# Exercise 1

Consider the model of default with uncertainty studied in class. We examined a numerical example with $r = 0.02$, $Y_2^H = 100$, $\rho = 0.05$, and $\phi = 0.5$.

(a) The optimal level of borrowing $D_1$ and the equilibrium interest rate $r^s$ are determined by the supply of funds from international capital markets and the demand for funds by domestic residents. Denote the gross interest rate $R^s = 1 + r^s$. Derive the supply and demand functions $R^s(D_1)$.

- Derive the supply function for loanable funds from foreign lenders using the risk premium equation:

$$1 + r = \left(1 - \frac{(1 + r^s)D_1}{\phi Y_2^H}\right)(1 + r^s)$$

- Define the gross interest rate:

$$R^s := (1 + r^s) \quad ; \quad R := (1 + r)$$

- Substitute to obtain:

$$R = \left(1 - \frac{D_1}{\varphi Y_2^H} R^s\right) R^s$$

$$\rightarrow R = R^s - \frac{D_1}{\varphi Y_2^H} (R^s)^2$$

- Solve for $R^s$ to obtain the

(inverse) supply function:

$$\frac{D_1}{\varphi Y_2^H}\left(R^S\right)^2 - R^S + R = 0$$

$$\longrightarrow R^S = \frac{1 \pm \sqrt{1 - \frac{4D_1}{\varphi Y_2^H} R}}{\frac{2D_1}{\varphi Y_2^H}}$$

(... two positive roots, pick the smaller one)

- Hence, we have the supply function:

$$\boxed{R^S_S = \frac{1 - \sqrt{1 - \frac{4D_1}{\varphi Y_2^H} R}}{\frac{2D_1}{\varphi Y_2^H}}}$$

- Now, derive the demand function from the borrower's FOC:

$$[1 - \beta(1+r)] = \beta\phi p Y_2^H \frac{\partial p}{\partial D_1}$$

- Substitute inside the probability of default:

$$p = prob(Y_2 < Y_2^*) = \frac{Y_2^*}{Y_2^H} = \frac{(1+r^s)D_1}{\phi Y_2^H}$$

to obtain:

$$(1-\beta R) = \beta \varphi R^s \frac{D_1}{\varphi Y_2^H} Y_2^H \frac{R^s}{\varphi Y_2^H}$$

- Simplify:

$$(1-\beta R) = \beta (R^s)^2 \frac{D_1}{\varphi Y_2^H}$$

- Solve for $R^s$:

$$(R^s)^2 = (1-\beta R)(\varphi Y_2^H)(\beta D_1)^{-1}$$

$$\longrightarrow R^s = \pm \sqrt{(1-\beta R)(\varphi Y_2^H)(\beta D_1)^{-1}}$$

- Pick the positive root:

$$\boxed{R_D^s = \sqrt{\frac{(1-\beta R)\,\varphi\, Y_2^{+1}}{\beta D_1}}}$$

→ This is the demand function for loanable funds.

(*) Plug in the values we have been given by the exercise:

$$R_s^s = \frac{1 - \sqrt{1 - \frac{2 \cdot 1.02}{25} D_1}}{\frac{D_1}{25}}$$

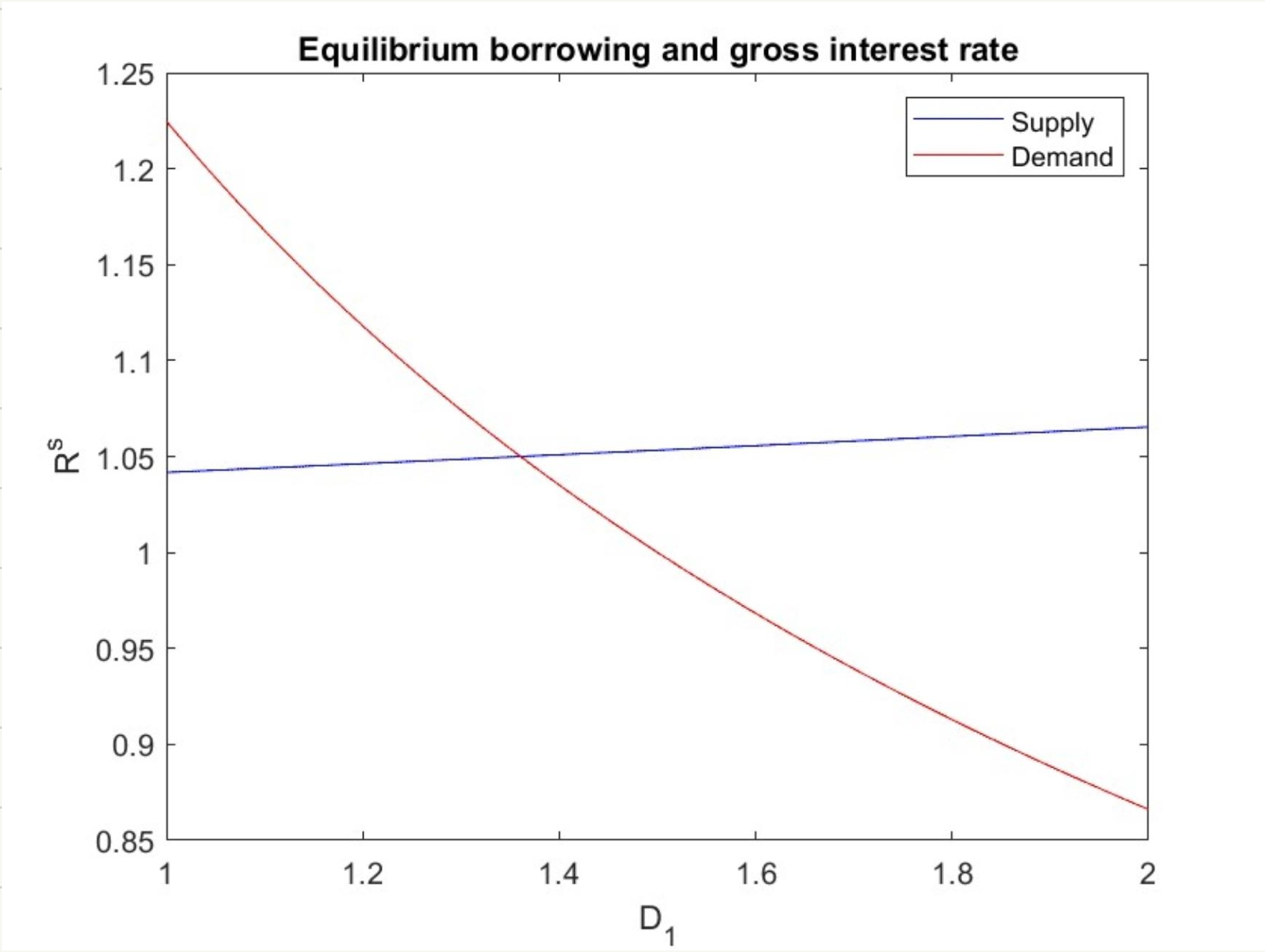
$$R_D^s = \sqrt{\frac{\left(1 - \frac{1.02}{1.05}\right) \cdot 50}{\frac{D_1}{1.05}}}$$

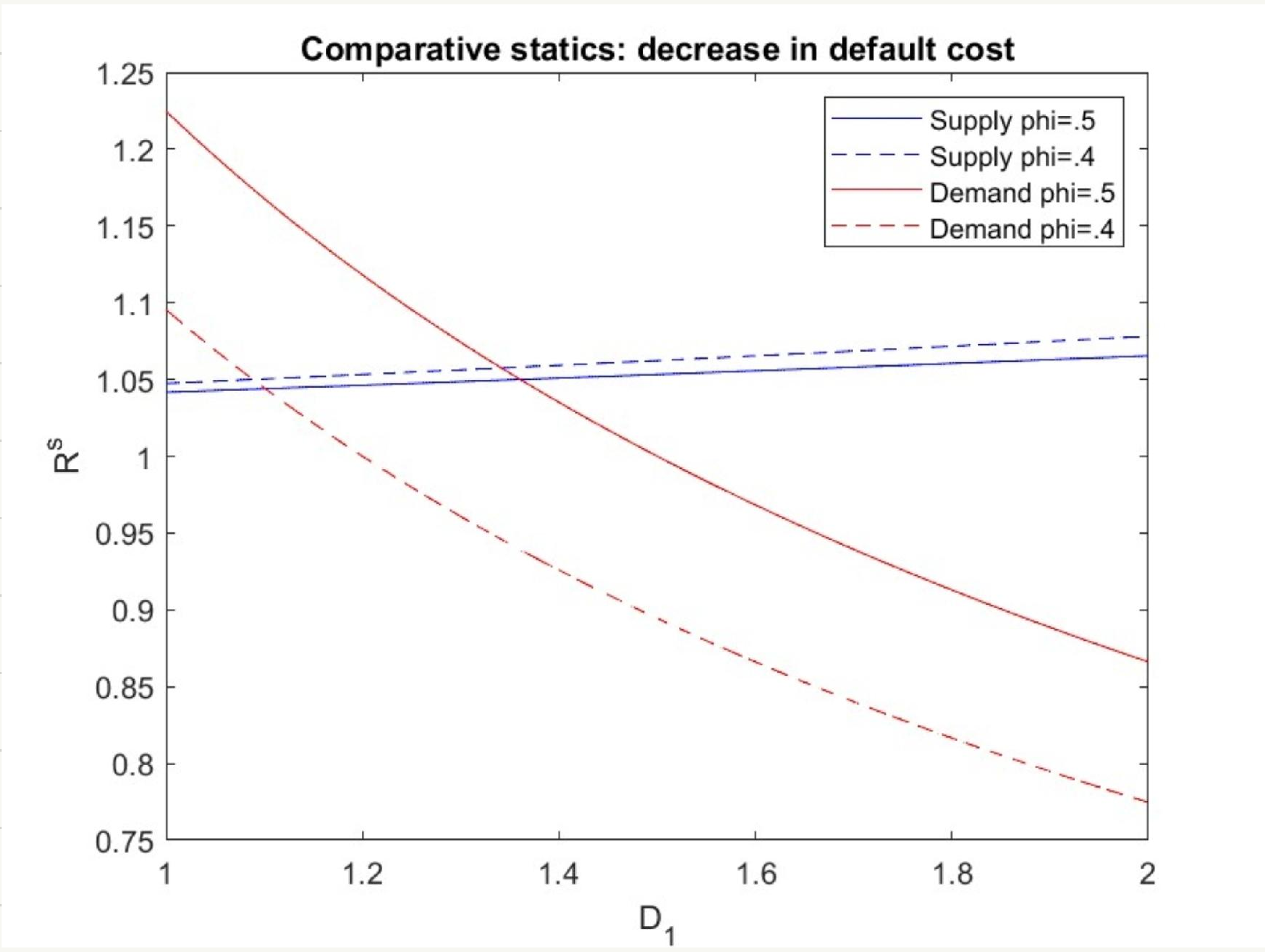
⇒ Solve for $R^{s*}$ and $D_1^*$:

$$R^{s*} = 1.05 \,; \quad D_1^* \simeq 1.361$$

(Matlab code for solution)

(b) Represent these two curves graphically and determine how they are affected by a reduction in $\phi$.




(*) INTUITION: default cost ↓ $\Rightarrow$ probability of default ↑ $\Rightarrow$ risk premium ↑ $\Rightarrow$ supply of funds ↓. Moreover, borrower's expected loss ↑ $\Rightarrow$ demand for funds ↓.

(c) The results were derived under the assumption of a full haircut in case of default. How would the results change if the haircut was only 50%?

- Use the equation for the supply of foreign funds with the haircut formula:

$$1 + r = (1 - p)(1 + r^s) + pz(1 + r^s) = (1 - p(1 - z))(1 + r^s)$$

and substitute for $p \equiv R^S \frac{D_1}{\varphi Y_2^H}$

to obtain:

$$R = \left(1 - R^S \frac{D_1}{\varphi Y_2^H}(1 - z)\right) R^S$$

$$\Rightarrow R = R^S - (R^S)^2 \frac{D_1}{\varphi Y_2^H}(1 - z)$$

↳ New supply function:

$$\boxed{R^S_s = \frac{1 - \sqrt{1 - \frac{4 D_1 (1-z) R}{\varphi Y_2^H}}}{\frac{2 D_1 (1-z)}{\varphi Y_2^H}}}$$

(Demand schedule remains unchanged)

- Plug in the numerical values into the supply function:

$$R_5^S = \frac{1 - \sqrt{1 - \frac{1.02}{25} D_1}}{\frac{D_1}{50}}$$

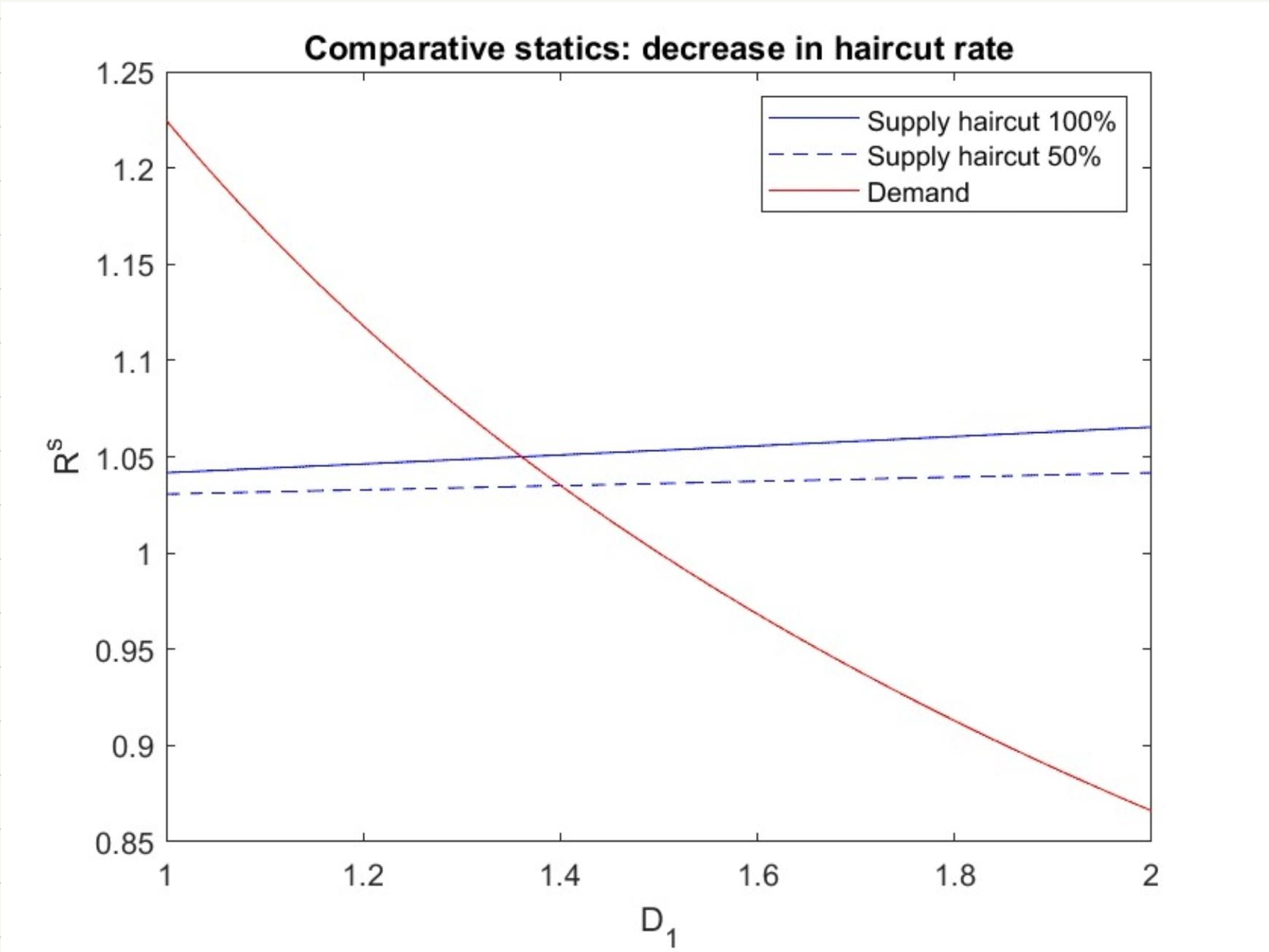
$\Rightarrow$ New solution: $R_2^{S*} = 1.035$

$$D_{1,2}^* \simeq 1.4$$

$\rightarrow$ We see that if the haircut decreases, then also the risk premium decreases, hence borrowing increases.

(*) INTUITION: lenders are willing to lend more for a given price if their value at risk decreases.

Graphical representation :



(d) The results were derived assuming $Y_2 \in \left[Y_2^L, Y_2^H\right]$ and $Y_2^L = 0$. How would the results change if $Y_2^L = 20$? (with full haircut)

- To answer this question, recall the property of the uniform distribution:

Remark about uniform distribution: If $x \in [a, b]$, $prob(x < x^*) = \frac{x^* - a}{b - a}$

→ In this case, we have that:

$$p = \mathbb{P}(Y_2 < Y_2^*) = \frac{Y_2^* - Y_2^L}{Y_2^H - Y_2^L}$$

→ By recalling that $Y_2^* = \frac{R^S D_1}{\varphi}$, substitute to have:

$$p = \frac{R^S D_1}{\varphi (Y_2^H - Y_2^L)} - \frac{Y_2^L}{Y_2^H - Y_2^L}$$

Plug in the new values:

$$p = \frac{R^S D_1}{.5(80)} - \frac{20}{80} = \frac{R^S D_1}{40} - \frac{1}{4}$$

- Now, plug in the new values into the borrower's FOC:

(Recall: $1-\beta R = \beta \varphi P Y_2^H \frac{\partial P}{\partial D_1}$)

$$\left(1-\beta R\right) = \beta\varphi\left(\frac{R^S D_1}{40} - \frac{1}{4}\right) Y_2^H \frac{R^S}{40}$$

$$\left(1-\frac{1.02}{1.05}\right) = \frac{\cancel{.5}\;5}{1.05}\,\frac{10\cancel{0}}{\cancel{40}\;4} R^S \left(\frac{R^S D_1}{40} - \frac{1}{4}\right)$$

$$\left(1-\frac{1.02}{1.05}\right) = \frac{1}{0.84} R^S \left(\frac{R^S D_1}{40} - \frac{1}{4}\right)$$

$$\left(1-\frac{1.02}{1.05}\right) = \left(R^S\right)^2 \frac{D_1}{33.6} - \frac{R^S}{3.36}$$

→ Solve for $R^S$:

$$\frac{D_1}{33.6}\left(R^S\right)^2 - \frac{R^S}{3.36} - \left(1-\frac{1.02}{1.05}\right) = 0$$

$$\boxed{R_D^S = \frac{\frac{1}{3.36} + \sqrt{\left(\frac{1}{3.36}\right)^2 - \frac{4D_1}{33.6}\left(\frac{1.02}{1.05}-1\right)}}{\frac{2D_1}{33.6}}}$$

• Do the same for supply (recall that $R = (1-p)R^S$, and that

$$p = \frac{R^S D_1}{q(Y_2^H - Y_2^L)} - \frac{Y_2^L}{Y_2^H - Y_2^L} = \frac{R^S D_1}{40} - \frac{1}{4}) .$$

$$\Rightarrow R = \left(1 - \frac{R^S D_1}{40} + \frac{1}{4}\right) R^S$$

$$\rightarrow R = \frac{5}{4} R^S - \frac{D_1}{40} (R^S)^2$$

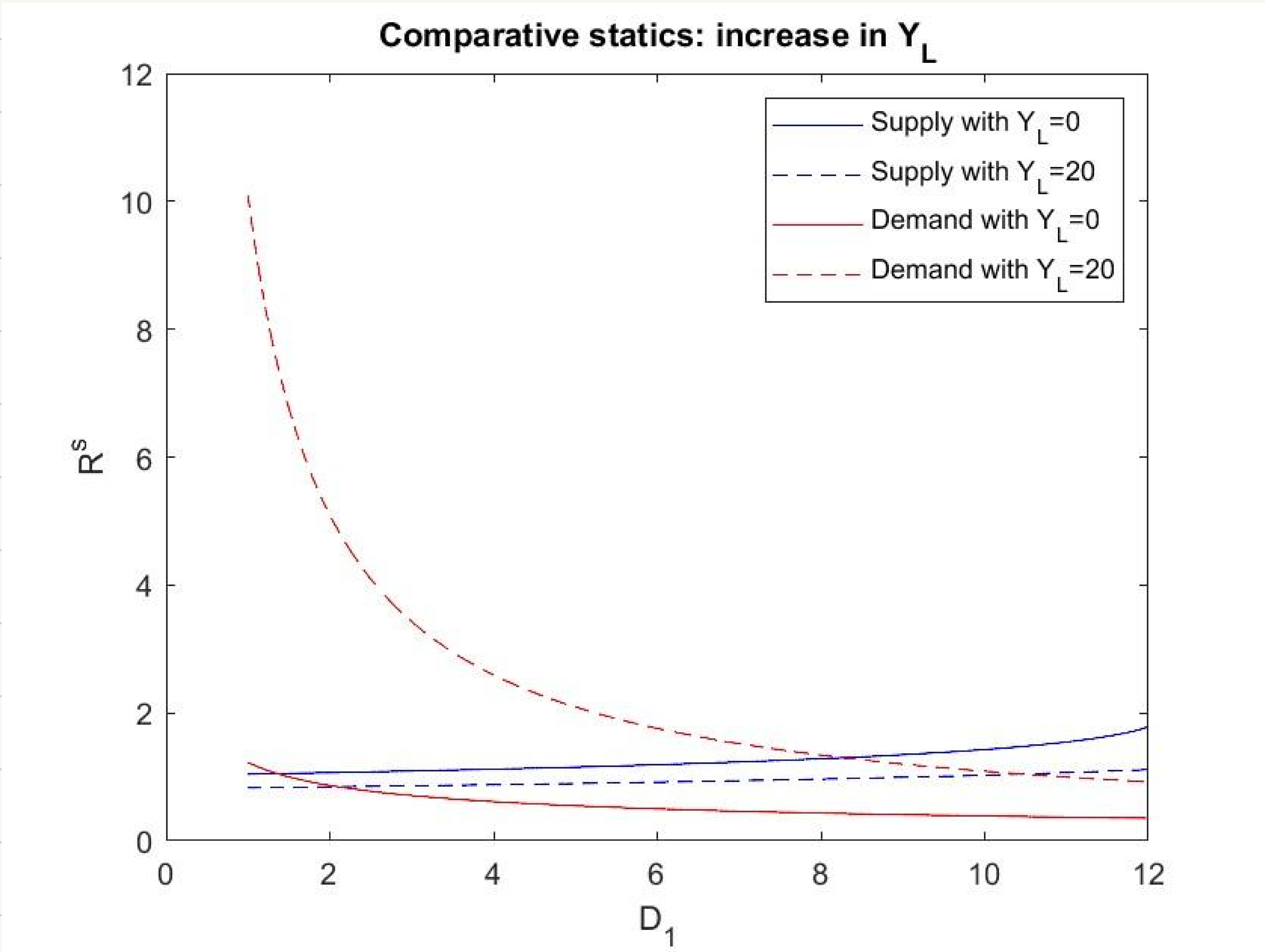
Solve for $R^S$:

$$(R^S)^2 \frac{D_1}{40} - \frac{5}{4} R^S + R = 0$$

$$\boxed{\rightarrow R^S_S = \frac{\frac{5}{4} - \sqrt{\frac{25}{16} - \frac{D_1 \cdot 1.02}{10}}}{\frac{D_1}{20}}}$$

Now use to solve for equilibrium values: $D_1^* \simeq 10.46$ ; $R^{S*} = 1.044$ .

# Graphical representation:




- Both demand and supply curve shift to the right:

INTUITION: expected value of income ↑ $\Rightarrow$ lender's value at risk ↓ $\Rightarrow$ increase in supply of funds. Moreover, probability of default ↓ $\Rightarrow$ borrower's expected loss ↓ $\Rightarrow$ increase in demand for funds.