Macroeconomics B, EI060

Class 4

Uncertainty in open economy

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What you will get from today class

- Intertemporal allocation under uncertainty.
  - Precautionary savings (Harms IV.5.1-5.3).
  - Complete asset markets (Obstfeld and Rogoff 5.1.1-5.1.7, 5.2.1).
- International portfolio choice.
  - Simple model of portfolio and consumption allocation (Harms IV.5.4-5.7, OR (secondary) 5.3-5.4.1).
  - Home bias in portfolio.

A question to start

DIVERSIFY (OIL RICH)
GLOBAL CRASH

SYSTEMATIC RISK

When countries can access a very broad range of assets, they can insure away the risk of their income and everyone is equally well off.

FRICTIONS
(K CONT, FX RISK)

Do you agree? Why or why not?

INCOME LEVEL

OPTIMIZATION UNDER UNCERTAINTY

States of nature

- Two period model, with uncertainty about output in 2.
- Various possible states of nature k, with probability $\pi(k)$. Consumption in a state is denoted by $C_2(k)$.
- Intertemporal expected utility:

$$u(C_1) + \beta E u(C_2) = u(C_1) + \beta \sum_k \pi(k) u(C_2(k))$$

- Can invest only in a bond giving and interest rate r. One budget constraint for period 1 and for for each state in period 2:

$\frac{\partial \mathcal{L}}{\partial C_1} \to \checkmark$

$\frac{\partial \mathcal{L}}{\partial C_2(k)} \to \# k$

$\frac{\partial \mathcal{L}}{\partial B_2} \to P'$, $\sum P_n 2$

$$\begin{aligned} C_1 + B_2 &= Y_1 \\ C_2(k) &= (1+r) B_2 + Y_2(k) \end{aligned}$$

- Euler condition:

$$\begin{aligned} u'(C_1) &= \sum_k (1+r) \beta \pi(k) u'(C_2(k)) \\ u'(C_1) &= \beta (1+r) E\left[u'(C_2)\right] \end{aligned}$$

Precautionary savings

- Two states of nature with equal probability. One with high output $Y_2(k) = Y_1 + \Delta$, one with low output, $Y_2(k) = Y_1 - \Delta$.
- Consider $\beta(1+r) = 1$. Without uncertainty ($\Delta = 0$), we have: $C_1 = C_2 = Y_1$ and $B_2 = 0$.
- Euler with log utility:

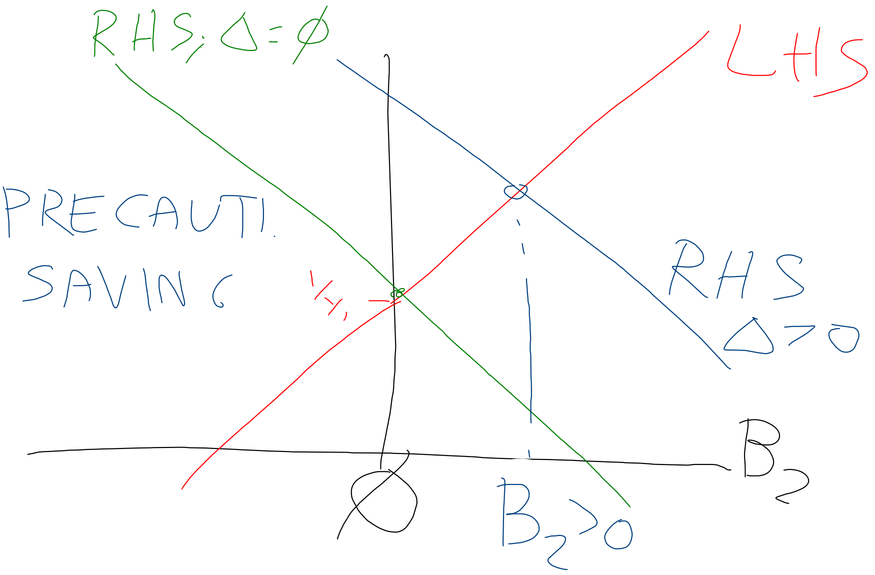
$C_2(k)$

$\pi(k)$

$$\frac{1}{Y_1 - B_2} = \frac{1}{2}\left[\frac{1}{(1+r)B_2 + Y_1 + \Delta} + \frac{1}{(1+r)B_2 + Y_1 - \Delta}\right]$$

$$\frac{1}{Y_1 - B_2} = \frac{(1+r)B_2 + Y_1}{[(1+r)B_2 + Y_1]^2 - [\Delta]^2}$$

  - The left-hand side is an increasing function of B_2, and the right-hand side a decreasing function of B_2.
  - The right-hand side is an increasing function of Δ.
- As Δ increases, the left-hand side must increase and/or the right-hand side must decrease.
- Both imply $B_2 > 0$, hence $C_1 < Y_1$.


RHS; Δ=∅
LHS
PRECAUTI.
SAVING
RHS
Δ>0
B2
∅
B2>0

Certainty equivalence

- Linear-quadratic utility:

$\ln(C)$

$u' = \frac{1}{C}$ $\quad u'' = -\frac{1}{C^2}$

$u''' = \frac{1}{C^3} > 0$

$$
\begin{aligned}
u(C) &= C - \frac{a}{2}C^2 \\
u'(C) &= 1 - aC \quad > 0
\end{aligned}
$$

$u'' = -a < 0 \qquad u''' = 0$

- The Euler condition is then:

$$
\begin{aligned}
1 - aC_1 &= \beta(1+r)\,E\left[1 - aC_2\right] \\
EC_2 &= \frac{\beta(1+r) - 1}{a\beta(1+r)} + \frac{1}{\beta(1+r)}C_1
\end{aligned}
$$

- If $\beta(1+r) = 1$ this simplifies to $EC_2 = C_1$. Consumption follows a random walk: future expected consumption is equal to current one.

COMPLETE ASSET MARKETS

Richer asset set

- In addition to the bond, the consumer can purchase contingent securities.
  - State k security pays off 1 unit if that state occurs, zero otherwise.
  - The cost of the security in the first period is $p_1(k)/(1+r)$.
  - No arbitrage implies $\sum_k p_1(k) = 1$. Purchasing the bond or purchasing one of each securities both deliver a certain payoff.
- Complete financial market: there is a security for each possible state (Arrow-Debreu set of complete contingent securities).
- Budget constraints are:

$$\begin{aligned} C_1 &= Y_1 - \sum_k \frac{p_1(k)}{1+r} B_2(k) - B_2 \\ C_2(k) &= (1+r) B_2 + B_2(k) + Y_2(k) \end{aligned}$$

- Value of consumption is equal to value of output, when future valued at state contingent prices:

$$C_1 + \sum_k \frac{p_1(k)}{1+r} C_2(k) = Y_1 + \sum_k \frac{p_1(k)}{1+r} (Y_2(k))$$

Euler conditions

- Euler condition for the bond:

$$u'(C_1) = \beta(1+r) E[u'(C_2)]$$

- Euler condition for state k security:

$$u'(C_1) \quad \frac{p_1(k)}{1+r} = \pi(k) \frac{\beta u'(C_2(k))}{u'(C_1)}$$

- Utility-discount factor (pricing kernel): $m_{1,2}(k) = \beta u'(C_2(k)) / u'(C_1)$.
- Expected discounted returns are equalized across assets:

$$E[m_{1,2}(1+r)] = E\left[m_{1,2} \boxed{\frac{1}{p_1/(1+r)}}\right]$$

RER sec

$1 + R(k)$

Gain from asset trades

- Autarky: consumption = output. Euler gives the autarky asset price:

$$\frac{p_1^A(k)}{1+r^A} = \pi(k)\frac{\beta u'(Y_2(k))}{u'(Y_1)}$$

- If agents can trade financial assets, it must be that the resulting consumption allocation is not affordable at the autarky asset prices (otherwise no gains from trade):

$$C_1^{\text{asset trade}} + \sum_k \frac{p_1(k)}{1+r} C_2^{\text{asset trade}}(k) = Y_1 + \sum_k \frac{p_1(k)}{1+r} Y_2(k)$$

$$C_1^{\text{asset trade}} + \sum_k \frac{p_1^A(k)}{1+r^A} C_2^{\text{asset trade}}(k) \geq Y_1 + \sum_k \frac{p_1^A(k)}{1+r^A} Y_2(k)$$

- Net buyer of securities in states of nature where such securities are expensive (i.e. consumption is valuable) under autarky:

$$\sum_k \left[\frac{p_1^A(k)}{1+r^A} - \frac{p_1(k)}{1+r}\right]\left[C_2^{\text{asset trade}}(k) - Y_2(k)\right] \geq 0$$

General equilibrium

- Label the country with H (Home), and introduce a Foreign country F.
- Home and Foreign Eulers for bonds and secutiries:

$$\frac{1}{1+r} = E\left[\frac{\beta u'\left(C_2^H\right)}{u'\left(C_1^H\right)}\right] = E\left[\frac{\beta u'\left(C_2^F\right)}{u'\left(C_1^F\right)}\right]$$

$$\frac{p_1\left(k\right)}{1+r} = \pi\left(k\right)\frac{\beta u'\left(C_2^H\left(k\right)\right)}{u'\left(C_1^H\right)} = \pi\left(k\right)\frac{\beta u'\left(C_2^F\left(k\right)\right)}{u'\left(C_1^F\right)}$$

- Conditions for securities show equal consumption ratios across states and time, hence similar growth:

$$\frac{u'\left(C_2^H\left(k\right)\right)}{u'\left(C_2^F\left(k\right)\right)} = \frac{u'\left(C_1^H\right)}{u'\left(C_1^F\right)}$$

$$\frac{C_2^H\left(k\right)}{C_1^H} = \frac{C_2^F\left(k\right)}{C_1^F} = \left[\frac{\beta\left(1+r\right)}{p_1\left(k\right)}\pi\left(k\right)\right]^{\frac{1}{\sigma}}$$

Interpretation

- In each period, each country consumes the same share of world output: $C_2(k)/Y_2^W(k) = C_1/Y_1^W$.
- Complete asset markets make the two countries evolve in parallel
- Two important points should be noted:
  - Trading only removes idiosyncratic risk (both countries move in step), not aggregate risk (consumption differs across the two periods).
  - Trading equalize the growth rates of consumption, not their levels (life is always more pleasant for the rich).
- We don't necessarily need the full set of securities.
  - Some assets have with payoffs proportional to shocks (Home equity for shocks to the Home endowment).
  - Having ~~h~~as many assets as shocks delivers complete markets.
  - Relative good prices can deliver completeness. This is the case with endowment shocks and a unit elasticity of substitution between Home and Foreign goods (Cole & Obstfeld).

Security prices

- Clearing of goods markets:

$$
\begin{aligned}
C_1^H + C_1^F &= Y_1^H + Y_1^F = Y_1^W \\
C_2^H(k) + C_2^F(k) &= Y_2^H(k) + Y_2^F(k) = Y_2^W(k)
\end{aligned}
$$

- As consumption growth is the same in both countries, Euler and good markets clearing imply:

$\sigma = 1 = \ln(c)$

$$
\begin{aligned}
\left[\frac{\beta(1+r)}{p_1(k)}\pi(k)\right]^{\frac{1}{\sigma}}\left(C_1^H + C_1^F\right) &= C_2^H(k) + C_2^F(k) \\
\left[\frac{\beta(1+r)}{p_1(k)}\pi(k)\right]^{\frac{1}{\sigma}} Y_1^W &= Y_2^W(k) \\
\frac{p(k)}{1+r} &= \beta\pi(k)\left[\frac{Y_2^W(k)}{Y_1^W}\right]^{-\sigma}
\end{aligned}
$$

SCARCE

- A security is expensive $(p(k)$ is high) if:
  - The state is likely to happen ($\pi(k)$ is high).
  - In that state output is ~~abundant~~ ($Y_2^W(k)/Y_1^W$ is ~~high~~). Low

Return on the bond

- Sum the security price across states:

$$\sum_k \frac{p(k)}{1+r} = \beta \sum_k \pi(k) \left[\frac{Y_2^W(k)}{Y_1^W}\right]^{-\sigma}$$

$$\beta(1+r) = \frac{\left[Y_1^W\right]^{-\sigma}}{E\left[\left(Y_2^W\right)^{-\sigma}\right]}$$

$h(c) \Rightarrow \beta = 1$

- The bond delivers a high return when future output is expected to be high relative to current output (more exactly, its valuation is expected to be low.
- First-order log approximation ($\tilde{r} = ln(1+r)$, $y^W = ln(Y^W)$) :

$$\tilde{r} = \sigma E\left(y_2^W - y_1^W\right)$$

In nominal terms

- Some state-contingent securities pay off in Home currency, and others that pay off in Foreign currency.
  - Security paying off one unit of Home currency in state k_{t+s} at time $t+s$ costs $p_t^H(k_{t+s})$ units of Home currency.
  - Security paying off one unit of Foreign currency in state k_{t+s} at time $t+s$ costs $p_t^F(k_{t+s})$ units of Foreign currency.
- S : nominal exchange rate,in units of Home currency per unit of Foreign currency (a higher value is a weaker Home currency)
- CPI in the two countries, in local currency: P^H and P^F.

Home agent

- Infinite horizon, with all securities bought initially. Expected utility:

$$u\left(C_t^H\right) + \sum_{s=1}^{\infty} \beta^s \left[\sum_{k_{t+s}} \pi\left(k_{t+s}\right) u\left(C_{t+s}^H\left(k_{t+s}\right)\right)\right]$$

- Budget constraints at t and $t+1$

$$\begin{aligned}
P_t^H C_t^H &= P_t^H Y_t^H - \sum_{s=1}^{\infty} \sum_{k_{t+s}} \frac{p_t^H\left(k_{t+s}\right)}{1+r} B_t^H\left(k_{t+s}\right) \\
&\quad - \sum_{s=1}^{\infty} \sum_{k_{t+s}} \frac{S_t p_t^F\left(k_{t+s}\right)}{1+r} B_t^F\left(k_{t+s}\right) - B_{t+1}^H \\
P_{t+1}^H\left(k_{t+1}\right) C_{t+1}^H\left(k_{t+1}\right) &= (1+r) B_{t+1} + B_t^H\left(k_{t+1}\right) \\
&\quad + S_{t+1} B_t^F\left(k_{t+1}\right) + P_{t+1}^H\left(k_{t+1}\right) Y_{t+1}^H\left(k_{t+1}\right)
\end{aligned}$$

- Euler for the Home security, the Foreign security, and the bond:

$$\frac{p_t^H(k_{t+1})}{1+r} = \pi(k_{t+1})\frac{\beta u'\left(C_{t+1}^H(k_{t+1})\right)/P_{t+1}^H(k_{t+1})}{u'\left(C_t^H\right)/P_t^H}$$

$$\frac{S_t p_t^F(k_{t+1})}{1+r} = S_{t+1}(k_{t+1})\,\pi(k_{t+1})\frac{\beta u'\left(C_{t+1}^H(k_{t+1})\right)/P_{t+1}^H(k_{t+1})}{u'\left(C_t^H\right)/P_t^H}$$

$$1 = (1+r)\sum_{k_{t+1}}\pi(k_{t+1})\frac{\beta u'\left(C_{t+1}^H(k_{t+1})\right)/P_{t+1}^H(k_{t+1})}{u'\left(C_t^H\right)/P_t^H}$$

- Similar relations for the Foreign agent.

Complete risk sharing

- Combine the Euler for the Home currency securities for Home and Foreign consumers:

$$\frac{u'\left(C^H_{t+1}\left(k_{t+1}\right)\right)/P^H_{t+1}\left(k_{t+1}\right)}{u'\left(C^H_t\right)/P^H_t} = \frac{S_t}{S_{t+1}\left(k_{t+1}\right)}\frac{u'\left(C^F_{t+1}\left(k_{t+1}\right)\right)/P^F_{t+1}\left(k_{t+1}\right)}{u'\left(C^F_t\right)/P^F_t}$$

- This implies that in any any state of nature:

$$\frac{u'\left(C^H_{t+1}\right)}{u'\left(C^F_{t+1}\right)}\overbrace{\frac{S_{t+1}P^F_{t+1}}{P^H_{t+1}}}^{\text{RER}} = \frac{u'\left(C^H_t\right)}{u'\left(C^F_t\right)}\frac{S_tP^F_t}{P^H_t} = \Gamma_t$$

- Γ_t captures any difference between Home and Foreign agents that cannot be handled through asset trading (different initial wealth, different volatility of income).
- Ratio of marginal utilities is in line with the real exchange rate. Giving an extra franc to the Home agent generates the same utility than giving it to the Foreign agent:

$$\frac{1}{P\left(k_{t+s}\right)}u'\left(C\left(k_{t+s}\right)\right) = \Gamma_t\frac{1}{S\left(k_{t+s}\right)P^{\overset{F}{*}}\left(k_{t+s}\right)}u'\left(C^{\overset{F}{*}}\left(k_{t+s}\right)\right)$$

Interpretation

- The ratio of relative consumption is proportional to the real exchange rate:

$$\frac{u'\left(C_{t+1}^{F}\right)}{u'\left(C_{t+1}^{H}\right)} = \frac{1}{\Gamma_t}\frac{S_{t+1}P_{t+1}^{F}}{P_{t+1}^{H}}$$

  - Giving an extra franc to the Home agent generates the same utility than giving it to the Foreign agent:

$$\frac{1}{P_{t+1}^{H}}u'\left(C_{t+1}^{H}\right) = \Gamma_t\frac{1}{S_{t+1}P_{t+1}^{F}}u'\left(C_{t+1}^{F}\right)$$

- Γ_t captures any difference between Home and Foreign agents that cannot be handled through asset trading (different initial wealth, different volatility of income).
- Risk sharing does not make everyone equal, but makes everyone proportional.

PORTFOLIO CHOICE

Trading in bond and equity

- Endowment in period 1: output Y_1^H and today's value of the future output V_1^H.
- Invest in a bond and equity: claims on each country's output. Home households buys:
  - x_2^{HH} units of the Home equity. Price V_1^H, each unit pays of the endowment Y_2^H.
  - x_2^{HF} units of the Foreign equity. Price V_1^F, pays of the endowment Y_2^F.
- The Foreign household purchases $x_2^{FH} = 1 - x_2^{HH}$ and $x_2^{FF} = 1 - x_2^{HF}$ units of equity.
- Budget constraints (before we imposed $x_2^{HH} = 1$ and $x_2^{HF} = 0$):

2

$$\begin{aligned} C_1^H &= Y_1^H + V_1^H - B_2^H - x_2^{HH} V_1^H - x_2^{HF} V_1^F \\ C_2^H(k) &= (1+r) B_2^H + x_2^{HH} Y_2^H(k) + x_2^{HF} Y_2^F(k) \end{aligned}$$

Optimal allocation

- Three Euler conditions: bond, Home equity, Foreign equity:

$$
\begin{aligned}
u'\left(C_1^H\right) &= \beta(1+r)\,E\left[u'\left(C_2^H\right)\right] \\
u'\left(C_1^H\right) &= \beta E\left[u'\left(C_2^H\right)\frac{Y_2^H}{V_1^H}\right] \\
u'\left(C_1^H\right) &= \beta E\left[u'\left(C_2^H\right)\frac{Y_2^F}{V_1^F}\right]
\end{aligned}
$$

$\rightarrow 1 + R^{H}$
DIV YIELD

- Expected discounted excess return of an asset is zero:

$$
0 = E\left[m_{1,2}^H\left(\frac{Y_2^H}{V_1^H} - (1+r)\right)\right] \;\; ; \;\; 0 = E\left[m_{1,2}^H\left(\frac{Y_2^F}{V_1^F} - (1+r)\right)\right]
$$

Asset pricing

- Asset price indicates a hedging property:

$$V_1^H = E\left[m_{1,2}^H Y_2^H\right] \quad ; \quad V_1^F = E\left[m_{1,2}^H Y_2^F\right]$$

- As $E(ab) = E(a)E(b) + Cov(a,b)$ we write:

$$V_1^F = E\left[m_{1,2}^H\right] E\left[Y_2^F\right] + Cov\left[m_{1,2}^H, Y_2^F\right]$$

QUADR APPR

- Foreign equity is more valuable if:
  - Foreign output is expected to be high on average
  - Foreign output is abundant when consumption is valued ($m_{1,2}^H$ is high), it is a good hedge.

Portfolio shares

- CRRA utility, $u(C) = \frac{(C)^{1-\sigma}}{1-\sigma}$. Both agents have the same portfolio shares, including for bonds.
  - Bonds are in zero net supply, so $B_2^H = 0$.
- Saving is endowment minus consumption: $V_1^H + Y_1^H - C_1^H$ and $V_1^F + Y_1^F - C_1^F$.
- With equal portfolio allocation ($C_1^H = \mu_1^H Y_1^W$):

IN H ST MK   IN H ST MKT

$$S_1^H \quad \frac{x_2^{HH} V_1^H}{V_1^H + Y_1^H - \mu_1^H Y_1^W} = \frac{x_2^{FH} V_1^H}{V_1^F + Y_1^F - \mu_1^F Y_1^W} \quad S^F$$

$$\frac{x_2^{HF} V_1^F}{V_1^H + Y_1^H - \mu_1^H Y_1^W} = \frac{x_2^{FF} V_1^F}{V_1^F + Y_1^F - \mu_1^F Y_1^W}$$

- With asset market clearing, we can show the Home investor accounts for the same share of all equity investments: $x_2^{HH} = x_2^{HF} = x_2^H$.

Consumption share: dynamics

- The budget constraint in the second period implies that the consumption share is the portfolio share: $\mu_2^H = x_2^H$, which is set in period .1

$$C_2^H(k) = \mu_2^H Y_2^W(k)$$

SAME PRESENCE IN ANY MKT

- Combine the Euler for the Home equity for both investors shows that the Home country accounts for the same share of consumption in the two periods:

MOVE PARALLEL

$$E\left[\beta\left(\frac{C_2^H}{C_1^H}\right)^{\sigma} Y_2^H\right] = E\left[\beta\left(\frac{C_2^F}{C_1^F}\right)^{\sigma} Y_2^H\right]$$

$$\left(\frac{\mu_2^H}{\mu_1^H}\right)^{\sigma} E\left[\left(\frac{Y_2^W}{Y_1^W}\right)^{\sigma} Y_2^H\right] = \left(\frac{1-\mu_2^H}{1-\mu_1^H}\right)^{\sigma} E\left[\left(\frac{Y_2^W}{Y_1^W}\right)^{\sigma} Y_2^H\right]$$

$$\mu_2^H = \mu_1^H$$

Consumption level and current account

- Budget constraint from period 1 gives the consumption share.

$$\mu^H Y_1^W = Y_1^H + V_1^H - x^H \left(V_1^H + V_1^F \right)$$

$$\mu^H = \frac{Y_1^H + V_1^H}{Y_1^W + V_1^W}$$

  - The country's share of consumption is its share of world wealth: current endowment, and value of future endowment.

- The current account is output minus consumption:

$$CA_1 = Y_1^H - C_1^H$$

$$CA_1 = \frac{V_1^W}{Y_1^W + V_1^W} Y_1^H - \frac{Y_1^W}{Y_1^W + V_1^W} V_1^H$$

  - A country with high initial endowment or low value of future endowment runs a surplus.

Asset prices

- Equity prices reflect the growth rate of world output, and the future level of the specific country's output:

$$
\begin{aligned}
V_1^H &= E\left[\beta\left(\frac{C_2^H}{C_1^H}\right)^{\sigma} Y_2^H\right] = E\left[\beta\left(\frac{Y_2^W}{Y_1^W}\right)^{\sigma} Y_2^H\right] \\
V_1^F &= E\left[\beta\left(\frac{Y_2^W}{Y_1^W}\right)^{\sigma} Y_2^F\right]
\end{aligned}
$$

More general formulation

- Recall that expected discounted excess return of an asset is zero. For instance, comparing the two equities:

$$0 = E\left[m_{1,2}^{H}\left(\frac{Y_2^H}{V_1^H} - \frac{Y_2^F}{V_1^F}\right)\right] \quad ; \quad 0 = E\left[m_{1,2}^{F}\left(\frac{Y_2^H}{V_1^H} - \frac{Y_2^F}{V_1^F}\right)\right]$$

- Difference implies that covariance between cross-country difference in pricing kernel and difference in returns is zero:

PORT SH (X's)

$$0 = E\left[\left(m_{1,2}^{H} - m_{1,2}^{F}\right)\left(\frac{Y_2^H}{V_1^H} - \frac{Y_2^F}{V_1^F}\right)\right]$$

- $m_{1,2}^{H} - m_{1,2}^{F}$ reflect future consumptions, hence portfolio shares (through the budget constraint of period 2).
- To solve a general model, linearizing is not enough.
  - Linear approximation gives everything conditional on portfolio shares.
  - The Euler difference gives the portfolio shares. It is a covariance, so it has to be approximated with a quadratic approximation.

Portfolio Home bias

- Can it be that a country holds a share of domestic assets in its portfolio that exceed its share in the world?
  - Empirically it is clearly the case
- It can be the case only if the domestic asset is a better hedge than the foreign one.
  - Domestic assets has a higher return than the foreign one when the investors consumption is low (marginal utility is high).
  - Relation with labor income: invest more in the asset that pays off better (than the other asset) when labor income is low.
- Not easy to get. If all goods are traded, higher productivity at home leads to high consumption. Makes holding foreign equity more appealing.
  - With labor income: high productivity raises both labor income and domestic asset return. The domestic asset is a bad hedge.

Explaining the bias

- Rich literature on generating a tilt towards domestic assets.
- Introduce sticky prices. Output is then driven by demand (price do not fall after a productivity gain).
  - Higher productivity allows for output to be produced with less labor.
  - Revenue is then paid more as profits (dividends) than wages.
  - Higher productivity boosts dividends and lowers wage income. Domestic equity is a good hedge.
- Introduce bonds in different currencies in addition to equity.
  - Demand shocks (such as monetary policy) leads to real exchange rate risk. Equity does not connect well to it.
  - Bonds handles demand shocks. Equity can then be used to hedge other shocks.
  - Side benefit: equity portfolio is very sensitive to the parameters when it is the only asset. Bonds solve this problem.
- Asymmetric information, as local investors know their own assets better.