$$B = (1+r)B + \frac{1}{2}Y - C$$

$$B^* = (1+r)B^* + \frac{1}{2}Y - C^*,$$

with $B^*(1+r) = B^{high}$, so that

$$\frac{B^{high}}{1+r} = B^{high} + \frac{1}{2}Y - C^*$$

that becomes

$$C^* = \frac{1}{2}Y + \frac{r}{1+r}B^{high}$$

(recall that B is negative for the impatient agent). From the goods market equilibrium condition

$$C + C^* = Y$$

we obtain the consumption of the patient agent

$$C = \frac{1}{2}Y - \frac{r}{1+r}B^{high}.$$

To determine the equilibrium real interest rate, we note that the patient agent is not facing the borrowing limit and as such her/his optimal consumption plan satisfies the Euler equation. The Euler equation is

$$\frac{1}{C_t} = (1+r_t)\beta\frac{1}{C_{t+1}}$$

so that in steady state, $C_t = C_{t+1}$

$$r = \frac{1-\beta}{\beta}$$

b. Consider now the following experiment in which the debt limit is reduced from B^{high} to B^{low}. Determine the new steady state and discuss how it differs from the one computed in part (a).

ANSWER: the steady state is very similar to the one derived above. The only difference consists in replacing the new debt limit. We also note that the change in the debt limit does not influence the long-run real interest rate. So we have the following expressions:

$$C^* = \frac{1}{2}Y + \frac{r}{1+r}B^{low}$$

$$C = \frac{1}{2}Y - \frac{r}{1+r}B^{high}.$$

$$r = \frac{1-\beta}{\beta}$$

In the new steady state the impatient agents borrow less (as the debt limit is more restrictive) so it means that they are able to consume relatively more that in the previous steady state since in the long run they have to repay their debt (that is now lower).

c. Consider now the transition between the old and the new steady state. Assume that the transition takes only one period and determine the short-run values for the consumption of the patient and impatient agents and the equilibrium real interest rate. Under which conditions the real interest rate can be negative?

   ANSWER: This again follows from the lecture note. We need to determine the transition between the old and new steady state. From budget constraint of the impatient agent (borrower) we have

$$B_t^* = (1+r_{t-1})B_{t-1}^* + \frac{1}{2}Y - C_t^*,$$

$$B_S^* = B_{t-1}^{*High} + \frac{1}{2}Y - C_S^*$$

where $B_{t-1}^{High*} = (1+r_{t-1})B_{t-1}$.If we assume that deleveraging occurs within a period then $B_S = \frac{B^{low}}{1+r_S}$

$$\frac{1}{1+r_S}B^{low*} = B_{t-1}^{High*} + \frac{1}{2}Y - C_S^*$$

and the short-run consumption of the borrower is given by:

$$C_S^* = \frac{1}{2}Y - \frac{1}{1+r_S}B^{low*} + B^{high*}$$

Keeping into account the goods market equilibrium condition $(C + C^* = Y)$ we can find the long and short run consumption of the savers as

$$C_S = \frac{1}{2}Y + \frac{1}{1+r_S}B^{low*} - B^{high*}$$

$$C_L = \frac{1}{2}Y - \frac{r}{1+r}B^{low*}$$

Since savers behavior satisfies the Euler equation

$$C_L(s) = (1+r_S)\beta(s)C_S(s)$$

we obtain the short run interest rate as

$$1 + r_S = \frac{\frac{1}{2}Y - B^{low*}}{\beta\left(\frac{1}{2}Y - B^{high*}\right)}$$

In the short run the real interest rate depends on the amount of deleveraging (i.e. the reduction in the borrowing limit). The bigger the deleveraging the lower is the short-run real interest rate. In particular, the short-run real interest rate becomes negative when

$$1 + r_S = \frac{\frac{1}{2}Y - B^{low*}}{\beta\left(\frac{1}{2}Y - B^{high*}\right)} < 1$$

so that

$$\frac{1}{2}Y - B^{low*} < \beta\left(\frac{1}{2}Y - B^{high*}\right)$$

d. Suppose now that both agents issue nominal bonds in zero net supply and can be traded only within each economy. Denote with P_t the price level. Discuss how the equilibrium interest rate and consumptions change in this case, during the one-period deleveraging transition, if there is a zero lower bound on nominal interest rates.

ANSWER: We need to rewrite slightly the budget constraints of the agents to keep into account that now they can trade also nominal assets along with the real bond.

$$B_t^* + \frac{B_t^{nom}}{P_t} = (1 + r_{t-1})B_{t-1}^* + \frac{B_{t-1}^{nom}}{P_t}(1 + i_t) + \frac{1}{2}Y - C_t^*,$$

where B^{nom} is the nominal bond and we have written as before the budget constraint in real terms and similar for the patient agent. Note also that the borrowing constraint is on the real bond. The patient agent is going to price also the nominal bond through the associated Euler equation.

$$\frac{1}{C_t} = (1 + i_t)\beta\frac{1}{C_{t+1}}\frac{P_t}{P_{t+1}}$$

The bond is in zero net supply: this means that the oustanding holding of the bond is zero and one agent's position is balanced by the opposite holding of the other agent.

$$B^{nom} + B^{nom*} = 0$$

In terms of consumption and the characterization of the equilibrium, everything is the same as above. The only difference arises from the nominal side as we examine the Fisher equation that arbitrage real and nominal assets' return. In general we have

$$1 + r_t = (1 + i_t)\frac{P_t}{P_{t+1}}$$

So we now focus on the short-run to examine the implication of deleveraging for nominal interest rates in the short run. We impose that $i_t \geq 0$ and assume that in the long run prices reflect an environment in which the zero lower bound is not binding and price level will be stable: $P_L = P^*$.In the short run we have:

$$1 + r_S = (1 + i_S)\frac{P_S}{P^*}$$

and the right policy would imply $P_S = P^*$ if the nominal interest rate did not have a lower bound. If $i_t = 0$ and real interest rate are negative then

$$(1+i_S)\frac{P_S}{P^*} = \frac{P_S}{P^*} = \frac{\frac{1}{2}Y - B^{low}}{\beta\left(\frac{1}{2}Y - B^{high}\right)} < 1$$

as the zero lower bound constraint becomes binding. Since prices are flexible the way to achieve the equilibrium real interest rate is to have a drop in prices in the short run (create deflation). Also note that this deflation does not have any real effects (i.e. there are no real consequences from the zero lower bound constraint and deflation).

e. Suppose now that there are no real bonds and the only assets that is traded are nominal currency bonds. The debt limit now becomes

$$\frac{B_t^{nom}}{P_t}(1+r_t) > B^{high}$$

and

$$\frac{B_t^{nom*}}{P_t}(1+r_t) > B^{high}.$$

Consider as before the experiment in which the debt limit is reduced from B^{high} to B^{low} and assume that the transition from one steady state to the other takes only one period. Determine the short-run value of both consumptions and the equilibium interest rate and compare your results with the one obtained in part (3).

ANSWER: Let's first rewrite the impatient budget constraint. We first derive the initial consumption and the new long-run consumption functions for the patient and impatient agents.

As the patient agent is constraint we have

$$\frac{B^{nom}}{P}(1+r) = B^{high}$$

in steady state.

$$\frac{B^{nom*}}{P} = \frac{B^{nom*}}{P}(1+i) + \frac{1}{2}Y - C_t^*,$$

$$\frac{B^{high*}}{1+r} = B^{high*} + \frac{1}{2}Y - C_t^*,$$

since in steady state prices are stable we have that $i = r$ so that

$$C^* = \frac{1}{2}Y + \frac{r}{1+r}B^{high*}$$

so that the consumption of the patient agent is going to be the same as in the case above. The new long run equilibrium with lower nominal borrowing limit does not change as well.

So we now focus on the transition

$$\frac{B_t^{nom*}}{P_t} = \frac{B_{t-1}^{nom*}}{P_t}(1+i_t) + \frac{1}{2}Y - C_t^*,$$

In the transition we have

$$\frac{B_S^{nom*}}{P_S} = \frac{B_{t-1}^{nom*}}{P_S}(1+i_S) + \frac{1}{2}Y - C_S^*,$$

with

$$\frac{B_t^{nom*}}{P_t}(1+r_t) = B^{high}.$$

so that

$$\frac{B_{t-1}^{nom}}{P_S} = \frac{B^{high}}{(1+r_S)}$$

Substituing in:

$$\frac{B_S^{low*}}{1+r_S} = \frac{B^{high}}{(1+r_S)}(1+i_S) + \frac{1}{2}Y - C_S^*,$$

Since

$$(1+r_S) = (1+i_S)\frac{P_S}{P_L}$$

we have

$$\frac{B_S^{low*}}{1+r_S} = \frac{B^{high}}{P_S}P_L + \frac{1}{2}Y - C_S^*,$$

Te consumption of the patient agent is

$$C_S = \frac{1}{2}Y - \frac{B^{high}}{P_S}P_L + \frac{B_S^{low}}{1+r_S},$$

We can normalize $P_L = 1$ since that is the long-run price level that we can set in an arbitrary way as long as the zero lower bound constraint does not bind in the long-run. We now need to compute that new real interest rate in the transition:

$$C_L = (1+r_S)\beta C_S$$

with

$$C_L = \frac{1}{2}Y - \frac{r}{1+r}B^{low}$$

and

$$C_S = \frac{1}{2}Y - \frac{B^{high}}{P_S}P_L + \frac{B^{low}}{1+r_S}$$

In deriving the final expression for the short-run real interest rate recall that in the long run $\beta(1+r) = 1$ so that $\frac{r}{1+r} = 1-\beta$.

$$\frac{1}{2}Y - \frac{r}{1+r}B^{low} = (1+r_S)\beta\left[\frac{1}{2}Y - \frac{B^{high}}{P_S}P_L + \frac{B^{low}}{1+r_S}\right]$$

$$\frac{1}{2}Y - (1-\beta)\,B^{low} = (1+r_S)\beta\left[\frac{1}{2}Y - \frac{B^{high}}{P_S}P_L\right] + \beta B^{low}$$

that becomes

$$1 + r_S = \frac{\frac{1}{2}Y - B^{low}}{\beta\left(\frac{1}{2}Y - \frac{B^{high}}{P_S}P_L\right)}$$