Macroeconomics A; EI056

Short problems

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1 Allocation of consumption through time

1.1 Euler condition

Question: Consider a two-period model without uncertainty, with the following utility of consumption:

$$U_1 = [\ln (C_1) + b \ln (1 - L_1)] + \frac{1}{1+\rho} [\ln (C_2) + b \ln (1 - L_2)]$$

The households earn a wage, rents out capital, and can invest. Initial capital K_1 is given. Capital does not depreciate, and the capital remaining at the end of period 2 is consumed. The budget constraints are:

$$\begin{aligned} C_1 + K_2 &= w_1 L_1 + (1 + r_1) K_1 \\ C_2 &= w_2 L_2 + (1 + r_2) K_2 \end{aligned}$$

Show that the Euler condition is:

$$\frac{1}{C_1} = \frac{1}{C_2} \frac{1 + r_2}{1 + \rho}$$

Answer: The Lagrangian for the optimization is:

$$\begin{aligned} \mathcal{L} &= [\ln (C_1) + b \ln (1 - L_1)] + \frac{1}{1+\rho} [\ln (C_2) + b \ln (1 - L_2)] \\ &\quad -\varphi_1 \left[\; C_1 + K_2 - w_1 L_1 - (1 + r_1) K_1 \; \right] \\ &\quad -\frac{1}{1+\rho} \varphi_2 \left[\; C_2 - w_2 L_2 - (1 + r_2) K_2 \; \right] \end{aligned}$$

The optimality condition with respect to C_1 is:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial C_1} \\ 0 &= \frac{1}{C_1} - \varphi_1 \\ \frac{1}{C_1} &= \varphi_1 \end{aligned}$$

The optimality condition with respect toC_2 is:

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial C_2} \\
0 &= \frac{1}{1+\rho}\frac{1}{C_2} - \frac{1}{1+\rho}\varphi_2 \\
\frac{1}{C_2} &= \varphi_2
\end{aligned}
$$

The optimality condition with respect toK_2 is:

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial K_2} \\
0 &= -\varphi_1 + \frac{1}{1+\rho}\varphi_2 (1+r_2) \\
\varphi_1 &= \frac{1}{1+\rho}\varphi_2 (1+r_2)
\end{aligned}
$$

Combining we get the Euler condition:

$$
\begin{aligned}
\varphi_1 &= \frac{1}{1+\rho}\varphi_2 (1+r_2) \\
\frac{1}{C_1} &= \frac{1}{1+\rho}\frac{1}{C_2}(1+r_2) \\
\frac{1}{C_1} &= \frac{1}{C_2}\frac{1+r_2}{1+\rho}
\end{aligned}
$$

1.2 Propensity to consume

Question: We can think of the wealth of an agent as the return on her initial capital, K_1, plus the discounted value of her wage earnings.

Combine the Euler and the budget constraint to solve for the level of consumption in period 1. How is the sensitivity of consumption to wealth affected by the interest rate?

Answer: Using the second period budget constraint, the Euler condition becomes:

$$
\begin{aligned}
\frac{1}{C_1} &= \frac{1}{C_2}\frac{1+r_2}{1+\rho} \\
\frac{1}{C_1} &= \frac{1}{w_2 L_2 + (1+r_2) K_2}\frac{1+r_2}{1+\rho} \\
w_2 L_2 + (1+r_2) K_2 &= C_1 \frac{1+r_2}{1+\rho} \\
w_2 L_2 + (1+r_2)(w_1 L_1 + (1+r_1) K_1 - C_1) &= C_1 \frac{1+r_2}{1+\rho} \\
w_2 L_2 + (1+r_2)(w_1 L_1 + (1+r_1) K_1) &= C_1 \frac{1+r_2}{1+\rho} + (1+r_2) C_1 \\
w_2 L_2 + (1+r_2)(w_1 L_1 + (1+r_1) K_1) &= \frac{2+\rho}{1+\rho}(1+r_2) C_1 \\
\frac{2+\rho}{1+\rho} C_1 &= w_1 L_1 + (1+r_1) K_1 + \frac{w_2 L_2}{1+r_2} \\
C_1 &= \frac{1+\rho}{2+\rho}\left[w_1 L_1 + (1+r_1) K_1 + \frac{w_2 L_2}{1+r_2}\right]
\end{aligned}
$$

Initial consumption is a constant fraction, $(1+\rho)/(2+\rho)$, of wealth. This fraction is not affected by the interest rate, and only reflects the degree of impatience. This is a property of the log utility of consumption.

2 Labor supply sensitivity

2.1 Representative household

Question: Consider a static model for simplicity. It is inhabited by a representative household with the following utility over consumption and labor:

$$(C, L) = \ln(C) + \frac{1}{1-\gamma}(1-L)^{1-\gamma}$$

The budget constraint is (w is the real wage and I other income sources, such as capital income):

$$C = wL + I$$

Show that the labor supply is:

$$(1-L)^{\gamma} = \frac{C}{w}$$

How sensitive is the marginal utility of leisure to the quantity of labor? What would you conclude regarding the sensitivity of labor supply to movements in the wage? To answer this, bear in mind that micro studies find a high value for γ.

Answer: The Lagrangian is written as follows:

$$\mathcal{L} = \ln(C) + \frac{1}{1-\gamma}(1-L)^{1-\gamma} - \varphi[C - wL - I]$$

The first order condition with respect to consumption is:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial C} \\ 0 &= \frac{1}{C} - \varphi \\ \frac{1}{C} &= \varphi \end{aligned}$$

The first order condition with respect to labor is:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial L} \\ 0 &= -(1-L)^{-\gamma} + \varphi w \\ (1-L)^{-\gamma} &= \varphi w \end{aligned}$$

Combining we get:

$$\begin{aligned} \frac{1}{C} &= (1-L)^{-\gamma}\frac{1}{w} \\ (1-L)^{\gamma} &= \frac{C}{w} \end{aligned}$$

The marginal utility of leisure is $(1-L)^{-\gamma}$. It increases as the household works more (leisure is lower), and the magnitude of this increase is proportional to γ. When γ is high, the utility of leisure is very concave. Empirically, studies finds that this is the case. People are thus reluctant to substantially increase their labor supply when the wage goes up.

To get the elasticity of the labor supply, we differentiate the labor supply with respect to wage and labor:

$$\begin{aligned}
-\gamma(1-L)^{\gamma-1}\,dL &= -\frac{C}{w^2}dw \\
\gamma L(1-L)^{\gamma-1}\frac{dL}{L} &= \frac{C}{w}\frac{dw}{w} \\
\gamma L(1-L)^{\gamma-1}\frac{dL}{L} &= (1-L)^{\gamma}\frac{dw}{w} \\
\gamma\frac{L}{1-L}\frac{dL}{L} &= \frac{dw}{w} \\
\frac{dL}{L} &= \tfrac{1}{\gamma}\tfrac{1-L}{L}\frac{dw}{w}
\end{aligned}$$

The elasticity is lower when γ is high.

2.2 Divisible household

Question: Consider now that the representative household is made of a large number of people. We normalize the total size to 1.

Each person either works H hours ($H<1$) or no hours. The probability that a person works is thus the ratio of total hours worked L to the number of hours worked by each employed person H.

Each person has the same utility function as above. We assume that all the members of the household pool their resources (whether they work or not), so that each consumes the same amount C.

Show that the average utility across the members of the households is:

$$\begin{aligned}
U^{avg} &= \ln(C) - \frac{L}{H}\left(\frac{1}{1-\gamma}(1)^{1-\gamma} - \frac{1}{1-\gamma}(1-H)^{1-\gamma}\right) + \frac{1}{1-\gamma}(1)^{1-\gamma} \\
&= \ln(C) - \frac{L}{H}\Omega_1 + \Omega_0
\end{aligned}$$

Treating this average household as a true person, how sensitive is the marginal utility of leisure to the quantity of labor? What would you conclude regarding the sensitivity of labor supply to movements in the wage?

Why can this alternative modelisation make the RBC model more realistic?

Answer: The utility of a person working is:

$$\ln(C) + \frac{1}{1-\gamma}(1-H)^{1-\gamma}$$

The utility of a person not working is:

$$\ln(C) + \frac{1}{1-\gamma}(1-0)^{1-\gamma}$$

There are L/H persons working. The average utility is then:

$$U^{avg} = \frac{L}{H}\left[\ln(C) + \frac{1}{1-\gamma}(1-H)^{1-\gamma}\right] + \left(1-\frac{L}{H}\right)\left[\ln(C) + \frac{1}{1-\gamma}(1)^{1-\gamma}\right]$$

$$
\begin{aligned}
&= \ln(C) + \frac{L}{H}\frac{1}{1-\gamma}(1-H)^{1-\gamma} + \left(1-\frac{L}{H}\right)\frac{1}{1-\gamma}(1)^{1-\gamma} \\
&= \ln(C) + \frac{L}{H}\left(\frac{1}{1-\gamma}(1-H)^{1-\gamma} - \frac{1}{1-\gamma}(1)^{1-\gamma}\right) + \frac{1}{1-\gamma}(1)^{1-\gamma} \\
&= \ln(C) - \frac{L}{H}\left(\frac{1}{1-\gamma}(1)^{1-\gamma} - \frac{1}{1-\gamma}(1-H)^{1-\gamma}\right) + \frac{1}{1-\gamma}(1)^{1-\gamma}
\end{aligned}
$$

The Lagrangian of this average household is written as :

$$
\mathcal{L} = \ln(C) - \frac{L}{H}\Omega_1 + \Omega_0 - \varphi\left[C - wL - I\right]
$$

The first order condition with respect to consumption is:

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial C} \\
0 &= \frac{1}{C} - \varphi \\
\frac{1}{C} &= \varphi
\end{aligned}
$$

The first order condition with respect to labor is:

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial L} \\
0 &= -\frac{1}{H}\Omega_1 + \varphi w \\
\frac{1}{H}\Omega_1 &= \varphi w
\end{aligned}
$$

Combining we get:

$$
\frac{1}{C} = \frac{1}{H}\Omega_1\frac{1}{w}
$$

which is not affected by L.

The average utility is linear in leisure:

$$
\begin{aligned}
U^{avg} &= \ln(C) - \frac{L}{H}\Omega_1 + \Omega_0 \\
U^{avg} &= \ln(C) + \frac{1-L}{H}\Omega_1 - \frac{1}{H}\Omega_1 + \Omega_0 \\
U^{avg} &= \ln(C) + (1-L)\frac{\Omega_1}{H} - \frac{1}{H}\Omega_1 + \Omega_0 \\
U^{avg} &= \ln(C) + (1-L)\Omega_3 + \Omega_4
\end{aligned}
$$

The marginal utility of leisure is thus always Ω_3. It does *not* increase as the household works more (leisure is lower). People do not mind substantially increasing their labor supply when the wage goes up.

As the average person has a constant marginal utility of leisure, it is more willing to adjust the labor input in response to wages that a true person would be, as a true person faces a rapidly changing marginal utility.

The model where the adjustment of hours takes place through fewer or more people working H hours each (adjustment of labor through the extensive margin) generates more labor volatility than a realistically calibrated model when hours adjust through the number of hours worked by each person (adjustment through the intensive margin). The model can thus generate more volatility of labor. In addition, labor empirically adjust though the number of employed people, not through the number of hours worked by each employed person.