PS1 Solutions

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1 Consumption Allocation

Problem Setup

The Home agent's consumption basket is given by

$$C_t = \left(\frac{C_{T,t}}{\gamma}\right)^{\gamma} \left(\frac{C_{N,t}}{1-\gamma}\right)^{1-\gamma},$$

where:

- $C_{T,t}$ is the quantity of the traded good (its price is normalized to 1),
- $C_{N,t}$ is the quantity of the domestic non-traded good (price $P_{N,t}$),
- γ is the expenditure share on the traded good.

The consumer minimizes total expenditure subject to attaining a given consumption level C_t. The problem is

$$\begin{aligned} \min_{C_{T,t},C_{N,t}} \; & P_t C_t = C_{T,t} + P_{N,t} C_{N,t} \\ \text{s.t.} \quad & C_t = \left(\frac{C_{T,t}}{\gamma}\right)^{\gamma} \left(\frac{C_{N,t}}{1-\gamma}\right)^{1-\gamma}. \end{aligned}$$

Define the Lagrangian function:

$$\mathcal{L} = C_{T,t} + P_{N,t} C_{N,t} + \lambda\Big(C_t - \Big(\frac{C_{T,t}}{\gamma}\Big)^{\gamma}\Big(\frac{C_{N,t}}{1-\gamma}\Big)^{1-\gamma}\Big).$$

The FOCs with respect to $C_{T,t}$ and $C_{N,t}$ are:

$$\begin{aligned} \mathcal{L}_{C_{T,t}} &= 1 - \lambda\gamma\Big(\frac{C_{T,t}}{\gamma}\Big)^{\gamma-1}\frac{1}{\gamma}\Big(\frac{C_{N,t}}{1-\gamma}\Big)^{1-\gamma} = 0, \\ \mathcal{L}_{C_{N,t}} &= P_{N,t} - \lambda(1-\gamma)\Big(\frac{C_{T,t}}{\gamma}\Big)^{\gamma}\frac{1}{1-\gamma}\Big(\frac{C_{N,t}}{1-\gamma}\Big)^{-\gamma} = 0 \\ \Rightarrow \quad \frac{1}{P_{N,t}} &= \frac{\gamma}{1-\gamma}\frac{C_{N,t}}{C_{T,t}}. \end{aligned}$$

The dual (expenditure) minimization problem yields the unit cost function (composite price index) for the consumption bundle:

$$P_tC_t = \min\left\{C_{T,t} + P_{N,t}C_{N,t} : \; C_t = \left(\frac{C_{T,t}}{\gamma}\right)^{\gamma}\left(\frac{C_{N,t}}{1-\gamma}\right)^{1-\gamma}\right\}.$$

So, we have:

$$\begin{aligned}
P_tC_t &= C_{T,t} + P_{N,t}C_{N,t} \\
&= C_{T,t} + \frac{1-\gamma}{\gamma}C_{T,t} \\
\Rightarrow C_{T,t} &= \gamma P_tC_t && \text{(1a)} \\
\Rightarrow C_{N,t} &= \frac{1-\gamma}{\gamma}\frac{C_{T,t}}{P_{N,t}} \\
&= (1-\gamma)P_tC_t. && \text{(1b)}
\end{aligned}$$

$$\begin{aligned}
&\left(\frac{C_{T,t}}{\gamma}\right)^{\gamma}\left(\frac{C_{N,t}}{1-\gamma}\right)^{1-\gamma} = C_t \\
\Rightarrow \quad &(P_tC_t)^{\gamma}\left(\frac{P_tC_t}{P_N,t}\right)^{1-\gamma} = C_t \\
\Rightarrow \quad &P_t = (P_{N,t})^{1-\gamma}. && \text{(1c)}
\end{aligned}$$

Analogously, for the Foreign agent, we have

$$\begin{aligned}
C^*_{T,t} &= \gamma P^*_tC^*_t && \text{(1d)} \\
C^*_{N,t} &= (1-\gamma)P^*_tC^*_t && \text{(1e)} \\
P^*_t &= (P^*_{N,t})^{1-\gamma}. && \text{(1f)}
\end{aligned}$$

2 Market Clearing

Under market clearing, the quantities of traded and non-traded goods produced must equal the quantities consumed. We have:

Non-Traded Goods Market

For Home, market clearing in non-traded goods is:

$$nC_{N,t} = A_{N,t}(L_{N,t})^{1-\alpha}.$$

Substituting $C_{N,t} = (1-\gamma)\frac{P_tC_t}{P_{N,t}}$ with $P_t = (P_{N,t})^{1-\gamma}$, we obtain:

$$n(1-\gamma)(P_{N,t})^{-\gamma}C_t = A_{N,t}(L_{N,t})^{1-\alpha}. \tag{2a}$$

For Foreign, the market clearing condition is: $(1-n)C^*_{N,t} = A^*_{N,t}(L^*_{N,t})^{1-\alpha}$. Following the same method, we have:

$$(1-n)(1-\gamma)(P^*_{N,t})^{-\gamma}C^*_t = A^*_{N,t}(L^*_{N,t})^{1-\alpha}. \tag{2b}$$

Traded Goods Market

Global market clearing for traded goods is:

$$nC_{T,t} + (1-n)C^*_{T,t} = A_{T,t}(n - L_{N,t})^{1-\alpha} + A^*_{T,t}\big((1-n) - L^*_{N,t}\big)^{1-\alpha}.$$

Substituting $C_{T,t} = \gamma P_t C_t$ with $P_t = (P_{N,t})^{1-\gamma}$ (and similarly for Foreign), we have:

$$n\gamma(P_{N,t})^{1-\gamma}C_t + (1-n)\gamma(P^*_{N,t})^{1-\gamma}C^*_t = A_{T,t}(n - L_{N,t})^{1-\alpha} + A^*_{T,t}\big((1-n) - L^*_{N,t}\big)^{1-\alpha}. \tag{2c}$$

3 Intertemporal Allocation

Home Optimization

The Home agent maximizes:

$$U_t = \sum_{s=0}^{\infty}(\beta_{H,t+s})^s \ln C_{t+s},$$

subject to the period-t budget constraint:

$$nP_tC_t + nB_{t+1} = A_{T,t}(n - L_{N,t})^{1-\alpha} + P_{N,t}A_{N,t}(L_{N,t})^{1-\alpha} + n(1+r_t)B_t.$$

Focusing on a single period (the infinite-horizon structure is standard), we write:

$$\mathcal{L}_t = \ln C_t + \beta_{H,t+1}\ln C_{t+1} - \lambda_t\Big[A_{T,t}(n - L_{N,t})^{1-\alpha} + P_{N,t}A_{N,t}(L_{N,t})^{1-\alpha} + n(1+r_t)B_t - nP_tC_t - nB_{t+1}\Big].$$

Take the FOCs:

$$\frac{\partial \mathcal{L}_t}{\partial C_t} = \frac{1}{C_t} - \lambda_t n P_t = 0 \quad \Rightarrow \quad \lambda_t = \frac{1}{nP_tC_t}$$
$$\frac{\partial \mathcal{L}_t}{\partial B_{t+1}} = -\lambda_t + \beta_{H,t+1}(1+r_{t+1})\lambda_{t+1} = 0.$$

Substitute the expressions for λ_t and λ_{t+1}:

$$\frac{1}{nP_tC_t} = \beta_{H,t+1}(1+r_{t+1})\frac{1}{nP_{t+1}C_{t+1}}.$$

Cancel n and rearrange:

$$\frac{1}{C_t} = \beta_{H,t+1}(1+r_{t+1})\frac{P_t}{P_{t+1}}\frac{1}{C_{t+1}}.$$

Set:

$$1 + r^C_{t+1} \equiv (1+r_{t+1})\frac{P_t}{P_{t+1}},$$

so the Euler equation becomes:

$$C_{t+1} = \beta_{H,t+1}(1 + r^C_{t+1})C_t. \tag{3a}$$

From Question (1), we know that $P_t = (P_{N,t})^{(1-\gamma)}$, so, we have:

$$1 + r^C_{t+1} = (1 + r_{t+1})\frac{P_t}{P_{t+1}} = (1 + r_{t+1})\Big(\frac{P_{N,t}}{P_{N,t+1}}\Big)^{1-\gamma}. \tag{3b}$$

Since

$$C_{T,t} = \gamma P_t C_t,$$

it follows that

$$C_{T,t+1} = \gamma P_{t+1}C_{t+1} = \gamma P_{t+1}\beta_{H,t+1}(1 + r^C_{t+1})C_t.$$

As we note that

$$\beta_{H,t+1}(1 + r^C_{t+1}) = \beta_{H,t+1}(1 + r_{t+1})\frac{P_t}{P_{t+1}},$$

so simplifying, we obtain:

$$C_{T,t+1} = \beta_{H,t+1}(1 + r_{t+1})C_{T,t}. \tag{3c}$$

Remark. The Foreign agent's optimization yields analogous Euler equations:

$$C^*_{t+1} = \beta_{F,t+1}(1 + r^*_{C,t+1})C^*_t, \quad C^*_{T,t+1} = \beta_{F,t+1}(1 + r_{t+1})C^*_{T,t},$$

with

$$1 + r^{*C}_{t+1} = (1 + r_{t+1})\frac{P^*_t}{P^*_{t+1}}.$$

Foreign Optimization

Similarly, for the Foreign country:

$$C^*_{t+1} = \beta_{F,t+1}(1 + r^*_{C,t+1})C^*_t \tag{3d}$$
$$C^*_{T,t+1} = \beta_{F,t+1}(1 + r_{t+1})C^*_{T,t} \tag{3e}$$

with

$$1 + r^{*C}_{t+1} = (1 + r_{t+1})\left(\frac{P^*_{N,t}}{P^*_{N,t+1}}\right)^{1-\gamma}. \tag{3f}$$

4 Labor Allocation

The production functions are given by:

$$Y_{T,t} = A_{T,t}(n - L_{N,t})^{1-\alpha}, \quad Y_{N,t} = A_{N,t}(L_{N,t})^{1-\alpha}.$$

Compute the marginal product of labor in each sector:

$$\frac{\partial Y_{T,t}}{\partial(n - L_{N,t})} = (1 - \alpha)A_{T,t}(n - L_{N,t})^{-\alpha},$$

$$\frac{\partial Y_{N,t}}{\partial L_{N,t}} = (1-\alpha)A_{N,t}(L_{N,t})^{-\alpha}.$$

The Home agent allocates labor so that the marginal value product in the traded sector equals the marginal value product (adjusted by the non-traded price) in the non-traded sector:

$$(1-\alpha)A_{T,t}(n-L_{N,t})^{-\alpha} = P_{N,t}(1-\alpha)A_{N,t}(L_{N,t})^{-\alpha}.$$

Cancel the common factor $1-\alpha$ and rearrange:

$$A_{T,t}(n-L_{N,t})^{-\alpha} = P_{N,t}A_{N,t}(L_{N,t})^{-\alpha}. \tag{4a}$$

As $Y^*_{T,t} = A^*_{T,t}\Big(1-n-L^*_{N,t}\Big)^{-\alpha}$, the analogous condition for the Foreign country is:

$$A^*_{T,t}\big((1-n)-L^*_{N,t}\big)^{-\alpha} = P^*_{N,t}A^*_{N,t}(L^*_{N,t})^{-\alpha}. \tag{4b}$$

5 Resource Constraints and the Real Exchange Rate

Resource Constraints

Recall the Home budget constraint:

$$nP_tC_t + nB_{t+1} = A_{T,t}(n-L_{N,t})^{1-\alpha} + P_{N,t}A_{N,t}(L_{N,t})^{1-\alpha} + n(1+r_t)B_t.$$

Given (2a),

$$P_{N,t}A_{N,t}\big(L_{N,t}\big)^{1-\alpha} = n(1-\gamma)\big(P_{N,t}\big)^{1-\gamma}C_t \tag{5.1}$$

By (1a) and (5.1), we have:

$$nP_tC_t - P_{N,t}A_{N,t}\big(L_{N,t}\big)^{1-\alpha} = n\big(P_{N,t}\big)^{1-\gamma}C_t\big(1-(1-\gamma)\big) = n\gamma(P_{N,t})^{1-\gamma}C_t \tag{5.2}$$

Bring (5.2) back to the budget constraint, we have:

$$n\gamma\big(P_{N,t}\big)^{1-\gamma}C_t + nB_{t+1} = A_{T,t}(n-L_{N,t})^{1-\alpha} + n(1+r_t)B_t. \tag{5a}$$

Similarly, for Foreign we obtain:

$$(1-n)\gamma(P^*_{N,t})^{1-\gamma}C^*_t - nB_{t+1} = A^*_{T,t}\big((1-n)-L^*_{N,t}\big)^{1-\alpha} - n(1+r_t)B_t. \tag{5b}$$

Then, we define the real exchange rate as the ratio of Foreign to Home consumption price indices:

$$Q_t \equiv \frac{P^*_t}{P_t}.$$

Since

$$P_t = (P_{N,t})^{1-\gamma} \quad \text{and} \quad P^*_t = (P^*_{N,t})^{1-\gamma},$$

we have:

$$Q_t = \left(\frac{P^*_{N,t}}{P_{N,t}}\right)^{1-\gamma}. \tag{5c}$$

6 Steady State

In steady state, consumption is constant so that $C_{t+1} = C_t$. The Euler equation for the Home agent is

$$C_{t+1} = \beta_{H,t+1}(1 + r^C_{t+1})C_t. \tag{6.1}$$

But by definition the real return in consumption units is

$$1 + r^C_{t+1} = (1 + r_{t+1})\left(\frac{P_t}{P_{t+1}}\right)^{1-\gamma}. \tag{6.2}$$

In steady state prices do not change ($P_t = P_{t+1}$) so that

$$1 + r^C_{t+1} = 1 + r_{t+1} \quad \Rightarrow \quad C_t = \beta_t(1 + r_t)C_t.$$

Dividing by $C_t > 0$ and take $t = 0$, yields:

$$1 = \beta_0(1 + r_0). \tag{6a}$$

Step 2. Price Normalization

By convention we normalize the steady state price of non-tradables to one:

$$P_{N,0} = 1, \quad \text{and similarly} \quad P^*_{N,0} = 1.$$

Then the consumption price indices become

$$P_0 = (P_{N,0})^{1-\gamma} = 1, \quad {}_0P^* = 1. \tag{6b}$$

Step 3. Labor Allocation in the Home Country

The Home intratemporal labor allocation condition is:

$$A_{T,0}(n - L_{N,0})^{-\alpha} = P_{N,0}A_{N,0}(L_{N,0})^{-\alpha}.$$

Since $P_{N,0} = 1$, this simplifies to:

$$A_{T,0}(n - L_{N,0})^{-\alpha} = A_{N,0}(L_{N,0})^{-\alpha}.$$

Rearrange by dividing both sides by $A_{T,0}$ and by $(L_{N,0})^{-\alpha}$:

$$\left(\frac{n - L_{N,0}}{L_{N,0}}\right)^{-\alpha} = \frac{A_{N,0}}{A_{T,0}}.$$

Taking the reciprocal and then the $1/\alpha$-th root,

$$\frac{n - L_{N,0}}{L_{N,0}} = \left(\frac{A_{T,0}}{A_{N,0}}\right)^{1/\alpha}.$$

Now, using the calibration

$$A_{N,0} = A_{T,0}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha},$$

we have

$$\frac{A_{T,0}}{A_{N,0}} = \left(\frac{\gamma}{1-\gamma}\right)^{\alpha}.$$

Then,

$$\frac{n - L_{N,0}}{L_{N,0}} = \left[\left(\frac{\gamma}{1-\gamma}\right)^{\alpha}\right]^{1/\alpha} = \frac{\gamma}{1-\gamma}.$$

Thus,

$$L_{N,0} = n(1-\gamma). \tag{6c}$$

By symmetry, the corresponding condition for Foreign is:

$$A^*_{T,0}\Big((1-n) - L^*_{N,0}\Big)^{-\alpha} = P^*_{N,0}A^*_{N,0}(L^*_{N,0})^{-\alpha}.$$

Since $P^*_{N,0} = 1$, the same steps lead to:

$$\frac{(1-n) - L^*_{N,0}}{L^*_{N,0}} = \frac{\gamma}{1-\gamma},$$

so that

$$L^*_{N,0} = (1-n)(1-\gamma). \tag{6d}$$

We derive the steady-state consumption from the market clearing conditions for non-traded and traded goods. The non-traded goods clearing condition is

$$nC_{N,0} = A_{N,0}(L_{N,0})^{1-\alpha}.$$

From Question 1 we have

$$C_{N,0} = (1-\gamma)\frac{P_0}{P_{N,0}}C_0.$$

Since $P_0 = 1$ and $P_{N,0} = 1$, it follows that

$$C_{N,0} = (1-\gamma)C_0.$$

Substitute into the clearing condition:

$$n(1-\gamma)C_0 = A_{N,0}(L_{N,0})^{1-\alpha}.$$

Recall that $L_{N,0} = n(1-\gamma)$, so

$$n(1-\gamma)C_0 = A_{N,0}\big[n(1-\gamma)\big]^{1-\alpha}.$$

Solve for C_0(as it is an expression of $A_{N,0}$, we denote by C_0^N):

$$C_0^N = A_{N,0}\big[n(1-\gamma)\big]^{-\alpha}.$$

We then derive from the traded goods clearing condition:

$$\begin{aligned}
&n\gamma(P_{N,t})^{1-\gamma}C_t + (1-n)\gamma(P_{N,t}^*)^{1-\gamma}C_t^* = A_{T,t}(n-L_{N,t})^{1-\alpha} + A_{T,t}^*\big((1-n)-L_{N,t}^*\big)^{1-\alpha}\\
\Rightarrow &n\gamma C_0 + \frac{\gamma}{1-\gamma}A_{N,0}^*(L_{N,0}^*)^{1-\alpha} = A_{T,0}(n-L_{N,0})^{1-\alpha} + A_{T,0}^*(1-n-L_{N,0}^*)^{1-\alpha}\\
\Rightarrow &n\gamma C_0 + \frac{\gamma}{1-\gamma}A_{N,0}^*(1-n)^{1-\alpha}(1-\gamma)^{1-\alpha} = A_{T,0}(n-n(1-\gamma))^{1-\alpha}\\
&\qquad\qquad\qquad\qquad\qquad\qquad + A_{T,0}^*(1-n-(1-n)(1-\gamma))^{1-\alpha}\\
\Rightarrow &n\gamma C_0 + \frac{\gamma}{1-\gamma}A_{N,0}^*(1-n)^{1-\alpha}(1-\gamma)^{1-\alpha} = A_{T,0}(n\gamma)^{1-\alpha} + A_{T,0}^*\Big[(1-n)\gamma\Big]^{1-\alpha}\\
\Rightarrow &n\gamma C_0 + \frac{\gamma}{1-\gamma}A_{T,0}\Big(\frac{1-n}{n}\Big)^{\alpha}\Big(\frac{1-\gamma}{\gamma}\Big)^{\alpha}(1-n)^{1-\alpha}(1-\gamma)^{1-\alpha} = A_{T,0}(n\gamma)^{1-\alpha}+\\
&A_{T,0}\Big(\frac{1-n}{n}\Big)^{\alpha}\Big[(1-n)\gamma\Big]^{1-\alpha}\\
\Rightarrow &n\gamma C_0 + A_{T,0}(1-n)n^{-\alpha}\gamma^{1-\alpha} = A_{T,0}(n\gamma)^{1-\alpha} + A_{T,0}(1-n)n^{-\alpha}\gamma^{1-\alpha}\\
\Rightarrow &C_0^T = A_{T,0}(n\gamma)^{-\alpha}.
\end{aligned}$$

We take a weighted geometric mean with weights γ and $1-\gamma$. That is,

$$C_0 = (C_0^N)^{1-\gamma}\cdot(C_0^T)^{\gamma},$$

so that

$$C_0 = \left[A_{N,0}n^{-\alpha}(1-\gamma)^{-\alpha}\right]^{1-\gamma}\left[A_{T,0}n^{-\alpha}\gamma^{-\alpha}\right]^{\gamma}.$$

We obtain:

$$C_0 = (A_{T,0})^{\gamma}(A_{N,0})^{1-\gamma}\left[n\gamma^{\gamma}(1-\gamma)^{1-\gamma}\right]^{-\alpha}. \tag{6e}$$

A similar derivation for Foreign (noting that the population is $1-n$) gives:

$$C_0^* = (A_{T,0}^*)^{\gamma}(A_{N,0}^*)^{1-\gamma}\left[(1-n)\gamma^{\gamma}(1-\gamma)^{1-\gamma}\right]^{-\alpha}. \tag{6f}$$

Because of the calibration and the fact that the relative productivities satisfy

$$\frac{A_{N,0}}{A_{T,0}} = \left(\frac{1-\gamma}{\gamma}\right)^{\alpha} \quad \text{and} \quad \frac{A_{N,0}^*}{A_{T,0}^*} = \left(\frac{1-n}{n}\right)^{\alpha}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha},$$

we can check that indeed

$$\frac{C_0}{C_0^*} = 1. \tag{6g}$$

7 Log-Linear Approximation

A. Non-Traded Goods Market

The Home non-traded goods market clearing condition is:

$$n(1-\gamma)(P_{N,t})^{-\gamma}C_t = A_{N,t}(L_{N,t})^{1-\alpha}.$$

Taking logarithms, we have

$$\ln n + \ln(1-\gamma) - \gamma \ln P_{N,t} + \ln C_t = \ln A_{N,t} + (1-\alpha)\ln L_{N,t}.$$

Linearizing around the steady state (and noting that constants vanish in the difference), we obtain:

$$-\gamma \widehat{P_{N,t}} + \widehat{C}_t = \widehat{A_{N,t}} + (1-\alpha)\widehat{L_{N,t}}. \tag{7a}$$

For foreign non-traded goods, we have:

$$\ln(1-n) + \ln(1-\gamma) - \gamma \ln P^*_{N,t} + \ln C^*_t = \ln A^*_{N,t} + (1-\alpha)\ln L^*_{N,t}.$$

Linearizing around the steady state, we obtain:

$$-\gamma \widehat{P^*_{N,t}} + \widehat{C^*_t} = \widehat{A^*_{N,t}} + (1-\alpha)\widehat{L^*_{N,t}}. \tag{7b}$$

B. Resource Constraint

The Home resource constraint is:

$$n\gamma(P_{N,t})^{1-\gamma}C_t + nB_{t+1} = A_{T,t}(n - L_{N,t})^{1-\alpha} + n(1+r_t)B_t.$$

Divide both sides by $n\gamma C_0$, we get:

$$\frac{(P_{N,t})^{1-\gamma}C_t}{C_0} + \widehat{B_{t+1}} = \frac{A_{T,t}(n-L_{N,t})^{1-\alpha}}{n\gamma C_0} + (1+r_t)\widehat{B_t}.$$

Taking logs and linearizing, we have:

$$(1-\gamma)\widehat{P_{N,t}} + \widehat{C}_t + \widehat{B_{t+1}} = \widehat{A_{T,t}} - \frac{(1-\alpha)(L_{N,t}-L_{N,0})}{n - L_{N,0}}\widehat{L_{N,t}} + \frac{1}{\beta_0}\widehat{B_t}.$$

Since in steady state $n - L_{N,0} = n\gamma$, and that $\widehat{L_{N,t}} = \frac{L_{N,t}-L_{N,0}}{L_{N,0}}$. Thus,

$$(1-\gamma)\widehat{P_{N,t}} + \widehat{C}_t + \widehat{B_{t+1}} = \widehat{A_{T,t}} - (1-\alpha)\frac{1-\gamma}{\gamma}\widehat{L_{N,t}} + \frac{1}{\beta_0}\widehat{B_t}. \tag{7c}$$

Recall foreign budget constraint (5b):

$$(1-n)\gamma(P^*_{N,t})^{1-\gamma}C^*_t - nB_{t+1} = A^*_{T,t}\big((1-n) - L^*_{N,t}\big)^{1-\alpha} - n(1+r_t)B_t.$$

Following similar steps, we have:

$$(1-n)(1-\gamma)\big(\widehat{P^*_{N,t}}+\widehat{C^*_t}\big)-n\widehat{B_{t+1}}=\widehat{A^*_{T,t}}-(1-\alpha)\frac{1-\gamma}{\gamma}\widehat{L^*_{N,t}}-\frac{n}{\beta_0}\widehat{B_t}.$$

Divide both sides by $1-n$, we get:

$$(1-\gamma)\widehat{P^*_{N,t}}+\widehat{C^*_t}-\frac{n}{1-n}\widehat{B_{t+1}}=\widehat{A^*_{T,t}}-(1-\alpha)\frac{1-\gamma}{\gamma}\widehat{L^*_{N,t}}-\frac{1}{\beta_0}\frac{n}{1-n}\widehat{B_t}. \tag{7d}$$

C. Euler Equation

Recall that:

$$C_{t+1}=C_t\beta_{H,t+1}(1+r_{t+1})\Big(\frac{P_{N,t}}{P_{N,t+1}}\Big)^{1-\gamma}$$

Taking logs and linearizing, we have:

$$\widehat{C_{t+1}}=\widehat{C}_t+\widehat{\beta_{H,t+1}}+\frac{r_{t+1}-r_0}{1+r_0}+(1-\gamma)(\widehat{P_{N,t}}-\widehat{P_{N,t+1}}).$$

As $\beta_0(1+r_0)=1$,

$$\begin{aligned}\widehat{r_t}&=r_t-\frac{1-\beta_0}{\beta_0}\\&=r_t-\frac{1-\frac{1}{1+r_0}}{\frac{1}{1+r_0}}\\&=r_t-r_0\end{aligned}$$

So, the Euler equation becomes:

$$\widehat{C_{t+1}}=\widehat{C}_t+\widehat{\beta_{H,t+1}}+\beta_0\widehat{r_{t+1}}+(1-\gamma)(\widehat{P_{N,t}}-\widehat{P_{N,t+1}}). \tag{7e}$$

Similarly for Foreign:

$$\widehat{C^*_{t+1}}=\widehat{C^*_t}+\widehat{\beta_{F,t+1}}+\beta_0\widehat{r_{t+1}}^*+(1-\gamma)(\widehat{P^*_{N,t}}-\widehat{P_{N,t+1}}^*). \tag{7f}$$

D. Labor Allocation

The intratemporal condition is:

$$A_{T,t}(n-L_{N,t})^{-\alpha}=P_{N,t}A_{N,t}(L_{N,t})^{-\alpha}.$$

Taking logarithms:

$$\ln A_{T,t}-\alpha\ln(n-L_{N,t})=\ln P_{N,t}+\ln A_{N,t}-\alpha\ln L_{N,t}.$$

Linearize around steady state:

$$\widehat{A_{T,t}} - \alpha\frac{L_{N,t} - L_{N,0}}{n - L_{N,0}} = \widehat{P_{N,t}} + \widehat{A_{N,t}} - \alpha\widehat{L_{N,t}}.$$

Using the fact that in steady state $n - L_{N,0} = n\gamma$:

$$\widehat{A_{T,t}} - \alpha\frac{\widehat{L_{N,t}}n(1-\gamma)}{n\gamma} = \widehat{P_{N,t}} + \widehat{A_{N,t}} - \alpha\widehat{L_{N,t}}.$$

$$\widehat{A_{T,t}} + \frac{\alpha}{\gamma}\widehat{L_{N,t}} = \widehat{P_{N,t}} + \widehat{A_{N,t}}. \tag{7g}$$

Similarly for Foreign:

$$\widehat{A^*_{T,t}} + \frac{\alpha}{1-\gamma}\widehat{L^*_{N,t}} = \widehat{P^*_{N,t}} + \widehat{A^*_{N,t}}. \tag{7h}$$

E. Real Exchange Rate

As we know:

$$1 + r^C_{t+1} = (1 + r_{t+1})\Big(\frac{P_{N,t}}{P_{N,t+1}}\Big)^{1-\gamma},$$

take logs and linearize both sides, we have:

$$\frac{r^C_{t+1} - r^C_0}{1 + r^C_0} = \frac{r_{t+1} - r_0}{1 + r_0} + (1-\gamma)(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}).$$

$$\Rightarrow \beta_0\widehat{r^C_{t+1}} = \beta_0\widehat{r_{t+1}} + (1-\gamma)(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}).$$

$$\widehat{r^C_{t+1}} = \widehat{r_{t+1}} + (1-\gamma)\frac{1}{\beta_0}(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}). \tag{7i}$$

Similarly for Foreign:

$$\widehat{r^{*C}_{t+1}} = \widehat{r^*_{t+1}} + (1-\gamma)\frac{1}{\beta_0}(\widehat{P^*_{N,t}} - \widehat{P^*_{N,t+1}}). \tag{7j}$$

8 Worldwide Equilibrium

Define world aggregates as population-weighted averages (e.g., $\widehat{C^W_t} = n\widehat{C_t} + (1-n)\widehat{C^*_t}$). Then, from the above log-linearized equations one can show:

- **Non-Traded Goods Market:**

$$n \times (7a) + (1-n) \times (7b) \Rightarrow -\gamma\widehat{P^W_{N,t}} + \widehat{C^W_t} = \widehat{A^W_{N,t}} + (1-\alpha)\widehat{L^W_{N,t}}. \tag{8a}$$

- **Resource Constraint:**

$$n \times (7c) + (1-n) \times (7d) \Rightarrow (1-\gamma)\widehat{P^W_{N,t}} + \widehat{C^W_t} = \widehat{A^W_{T,t}} - (1-\alpha)\frac{1-\gamma}{\gamma}\widehat{L^W_{N,t}}. \tag{8b}$$

- **Euler Equation:**

$$n \times (7e) + (1-n) \times (7f) \Rightarrow \widehat{C^W_{t+1}} = \widehat{C^W_t} + (1-\gamma)(\widehat{P^W_{N,t}} - \widehat{P^W_{N,t+1}}) + \widehat{\beta^W_{H,t+1}} + \beta_0 \widehat{r_{t+1}}. \tag{8c}$$

- **Labor Allocation:**

$$n \times (7g) + (1-n) \times (7h) \Rightarrow \widehat{A^W_{T,t}} + \frac{\alpha}{\gamma} \widehat{L^W_{N,t}} = \widehat{P^W_{N,t}} + \widehat{A^W_{N,t}}. \tag{8d}$$

- **Real Exchange Rate:**

$$n \times (7i) + (1-n) \times (7j) \Rightarrow \beta_0 \widehat{r^{CW}_{t+1}} = (1-\gamma)(\widehat{P^W_{N,t}} - \widehat{P^W_{N,t+1}}) + \beta_0 \widehat{r_{t+1}}. \tag{8e}$$

Let (8a)-(8d), we get:

$$(1-\gamma)\widehat{P^W_{N,t}} + \widehat{C^W_t} = \widehat{A^W_{T,t}} + \left(1 - \alpha + \frac{\alpha}{\gamma}\right) \widehat{L^W_{N,t}}.$$

Combine with (8b), we have:

$$\begin{aligned}
\left(1 - \alpha + \frac{\alpha}{\gamma}\right) \widehat{L^W_{N,t}} &= -(1-\alpha)\frac{1-\gamma}{\gamma} \widehat{L^W_{N,t}} \\
&= (1-\alpha)\widehat{L^W_{N,t}} - \frac{1-\alpha}{\gamma} \widehat{L^W_{N,t}} \\
\Rightarrow \frac{1}{\gamma} \widehat{L^W_{N,t}} &= 0.
\end{aligned} \tag{8f}$$

Let (8b)-(8a), we have:

$$\begin{aligned}
\widehat{P^W_{N,t}} &= \widehat{A^W_{T,t}} - \widehat{A^W_{N,t}} - (1-\alpha)\left(\frac{1-\gamma}{\gamma} + 1\right) \widehat{L^W_{N,t}} \\
&= \widehat{A^W_{T,t}} - \widehat{A^W_{N,t}}.
\end{aligned} \tag{8g}$$

Take $(1-\gamma) \times (8a) + \gamma(8b)$, we have:

$$(1-\gamma)\widehat{C^W_t} + \gamma \widehat{C^W_t} = \widehat{C^W_t} = (1-\gamma)\widehat{A^W_{N,t}} + \gamma \widehat{A^W_{T,t}}. \tag{8h}$$

Finally, from (8c), we know that:

$$\begin{aligned}
\beta_0 \widehat{r_{t+1}} + \widehat{\beta^W_{H,t+1}} &= \widehat{C^W_{t+1}} - \widehat{C^W_t} - (1-\gamma)\left(\widehat{P^W_{N,t}} - \widehat{P^W_{N,t+1}}\right) \\
&= \gamma \hat{A}^W_{t+1} + (1-\gamma)\widehat{A^W_{N,t+1}} - \gamma \widehat{A^W_{T,t+1}} - (1-\gamma)\widehat{A^W_{N,t+1}} \\
&\quad - (1-\gamma)\left(\widehat{A^W_{T,t}} - \widehat{A^W_{N,t}}\right) + (1-\gamma)\left(\widehat{A^W_{T,t+1}} - \widehat{A^W_{N,t+1}}\right) \\
&= \widehat{A^W_{T,t+1}} - \widehat{A^W_{T,t}}.
\end{aligned} \tag{8i}$$

Relative Price of Goods:

The relative price of goods in our model is captured by the real exchange rate

$$Q_t = \left(\frac{P_{N,t^*}}{P_{N,t}}\right)^{1-\gamma}.$$

Its log-deviation is

$$\widehat{Q_t} = (1-\gamma)(\widehat{P^*_{N,t}} - \widehat{P_{N,t}}).$$

This relative price is driven by:

- **Productivity Differences:** Differences in sector-specific productivity. An increase in Home's traded productivity (a higher $\widehat{A_{T,t}}$) relative to non-traded productivity (or relative to Foreign's productivity) lowers the relative cost of producing tradables. In our derivations, the term $\widehat{A_{T,t}} - \widehat{A^*_{T,t}}$ appears with a negative sign in $\widehat{Q_t}$, indicating that higher Home traded productivity tends to lower Home's non-tradable price relative to Foreign's.

- **Relative Demand Effects:** Shocks that affect consumption (e.g. through changes in discount factors) alter demand for non-traded goods, hence influencing $P_{N,t}$.

Consumption:
Aggregate consumption in each country is determined by both traded and non-traded output. In log-linear terms, world consumption is given by

$$\widehat{C^W_t} = \gamma\,\widehat{A^W_{T,t}} + (1-\gamma)\,\widehat{A^W_{N,t}},$$

where $\widehat{A^W_{T,t}}$ and $\widehat{A^W_{N,t}}$ are the population-weighted productivity shocks in the traded and non-traded sectors. Thus, consumption is driven by overall improvements in productivityimprovements in traded productivity increase income (and hence consumption) by γ while improvements in non-traded productivity raise consumption by $1-\gamma$.

Real Interest Rate:
The consumption-based real interest rate is determined by the Euler equation

$$1 = \beta_0(1+r_0),$$

and its log-linear deviations are influenced by both the intertemporal preference parameter β and by the relative price changes in non-tradables (through $\widehat{P}_{N,t}$). Specifically, we have

$$\widehat{r^C_{t+1}} = \widehat{r_{t+1}} + \frac{1-\gamma}{\beta_0}\left(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}\right).$$

Thus, the real interest rate responds to:

- **Household Patience:** A higher β_0 (or a more favorable $\widehat{\beta_{H,t}}$) implies lower required returns to equate current and future consumption.

- **Expected Price Changes:** Changes in $\widehat{P_{N,t}}$ affect the real return by altering the consumption price index.

<u>What Drives Relative Price:</u>
This relative price is driven by the productivity gap between the traded and non-traded sectors.

When traded-goods productivity rises relative to non-traded-goods productivity, the price of non-traded goods increases (since traded goods become relatively cheaper to produce). Conversely, higher productivity in the non-traded sector (relative to traded) makes non-traded goods relatively cheaper, lowering their relative price.

What Drives Consumption:

Aggregate consumption is driven by productivity improvements in both sectors, weighted by their share in consumption. In particular, higher productivity in the traded sector raises consumption (through more output/income of tradables), and higher productivity in the non-traded sector also raises consumption (more domestic goods), each in proportion to their expenditure share (γ for traded, $1-\gamma$ for non-traded).

What Drives the Real Exchange Rate:

The real interest rate is determined by households' patience and expected productivity growth. An increase in the Home discount factor (greater patience) lowers the real interest rate, since more patient consumers save more (reducing the return needed to equilibrate) In addition, if future traded-sector productivity is expected to grow faster than present (a positive productivity growth outlook), the real interest rate tends to rise (reflecting higher expected returns to saving/investing when future output is higher). In summary, a shock that makes consumers more patient drives down the consumption-based real interest rate, while faster productivity growth prospects push it up.

9 Cross-Country Differences

As we know that $Q_t = \left(\frac{P^*_{N,t}}{P_{N,t}}\right)^{1-\gamma}$, log-linearize the equation, we have:

$$\widehat{Q_t} = (1-\gamma)(\widehat{P^*_{N,t}}) - \widehat{P_{N,t}}.$$

Use (7a) - (7b), we get:

$$\begin{aligned}
&\widehat{C_t} - \widehat{C^*_t} - \gamma(\widehat{P_{N,t}} - \widehat{P^*_{N,t}}) = (\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) + (1-\alpha)(\widehat{L_{N,t}} - \widehat{L^*_{N,t}}) \\
&\Rightarrow \widehat{C_t} - \widehat{C^*_t} + \frac{\gamma}{1-\gamma}\widehat{Q_t} = (\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) + (1-\alpha)(\widehat{L_{N,t}} - \widehat{L^*_{N,t}}).
\end{aligned} \tag{9a}$$

Use (7c) - (7d), we get:

$$\begin{aligned}
&\text{LHS} = (1-\gamma)(\widehat{P_{N,t}} - \widehat{P^*_{N,t}}) + \widehat{C_t} - \widehat{C^*_t} + \frac{\widehat{B_{t+1}}}{1-n} \\
&\text{RHS} = (\widehat{A_{T,t}} - \widehat{A^*_{T,t}}) - (1-\alpha)\frac{1-\gamma}{\gamma}(\widehat{L_{N,t}} - \widehat{L^*_{N,t}}) + \frac{1}{\beta_0}\frac{\widehat{B_t}}{1-n} \\
&\text{As } \widehat{Q_t} = (1-\gamma)(\widehat{P^*_{N,t}} - \widehat{P_{N,t}}), \text{ we have} \\
&\text{LHS} = -\widehat{Q_t} + (\widehat{C_t} - \widehat{C^*_t}) + \frac{\widehat{B_{t+1}}}{1-n} = \text{RHS}
\end{aligned} \tag{9b}$$

Use (7e) - (7f), we get:

$$(\widehat{C_{t+1}} - \widehat{C_{t+1}^*}) = (\widehat{C}_t - \widehat{C_t^*}) + (1-\gamma)(\widehat{P_{N,t}} - \widehat{P_{N,t}^*}) + (1-\gamma)(\widehat{P_{N,t+1}^*} - \widehat{P_{N,t+1}}) + (\widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}})$$
$$= (\widehat{C}_t - \widehat{C_t^*}) - \widehat{Q_t} + \widehat{Q_{t+1}} + \widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}}. \tag{9c}$$

Use (7g) - (7h), we get:

$$(\widehat{A_{T,t}} - \widehat{A_{T,t}^*}) + \frac{\alpha}{\gamma}(\widehat{L_{N,t}} - \widehat{L_{N,t}^*}) = (\widehat{P_{N,t}} - \widehat{P_{N,t}^*}) + (\widehat{A_{N,t}} - \widehat{A_{N,t}^*})$$
$$= -\frac{1}{1-\gamma}\widehat{Q_t} + (\widehat{A_{N,t}} - \widehat{A_{N,t}^*}). \tag{9d}$$

Use (7i) - (7j), we get:

$$\beta_0(\widehat{r_{t+1}^C} - \widehat{r_{t+1}^{C*}}) = (1-\gamma)(\widehat{P_{N,t}} - \widehat{P_{N,t}^*}) + (1-\gamma)(\widehat{P_{N,t}^*} - \widehat{P_{N,t}})$$
$$= \widehat{Q_{t+1}} - \widehat{Q_t}. \tag{9e}$$

10 Long-Run Allocation (Period $t+1$)

Assume that from $t+1$ onward the economy reaches a new steady state with no further discount factor shocks ($\widehat{\beta_{H,t+2}} = \widehat{\beta_{F,t+2}} = 0$). In the steady state, the consumption growth rate is zero, the asset position is fixed and the real exchange rate is stable.
Using the labor allocation equation at $t+1$, we have:

$$\frac{\alpha}{\gamma}(\widehat{L_{N,t}} - \widehat{L_{N,t}^*}) = -\frac{1}{1-\gamma}\widehat{Q_{t+1}} + \Big[(\widehat{A_{N,t}} - \widehat{A_{N,t}^*}) - (\widehat{A_{T,t}} - \widehat{A_{T,t}^*})\Big]. \tag{10.1}$$

Then, we use the market clearing condition for non-traded goods and the resource allocation constraints at $t+1$:

$$\frac{\gamma}{1-\gamma}\widehat{Q_{t+1}} + (\widehat{C_{t+1}} - \widehat{C_{t+1}^*}) = (\widehat{A_{N,t+1}} - \widehat{A_{N,t+1}^*}) + (1-\alpha)(\widehat{L_{N,t+1}} - \widehat{L_{N,t+1}^*}) \tag{10.2}$$
$$-\widehat{Q_{t+1}} + (\widehat{C_{t+1}} - \widehat{C_{t+1}}^*) + \frac{\widehat{B_{t+2}}}{1-n} = (\widehat{A_{T,t+1}} - \widehat{A_{T,t+1}^*}) - (1-\alpha)\frac{1-\gamma}{\gamma}(\widehat{L_{N,t+1}} - \widehat{L_{N,t+1}^*}) + \frac{1}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}. \tag{10.3}$$

As $\widehat{B_{t+2}} = \widehat{B_{t+1}}$, using (10.2)-(10.3), we get:

$$\Big(\frac{1-\gamma}{\gamma} + 1\Big)\widehat{Q_{t+1}} = \Big[(\widehat{A_{N,t+1}} - \widehat{A_{N,t+1}^*}) - (\widehat{A_{T,t+1}} - \widehat{A_{T,t+1}^*})\Big]$$
$$+ (1-\alpha)\Big(1 + \frac{1-\gamma}{\gamma}\Big)(\widehat{L_{N,t+1}} - \widehat{L_{N,t+1}^*}) - \frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}$$
$$\Rightarrow \quad \widehat{Q_{t+1}} = (1-\gamma)\Big[(\widehat{A_{N,t+1}} - \widehat{A_{N,t+1}^*}) - (\widehat{A_{T,t+1}} - \widehat{A_{T,t+1}^*})\Big] + (1-\alpha)\frac{1-\gamma}{\gamma}(\widehat{L_{N,t+1}} - \widehat{L_{N,t+1}^*})$$
$$- (1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}. \tag{10.4}$$

Replacing $\widehat{L_{N,t+1}} - \widehat{L^*_{N,t+1}}$ using (10.1), we get:

$$\begin{aligned}
\widehat{Q_{t+1}} &= (1-\gamma)\Big[(\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}})\Big] - \frac{1-\alpha}{\alpha}\widehat{Q_{t+1}} \\
&\quad + \frac{1-\alpha}{\alpha}(1-\gamma)\Big[(\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}})\Big] - (1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
\Rightarrow \quad \Big(1+\frac{1-\alpha}{\alpha}\Big)\widehat{Q_{t+1}} &= (1-\gamma)\Big(1+\frac{1-\alpha}{\alpha}\Big)\Big[(\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}})\Big] \\
&\quad - (1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
\Rightarrow \quad \widehat{Q_{t+1}} &= -(1-\gamma)\Big[(\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) - (\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}})\Big] - \alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}. \qquad \text{(10a)}
\end{aligned}$$

Comparing (10.4) and (10a), we have:

$$\begin{aligned}
(1-\alpha)\frac{1-\gamma}{\gamma}(\widehat{L_{N,t+1}} - \widehat{L^*_{N,t+1}}) - (1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} &= -\alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
\Rightarrow \quad \widehat{L_{N,t+1}} - \widehat{L^*_{N,t+1}} &= \gamma\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}. \qquad \text{(10b)}
\end{aligned}$$

Implementing (10a) and (10b) back into (10.2), we get:

$$\begin{aligned}
\widehat{C_{t+1}} - \widehat{C^*_{t+1}} &= \widehat{Q_{t+1}} + (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) - \frac{(1-\alpha)(1-\gamma)}{\gamma}\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} + \frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
&= -(1-\gamma)(\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) + (1-\gamma)(\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) + (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) \\
&\quad - \alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} - (1-\alpha)(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} + \frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
&= \gamma(\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) + (1-\gamma)(\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) + \gamma\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \qquad \text{(10c)}
\end{aligned}$$

(1) Impact of Home Being Wealthier ($\widehat{B_{t+1}} > 0$)

When $\widehat{B} > 0$, Home has accumulated net foreign assets. In the long run, this wealth effect leads to:

- A higher steady-state consumption level in Home relative to Foreign, since net asset income augments Home's overall resources.
- An upward pressure on domestic (Home) demand, which, via the resource constraints, increases the domestic price of non-tradables. This results in a real appreciation of Home's currency (i.e. a lower Q_{t+1}, meaning Home's goods become relatively more expensive).

Thus, being wealthier (with $\widehat{B_{t+1}} > 0$) directly boosts Home's consumption while causing a real appreciation.

(2) Long-Run Impact of Productivity (Excluding the Asset Channel)

- An increase in Home's traded productivity ($\widehat{A_{T,t}} > 0$) raises the output in the traded sector. This increases Home's income and consumption by the factor γ and, via the Balassa–Samuelson mechanism, raises wages and non-traded prices, leading to a real appreciation.

- Conversely, an increase in Home's non-traded productivity ($\widehat{A_{N,t}} > 0$) improves domestic output in non-tradables, boosting consumption by the factor $1-\gamma$ and reducing the relative price of non-tradables, which tends to depreciate Home's currency.

In either case, productivity improvements raise long-run consumption directly, and the real exchange rate adjusts according to the sector in which the productivity shock occurs.
Impact of home country wealthier:
If Home has a positive net foreign asset position (Home is wealthier), it can afford higher consumption relative to Foreign. Home's steady-state consumption will be higher than Foreign's, and this higher demand for Home goods makes Home's goods relatively more expensive. In other words, a wealthier Home leads to a real appreciation of Home's goods (a lower Q_{t+1}, meaning Foreign goods become cheap relative to Home goods). Home runs a trade balance surplus (earning income from its assets), which supports its higher consumption.
Longrun impact of productivity:
Permanent productivity differences translate directly into differences in consumption and relative prices in steady state. A country with higher productivity (Home, in this case) will enjoy higher long-run consumption levels and a stronger relative price for its goods independent of net foreign assets. In effect, Home's higher productivity raises its real income and consumption, and causes a direct real exchange rate adjustment: for example, higher traded-sector productivity in Home leads to higher Home wages and non-traded prices (Balassa-Samuelson effect), appreciating Home's real exchange rate; higher non-traded productivity would raise Home consumption and lower Home's non-traded prices, depreciating the real exchange rate. In all cases, the more productive economy has a higher standard of living (consumption) and its price levels adjust accordingly in the long run, aside from any asset accumulation.

11 Short-Run Allocation (Period t)

To simplify notation, we denote $\widehat{A_{N,t}} - \widehat{A^*_{N,t}} = \widetilde{A_{N,t}}$, $\widehat{A_{T,t}} - \widehat{A^*_{T,t}} = \widetilde{A_{T,t}}$, $\widehat{L_{N,t}} - \widehat{L^*_{N,t}} = \widetilde{L_{N,t}}$.
Combining (9a) and (9d), we can get:

$$\begin{aligned}\widetilde{C}_t + \frac{\gamma}{1-\gamma}\widehat{Q_t} - \gamma\widetilde{A_{T,t}} - \alpha\widetilde{L_{N,t}} &= \widetilde{A_{N,t}}(1-\gamma) + (1-\alpha)\widetilde{L_{N,t}} + \frac{\gamma}{1-\gamma}\widehat{Q_t} \\ \Rightarrow \widetilde{L_{N,t}} &= \widetilde{C}_t - \gamma\widetilde{A_{T,t}} - (1-\gamma)\widetilde{A_{N,t}} \qquad (11.1)\end{aligned}$$

and that

$$(1-\alpha)\gamma\widetilde{A_{T,t}} + \alpha(1-\alpha)\widetilde{L_{N,t}} + \frac{\alpha\gamma}{1-\gamma}\widehat{Q_t} + \alpha\widetilde{C}_t = -\frac{\gamma(1-\alpha)}{1-\gamma}\widehat{Q_t} + (1-\alpha)\gamma\widetilde{A_{N,t}} + \alpha\widetilde{A_{N,t}} + \alpha(1-\alpha)\widetilde{L_{N,t}}$$

$$\Rightarrow \widehat{Q_t} = \left(\frac{1-\gamma}{\gamma}\right)[(1-\alpha)\gamma + \alpha]\widetilde{A_{N,t}} - (1-\alpha)(1-\gamma)\widetilde{A_{T,t}} - \frac{\alpha(1-\gamma)}{\gamma}\widetilde{C}_t \qquad (11.2)$$

Implementing (11.1) and (11.2) back into (9b), we get:

$$
\begin{aligned}
-\Big(\frac{1-\gamma}{\gamma}\Big)[(1-\alpha)\gamma+\alpha]\widetilde{A_{N,t}}+(1-\alpha)(1-\gamma)\widetilde{A_{T,t}}+\frac{\alpha(1-\gamma)}{\gamma}\widetilde{C_t}+\widetilde{C_t}+\frac{\widehat{B_{t+1}}}{1-n}\\
=\widehat{A_{T,t}}-\frac{(1-\alpha)(1-\gamma)}{\gamma}\widetilde{C_t}+(1-\alpha)(1-\gamma)\widetilde{A_{T,t}}+\frac{(1-\alpha)(1-\gamma)^2}{\gamma}\widetilde{A_{N,t}}\\
\Rightarrow\frac{\widehat{B_{t+1}}}{1-n}=\widetilde{A_{T,t}}+\frac{1-\gamma}{\gamma}\widetilde{A_{N,t}}-\frac{1}{\gamma}\widetilde{C_t}\\
\Rightarrow\widetilde{C_t}=\gamma\widetilde{A_{T,t}}+(1-\gamma)\widetilde{A_{N,t}}-\gamma\frac{\widehat{B_{t+1}}}{1-n}
\end{aligned}
\tag{11.3}
$$

Bring (11.3) back to (11.2), we get:

$$
\widehat{Q_t}=(1-\gamma)(\widetilde{A_{N,t}}-\widetilde{A_{T,t}})+\gamma\frac{\widehat{B_{t+1}}}{1-n}
\tag{11.4}
$$

Subtracting (11.4) from (10a), we get:

$$
\widehat{Q_{t+1}}-\widehat{Q_t}=(1-\gamma)(\widetilde{A_{N,t+1}}-\widetilde{A_{N,t}})-(1-\gamma)(\widetilde{A_{T,t+1}}-\widetilde{A_{T,t}})-\alpha(1-\gamma)\frac{1}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}
\tag{11.5}
$$

Subtracting (11.3) from (10c), we get:

$$
\widetilde{C_{t+1}}-\widetilde{C_t}=\gamma(\widetilde{A_{T,t+1}}-\widetilde{A_{T,t}})+(1-\gamma)(\widetilde{A_{N,t+1}}-\widetilde{A_{N,t}})+\frac{\gamma}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}
\tag{11.6}
$$

Bring (11.5) and (11.6) back to (9c), we get:

$$
\begin{aligned}
\gamma(\widetilde{A_{T,t+1}}-\widetilde{A_{T,t}})+(1-\gamma)(\widetilde{A_{N,t+1}}-\widetilde{A_{N,t}})+\frac{\gamma}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}\\
=(1-\gamma)(\widetilde{A_{N,t+1}}-\widetilde{A_{N,t}})-(1-\gamma)(\widetilde{A_{T,t+1}}-\widetilde{A_{T,t}})-\alpha(1-\gamma)\frac{1}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}+\widehat{\beta_{H,t+1}}-\widehat{\beta_{F,t+1}}\\
\Rightarrow\Big[\gamma+\alpha(1-\gamma)\Big]\frac{1}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}=\widehat{\beta_{H,t+1}}-\widehat{\beta_{F,t+1}}-(\widetilde{A_{T,t+1}}-\widetilde{A_{T,t}})\\
\Rightarrow\frac{\widehat{B_{t+1}}}{1-n}=\frac{\beta_0}{\gamma+\alpha(1-\gamma)}\Big[\widehat{\beta_{H,t+1}}-\widehat{\beta_{F,t+1}}-(\widehat{A_{T,t+1}}-\widehat{A^*_{T,t+1}})+(\widehat{A_{T,t}}-\widehat{A^*_{T,t}})\Big]
\end{aligned}
\tag{11a}
$$

Bring (11a) back to (11.3), we get:

$$
\begin{aligned}
\widehat{C_{t+1}}-\widehat{C_t}=\gamma(\widehat{A_{T,t}}-\widehat{A^*_{T,t}})+(1-\gamma)(\widehat{A_{N,t}}-\widehat{A^*_{N,t}})\\
-\frac{\gamma\beta_0}{\gamma+\alpha(1-\gamma)}\Big[\widehat{\beta_{H,t+1}}-\widehat{\beta_{F,t+1}}-(\widehat{A_{T,t+1}}-\widehat{A^*_{T,t+1}})+(\widehat{A_{T,t}}-\widehat{A^*_{T,t}})\Big]
\end{aligned}
\tag{11b}
$$

Bring (11a) back to (11.4), we get:

$$\begin{aligned}\widehat{Q_t} &= -(1-\gamma)(\widehat{A_{T,t}} - \widehat{A^*_{T,t}}) + (1-\gamma)(\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) \\ &\quad + \frac{\alpha(1-\gamma)\beta_0}{\gamma+\alpha(1-\gamma)}\Big[\widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}} - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) + (\widehat{A_{T,t}} - \widehat{A^*_{T,t}})\Big]\end{aligned} \tag{11c}$$

Bring (11b) and (11c) back to (11.1), we get:

$$\widehat{L_{N,t}} - \widehat{L^*_{N,t}} = -\frac{\gamma\beta_0}{\gamma+\alpha(1-\gamma)}\Big[\widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}} - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) + (\widehat{A_{T,t}} - \widehat{A^*_{T,t}})\Big] \tag{11d}$$

By (9c), we know that:

$$\beta_0(\widehat{r^C_{t+1}} - \widehat{r^{C*}_{t+1}}) = \widehat{Q_{t+1}} - \widehat{Q_t}$$

Implementing (10.4) and (11c) back to the equation, we get:

$$\begin{aligned}\beta_0(\widehat{r^C_{t+1}} - \widehat{r^{C*}_{t+1}}) &= -(1-\gamma)\Big[(\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) - (\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}})\Big] - \alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\ &\quad - \big\{-(1-\gamma)(\widehat{A_{T,t}} - \widehat{A^*_{T,t}}) + (1-\gamma)(\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) + \alpha(1-\gamma)\frac{\widehat{B_{t+1}}}{1-n}\big\}\end{aligned}$$

Bring (11a) back to the equation, we get:

$$\begin{aligned}\beta_0(\widehat{r^C_{t+1}} - \widehat{r^{C*}_{t+1}}) &= -(1-\gamma)\Big[(\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) - (\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}})\Big] - \alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\ &\quad + (1-\gamma)\Big[(\widehat{A_{T,t}} - \widehat{A^*_{T,t}}) - (\widehat{A_{N,t}} - \widehat{A^*_{N,t}})\Big] \\ &\quad - \frac{\alpha(1-\gamma)}{\gamma+\alpha(1-\gamma)}\Big[\widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}} - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) + (\widehat{A_{T,t}} - \widehat{A^*_{T,t}})\Big]\end{aligned} \tag{11e}$$

(1) Impact of a Temporary Increase in Home Patience ($\widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}} > 0$)

- Home households postpone current consumption and save more, leading to a positive net asset accumulation, i.e., $\widehat{B_{t+1}} > 0$.
- The current account surplus (positive $\widehat{B_{t+1}}$) reduces Home's current consumption relative to Foreign, so that the cross-country consumption gap $\widetilde{C}_t$ decreasaes.
- As Home saves more, there is a reallocation of labor away from the non-traded sector (since non-tradables are largely consumed domestically) and the real exchange rate adjusts accordingly. Typically, increased saving puts downward pressure on domestic non-tradable prices, leading to a real depreciation, which helps balance the external sector.

(2) Impact of a Temporary Shock in Home Traded Productivity ($\widehat{A_{T,t}} > 0$) versus Non-Traded Productivity ($\widehat{A_{N,t}} > 0$)
Temporary Traded Productivity Shock ($\widehat{A_{T,t}} > 0$):

- A positive temporary shock in $\widehat{A_{T,t}}$ increases Home's output of tradable goods.

- Because tradables are internationally traded, Home can export the surplus. However, households do not instantaneously increase their consumption by the full amount of the productivity gain; instead, some of the extra income is saved.
- As a result, Home runs a current account surplus ($\widehat{B_{t+1}} > 0$) and, by the mechanism of reallocation, its non-traded goods sector experiences higher demandleading to higher domestic prices and a real appreciation of Home's currency.

Temporary Non-Traded Productivity Shock ($\widehat{A_{N,t}} > 0$):

- A temporary shock in $\widehat{A_{N,t}}$ boosts the production of non-tradable goods, which are consumed domestically.
- Since non-tradables are not exported, the entire output effect is absorbed by domestic consumption, raising Home's real consumption without significantly altering external trade.
- The increased supply of non-tradables tends to lower their relative price, so the domestic consumption price index falls. This produces a real depreciation of Home's currency.
- Additionally, with most of the income effect confined to domestic consumption, there is little pressure for a current account surplus or deficit.

(3) Impact of Permanent Productivity Shocks:

For permanent shocks (i.e. when $\widehat{A_{T,t+1}} = \widehat{A_{T,t}}$ or $\widehat{A_{N,t+1}} = \widehat{A_{N,t}}$), the following occurs:

- The economy adjusts immediately to the new steady state, and by the intertemporal Euler condition no net asset accumulation is required ($\widehat{B_{t+1}} = 0$).
- Home's consumption jumps to a higher level that reflects the permanent increase in productivity.
- The relative price adjustments occur immediately so that the real exchange rate settles at its new steady-state value.

In essence, permanent productivity shocks affect the long-run levels of consumption and relative prices without generating a temporary current account imbalance.