Macroeconomics A: EI056

Problem set 2

Cédric Tille

Due in the class of November 28, 2023

1 Cash in advance model

1.1 Constraints

Consider that the household maximizes an intertemporal utility of consumption over two goods, c^m and c^d:

$$\sum_{s=0}^{\infty} \beta^s u\left(c_{t+s}^m, c_{t+s}^d\right)$$

where $\beta < 1$ is the discount factor. The price of good d is P_{t+s}, while the price of good m is subject to a tax, so the consumer pays $\left(1+\kappa_{t+s}\right) P_{t+s}$.

In period t the household gets a real endowment of goods y_t (with price P_t), and a nominal transfer T_t. She enters the period with cash holdings M_t and $B_{t.}$ units of a nominal bond paying the interest i_t.

To consume the good c_t^m, the household must use cash and transfer, so that the nominal spending on the good cannot exceed $T_t + M_t$. This is the cash in advance constraint.

Show that the constraints in real terms are:

$$\begin{aligned}
\left(1+\kappa_t\right) c_t^m &= \tau_t + m_t \\
\left(1+\kappa_t\right) c_t^m + c_t^d + b_t + m_t &= y_t + \tau_t + \frac{1}{1+\pi_t}\left(m_{t-1} + \left(1+i_{t-1}\right) b_{t-1}\right)
\end{aligned}$$

where $b_t = B_t/P_t$, $m_t = M_t/P_t$, $\tau_t = T_t/P_t$ and $\pi_t = \left(P_t - P_{t-1}\right)/P_{t-1}$.

1.2 Optimal intertemporal choice

Set up the Lagrangian with multipliers $\beta^s \lambda_{t+s}$ on the period $t+s$ budget constraint, and $\beta^s \mu_{t+s}$ on the period $t+s$ cash-in-advance constraint.

Show that at the optimal allocation, the multipliers are linked by the interest rate:

$$\mu_t = \lambda_t \frac{i_t}{1 + i_t}$$

1.3 Interest rate as a tax

Show that the optimal allocation of consumption is such that the nominal interest rate acts as a tax on good m, i.e that it drives a wedge between the marginal utility of consumption of the two goods

What is the economic intuition?.

2 Money and productivity shocks

2.1 Household optimization

A representative household maximizes a utility that is affected by consumption, C_t, real money balances, M_t/P_t, and effort, N_t:

$$U_t = \sum_{s=0}^{\infty} \beta^s \left[\frac{(C_{t+s})^{1-\sigma}}{1-\sigma} \left(\frac{M_{t+s}}{P_{t+s}} \right)^{1-b} - \frac{N_{t+s}^{1+\eta}}{1+\eta} \right]$$

where $b < 1$.

The household can purchase bonds that earn a nominal interest rate i, as well as money. The budget constraint at time t:

$$P_t C_t + B_t + M_t = W_t N_t - T_t - M_{t-1} - (1 + i_{t-1}) B_{t-1} + T_t$$

where T is a lump sum source of income, including firms' profits and any payment from the central bank.

Solve for the household optimization, and show that we get a money demand and a labor supply:

$$\begin{aligned} \frac{M_t}{P_t} &= \frac{1-b}{1-\sigma} \frac{1+i_t}{i_t} C_t \\ \frac{W_t}{P_t} &= N_t^{\eta} (C_t)^{\sigma} \left(\frac{M_t}{P_t} \right)^{-(1-b)} \end{aligned}$$

2.2 Firm optimization, and linearization

Firms use a production function that has constant returns to scale with labor as the only input, and productivity Z_t: $Y_t = Z_t N_t$.

The firm chooses labor so that the marginal cost is equal to the marginal product. Derive the labor demand.

The model can be expressed in log-linear approximations around a steady state, with the notation $\hat{x}_t = (x_t - x_{ss}) / x_{ss}$ except for $\hat{i}_t = (i_t - i_{ss}) / (1 + i_{ss})$.

Show that the labor supply, money demand, labor demand and output are written as (recall that output is equal to consumption as there is no capital):

$$\begin{aligned} \hat{W}_t - \hat{P}_t &= \eta \hat{N}_t + \sigma \hat{Y}_t - (1-b) \left(\hat{M}_t - \hat{P}_t \right) \\ \hat{M}_t - \hat{P}_t &= \hat{Y}_t - \frac{1}{i_{ss}} \hat{i}_t \\ \hat{Z}_t &= \hat{W} - \hat{P}_t \\ \hat{Y}_t &= \hat{Z}_t + \hat{N}_t \end{aligned}$$

2.3 Output and interest rate

Using the four expressions above, show that:

$$\hat{Y}_t = \frac{(1+\eta)\hat{Z}_t + (1-b)\left(\hat{M}_t - \hat{P}_t\right)}{\sigma+\eta}$$
$$\hat{Y}_t = \frac{1}{\sigma+\eta-(1-b)}\left[(1+\eta)\hat{Z}_t - \frac{1-b}{i_{ss}}\hat{i}_t\right]$$

Explain the mechanism by which the interest rate affects output.

2.4 Alternative utility

We now consider a different utility, splitting consumption and real balance:

$$U_t = \sum_{s=0}^{\infty}\beta^s\left[\frac{(C_{t+s})^{1-\sigma}}{1-\sigma} + \frac{1}{1-b}\left(\frac{M_{t+s}}{P_{t+s}}\right)^{1-b} - \frac{N_{t+s}^{1+\eta}}{1+\eta}\right]$$

Show that (hint: many relations are unchanged from the previous questions):

$$\hat{Y}_t = \frac{1+\eta}{\sigma+\eta}\hat{Z}_t$$

What is the effect of the interest rate on output? Why does the results differs from the one in the previous question?

3 Government deficits with the inflation tax

3.1 Inflation and money growth

Consider an economy that is on a steady nominal growth path, along which real variables are constant.

We denote the level of money by M_t, and the level of real money by $m_t = M_t/P_t$. Money grows at a rate $\mu_t = (M_t - M_{t-1})/M_{t-1}$, which is constant. Inflation is $\pi_t = (P_t - P_{t-1})/P_{t-1}$,

Show that $\pi = \mu$.

3.2 Monetary financing of government spending

Consider that the government needs to finance a constant amount g of real deficit. The only source of revenue is money printing, and the real revenue is inflation times the real balances: $g = \pi m$.

The money demand is an inverse relation between real balances and expected inflation:

$$m_t = \exp\left[-\pi_{t+1}^e\right]$$

For now, we focus on a steady state where inflation and real variables are constant.

Show that one can write expected inflation π^e needed to finance the government spending as a concave function of the growth rate of money μ, conditional on g. Draw this line for $g = 0.25$.

3.3 Equilibrium

In a steady state, inflation is fully expected.

Using the figure you drew in the previous point, what are the equilibria in the system?

What is the intuition?

3.4 Feasibility of monetary finance

Show that there is a limit of the amount g that can be financed in this way.

What are the values of g and μ at this limit?

Hint: think of how the various lines in your figure interact at the limit (in terms of levels and slopes).

3.5 Dynamics of expected and actual inflation

Consider that $g = 0.25$, and that inflation expectation react in an adaptive way to any error:

$$\pi_{t+1}^e - \pi_t^e = \eta\left(\pi_t - \pi_t^e\right)$$

We analyze the model in terms of deviation around the steady growth path. On that path, real variables and growth rates are constant: $m_{t.ss} = m_{ss}$ (where the t, ss subscript indicates the value at time t along the path), $\pi_{t,ss} = \pi_{ss} = \mu$.

We denote log deviations around the growth path by hatted variables:

$$\hat{m}_t = \frac{m_t - m_{ss}}{m_{ss}} \qquad \hat{M}_t = \frac{M_t - M_{t.ss}}{M_{t.ss}} \qquad \hat{P}_t = \frac{P_t - P_{t.ss}}{P_{t.ss}}$$
$$\hat{\pi}_t = \frac{\pi_t - \mu}{1+\mu} = \hat{P}_t - \hat{P}_{t-1}$$
$$\hat{\mu}_t = \frac{\mu_t - \mu}{1+\mu} = \hat{M}_t - \hat{M}_{t-1}$$

Show that the money demand, $m_t = \exp\left[-\pi^e_{t+1}\right]$, is approximated as:

$$\hat{m}_t \quad = \quad -(1+\mu)\,\hat{\pi}^e_{t+1}$$

Show that the dynamics of inflation expectations are (we set $\hat{\mu}_t = 0$ for all t):

$$\hat{\pi}_t \quad = \quad -\frac{(1+\mu)\,\eta}{1-(1+\mu)\,\eta}\hat{\pi}^e_t$$

and:

$$\hat{\pi}^e_{t+1} - \hat{\pi}^e_t \quad = \quad -\frac{\eta}{1-(1+\mu)\,\eta}\hat{\pi}^e_t$$

3.6 Stability

If a shock moves inflation expectations away from a long run equilibrium, do they revert to equilibrium (in which case expectations ultimately stabilize)?

Do the values of parameters matter?

To assess this question, take two values of the growth rate of money: $\mu = 0.357$ and $\mu = 2.153$.

Start with $\mu = 0.357$. Draw a chart with 3 panels, each for a different value of η (namely $\eta = 0.2$, $\eta = 0.5$, $\eta = 1($. In each of the 3 panels show the evolution of inflation and expected inflation after a shock that puts $\hat{\pi}^e_t = 1$.

Then do the same thing for $\mu = 2.153$.

What message emerges from you analysis regarding the policy of financing government spending with money creation?

4 Intertemporal allocation under uncertainty

4.1 Euler condition

Consider a two period model where a household gets an income Y_1 in the first period and $Y_1(1+\varepsilon_2)$ in the second period where ε_2 is a shock of expected value zero.

The household pays taxes at rates at τ_1 and τ_2 in periods 1 and 2, with both rates known at period 1. The household can lend and borrow at zero interest rate (so one unit invested in period 1 gives one unit in period 2). We index the states of nature in the second period by k_2, and denote the respective probability by $\pi(k_2)$.

The intertemporal resource constraint for a given realization of output in the second period is:

$$C_2 + C_1 = (1-\tau_1)Y_1 + (1-\tau_2)Y_2(k_2)$$

Notice that there is one such constraint for each possible value of ε_2.

The household maximizes the following utility of consumption:

$$\ln(C_1) + E[\ln(C_2)]$$

Show that the optimal path of consumption is given by:

$$\frac{1}{C_1} = E\left[\frac{1}{C_2}\right]$$

Interpret this relation in terms of intuition.

4.2 Taxes under certainty

Consider that the government lowers the tax rate in the first period and increases it in the second period ($d\tau_1 < 0$). The change is calibrated so that the impact on the expected tax return, $\tau_1 Y_1 + \tau_2 E[Y_1(1+\varepsilon_2)] = \tau_1 Y_1 + \tau_2 Y_1$, is zero. In other words $d\tau_1 = -d\tau_2$.

Consider that there is no uncertainty ($\varepsilon_2 = 0$ always). What is the impact of the tax cut?

4.3 Effect of uncertainty

Now consider that there is uncertainty. Remember that if $F(x)$ is a convex function ($F'' > 0$) then $E[F(x)] > F(E[x])$.

Show that:

$$C_1 < E[C_2]$$

Interpret this relation.

4.4 Differentiation of consumption dynamics

Now consider the impact of the tax switch ($d\tau_1 = -d\tau_2$) in the presence of uncertainty.

Differentiate the dynamics of consumption derived in in the first part of the question to show that:

$$\left[E\frac{1}{(C_2)^2}+\frac{1}{(C_1)^2}\right]dC_1=Y_1d\tau_1E\left[\frac{\varepsilon_2}{(C_2)^2}\right]$$

4.5 Taxes under uncertainty

Remember that if x and y are random variables, then $E(xy)=E(x)E(y)+covar(x,y)$. Show that:

$$\frac{dC_1}{d\tau_1}=\frac{Y_1}{E\frac{1}{(C_2)^2}+\frac{1}{(C_1)^2}}cov\left(\left(\frac{1}{C_2}\right)^2,\varepsilon_2\right)$$

What is the sign of $dC_1/d\tau$? Explain the economic intuition. What does this tell us about the fiscal multiplier depending on the economic environment?