PS5 Solutions

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We use the following notation:

$$\Phi_1 = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}, \quad \Phi_\varepsilon = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad \Sigma_u := \mathbb{V}[u_t] = \mathbb{E}[u_t u_t'] = \Phi_\varepsilon \Phi_\varepsilon'.$$

Solution (a).
The VAR(1) is weakly stationary iff all eigenvalues of Φ_1 lie strictly inside the unit circle,

$$\rho(\Phi_1) = \max_i \left|\lambda_i(\Phi_1)\right| < 1.$$

Solution (b).
Assuming y_t is stationary, $\mathbb{E}[y_t] = \mathbb{E}[\Phi_1 y_{t-1} + \Phi_\varepsilon \varepsilon_t] = \Phi_1 \mathbb{E}[y_{t-1}]$, as $\mathbb{E}[y_t] = \mu, \forall t$, $\mu = 0$.

$$\begin{aligned}
\Gamma_{yy}(0) = \mathbb{E}[y_t y_t'] &= \mathbb{E}\left[(\Phi_1 y_{t-1} + u_t)(\Phi_1 y_{t-1} + u_t)'\right] \\
&= \mathbb{E}\left[\Phi_1 y_{t-1} y_{t-1}' \Phi_1' + \Phi_1 y_{t-1} u_t' + u_t y_{t-1}' \Phi_1' + u_t u_t'\right] \\
&= \Phi_1 \mathbb{E}[y_{t-1} y_{t-1}'] \Phi_1' + \Phi_1 \mathbb{E}[y_{t-1} u_t'] + \mathbb{E}[u_t y_{t-1}'] \Phi_1' + \mathbb{E}[u_t u_t'] \\
&= \Phi_1 \Gamma_{yy}(0) \Phi_1' + \mathbb{E}[u_t u_t'] \\
&= \Phi_1 \Gamma_{yy}(0) \Phi_1' + \Sigma_u
\end{aligned}$$

Since u_t contains contemporaneous shocks ε_t which are independent of past y_{t-1}, $\mathbb{E}[y_{t-s} u_t'] = 0$ for $s \geq 0$, $\mathbb{E}[y_{t-1} u_t'] = 0$ and $\mathbb{E}[u_t y_{t-1}'] = 0$. As $\Sigma_u' = (\Phi_\varepsilon \Phi_\varepsilon')' = \Phi_\varepsilon \Phi_\varepsilon' = \Sigma_u$ is Hermitian, this is a discrete Lyapunov equation, which can be solved for $\Gamma_{yy}(0)$ using the vectorization operator:

$$\operatorname{vec}(\Gamma_{yy}(0)) = (I_{k^2} - \Phi_1 \otimes \Phi_1)^{-1} \operatorname{vec}(\Sigma_u)$$

where $k = 2$ is the dimension of y_t (so $k^2 = 4$), and $\otimes$ is the Kronecker product.

$$\begin{aligned}
\Gamma_{yy}(1) = \mathbb{E}[y_t y_{t-1}'] &= \mathbb{E}[(\Phi_1 y_{t-1} + u_t) y_{t-1}'] \\
&= \Phi_1 \mathbb{E}[y_{t-1} y_{t-1}'] + \mathbb{E}[u_t y_{t-1}']
\end{aligned}$$

Again, $\mathbb{E}[u_t y_{t-1}'] = 0$. So, $\Gamma_{yy}(1) = \Phi_1 \Gamma_{yy}(0)$.

Solution (c).

Starting from $y_t = \Phi_1 y_{t-1} + \Phi_\varepsilon \varepsilon_t$. By repeated substitution, we can write y_t in its MA(∞) representation (assuming stationarity):

$$y_t = \sum_{j=0}^{\infty} \Phi_1^j \Phi_\varepsilon \varepsilon_{t-j} + \lim_{k\to\infty} \Phi_1^k y_{t-k}.$$

(Here $\Phi_1^0 = I_k$, where $k = 2$). As we assume that y_t is stationary, we know that Φ_1 has all the eigenvalues in the unit circle, and the last term vanishes as $k \to \infty$. Define $e_1 = (1,0)'$, $e_2 = (0,1)'$. A one-unit structural shock at t affects y_{t+h} by

$$\Psi(h) := \frac{\partial y_{t+h}}{\partial \varepsilon_t} = \Phi_1^h \Phi_\varepsilon, \quad h = 0, 1, \ldots$$

Impact of a labour-supply shock three periods ago on log wages

$$\frac{\partial w_t}{\partial \varepsilon_{b,t-3}} = e_1' \Psi(3)\, e_2 = e_1' \Phi_1^3 \Phi_\varepsilon e_2 = \left(\Phi_1^3\right)_{1\bullet} b_{12}.$$

where $\left(\Phi_1^3\right)_{1\bullet}$ denotes the first row of Φ_1^3.

Solution (d).
From the reduced-form VAR, we can consistently estimate Φ_1 and $\Sigma_u = \Phi_\varepsilon \Phi_\varepsilon'$.

$$\Sigma_u = \begin{bmatrix} b_{11}^2 + b_{12}^2 & b_{11}b_{21} + b_{12}b_{22} \\ \cdot & b_{21}^2 + b_{22}^2 \end{bmatrix},$$

provides three distinct equations for the four unknowns in Φ_ε. If Φ_ε is a solution, then for any $k \times k$ (here we have $k = 2$) orthogonal matrix P (such that $PP' = I_k$), $\Phi_\varepsilon^* = \Phi_\varepsilon P$ is also a solution because $\Phi_\varepsilon^*(\Phi_\varepsilon^*)' = (\Phi_\varepsilon P)(\Phi_\varepsilon P)' = \Phi_\varepsilon P P' \Phi_\varepsilon' = \Phi_\varepsilon I_k \Phi_\varepsilon' = \Phi_\varepsilon \Phi_\varepsilon' = \Sigma_u$. The identification problem is to find restrictions to pin down P.

Solution (e).
As $u_t = (u_{w,t}, u_{h,t})' = \Phi_\varepsilon \varepsilon_t$, we know:

$$\begin{aligned} u_{w,t} &= b_{11}\varepsilon_{a,t} + b_{12}\varepsilon_{b,t} \\ u_{h,t} &= b_{21}\varepsilon_{a,t} + b_{22}\varepsilon_{b,t} \end{aligned}$$

and the asusmption gives that $u_{h,t}$ is only affected by $\varepsilon_{b,t}$, so $b_{21} = 0$.
Since we need only 1 restriction for 2×2 matrix, this is exactly enough for identification (up to sign normalizations). With $b_{21} = 0$, Φ_ε becomes upper triangular:

$$\Phi_\varepsilon = \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix}$$

Then $\Sigma_u = \Phi_\varepsilon \Phi_\varepsilon' = \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix} \begin{bmatrix} b_{11} & 0 \\ b_{12} & b_{22} \end{bmatrix} = \begin{bmatrix} b_{11}^2 + b_{12}^2 & b_{12}b_{22} \\ b_{12}b_{22} & b_{22}^2 \end{bmatrix}$.

Let $\Sigma_u = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$ (where $\sigma_{12} = \sigma_{21}$).

1. From $\sigma_{22} = b_{22}^2$, we get $b_{22} = \sqrt{\sigma_{22}}$ (by convention, positive).
2. From $\sigma_{12} = b_{12}b_{22}$, we get $b_{12} = \sigma_{12}/b_{22}$ (assuming $b_{22} \neq 0$).
3. From $\sigma_{11} = b_{11}^2 + b_{12}^2$, we get $b_{11} = \sqrt{\sigma_{11} - b_{12}^2}$ (by convention, positive, and assuming $\sigma_{11} - b_{12}^2 \geq 0$).

This uniquely identifies Φ_ε (given sign normalizations for diagonal elements). This procedure is equivalent to finding an upper Cholesky factor of Σ_u.

Solution (f).

1. Labor supply shock ($\varepsilon_{b,t}$) moves wages (w_t) and hours (h_t) in opposite directions upon impact: $\frac{\partial w_t}{\partial \varepsilon_{b,t}} = b_{12}$ and $\frac{\partial h_t}{\partial \varepsilon_{b,t}} = b_{22}$. So, $b_{12} \cdot b_{22} < 0$.
2. Demand shock ($\varepsilon_{a,t}$) moves wages (w_t) and hours (h_t) in the same direction upon impact: $\frac{\partial w_t}{\partial \varepsilon_{a,t}} = b_{11}$ and $\frac{\partial h_t}{\partial \varepsilon_{a,t}} = b_{21}$. So, $b_{11} \cdot b_{21} > 0$.

These are inequality restrictions. They do not typically lead to point identification. Let $\Phi_{\varepsilon,0}$ be any matrix such that $\Phi_{\varepsilon,0}\Phi_{\varepsilon,0}' = \Sigma_u$ (e.g., from a Cholesky decomposition of Σ_u). Then any other valid matrix is $\Phi_\varepsilon = \Phi_{\varepsilon,0}P$, where P is an orthogonal matrix.
For $k = 2$, P can be a rotation matrix $P(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$ (or $P(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$, depending on convention). The sign restrictions define a set of admissible rotation angles θ. If this set is not a singleton (or two points corresponding to P and $-P$ after sign normalizations), then Φ_ε is not uniquely identified. Generally, sign restrictions lead to set identification, meaning there is a range of θ values (and thus a set of Φ_ε matrices) consistent with the restrictions. So, this is not enough to uniquely identify Φ_ε, the model is only *set-identified.*

Solution (g).

Solution (h).
The assumption is: "Labor demand is only affected by the technology shock ($\varepsilon_{a,t}$), not the preference shock ($\varepsilon_{b,t}$), whereas labor supply is affected by both shocks." This is a restriction on the underlying structural economic model. Consider a linear structural model for the innovations:

$$\begin{aligned} \text{Demand:} \quad & a_{11}u_{w,t} + a_{12}u_{h,t} = \gamma_{1a}\varepsilon_{a,t} \quad (\text{no } \varepsilon_{b,t} \text{ term, so } \gamma_{1b} = 0) \\ \text{Supply:} \quad & a_{21}u_{w,t} + a_{22}u_{h,t} = \gamma_{2a}\varepsilon_{a,t} + \gamma_{2b}\varepsilon_{b,t} \end{aligned}$$

In matrix form, $A_0 u_t = \Gamma \varepsilon_t$, where $A_0 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\Gamma = \begin{bmatrix} \gamma_{1a} & 0 \\ \gamma_{2a} & \gamma_{2b} \end{bmatrix}$. The reduced form innovations are $u_t = A_0^{-1}\Gamma\varepsilon_t$. So $\Phi_\varepsilon = A_0^{-1}\Gamma$. Let $A_0^{-1} = B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. Then $\Phi_\varepsilon = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \gamma_{1a} & 0 \\ \gamma_{2a} & \gamma_{2b} \end{bmatrix} = \begin{bmatrix} b_{11}\gamma_{1a} + b_{12}\gamma_{2a} & b_{12}\gamma_{2b} \\ b_{21}\gamma_{1a} + b_{22}\gamma_{2a} & b_{22}\gamma_{2b} \end{bmatrix}$. So, $b_{12} = b_{12}\gamma_{2b}$ and $b_{22} = b_{22}\gamma_{2b}$. This implies $b_{12}/b_{22} = b_{12}/b_{22}$ (if $\gamma_{2b} \neq 0$ and $b_{22} \neq 0$). The ratio b_{12}/b_{22} depends on the elements of A_0, which are structural parameters (related to slopes of demand/supply curves). For instance, if the demand equation (Equation 1) is $u_{w,t} + \alpha_D u_{h,t} = \dots$ (so $a_{11} = 1, a_{12} = \alpha_D$) and the supply equation is $u_{w,t} + \alpha_S u_{h,t} = \dots$ (so $a_{21} = 1, a_{22} = \alpha_S$), then $A_0 = \begin{bmatrix} 1 & \alpha_D \\ 1 & \alpha_S \end{bmatrix}$. Then $A_0^{-1} = \frac{1}{\alpha_S - \alpha_D} \begin{bmatrix} \alpha_S & -\alpha_D \\ -1 & 1 \end{bmatrix}$. So $b_{12} = \frac{-\alpha_D}{\alpha_S - \alpha_D}$ and $b_{22} = \frac{1}{\alpha_S - \alpha_D}$. Thus $b_{12}/b_{22} = -\alpha_D$. The restriction becomes $b_{12} = (-\alpha_D)b_{22}$. This is one restriction on the elements of Φ_ε, but it involves an unknown structural parameter α_D. Without knowing α_D, this restriction is not sufficient to identify Φ_ε from Σ_u. Thus, this is not enough to uniquely identify Φ_ε. The hint "remember that demand must equal supply at all times" is implicitly used by solving for equilibrium w_t, h_t which give rise to u_t and thus Φ_ε.

Solution (i).

Let L be the lower Cholesky factor of Σ_u, such that $LL' = \Sigma_u$. $L = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix}$, where $l_{11} = \sqrt{\sigma_{11}}$, $l_{21} = \sigma_{12}/l_{11}$ (if $l_{11} \neq 0$), $l_{22} = \sqrt{\sigma_{22} - l_{21}^2}$. Any Φ_ε such that $\Phi_\varepsilon\Phi_\varepsilon' = \Sigma_u$ can be written as $\Phi_\varepsilon = LP(\theta)$ for some orthogonal matrix $P(\theta)$. For $k = 2$, a common rotation matrix is $P(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

$$\Phi_\varepsilon = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} l_{11}\cos(\theta) & -l_{11}\sin(\theta) \\ l_{21}\cos(\theta) + l_{22}\sin(\theta) & -l_{21}\sin(\theta) + l_{22}\cos(\theta) \end{bmatrix}$$

So, $b_{12} = -l_{11}\sin(\theta)$ and $b_{22} = -l_{21}\sin(\theta) + l_{22}\cos(\theta)$. The restriction $b_{12} = (\alpha - 1)b_{22}$ becomes:

$$-l_{11}\sin(\theta) = (\alpha - 1)[-l_{21}\sin(\theta) + l_{22}\cos(\theta)]$$

$$[(\alpha - 1)l_{21} - l_{11}]\sin(\theta) = (\alpha - 1)l_{22}\cos(\theta)$$

If $(\alpha - 1)l_{22} \neq 0$ and the coefficient of $\sin(\theta)$ is not zero (and $\cos(\theta) \neq 0$ to avoid division by zero for $\tan(\theta)$):

$$\tan(\theta) = \frac{(\alpha - 1)l_{22}}{(\alpha - 1)l_{21} - l_{11}}$$

As Cholesky decomposition is unique (up to sign normalizations), l_{11}, l_{21}, l_{22} are uniquely determined, hence $\tan(\theta)$ is uniquely determined. Hence this equation determines θ up to a multiple of π. For example, if θ_0 is a solution, then $\theta_0 + \pi$ is also a solution. Adding π to θ changes $P(\theta)$ to $-P(\theta)$, which flips the sign of all elements in Φ_ε. This means Φ_ε is identified up to an overall sign change.

Final answer for (i): Yes, if α is given, it is possible to uniquely identify the elements of Φ_ε (up to conventional sign normalizations). The steps above show how Φ_ε can be solved for based on α and the elements of Σ_u (which are estimated from the reduced-form VAR).