B_{t-1} = T_t + \frac{B_t}{R_t}$$

- where T_t are nominal lump-sum taxes (or subsidies) that also appear in household's budget constraint
- SW assume that Government purchases G_t are exogenous
- They express ***government spending*** relative to steady state output path ("trend output") as

$$\varepsilon_t^g = G_t / (Y\gamma^t)$$

- Path of ε_t^g is assumed to be given by

$$\ln \varepsilon_t^g = (1 - \rho_g) \ln \varepsilon^g + \rho_g \ln \varepsilon_{t-1}^g + \rho_{ga} \ln \varepsilon_t^a - \rho_{ga} \ln \varepsilon_{t-1}^a + \eta_t^g, \eta_t^g \sim N(0, \sigma_g)$$

- This allows for a reaction of government spending to productivity process γ^t
- Government purchases have no effect on marginal utility of private consumption, nor do they serve as an input into goods production

Market Equilibrium

- Finally, ***Market equilibrium***
- Final goods market is in equilibrium if production equals demand by households for consumption and investment and by government

$$C_t + I_t + G_t + a(Z_t)K_{t-1} = Y_t$$

- ***Capital rental market*** is in equilibrium when demand for capital by intermediate goods producers equals supply by households
- ***Labour market*** is in equilibrium if firms' demand for labour equals labour supply at wage level set by households
- In capital market, equilibrium means that ***government debt*** is held by domestic investors at market interest rate, R_t

Recall: General NK model introduced into basic RBC more realistic specifications:

- *imperfect* competition ⇒ Last week
- *frictions* in *prices* and in *wages* ⇒ Last week
- *habit* formation in consumption ⇒ Last week
- *non-Ricardian* agents ⇒ Last week
- *adjustment costs* in investment ⇒ Last week
- *capacity* (non-)utilisation *costs* ⇒ Last week
- *government* sector (*monetary* and *fiscal*) ⇒ Last week
- *financial* sector ⇒ Week 6
- *housing* sector ⇒ Week 6
- *unemployment* ⇒ Week 7
- *environment* ⇒ Week 7
- *foreign* trade ⇒ Week 8