Midterm Exam

Yuan Zi

EI037 Microeconomics

1. Weak Axiom of Revealed Preference and the Compensated Law of Demand

The choice structure $(\mathcal{B}, C(\cdot))$ satisfies the **weak axiom of revealed preference (WA)**, iff the following statement holds:

> If for some $B \in \mathcal{B}$ with $x, y \in B$ we have $x \in C(B)$, then for any $B' \in \mathcal{B}$ with $x, y \in B'$ and $y \in C(B')$, we must also have $x \in C(B')$.

1.a. Can you explain the definition of the WA in words?

1.b. Can you explain the definition of WA in graphs? Can you give give one graphical example when WA is violated, and one example when WA is not violated?

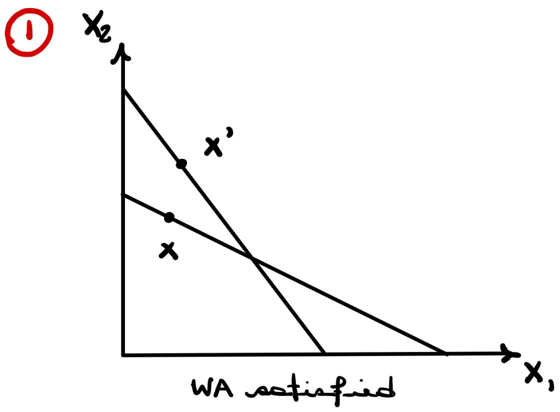
1.c. Show that the WA is equivalent to the following property holding:

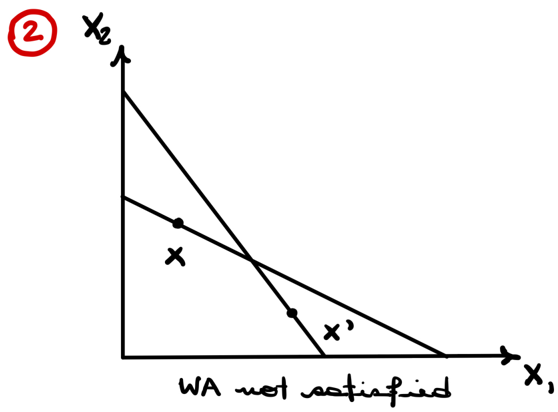
> Suppose that $B, B' \in \mathcal{B}$, that $x, y \in B$ and $x, y \in B'$. Then if $x \in C(B)$, $y \in C(B')$, we must have that $\{x, y\} \subset C(B)$ and $\{x, y\} \subset C(B')$.

Answer:

a. The weak axiom of revealed preference (WA) states that if a consumption bundle x is ever chosen when another consumption bundle y is available, then there can be no budget set containing both alternatives for which bundle y is chosen and bundle x is not.

b. Graphs 1 and 2 below represent situations in which the WA is satisfied and not satisfied, respectively. For clarity, in both graphs the bundle x represents the individual's observed consumption choice under a price-wealth situation (p, w), while the bundle x' represents the individual's observed consumption choice under a price-wealth situation (p', w').





Graph 1: in price-wealth situation (p', w'), the consumer chooses bundle x', although bundle x was also affordable to them. Conversely, in price-wealth situation (p, w), the consumer could

not have afforded bundle x', and chooses bundle x instead - that is, bundle x' is outside of the consumer's budget set for (p, w). The WA is therefore satisfied.

Graph 2: in price-wealth situation (p, w), the consumer could have chosen either bundle x or x', as both were affordable to them. But the consumer chosen x, which implies x is revealed preferable compared to x'. In price-wealth situation (p', w'), both consumption bundles x or x' were affordable, but the consumer chosen x' and not x. The WA is therefore violated.
c. The property states that the bundles x and y are both available for a consumer under two different budget sets, B and B'. If under budget set B with $\{x, y\} \subset B$, the consumer chooses the bundle x, i.e., $x \in C(B)$.

Then, when facing another budget set $B' = \{x, y\}$ in which both bundles x and y are also available for the consumer, if the consumer were to choose y but not x, their choice structure $C(B')$ would contradict the preferences observed in $C(B)$. Given $y \in C(B')$, it must therefore be that both x and y are chosen by the consumer under budget set B', i.e., $\{x, y\} \subset C(B')$.

Similarly, we can prove that it must be that both x and y are chosen by the consumer under budget set B, i.e., $\{x, y\} \subset C(B)$. Hence we concludes the proof.

2. Classic Demand Theory

Consider the utility function

$$u = 2x_1^{\frac{1}{2}} + 4x_2^{\frac{1}{2}}.$$

2.a. Find the demand functions for goods 1 and 2 as a function of prices and wealth.

2.b. Find the Hicksian (compensated) demand function $h(p, u)$.

2.c. Find the expenditure function, and verify that $h(p, u) = \nabla_p e(p, u)$.

2.d. Find the indirect utility function, and verify Roy's identity.

Answer:

a. We solve for the consumer's utility maximization problem (UMP)

$$\max_{x_1,x_2\geq 0} \quad 2x_1^{\frac{1}{2}}+4x_2^{\frac{1}{2}} \quad s.t.\ p_1x_1+p_2x_2\leq w$$

The consumer's Lagrangian is set up as

$$\mathcal{L}=2x_1^{\frac{1}{2}}+4x_2^{\frac{1}{2}}+\lambda(w-p_1x_1-p_2x_2)$$

Take the first order derivative w.r.t x_1, x_2,respectively:

$$\begin{matrix}\frac{\partial\mathcal{L}}{\partial x_1}=\frac{1}{x_1^{\frac{1}{2}}}-\lambda p_1=0\\ \frac{\partial\mathcal{L}}{\partial x_2}=\frac{2}{x_2^{\frac{1}{2}}}-\lambda p_2=0\end{matrix} \quad\Rightarrow\quad \frac{p_1}{p_2}=\frac{x_2^{\frac{1}{2}}}{2x_1^{\frac{1}{2}}} \quad\Rightarrow\quad \frac{p_1^2}{p_2^2}=\frac{x_2}{4x_1} \quad\Rightarrow\quad x_2=\frac{4p_1^2x_1}{p_2^2}$$

Substituting x_2 into the consumer's budget constraint, we have

$$p_1x_1+p_2\frac{4p_1^2x_1}{p_2^2}=w \quad\Rightarrow\quad p_1p_2x_1+4p_1^2x_1=p_2w \quad\Rightarrow\quad x_1^*(p,w)=\frac{p_2w}{p_1p_2+4p_1^2}$$

$$and\ \ x_2^*(p,w)=\frac{4p_1w}{4p_1p_2+p_2^2}$$

b. We solve for the consumer's expenditure minimization problem (EMP)

$$\min_{x_1,x_2\geq 0} \quad p_1x_1+p_2x_2 \quad s.t. \quad 2x_1^{\frac{1}{2}}+4x_2^{\frac{1}{2}}\geq\bar{u}$$

The consumer's Lagrangian is set up as

$$\mathcal{L}=p_1x_1+p_2x_2-\lambda(2x_1^{\frac{1}{2}}+4x_2^{\frac{1}{2}}-\bar{u})$$

Take the first order derivative w.r.t x_1, x_2,respectively:

$$\begin{matrix}\frac{\partial\mathcal{L}}{\partial x_1}=p_1-\lambda\frac{1}{x_1^{\frac{1}{2}}}=0\\ \frac{\partial\mathcal{L}}{\partial x_2}=p_2-\lambda\frac{2}{x_2^{\frac{1}{2}}}=0\end{matrix} \quad\Rightarrow\quad \frac{p_1}{p_2}=\frac{x_2^{\frac{1}{2}}}{2x_1^{\frac{1}{2}}} \quad\Rightarrow\quad x_2^{\frac{1}{2}}=\frac{2p_1x_1^{\frac{1}{2}}}{p_2}$$

Substituting $x_2^{\frac{1}{2}}$ into the consumer's utility constraint, we have

$$2x_1^{\frac{1}{2}}+4\frac{2p_1x_1^{\frac{1}{2}}}{p_2}=\bar{u} \quad\Rightarrow\quad 2p_2x_1^{\frac{1}{2}}+8p_1x_1^{\frac{1}{2}}=p_2\bar{u} \quad\Rightarrow\quad h_1^*(p,u)=\left(\frac{\bar{u}}{2}\frac{p_2}{4p_1+p_2}\right)^2$$

$$and\ \ h_2^*(p,u)=\left(\bar{u}\frac{p_1}{4p_1+p_2}\right)^2$$

c. The consumer's expenditure function, $e(p,u)$ is given by

$$\begin{aligned}
e(p,u) &= p_1 h_1(p,u) + p_2 h_2(p,u) \\
&= p_1 \left(\frac{\bar{u}}{2}\frac{p_2}{4p_1+p_2}\right)^2 + p_2\left(\bar{u}\frac{p_1}{4p_1+p_2}\right)^2 = \left(\frac{\bar{u}}{2}\right)^2 \frac{p_1p_2^2}{(4p_1+p_2)^2} + \bar{u}^2 \frac{p_1^2p_2}{(4p_1+p_2)^2} \\
&= \frac{\bar{u}^2}{(4p_1+p_2)^2}\left[\frac{p_1p_2^2}{4} + p_1^2p_2\right] = \left(\frac{\bar{u}}{2}\right)^2\left[\frac{p_1p_2^2+4p_1^2p_2}{(4p_1+p_2)^2}\right] = \left(\frac{\bar{u}}{2}\right)^2\left[\frac{p_1p_2(4p_1+p_2)}{(4p_1+p_2)^2}\right] \\
e(p,u) &= \left(\frac{\bar{u}}{2}\right)^2\left[\frac{p_1p_2}{4p_1+p_2}\right] = \frac{\bar{u}^2p_1p_2}{4(4p_1+p_2)}
\end{aligned}$$

We are asked to verify that $h(p,u) = \nabla_p e(p,u)$.

$$\begin{aligned}
\frac{\partial e(p,u)}{\partial p_1} &= \frac{\bar{u}^2p_2\times 4(4p_1+p_2) - \bar{u}^2p_1p_2\times 16}{16(4p_1+p_2)^2} \\
&= \frac{16\bar{u}^2p_1p_2 + 4\bar{u}^2p_2^2 - 16\bar{u}^2p_1p_2}{16(4p_1+p_2)^2} = \frac{4\bar{u}^2p_2^2}{16(4p_1+p_2)^2} \\
&= \frac{\bar{u}^2p_2^2}{4(4p_1+p_2)^2} = \left(\frac{\bar{u}}{2}\frac{p_2}{4p_1+p_2}\right)^2 = h_1^*(p,u)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial e(p,u)}{\partial p_2} &= \frac{\bar{u}^2p_1\times 4(4p_1+p_2) - \bar{u}^2p_1p_2\times 4}{16(4p_1+p_2)^2} \\
&= \frac{16\bar{u}^2p_1^2 + 4\bar{u}^2p_1p_2 - 4\bar{u}^2p_1p_2}{16(4p_1+p_2)^2} = \frac{16\bar{u}^2p_1^2}{16(4p_1+p_2)^2} \\
&= \frac{\bar{u}^2p_1^2}{(4p_1+p_2)^2} = \left(\bar{u}\frac{p_1}{4p_1+p_2}\right)^2 = h_2^*(p,u)
\end{aligned}$$

d. By the properties of the dual problem, we invert the consumer's expenditure function $e(p,u)$ we found in item (c) to obtain the consumer's indirect utility function $v(p,w)$

$$\begin{aligned}
e(p,u) &= \frac{\bar{u}^2p_1p_2}{4(4p_1+p_2)} = w \\
\bar{u}^2 &= \frac{4(4p_1+p_2)}{p_1p_2}w \\
\bar{u} &= 2\left[\frac{(4p_1+p_2)w}{p_1p_2}\right]^{\frac{1}{2}} = 2\left(\frac{4p_1w+p_2w}{p_1p_2}\right)^{\frac{1}{2}} \\
v(p,w) &= 2\left(\frac{w}{p_1}+\frac{4w}{p_2}\right)^{\frac{1}{2}}
\end{aligned}$$

To verify Roy's identity, we first derive each of it's elements separately

$$\frac{\partial v(p,w)}{\partial p_1} = 2\times\frac{1}{2}\left(\frac{w}{p_1}+\frac{4w}{p_2}\right)^{-\frac{1}{2}}\left(-\frac{w}{p_1^2}\right) = -\left(\frac{w}{p_1}+\frac{4w}{p_2}\right)^{-\frac{1}{2}}\frac{w}{p_1^2}$$

$$\frac{\partial v(p,w)}{\partial p_2} = 2 \times \frac{1}{2}\left(\frac{w}{p_1} + \frac{4w}{p_2}\right)^{-\frac{1}{2}}\left(-\frac{4w}{p_2^2}\right) = -\left(\frac{w}{p_1} + \frac{4w}{p_2}\right)^{-\frac{1}{2}}\frac{4w}{p_2^2}$$

$$\frac{\partial v(p,w)}{\partial w} = 2 \times \frac{1}{2}\left(\frac{w}{p_1} + \frac{4w}{p_2}\right)^{-\frac{1}{2}}\left(\frac{1}{p_1} + \frac{4}{p_2}\right) = \left(\frac{w}{p_1} + \frac{4w}{p_2}\right)^{-\frac{1}{2}}\left(\frac{1}{p_1} + \frac{4}{p_2}\right)$$

Then

$$\begin{aligned}
-\frac{\frac{\partial v(p,w)}{\partial p_1}}{\frac{\partial v(p,w)}{\partial w}} &= -\frac{-\left(\frac{w}{p_1} + \frac{4w}{p_2}\right)^{-\frac{1}{2}}\frac{w}{p_1^2}}{\left(\frac{w}{p_1} + \frac{4w}{p_2}\right)^{-\frac{1}{2}}\left(\frac{1}{p_1} + \frac{4}{p_2}\right)} = \frac{\frac{w}{p_1^2}}{\frac{p_2+4p_1}{p_1p_2}} \\
&= \frac{p_2w}{p_1p_2 + 4p_1^2} = x_1^*(p,w)
\end{aligned}$$

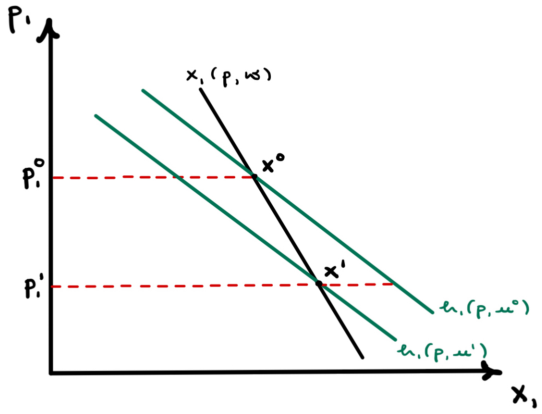
$$\begin{aligned}
-\frac{\frac{\partial v(p,w)}{\partial p_2}}{\frac{\partial v(p,w)}{\partial w}} &= -\frac{-\left(\frac{w}{p_1} + \frac{4w}{p_2}\right)^{-\frac{1}{2}}\frac{4w}{p_2^2}}{\left(\frac{w}{p_1} + \frac{4w}{p_2}\right)^{-\frac{1}{2}}\left(\frac{1}{p_1} + \frac{4}{p_2}\right)} = \frac{\frac{4w}{p_2^2}}{\frac{p_2+4p_1}{p_1p_2}} \\
&= \frac{4p_1w}{4p_1p_2 + p_2^2} = x_2^*(p,w)
\end{aligned}$$

3. Welfare Economics

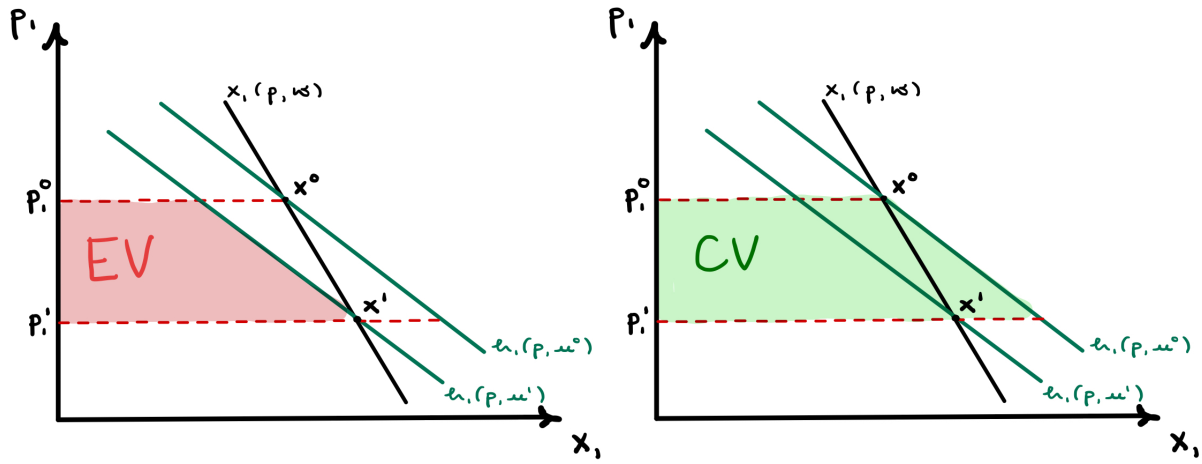
a. Consider a price change from the initial price vector p^0 to a new price vector $p^1 \leq p^0$ in which only the price of good l changes. Show that $CV \geq EV$ if good l is inferior.

b. Patrick's utility function is $u(x_1, x_2) = x_1 \cdot x_2$, where good 1 is food and good 2 is housing. Patrick gets a monthly salary of \$3000. The price of good 1 and the price of good 2 are $p_1 = p_2 = 1$. Patrick's boss is thinking of sending him to another town where the price of food is the same, but the price of housing is 2.25. The boss offers no raise in pay. Patrick, who understands compensating and equivalent variation perfectly, complains bitterly. He says that although he doesn't mind moving for its own sake and the new town is just as pleasant as the old, having to move is as bad as a cut in pay of \$A. He also says he wouldn't mind moving if - when he moved - he got a raise of \$B. What are A and B equal to?

Answer:
a. We observe a change in prices from p^0 to p^1 ($p^0 \geq p^1$), where only the price of good l changes in the price vector. If l is an inferior good, we know that the slope of the consumer's Hicksian demand function for good l will be flatter than that of the Marshallian demand for the same good. Graphically, we then have



We know that the equivalent variation (EV) can be calculated as the area under the Hicksian demand function for the "new" utility level, $h_l(p,u^1)$, while the compensating variation (CV) will be the area under the Hicksian demand function for the "old" utility, $h_l(p,u^0)$. Graphically, we have



We know that when prices decrease, both EV and CV are positive. From the figure above, we can therefore conclude that $CV \geq EV$ for the inferior good l.

b. \$A represents Patrick's equivalent variation (EV), while \$B represents their compensating variation (CV). We know that EV and CV are given by

$$
\begin{aligned}
EV &= e(p^0,u^1) - e(p^0,u^0) \quad or \quad v(p^0, w+EV) = v(p^1,w) = u^1 \\
CV &= e(p^1,u^1) - e(p^1,u^0) \quad or \quad v(p^0, w) = v(p^1,w-CV) = u^0
\end{aligned}
$$

We begin by solving Patrick's utility maximization problem, in order to obtain their optimal demand of each good. We have

$$\max_{x_1,x_2 \geq 0} \quad x_1 x_2 \quad s.t. \; p_1x_1 + p_2x_2 \leq w$$

The problem's Lagrangian is therefore

$$\mathcal{L} = x_1x_2 - \lambda(p_1x_1 + p_2x_2 - w)$$

Optimizing the Lagrangian with respect to goods 1 and 2,

$$\frac{\partial \mathcal{L}}{\partial x_1} = x_2 - \lambda p_1 = 0$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = x_1 - \lambda p_2 = 0$$

Since the consumer's utility function is of the Cobb-Douglas form ($x_1^a x_2^b$ with $a = b = 1$), we know that Patrick's optimal demand for goods 1 and 2 will be, respectively

$$x_1^* = \frac{w}{2p_1} \quad \text{and } x_2^* = \frac{w}{2p_2}$$

We can derive Patrick's indirect utility function, $v(p, w)$, from their optimal demand functions

$$v(p, w) = x_1^* x_2^* = \frac{w}{2p_1}\frac{w}{2p_2} = \frac{w^2}{4p_1p_2}$$

We are given that $w = 3000$, $p_1^0 = p_2^0 = 1$, $p_1^1 = 1$ and $p_2^1 = 2.25$. We can then calculate Patrick's EV and CV as follows

EV: $v(p^0, w + EV) = v(p^1, w) = u^1$

$$\frac{(w + EV)^2}{4p_1^0p_2^0} = \frac{w^2}{4p_1^1p_2^1} \quad \Rightarrow \quad \frac{(3000 + EV)^2}{4 \times 1 \times 1} = \frac{3000^2}{4 \times 1 \times 2.25} \quad \Rightarrow \quad \frac{3000 + EV}{2} = \frac{3000}{2 \times 1.5}$$
$$\Rightarrow \quad (3000 + EV) \times 1.5 = 3000 \quad \Rightarrow \quad EV = -1000$$

CV: $v(p^0, w) = v(p^1, w - CV) = u^0$

$$\frac{w^2}{4p_1^0p_2^0} = \frac{(w - CV)^2}{4p_1^1p_2^1} \quad \Rightarrow \quad \frac{3000^2}{4 \times 1 \times 1} = \frac{(3000 - CV)^2}{4 \times 1 \times 2.25} \quad \Rightarrow \quad \frac{3000}{2} = \frac{3000 - CV}{2 \times 1.5}$$
$$\Rightarrow \quad 3000 \times 1.5 = 3000 - CV \quad \Rightarrow \quad CV = -1500$$

Hence for Patrick, having to move is as bad as a cut in pay of \$1000. He wouldn't mind moving if he got a raise of \$1500. In other words, A equals to 1000 and B equals to 1500.