PS2 Solutions

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Optimisation Theory

Solution 1.

1. *Differentiate both sides of the implicit function, we have*

$$3x^2 - 3(y^2 + 2xy\frac{dy}{dx}) = 0$$

thus, $\frac{dy}{dx} = \frac{3x^2-3y^2}{6xy} = \frac{x^2-y^2}{2xy}$.
Set $\frac{dy}{dx} = 0$ *for horizontal tangent lines, thus* $y = \pm x$.
Substitute $y^2 = x^2$ *back into the original function, we have* $2x^3 = 1$*, thus,* $x = -\frac{1}{\sqrt[3]{2}}$, $y = \pm\frac{1}{\sqrt[3]{2}}$
2. *We have* $F_x = 3x^2 - 3y^2$, $F_y = -6xy$, $F_{xx} = 6x$, $F_{xy} = -6y$, $F_{yy} = -6x$.
Thus

$$D = F_{xx}F_{yy} - F_{xy}^2 = -36x^2 - 36y^2 < 0$$

Since $x < 0$, $D < 0$*, both points are saddle points.*

Solution 2.

Define the Lagrangian

$$\mathcal{L} = [\alpha x_1^\rho + \beta x_2^\rho]^{\frac{1}{\rho}} - \lambda(p_1x_1 + p_2x_2 - M)$$

Thus, we have the FOCs:

$$\frac{\partial \mathcal{L}}{\partial x_1} = \alpha x_1^{\rho-1}\left[\alpha x_1^\rho + \beta x_2^\rho\right]^{\frac{1}{\rho}-1} - \lambda p_1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = \beta x_2^{\rho-1}\left[\alpha x_1^\rho + \beta x_2^\rho\right]^{\frac{1}{\rho}-1} - \lambda p_2 = 0$$

$$\Rightarrow \frac{p_1}{p_2} = \frac{\alpha x_1^{\rho-1}}{\beta x_2^{\rho-1}} \Leftrightarrow \frac{p_1 x_1}{p_2 x_2} = \frac{\alpha x_1^{\rho}}{\beta x_2^{\rho}}$$

Let $x_1 = \left(\frac{\beta p_1}{\alpha p_2}\right)^{\frac{1}{\rho-1}} x_2$*. Substitute this into the budget constraint:*

$$p_1 \left(\frac{\beta p_1}{\alpha p_2}\right)^{\frac{1}{\rho-1}} x_2 + p_2 x_2 = M$$

Solve for x_2*:*

$$x_2 \left(p_1 \left(\frac{\beta p_1}{\alpha p_2}\right)^{\frac{1}{\rho-1}} + p_2\right) = M$$

$$x_2 = \frac{M}{p_2 + p_1 \left(\frac{\beta p_1}{\alpha p_2}\right)^{\frac{1}{\rho-1}}}$$

Substitute x_2 *back to find* x_1*:*

$$x_1 = \left(\frac{\beta p_1}{\alpha p_2}\right)^{\frac{1}{\rho-1}} x_2 = \frac{\left(\frac{\beta p_1}{\alpha p_2}\right)^{\frac{1}{\rho-1}} M}{p_2 + p_1 \left(\frac{\beta p_1}{\alpha p_2}\right)^{\frac{1}{\rho-1}}}$$

Hence,

$$\begin{aligned}
\mathcal{L}_{\max} &= \left[\frac{\beta x_2^{\rho-1} M}{p_2}\right]^{\frac{1}{\rho}} \\
&= \left(\frac{\beta M}{p_2}\right)^{\frac{1}{\rho}} \left[\frac{M}{p_2 + p_1 \left(\frac{\beta p_1}{\alpha p_2}\right)^{\frac{1}{\rho-1}}}\right]^{\frac{1}{\rho}} \\
&= \left(\frac{\beta M}{p_2}\right)^{\frac{1}{\rho}} \left[\frac{\alpha^{\frac{1}{\rho-1}} p_2^{\frac{1}{\rho-1}} M}{\alpha^{\frac{1}{\rho-1}} p_2^{\frac{\rho}{\rho-1}} + \beta^{\frac{1}{\rho-1}} p_1^{\frac{\rho}{\rho-1}}}\right]^{\frac{1}{\rho}} \\
&= \left(\frac{\beta M}{p_2}\right)^{\frac{1}{\rho}} \left[\frac{(\alpha p_2)^{-\sigma} M}{\alpha^{-\sigma} p_2^{1-\sigma} + \beta^{-\sigma} p_2^{1-\sigma}}\right]^{\frac{1}{\rho}}
\end{aligned}$$

Solution 3.

1.

$$\mathcal{L} = p\sqrt{x_1 x_2} - (w_1 x_1 + w_2 x_2) - \lambda(w_1 x_1 + w_2 x_2 - B)$$

Then, we have the FOCs:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x_1} &= \frac{p}{2}\sqrt{\frac{x_2}{x_1}} - w_1(\lambda+1) = 0\\ \frac{\partial \mathcal{L}}{\partial x_2} &= \frac{p}{2}\sqrt{\frac{x_1}{x_2}} - w_2(\lambda+1) = 0\\ \frac{\partial \mathcal{L}}{\partial \lambda} &= w_1x_1 + w_2x_2 - B = 0\end{aligned}$$

$$\Rightarrow \frac{x_1}{x_2} = \frac{w_2}{w_1}$$

Since $w_1x_1 + w_2x_2 = B$, we have:

$$x_1^* = \frac{B}{2w_1}, x_2^* = \frac{B}{2w_2}$$

2.

$$\Pi^* = p\sqrt{\frac{B^2}{4w_1w_2}} - B = \frac{pB}{2}(w_1w_2)^{-\frac{1}{2}} - B$$

3.

$$\frac{\partial \mathcal{L}}{\partial p} = \sqrt{x_1x_2} = \frac{B}{2\sqrt{w_1w_2}} = \frac{\partial \Pi^*}{\partial p}$$

This result shows that an increase in the output price p leads to an increase in the maximum profit, with the rate of increase being $\frac{B}{2\sqrt{w_1w_2}}$.

Solution 4.

Define the Lagrangian

$$\mathcal{L} = 5x_1x_2 - x_1^2 - x_2^2 - \lambda_1(2x_1 + x_2 - 10) - \lambda_2(4 - x_1 - x_2)$$

Then we have the FOCs:

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial x_1} &= 5x_2 - 2x_1 - 2\lambda_1 + \lambda_2\\ \frac{\partial \mathcal{L}}{\partial x_2} &= 5x_1 - 2x_2 - \lambda_1 + \lambda_2\\ \frac{\partial \mathcal{L}}{\partial \lambda_1} &= 2x_1 + x_2 - 10\\ \frac{\partial \mathcal{L}}{\partial \lambda_2} &= x_1 + x_2 - 4\end{aligned}$$

Since $x_1, x_2 \geq 0$, due to complementarity slackness conditions, we have

$$\frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial \mathcal{L}}{\partial x_2} = 0$$

Hence $\lambda_1^* = 7x_2 - 7x_1$, $\lambda_2^* = 9x_2 - 12x_1$.

Due to the complementarity slackness conditions of inequality constraints, we also have:

$$\lambda_1^*(2x_1 + x_2 - 10) = 0, \lambda_2^*(x_1 + x_2 - 4) = 0.$$

Firstly, if $\lambda_1^* = 0$, then $x_1 = x_2$, thus $\lambda_2^* \neq 0$, otherwise $x_1 = x_2 = 0$, the second constraint doesn't hold. Then, we have $x_1 + x_2 = 4 \Rightarrow x_1 = x_2 = 2$, but $\lambda_2^* = -6 \leq 0$, contradicts the slackness conditions. Thus, $2x_1 + x_2 - 10 = 0$, if $x_1 + x_2 - 4 = 0$, then $x_1 = 6$, $x_2 = -2$, contradicts $x_2 \geq 0$. Hence we know it that $\lambda_2^* = 0$, $x_2 = \frac{4}{3}x_1$, $x_1 = 3$, $x_2 = 4$. At this point $\Pi^* = 35$.

Solution 5.

Let $\Pi = p_1F_1(K_1, L_1) + p_2F_2(K_2, L_2)$. Define the Lagrangian

$$\mathcal{L} = p_1F_1(K_1, L_1) + p_2F_2(K_2, L_2) - \lambda(K_1 + K_2 - K) - \mu(L_1 + L_2 - L)$$

***1.** Using the **Envelop Themrem**, we have:*

$$\frac{\partial V}{\partial p_i} = \frac{\partial \mathcal{L}}{\partial p_i} = F_i(K_i, L_i)$$

***2.** Using the **Envelop Themrem**, we have:*

$$\frac{\partial V}{\partial K} = \frac{\partial \mathcal{L}}{\partial K} = \lambda$$

Take the FOCs of $\mathcal{L}$ with respect to K_i:

$$\frac{\partial \mathcal{L}}{\partial K_1} = p_1\frac{\partial F_1}{\partial K_1} - \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial K_1} = p_2\frac{\partial F_2}{\partial K_2} - \lambda = 0$$

Thus,

$$\frac{\partial V}{\partial K} = \lambda = p_1\frac{\partial F_1}{\partial K_1} = p_2\frac{\partial F_2}{\partial K_2}$$

3. *Using the* ***Envelop Themrem****, we have:*

$$\frac{\partial V}{\partial L} = \frac{\partial \mathcal{L}}{\partial L} = \mu$$

Take the FOCs of $\mathcal{L}$ with respect to L_i:

$$\frac{\partial \mathcal{L}}{\partial L_1} = p_1 \frac{\partial F_1}{\partial L_1} - \mu = 0$$
$$\frac{\partial \mathcal{L}}{\partial L_2} = p_2 \frac{\partial F_2}{\partial L_2} - \mu = 0$$

Thus,

$$\frac{\partial V}{\partial L} = \mu = p_1 \frac{\partial F_1}{\partial L_1} = p_2 \frac{\partial F_2}{\partial L_2}.$$

Probability and Statistics

Solution 6.

1. *Firstly, we know that $\lim_{x \to -\infty} F(x) = 0$, and that $\lim_{x \to \infty} F(x) = \lim_{x \to \infty} (1 - e^{-x}) = 1$. And for $x, y \in \mathbb{R}_+$, if $x \leq y$*

$$1 - e^{-x} - (1 - e^{-y}) = e^{-y} - e^{-x} = e^{-y}(1 - e^{y-x}) < 0$$

Thus, $F(x)$ is a CDF.

2. *By definition, $f(x)$ satisfies that*

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{0}^{x} f(x)dx$$

Since $F(x) = 0$ for $x < 0$. Thus,

$$f(x) = \frac{d}{dx} F(x) = \exp(-x)$$

3.

$$\begin{aligned}\mathbb{E}[X] &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_{0}^{\infty} x e^{-x} dx \\ &= -(x+1)e^{-x} \mid_{x=\infty} + (x+1)e^{-x} \mid_{x=0} \\ &= 1\end{aligned}$$

4. We use the change of variables formula:

$$f_Y(y) = f_X(y^2) \cdot \left| \frac{d}{dy} y^2 \right| = \exp(-y^2) \cdot 2y$$

for $y \geq 0$. So, the PDF of Y is:

$$f_Y(y) = \begin{cases} 2y \exp(-y^2) & \textit{if } y \geq 0 \\ 0 & \textit{if } y < 0 \end{cases}$$

Solution 7.

$$\begin{aligned}F_Y(y) &= F_X(g^{-1}(y)) \\ &= F_X(F_X^{-1}(y) \\ &= y\end{aligned}$$

Since cumulative distribution functions have codomain $[0, 1]$, it follows that $y \in [0, 1]$. Hence, $Y \sim \mathcal{U}[0, 1]$.

Solution 8.

First, we find Z for both situations:

$$Z_{60} = \frac{X - \mu}{\sigma} = \frac{60 - 75}{10} = -1.5$$

$$Z_{70} = \frac{70 - 75}{10} = -0.5$$

$$Z_{100} = \frac{100 - 75}{10} = 2.5$$

Then, we find the probability by table:

$$P(X < 60) = P(Z < -1.5) = 1 - 0.9332 = 0.0668$$

$$\begin{aligned} P(70 < x < 100) &= P(-0.5 < Z < 2.5) \\ &= P(Z = 2.5) - P(Z = -0.5) \\ &= 0.9938 - (1 - 0.6915) = 0.6853 \end{aligned}$$

Solution 9.

We know that:

$$P(Z < \frac{89 - \mu}{\sigma}) = 0.9$$
$$P(Z < \frac{94 - \mu}{\sigma}) = 0.95$$

From the table we know that:

$$\frac{89 - \mu}{\sigma} = 1.28$$
$$\frac{94 - \mu}{\sigma} = 1.645$$

Hence, $\frac{5}{\sigma} = 0.365$, $\sigma \approx 13.7$, $\sigma^2 \approx 187.69$, $\mu \approx 71.46$.

Solution 10.

1. *From the conditions, we know that* $\bar{X} = 1011$, $\mu = 1000$, $\sigma = 20$, $n = 10$. *Firstly, we get the* Z*-score of the hypothesis, which is*

$$Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} = \frac{1011 - 1000}{20/\sqrt{10}} \approx 1.739$$

   The t-test is 2 two-tail test, thus the p-value is

$$p = 2 \times P(Z > |1.739|) = 2 \times (1 - 0.9591) = 0.0818 > 0.01$$

   Since $p > 0.01$ *and* $1.739 < 2.57$, *we can't reject* $\mathcal{H}_0$ *at* 1% *level.*

2. $P(Z > 1.739) = 0.0409 > 0.01$, *we can't reject* $\mathcal{H}_0$ *at* 1% *level.*

3. *This time, set* $n = 100$, $\mu = 1005$. *We have*

$$Z = \frac{1011 - 1005}{20/\sqrt{100}} = 3$$

$$p = 2 \times P(Z > 3) = 2 \times (1 - 0.9987) = 0.0026 < 0.01$$

Reject the null hypothesis.

4. *Set two new hypotheses:* $\mathcal{H}_0$: *The variance is 400g*^{2}.

$\mathcal{H}_1$: *The variance is not 400g*^{2}.

$$\chi^2 = \frac{(n-1)s^2}{\sigma^2} = \frac{9 \times 32^2}{20^2} = 23.04$$

Check the table of χ^2, *we know that with df* $= 9$, $\chi^2_{0.01} = 21.666$, *thus* $\chi^2 = 23.04 > 21.666$, *thus we reject the null hypothesis.*

5. $\mathcal{H}_0$: *The mean weight is 1 kg*

$$t = \frac{\bar{X} - \mu}{s/\sqrt{n}} = \frac{1011 - 1000}{32/\sqrt{10}} = 1.087$$

At 1% *level, with degree of freedom df* $= 9$, $t_{0.01} = 2.821 > 1.087$, *thus, we don't reject the null hypothesis.*