Lecture 7: Economic geography

Treb Allen and Costas Arkolakis

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1 Introduction

Up until now we have taken the labor supply as fixed. Today, we examine what happens when labor can move across different regions. Today's lecture is based upon Allen and Arkolakis (2014).

2 Model

2.1 Setup

For simplicity, let us begin using the Armington set-up. Let there be a compact set $i \in S$ of regions inhabited by a measure $\bar{L}$ of workers. Each location $i \in S$ produces a unique variety. Consumers have identical CES preferences, so that in equilibrium all inhabited locations will export to all other destinations.

Suppose that each location $i \in S$ has an *exogenous productivity* $\bar{A}_i \geq 0$ and an *exogenous amenity* $\bar{u}_i \geq 0$. Furthermore, between any two regions $i, j \in S$, suppose there exists iceberg trade costs $\tau_{ij} \geq 1$. We define the **geography** of S to be the set of functions $\bar{A} : S \to \mathbb{R}_{++}$, $\bar{u} : S \to \mathbb{R}_{++}$, and $\tau : S \times S \to [1, \infty)$. We say a geography is **regular** if $\bar{A}$, $\bar{u}$, and τ are continuous and bounded above and below by positive real numbers.

2.2 Agglomeration and dispersion forces

An important aspect in the study of the economic geography literature is how the distribution of labor affects the conditions of living in a particular location. (These forces are often called "second nature" forces, in contrast to the "first nature" forces that are exogenously given by the geography of S; see Cronon (1992)). In general, there are two types of forces: agglomeration forces, which all else equal cause people to want to live closer together, and dispersion forces, which all else equal cause people to want to live further apart. There has been a large literature providing micro-foundations for these forces (see e.g. Lucas and

Rossi-Hansberg (2003)). Today, however, we will abstract from the particular source of these forces and instead suppose that the *total productivity* of a location $i \in S$, A_i, is determined by a function of its exogenous productivity and the distribution of labor across space:

$$A_i = f_A\left(\bar{A}_i, \{L_j\}_{j \in S}\right), \tag{1}$$

where L_j is the (equilibrium) number of workers living in $j \in S$. Similarly, we suppose that the *total amenity* of a particular location $i \in S$, u_i, is determined by a function of its exogenous amenity and the distribution of labor across space:

$$u_i = f_u\left(\bar{u}_i, \{L_j\}_{j \in S}\right). \tag{2}$$

As we will see, one particularly attractive functional form for equations (1) and (2) is to assume that the total productivity (or amenity) of a location is a log linear function of its exogenous productivity (or amenity) and its own population:

$$A_i = \bar{A}_i L_i^{\alpha} \tag{3}$$

$$u_i = \bar{u}_i L_i^{\beta}, \tag{4}$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ govern the strength of the agglomeration and dispersion forces. In what follows, we refer to α and β as the strength of spillovers. Note, however, that equations (3) and (4) assume that only the local population affects productivities and amenities, i.e. the spillovers in the model are entirely local.

2.3 Trade & Welfare

As we have seen, in the Armington model, the value of bilateral trade from $i \in S$ to $j \in S$ can be written as:

$$X_{ij} = \tau_{ij}^{1-\sigma} A_i^{\sigma-1} w_i^{1-\sigma} P_j^{\sigma-1} Y_j, \tag{5}$$

where σ is the elasticity of substitution (assumed to be greater than one), w_i is the wage in $i \in S$, Y_j is the income in $j \in S$ and P_j is the Dixit-Stiglitz price index:

$$P_j \equiv \left(\sum_{k \in S} \tau_{kj}^{1-\sigma} A_k^{\sigma-1} w_k^{1-\sigma}\right)^{\frac{1}{1-\sigma}}. \tag{6}$$

Furthermore, if we suppose that the local amenity enters worker's utility multiplicatively, from CES preferences the welfare of a worker in location $j \in S$ is:

$$W_j = \left(\sum_{i \in S} \left(\frac{Q_{ij}}{L_j}\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} u_j, \tag{7}$$

where Q_{ij} is the quantity of the variety produced in $i \in S$ that is consumed in $j \in S$. As you have shown in your first problem set, given consumer optimization, the welfare can be written in the indirect utility form as a function of wages, the price index, and the amenity:

$$W_j = \frac{w_j}{P_j} u_j. \tag{8}$$

2.4 Equilibrium

For any geography, we define a **spatial equilibrium** to be the set of the wage function $w : S \to \mathbb{R}_+$ and population function $L : S \to \mathbb{R}_+$ such that:

1. The goods market clears, i.e. for all $i \in S$:

$$Y_i = \sum_{j \in S} X_{ij}. \tag{9}$$

2. Trade is balanced, i.e. for all $i \in S$:

$$Y_i = \sum_{j} X_{ji}. \tag{10}$$

3. The labor market clears, i.e. for all $i \in S$:

$$Y_i = w_i L_i. \tag{11}$$

4. Welfare equalized, i.e. there exists a $W > 0$ such that for all $i \in S$ such that $L_i > 0$, $W_i = W$ and for all $i \in S$ such that $L_i = 0$, $W_i \leq W$.

5. The population function is constrained by the total world population, i.e.:

$$\bar{L} = \sum_{i \in S} L_i. \tag{12}$$

We say that a spatial equilibrium is **regular** if the wage function $w : S \to \mathbb{R}_+$ and population function $L : S \to \mathbb{R}_+$ are continuous and bounded both above and below by strictly positive numbers.

As an aside, note that we have already (implicitly) assumed balanced trade when we wrote down the gravity equation and Dixit-Stiglitz price index above:

$$\begin{aligned}
P_j \equiv \left(\sum_{k \in S} \tau_{kj}^{1-\sigma} A_k^{\sigma-1} w_k^{1-\sigma} \right)^{\frac{1}{1-\sigma}} &\iff \\
1 = \sum_{k \in S} \tau_{kj}^{1-\sigma} A_k^{\sigma-1} w_k^{1-\sigma} P_j^{\sigma-1} &\iff \\
Y_j = \sum_{k \in S} \tau_{kj}^{1-\sigma} A_k^{\sigma-1} w_k^{1-\sigma} P_j^{\sigma-1} Y_j &\iff \\
Y_j = \sum_{k \in S} X_{kj}.
\end{aligned}$$

3 Solving the equilibrium

3.1 Equilibrium without spillovers

Let us first consider the equilibrium of the model in the absence of spillovers, i.e. where $\alpha = \beta = 0$. In this case, we can treat the productivities and amenities as entirely exogenous. To begin, let us consider the goods market clearing condition (9). Substituting in the labor market clearing condition (11) and the gravity expression (5) yields:

$$\begin{aligned}
Y_i &= \sum_{j \in S} X_{ij} \iff \\
w_i L_i &= \sum_{j \in S} X_{ij} \iff \\
w_i L_i &= \sum_{j \in S} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} w_i^{1-\sigma} P_j^{\sigma-1} Y_j \iff \\
w_i L_i &= \sum_{j \in S} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} w_i^{1-\sigma} P_j^{\sigma-1} w_j L_j \iff \\
w_i^{\sigma} L_i &= \sum_{j \in S} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} P_j^{\sigma-1} w_j L_j \qquad (13)
\end{aligned}$$

From the indirect utility function (8) $W_j = \frac{w_j}{P_j} u_j$, we have $P_j^{\sigma-1} = W_j^{1-\sigma} w_j^{\sigma-1} u_j^{\sigma-1}$, so that equation (13) can be written as:

$$\begin{aligned}
w_i^{\sigma} L_i &= \sum_{j \in S} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} \left(W_j^{1-\sigma} w_j^{\sigma-1} u_j^{\sigma-1} \right) w_j L_j \iff \\
w_i^{\sigma} L_i &= \sum_{j \in S} W_j^{1-\sigma} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} u_j^{\sigma-1} w_j^{\sigma} L_j. \qquad (14)
\end{aligned}$$

By imposing welfare equalization in equation (14), we then get:

$$w_i^{\sigma} L_i = W^{1-\sigma} \sum_{j \in S} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} u_j^{\sigma-1} w_j^{\sigma} L_j. \qquad (15)$$

Equation (15) is one of the two sets of equilibrium equations.

The other set comes from welfare equalization directly. Now let us combine welfare equalization with the indirect utility function (8) using the functional form of the Dixit-Stiglitz price index from equation (6):

$$\begin{aligned}
W &= \frac{w_i}{P_i} u_i \iff \\
w_i^{1-\sigma} &= W^{1-\sigma} u_i^{1-\sigma} P_i^{1-\sigma} \iff \\
w_i^{1-\sigma} &= W^{1-\sigma} \sum_{j \in S} \tau_{ji}^{1-\sigma} u_i^{1-\sigma} A_j^{\sigma-1} w_j^{1-\sigma} \qquad (16)
\end{aligned}$$

Since A and u are assumed to be exogenous, note that equations (15) and (16) can be written as eigenfunctions:

$$\mathbf{x} = \lambda \mathbf{A} \mathbf{x} \tag{17}$$

$$\mathbf{y} = \lambda \mathbf{A}^T \mathbf{y}, \tag{18}$$

where $\mathbf{x}_i \equiv w_i^\sigma L_i$, $\mathbf{y}_i \equiv w_i^{1-\sigma}$, $\mathbf{A}_{ij} \equiv \tau_{ij}^{1-\sigma} A_i^{\sigma-1} u_j^{\sigma-1}$, and $\lambda \equiv W^{1-\sigma}$. Hence, if we find an $\mathbf{x}$ and $\mathbf{y}$ that solve equations (17) and (18), we will have found a spatial equilibrium. If the geography is regular, we know that $\mathbf{A}_{ij} > 0$ for all $i \in S$ and $j \in S$. Hence from the Perron-Frobenius theorem, there exists a unique (to-scale) strictly positive vector $\mathbf{x}$ that solves equation (17) and a unique (to-scale)[1] strictly positive vector $\mathbf{y}$ that solves equation (18), both corresponding to the largest eigenvalue of the system λ. Furthermore, because the kernels are transposes of each other, the eigenvalues of the two systems are identical[2], so that the largest eigenvalue of both systems is the same. This proves the following theorem:

Theorem 1. *For any regular geography when $\alpha = \beta = 0$, there exists a unique spatial equilibrium. Furthermore, that spatial equilibrium is regular.*

3.2 Equilibrium with spillovers

When $\alpha \neq 0$ or $\beta \neq 0$ (i.e. when there are spillovers), we can no longer treat the total productivity and/or amenity in a location as exogenous. Substituting equations (3) and (4) governing the relationship between total productivity and local population into the equilibrium conditions (15) and (16) yields:

$$\begin{aligned}
w_i^\sigma L_i &= W^{1-\sigma} \sum_{j \in S} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} u_j^{\sigma-1} w_j^\sigma L_j \iff \\
w_i^\sigma L_i &= W^{1-\sigma} \sum_{j \in S} \tau_{ij}^{1-\sigma} \left(\bar{A}_i L_i^\alpha\right)^{\sigma-1} \left(\bar{u}_j L_j^\beta\right)^{\sigma-1} w_j^\sigma L_j \iff \\
w_i^\sigma L_i^{1-\alpha(\sigma-1)} &= W^{1-\sigma} \sum_{j \in S} \tau_{ij}^{1-\sigma} \bar{A}_i^{\sigma-1} \bar{u}_j^{\sigma-1} w_j^\sigma L_j^{1+\beta(\sigma-1)}.
\end{aligned} \tag{19}$$

and:

$$\begin{aligned}
w_i^{1-\sigma} &= W^{1-\sigma} \sum_{j \in S} \tau_{ji}^{1-\sigma} u_i^{1-\sigma} A_j^{\sigma-1} w_j^{1-\sigma} \iff \\
w_i^{1-\sigma} &= W^{1-\sigma} \sum_{j \in S} \tau_{ji}^{1-\sigma} \left(\bar{u}_i L_i^\beta\right)^{1-\sigma} \left(\bar{A}_j L_j^\alpha\right)^{\sigma-1} w_j^{1-\sigma} \iff \\
w_i^{1-\sigma} L_i^{\beta(\sigma-1)} &= W^{1-\sigma} \sum_{j \in S} \tau_{ji}^{1-\sigma} \bar{u}_i^{1-\sigma} \bar{A}_j^{\sigma-1} w_j^{1-\sigma} L_j^{\alpha(\sigma-1)}
\end{aligned} \tag{20}$$

[1] Note that w_i has an arbitrary normalization and the scale of L_i is pinned down by the total world population $\bar{L}$.

[2] Suppose that $\lambda \mathbf{x} = \mathbf{A}\mathbf{x} \iff (\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$. Note that $\det(\mathbf{A} - \lambda_1 \mathbf{I}) = \det(\mathbf{A} - \lambda_1 \mathbf{I})^T = \det\left(\mathbf{A}^T - \lambda_1 \mathbf{I}\right)$, so that $\det(\mathbf{A} - \lambda_1 \mathbf{I}) = 0 \iff \det\left(\mathbf{A}^T - \lambda_1 \mathbf{I}\right) = 0$. Hence, λ_1 is an eigenvalue for the matrix $\mathbf{A}^T$.

Equations (19) and (20) are inter-related non-linear sets of equation that bear a strong resemblance to the non-linear equations we saw for gravity trade models, except that there is an unknown characteristic value $W^{1-\sigma}$. It turns out that the proof of existence and uniqueness from Allen, Arkolakis, and Takahashi (2014) can be extended to include such a characteristic value. It is a little bit annoying, but one can write the α and β governing spillovers in the universal gravity framework with α^U and β^U such that $Y_i = B_i \gamma_i^{\alpha^U} \delta_i^{\beta^U}$:

$$\alpha^U = \frac{1-\beta}{1+\beta(\sigma-1)+\sigma\alpha}$$
$$\beta^U = \frac{1+\alpha}{1+\beta(\sigma-1)+\sigma\alpha}$$

This lets us prove the following result:

Theorem 1. *Consider any regular geography. Then:*

(i) as long as $\alpha^U + \beta^U = 1$ (or equivalently, $1 \neq \beta\sigma + (\sigma-1)\alpha$), a regular spatial equilibrium exists.

(ii) The regular equilibrium is unique if either $\alpha^U, \beta^U \leq 0$ or $\alpha^U, \beta^U \geq 1$ (in particular, $\alpha + \beta \leq 0$ and $\alpha, \beta \in (-1, 1)$ ensure this condition holds).

Proof. See Corollary 2 of Allen, Arkolakis, and Takahashi (2014). □

Hence, we get a unique equilibrium if the dispersion forces are at least as strong as the agglomeration forces; intuitively, when all else equal people would prefer to not live near others, there is only one way that the population can be distributed across space.

4 Quasi-symmetry

If we assume trade costs are quasi-symmetric, we can push the equilibrium a little further. Suppose that trade costs are quasi-symmetric, i.e. for all $i \in S$ and $j \in S$:

$$\tau_{ij} = \tilde{\tau}_{ij} \tau_i^A \tau_j^B,$$

where $\tilde{\tau}_{ij} = \tilde{\tau}_{ji}$. From the previous lecture (and proven in Allen, Arkolakis, and Takahashi (2014)) we then know that because we have a gravity equation where in equilibrium the goods market clears, trade is balanced, and the labor market clears, then if trade costs are quasi-symmetric, we have for all $i \in S$:

$$\tau_i^A \gamma_i = \kappa \tau_i^B \delta_i, \tag{21}$$

where $\gamma_i = w_i^{1-\sigma} A_i^{\sigma-1}$ is the origin fixed effect, $\delta_i = P_i^{\sigma-1} w_i L_i$ is the destination fixed effect, and $\kappa > 0$ is a scalar. Equation (21), along with welfare equalization, allows us to write the equilibrium wages w_i as a function of the local labor supply:

$$\begin{aligned}
\tau_i^A \gamma_i = \kappa \tau_i^B \delta_i &\iff \\
\tau_i^A \left(w_i^{1-\sigma} A_i^{\sigma-1}\right) = \kappa \tau_i^B \left(P_i^{\sigma-1} w_i L_i\right) &\iff \\
\tau_i^A \left(w_i^{1-\sigma} A_i^{\sigma-1}\right) = \kappa \tau_i^B \left(\left(w_i^{\sigma-1} u_i^{\sigma-1} W^{1-\sigma}\right) w_i L_i\right) &\iff \\
w_i = \left(\kappa \frac{\tau_i^B}{\tau_i^A} W^{1-\sigma} A_i^{1-\sigma} u_i^{\sigma-1} L_i\right)^{\frac{1}{1-2\sigma}}&.
\end{aligned} \tag{22}$$

Substituting equation (22) into the equilibrium condition (15) then yields a non-linear equation where only the local population is unknown:

$$\begin{aligned}
w_i^{\sigma} L_i &= W^{1-\sigma} \sum_{j\in S} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} u_j^{\sigma-1} w_j^{\sigma} L_j \iff \\
\left(\kappa \frac{\tau_i^B}{\tau_i^A} W^{1-\sigma} A_i^{1-\sigma} u_i^{\sigma-1} L_i\right)^{\frac{\sigma}{1-2\sigma}} L_i &= W^{1-\sigma} \sum_{j\in S} \tau_{ij}^{1-\sigma} A_i^{\sigma-1} u_j^{\sigma-1} \left(\kappa \frac{\tau_j^B}{\tau_j^A} W^{1-\sigma} A_j^{1-\sigma} u_j^{\sigma-1} L_j\right)^{\frac{\sigma}{1-2\sigma}} L_j \iff \\
L_i^{\tilde{\sigma}} &= W^{1-\sigma} \sum_{j\in S} \tau_{ij}^{1-\sigma} \left(\frac{\tau_i^B \tau_j^A}{\tau_i^A \tau_j^B}\right)^{\frac{\sigma}{2\sigma-1}} A_j^{\sigma\tilde{\sigma}} A_i^{(\sigma-1)\tilde{\sigma}} u_j^{(\sigma-1)\tilde{\sigma}} u_i^{\sigma\tilde{\sigma}} L_j^{\tilde{\sigma}} \iff \\
L_i^{\tilde{\sigma}\gamma_1} &= W^{1-\sigma} \sum_{j\in S} \tau_{ij}^{1-\sigma} \left(\frac{\tau_i^B \tau_j^A}{\tau_i^A \tau_j^B}\right)^{\frac{\sigma}{2\sigma-1}} \bar{A}_j^{\sigma\tilde{\sigma}} \bar{u}_j^{(1-\sigma)\tilde{\sigma}} \bar{A}_i^{(\sigma-1)\tilde{\sigma}} \bar{u}_i^{\sigma\tilde{\sigma}} L_j^{\tilde{\sigma}\gamma_2}
\end{aligned} \tag{23}$$

where $\tilde{\sigma} \equiv \frac{\sigma-1}{2\sigma-1}$, $\gamma_1 \equiv 1 - \alpha(\sigma-1) - \beta\sigma$, and $\gamma_2 \equiv 1 + \alpha\sigma + \beta(\sigma-1)$. Equation (23) can be written as:

$$x_i = \lambda \sum_{j\in S} F_{ij} x_j^{\frac{\gamma_2}{\gamma_1}}, \tag{24}$$

where $x_i \equiv L_i^{\tilde{\sigma}\gamma_1}$, $\lambda \equiv W^{1-\sigma}$, and $F_{ij} \equiv \tau_{ij}^{1-\sigma} \left(\frac{\tau_i^B \tau_j^A}{\tau_i^A \tau_j^B}\right)^{\frac{\sigma}{2\sigma-1}} \bar{A}_j^{\sigma\tilde{\sigma}} \bar{u}_j^{(1-\sigma)\tilde{\sigma}} \bar{A}_i^{(\sigma-1)\tilde{\sigma}} \bar{u}_i^{\sigma\tilde{\sigma}}$. If the geography is regular, F_{ij} will be continuous and bounded above and below by strictly positive numbers. As a result, we know from two classes ago that there will exist a continuous and strictly positive solution to equation (24). Furthermore, that solution will be unique (to-scale) if $\frac{\gamma_2}{\gamma_1} \in [-1, 1]$. (The scale, however, is pinned down by the world population $\bar{L}$). Hence, we have proved the following theorem:

Theorem 2. *For any regular geography with quasi-symmetric trade costs, there exists a regular spatial equilibrium. Furthermore, if* $\frac{\gamma_2}{\gamma_1} \in [-1, 1]$*, that equilibrium is unique.*

With a little bit of algebra, you can show that $\frac{\gamma_2}{\gamma_1} \in [-1, 1] \iff \alpha + \beta \leq 0$. Hence, we get a unique equilibrium if the dispersion forces are at least as strong as the agglomeration forces; intuitively, when all else equal people would prefer to not live near others, there is only one way that the population can be distributed across space.

4.1 The topography of the spatial economy

In addition to existence and uniqueness, we can say something concrete about how the geography of the world shapes the equilibrium distribution of where people live. Substituting in equation (22) (which recall came from the fact that the origin and destination fixed effects are equal up to scale) into utility equalization yields:

$$
\begin{aligned}
\frac{w_i}{P_i} u_i = W &\iff \\
w_i = \frac{W}{u_i} P_i &\iff \\
\left(\kappa \frac{\tau_i^B}{\tau_i^A} W^{1-\sigma} A_i^{1-\sigma} u_i^{\sigma-1} L_i \right)^{\frac{1}{1-2\sigma}} = \frac{W}{u_i} P_i &\iff \\
L_i = \frac{1}{\kappa} W^{\sigma} \frac{\tau_i^A}{\tau_i^B} u_i^{\sigma} A_i^{\sigma-1} P_i^{1-2\sigma} &\iff \\
L_i^{\gamma_1} = \frac{1}{\kappa} \frac{\tau_i^A}{\tau_i^B} W^{\sigma} \bar{u}_i^{\sigma} \bar{A}_i^{\sigma-1} P_i^{1-2\sigma} &\iff \\
\gamma_1 \ln L_i = C + \ln \left(\frac{\tau_i^A}{\tau_i^B} \right) + \sigma \ln \bar{u}_i + (\sigma - 1) \ln \bar{A}_i - (2\sigma - 1) \ln P_i&,
\end{aligned} \tag{25}
$$

where $C \equiv \sigma \ln W - \ln \kappa$ is a constant that is pinned down by the world population $\bar{L}$. Equation (25) shows how the spatial distribution of population is influenced by the geography of the world. If $\gamma_1 > 0$ (as it will be for the "well-behaved" equilibria), then locations with higher amenities and higher productivities will have greater populations (all else equal). More remote locations (as captured by the price index), in contrast, will have lower populations. Finally, the only direct effect of the spillovers on the distribution of economic activity is to affect the elasticity of the distribution of the population to the underlying geography. Since $\gamma_1 \equiv 1 - \alpha(\sigma - 1) - \beta\sigma$, as either α or β get larger (i.e. as the agglomeration forces increase), the elasticity of the equilibrium population to the geography increases. Intuitively, the greater the agglomeration forces, smaller differences in the underlying productivities or amenities yield a greater response in the equilibrium distribution of population because the agglomeration forces provide a feedback loop. For example, "initially" people may choose to live in a particular location because it has a nice climate; if, however, this increases the total productivity of that location, even more people will choose to settle there.

4.2 Theory consistent regressions

Oftentimes, empiricists are interested in how shocks to a location fundamental will affect the income of a particular location. (For example, what is the effect of opening a new factory on income?). The traditional way of estimating such a shock is to regress income in location i on the observed shock S_i:

$$\ln Y_i = \tilde{\gamma} \ln S_i + C + \nu_i, \tag{26}$$

where $\tilde{\gamma}_i$ is a reduced form coefficient that (loosely) captures the elasticity of local income to the local shock S_i.

The discussion above lets us do something a little more model consistent. First, let us use equation (22) to derive a structural relationship between income in a location and its productivity:

$$\begin{aligned}
w_i^{1-2\sigma} &= \kappa \frac{\tau_i^B}{\tau_i^A} W^{1-\sigma} A_i^{1-\sigma} u_i^{\sigma-1} L_i \iff \\
w_i^{1-2\sigma} &= \kappa \frac{\tau_i^B}{\tau_i^A} W^{1-\sigma} \bar{A}_i^{1-\sigma} \bar{u}_i^{\sigma-1} L_i^{1+(\sigma-1)(\beta-\alpha)} \iff \\
w_i &= \kappa^{\frac{1}{1-2\sigma}} \left(\frac{\tau_i^A}{\tau_i^B}\right)^{\frac{1}{2\sigma-1}} W^{\frac{\sigma-1}{2\sigma-1}} \bar{A}_i^{\frac{\sigma-1}{2\sigma-1}} \bar{u}_i^{\frac{1-\sigma}{2\sigma-1}} L_i^{\frac{(\sigma-1)(\alpha-\beta)-1}{2\sigma-1}} \iff \\
w_i L_i &= \kappa^{\frac{1}{1-2\sigma}} W^{\frac{\sigma-1}{2\sigma-1}} \left(\frac{\tau_i^A}{\tau_i^B}\right)^{\frac{1}{2\sigma-1}} \bar{A}_i^{\frac{\sigma-1}{2\sigma-1}} \bar{u}_i^{\frac{1-\sigma}{2\sigma-1}} L_i^{\frac{\sigma-1}{2\sigma-1}(\alpha-\beta+2)} \iff \\
\ln Y_i &= C + \frac{\tilde{\sigma}}{\sigma-1} \ln \frac{\tau_i^A}{\tau_i^B} + \tilde{\sigma} \ln \bar{A}_i - \tilde{\sigma} \ln \bar{u}_i + \tilde{\sigma}\left(\alpha+\beta-2\right) \ln L_i,
\end{aligned} \tag{27}$$

where $C \equiv \frac{1}{1-2\sigma} \ln \kappa + \frac{\sigma-1}{2\sigma-1} \ln W$ and $\tilde{\sigma} \equiv \frac{\sigma-1}{2\sigma-1}$. Second, we can assume that shock S_i affects the productivity of location i, in the following way:

$$\ln \bar{A}_i = \gamma \ln S_i + \varepsilon_i \tag{28}$$

Then combining equation (27) and (28) yields the structural analog of the reduced form equation (26):

$$\ln Y_i = C + \tilde{\sigma}\gamma \ln S_i + \frac{\tilde{\sigma}}{\sigma-1} \ln \frac{\tau_i^A}{\tau_i^B} - \tilde{\sigma} \ln \bar{u}_i + \tilde{\sigma}\left(\alpha+\beta-2\right) \ln L_i + \tilde{\sigma}\varepsilon_i. \tag{29}$$

Equation (29) is helpful in evaluating both what the reduced form equation (26) is actually estimating and some potential pitfalls of such a reduced form estimation. First, comparing the coefficients on $\ln S_i$, we see immediately that $\tilde{\gamma} = \tilde{\sigma}\gamma$, i.e the reduced form coefficient combines both the direct productivity effect and $\tilde{\sigma}$. Note that $\tilde{\sigma} = 1$ only when $\sigma = 0$, [Class question: what is the intuition for this?] so that the reduced form regression does not usually recover the parameter of interest. Second, we can see that equation (26) suffers from an omitted variable problem, since:

$$\nu_i = \frac{\tilde{\sigma}}{\sigma-1} \ln \frac{\tau_i^A}{\tau_i^B} - \tilde{\sigma} \ln \bar{u}_i + \tilde{\sigma}\left(\alpha+\beta-2\right) \ln L_i + \tilde{\sigma}\varepsilon_i.$$

Hence, if the productivity shock $\ln S_i$ is correlated with any element of ν_i, then the estimation of $\tilde{\gamma}$ will be biased. While it perhaps is acceptable to assume that amenities and trade costs (i.e. $\frac{\tilde{\sigma}}{\sigma-1} \ln \frac{\tau_i^A}{\tau_i^B} - \tilde{\sigma} \ln \bar{u}_i$) are orthogonal to productivity shocks, from equation (25) the model tells us that $\ln L_i$ is positively correlated with $\ln \bar{A}_i$. If $\alpha+\beta-2 \leq 0$, this tells us that ν_i and S_i are negatively correlated, which implies that $\tilde{\gamma}$ will be biased downwards. Intuitively, a positive productivity shock will not increasing income as much because the dispersion forces will cause less than unity corresponding increase in the local labor supply.

You may think that the structural estimation suggests one just needs include the local population as a control variable. Unfortunately, this does not work either, as the labor supply is mechanically correlated with the error term. To see this, refer again to equation (25):

$$
\begin{aligned}
E\left[\varepsilon_i \times \ln L_i\right] &= E\left[\frac{1}{\gamma_1}\left(C + \ln\left(\frac{\tau_i^A}{\tau_i^B}\right) + \sigma \ln \bar{u}_i + (\sigma-1)\gamma \ln S_i - (2\sigma-1)\ln P_i + (\sigma-1)\varepsilon_i\right) \times \varepsilon_i\right] \iff \\
E\left[\varepsilon_i \times \ln L_i\right] &\frac{1}{\gamma_1} E\left[C\varepsilon_i + \ln\left(\frac{\tau_i^A}{\tau_i^B}\right)\varepsilon_i + \sigma \ln \bar{u}_i \varepsilon_i + \varepsilon_i \times (\sigma-1)\gamma \ln S_i - \varepsilon_i \times (2\sigma-1)\ln P_i + \varepsilon_i \times (\sigma-1)\varepsilon_i\right] \\
E\left[\varepsilon_i \times \ln L_i\right] &= \frac{\sigma-1}{\gamma_1}\sigma_\varepsilon^2 \neq 0.
\end{aligned}
$$

As a result, estimating equation (26) will necessarily be biased (without a suitable instrument for L_i).

4.3 The line

Let us consider a very simple example where the world is a finite interval, i.e. $S \equiv [-\pi, \pi]$ (why we make the interval a length of 2π will become apparent). Let us assume that $\alpha = \beta = 0$ and $A(i) = u(i) = 1$ for all $i \in S$, i.e. the line is homogeneous and there are no spillovers.[3] Finally, let us say that the trade cost between any two locations $i, j \in S$ is: $\tau(i,j) = \exp\{\tau(|i-s|)\}$, where $\tau > 0$. Then note that equation (25) implies that:

$$
\frac{\partial \ln L(i)}{\partial i} = -(2\sigma - 1)\frac{\partial \ln P(i)}{\partial i}. \tag{30}
$$

Because we are considering a line, it is apparent how moving along the line changes the price index; when $i = -\pi$, increasing i will reduce the price index (since moving to the right reduces the distance to all locations). However, as you move more and more to the right, the reduction in the price index becomes less and less (since you now are moving away from more and more locations). When $i = 0$, we have $\frac{\partial \ln P(0)}{\partial i} = 0$ since there are an equal number of locations to the left and to the right. For $i > 0$, moving to the right begins increasing the price index by more and more. From this intuition, we see that the population is increasing (at an increasingly slow rate) as you move left to right for $i < 0$ and decreasing (at an increasingly fast rate) as you move left to right for $i > 0$. Indeed, without going into the details, by differentiating equation (23) twice, we find that:

$$
\frac{\partial^2}{\partial i^2} L(i)^{\tilde{\sigma}} = k_1 L(i)^{\tilde{\sigma}},
$$

where $k_1 = (1-\sigma)^2 \tau^2 + 2(1-\sigma)\tau W^{1-\sigma}$. It turns out that the solution to this second order differential equation is that:

$$
L(i) = k_2 \cos\left(i\sqrt{k_1}\right),
$$

for a $k_2 > 0$ that depends on $\bar{L}$. Hence, in this (very simple case), we can actually derive a closed form solution!

[3] Because we are considering a continuum of locations, I will use continuous notation.

5 Next steps

We are rapidly concluding the theory portion of this course. Next class, we will discuss optimal policy. After that, we are on to empirics!

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