# Final exam

Mathematics and Statistics for Economists

September 11, 2023

**Exercise 1** Evaluate $\det(L)$ if $L = \begin{pmatrix} 0 & 1 & 2 & c \\ 0 & 6 & 13 & c \\ 0 & 0 & 1 & b \\ a & 1 & -3 & c \end{pmatrix}$ in terms of $a$, $b$ and $c$.

Solution: $\det(L) = -a \det \begin{pmatrix} 1 & 2 & c \\ 6 & 13 & c \\ 0 & 1 & b \end{pmatrix} = -a\Big(1 \det \begin{pmatrix} 13 & c \\ 1 & b \end{pmatrix} - 6 \det \begin{pmatrix} 2 & c \\ 1 & b \end{pmatrix}\Big) = -a\Big((13b - c) - 6(2b - c)\Big) = -a\Big(b + 5c\Big)$

**Exercise 2** Consider the quadratic form

$$q(x_1, x_2, x_3) = x_1^2 + 2x_1x_3 + 2x_2^2 + x_3^2$$

(a) Find the matrix $\mathbf{A}$ such that $q(x_1, x_2, x_3) = \mathbf{z}^T\mathbf{A}\mathbf{z}$ where $\mathbf{z} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

(b) Determine the definiteness of $q$ (equivalently, the definiteness of $\mathbf{A}$).

Solution:

a) $\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$

b) $\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 2-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{pmatrix} = (2-\lambda) \det \begin{pmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = (2-\lambda)\Big[(1-\lambda)(1-\lambda) - 1\Big] = (2-\lambda)\Big[\lambda^2 - 2\lambda + 1 - 1\Big] = \lambda(2-\lambda)(\lambda - 2) = 0.$

Eigenvalues are $\lambda = 2, 0$. Therefore, positive semidefinite.

**Exercise 3** Let

$$y = -x^4 + 4x^3 - 6x^2 + 8x + 3$$

(a) Show that the function is concave, and that its slope where $x = 2$. Deduce the coordinates of the global maximum point.

(b) Suppose that the function is now defined for $x \geq 0$. Write down the coordinates of the local minimum point and the global maximum point.

Solution: a) $f'(x) = -4x^3 + 12x^2 - 12x + 8$

$f''(x) = -12x^2 + 24x - 12 = -12(x^2 - 2x + 1) = -12(x-1)^2 \leq 0$ for all $x$, therefore $f$ is concave.

$f'(2) = 0$. Since we know that $f$ concave, then critical point is the global maximum. $(2, 11)$ global max.

b) Global max is still (2,11). Local max now at the boundary, (0,3).

**Exercise 4** In a model of a small open economy, the demand for money is given by

$$M = f(Y, i) \tag{1}$$

where $Y$ is national income and $i$ is the rate of interest. Suppose that national income is determined by the aggregate demand function

$$Y = g(i, u) \tag{2}$$

where $u$ is the exchange rate. Substituting (2) into (1), we have

$$M = f(g(i, u), i) \tag{3}$$

Let the right-hand side of (3) be denoted by $H(i, u)$.

(a) Show how the expression $\partial H/\partial i$ and $\partial H/\partial u$ are related to the partial derivatives of the functions $f$ and $g$.

(b) Verify your answers in the case where

$$f(Y, i) = AYe^{-ai}, \quad g(i, u) = Be^{-bi}u^c$$

for some positive constants $A$, $B$, $a$, $b$, $c$.

Solution: Using chain rule,

$$\frac{\partial H}{\partial i} = \frac{\partial f}{\partial Y}\frac{\partial g}{\partial i} + \frac{\partial f}{\partial i} \tag{4}$$

$$\frac{\partial H}{\partial u} = \frac{\partial f}{\partial Y}\frac{\partial g}{\partial u} \tag{5}$$

In the particular case given, $H(i, u) = ABe^{-(a+b)i}u^c$. Then $\partial H/\partial i = -(a+b)H(i, u) = -(a+b)M$ and $\partial H/\partial u = (c/u)H(i, u) = cM/u$. Also $\partial f/\partial Y = f(y, i)/Y = M/Y$, $\partial f/\partial i = -af(y, i) = -aM$, $\partial g/\partial i = -bg(i, u) = -bY$ and $\partial g/\partial u = (c/u)g(i, u) = cY/u$. Hence. $RHS(4) = \frac{M}{Y} \times (-bY) - aM = -(a+b)M = LHS(4)$, $RHS(5) = \frac{M}{Y} \times \frac{cY}{u} = \frac{cM}{u} = LHS(5)$

**Exercise 5** Find and classify the critical points of the following function:

$$f(x,y) = xe^{-x}(y^2 - 4y)$$

Solution:

$$\text{Gradient} = e^{-x}\begin{pmatrix} (1-x)(y^2-4y) \\ x(2y-4) \end{pmatrix}$$

The critical points are solutions to the following system of equations

$$\begin{aligned} (1-x)(y^2-4y) &= 0 \\ x(2y-4) &= 0 \end{aligned}$$

that is (0,0), (0,4) and (1,2). The Hessian is

$$Hf(x,y) = e^{-x}\begin{pmatrix} (x-2)(y^2-4y) & (1-x)(2y-4) \\ (1-x)(2y-4) & 2x \end{pmatrix}$$

At the critical points we obtain

$$\begin{aligned} Hf(0,0) &= \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix} \text{(indefinite)} \\ Hf(0,4) &= \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix} \text{(indefinite)} \\ Hf(1,2) &= \begin{pmatrix} 4e^{-1} & 0 \\ 0 & 2e^{-1} \end{pmatrix} \text{(positive definite)} \end{aligned}$$

so (0,0), (0,4) are saddle points and (1,2) is a local minimum.

**Exercise 6** Let $A, \alpha, \beta$ be positive constants. Show that the Cobb-Douglas production function

$$Q(K,L) = AK^\alpha L^\beta$$

is concave if and only if $\alpha + \beta \leq 1$.

Solution: Denote the Hessian by $\mathbf{H}$.

$$\mathbf{H} = \begin{pmatrix} \alpha(\alpha-1)AK^{\alpha-2}L^\beta & \alpha\beta AK^{\alpha-1}L^{\beta-1} \\ \alpha\beta AK^{\alpha-1}L^{\beta-1} & \beta(\beta-1)AK^\alpha L^{\beta-2} \end{pmatrix}$$

One diagonal entry of H has the same sign as $\alpha(\alpha-1)$. the other has the same sign as $\beta(\beta-1)$ and

$\det \mathbf{H}$ has the sign of

$$\alpha\beta(\alpha-1)(\beta-1)-\alpha^2\beta^2 = \alpha\beta(1-\alpha-\beta)$$

Since $\alpha$ and $\beta$ are positive, $\det H$ has the same sign as $(1-\alpha-\beta)$. Thus if $\alpha+\beta>1$ the function is not concave; if $\alpha+\beta\leq 1$ the function will be concave provided the diagonal entries of $\mathbf{H}$ are non-positive. But the three inequalities $\alpha>0$, $\beta>0$ and $\alpha+\beta\leq 1$ imply that $0<\alpha<1$ and $0<\beta<1$, and hence that the diagonal entries of $\mathbf{H}$ are negative; therefore $\alpha+\beta\leq 1$ is sufficient as well as necessary for concavity.

**Exercise 7** Log-linearize the following Cobb-Douglas production function:

$$Y = A_t K_t^{\alpha} H_t^{1-\alpha}$$

Solution: see exercise in class.

**Exercise 8** The value $x$ is sampled from a normal distribution with $2\theta$ and variance 3 and the value $y$ is independently sampled from a normal distribution with mean $\theta$ and variance 2. Consider estimators of the form $\hat{\theta} = ax + by$

(a) Find $\mathbb{E}[\hat{\theta}]$ and $\text{var}[\hat{\theta}]$.

(b) If $\hat{\theta} = ax + by$ is an unbiased estimator of $\theta$, what is the value of $b$ in terms of $a$?

(c) Find the values of $a$ and $b$ such that $\hat{\theta} = ax + by$ is unbiased and has the minimum possible variance.

Solution: a) Using expected value and variance properties, we have $E(\hat{\theta} = E(ax+by) = aE(x)+bE(y) = a2\theta + b\theta = (2a+b)\theta$, $var(\hat{\theta}) = var(ax+by) = a^2 var(x) + b^2 var(y) = 3a^2 + 2b^2$.

b) In order to be unbiased we require $E(\hat{\theta}) = \theta$ for all $\theta$. This means $(2a+b)\theta = \theta$ and so $2a+b=1$, then $b = 1-2a$.

c) Using the value found in b), the variance is $3a^2+2b^2 = 3a^2+2(1-2a)^2 = 11a^2-8a+2$, with derivative $22a-8$. This is zero when $a = 8/22 = 4/11$. Minimum occurs for $a = 4/11$ and $b = 3/11$.

**Exercise 9** The continuous random variable $X$ has probability density function

$$f_X(x) = \begin{cases} \frac{a}{x^4} & \text{if } x \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant $a$

(b) Find $\mathbb{P}[X \geq 2]$ (If you were unable to solve part (a), leave $a$ in your answer).

(c) Find the expected value and variance of $X$.

Solution: a) To be valid, the pdf must integrate to 1 on its domain. Thus $\int_1^\infty \frac{a}{x^4} dx = 1$, but the integral is $\int_1^\infty \frac{a}{x^4} dx = [-\frac{a}{3}x^{-2}]_{x=1}^\infty = \frac{a}{3}$ and so $a = 3$.

b) We have $P(X \geq 2) = \int_2^\infty \frac{3}{x^4} dx = [-x^{-3}]_{x=2}^\infty = 1/8$.

c) We have $E(X) = \int_1^\infty x \frac{3}{x^4} dx = [-\frac{3}{2}x^{-2}]_{x=1}^\infty = 3/2$ and also $E(X^2) = \int_1^\infty x^2 \frac{3}{x^4} dx = [3x^{-1}]_{x=1}^\infty = 3$. Then $var(X) = E(X^2) - E(X)^2 = 3 - (3/2)^2 = 3/4$.

**Exercise 10** Give and justify (shortly) the correct response for each of the following:

Laziness values are normally distributed with a known standard deviation of 2.7. A sample of 30 values is collected with mean 19.1, and the null hypothesis $H_0 : \mu = 20$ is tested against the alternative $H_a : \mu < 20$. The $p$-value for the resulting test is $p = 0.03394$.

(a) At a significance level $\alpha = 3\%$,

(1) $H_0$ should be rejected.

(2) $H_a$ should be accepted.

(3) $H_0$ should not be rejected.

Since the $p$-value is above the significance level, (3) $H_0$ should not be rejected.

(b) If the sample size were doubled,

(1) the power would increase.

(2) the power would decrease.

(3) the power would stay the same.

If we increase the sample size, the results of the test will match the reality more closely, meaning that (1) the power would increase (you can also see that if $n$ increases, the variance decreases)

(c) If the null hypothesis is rejected, and the true mean is 20, then

(1) this is a correct decision.

(2) this is a type I error.

(3) this is a type II error.

Here the null hypothesis is true but it has been rejected so (2) this is a type I error.

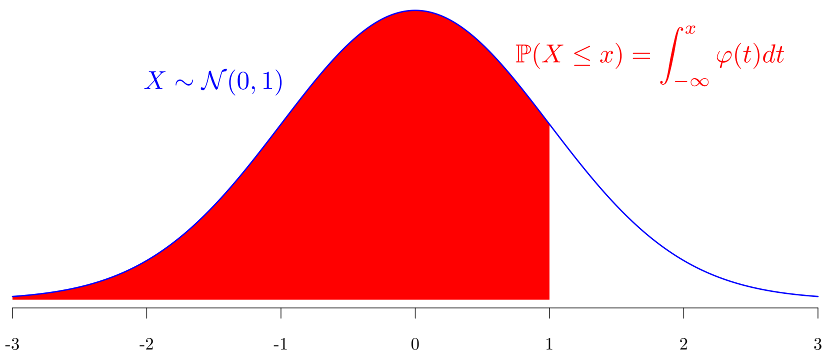
(d) If the null hypothesis is rejected, and the true mean is 19.1, then

(1) this is a correct decision.

(2) this is a type I error.

(3) this is a type II error.

Here the null hypothesis is false but it has been rejected so (1) this is a correct decision.




| $x$ | 0.00 | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 0.5000 | 0.5040 | 0.5080 | 0.5120 | 0.5160 | 0.5199 | 0.5239 | 0.5279 | 0.5319 | 0.5359 |
| 0.1 | 0.5398 | 0.5438 | 0.5478 | 0.5517 | 0.5557 | 0.5596 | 0.5636 | 0.5675 | 0.5714 | 0.5753 |
| 0.2 | 0.5793 | 0.5832 | 0.5871 | 0.5910 | 0.5948 | 0.5987 | 0.6026 | 0.6064 | 0.6103 | 0.6141 |
| 0.3 | 0.6179 | 0.6217 | 0.6255 | 0.6293 | 0.6331 | 0.6368 | 0.6406 | 0.6443 | 0.6480 | 0.6517 |
| 0.4 | 0.6554 | 0.6591 | 0.6628 | 0.6664 | 0.6700 | 0.6736 | 0.6772 | 0.6808 | 0.6844 | 0.6879 |
| 0.5 | 0.6915 | 0.6950 | 0.6985 | 0.7019 | 0.7054 | 0.7088 | 0.7123 | 0.7157 | 0.7190 | 0.7224 |
| 0.6 | 0.7257 | 0.7291 | 0.7324 | 0.7357 | 0.7389 | 0.7422 | 0.7454 | 0.7486 | 0.7517 | 0.7549 |
| 0.7 | 0.7580 | 0.7611 | 0.7642 | 0.7673 | 0.7704 | 0.7734 | 0.7764 | 0.7794 | 0.7823 | 0.7852 |
| 0.8 | 0.7881 | 0.7910 | 0.7939 | 0.7967 | 0.7995 | 0.8023 | 0.8051 | 0.8078 | 0.8106 | 0.8133 |
| 0.9 | 0.8159 | 0.8186 | 0.8212 | 0.8238 | 0.8264 | 0.8289 | 0.8315 | 0.8340 | 0.8365 | 0.8389 |
| 1.0 | 0.8413 | 0.8438 | 0.8461 | 0.8485 | 0.8508 | 0.8531 | 0.8554 | 0.8577 | 0.8599 | 0.8621 |
| 1.1 | 0.8643 | 0.8665 | 0.8686 | 0.8708 | 0.8729 | 0.8749 | 0.8770 | 0.8790 | 0.8810 | 0.8830 |
| 1.2 | 0.8849 | 0.8869 | 0.8888 | 0.8907 | 0.8925 | 0.8944 | 0.8962 | 0.8980 | 0.8997 | 0.9015 |
| 1.3 | 0.9032 | 0.9049 | 0.9066 | 0.9082 | 0.9099 | 0.9115 | 0.9131 | 0.9147 | 0.9162 | 0.9177 |
| 1.4 | 0.9192 | 0.9207 | 0.9222 | 0.9236 | 0.9251 | 0.9265 | 0.9279 | 0.9292 | 0.9306 | 0.9319 |
| 1.5 | 0.9332 | 0.9345 | 0.9357 | 0.9370 | 0.9382 | 0.9394 | 0.9406 | 0.9418 | 0.9429 | 0.9441 |
| 1.6 | 0.9452 | 0.9463 | 0.9474 | 0.9484 | 0.9495 | 0.9505 | 0.9515 | 0.9525 | 0.9535 | 0.9545 |
| 1.7 | 0.9554 | 0.9564 | 0.9573 | 0.9582 | 0.9591 | 0.9599 | 0.9608 | 0.9616 | 0.9625 | 0.9633 |
| 1.8 | 0.9641 | 0.9649 | 0.9656 | 0.9664 | 0.9671 | 0.9678 | 0.9686 | 0.9693 | 0.9699 | 0.9706 |
| 1.9 | 0.9713 | 0.9719 | 0.9726 | 0.9732 | 0.9738 | 0.9744 | 0.9750 | 0.9756 | 0.9761 | 0.9767 |
| 2.0 | 0.9772 | 0.9778 | 0.9783 | 0.9788 | 0.9793 | 0.9798 | 0.9803 | 0.9808 | 0.9812 | 0.9817 |
| 2.1 | 0.9821 | 0.9826 | 0.9830 | 0.9834 | 0.9838 | 0.9842 | 0.9846 | 0.9850 | 0.9854 | 0.9857 |
| 2.2 | 0.9861 | 0.9864 | 0.9868 | 0.9871 | 0.9875 | 0.9878 | 0.9881 | 0.9884 | 0.9887 | 0.9890 |
| 2.3 | 0.9893 | 0.9896 | 0.9898 | 0.9901 | 0.9904 | 0.9906 | 0.9909 | 0.9911 | 0.9913 | 0.9916 |
| 2.4 | 0.9918 | 0.9920 | 0.9922 | 0.9925 | 0.9927 | 0.9929 | 0.9931 | 0.9932 | 0.9934 | 0.9936 |
| 2.5 | 0.9938 | 0.9940 | 0.9941 | 0.9943 | 0.9945 | 0.9946 | 0.9948 | 0.9949 | 0.9951 | 0.9952 |
| 2.6 | 0.9953 | 0.9955 | 0.9956 | 0.9957 | 0.9959 | 0.9960 | 0.9961 | 0.9962 | 0.9963 | 0.9964 |
| 2.7 | 0.9965 | 0.9966 | 0.9967 | 0.9968 | 0.9969 | 0.9970 | 0.9971 | 0.9972 | 0.9973 | 0.9974 |
| 2.8 | 0.9974 | 0.9975 | 0.9976 | 0.9977 | 0.9977 | 0.9978 | 0.9979 | 0.9979 | 0.9980 | 0.9981 |
| 2.9 | 0.9981 | 0.9982 | 0.9982 | 0.9983 | 0.9984 | 0.9984 | 0.9985 | 0.9985 | 0.9986 | 0.9986 |
| 3.0 | 0.9987 | 0.9987 | 0.9987 | 0.9988 | 0.9988 | 0.9989 | 0.9989 | 0.9989 | 0.9990 | 0.9990 |