Game Theory

Static Game of Incomplete Information

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Outline

- Static Game with Incomplete Information
- Bayesian Nash Equilibrium (BNE) 贝叶斯纳什均衡
- Harsanyi's interpretation of Mixed-Strategies

完静 NE
完动 SPNE
$\boxed{\text{不完静 BNE}}$
不完动 PBNE

Example: Sheriff's Dilemma

- A sheriff faces an armed suspect. Both must simultaneously decide whether to shoot the other or not.
- The suspect can either be of type "criminal" or type "civilian." The sheriff has only one type.
- The suspect knows its type and the Sheriff's type, but the Sheriff does not know the suspect's type—and the above messages are common knowledge. Thus, there is incomplete information (because the suspect has private information), making it a Bayesian game.
- The sheriff would rather defend himself and shoot if the suspect shoots, or not shoot if the suspect does not (even if the suspect is a criminal).
- The suspect would rather shoot if he is a criminal, even if the sheriff does not shoot; The suspect would rather not shoot if he is a civilian, even if the sheriff shoots.

The payoff matrix of this Normal-form game for both players depends on the **type** (类型) of the suspect

Type: Civilian w.p. 1-*p*		Suspect	
		~~Shoot~~	Not
Sheriff	Shoot	-1, -3	-1, -2
	Not	-2, -1	0, 0

Type: Criminal w.p. *p*		Suspect	
		Shoot	~~Not~~
Sheriff	Shoot	0, 0	-1, -2
	Not	-2, 2	1, -1

Civilian Type
- suspect: “Not” is a dominant strategy
  - NE: (not, not)

Criminal Type
- suspect: “Shoot” is a dominant strategy
- NE: (shoot, shoot)

{1, 4}

However, suspect’s type is private information, i.e., from the view of the sheriff; information partition: {civilian, criminal} cannot be distinguished. The sheriff does not know which game he is playing.

Harsanyi's Transformation (海萨尼转换)

$1-p$

Type: Civilian w.p. 1-*p*		Suspect	
		~~Shoot~~	Not
Sheriff	Shoot	-1, -3	-1, -2
	Not	-2, -1	0, 0

p

Type: Criminal w.p. *p*		Suspect	
		Shoot	~~Not~~
Sheriff	Shoot	0, 0	-1, -2
	Not	-2, 2	1, -1

- In order to capture such uncertainty, we **introduce the move of "Nature"**
  - Nature decides whether the suspect is civilian (with probability $1-p$) or criminal (w.p., p)
- From the view of the sheriff: after eliminating the strictly dominated strategies (of the suspect):
  - the expected payoff of shooting is $-1 \cdot (1-p) + 0 \cdot = p - 1$
  - the expected payoff of not shooting is $0 \cdot (1-p) - 2 \cdot p = -2p$
- The sheriff should shoot if $p > \frac{1}{3}$

Strategic Representation of Bayesian Games

- Incomplete information:
  - a player's preferences are associated with his/her type
  - uncertainty over types is described by nature choosing types for different players
  - common knowledge (prior) about the probability distribution of types
- The normal-form representation of an n-player static Bayesian game of incomplete information

$$\langle N, \{A_i\}_{i=1}^n, \underbrace{\{\Theta_i\}_{i=1}^n}_{\theta}, \{v_i(\cdot;\theta_i), \theta_i \in \Theta_i\}_{i=1}^n, \{\phi_i\}_{i=1}^n \rangle$$

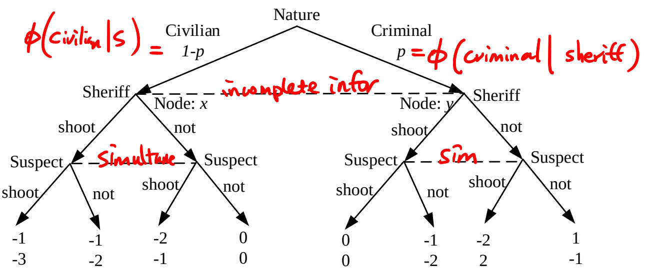
θ theta type

  - $N = \{1, 2, ..., n\}$: set of players
  - A_i: action set
  - $\Theta_i = \{\theta_{i1}, ..., \theta_{ik}\}$: type space of player i
  - v_i: payoff of player i $-i$: others
  - $\phi_i = \phi_i(\theta_{-i}|\theta_i)$: belief of player i with respect to the uncertainty over others, i.e., posterior conditional distribution on θ_{-i}

Steps of Static Bayesian Games

- A static Bayesian game proceeds through the following:
  1. Nature chooses a profile of types $(\theta_1, ..., \theta_n)$
  2. Each player i learns his/her own type θ_i, which is private information, and then uses his/her prior ϕ_i to form posterior beliefs over the other types of players
  3. Players simultaneously choose actions $a_i \in A_i$
  4. Given the players' choices $a = (a_1, ..., a_n)$, the payoffs $v_i(a; \theta_i)$ are realized for each player i.

Introduction of "Nature" and the Extensive-Form



- Nature chooses the types of the suspect
  - Sheriff: $\{x, y\}$
  - Suspect: $\{x\}, \{y\}$
- The strategy set of sheriff is $\{$shoot,not$\}$ → set
- The suspect has four pure strategies (conditional on his type (x, y)): $\{$(shoot,shoot), (not,not), (shoot,not), (not,shoot)$\}$.

Normal-Form Matrix Representation

Type: Civilian w.p. 1-p		Suspect	
		~~Shoot~~	Not
Sheriff	Shoot	-1,-3	-1,-2
	Not	-2,-1	0, 0

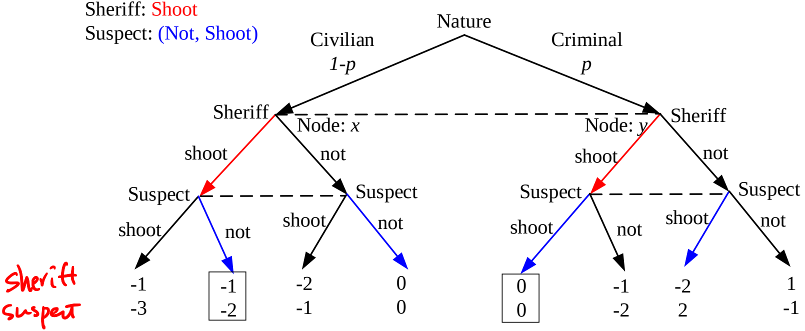
Type: Criminal w.p. p		Suspect	
		Shoot	~~Not~~
Sheriff	Shoot	0, 0	-1,-2
	Not	-2, 2	1,-1

		Suspect (Civilian, Criminal)			
		x y	x y	x y	x y
		Shoot, Shoot	Not, Not	Shoot, Not	Not, Shoot
Sheriff	Shoot	-(1-p), -3(1-p)	-1, -2	-1, p-3	-(1-p), -2(1-p)
	Not	-2, 3p-1	p, -p	3p-2, -1	-2p, 2p

- Nature chooses the types of the suspect: the sheriff cannot distinguish node x and y.
- Sheriff chooses from {shoot,not}
- Conditional on the sheriff's belief, the suspect has four pure strategies: {(shoot,shoot), (not,not), (shoot,not), (not,shoot)}, where (xy) means: x describes what a civilian does and y what a criminal does.
- To obtain the equilibrium, use the matrix to represent the extensive-form, and then find the "intersections" of best responses.

		Suspect (Civilian, Criminal)			x y
		Shoot, Shoot	Not, Not	Shoot, Not	Not, Shoot
Sheriff	Shoot	$\underline{-(1-p)}$, $-3(1-p)$	-1, -2	-1, p-3	$-(1-p)$, $\underline{-2(1-p)}$
	Not	-2, $3p$-1	$\underline{p}$, $-p$	$3p$-2, -1	$-2p$, $\underline{2p}$

Sheriff: Shoot
Suspect: (Not, Shoot)

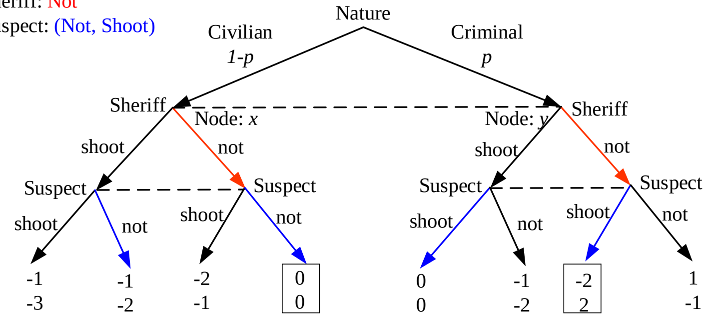


Example: find the payoffs for $\{$shoot, (not, shoot)$\}$:

- Sheriff: $-1 \cdot (1-p) + 0 \cdot p$ $= -(1-p)$
- Suspect: $-2 \cdot (1-p) + 0 \cdot p$ $= -2(1-p)$

		Suspect (Civilian, Criminal)			
		Shoot, Shoot	Not, Not	Shoot, Not	Not, Shoot
Sheriff	Shoot	$\underline{-(1-p)}$, $-3(1-p)$	-1, -2	-1, $p-3$	$-(1-p)$, $\underline{-2(1-p)}$
	Not	-2, $3p-1$	$\underline{p}$, $-p$	$3p-2$, -1	$-2p$, $\underline{2p}$

Sheriff: Not
Suspect: (Not, Shoot)



Example: find the payoffs for $\{\text{not}, (\text{not}, \text{shoot})\}$:

- Sheriff: $0 \cdot (1-p) - 2 \cdot p$
- Suspect: $0 \cdot (1-p) + 2 \cdot p$

Type: Civilian w.p. 1-p		Suspect	
		~~Shoot~~	Not
Sheriff	Shoot	-1,-3	-1,-2
	Not	-2,-1	0, 0

Type: Criminal w.p. p		Suspect	
		Shoot	~~Not~~
Sheriff	Shoot	0, 0	-1,-2
	Not	-2, 2	1,-1

		Suspect (Civilian, Criminal)			1-p x y p
		Shoot, Shoot	Not, Not	Shoot, Not	Not, Shoot
Sheriff	Shoot	-(1-p), -3(1-p)	-1, -2	-1, p-3	-(1-p), -2(1-p)
	Not	-2, 3p-1	p, -p	3p-2, -1	-2p, 2p

- E.g., for the outcome {shoot, (not, shoot)}: the sheriff believes that the suspect is using the pure strategy (not, shoot), which can be described as

$$s_2(\theta_2) = \begin{cases} \text{not}, & \text{if } \theta_2 = \text{civilian} \\ \text{shoot}, & \text{if } \theta_2 = \text{criminal} \end{cases}$$

- If the suspect ($i = 2$) is using a pure strategy (type-dependent), while nature chooses 2's type randomly, then the sheriff ($i = 1$) is facing a suspect who is using a "mixed" strategy: if suspect uses the strategy (not, shoot), it is as if the suspect is choosing "not" with probability $1 - p$ and "shoot" with probability p.

$p=0.5$ $p=0.1$ 坏: p | $1-p$ 好

Type: Civilian w.p. 1-p		Suspect	
		~~Shoot~~	Not
Sheriff	Shoot	-1,-3	-1,-2
	Not	-2,-1	0, 0

Type: Criminal w.p. p		Suspect	
		Shoot	~~Not~~
Sheriff	Shoot	0, 0	-1,-2
	Not	-2, 2	1,-1

		Suspect (Civilian, Criminal)			
		Shoot, Shoot	Not, Not	Shoot, Not	Not, Shoot
Sheriff	Shoot	-(1-p), -3(1-p)	-1, -2	-1, p-3	-(1-p), -2(1-p)
	Not	-2, 3p-1	p, -p	3p-2, -1	-2p, 2p

$(0<p<1)$

- When the sheriff chooses shoot, the best response of the suspect is (not, shoot): $-2(1-p)$ is the greatest number among the second arguments in the first row; when the sheriff chooses not, the best response is also (not, shoot): $2p$ is the greatest
- Similarly, after underlining the best responses of the sheriff: the "intersection" is obtained by comparing the last column (red):
  - $-(1-p) > -2p \Leftrightarrow p > \frac{1}{3}$: NE is {shoot, (not,shoot)}
  - $-(1-p) < -2p \Leftrightarrow p < \frac{1}{3}$: NE is {not, (not,shoot)}

$p-3>-2(1-p)$
$p-3>-2+2p$
$-1>p$

$3p-1>2p$
$p>1$

$-(1-p)>-2p$
$-1+p>-2p$
$3p>1$
$p>\frac{1}{3}$

Pure-Strategy Bayesian Nash Equilibrium (BNE)

i 我

$-i$ 别人

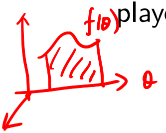
$$E(X) = \overset{v_1}{p_1 X_1} + \overset{v_2}{p_2 X_2} + \cdots + \overset{v_n}{p_n X_n}$$

x_k

- In the Bayesian game

$$\langle N, \{A_i\}_{i=1}^n, \{\Theta_i\}_{i=1}^n, \{v_i(\cdot;\theta_i), \theta_i \in \Theta_i\}_{i=1}^n, \{\phi_i\}_{i=1}^n \rangle$$

a strategy profile $s^* = (s_1^*, \ldots s_n^*)$ is a pure-strategy Bayesian Nash equilibrium (BNE) if, for every player i, for each of player i's types $\theta_i \in \Theta_i$, and for every $a_i \in A$, s_i^* solves



$$\sum_{\theta_{-i} \in \Theta_{-i}} \underline{\phi_i(\theta_{-i}|\theta_i)} \overset{s^*}{\underline{v_i\left(s_i^*(\theta_i), s_{-i}^*(\theta_{-i}); \theta_i\right)}}$$

$$\geq \sum_{\theta_{-i} \in \Theta_{-i}} \phi_i(\theta_{-i}|\theta_i) v_i\left(\overset{a_i}{a_i}, s_{-i}^*(\theta_{-i}); \theta_i\right)$$

$EV = \int \underline{v} \cdot f(\theta) d\theta$

$\overset{p}{(\cdots)} + \overset{1-p}{(\cdots)}$ shoot

$\overset{p}{(\cdots)} + \overset{1-p}{(\cdots)}$ not

- Playing a best response as (for all $a_i \in A_i$)

$$E_{\boxed{\theta_{-i}}}\left[v_i\left(\underline{s_i^*(\theta_i)}, s_{-i}^*(\theta_{-i}); \theta_i\right) |\theta_i\right] \geq E_{\theta_{-i}}\left[v_i\left(\underline{a_i}, s_{-i}^*(\theta_{-i}); \theta_i\right) |\theta_i\right]$$

BNE of the Sheriff's Dilemma

		Suspect (Civilian, Criminal)			
		Shoot, Shoot	Not, Not	Shoot, Not	Not, Shoot
Sheriff	Shoot	-(1-*p*), -3(1-*p*)	-1, -2	-1, *p*-3	-(1-*p*), -2(1-*p*)
	Not	-2, 3*p*-1	*p*, -*p*	3*p*-2, -1	-2*p*, 2*p*

- The suspect plays (not, shoot):

$$s_2(\theta_2) = \begin{cases} \text{not}, & \text{if } \theta_2 = \text{civilian} \\ \text{shoot}, & \text{if } \theta_2 = \text{criminal} \end{cases}$$

- BNE is {shoot, (not,shoot)} if
$E_{\theta_2}[v_1(s_1 = \text{shoot}, s_2 = (\text{not,shoot}); \theta_1|\theta_1] =$
$(1-p)v_1(\text{shoot}, s_2(\text{civilian})) + pv_1(\text{shoot}, s_2(\text{criminal})) = -(1-p) \geq$
$E_{\theta_2}[v_1(s_1 = \text{not}, s_2 = (\text{not,shoot})); \theta_1|\theta_1] =$
$(1-p)v_1(\text{shoot}, s_2(\text{civilian})) + pv_1(\text{shoot}, s_2(\text{criminal})) = -2p$
- BNE is {not, (not,shoot)} if $-(1-p) < -2p$
- "shoot" if $p > \frac{1}{3}$; "not" if $p < \frac{1}{3}$.

- BNE is determined by the sheriff's **belief** about the type of the suspect: $\Pr(\theta_{\text{suspect}} = \text{criminal} | \theta_{\text{sheriff}}) \gtrless \text{threshold}$

(a) sheriff: "shoot", suspect: not

(b) sheriff: shoot, suspect: shoot

Nature

(c) sheriff: not, suspect: shoot

“sheriff: not, suspect: not”

Example: Teenagers and the Game of Chicken

duel

- Two teenagers, player 1 and 2, borrowed their parents' cars and decided to play the game of chicken as follows:
  - drive toward each other; simultaneously choose whether to be chicken, i.e., swerve to the right (C); or continue driving head on (D)
    Chicken
  - both chicken: both gain no respect and suffer no loses: 0
  - i drive while j plays chicken, then i gains respect R and j gets 0
  - both drive head on: split the respect $\frac{R}{2}$, but suffer from an accident $\boxed{k}$ which is measured by punishments from parents
  - the punishment k depends on whether parents are harsh (H) or lenient (L) with equal probability, and $H > L$
  - each kid knows the type of his/her own parents but the type of the opponent's parents is private information

LH: k

12

Type: LL w.p. *1/4*		Player 2	
		~~c~~	d
Player 1	~~C~~	0, 0	0, $\underline{R}$
	D	$\underline{R}$, 0	$\underline{0.5R\text{-}L}$, $\underline{0.5R\text{-}L}$

12

Type: HH w.p. *1/4*		Player 2	
		c	d
Player 1	C	0, 0	$\underline{0}$, $\underline{R}$
	D	$\underline{R}$, $\underline{0}$	0.5R-H, 0.5R-H

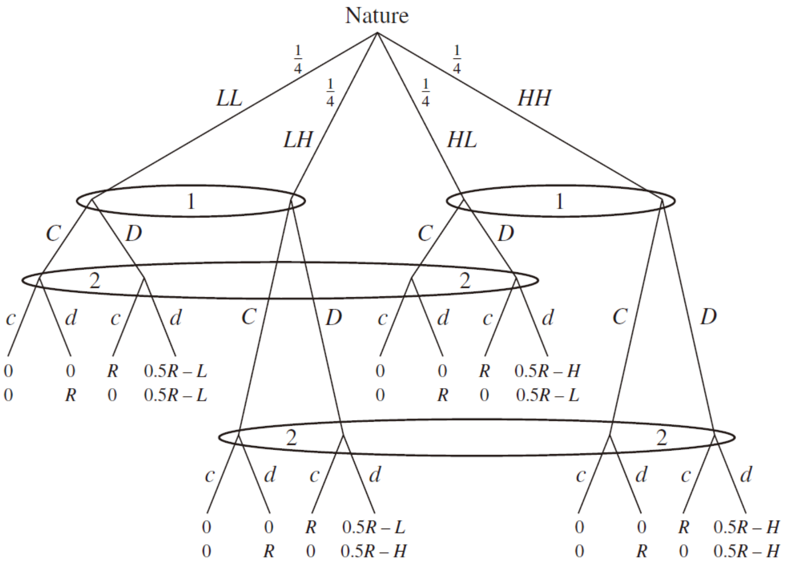
12

Type: LH w.p. *1/4*		Player 2	
		c	d
Player 1	~~C~~	0, 0	0, $\underline{R}$
	D	$\underline{R}$, $\underline{0}$	$\underline{0.5R\text{-}L}$, 0.5R-H

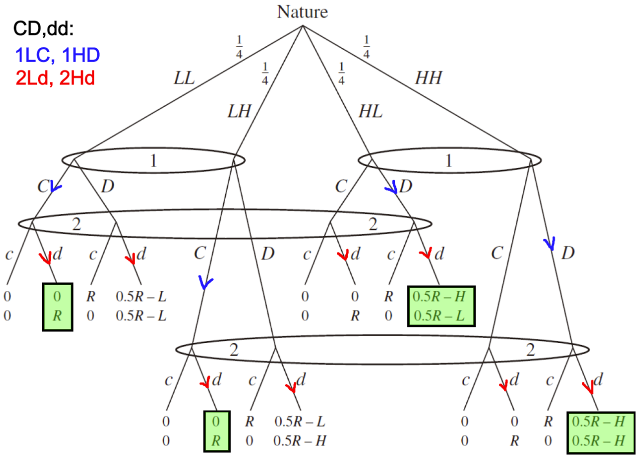
12

Type: HL w.p. *1/4*		Player 2	
		~~c~~	d
Player 1	C	0, 0	$\underline{0}$, $\underline{R}$
	D	$\underline{R}$, 0	0.5R-H, $\underline{0.5R\text{-}L}$

- "Nature" moves first and picks one of the above four states, with probability 0.25 each.
- The payoffs are assumed to be: $R = 8, H = 16, L = 0$


Nature
1/4
1/4
1/4
1/4
LL
LH
HL
HH
1
1
C
D
C
D
2
2
c
d
c
d
C
D
c
d
c
d
C
D
0 0
0 R
R 0
0.5R – L 0.5R – L
0 0
0 R
R 0
0.5R – H 0.5R – L
2
2
c
d
c
d
c
d
c
d
0 0
0 R
R 0
0.5R – L 0.5R – H
0 0
0 R
R 0
0.5R – H 0.5R – H

- Four states of nature: $\theta_1\theta_2 \in \{LL, HH, HL, LH\}$ with probability 0.25 each. The first (resp., second) letter denotes for player 1's (resp., 2's) type.
    - Player 1's information function is $P1 : \{LL, LH\}, \{HL, HH\}$
    - Player 2's information function is $P2 : \{LL, HL\}, \{LH, HH\}$
- Action set: $A_1 = \{C, D\}$; $A_2 = \{c, d\}$
- Strategy set $S_1 = \{CC, CD, DC, DD\}$ and $S_2 = \{cc, cd, dc, dd\}$, where the first letter denotes the choice when his/her own type is L while the second letter denotes the choice when one's own type is H.
    - e.g., (CD, dd) means if player 1's type is L then 1 chooses C and if player 1's type is H then 1 chooses D; if player 2's type is L then 2 chooses d and if player 2's type is H then 2 chooses d.



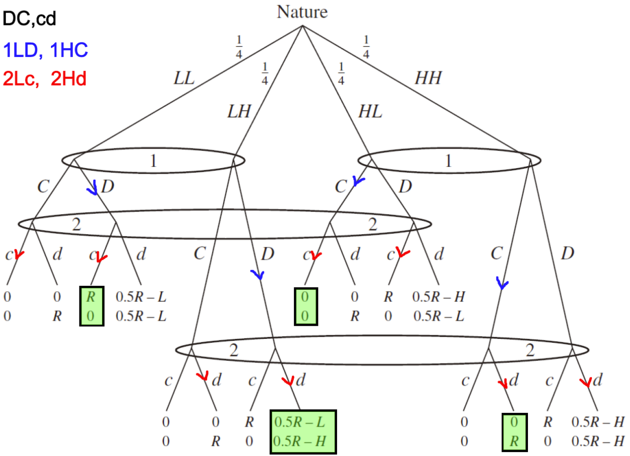
For (CD, dd):

- 1's expected payoff: $\frac{1}{4} \cdot 0 + \frac{1}{4} \cdot 0 + \frac{1}{4}\left(\frac{1}{2}R - H\right) + \frac{1}{4}\left(\frac{1}{2}R - H\right) = \frac{1}{4}R - \frac{1}{2}H$
- 2's expected payoff: $\frac{1}{4}R + \frac{1}{4}R + \frac{1}{4}\left(\frac{1}{2}R - L\right) + \frac{1}{4}\left(\frac{1}{2}R - H\right) = \frac{3}{4}R - \frac{1}{4}L - \frac{1}{4}H$

DC,cd

1LD, 1HC

2Lc, 2Hd



For (DC, cd):

- 1's expected payoff: $\frac{1}{4}R + \frac{1}{4}\left(\frac{1}{2}R - L\right) = \frac{3}{8}R - \frac{1}{4}L$
- 2's expected payoff: $\frac{1}{4}\left(\frac{1}{2}R - H\right) + \frac{1}{4}R = \frac{3}{8}R - \frac{1}{4}H$

Strategy $S_1 = \{CC, CD, DC, DD\}$ and $S_2 = \{cc, cd, dc, dd\}$. There are 16 possible outcomes.

		Player 2			
		cc	cd	dc	dd
Player 1	CC	$0, 0$	$0, \frac{R}{2}$	$0, \frac{R}{2}$	$0, R$
	CD	$\frac{R}{2}, 0$	$\frac{3R}{8} - \frac{H}{4}, \frac{3R}{8} - \frac{H}{4}$	$\frac{3R}{8} - \frac{H}{4}, \frac{3R}{8} - \frac{L}{4}$	$\frac{R}{4} - \frac{H}{2}, \frac{3R}{4} - \frac{L}{4} - \frac{H}{4}$
	DC	$\frac{R}{2}, 0$	$\frac{3R}{8} - \frac{L}{4}, \frac{3R}{8} - \frac{H}{4}$	$\frac{3R}{8} - \frac{L}{4}, \frac{3R}{8} - \frac{L}{4}$	$\frac{R}{4} - \frac{L}{2}, \frac{3R}{4} - \frac{L}{4} - \frac{H}{4}$
	DD	$R, 0$	$\frac{3R}{4} - \frac{L}{4} - \frac{H}{4}, \frac{R}{4} - \frac{H}{2}$	$\frac{3R}{4} - \frac{L}{4} - \frac{H}{4}, \frac{R}{4} - \frac{L}{2}$	$\frac{R}{2} - \frac{L}{2} - \frac{H}{2}, \frac{R}{2} - \frac{L}{2} - \frac{H}{2}$

- Assume $R = 8$, $H = 16$ and $L = 0$, the matrix-form Bayesian game becomes

		Player 2			
		cc	cd	dc	dd
	CC	0, 0	0, 4	0, 4	$\overline{0, 8}$
	CD	4, 0	−1, −1	$\overline{-1, 3}$	−6, 2
Player 1	DC	4, 0	$\underline{3, -1}$	$\underline{\overline{3, 3}}$	$\underline{2, 2}$
	DD	$\underline{8, 0}$	2, −6	$\overline{2, 2}$	−4, −4

- A unique pure-strategy Bayesian Nash equilibrium (DC, dc): the kid with lenient parents continues to drive on; the kid with harsh parents will swerve

Continuous Actions: Cournot Game

Cournot Competition under Asymmetric Information

- Recall the Cournot model: two firms, 1 and 2 offer quantities simultaneously
- $p(Q) = a - Q$, $Q = q_1 + q_2$
- Firm 1's cost function is $c_1 q_1$, which is common knowledge
- Firm 2's "type" is privately known:

$$c_2 = \begin{cases} c_H, & \text{w.p. } \theta \\ c_L, & \text{w.p. } 1-\theta \end{cases}$$

Assume that $c_L < c_H$ is private information. But the probability distribution of c_2 is common knowledge.

- Firm 1's strategies: q_1; firm 2's strategies: $q_2(c_H)$ and $q_2(c_L)$
- If $c_2 = c_H$, firm 2 solves $\max_{q_2}[a - q_1 - q_2 - c_H]q_2$.
  - FOC2H: $q_2(c_H) = \frac{a-q_1-c_H}{2}$
- If $c_2 = c_L$, firm 2 solves $\max_{q_2}[a - q_1 - q_2 - c_L]q_2$
  - FOC2L: $q_2(c_L) = \frac{a-q_1-c_L}{2}$
- Firm 1 solves $\max_{q_1} \theta[a - q_1 - q_2(c_H) - c_1]q_1 + (1-\theta)[a - q_1 - q_2(c_L) - c_1]q_1$
  - FOC1: $q_1 = \frac{\theta[a-q_2(c_H)-c_1]+(1-\theta)[a-q_2(c_L)-c_1]}{2}$
- The BNE $\{q_1^*, q_2^*(c_H), q_2^*(c_L)\}$ is determined by FOC1, FOC2H and FOC2L
  - $q_2^*(c_H) = \frac{a-2c_H+c_1}{3} + \frac{1-\theta}{6}(c_H - c_L)$
  - $q_2^*(c_L) = \frac{a-2c_L+c_1}{3} - \frac{\theta}{6}(c_H - c_L)$
  - $q_1^* = \frac{a-2c_1+\theta c_H+(1-\theta)c_L}{3}$

Mixed Strategies Revisited

Battle of the Sexes		Chris	
		O	F
Alex	O	2,1	0,0
	F	0,0	1,2

Battle of the Sexes		Chris	
		O	F
Alex	O	2+t_A,1	0,0
	F	0,0	1,2+t_C

- In Lecture 2, we show that a mixed-strategy Nash equilibrium is that Alex plays O with probability $\frac{2}{3}$ and Chris plays F with probability $\frac{2}{3}$.
- We can generalize the result by transforming the game into a Bayesian game, assuming the payoff of each player is private information.
  - If O is realized, Alex's payoff is $2 + t_A$.
  - If F is realized, Chris's payoff is $2 + t_C$.
- t_A and t_C are independently distributed according to a uniform distribution on $[0, \bar{t}]$
- Actions: $\{O, F\}$; Type spaces: $\theta_A, \theta_C \in [0, \bar{t}]$

Battle of the Sexes		Chris	
		O	F
Alex	O	<u>2</u>,<u>1</u>	0,0
	F	0,0	<u>1</u>,<u>2</u>

Battle of the Sexes		Chris	
		O	F
Alex	O	2+t_A,1	0,0
	F	0,0	1,2+t_C

- Harsanyi (1973): a mixed-strategy Nash equilibrium in a game of complete information can be interpreted as a pure-strategy Bayesian Nash equilibrium with incomplete information.
- Pure-strategy BNE: Alex plays O if t_A exceeds a threshold t_1; Chris plays F if t_C exceeds a threshold t_2.
- Alex plays O with probability $\Pr(t_A > t_1) = \int_{t_1}^{\bar{t}} \frac{dt_A}{\bar{t}} = \frac{\bar{t}-t_1}{\bar{t}}$; Chris plays F w.p. $\Pr(t_C > t_2) = \frac{\bar{t}-t_2}{\bar{t}}$
- We will show that as incomplete information disappears, i.e., $\bar{t} \to 0$, $\frac{\bar{t}-t_i}{\bar{t}} \to \frac{2}{3}$ $(i = 1, 2)$

Battle of the Sexes		Chris	
		O	F
Alex	O	2,1	0,0
	F	0,0	1,2

Battle of the Sexes		Chris	
		O	F
Alex	O	2+t_A,1	0,0
	F	0,0	1,2+t_C

- Alex:
  - expected payoff from O: $E_{t_C} v_1(O, s_2; t_A) = \frac{t_2}{\bar{t}}(2 + t_A)$
  - expected payoff from F: $E_{t_C} v_1(F, s_2; t_A) = \frac{\bar{t} - t_2}{\bar{t}}$
  - playing O is better if $\frac{t_2}{\bar{t}}(2 + t_A) \geq \frac{\bar{t} - t_2}{\bar{t}} \Rightarrow t_A \geq \frac{\bar{t}}{t_2} - 3$
  - When t_A exceeds the threshold t_1, Alex chooses O, i.e., $\frac{\bar{t}}{t_2} - 3 = t_1$
- Chris:
  - from O: $\frac{\bar{t} - t_1}{\bar{t}}$
  - from F: $\frac{t_1}{\bar{t}}(2 + t_C)$
  - playing F is optimal if $\frac{t_1}{\bar{t}}(2 + t_C) \geq \frac{\bar{t} - t_1}{\bar{t}} \Rightarrow t_C \geq \frac{\bar{t}}{t_1} - 3$; and Chris plays F if $t_C \geq t_2$. Hence $\frac{\bar{t}}{t_1} - 3 = t_2$

Battle of the Sexes		Chris	
		O	F
Alex	O	2,1	0,0
	F	0,0	1,2

Battle of the Sexes		Chris	
		O	F
Alex	O	2+t_A,1	0,0
	F	0,0	1,2+t_C

- Solving $\frac{\bar{t}}{t_2} - 3 = t_1$ and $\frac{\bar{t}}{t_1} - 3 = t_2$, we have $t_1 = t_2 = t \Rightarrow t^2 + 3t - \bar{t} = 0 \Rightarrow$

$$t_1 = t_2 = \frac{-3 + \sqrt{9 + 4\bar{t}}}{2}$$

- Alex plays O with probability $\Pr(t_A > t_1) = \frac{\bar{t} - t_1}{\bar{t}} = 1 - \frac{-3+\sqrt{9+4\bar{t}}}{2\bar{t}}$
- Recall: $(1+x)^n - 1 \sim nx \; (x \to 0)$
- $\lim_{\bar{t} \to 0} \left(1 - \frac{-3+\sqrt{9+4\bar{t}}}{2\bar{t}}\right) = \frac{2}{3}$