PS2 Solutions

Jingle Fu
Group members: Yingjie Zhang, Irene Licastro

1 First generation crisis model

1.1 Consumption under a peg

We begin with the Euler equation. Under fixed exchange rate, the exchange depreciation rate $\varepsilon_t = 0$. Thus the interest parity gives that $i = r$, so that $c_t^{-\frac{1}{\sigma}} = \lambda(1+\theta+\alpha r)$, as we know that $r = \beta$ and θ is the tax rate on consumption, which is constant, consumption will be constant, we can write it as $\tilde{c}$.
We then identify the constant by the intertemporal budget constraint:

$$\begin{aligned}
\alpha_0 + \frac{y}{r} &= \int_0^\infty e^{-rt}\left(\tilde{c}(1+\theta) + rm_t\right)dt \\
&= \int_0^\infty e^{-rt}\left(\tilde{c}(1+\theta) + r\alpha c_t\right)dt \\
&= \int_0^\infty e^{-rt}\tilde{c}(1+\theta+r\alpha)dt \\
&= \tilde{c}(1+\theta+\alpha r)\int_0^\infty e^{-rt}dt \\
&= \tilde{c}(1+\theta+\alpha r)\frac{1}{r}
\end{aligned}$$

Thus we have:

$$\tilde{c} = \frac{\alpha_0 r + y}{1+\theta+\alpha r}$$

1.2 Unsustainable peg

From Question 1.1, we know that the consumption is a constant $\tilde{c} = \frac{\alpha_0 r + y}{1+\theta+\alpha r}$, thus the government tax income would be $\theta\tilde{c} = \frac{\theta(\alpha_0 r + y)}{1+\theta+\alpha r}$.

If the peg holds forever, then $\epsilon_t = 0$, $i = r$, $m_t = \alpha\tilde{c}$:

$$\begin{aligned}
\frac{g}{r} &= h_0 + \int_0^\infty e^{-rt}\left[\theta c_t + \dot{m}_t + \varepsilon_t m_t\right]dt + e^{-rT}(m_T - m_{T-}) \\
&= h_0 + \theta\tilde{c}\int_0^\infty e^{-rt}dt \\
&= h_0 + \frac{\theta\tilde{c}}{r} \\
\Rightarrow g &= rh_0 + \frac{\theta(\alpha_0 r + y)}{1+\theta+\alpha r}
\end{aligned}$$

If government spending g exceeds the threshold as given, these two results contradicts. The exchange rate peg cannot hold forever.
The government tax revenue is:

$$s^p = \theta\tilde{c} - g = \frac{\theta(\alpha_0 r + y)}{1+\theta+\alpha r} - g < -rh_0 < 0$$

Under a fixed exchange rate, $\varepsilon_t = 0$ and at the steady state, the real balance is constant, $\dot{m}_t = 0$, we know that the foreign reserves change is:

$$\dot{h}_t = rh_t + s_t^p < rh_t - rh_0$$

at $t = 0$, $h_0 < 0$. As time passes, the foreign reserves will be negative and the government will default on its debt. Hence the government cannot continue to run a peg, and the peg is unsustainable.

1.3 Consumption and depreciation pre- and post-break

From the Euler equation, before the abandon of the peg, we have:

$$c_1^{-\frac{1}{\sigma}} = \lambda(1+\theta+\alpha r)$$

and after the abandon, we have:

$$c_2^{-\frac{1}{\sigma}} = \lambda(1+\theta+\alpha(r+\varepsilon)).$$

So,

$$\begin{aligned}
\left(\frac{c_1}{c_2}\right)^{\frac{1}{\sigma}} &= \frac{1+\theta+\alpha(r+\varepsilon)}{1+\theta+\alpha r} \\
&= 1 + \frac{\alpha\epsilon}{1+\theta+\alpha r}
\end{aligned}$$

Now we use that budget constraint and the cash in advance constraint, we have:
With m constant and in steady state of reserves: $\dot{h}_t = 0$,

$$0 = rh_t + (\theta c_2 - g) + \varepsilon m_2 = \theta(c_2 - c_1) + \varepsilon\alpha c_2$$

since $\theta c_1 = g$, and we can solve:

$$\varepsilon = \frac{\theta}{\alpha}\left(\frac{c_1}{c_2} - 1\right)$$

Denote $c = \frac{c_1}{c_2}$ and $D = 1 + \theta + \alpha r$, we have:

$$c^{\frac{1}{\sigma}} = 1 + \frac{\alpha\varepsilon}{D} = 1 + \frac{\alpha}{D}\frac{\theta}{\alpha}(c-1) = 1 + \frac{\theta}{D}(c-1)$$

Define $f(c) = c^{\frac{1}{\sigma}} - 1 - \frac{\theta}{D}(c-1)$, we have:

$$\begin{aligned} f(1) &= 0 \\ f'(c) &= \frac{1}{\sigma}c^{\frac{1}{\sigma}-1} - \frac{\theta}{D} \\ f''(c) &= \frac{1}{\sigma}\left(\frac{1}{\sigma} - 1\right)c^{\frac{1}{\sigma}-2} < 0 \end{aligned}$$

Thus f is strictly concave, and we only need to show that $f'(1) > 0$.
If $f'(1) \leq 0$, then by concavity, the only solution to $f(c) = 0$ is $c = 1$, $c_1 = c_2$ and $m_1 = m_2$. In this case, the abandon of peg would generate no depreciation, thus $g = \theta c_2$.
As in question 1.2, we have:

$$g > rh_0 + \frac{\theta(\alpha_0 r + y)}{1 + \theta + \alpha r} = rh_0 + \theta c_1 = rh_0 + \theta c_2 > \theta c_2$$

This contradicts our assumption that $g = \theta c_2$. Hence $f'(1) > 0$.
By concavity, we know that at the equilibrium $c > 1$, which is $\frac{c_1}{c_2} > 1$, and thus $\varepsilon = \frac{\theta}{\alpha}\left(\frac{c_1}{c_2} - 1\right) > 0$.

1.4 Dynamics of reserves and assets

Before the break, $s_t^p = \theta c_1 - g = 0$, $\dot{m}_t = 0$ and $\varepsilon_t = 0$, so $\dot{h}_t = rh_t$.
For foreign assets, we have:

$$\dot{a}_t = ra_t + y - c_1(1+\theta) - rm_1 = ra_t + y - c_1(1 + \theta + \alpha r)$$

and the uverall position is:

$$\dot{a}_t + \dot{h}_t = r(a_t + h_t) + y - c_1(1 + \theta + \alpha r) = r(a_t + h_t) + y - \frac{g(1 + \theta + \alpha r)}{\theta}$$

since $c_1 = \frac{g}{\theta}$.
Evaluated at $t = 0$, we have:

$$\begin{aligned} \dot{h}_0 &= rh_0 \\ \dot{a}_0 + \dot{h}_0 &= r(a_0 + h_0) + y - \frac{g(1 + \theta + \alpha r)}{\theta} \end{aligned}$$

which is exactly the current-account balance.

If g exceeds the threshold, we will have:

$$\begin{aligned}
CA_0 &\equiv \dot{a}_0 + \dot{h}_0 \\
&< r(a_0 + h_0) + y - \left[r h_0 + \frac{\theta(r a_0 + y)}{1+\theta+\alpha r} \right] \frac{(1+\theta+\alpha r)}{\theta} \\
&= r h_0 \left(1 - \frac{1+\theta+\alpha r}{\theta} \right) \\
&< 0
\end{aligned}$$

This is a classic Krugman-style "first-generation" crisis: persistent fiscal deficits erode reserves, and the peg becomes internally inconsistent with solvency.
Looking at the dynamics, we know tha tthe the reserves keeps growing, but the deterioration of the currrent account indicates a deterioration of the net foreign asset position.
Once that inequality holds, the model guarantees that net foreign assets will start declining, reserves will be gradually depleted (after private assets fall), and the peg will become unsustainable. So a persistent current account deficituncovered by reserve gainswould be a clear, quantitative early-warning of the pending crisis.

1.5 Timing of break

Recall that the intertemporal budget constraint is:

$$\frac{g}{r} = h_0 + \int_0^\infty e^{-rt} \left[\theta c_t + \dot{m}_t + \varepsilon_t m_t \right] dt + e^{-rT} \left[m_T - m_{T-} \right].$$

Before the break fo the peg($0 \leq t \leq T$):

$$c_t = c_1, \quad \dot{m}_t = 0, \quad \varepsilon_t = 0$$

and after the break, ($t > T$):

$$c_t = c_2, \quad \dot{m}_t = 0, \quad \varepsilon_t = \varepsilon > 0$$

Bringing these conditions into the intertemporal budget constraint, we have:

$$\frac{g}{r} = h_0 + \int_0^T e^{-rt} \theta c_1 dt + \int_T^\infty e^{-rt} \left[\theta c_2 + \varepsilon m_2 \right] dt + e^{-rT} \left[m_T - m_{T-} \right].$$

As $\theta c_1 = g$, we can write:

$$\int_0^T e^{-rt} \theta c_1 dt = g \int_0^T e^{-rt} dt = \frac{g \left(1 - e^{-rT} \right)}{r}$$

and that the second term is:

$$\int_T^\infty e^{-rt} \left[\theta c_2 + \varepsilon m_2 \right] dt = \left[\theta c_2 + \varepsilon m_2 \right] \frac{e^{-rT}}{r}$$

Thus,

$$\frac{g}{r} = h_0 + \frac{g\left(1 - e^{-rT}\right)}{r} + \frac{[\theta c_2 + \varepsilon m_2]e^{-rT}}{r} + e^{-rT}\left[m_T - m_{T-}\right].$$

As $m_2 = \alpha c_2$, $\varepsilon = \frac{\theta}{\alpha}\left(\frac{c_1}{c_2} - 1\right)$, we can further simplify that:

$$\theta c_2 + \varepsilon m_2 = \theta c_2 + \frac{\theta}{\alpha}\left(\frac{c_1}{c_2} - 1\right)\alpha c_2 = \theta c_2 + \theta(c_1 - c_2) = \theta c_1 = g$$

Thus the budget constraint is reduced to:

$$\begin{aligned}
\frac{g}{r} &= h_0 + \frac{g\left(1 - e^{-rT}\right)}{r} + \frac{ge^{-rT}}{r} + e^{-rT}\left[m_T - m_{T-}\right] \\
&= h_0 + \frac{g}{r} + e^{-rT}[m_T - m_{T-}] \\
\Rightarrow h_0 &= -e^{-rT}[m_T - m_{T-}] \\
\Rightarrow T &= \frac{1}{r}\ln\left(\frac{m_{T-} - m_T}{h_0}\right)
\end{aligned}$$

2 Choice of policy regime

2.1 Constant money

The shocks ϵ_t and v_t have zero expected value, $\mathbb{E}_{t-1}[\varepsilon_t] = \mathbb{E}_{t-1}[v_t] = 0$. Taking hte expectation of uncovered interest parity, aggregate supply, aggregate demand, and money demand at time $t-1$, we have:

$$\begin{aligned}
\mathbb{E}_{t-1}[i_{t+1}] &= \mathbb{E}_{t-1}[\mathbb{E}_t[e_{t+1}]] - \mathbb{E}_{t-1}[e_t] = \mathbb{E}_{t-1}[e_{t+1}] - \mathbb{E}_{t-1}[e_t] \\
\mathbb{E}_{t-1}[y_t] &= \theta\left(\mathbb{E}_{t-1}[p_t] - \mathbb{E}_{t-1}[p_t]\right) = 0 \\
\mathbb{E}_{t-1}[y_t] &= \delta(\mathbb{E}_{t-1}[e_t] - \mathbb{E}_{t-1}[p_t]) \\
\overline{m} - \mathbb{E}_{t-1}[p_t] &= -\eta\mathbb{E}_{t-1}[i_{t+1}] + \phi\mathbb{E}_{t-1}[y_t]
\end{aligned}$$

Solve the four equations, we have

$$\mathbb{E}_{t-1}[e_t] = \mathbb{E}_{t-1}[p_t] = \overline{m}$$

Similarly, we take the expectations of the four equations at time t, we can obtain: $\mathbb{E}_t[e_{t+1}] = \overline{m}$. So, we have $i_{t+1} = \overline{m} - e_t$, and we implement the total output function, we can get:

$$\begin{aligned}
\overline{m} - p_t &= -\eta(\overline{m} - e_t) + \phi y_t + v_t \\
&= -\eta\overline{m} + \eta e_t + \phi\left[\delta(e_t - p_t) + \varepsilon_t\right] + v_t \\
&= -\eta\overline{m} + \eta e_t + \phi\delta(e_t - p_t) + \phi\varepsilon_t + v_t \\
\Rightarrow (\phi\delta - 1)p_t &= (\eta + \phi\delta)e_t + \phi\varepsilon_t + v_t - (1+\eta)\overline{m} \qquad (2.1.1)
\end{aligned}$$

We bring this back to the total supply function, with $\mathbb{E}_{t-1}[p_t] = \overline{m}$(expectation price is equal to the long-term equilibrium), we have:

$$\begin{aligned}
y_t = \theta(p_t - \overline{m}) &\Rightarrow p_t = \frac{y_t}{\theta} + \overline{m} \\
\Rightarrow y_t &= \delta\left(e_t - \overline{m} - \frac{y_t}{\theta}\right) + \varepsilon_t \\
\Rightarrow y_t &= \frac{\theta\left[\delta(e_t - \overline{m}) + \varepsilon_t\right]}{\theta + \delta} \\
\Rightarrow p_t &= \overline{m} + \frac{\delta(e_t - \overline{m}) + \varepsilon_t}{\theta + \delta}
\end{aligned}$$

Bring this back to equation 2.1.1, we have:

$$\begin{aligned}
(\phi\delta - 1)\left[\overline{m} + \frac{\delta(e_t - \overline{m}) + \varepsilon_t}{\theta + \delta}\right] &= (\eta + \phi\delta)e_t + \phi\varepsilon_t + v_t - (1+\eta)\overline{m} \\
\Rightarrow (\phi\delta + \eta)\overline{m} + \frac{\phi\delta - 1}{\theta + \delta}\left[\delta(e_t - \overline{m}) + \varepsilon_t\right] &= (\eta + \phi\delta)e_t + \phi\varepsilon_t + v_t \\
\Rightarrow \left[\phi\delta + \eta - \frac{(\phi\delta - 1)\delta}{\theta + \delta}\right]\overline{m} + \left[\frac{\phi\delta - 1}{\theta + \delta} - \phi\right]\varepsilon_t &= \left[\eta + \phi\delta - \frac{(\phi\delta - 1)\delta}{\theta + \delta}\right]e_t + v_t \\
\Rightarrow \frac{\phi\delta\theta + \eta(\theta + \delta) + \delta}{\theta + \delta}\overline{m} - \frac{1 + \phi\theta}{\theta + \delta}\varepsilon_t &= \frac{\phi\delta\theta + \eta(\theta + \delta) + \delta}{\theta + \delta}e_t + v_t \\
\Rightarrow \left[(\phi\theta + 1)\delta + \eta(\theta + \delta)\right]\overline{m} - (1 + \phi\theta)\varepsilon_t &= \left[(\phi\theta + 1)\delta + \eta(\theta + \delta)\right]e_t + v_t \\
\Rightarrow e_t &= \overline{m} - \frac{(1 + \phi\theta)\varepsilon_t + (\theta + \delta)v_t}{(\phi\theta + 1)\delta + \eta(\theta + \delta)} \qquad (2.1.2)
\end{aligned}$$

We bring this result back to p_t, we have:

$$\begin{aligned}
p_t &= \overline{m} + \frac{\delta(e_t - \overline{m}) + \varepsilon_t}{\theta + \delta} \\
&= \overline{m} + \frac{\varepsilon_t - \frac{\delta(1+\phi\theta)\varepsilon_t + \delta(\theta+\delta)v_t}{(\phi\theta+1)\delta + \eta(\theta+\delta)}}{\theta + \delta} \\
&= \overline{m} + \frac{\eta(\theta + \delta)\varepsilon_t - \delta(\theta + \delta)}{(\theta + \delta)\left[(\phi\theta + 1)\delta + \eta(\theta + \delta)\right]} \\
&= \overline{m} + \frac{\eta\varepsilon_t - \delta v_t}{(1 + \phi\theta)\delta + \eta(\theta + \delta)} \qquad (2.1.3)
\end{aligned}$$

and that

$$\begin{aligned}
y_t &= \frac{\theta\left[\delta(e_t - \overline{m}) + \varepsilon_t\right]}{\theta + \delta} \\
&= \frac{\eta\theta\varepsilon_t - \delta\theta v_t}{(\phi\theta + 1)\delta + \eta(\theta + \delta)} \qquad (2.1.4)
\end{aligned}$$

Then, the variance of output should be:

$$\mathbb{E}_{t-1}[y_t^2] = \left(\frac{\eta\theta}{(\phi\theta+1)\delta+\eta(\theta+\delta)}\right)^2 \sigma_\varepsilon^2 + \left(\frac{\delta\theta}{(\phi\theta+1)\delta+\eta(\theta+\delta)}\right)^2 \sigma_v^2$$

denote $A = (\phi\theta+1)\delta+\eta(\theta+\delta)$, we have:

$$\mathbb{E}_{t-1}[y_t^2] = \frac{\theta^2\eta^2\sigma_\varepsilon^2 + \theta^2\delta^2\sigma_v^2}{A^2}.$$

2.2 Exchange rate peg

Under a fixed exchange rate, $e_t = \overline{e} = \overline{m}$.
From the total demand function, we have:

$$y_t = \delta(e_t - p_t) + \varepsilon_t = \delta(\overline{m} - p_t) + \varepsilon_t$$

From the total supply function, we have:

$$y_t = \theta(p_t - \mathbb{E}_{t-1}[p_t]) = \theta(p_t - \overline{m})$$

Combining these two equations, we have:

$$\begin{aligned} \theta(p_t - \overline{m}) &= \delta(\overline{m} - p_t) + \varepsilon_t \\ \Rightarrow p_t &= \overline{m} + \frac{\varepsilon_t}{\theta+\delta} \end{aligned}$$

Bring p_t back to the total supply function, we have:

$$y_t = \theta(p_t - \overline{m}) = \theta\frac{\varepsilon_t}{\theta+\delta} = \frac{\theta}{\theta+\delta}\varepsilon_t$$

As the exchange rate is pegged, $i_{t+1} = \mathbb{E}_t[e_{t+1}] - e_t = 0$. From the money demand function, we could get:

$$\begin{aligned} m_t - p_t &= \phi y_t + v_t = \phi\frac{\theta}{\theta+\delta}\varepsilon_t + v_t \\ \Rightarrow m_t &= p_t + \phi\frac{\theta}{\theta+\delta}\varepsilon_t + v_t \\ \Rightarrow m_t &= \overline{m} + \frac{\varepsilon_t}{\theta+\delta} + \phi\frac{\theta}{\theta+\delta}\varepsilon_t + v_t \\ &= \overline{m} + \frac{(1+\phi\theta)\varepsilon_t}{\theta+\delta} + v_t \end{aligned}$$

For the variance:

$$\mathbb{E}_{t-1}[y_t^2] = \left(\frac{\theta}{\theta+\delta}\right)^2 \sigma_\varepsilon^2$$

2.3 Regime choice

The relative volatility is given by the ratio of variance under two policies, given by:

$$\frac{\mathbb{V}[y_t]_{peg}}{\mathbb{V}[y_t]_{money}} = \frac{[\eta(\theta+\delta)+\delta(1+\phi\theta)]^2}{(\theta+\delta)^2}\frac{\sigma_\varepsilon^2}{\eta^2\sigma_\varepsilon^2+\delta^2\sigma_v^2}$$

So, if $\sigma_\varepsilon^2/\sigma_v^2 \to 0$, the relative volativity is determined by $\sigma_v^2/\sigma_\varepsilon^2 \to \infty$, thus the ratio is close to 0, thus a peg policy is less volatile than a fixed money supply.
When the main source of volatility is a money demand shock (v_t) rather than a real economy shock (ε_t), a fixed exchange rate regime stabilizes output by allowing the money supply to adjust automatically to offset money demand shocks. Therefore, the output equation only relies on the real shock ϵ_t.
In contrast, a fixed money supply regime is unable to adjust in the face of money demand shocks, leading to higher output volatility.

2.4 Optimal rule

At steady state, we have:

$$\begin{aligned} i &= \mathbb{E}_t\overline{e} - \overline{e} = 0 \\ \overline{y} &= \theta(\overline{p}-\overline{p}) = 0 \\ \overline{y} &= \delta(\overline{e}-\overline{p}) + 0 = 0 \implies \overline{e} = \overline{p} \\ \overline{m} - \overline{p} &= -\eta i + \phi\overline{y} + 0 \implies \overline{m} = \overline{p} \end{aligned}$$

which means $\overline{e} = \overline{p} = \overline{m}$ and $i_{t+1} = \mathbb{E}_t e_{t+1} - e_t = \overline{e} - e_t = \overline{m} - e_t$. Therefore:

$$\begin{aligned} m_t - p_t &= -\eta i_{t+1} + \phi y_t + v_t \\ \overline{m} + \Phi(\overline{e}-e_t) - p_t &= -\eta(\overline{e}-e_t) + \phi y_t + v_t \\ \overline{m} - p_t &= -(\eta+\Phi)(\overline{e}-e_t) + \phi y_t + v_t \end{aligned}$$

Compare with section 2.1, we can see that the only difference is the Φ term, which is the Taylor rule coefficient. Thus we only need to replace the η with $(\eta+\Phi)$ in the previous equations, we can get:

$$\begin{aligned} p_t &= \overline{m} + \frac{(\eta+\Phi)\varepsilon_t - \delta v_t}{(1+\phi\theta)\delta + (\eta+\Phi)(\theta+\delta)} \\ y_t &= \frac{\theta(\eta+\Phi)\varepsilon_t - \theta\delta v_t}{[(\phi\theta+1)\delta + (\eta+\Phi)(\theta+\delta)]^2} \\ e_t &= \overline{m} - \frac{(1+\phi\theta)\varepsilon_t + (\theta+\delta)v_t}{(\phi\theta+1)\delta + (\eta+\Phi)(\theta+\delta)} \end{aligned}$$

Then, the variance of output should be:

$$\mathbb{E}_{t-1}[y_t^2] = \frac{\theta^2(\eta+\Phi)^2\sigma_\varepsilon^2 + \theta^2\delta^2 v_t^2}{[(\phi\theta+1)\delta + (\eta+\Phi)(\theta+\delta)]^2}$$

To solve the optimal value of Φ, we aim to minimizes tha variance of output. First we denote $A = \eta + \Phi$ and $D = A(\theta + \delta) + \delta(1 + \phi\theta)$, then we know that $\frac{dA}{d\Phi} = 1$ and $\frac{dD}{d\Phi} = \theta + \delta$. Take the FOC of $\mathbb{E}_{t-1}[y_t^2]$, we have:

$$\frac{\partial \mathbb{E}_{t-1}[y_t^2]}{\partial \Phi} = \theta^2 \frac{2A\sigma_\varepsilon^2 D^2 - [A^2\sigma_\varepsilon^2 + \delta^2\sigma_v^2]\, 2DD'}{D^4} = 0$$

Then, we have:

$$\begin{aligned}
&A\sigma_\varepsilon^2 D = \left(A^2\sigma_\varepsilon^2 + \delta^2\sigma_v^2\right)(\theta + \delta) \\
\Rightarrow &\left[A^2(\theta + \delta) + A\delta(1 + \phi\theta)\right]\sigma_\varepsilon^2 = A^2(\theta + \delta)\sigma_\varepsilon^2 + (\theta + \delta)\delta^2\sigma_v^2 \\
\Rightarrow &A = \frac{(\theta + \delta)\delta\sigma_v^2}{(1 + \phi\theta)\sigma_\varepsilon^2}
\end{aligned}$$

So, we have:

$$\Phi^* = \frac{(\theta + \delta)\delta\sigma_v^2}{(1 + \phi\theta)\sigma_\varepsilon^2} - \eta.$$

When $\sigma_\varepsilon^2/\sigma_v^2 \to 0$, $\Phi \to \infty$, this implies that the central bank should fix the exchange rate completely. This is consistent with our conclusion in 2.3.
When $\sigma_v^2/\sigma_\varepsilon^2 \to 0$, $\Phi \to -\eta$. This implies that the central bank should implement a fully floating exchange rate regime because at this point $\Phi + \eta \approx 0$, which is equivalent to not reacting to exchange rate fluctuations.
When money demand shocks dominate, it is better to fix the exchange rate to offset these shocks; when real economy shocks dominate, it is better to let the exchange rate float freely to absorb these shocks.

3 Taxation of debt

3.1 Decentralized and centralized choice

Under decentralized allocation, the household's budget constraint is:

$$C_2 = Y_2 - (1 + r_s)D_1 = Y_2 - (1 + r + \alpha D_1)D_1$$

while the household maximizes the utility function:

$$\max_{D_1} U(D_1) = \max_{D_1} \left\{ D_1 + \frac{1}{1 + \delta}\left[Y_2 - (1 + r_s)D_1\right] \right\}$$

Take the first derivative w.r.t. D_1 and set it to 0, we have:

$$
\begin{aligned}
&1 - \frac{1+r_s}{1+\delta} = 0 \\
\Rightarrow &\delta = r_s = r + \alpha D_1 \\
\Rightarrow &D_1^{decentralized} = \frac{\delta - r}{\alpha}
\end{aligned}
$$

$r_s^{decentralized} = \delta$. For planner's decision, we have:

$$\max_{D_1} U(D_1) = \max_{D_1} \left\{ D_1 + \frac{1}{1+\delta} \left[Y_2 - (1 + r + \alpha D_1) D_1 \right] \right\}$$

Take the first derivative w.r.t. D_1 and set it to 0, we have:

$$
\begin{aligned}
&1 - \frac{1 + r + \alpha D_1 + \alpha D_1}{1+\delta} = 0 \\
\Rightarrow &\delta = r + 2\alpha D_1 \\
\Rightarrow &D_1^{centralized} = \frac{\delta - r}{2\alpha}
\end{aligned}
$$

$r_s^{centralized} = r + \alpha D_1^{centralized} = \frac{\delta + r}{2}$.

The speaking order decentralized decision leads to overborrowing ($D_1^{decentralized} > D_1^{planner}$) because individual households do not take into account the fact that their own borrowing behavior increases the cost of borrowing for the whole economy by raising interest rates. This is a classic externality problem.

The planner takes this externality into account and therefore chooses a lower level of borrowing, which reduces the overall cost of borrowing. Under the planner's allocation, the interest rate is lower than the decentralized allocation ($r_s^{planner} < r_s^{decentralized}$), which reflects an internalization of the borrowing externality.

3.2 Taxes

Under the first tax regime, the interest rate with tax is:

$$r_s + \tau^{variable} = r_s + \gamma D_1$$

So, the household's utility maximization is:

$$\max_{D_1} U(D_1) = \max_{D_1} \left\{ D_1 + \frac{1}{1+\delta} \left[Y_2 - (1 + r_s + \tau^{variable}) D_1 \right] \right\}$$

where r_s is still regarded as a constant. Take the first order derivative w.r.t. D_1 and set it to 0, we have:

$$
\begin{aligned}
&1 - \frac{1 + r_s + 2\tau^{variable} D_1}{1+\delta} = 0 \\
\Rightarrow &\delta = r_s + \gamma D_1
\end{aligned}
$$

If we want the household to choose $D_1^{centralized}$, we bring this back to the tax rate:

$$\begin{aligned} \delta &= r + \frac{\delta - r}{2} + \gamma \frac{\delta - r}{2\alpha} \\ \Rightarrow \gamma &= \alpha \\ \Rightarrow \tau^{variable} &= \alpha D_1 \end{aligned}$$

Under a flat tax rate, we have the household's utility maximization problem as:

$$\max_{D_1} U(D_1) = \max_{D_1} \left\{ D_1 + \frac{1}{1+\delta} \left[Y_2 - (1 + r_s + \tau^{flat}) D_1 \right] \right\}$$

Take the first order derivative w.r.t. D_1 and set it to 0, we have:

$$\begin{aligned} 1 - \frac{1 + r_s + \tau^{flat}}{1+\delta} &= 0 \\ \Rightarrow \delta = r_s + \tau^{flat} \end{aligned}$$

As we still require $D_1 = D_1^{centralized}$, we bring it back:

$$\begin{aligned} \tau^{flat} &= \delta - r_s \\ &= \delta - (r + \alpha \frac{\delta - r}{2\alpha}) \\ &= \frac{\delta - r}{2} \end{aligned}$$

From a feasibility point of view, there are advantages and disadvantages to each of these two tax systems:
Variable tax rate ($\tau_{variable} = \gamma D_1$ where $\gamma = \alpha$):

- Advantages: adjusts with debt level, more accurate treatment of externalities
- Disadvantages: governments need to accurately estimate the α parameter, which can be challenging in practice

Fixed tax rate ($\tau^{flat} = \frac{\delta - r}{2}$):

- Advantages: simple to implement, no need to monitor each household's debt level
- Disadvantages: government needs to accurately estimate δ, r parameters

A flat tax rate may be easier to implement because it does not require constant monitoring of debt levels and adjusting the tax rate accordingly. The risk-free interest rate r can be directly observed in financial markets. However, it relies on an accurate estimate of the time preference rate δ, which depends on the model used and can be challenging in heterogeneous household settings.
A debt-dependent tax rate may be more precise in theory, but is more difficult to implement because it requires the government to understand the precise nature of the debt-interest rate relationship and would need frequent recalculation as debt levels change.
Overall, flat tax rates may be more feasible in practical policy settings, especially when governments face information constraints and administrative costs.