ECON 39: Undergraduate International Trade
Problem Set #2 solutions
Professor Treb Allen

Preliminaries

This week, we are learning about two modern extensions of comparative advantage theory: one to many goods (and two countries, possibly with trade costs), and one to many goods, many countries, and arbitrary trade costs. This problem set is going to be tricky, but do your best!

Questions

1. [DFS '77] Consider a world with two countries $i \in \{1, 2\}$, each inhabited by L_i consumers. Suppose that each country has the technology to produce a continuum $g \in [0, 1]$ of goods. Let $\alpha_i(g)$ be the unit labor cost of producing good $g \in [0, 1]$ in country $i \in \{1, 2\}$. Normalize the wage in country 2 to one and let w denote the relative wage in country 1.

   (a) Suppose that $\alpha_1(g) = g^2$ and $\alpha_2(g) = g$. If the equilibrium wage is $w = 2$, which country would produce which good?

   - *Answer: The price of producing good g in country 1 is $p_1(g) = \alpha_1(g)w = g^2 w$. The price of producing good g in country 2 is $p_2(g) = \alpha_2(g) = g$. Country 1 will produce the good as long as:*

   $$\begin{aligned} p_1(g) \leq p_2(g) &\iff \\ g^2 w \leq g &\iff \\ g \leq \frac{1}{w} \end{aligned}$$

   If $w = 2$, country 1 will produce all $g \in [0, \frac{1}{2})$ and country 2 will produce all $g \in (\frac{1}{2}, 1]$, with consumers being indifferent between the two sources for $g = \frac{1}{2}$.

   (b) Suppose consumers have Cobb-Douglas preferences with equal demand shifters for all goods, i.e.:

   $$U_i = \int_0^1 \log C_i(g)\, dg$$

   Solve for the equilibrium quantity consumed of good g by a consumer in i as a function of the good's price $p_i(g)$ and the total income in country i, Y_i.

   - *Answer: The utility maximization problem is:*

   $$\max_{C_i(g)} \int_0^1 \log C_i(g)\, dg \text{ subject to } \int_0^1 p_i(g) C_i(g) \leq Y_i$$

   For the purposes of our class, we will take a short-cut and treat the integrals like summations, so that we can write the Lagrangian in the standard way we are used to:[1]

   $$\mathcal{L} : \int_0^1 \log C_i(g)\, dg - \lambda \left(\int_0^1 p_i(g) C_i(g)\, dg - Y_i \right)$$

   First order conditions with respect to $C_i(g)$ are:

   $$\begin{aligned} \frac{1}{C_i(g)} = \lambda p_i(g) &\iff \\ \frac{1}{\lambda} = p_i(g) C_i(g) \end{aligned}$$

[1] *Strictly speaking, since we are maximizing a function, we ought to use calculus of variations to solve the problem.*

Since these first order conditions hold for all $g \in [0,1]$ and λ is constant, this implies that we can integrate over all commodities, yielding:

$$\int_0^1 \frac{1}{\lambda} dg = \int_0^1 p_i(g) C_i(g)\, dg \iff$$
$$\frac{1}{\lambda} = Y$$

so that we have:

$$p_i(g) C_i(g) = Y \iff$$
$$C_i(g) = \frac{Y}{p_i(g)}$$

(c) Now let us focus on the equilibrium:

i. List the exogenous parameters of the model.

- *Answer: Preferences $U_i(\cdot)$, unit labor costs $\alpha_i(g)$, and populations L_i for all $g \in [0,1]$ and $i \in \{1,2\}$.*

ii. List the endogenous outcomes of the model.

- *Answer: The relative wage w, the quantity produced of each good in each location $Q_i(g)$, and the quantity consumed of each good in each location $C_i(g)$.*

iii. State the equilibrium conditions.

- *Answer: For any preferences $U_i(\cdot)$, unit labor costs $\alpha_i(g)$, and populations L_i, equilibrium is the relative wage w, the quantity produced of each good in each location $Q_i(g)$, and the quantity consumed of each good in each location $C_i(g)$ for all $g \in [0,1]$ and $i \in \{1,2\}$ such that:*
  - *Given prices, producers choose which products to produce to maximize their profits;*
  - *Given prices and incomes, consumers choose where to purchase a product and the quantity to purchase to maximize their utility; and*
  - *Markets clear, i.e. income spent by consumers in each country is equal to that countries total sales.*

(d) Suppose $L_1 = 1$ and $L_2 = 2$, unit labor costs are as in 1(a) and preferences are as in 1(b). Find the equilibrium relative wage, incomes in the two countries and pattern of specialization.

- *Answer: Let g^* denote the threshold at which country 1 produces all goods $g \in [0, g^*]$ and country 2 produces all goods $g \in [g^*, 1]$. From 1(a), we can write g^* as a function of the equilibrium wage:*

$$g^* = \frac{1}{w} \iff$$
$$w = \frac{1}{g^*}$$

This is our factor demand curve.

- *Given relative wage w, income in country 1 is $Y_1 = wL_1 = w$ and income in country 2 is $Y_2 = L_2 = 2$. Hence, total world income is $Y^{world} = Y_1 + Y_2 = 2 + w$. From 1(b), expenditure is equal across all goods. Hence. if goods $g \in [0, g^*]$ are produced in country 1, country 1 will earn a fraction g^* of world income. In order for markets to clear, this income has to equal the wages paid to workers in country 1, i.e.:*

$$Y_1 = g^* Y^{world} \iff$$
$$w = g^*(2 + w) \iff$$
$$w = \frac{2g^*}{1 - g^*}$$

This is our factor supply curve.

- *To solve for the equilibrium relative wage, we simply set the supply curve equal to the demand curve:*

$$\begin{aligned} \frac{1}{g^*} = \frac{2g^*}{1-g^*} &\iff \\ 1 - g^* = 2\left(g^*\right)^2 &\iff \\ 2\left(g^*\right)^2 + g^* - 1 = 0 & \end{aligned}$$

The solution to this quadratic equation is $g^* = 0.5$, so country 1 produces goods $g \in [0, 0.5]$ and country 2 produces goods $g \in [0.5, 1]$. Substituting in $g^* = 0.5$ into either the factor supply or factor demand equations yields a relative wage of $w = 2$. This in turn implies the income in country 1 is 2 and the income in country 2 is 2.

(e) Suppose the population of country 1 doubled to $L_1 = 2$. Show using both math and a figure how equilibrium wages would change. Would the equilibrium pattern of specialization change?

- *Answer: As in part (d) we have:*

$$w = \frac{1}{g^*}$$

However, now the factor supply curve changes:

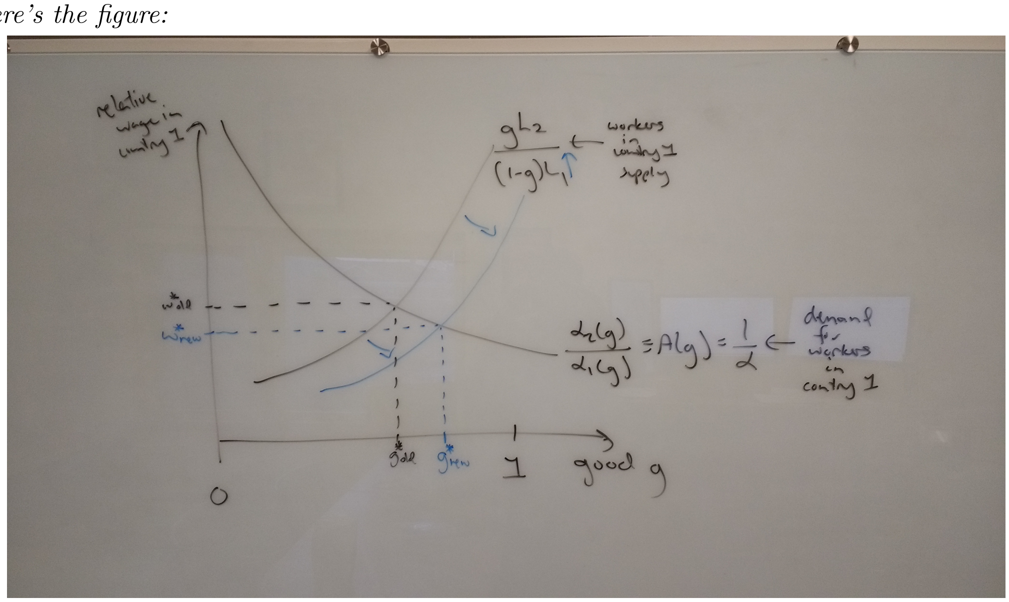
$$\begin{aligned} Y_1 = g^* Y^{world} &\iff \\ 2w = g^*\left(2 + 2w\right) &\iff \\ w = \frac{g^*}{1-g^*} & \end{aligned}$$

When we combine the two equations to solve for g^* we get:

$$\begin{aligned} \frac{1}{g^*} = \frac{g^*}{1-g^*} &\iff \\ \left(g^*\right)^2 + g^* - 1 = 0 & \end{aligned}$$

The solution of this equation is $g^* \approx 0.62$. Hence, the equilibrium relative wage $w \approx 1.6$, i.e. the relative wage in country 1 falls. Intuitively, since there is more labor in country 1, it will increase the set of goods it produces. But these new goods it is not as good at producing, so its average wage must fall.

- *Here's the figure:*



2. [EK '02]. Consider a world with N countries with iceberg trade costs from i to j $\tau_{ij} \geq 1$. Let A_i be the aggregate productivity in country i and let w_i be its wage. Recall that the value of trade flows from i to j can be written as:
$$X_{ij} = \tau_{ij}^{-\theta} w_i^{-\theta} A_i^{\theta} P_j^{\theta} w_j L_j,$$
where $P_j \equiv \left(\sum_{k=1}^{K} \tau_{kj}^{-\theta} w_k^{-\theta} A_k^{\theta}\right)^{-\frac{1}{\theta}}$ is the price index. Recall too that the welfare of a worker in location j can be written as $U_j = \frac{w_j}{P_j}$. Suppose that the wages are exogenous. [Note: This assumption is sometimes made by assuming there is an "outside sector" that is costlessly traded that all countries produce. I am not a fan of this assumption.] Derive the elasticity of welfare in country j to a change in the productivity in country i, i.e. $\frac{\partial \ln U_j}{\partial \ln A_i}$. Is this variable observed in the data? What is the intuition for this result?

- *Answer: First, we substitute in the price index and take logs:*

$$\begin{aligned}
U_j &= \frac{w_j}{P_j} \iff \\
\ln U_j &= \ln w_j - \ln P_j \iff \\
\ln U_j &= \ln w_j - \ln\left(\left(\sum_{k=1}^{K} \tau_{kj}^{-\theta} w_k^{-\theta} A_k^{\theta}\right)^{-\frac{1}{\theta}}\right) \iff \\
\ln U_j &= \ln w_j + \frac{1}{\theta}\ln\left(\sum_{k=1}^{K} \tau_{kj}^{-\theta} w_k^{-\theta} A_k^{\theta}\right)
\end{aligned}$$

Then, we replace the variable A_k with $\exp(\log A_k)$*:*

$$\ln U_j = \ln w_j + \frac{1}{\theta}\ln\left(\sum_{k=1}^{K} \tau_{kj}^{-\theta} w_k^{-\theta} \exp(\theta \ln A_k)\right)$$

Now we differentiate:

$$\begin{aligned}
\frac{\partial \ln U_j}{\partial \ln A_i} &= \frac{1}{\theta}\frac{1}{\sum_{k=1}^{K} \tau_{kj}^{-\theta} w_k^{-\theta} \exp(\theta \ln A_k)} \times \tau_{ij}^{-\theta} w_i^{-\theta} \exp(\theta \ln A_i) \times \theta \iff \\
\frac{\partial \ln U_j}{\partial \ln A_i} &= \frac{\tau_{ij}^{-\theta} w_i^{-\theta} A_i^{\theta}}{\sum_{l=1}^{K} \tau_{kj}^{-\theta} w_k^{-\theta} A_k^{\theta}} \iff \\
\frac{\partial \ln U_j}{\partial \ln A_i} &= \frac{X_{ij}}{Y_j}
\end{aligned}$$

That is, the elasticity of welfare in j to a productivity shock in i is equal to the fraction of expenditure that consumers in j spend on goods from i. Intuitively, the more consumers in j purchase from i, the greater their increase in welfare from a productivity increase in country i.