Intermediate Microeconomics

Choice

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Outline

- Utility maximization and Mashallian demand
  - Lagrangian and interior solutions
  - Corner solutions
- Expenditure minimization and Hicksian demand
- Slutsky equation
  - Duality
  - Decomposition: substitution & income effects
- Envelop theorem
  - Roy's identity
  - Shephard's lemma

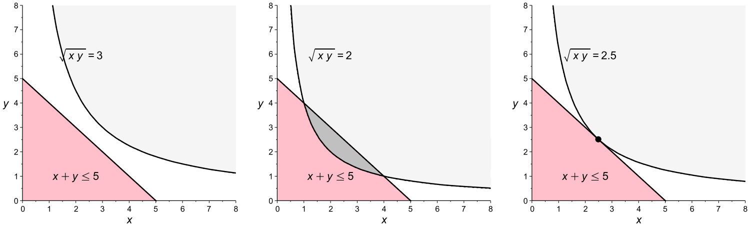
Review

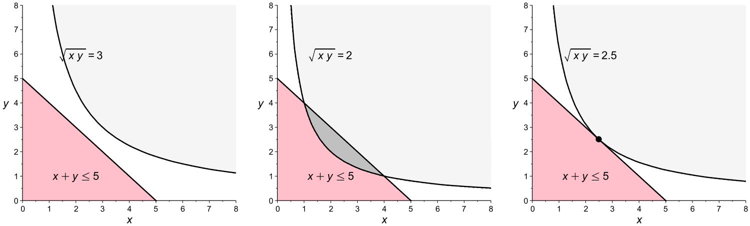
- In the last lecture, we have discussed about two important elements in consumer theory:
  - utility
  - budget
- There are two ways of measuring the "optimal choice" of a consumer.
  1. Given prices and income, what is the optimal amount x and y that should be bought to maximize your utility?
  2. Fixing a particular level of utility, what is the optimal amount of x and y that should be bought to minimize your expenditure?

- The optimization problem of the first question, is called "utility maximization problem" (效用最大化), or UMP.
  - The solutions of UMP, denoted as (x^*, y^*), are "Marshallian demand" for x and y.
- The optimization problem of the second question, is called "expenditure minimization problem" (支出最小化), or EMP.
  - The solutions of EMP, denoted as (h_x, h_y), are "Hicksian demand" for x and y.

Utility maximization problem (效用最大化问题， UMP)

- Assume that the utility function is $U = x^a y^b$ where $a = b = 1/2$
- The budget set is $p_x x + p_y y \leq I$ where $p_x = p_y = 1$ and $I = 5$.





- The consumption bundle along the indifference curve $3 = \sqrt{xy}$ is not feasible.
- You could achieve a utility level at $2 = \sqrt{xy}$, but you can do better.
- The optimal choice: the tangent point.

A simple approach of UMP

- The UMP is formally written as

$$\max_{x,y} U(x,y)$$
$$\text{subjected to } p_x x + p_y y \leq I$$ 预算 (income, ...)

- Observe that: all the money should be spent, i.e., the choice should be made somewhere along the budget line.

$$p_x x + p_y y = I.$$

The budget line can be expressed as $y = -\frac{p_x}{p_y}x + \frac{I}{p_y}$.

- Plug the budget line into the utility, then you solve

$$\max_{x} U\left(x, -\frac{p_x}{p_y}x + \frac{I}{p_y}\right)$$

- The first-order condition of

$$\max_x U\left(x, -\frac{p_x}{p_y}x + \frac{I}{p_y}\right)$$

with respect to x, gives

$$U_x' + U_y' \cdot \left(-\frac{p_x}{p_y}\right) = 0 \Rightarrow \frac{U_x'}{U_y'} = \frac{p_x}{p_y}$$

- There is only one variable x in the above equation: $x^*(p_x, p_y, I)$
- Plug $x^*(p_x, p_y, I)$ back into the budget line $p_x x^* + p_y y = I$, you can solve $y^*(p_x, p_y, I)$.
- Recall the definition of $MRS = \frac{U_x'}{U_y'}$:

Theorem

At optimum, the marginal rate of substitution is equal to the relative prices:

$$MRS = \frac{U_x'(x^*, y^*)}{U_y'(x^*, y^*)} = \frac{p_x}{p_y}$$

Example ($U = \sqrt{xy}$)

- *The budget line:* $y = -\frac{p_x}{p_y}x + \frac{I}{p_y}$.
- *Choose* x *to maximize* $U = \sqrt{xy} = \sqrt{x\left(-\frac{p_x}{p_y}x + \frac{I}{p_y}\right)}$.
  - $\frac{dU}{dx} = \frac{-2\frac{p_x}{p_y}x + \frac{I}{p_y}}{2\sqrt{x\left(-\frac{p_x}{p_y}x + \frac{I}{p_y}\right)}} = 0 \Rightarrow x = \frac{I}{2p_x}$
- *Plug* $x = \frac{I}{2p_x}$ *into the budget line:* $y = -\frac{p_x}{p_y} \cdot \frac{I}{2p_x} + \frac{I}{p_y} = \frac{I}{2p_y}$.

You can confirm that

$$MRS = \left.\frac{U_x'}{U_y'}\right|_{x=\frac{I}{2p_x}, y=\frac{I}{2p_y}} = \left.\frac{\frac{\sqrt{y}}{2\sqrt{x}}}{\frac{\sqrt{x}}{2\sqrt{y}}}\right|_{x=\frac{I}{2p_x}, y=\frac{I}{2p_y}} = \frac{p_x}{p_y}.$$

The Lagrangian (拉格朗日) Approach

- Mathematically, if we want to maximize $U(x, y)$ subjected to the constraint $p_x x + p_y y \leq I$, we can use the "Lagrangian approach," i.e., constraint optimization (带有约束条件的最优化).
- The Lagrangian is

$$\mathcal{L}(x, y, \lambda) = U(x, y) + \lambda(I - p_x x - p_y y)$$

  where λ is called "multiplier," i.e., the marginal value of an additional unit of money.
- We maximize $\mathcal{L}$ with three variables: x, y, λ. The first-order conditions are

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x} &= U_x' - \lambda p_x = 0 \\ \frac{\partial \mathcal{L}}{\partial y} &= U_y' - \lambda p_y = 0 \qquad \Rightarrow x, y, \lambda \\ \frac{\partial \mathcal{L}}{\partial \lambda} &= I - p_x x - p_y y = 0 \end{aligned}$$

  Three unknowns (x, y, λ) are determined by three equations.

Second-order conditions

- By so far, we have obtained the solutions (x^*, y^*, λ^*) through the first-order conditions

$$\frac{\partial \mathcal{L}}{\partial x} = 0, \frac{\partial \mathcal{L}}{\partial y} = 0, \frac{\partial \mathcal{L}}{\partial \lambda} = 0$$

- We need to verify whether they are maximum or minimum, by using the second-order conditions.
- In a constrained optimization, the second-order matrix, is called "boarded Hessian"

$$H_b = \begin{bmatrix} 0 & \mathcal{L}''_{\lambda x} & \mathcal{L}''_{\lambda y} \\ \mathcal{L}''_{\lambda x} & \mathcal{L}''_{xx} & \mathcal{L}''_{xy} \\ \mathcal{L}''_{\lambda y} & \mathcal{L}''_{yx} & \mathcal{L}''_{yy} \end{bmatrix} = \begin{bmatrix} 0 & -p_x & -p_y \\ -p_x & U''_{xx} & U''_{xy} \\ -p_y & U''_{yx} & U''_{yy} \end{bmatrix}$$

  - Maximization: $(-1)H_b$ is negative semidefinite
  - Minimization: $(-1)H_b$ is positive semidefinite

$$H_b = \begin{bmatrix} 0 & \mathcal{L}''_{\lambda x} & \mathcal{L}''_{\lambda y} \\ \mathcal{L}''_{\lambda x} & \mathcal{L}''_{xx} & \mathcal{L}''_{xy} \\ \mathcal{L}''_{\lambda y} & \mathcal{L}''_{yx} & \mathcal{L}''_{yy} \end{bmatrix} = \begin{bmatrix} 0 & -p_x & -p_y \\ -p_x & U''_{xx} & U''_{xy} \\ -p_y & U''_{yx} & U''_{yy} \end{bmatrix}$$

- Maximization: $(-1)H_b$ is negative semidefinite: starting from the second minor of H_b, the signs of the determinants are $-, +, -, +, \ldots$

$$\det \begin{bmatrix} 0 & -p_x \\ -p_x & U''_{xx} \end{bmatrix} = -p_x^2 < 0, \ \ \det(H_b) \geq 0.$$

- Minimization: $(-1)H_b$ is positive semidefinite: starting from the second minor of H_b, the signs of the determinants are negative (or non-positive).
    - Clearly, UMP is associated with maximization. We will see a positive semidefinite $(-1)H_b$ in the expenditure minimization problem.

Example: $U = \sqrt{xy}$

The Lagrangian for UMP is

$$\mathcal{L}(x, y, \lambda) = \sqrt{xy} + \lambda(I - p_x x - p_y y)$$

The first-order conditions:

$$\mathcal{L}'_x = \frac{\sqrt{y}}{2\sqrt{x}} - \lambda p_x = 0$$

$$\mathcal{L}'_y = \frac{\sqrt{x}}{2\sqrt{y}} - \lambda p_y = 0$$

$$\mathcal{L}'_\lambda = I - p_x x - p_y y = 0$$

$$\Rightarrow x^* = \frac{I}{2p_x}, \; y^* = \frac{I}{2p_y}, \; \lambda = \frac{U'_x}{p_x} = \frac{U'_y}{p_y}$$

Example: $p_x = p_y = 1, I = 5,$ then $x^* = y^* = 2.5.$

The boarded-Hessian for second-order derivatives:

$$H_b = \begin{bmatrix} 0 & -p_x & -p_y \\ -p_x & -\frac{1}{4}x^{-3/2}y^{1/2} & \frac{1}{4}x^{-1/2}y^{-1/2} \\ -p_y & \frac{1}{4}x^{-1/2}y^{-1/2} & -\frac{1}{4}x^{1/2}y^{-3/2} \end{bmatrix} = \begin{bmatrix} 0 & -1 & -1 \\ -1 & -\frac{1}{10} & \frac{1}{10} \\ -1 & \frac{1}{10} & -\frac{1}{10} \end{bmatrix}$$

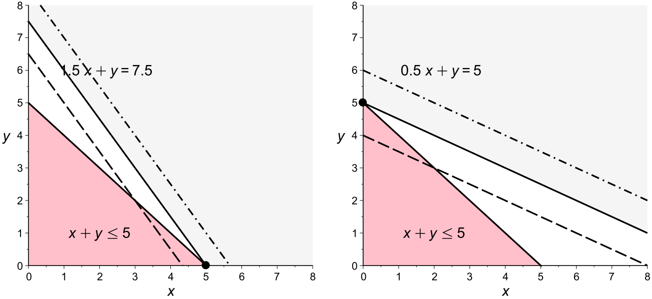
(You should verify that $(-1)H_b$ is negative semidefinite)

Interior & Corner Solutions (内点解与角点解)

- In the previous example, the solution of the UMP is obtained by "first-order conditions." We call such solutions "interior solutions."
- However, for some utility functions, we cannot use derivatives to solve the optimum.
- Except the Cobb-Douglas utility, you should be cautious with respect to the following three types of utilities:
  - Perfect substitutes
  - Perfect complements
  - Quasi-linear utility
- For perfect substitutes and complements, the indifference curves are "straight lines."
  - You should plot graphs first, and then check the point that maximize the utility.
- For quasi-linear utility:
  - Under certain conditions, the optimum corresponds to interior solutions.
  - Under some other conditions, the optimum corresponds to corner solutions.

Perfect Substitutes

- Utility function: $U(x, y) = ax + by$. The indifference curve is a straight line.
- Budget line: $p_x x + p_y y = I$.
- The optimum is determined by the relative slopes of the two straight lines.



- Fixing a particular utility level u_0, the indifference curve is

$$y(x) = \underbrace{-\frac{a}{b}}_{\text{slope}} x + \underbrace{\frac{u_0}{b}}_{\text{intercept}}$$

- The budget line is

$$y = \underbrace{-\frac{p_x}{p_y}}_{\text{slope}} x + \frac{I}{p_y}$$

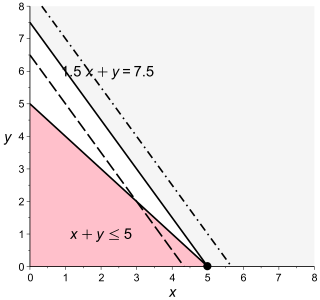
  - Recall that $MRS = \frac{U'_x}{U'_y} = \frac{a}{b}$.
  - If $MRS = \frac{a}{b} > \frac{p_x}{p_y}$, you should spend all your money on x and buy zero y. Plug $y^* = 0$ into the budget line: $p_x x = I \Rightarrow x^* = \frac{I}{p_x}$.
  - If $MRS = \frac{a}{b} < \frac{p_x}{p_y}$, you should buy zero x and spend all your money on y. Plug $x^* = 0$ into the budget line: $p_y y = I \Rightarrow y^* = \frac{I}{p_y}$

Example

$$\max_{x,y} 1.5x + y$$

$$s.t.\ x + y \leq 5$$

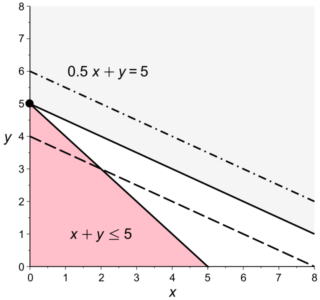
The slope of indifference curve: $MRS = \frac{1.5}{1}$. *The slope of the budget line:* $-\frac{p_x}{p_y} = 1$. *Then you should spend all your money on* x:
$y^* = 0 \Rightarrow x + 0 = 5 \Rightarrow x^* = 5$



Example

$$\max_{x,y} 0.5x + y$$
$$s.t.\ x + y \leq 5$$

The slope of indifference curve: $MRS = \frac{0.5}{1}$. *The slope of the budget line:* $-\frac{p_x}{p_y} = 1$. *Then you should spend all your money on* y*:*
$x^* = 0 \Rightarrow 0 + y = 5 \Rightarrow y^* = 5$



Perfect Complements

- Utility function: $U(x,y) = \min\{ax, by\}$
- Budget line: $p_x x + p_y y = I$
  - If you buy some amount of x and y such that $ax > by$, then you obtain utility $U = \min\{ax, by\} = by$. You should not buy too many x that is greater than $x > \frac{b}{a}y$ because you have to pay for what you buy, without obtaining additional utility.
  - Similarly, if you choose $ax < by$, then you obtain $U = ax$. Then you should reduce the amount of y such that $ax = by$ because you need to pay for the additional y that brings no additional benefits.
  - Therefore, the optimal choice is $ax = by$
- Plug $ax = by$ into your budget line: $p_x x + p_y y = I$:

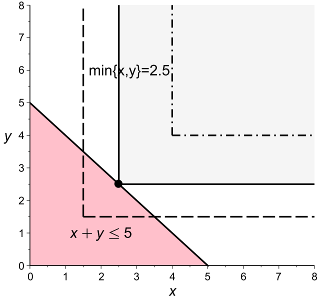
$$p_x x + p_y \left(\frac{a}{b}x\right) = I \Rightarrow x^* = \frac{bI}{bp_x + ap_y}, y^* = \frac{aI}{bp_x + ap_y}$$

Example

$$\max_{x,y} \min\{x, y\}$$

$$s.t.\ x + y \leq 5$$

The optimal choice is $x = y$. *Plug* $x = y$ *into the budget line:* $x + y = 5 \Rightarrow 2x = 5 \Rightarrow x^* = y^* = 2.5$.



Quasi-linear Utility

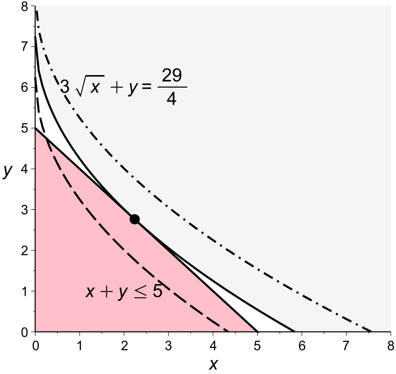
- Utility function: $U(x,y) = u(x) + y$, i.e., concave in x while linear in y.
  - Sometimes we implicitly assume that $u'(0) \to +\infty$.
- Budget: $p_x x + p_y y \leq I$.
- You should be careful about quasi-linear because it is possible that
  - The UMP gives an interior solution if $MRS = \frac{U'_x}{U'_y} = u'(x) = \frac{p_x}{p_y}$.
  - The UMP gives a corner solution if $MRS = \frac{U'_x}{U'_y} = u'(x) > \frac{p_x}{p_y}$.
- There are two ways to specify whether the solution is interior or corner:
  1. Use the first-order condition to solve x^*, and check whether $u'(x^*) =$ or $> \frac{p_x}{p_y}$
  2. Use the first-order condition to solve x^*, and plug x^* into your budget line and check whether $y^* \geq 0$ or $y^* < 0$.

Example

$$\max_{x,y} 3\sqrt{x} + y$$
$$s.t.\ x + y \leq 5$$

- *Because* $x + y = 5$, *then* $y = 5 - x$. *Plug* $y = 5 - x$ *into your objective.*
- *You maximize* $U(x, y) = 3\sqrt{x} + 5 - x$
- *The first-order condition is* $U'_x = 3\frac{1}{2\sqrt{x}} - 1 = 0 \Rightarrow x^* = 9/4$
  - *Check:* $MRS = u'(x^*) = \frac{3}{2\sqrt{x^*}} = 1 = \frac{p_x}{p_y} = \frac{1}{1}$
  - *Check:* $x^* + y = 5 \Rightarrow y = 5 - 9/4 > 0.$
- *Therefore, the interior solution is* $(x^*, y^*) = (9/4, 11/4)$

Interior Solution for Quasi-linear Utility



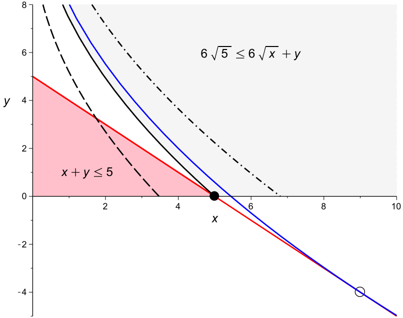
Example

$$\max_{x,y} 6\sqrt{x} + y$$

$$s.t.\ x + y \leq 5$$

- *Plug* $y = 5 - x$ *into your objective.*
- *Maximize* $U(x,y) = 6\sqrt{x} + 5 - x$
- $U'_x = \frac{6}{2\sqrt{x}} - 1 = 0 \Rightarrow x = 9$
- *However, if you plug* $x = 9$ *back into the budget:* $y = 5 - 9 < 0$. *You cannot buy a negative amount of* y.
- *Essentially, at current prices* $p_x/p_y = 1$, *because you prefer* x *"much more" than* y, *then you buy zero unit of* y, *i.e.,* $x + 0 = 5 \Rightarrow x = 5$. *Evaluated at* $x = 5$, $MRS = u'(x) = \frac{3}{\sqrt{5}} > 1 = p_x/p_y$.
  - *Even you buy zero* y *and spend all your money buying 5 units of* x, *if you are provided with an additional unit of* x, *the additional utility obtained from an additional unit of* x *is still greater than the relative prices* p_x/p_y.

Corner Solution for Quasi-linear Utility



You do not buy y. Hence $y^* = 0 \Rightarrow x^* = 5 - y = 5$. The slope of the indifference curve $MRS = U'_x = \frac{3}{\sqrt{5}}$ is steeper than the budget line (not tangent).

Expenditure Minimization Problem (支出最小化，EMP)

- Previously, we have discussed the question: given prices $\mathbf{p}$ and income I, the optimal choice of (x^*, y^*) that maximizes the utility:

$$\left.\begin{array}{l}\max\limits_{x,y} U(x,y) \\ s.t.\ p_x x + p_y y \leq I\end{array}\right. \Rightarrow (x^*, y^*)$$

  The solution (x^*, y^*) is called "Marshallian demand" (**马歇尔需求**).
- Now, let's "reverse" the problem: given a particular utility level u, the optimal choice of (x, y) that minimizes the total expenditure. The solution of this problem, denoted as (h_x, h_y), is "Hicksian demand." (**希克斯需求**)

$$\left.\begin{array}{l}\min\limits_{x,y} p_x x + p_y y \\ s.t.\ U(x,y) \geq u\end{array}\right. \Rightarrow (h_x, h_y)$$

- The process of EMP is similar to UMP.
- The Lagrangian is

$$\mathcal{L}(x, y, \lambda) = p_x x + p_y y + \lambda \left[u - U(x, y)\right]$$

- The three unknowns (x, y, λ) are solved from the three FOCs:
  - $\mathcal{L}'_x = p_x - \lambda U'_x = 0$
  - $\mathcal{L}'_y = p_y - \lambda U'_y = 0$
  - $\mathcal{L}'_\lambda = u - U(x, y) = 0$
- The Hicksian demand for x and y is

$$h_x(p_x, p_y, u), \;\; h_y(p_x, p_y, u)$$

Example (Cobb-Douglas $U(x,y) = \sqrt{xy}$)

$$\min_{x,y} p_x x + p_y y$$
$$s.t.\ \sqrt{xy} \geq u$$

The Lagrangian is

$$\mathcal{L} = p_x x + p_y y + \lambda(u - \sqrt{xy})$$

- $\mathcal{L}'_x = p_x - \lambda \frac{\sqrt{y}}{2\sqrt{x}} = 0$
- $\mathcal{L}'_y = p_y - \lambda \frac{\sqrt{x}}{2\sqrt{y}} = 0$
- $\mathcal{L}'_\lambda = u - \sqrt{xy} = 0$

The Hicksian demand is

$$h_x = \sqrt{\frac{p_y}{p_x}}u,\ h_y = \sqrt{\frac{p_x}{p_y}}u.$$

- Assume that $u = 2.5$ and $p_x = p_y = 1$, then $\lambda = 2$ and the Hicksian demand is

$$h_x = h_y = 2.5$$

- Check the second-order Hessian*:

$$H_b = \begin{bmatrix} 0 & \mathcal{L}''_{\lambda x} & \mathcal{L}''_{\lambda y} \\ \mathcal{L}''_{\lambda x} & \mathcal{L}''_{xx} & \mathcal{L}''_{xy} \\ \mathcal{L}''_{\lambda y} & \mathcal{L}''_{yx} & \mathcal{L}''_{yy} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{1}{2} & -\frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

- We can verify that all the determinants of the principal minors of H_b are negative, and hence $(-1)H_b$ is positive definite, i.e., (h_x, h_y) is a minimum.

UMP & EMP

- Now let's consider the relationship between UMP and EMP.
- For UMP:
  - We maximize $U(x, y)$ subjected to $p_x x + p_y y = I$, which gives the solution (x^*, y^*)
  - Plug (x^*, y^*) into the objective, the maximized utility $U(x^*, y^*)$, is called "indirect utility" or the "value function," denoted by V.
  - (x^*, y^*) are functions of p_x, p_y, I, so V is a function of p_x, p_y, I.
- For EMP:
  - We minimize $p_x x + p_y y$ subjected to $U(x, y) = u$, which gives the solution (h_x, h_y)
  - Plug (h_x, h_y) into the objective, the minimized expenditure $p_x h_x + p_y h_y$, is called "expenditure function, denoted by E.
  - (h_x, h_y) are functions of p_x, p_y, u, so E is a function of p_x, p_y, u.

Duality (对偶性)

- For UMP, the solutions are $x^*(p_x, p_y, I)$ and $y^*(p_x, p_y, I)$ with indirect utility $V(p_x, p_y, I) = U(x^*, y^*)$.
- For EMP, the solutions are $h_x(p_x, p_y, u)$ and $h_y(p_x, p_y, u)$ with expenditure function $E(p_x, p_y, u)$.
- Then the following conditions hold:
  - $E\left(p_x, p_y, u\right)|_{u=V(p_x, p_y, I)} = I$
  - $V\left(p_x, p_y, I\right)|_{I=E(p_x, p_y, u)} = u$
  - $x^*\left(p_x, p_y, I\right)|_{I=E(p_x, p_y, u)} = h_x(p_x, p_y, u)$
  - $h_x\left(p_x, p_y, u\right)|_{u=V(p_x, p_y, I)} = x^*(p_x, p_y, I)$

Example: $U = \sqrt{xy}$, $p_x = p_y = 1$

- For the UMP where $I = 5$, we have solved that

$$x^*(p_x, p_y, I) = \frac{I}{2p_x} = 2.5,\ y^* = (p_x, p_y, I) = \frac{I}{2p_x} = 2.5$$

Hence $V(p_x, p_y, I) = \sqrt{x^* y^*} = \frac{I}{2\sqrt{p_x p_y}} = 2.5.$

- For the EMP where $u = 2.5 (= V)$, we have solved that

$$h_x(p_x, p_y, u) = \sqrt{\frac{p_y}{p_x}} u = 2.5,\ h_y(p_x, p_y, u) = \sqrt{\frac{p_x}{p_y}} u = 2.5.$$

Hence $E(p_x, p_y, u) = p_x h_x + p_y h_y = 2\sqrt{p_x p_y} u = 5$

Comparative Statics

- Let's consider the effect of a marginal change in p_x on the optimal choice of x.
- By duality, we know that: fixing a particular utility level u, the Marshallian demand is equivalent with the Hicksian demand:

$$x^* \left[p_x, p_y, E(p_x, p_y, u)\right] = h_x(p_x, p_y, u)$$

- Differentiate the equation of both sides with respect to p_x:

$$\frac{\partial x^*}{\partial p_x} + \frac{\partial x^*}{\partial I}\frac{\partial E}{\partial p_x} = \frac{\partial h_x}{\partial p_x}$$

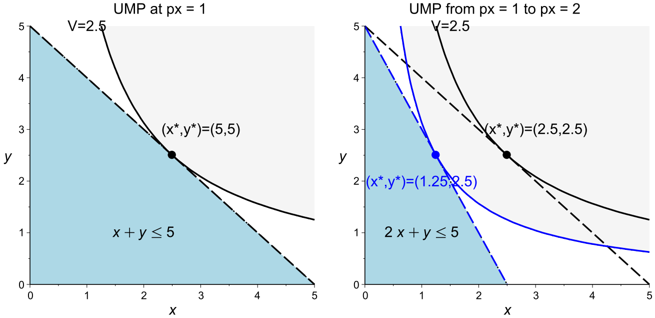
Recall that in calculus, if we want to differentiate a function $z = F(x_1(t), x_2(t))$ with respect to t:

$$\frac{dz}{dt} = F'_{x_1} x'_1(t) + F'_{x_2} x'_2(t)$$

Here $z = h_x$, $F = x^*$, $t = p_x$, $x_1(t) = t = p_x$, $x_2(t) = E$.

The Effect of Price Changes

- Recall the previous example: $U(x,y) = \sqrt{xy}$, $p_x = p_y = 1$ and $I = 5$
  - UMP gives $(x^*, y^*) = (2.5, 2.5)$.
- Consider that the price of good x increases, from $p_x = 1$ to $p_x = 2$.
  - UMP gives $(x^*, y^*) = (1.25, 1.25)$
- The consumption of x is reduced from 2.5 to 1.25.



The Decomposition of Price Changes

- Due to a price increase, the consumption of x is reduced by $2.5 - 1.25 = 1.25$.
  - We say -1.25 is the **total effect** due to an increase in p_x.
- We want to go one step further, by decomposing total effect into two types of effects:
  1. **substitution effect (替代效应)**: since x is more expensive **relative to** y, hence if you want to keep your original utility (before price change) unchanged, you should reduce the consumption of x whereby increase the consumption of y at the new price levels — the rate of exchange between the two goods is changed.
  2. **income effect (收入效应)**: the purchase power is reduced, and hence you should decrease your consumption on x.
- Total effect = substitution effect + income effect
- We have solved total effect. How to compute substitution and income effects?

Slutsky Identity (斯勒茨基恒等式)

- Recall the duality: $x^*(p_x, p_y, E(p_x, p_y, u)) = h_x(p_x, p_y, u)$
- Differentiate both sides with respect to p_x: $\frac{\partial x^*}{\partial p_x} + \frac{\partial x^*}{\partial I}\frac{\partial E}{\partial p_x} = \frac{\partial h_x}{\partial p_x}$. Rearranging, the equation becomes **Slutsky Identity** (斯勒茨基恒等式)

$$\underbrace{\frac{\partial x^*}{\partial p_x}}_{\text{total effect}} = \underbrace{\left.\frac{\partial h_x}{\partial p_x}\right|_{u=\text{const}}}_{\text{substitution effect}} \underbrace{- \frac{\partial x^*}{\partial I}\boxed{\frac{\partial E}{\partial p_x}}}_{\text{income effect}}$$

$= x^*$
Sheppard Lemma

- Our definition of "substitution effect" is: after the price change, the amount of x that shall be changed to keep the original utility unchanged.
  - The original utility is the indirect utility $V = 2.5$
  - At the new price $p_x = 2$, you should choose an amount of x "optimally" to keep your utility unchanged at $u = 5$.
  - That is, you solve an EMP, where the price is $p_x = 2$, and the constraint is $\sqrt{xy} = 2.5$.
  - The Hicksian demand you obtained from EMP, is h_x. The difference between the original $x^*(p_x = 1, p_y = 1, I = 5)$, and $h_x(p_x = 2, p_y = 1, u = 2.5)$ is the substitution effect.

Example: Compute Substitution Effect

- Before price change ($p_x = p_y = 1, I = 5$):

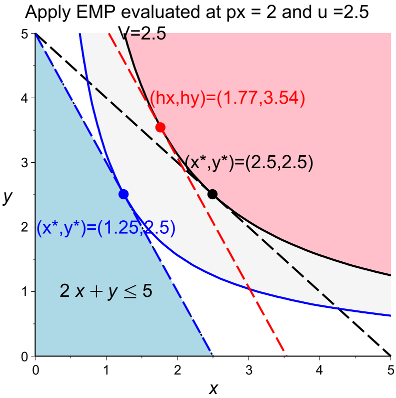
$$\left.\begin{array}{l} \max\limits_{x,y} \sqrt{xy} \\ s.t.\ x + y = I = 5 \end{array}\right. \Rightarrow x^* = 2.5, V = 2.5.$$

- After the price change ($p_x = 2, p_y = 1$), solve the optimal x that minimize your expenditure, while keeping your utility at $u = 5$, i.e.,

$$\left.\begin{array}{l} \min\limits_{x,y} 2x + y \\ s.t.\ \sqrt{xy} = u = 2.5 \end{array}\right. \Rightarrow h_x = \frac{5}{4}\sqrt{2} \approx 1.77$$

- Therefore, when p_x increases from 1 to 2, the substitution effect is $h_x - x^* = 1.77 - 2.5 = -0.73$, i.e., you decrease your consumption on x by 0.73 to keep your utility unchanged at the original level $V = 2.5$.

Total effect = substitution effect + income effect



- Total effect: $x^* \to x^*$
- Substitution effect: $x^* \to h_x$
- Income effect: $h_x \to x^*$

- In the above example, we compute total, substitution and income effects by considering a price change that jumps from 1 to 2.
- Now let's compute those effects by considering a locally, marginal increase in p_x.
- The UMP before price change:

$$\begin{aligned}&\max_{x,y} \sqrt{xy}\\ &s.t.\ p_x x + p_y y = I\end{aligned} \Rightarrow x^*(p_x, p_y, I) = \frac{I}{2p_x}, V = \frac{I}{2\sqrt{p_x p_y}}.$$

Total effect of p_x on x^* is $\frac{\partial x^*}{\partial p_x} = -\frac{I}{2p_x^2}$.

- To obtain substitution effect, we need to solve EMP:

$$\begin{aligned}&\min_{x,y} p_x x + p_y y\\ &s.t.\ \sqrt{xy} = u\end{aligned} \Rightarrow h_x(p_x, p_y, u) = \sqrt{\frac{p_y}{p_x}}u, E = 2\sqrt{p_x p_y}u.$$

Fixing the utility level u, $\frac{\partial h_x}{\partial p_x} = -\frac{1}{2}\frac{\sqrt{p_y}}{p_x\sqrt{p_x}}u$.

- $x^*(p_x, p_y, I) = \frac{I}{2p_x}$. Total effect: $\frac{\partial x^*}{\partial p_x} = -\frac{I}{2p_x^2}$
- $\frac{\partial h_x}{\partial p_x} = -\frac{1}{2}\frac{\sqrt{p_y}}{p_x\sqrt{p_x}}u$. Plug $u = V = \frac{I}{2\sqrt{p_x p_y}}$ into $\frac{\partial h_x}{\partial p_x}$, then the substitution effect is
$$\left.\frac{\partial h_x}{\partial p_x}\right|_{u=V} = -\frac{I}{4p_x^2}.$$
- The income effect is $-\frac{\partial x^*}{\partial I}\frac{\partial E}{\partial p_x}$.
    - $\frac{\partial x^*}{\partial I} = \frac{1}{2p_x}$
    - $E = 2\sqrt{p_x p_y}u$, $\frac{\partial E}{\partial p_x} = \frac{\sqrt{p_y}u}{\sqrt{p_x}}$.
    - Using $u = V = \frac{I}{2\sqrt{p_x p_y}}$, then $-\frac{\partial x^*}{\partial I}\frac{\partial E}{\partial p_x} = -\frac{1}{2p_x}\cdot\frac{\sqrt{p_y}}{\sqrt{p_x}}\frac{I}{2\sqrt{p_x p_y}} = -\frac{I}{4p_x^2}$
- That is, total effect $\frac{\partial x^*}{\partial p_x} = -\frac{I}{2p_x^2}$ is the sum of substitution effect $\left.\frac{\partial h_x}{\partial p_x}\right|_{u=V} = -\frac{I}{4p_x^2}$ and the income effect $-\frac{\partial x^*}{\partial I}\frac{\partial E}{\partial p_x} = -\frac{I}{4p_x^2}$.

Example: Perfect Substitutes

- Utility function is $U(x, y) = ax + by$
- Budget line is $p_x x + p_y y = I$
- Assume that $\frac{a}{b} > \frac{p_x}{p_y}$ hence

$$\text{UMP} \Rightarrow y^* = 0, x^* = \frac{I}{p_x},\ V = \frac{aI}{p_x}$$

- Consider a **locally marginal** increase in p_x (the relative slopes between the budget and the indifference curve is unchanged such that $y = 0$).
- Total effect on x: $\frac{\partial x^*}{\partial p_x} = -\frac{I}{p_x^2}$.
- To obtain substitution effect, we need to solve EMP

$$\text{EMP} \Rightarrow y = 0 \Rightarrow h_x = \frac{u}{a} \Rightarrow \frac{\partial h_x}{\partial p_x} = 0, E = \frac{p_x u}{a}$$

  Therefore, there is no substitution effect.
- Income effect: $-\frac{\partial x^*}{\partial I}\frac{\partial E}{\partial p_x} = -\frac{1}{p_x}\frac{u}{a}\Big|_{u=V=\frac{aI}{p_x}} = -\frac{I^2}{p_x} =$ total effect.

Example: Perfect Complements

- Utility function is $U(x, y) = \{ax, by\}$
- Budget line is $p_x x + p_y y = I$

$$\text{UMP} \Rightarrow x^* = \frac{bI}{bp_x + ap_y}, \ V = \frac{abI}{bp_x + ap_y}$$

- Consider a locally marginal increase in p_x
- Total effect on x: $\frac{\partial x^*}{\partial p_x} = -\frac{b^2 I}{(bp_x + ap_y)^2}$.
- To obtain substitution effect, we need to solve EMP

$$\text{EMP} \Rightarrow h_x = \frac{u}{a} \Rightarrow \frac{\partial h_x}{\partial p_x} = 0, \ E = p_x \frac{u}{a}$$

Therefore, there is no substitution effect.

- Income effect is
$-\frac{\partial x^*}{\partial I} \frac{\partial E}{\partial p_x} = -\frac{b}{bp_x + ap_y} \cdot \frac{u}{a} \Big|_{u = V = \frac{abI}{bp_x + ap_y}} = -\frac{b^2 I}{(bp_x + ap_y)^2} =$ total effect.

Example: Quasi-linear Utility (Interior Case)

- Utility function is $U(x,y) = u(x) + y$, where $u''(x) < 0$.
- Budget line is $p_x x + p_y y = I$, or $y = -\frac{p_x}{p_y}x + \frac{I}{p_y}$
- Let's consider the interior solution:

$$\text{UMP} \Rightarrow u'(x^*) = \frac{p_x}{p_y}.$$

  Notice that x^* is not a function of I!
- Consider a locally marginal increase in p_x
- Total effect on x: $u''(x^*)\frac{dx^*}{dp_x} = \frac{1}{p_y}$.
- To obtain substitution effect, we need to solve EMP

$$\text{EMP} \Rightarrow u'(h_x) = \frac{p_x}{p_y}$$

  $u''(h_x)\frac{dh_x}{dp_x} = \frac{1}{p_y}$.
- For the interior solution of quasi-linear utility, total effect = substitution effect, while there is no income effect.
- The above argument is valid only for the interior solution!

Roy's Identity (罗伊恒等式)

- There are some useful results you should keep in mind.
- Recall the Marshallian demand $x^*(p_x, p_y, I)$ obtained from UMP.
- Alternatively, $x^*(p_x, p_y, I)$ can be expressed as

$$x^*(p_x, p_y, I) = -\frac{\frac{\partial V(p_x, p_y, I)}{\partial p_x}}{\frac{\partial V(p_x, p_y, I)}{\partial I}}$$

  The above equation is called "Roy's identity."
- If you know $V(p_x, p_y, I)$ already, you can obtain x^* directly by using Roy's identity.

Proof of Roy's identity

- The solution x^* is obtained from the Lagrangian $\mathcal{L} = U(x,y) + \lambda(I - p_x x - p_y y)$, where the FOCs imply
$$U'_x = \lambda p_x, \ U'_y = \lambda p_y$$
- Evaluated at the optimal choices (x^*, y^*), the optimal point on the budget line is $p_x x^* + p_y y^* = I$. Differentiate both sides with respect to p_x, and I, respectively:
$$x^* + p_x \frac{\partial x^*}{\partial p_x} + p_y \frac{\partial y^*}{\partial p_x} = 0, \ p_x \frac{\partial x^*}{\partial I} + p_y \frac{\partial y^*}{\partial I} = 1$$
- Evaluated at (x^*, y^*), the indirect utility is $V = U(x^*, y^*)$. Differentiate V with respect to p_x, and I:
$$\frac{\partial V}{\partial p_x} = \underbrace{U'_x}_{=\lambda p_x} \frac{\partial x^*}{\partial p_x} + \underbrace{U'_y}_{=\lambda p_y} \frac{\partial y^*}{\partial p_x} = \lambda \left(p_x \frac{\partial x^*}{\partial p_x} + p_y \frac{\partial y^*}{\partial p_x} \right) = \lambda(-x^*)$$
$$\frac{\partial V}{\partial I} = \underbrace{U'_x}_{=\lambda p_x} \frac{\partial x^*}{\partial I} + \underbrace{U'_y}_{=\lambda p_y} \frac{\partial y^*}{\partial I} = \lambda \left(p_x \frac{\partial x^*}{\partial I} + p_y \frac{\partial y^*}{\partial I} \right) = \lambda \cdot 1$$

Shephard's Lemma (谢泼德引理)

- Recall the Hicksian demand $h_x(p_x, p_y, u)$ and $h_y(p_x, p_y, I)$ obtained from EMP.
- The expenditure function is $E(p_x, p_y, u) = p_x h_x + p_y h_y$.
- We can show that

$$\frac{\partial E}{\partial p_x} = h_x$$

Proof of Shephard's Lemma

- Differentiate $E = p_x h_x(p_x, p_y, u) + p_y h_y(p_x, p_y, u)$ with respect to p_x:

$$\frac{\partial E}{\partial p_x} = h_x + p_x \frac{\partial h_x}{\partial p_x} + p_y \frac{\partial h_y}{\partial p_x}$$

- The solution h_x and h_y are obtained from EMP using Lagrangian:

$$\mathcal{L} = p_x x + p_y y + \lambda \left(u - U(x, y) \right)$$

where the FOCs imply

$$p_x = \lambda U'_x, \; p_y = \lambda U'_y$$

- Because you minimize the expenditure evaluated at a particular utility level u, hence at optimum,

$$U(h_x, h_y) = u$$

Differentiate the above equation with respect to p_x on both sides:

$$\underbrace{U'_x}_{=p_x/\lambda} \frac{\partial h_x}{\partial p_x} + \underbrace{U'_y}_{=p_y/\lambda} \frac{\partial h_y}{\partial p_x} = 0 \Rightarrow \frac{1}{\lambda} \left(p_x \frac{\partial h_x}{\partial p_x} + p_y \frac{\partial h_y}{\partial p_x} \right) = 0.$$

Envelop Theorem* (包络定理)

- The Roy's identity, and Shephard's Lemma, are two examples of the "Envelop Theorem."
- Assume that we are choosing x to maximize $y = f(x, a)$, with a parameter a:

$$\max_x f(x, a)$$

- The first-order condition gives $\frac{dy}{dx} = f'_x(x, a) = 0$. The solution is $x^*(a)$.
- Plug the optimal $x^*(a)$ into y, we have the maximized y, denoted as y^*:

$$y^* = f(x^*(a), a)$$

- Now consider we want to see the effect from a change of a on the **optimized** y:

$$\frac{dy^*}{da} = f'_x \frac{dx^*}{da} + f'_a$$

Since x^* is obtained by condition $f'_x = 0$, the above equation can be simplified as $\frac{dy^*}{da} = f'_a$.