Intermediate Microeconomics

Competitive Equilibrium

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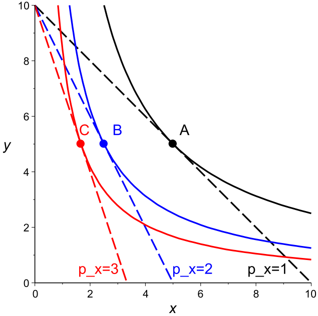
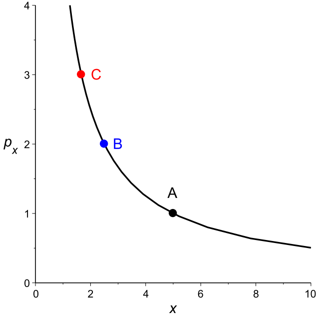
Overview

- Partial Equilibrium (局部均衡)
    - Aggregation: Demand and Supply
    - Equilibrium
    - Comparative Statics
- General Equilibrium (一般均衡)
    - Edgeworth Box (埃斯沃奇盒)
    - Pareto efficiency (帕累托效率)
    - Walrasian equilibrium (瓦尔拉斯均衡)
    - First theorem of welfare economics (福利经济学第一定理)

Individual Demand (个体需求函数) for One Good

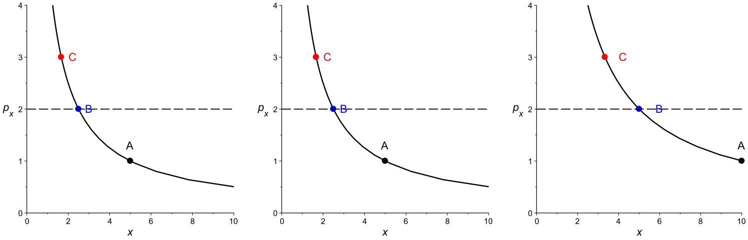
- Recall: the UMP gives the Marshallian demand $x^*(p_x, p_y, I)$. For example, $U = \sqrt{xy} \Rightarrow x^*(p_x) = \frac{I}{2p_x}$, we can plot the Marshallian demand as tangent points as p_x changes.
- Alternatively, for different p_x, we can also plot p_x as a function of x^* in another graph: demand curve.





Aggregated Demand (市场需求函数)

- Suppose that there are $i = 1, ..., n$ consumers. The demand of each consumer is $x_i^*(p_x, p_y, I_i)$. The market demand for good x, evaluated at price p_x, is $Q_D(p_x) = \sum_{i=1}^{n} x_i^*(p_x, p_y, I_i)$
- If we plot p_x in the vertical axis, then $p_x = Q_D^{-1}(\cdot)$ is the **inverse demand** (**反需求函数**).
- $Q_D(p_x)$ is the horizontally aggregation: e.g., there are two consumers: UMP of each is $x^* = \frac{I}{2p_x}$. The market demand is $Q_D = 2x^* = \frac{I}{p_x}$.



Short-Run Supply of A Price-Taking Firm

Given price p for good x, a price-taking firm chooses quantity q to maximize profit:

$$\max_q \underbrace{pq}_{\text{revenue}} - C(q) \Rightarrow \underbrace{p}_{\text{marginal revenue}} - \underbrace{C'(q)}_{\text{marginal cost}} = 0.$$

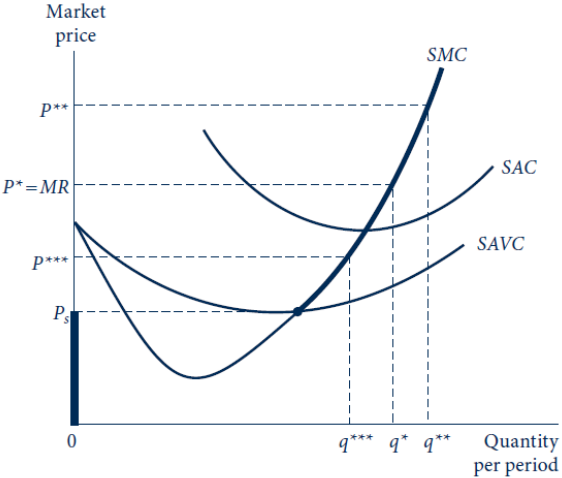
Therefore, the supply curve is obtained by $p = C'(q) = MC$. We can plot q in the horizontal axis and p in the vertical axis.

Remark: perfect-competitive market $\Leftrightarrow$ price-taking firms $\Leftrightarrow$ each firm cannot affect market price $\Leftrightarrow$ p is independent of *individually* chosen q.

- The second-order condition implies $\frac{\partial^2 \pi}{\partial q^2} = -C''(q) \leq 0 \Leftrightarrow -\frac{dMC}{dq} \leq 0 \Leftrightarrow \frac{dMC}{dq} \geq 0$.
- Non-negative profit: $\pi \geq 0 \Rightarrow pq - C \geq 0 \Rightarrow p \geq \frac{C}{q} = SAC$

$\Rightarrow$ In the short run, an individual firm is willing to supply at somewhere at $p = MC$, $\frac{dMC}{dq} > 0$ and $p > SAC$.

Short-Run Supply Curve

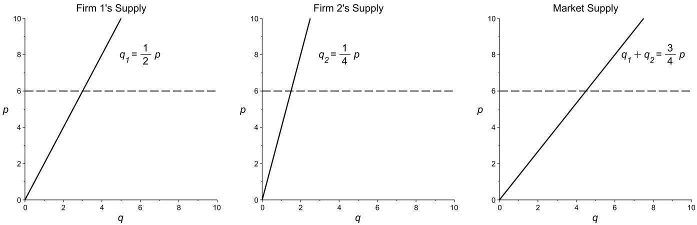


Aggregated Supply (市场供给函数)

- Assume that there are m firms. The aggregated (e.g., industry) supply is

$$Q_S = \sum_{i=1}^{m} q_i^*(p).$$

where each q_i^* should satisfy $p = MC_i$, $\pi \geq 0$ and $\frac{dMC_i}{dq} > 0$.



Example: Industry Supply

- Two firms with different cost functions: $C_1 = q_1^2$ and $C_2 = 2q_2^2$.
  - FOC of firm 1 is $p = 2q_1 \Rightarrow q_1(p) = \frac{p}{2}$, $pq_1 \geq q_1^2 \Rightarrow q_1 \leq p$. Hence individual supply $q_1(p) = \frac{p}{2} < p$.
  - FOC of firm 2 is $p = 4q_2 \Rightarrow q_2(p) = \frac{p}{4}$. $pq_2 - 2q_2^2 \geq 0 \Rightarrow q_2 \leq \frac{p}{2}$. Hence individual supply is $q_2(p) = \frac{p}{4} < \frac{p}{2}$.
  - Aggregation: $Q_S(p) = q_1(p) + q_2(p) = \frac{3}{4}p$.
- m firms with identical cost function: $C = q^2 + 1$.
  - FOC gives $p = 2q \Rightarrow q(p) = \frac{p}{2}$.
  - $AVC = q$, $pq \geq qAVC \Leftrightarrow pq \geq q^2 \Rightarrow p \geq q \Rightarrow q(p) = \frac{p}{2} < p$.
  - Aggregation: $Q_S(p) = mq(p) = \frac{mp}{2}$.
  - Note that $p = \frac{2Q_S}{m}$, which means that as the number of firms increases, the slope of supply curve becomes flatter.

Market Equilibrium (市场均衡)

Assume that there are n consumers and m firms. At equilibrium, total demand = total supply

$$Q_D(p) = \sum_{i=1}^{n} x_i^*(p) = \sum_{i=1}^{m} q_i^*(p) = Q_S(p)$$

Three equations: $Q_D = Q_D(p)$, $Q_S = Q_S(p)$ and $Q_D = Q_S$;
Three unknowns, Q_D, Q_S and p. At equilibrium, $Q = Q_S = Q_D$.
We can simply solve Q and p by equating $Q_S(p) = Q_D(p)$.

Comparative Statics: Numbers of Market Participants

For $Q_D(p) = nx^*(p)$ and $Q_S(p) = mq^*(p)$. At equilibrium: $nx^*(p^*) = mq^*(p^*)$, where the equilibrium price p^* as an intersection point, is a function of n and m.

- An increase in n (fixing m): $x^* + n\frac{\partial x^*}{\partial p}\frac{\partial p}{\partial n} = m\frac{\partial q^*}{\partial p}\frac{\partial p}{\partial n}$.
  $\frac{\partial p}{\partial n} = \frac{x^*}{mq^{*\prime}(p) - nx^{*\prime}(p)} > 0$. A higher price.
- An increase in m (fixing n):
  $nx^{*\prime}(p)\frac{\partial p}{\partial m} = q^* + mq^{*\prime}(p)\frac{\partial p}{\partial m} \Rightarrow \frac{\partial p}{\partial m} = \frac{q^*}{nx^{*\prime}(p) - mq^{*\prime}(p)} < 0$.
  A lower price.

Comparative Statics

- Demand curve shifts because
  - Incomes change I
  - Prices of the related good y (e.g., substitutes or complements): **NOT** the change of p_x along the demand curve.
  - Preferences change $U(\cdot)$
  - Number of buyers
- Supply Curves Shift Because
  - Input prices change r, w
  - Technology changes $f(\cdot)$
  - Number of producers changes

Let α (resp., β) be the parameter that shifts the demand (resp., supply) curve.

At equilibrium, $Q_D(p,\alpha) = Q_S(p,\beta)$

- Total differentiation: $\frac{\partial Q_D}{\partial p}dp + \frac{\partial Q_D}{\partial \alpha}d\alpha = \frac{\partial Q_S}{\partial p}dp + \frac{\partial Q_S}{\partial \beta}d\beta$
- Fixing the supply conditions, $d\beta = 0$: $\frac{dp}{d\alpha} = \frac{\frac{\partial Q_D}{\partial \alpha}}{\frac{\partial Q_S}{\partial p} - \frac{\partial Q_D}{\partial p}}$
- sign $\left(\frac{dp}{d\alpha}\right) =$ sign $\left(\frac{\partial Q_D}{\partial \alpha}\right)$
  - $\frac{\partial Q_D}{\partial \alpha} > 0$: a greater demand due to a larger $\alpha \Rightarrow$ upward shift of the demand curve $\Rightarrow$ a higher price and quantity.
  - $\frac{\partial Q_D}{\partial \alpha} < 0$: a lower demand due to a larger α (negative demand shock) $\Rightarrow$ downward shift of the demand curve $\Rightarrow$ lower price and quantity.

Example: Tax Incidence (税负归宿)

Let p be the price received by sellers, t be lump-sum tax, and hence consumer pays $p+t$. At equilibrium: $Q_D(p+t) = Q_S(p)$.

- Total differentiation: $\frac{\partial Q_D}{\partial p}(dp + dt) = \frac{\partial Q_S}{\partial p} dp$
- Effect on seller's price $\frac{dp}{dt} = \frac{\partial Q_D/\partial p}{\partial Q_S/\partial p - \partial Q_D/\partial p} < 0$
- Effect on buyer's price: $\frac{d(p+t)}{dt} = \frac{dp}{dt} + 1 = \frac{\partial Q_S/\partial p}{\partial Q_S/\partial p - \partial Q_D/\partial p} > 0$
- Buyers pay more; Sellers earn less. Both parties are worse-off.
- Who is more heavily harmed? Buyers of sellers?

Elasticity (弹性)

Elasticity measures the "responsiveness" in terms of an exogenous change.

Definition

- *Price elasticity of demand:*

$$\varepsilon_D = -\frac{\Delta Q_D / Q_D}{\Delta p / p} = -\frac{dQ_D}{dp}\frac{p}{Q_D}$$

- *Elasticity of supply:*

$$\varepsilon_S = \frac{\Delta Q_S / Q_S}{\Delta p / p} = \frac{dQ_S}{dp}\frac{p}{Q_S}$$

Demand Elasticity

Consider only one consumer with quasi-linear utility: $u(x)$.

- The interior solution is determined by $u'(x^*) = p$.
- $u''(x^*)\frac{dx^*}{dp} = 1 \Rightarrow \frac{dx^*}{dp} = \frac{1}{u''(x^*)}$.
- $\varepsilon_D = -\frac{dx^*}{dp}\frac{p}{x^*} = \frac{u'(x^*)}{-x^* u''(x^*)}$
- Example: iso-elastic (CRRA: constant relative risk-averse) utility:

$$u(x) = \begin{cases} \frac{x^{1-\sigma}-1}{1-\sigma}, & \sigma \neq 1 \\ \ln(x), & \sigma = 1 \end{cases}$$

$$\varepsilon_D = \frac{u'(x^*)}{-x^* u''(x^*)} = \frac{1}{\sigma}.$$

Tax Incidence: Continued

Using the definitions of elasticities, $\varepsilon_D = -\frac{dQ_D}{dp}\frac{p}{Q_D}$ and $\varepsilon_S = \frac{dQ_S}{dp}\frac{p}{Q_S}$:

- For seller: $\frac{dp}{dt} = \frac{\partial Q_D/\partial p}{\partial Q_S/\partial p - \partial Q_D/\partial p} = \frac{\partial Q_D/\partial p}{\partial Q_S/\partial p - \partial Q_D/\partial p}\frac{p/Q}{p/Q} = \frac{-\varepsilon_D}{\varepsilon_S+\varepsilon_D}$
- For buyer: $\frac{d(p+t)}{dt} = \frac{\partial Q_S/\partial p}{\partial Q_S/\partial p - \partial Q_D/\partial p}\frac{p/Q}{p/Q} = \frac{\varepsilon_S}{\varepsilon_S+\varepsilon_D}$
- Who suffers more?

$$\left|\frac{\text{seller's burden}}{\text{buyer's burden}}\right| = \frac{\varepsilon_D}{\varepsilon_S}$$

- Nominally, buyers pay the tax t. But if $\frac{\varepsilon_D}{\varepsilon_S} \to +\infty$ (nobody buys), sellers experience the price change.
- Conversely, if the demand is inelastic $\varepsilon_D \to 0$, then tax burdens are shifted to buyers.
- Those who "care more" will suffer.

Welfare Implications in a Competitive Market

Consider the consumption good x only with quasi-linear utility: $U(x, y) = u(x) + y$ where $p_y = 1$ and $p_x = p$. The good x is of out interest.

- n buyers and m sellers.
- The total amount paid by buyers is $\sum_{i=1}^{n} px_i^* = pQ_D$.
- The total amount received by sellers is $\sum_{i=1}^{m} pq_i^* = pQ_S$.
- Aggregation: payments cancel out at equilibrium because $Q_D = Q_S$.
- Buyers (assuming interior solution): $u'(x^*) = MU = p$
- Sellers: $p = C'(q^*) = MC$
- Market Equilibrium:

$$MU = p = MC$$

Competitive Equilibrium = Socially Optimum

- Now consider a benevolent planner (dictator) confiscates all equipments and allocates the consumption good x directly, in order to maximize total surplus on x (consumer surplus plus firm profits).
- Because the amount paid by consumers is equal to the amount received by sellers, hence the social objective can be simplified as the sum of utilities on x, minus production costs.

$$\max_{x,q} nu(x) - mC(q), \quad s.t.\ nx = mq$$
$$\Rightarrow \max_{x} nu(x) - mC\left(\frac{nx}{m}\right)$$
$$\Rightarrow nu'(x^{opt}) - m\frac{n}{m}C'\left(\frac{nx^{opt}}{m}\right) = 0$$
$$\Rightarrow u'(x^{opt}) = C'(q^{opt}) \Rightarrow MU = MC$$

Therefore, competitive equilibrium coincides with the socially optimal allocation.

Deadweight Loss (净损失) Generated by Tax

Assume that the authority imposes a lump-sum tax: for each unit of x traded, a buyer pays $p+t$ and while the seller receives p.

- Buyer: $u'(x^*) = p + t$
- Seller: $p = C'(q^*)$
- At equilibrium, $u'(x^*) = C'(q^*) + t \Leftrightarrow MU > MC$
- Differentiate the equilibrium with respect to t: $u''(x^*)\frac{dx^*}{dt} = C''(q^*)\frac{dq^*}{dt} + 1$; At equilibrium: $Q_D = Q_S \Rightarrow nx^* = mq^* \Rightarrow n\frac{dx^*}{dt} = m\frac{dq^*}{dt}$, we can solve $\frac{dx^*}{dt} < 0$ and $\frac{dq^*}{dt} < 0$.

We have shown that the first-best outcome is given by x^{opt}, q^{opt}. With tax, $\frac{dx^*}{dt} < 0 \Rightarrow x^* < x^{opt}$: output/quantity is downward distorted by tax. In other words, the optimal tax is no tax.

- "Total surplus" W is defined as the sum of consumer surplus, sellers' profits and tax revenue:

$$W = \underbrace{nu(x^*) - \underbrace{n(p+t)x^*}_{\text{paid by buyers}}}_{\text{consumer surplus}} + \underbrace{\underbrace{mpq^*}_{\text{received by sellers}} - mC(q^*)}_{\text{producer surplus}} + \underbrace{ntx^*}_{\text{tax revenue}}$$

$$= nu(x^*) - mC(q^*)$$

where $Q_D^* = Q_S^* \Rightarrow pQ_D^* = pQ_S^* \Rightarrow p \cdot nx^* = p \cdot mq^*$.

- Note that x^* and q^* are functions of tax t, and we have shown that $\frac{dx^*}{dt} < 0$, $\frac{dq^*}{dt} < 0$ and $n\frac{dx^*}{dt} = m\frac{dq^*}{dt}$.
- The effect of t on W:

$$\frac{dW}{dt} = n\underbrace{u'(x^*)}_{=p+t}\frac{dx^*}{dt} - m\underbrace{C'(q^*)}_{=p}\frac{dq^*}{dt}$$

$$= (p+t-p)n\frac{dx^*}{dt} < 0.$$

From Partial Equilibrium to General Equilibrium

In the partial-equilibrium market:

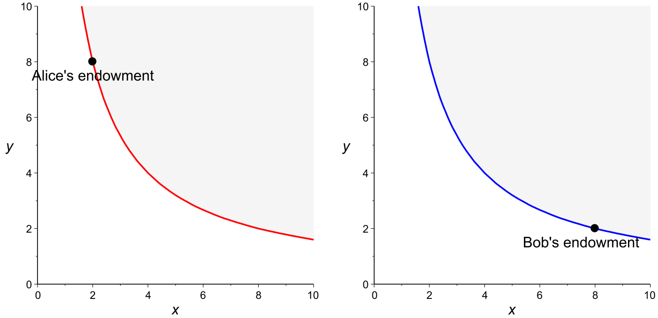
- We focus on good x, without considering the other good y.
- When a consumer chooses two goods x and y, the prices are assumed to be given.

In order to examine how the prices p_x and p_y are determined, consider a simple story:

- An island, with only two persons: Alice and Bob. The utility function is $U = \sqrt{xy}$.
- Alice is endowed with 2 units of x and 8 units of y; Bob is endowed with 8 units of x and 2 units of y.
- Assuming no production. Total amount of x and y available is (10, 10).
- Under autarky, the utility of each person is $\sqrt{2 \cdot 8} = 4$.
- Will they be better-off by making transactions? How will the "prices" of x and y be determined?

Autarky

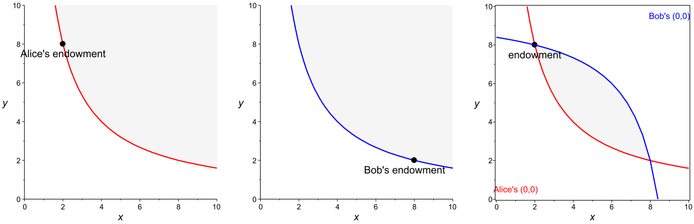
The indifference curves and utilities of Alice and Bob.



Alice is endowed with $(x_a, y_a) = (2, 8)$; Bob is endowed with $(x_b, y_b) = (8, 2)$. Without trade, each gets utility $\sqrt{16} = 4$.

Edgeworth Box

- Use the horizontal axis for x: the length is total endowments $x_a + y_a = 10$
- Use the vertical axis for y: the length is total endowments $y_a + y_b = 10$
- Rotate Bob's problem such that Bob's origin locates at the northeast corner of the box.



- The new diagram, called "Edgeworth Box," allows us to analyze the allocations (e.g., trade) between two persons.

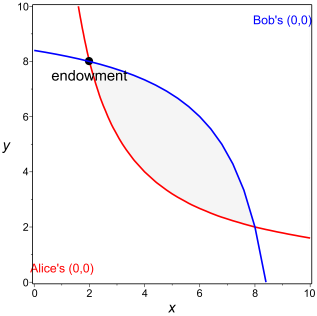
Potential Gains from Trade

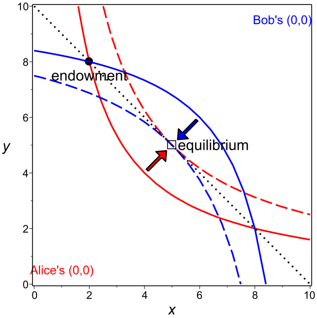
- Two person: Alice (A) and Bob (B).
- Two goods: x and y. $U = \sqrt{xy}$
- Endowments: Alice has $x_a = 2$ and $y_a = 8$; Bob has $x_b = 8$ and $y_b = 2$.
- Autarky: $U_A = \sqrt{x_a y_a} = 4$; $U_B = \sqrt{x_b y_b} = 4$.
- Let x_A and y_A be Alice's final choices; x_B and y_B be Bob's final choices.
- No production: $x_A + x_B = x_a + x_b = 10$; $y_A + y_B = y_a + y_b = 10$.
- Utilities: $U_A(x_A, y_A)$ and $U(x_B, y_B)$.
- Will they be better-off by trading with each other?

Equilibrium

- Here, prices p_x and p_y, and "income" seem to be missing.
- In an exchange economy, prices are determined by exchange; income is determined by endowments owned.
- For Alice:
  - The "budget line" is $p_x x_A + p_y y_A = p_x x_a + p_y y_b = 2p_x + 8p_y$
  - The UMP gives $x_A = \frac{2p_x+8p_y}{2p_x}$ and $y_A = \frac{2p_x+8p_y}{2p_y}$
- For Bob:
  - The budget line is $p_x x_B + p_y y_B = p_x x_b + p_y y_b = 8p_x + 2p_y$
  - the UMP gives $x_B = \frac{8p_x+2p_y}{2p_x}$ and $y_B = \frac{8p_x+2p_y}{2p_y}$.
- The resource constraint: $x_A + x_B = x_a + x_b = 10$ and $y_A + y_B = y_a + y_b = 10$
- Plug the 4 Marshallian demand into the two resource constraint, we can solve the **relative price**: $p_x = p_y$.
- Plug the relative prices into the Marshallian demand, the equilibrium allocation is $(x_A, y_A) = (5, 5)$ and $(x_B, y_B) = (5, 5)$.

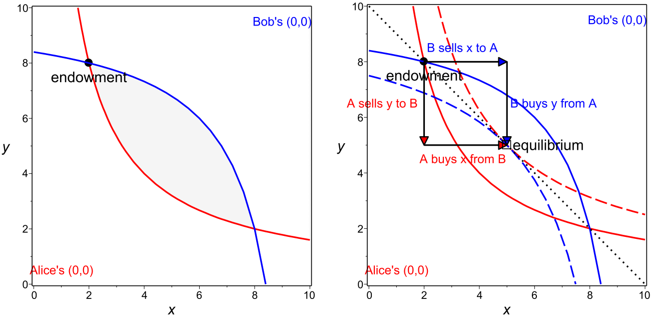
Edgeworth Box: Equilibrium





- Under Autarky, the utility of each is $\sqrt{16} = 4$.
- After trade at price $p_x = p_y$, the utility of each becomes $\sqrt{5 \cdot 5} = 5$: they are better-off.
- Each has a higher indifference curve.
- The new indifference curves are tangent, and both are tangent to the "budget line" $p_x x + p_y y = 2p_x + 8p_y = 8p_x + 2p_y \Rightarrow y = -x + 10$.

Edgeworth Box: Trading Profiles

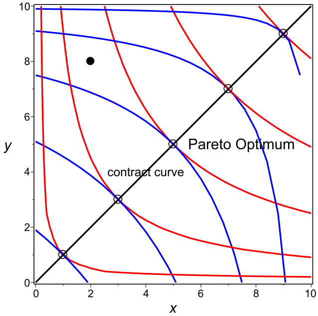


- Under Autarky, Alice's endowment is $(2, 8)$; Bob's endowment is $(8, 2)$.
- The relative price is $p_x = p_y = p$.
- Alice sells 3 units of y to Bob (obtaining $3p$), in exchange for 3 units of x from Bob (by paying $3p$).
- Bob buys 3 units of y from Alice (by paying $3p$), by selling 3 units of x to Alice (obtaining $3p$).

Mutually Beneficial Trade and Pareto (帕累托) Optimum

- After trade, Alice is better off whereby Bob is no worse off; Bob is better off whereby Alice is no worse off.
- A Pareto efficient allocation: each person is on his highest possible indifference curve, given the indifference curve of the other person.
    - There is no way to make all the people involved better off; or
    - there is no way to make some individual better off without making someone else worse off; or
    - all of the gains from trade have been exhausted; or
    - there are no mutually advantageous trades to be made, and so on.
- In our example: the allocation under autarky is not Pareto efficient because there exists Pareto improvement such that at least one person will be better-off without harming anyone else. The equilibrium after trade is Pareto efficient because we can no long find any other ways to make one person better-off without harming the other.

Pareto Optimum (帕累托最优) and Contract Curve (合约曲线)



- When two indifference curves are tangent, there is no way to make one person better-off without harming the other, i.e., each tangent point is a Pareto optimum allocation (circles).
- The line connecting all such tangent points is called contract curve (black curve).

Walrasian Equilibrium (瓦尔拉斯均衡)

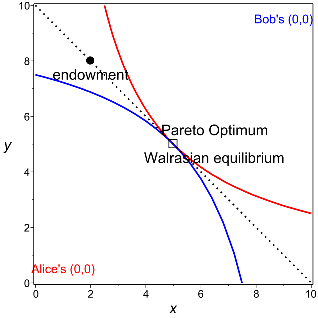
In an exchange economy (without production)

- $i = 1, ..., n$ individuals and $g = 1, ..., G$ goods.
- Each individual is endowed with $\boldsymbol{w} = (w_1, ..., w_G)^T$.
- Everyone is price-taker. Prices $\boldsymbol{p} = (p_1, ..., p_G)$ are given.
- Each agent's initial wealth is measured by $\boldsymbol{pw}$.
- They can choose to exchange, to consume $\boldsymbol{x} = (x_1, ..., x_G)^T$ finally.
- Utility $U_i(x_1, ..., x_G)$
- The Walrasian equilibrium is an allocation of resources and an associated price vector $\boldsymbol{p}$ such that

$$\sum_{i=1}^{n} x_g^*(\boldsymbol{p}, \boldsymbol{pw}) = \sum_{i=1}^{n} w_g \quad \forall g = 1, 2, ..., G$$

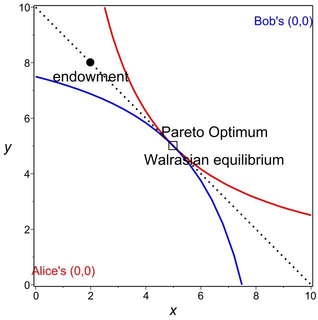
Walrasian Equilibrium in Our Example

- Equilibrium allocation (tangent point):
  - Alice: $x_A^*(p_x, p_y, 2p_x + 8p_y)$, $y_A^*(p_x, p_y, 2p_x + 8p_y)$
  - Bob: $x_B^*(p_x, p_y, 8p_x + 2p_y)$, $y_B^*(p_x, p_y, 8p_x + 2p_y)$
- Price vector: p_x and p_y (the slope of the dotted line)
- $x_A + x_B = 10 =$ total endowments of x, $y_A + y_B = 10 =$ total endowments of y



Walrasian Equilibrium in Our Example

- Walrasian equilibrium is Pareto efficient.
- The marginal rate of substitutions are equalized: $MRS_A = \frac{p_x}{p_y} = MRS_B$. (not necessarily true for other utilities, e.g., perfect substitutes)
- There exists a Pareto improvement from autarky to equilibrium allocation (gains from trade).



Free Market v.s. Centralized Planning

At Walrasian equilibrium (free market), the allocation is $(x_A^*, y_A^*) = (x_B^*, y_B^*) = (5, 5)$. Now consider that a benevolent authority can reallocate resources. To see whether the market outcome is Pareto efficient, let's check whether the authority can find some ways to make someone better-off whereby keeping the other no worse-off.

- The authority tries to maximize Alice's utility, without making Bob worse-off:

$$\begin{aligned}&\max_{x_A, y_A} \sqrt{x_A y_A}\\ &s.t. \sqrt{(10 - x_A)(10 - y_A)} \geq 5.\end{aligned}$$

- By solving the Lagrangian: $\mathcal{L} = \sqrt{x_A y_A} + \lambda(\sqrt{(10 - x_A)(10 - y_A)} - 5)$, the socially optimal solution is $x_A^{opt} = 5 = x_A^*$ and $y_A^{opt} = 5 = y_A^*$.
- Competitive equilibrium (Walrasian) and socially optimum are consistent.
- Any further adjustment will not make things better.

First Theorem of Welfare economics (福利经济学第一定理)

Definition (Pareto efficient allocation)

An allocation of the available goods in an exchange economy is efficient if it is not possible to devise an alternative allocation in which at least one person is better off and no one is worse off.

Proposition (First theorem of welfare economics)

Every Walrasian equilibrium is Pareto efficient.

The Socialist Calculation Debate

Maurice Dobb

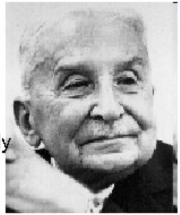
Ludwig von Mises

Friedrich von Hayek

Oskar Lange

The Socialist Calculation Debate

Government's could
End inefficiency...



But they can't adequately
price goods...

Why not?
It's a matter of
Calculating supply
And demand!...

No..Too much
informational
Requirement
is needed!

Quasi-linear Utilities and Aggregation*

In previous lectures, we measure "welfare" by using Pareto criterion. Sometimes we care about the aggregate level of utilities of different agents. Quasi-linear utility makes it simple to aggregate utilities of different agents.

- Consider good x and the numeraire y. There are n agents. Utility for agent i is $\theta_i u(x) + y$, $i = 1, 2, ..., n$. Assume that $u''(x) < 0$, $p_y = 1$, $p_x = p$, and the solutions are interior.
- Total amount of x is X, which is initially owned by m agents. Each is endowed with X/m units.
- Total amount of y is Y, which is equally endowed by every agent. Assume that Y/n is sufficiently large such that the initial wealth can support interior solutions for x at equilibrium.
- Initial owners are suppliers of x. Those who without endowments are buyers of x.
- For each agent, interior solution implies $\theta_i u'(x_i) = p$.
- For each seller i, consider how to use an additional unit of x: selling it gives p; consuming it gives $\theta_i u'(x_i)$. The last unit satisfies $p = \theta_i u'(x_i)$.

- The Walrasian equilibrium is characterized by

$$\begin{cases} \theta_i u'(x_i^*) = p & \forall i \\ \sum_{i=1}^{n} x_i^* = X \end{cases}$$

- MRS are equalized, i.e.,
$MRS_1 = \frac{\theta_1 u'(x_1^*)}{1} = p = ... = \frac{\theta_n u'(x_n^*)}{1} = MRS_n$
- The agent with a higher θ_i consumes more x_i: consider two persons, with $\theta_L < \theta_H$.
$\theta_L u'(x_L^*) = p = \theta_H u'(x_H^*) \Rightarrow u'(x_L^*) > u'(x_H^*)$. By $u''(\cdot) < 0$, then $x_L^* < x_H^*$.

Note that as long as the market for x is cleared, the market for y must also be cleared:

- The budget of each seller: $px^*_{\text{seller}} + y^*_{\text{seller}} = Y/n + pX/m$, hence $p\sum x^*_{\text{seller}} + \sum y^*_{\text{seller}} = mY/n + pX \Rightarrow \sum y^*_{\text{seller}} = mY/n + \underbrace{p\left(X - \sum x^*_{\text{seller}}\right)}_{\text{revenue received by selling } x}$.
- The budget of each buyer: $px^*_{\text{buyer}} + y^*_{\text{buyer}} = Y/n$, hence $p\sum x^*_{\text{buyer}} + \sum y^*_{\text{buyer}} = (n-m)Y/n \Rightarrow \sum y^*_{\text{buyer}} = (n-m)Y/n - \underbrace{p\sum x^*_{\text{buyer}}}_{\text{amount paid to buy } x}$.
- Because the consumption of all agents $\sum_{i=1}^{n} x^*_i = X$, hence transfers among buyers and sellers cancel out, hence by summing the budget of all agents: $\sum y^*_{\text{seller}} + \sum y^*_{\text{buyer}} = Y = Y + pX - p\sum x^*_{\text{seller}} - p\sum x^*_{\text{buyer}} = Y + pX - pX = Y$.

- Now suppose that all endowments of x are owned by a benevolent authority, who allocates $x_1, ..., x_n$ to different agents to maximize the sum of utilities of all agents.
- The Lagrangian is

$$\begin{array}{l} \max_{x_i} \sum_{i=1}^{n} \theta_i u(x_i) + Y \\ s.t. \ \sum_{i=1}^{n} x_i = X \\ \mathcal{L} = \sum_{i=1}^{n} \theta_i u(x_i) + Y + \lambda \left(X - \sum_{i=1}^{n} x_i \right) \\ \Rightarrow \theta_i u'(x_i) = \lambda \text{ and } \sum_{i=1}^{n} x_i = X \end{array}$$

which implies that MRS are equalized. The Walrasian equilibrium is equivalent with the welfare-maximizing allocation.

The Effect of Tax in General Equilibrium*

- Now suppose that all endowments are still privately owned, but the authority imposes the *ad valorem* tax to the sellers: for each unit of x to be sold at price p, the buyer pays p; the seller pays τp to the government and receives $(1-\tau)p$, where tax rate τ is proportional to price.
- For a buyer b, $\theta_b u'(x_b) = p$.
- For a seller s, consuming an additional unit of the endowed x gives $\theta_s u'(x_s)$, whereby losing the opportunity of selling this unit, which gives $(1-\tau)p$. That is, upon dealing with the last unit, self-consumption and selling should be equally profitable: $\theta_s u'(x_s) = (1-\tau)p$.
- At equilibrium, $\theta_s u'(x_s^*) = (1-\tau)\theta_b u'(x_b^*)$

- The Walrasian equilibrium with tax is featured by

$$\begin{cases} \theta_s u'(x_s^*) = (1-\tau)\theta_b u'(x_b^*) \\ \sum_{s=1}^{m} x_s^* + \sum_{b=1}^{n-m} x_b^* = X \end{cases}$$

- Differentiate the equilibrium with respect to τ:

$$\begin{aligned} &\theta_s u''(x_s^*)\frac{dx_s^*}{d\tau} = -\theta_b u'(x_b^*) + (1-\tau)\theta_b u''(x_b^*)\frac{dx_b^*}{d\tau} \\ &\sum_{s=1}^{m} \frac{dx_s^*}{d\tau} + \sum_{b=1}^{n-m} \frac{dx_b^*}{d\tau} = 0 \\ &\Rightarrow \frac{dx_s^*}{d\tau} > 0 \text{ and } \frac{dx_b^*}{d\tau} < 0. \end{aligned}$$

- Total surplus is decreasing in τ:

$$\begin{aligned} W(\tau) &= \sum_{s=1}^{m} \theta_s u(x_s^*) + \sum_{b=1}^{n-m} \theta_b u(x_b^*) + Y \\ W'(\tau) &= \sum_{s=1}^{m} \underbrace{\theta_s u'(x_s^*)}_{=(1-\tau)p} \frac{dx_s^*}{d\tau} + \sum_{b=1}^{n-m} \underbrace{\theta_b u'(x_b^*)}_{=p} \frac{dx_b^*}{d\tau} \\ &= p\underbrace{\left(\sum_{s=1}^{m} \frac{dx_s^*}{d\tau} + \sum_{b=1}^{n-m} \frac{dx_b^*}{d\tau}\right)}_{=0} - \tau p \sum_{s=1}^{m} \underbrace{\frac{dx_s^*}{d\tau}}_{+} < 0 \end{aligned}$$

- Interventions imposed on Walrasian equilibrium is not socially optimal.

Production in General Equilibrium

Now let's consider production in the island with Alice and Bob.

- Consumption goods: x and y;
- Alice and Bob build a factory, which uses q_x units of x as inputs to produce q_y units of y.
- The firm's profit is $\pi = p_y q_y - p_x q_x$.
- Alice owns α share of the profit and Bob owns β share of the profit ($\alpha + \beta = 1$).
- Feasibility: $x_A + x_B + q_x \leq x_a + x_b$ and $y_A + y_B \leq y_a + y_b + q_y$
- Alice's budget: $p_x x_A + p_y y_A \leq p_x x_a + p_y y_a + \alpha\pi$; Bob's budget: $p_x x_B + p_y y_B \leq p_x x_b + p_y y_b + \beta\pi$
- Each person maximizes his/her utility. Firm maximizes its profit.

Example: from Exchange to Production

- In our previous example, without production, the equilibrium is $(x_A^*, y_A^*) = (5,5)$ and $(x_B^*, y_B^*) = (5,5)$.
- Without production, $MRS_A = MRS_B = \frac{p_x}{p_y} = 1$.
- With production: let $\alpha = \beta = \frac{1}{2}$ and normalize $p_y = 1$.
    - The firm solves $\max_{q_x} p_y\sqrt{q_x} - p_x q_x \Rightarrow q_x^* = \frac{p_y^2}{4p_x^2}$, $q_y^* = \sqrt{q_x^*} = \frac{p_y}{2p_x}$, $\pi^* = \frac{p_y^2}{4p_x}$.
    - UMP gives: $x_A^* = \frac{2p_x+8p_y+0.5\pi^*}{2p_x}$, $y_A^* = \frac{2p_x+8p_y+0.5\pi^*}{2p_y}$, $x_B^* = \frac{8p_x+2p_y+0.5\pi^*}{2p_x}$ and $y_B^* = \frac{8p_x+2p_y+0.5\pi^*}{2p_y}$.
    - Feasibility: $x_A^* + x_B^* + q_x^* = x_a + x_b = 10$ and $x_B^* + y_B^* = \sqrt{q_x^*} + 10$.
    - We can solve the relative price $\frac{p_y}{p_x} = \frac{-20+2\sqrt{130}}{3} < 1$ from the two feasibility conditions.
    - Plug $\frac{p_y}{p_x}$ into the above expressions, we get the equilibrium level of consumption and production.

Quasi-linear Utilities with Production*

Recall the quasi-linear world with n heterogeneous agents, and $U(x, y) = \theta_i u(x) + y$. Assume that there's no endowments of x, and all x are produced by a competitive sector with convex-cost technology.

- An individual firm solves $\max_q pq - C(q) \Rightarrow p = C'(q)$.
- The supply function is $q(p) = MC^{-1}(p)$.
- There are k firms, and the industry supply is $S(p) = kq(p) = kMC^{-1}(p)$.
- Note that due to convex cost, $S'(p) = k\frac{dMC^{-1}}{dp} = k\frac{1}{MC'} > 0$.

- The Walrasian equilibrium is featured by

$$\begin{cases} \theta_i u'(x_i^*) = p \\ \sum_{i=1}^n x_i^* = S(p) = kq^* \\ p = C'(q^*) \end{cases}$$

- Now suppose that all factories are owned by a benevolent planner, who maximizes total surplus (the sum of consumer utilities and firm profits) by solving

$$\max_{x,q} \sum_{i=1}^n \theta_i u(x_i) - kC(q), \quad s.t. \ \sum_{i=1}^n x_i = kq$$

  - Note that the linear term Y does not affect the outcome (why?), and therefore can be omitted from the objective.
  - FOC w.r.t. x_i:
    $\theta_i u'(x_i) - kC'\left(\frac{1}{k}\sum_{i=1}^n x_i\right)\frac{1}{k} = 0 \Rightarrow \theta_i u'(x_i^{opt}) = C'(q^{opt})$
  - Combining the resource constraint, the competitive equilibrium is socially efficient: $x_i^* = x_i^{opt}$ and $q^* = q^{opt}$.

The Effect of Tax*

Now assume that all factories are privately owned, but producers are taxed by τp, where τ is the *ad valorem* tax rate.

- For each consumer, $\theta_i u'(x_i) = p$;
- For each supplier, $(1-\tau)p = C'(q) \Leftrightarrow q = MC^{-1}((1-\tau)p) \Rightarrow S((1-\tau)p) = kq$.
- The equilibrium is

$$\begin{cases} \theta_i u'(x_i^*) = p \\ C'(q^*) = (1-\tau)p \\ \sum_{i=1}^{n} x_i^* = kq^* \end{cases}$$

The Walrasian equilibrium is given by

$$\begin{cases} \theta_i u'(x_i^*) = p \\ C'(q^*) = (1-\tau)p \\ \sum_{i=1}^n x_i^* = kq^* \end{cases}$$

- Plug the first equation into the second equation: $C'(q^*) = (1-\tau)\theta_i u'(x_i^*)$. Differentiate it with respect to τ: $C''(q^*)\frac{dq^*}{d\tau} = -\theta_i u'(x_i^*) + (1-\tau)\theta_i u''(x_i^*)\frac{dx_i^*}{d\tau}$.
- From the third equation: $k\frac{dq^*}{d\tau} = \sum_{i=1}^n \frac{dx_i^*}{d\tau}$.
- Combining the above two equations, we have $C''(q^*)\frac{1}{k}\sum_{i=1}^n \frac{dx_i^*}{d\tau} = -\theta_i u'(x_i^*) + (1-\tau)\theta_i u''(x_i^*)\frac{dx_i^*}{d\tau}$.
- Suppose $\frac{dx_i^*}{d\tau} > 0$, then the RHS is negative. Hence the LHS should be negative, which means $\sum_{i=1}^n \frac{dx_i^*}{d\tau} < 0$, a contradiction. Therefore, $\frac{dx_i^*}{d\tau} < 0$.

Plug

$$\begin{cases} \theta_i u'(x_i^*) = p \\ C'(q^*) = (1-\tau)p \\ \sum_{i=1}^n x_i^* = kq^* \end{cases}$$

into total surplus, then

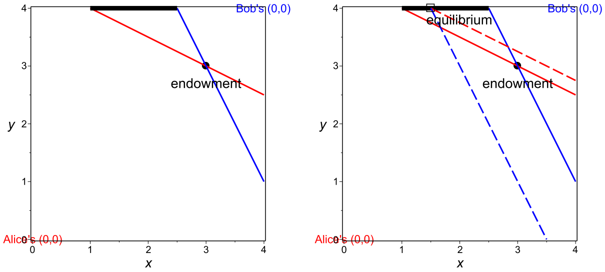
$$W(\tau) = \sum_{i=1}^n \theta_i u(x_i^*) - kC\left(\frac{\sum_{i=1}^n x_i^*}{k}\right)$$

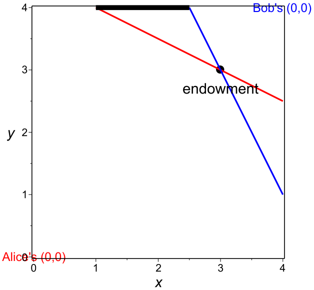
$$W'(\tau) = \sum_{i=1}^n \underbrace{\theta_i u'(x_i^*)}_{=p} \frac{dx_i^*}{d\tau} - \underbrace{C'(q^*)}_{=(1-\tau)p} \sum_{i=1}^n \frac{dx_i^*}{d\tau}$$

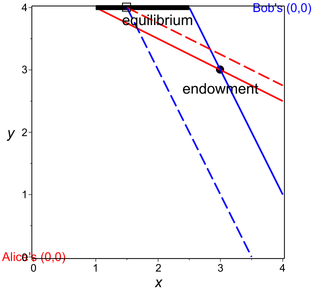
$$\Rightarrow W'(\tau) = \tau p \sum_{i=1}^n \frac{dx_i^*}{d\tau} < 0.$$

Perfect Substitutes (Exchange Economy)

- Alice: $U_A = x_A + 2y_A$; endowments: $(x_a, y_a) = (3, 3)$
- Bob: $U_B = 2x_B + y_B$; endowments: $(x_b, y_b) = (1, 1)$
- By comparing the relative slopes of the indifference curves, the Pareto improvement occurs at somewhere at $y_A = 4$ (the bold line).



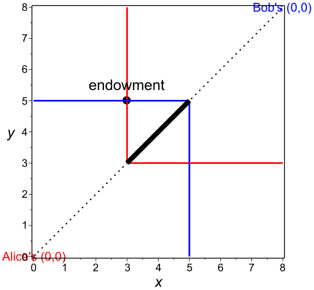


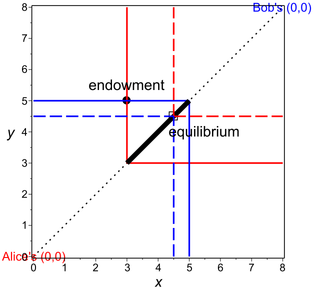


- $y_A + y_B = 3 + 1 = 4$, and $y_A^* = 4$, imply $y_B^* = 0$.
- $MRS_A = \frac{1}{2} < MRS_B = 2$, hence $\frac{1}{2} \leq \frac{p_x}{p_y} \leq 2$.
- Alice's budget: $p_x x_A + 4p_y = 3p_x + 3p_y \Rightarrow x_A = 3 - \frac{p_y}{p_x} \Rightarrow 1 \leq x_A \leq \frac{5}{2}$.
- Bob's budget: $p_x x_B = p_x + p_y \Rightarrow x_B = 1 + \frac{p_y}{p_x} \Rightarrow \frac{3}{2} \leq x_B \leq 3$.
- At equilibrium, $\frac{p_x}{p_y} \in [0.5, 2]$, $y_A^* = 4$, $y_B^* = 0$, $x_A^* + x_B^* = 4$ while $x_A^* \in [1, 2.5]$.

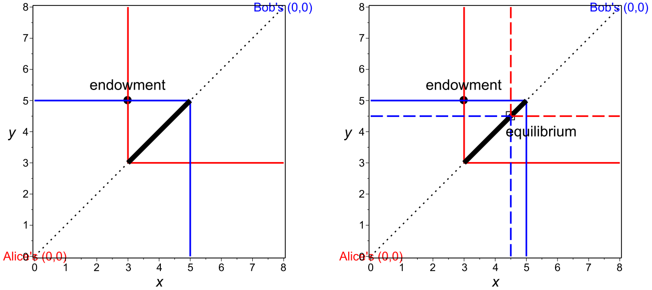
Perfect Complements (Exchange Economy)

- Alice: $U_A = \min\{x_A, y_A\}$, endowments: $(x_a, y_a) = (3, 5)$.
- Bob: $U_B = \min\{x_B, y_B\}$, endowments: $(x_b, y_b) = (5, 3)$.
- By observing the indifference curves, the Pareto improvement occurs at somewhere at $y = x$ and $x_A \in [3, 5]$.





Perfect Complements



- The equilibrium consumption satisfies $x_A = y_A$ and $x_B = y_B$.
- Alice's budget: $p_x x_A + p_y y_A = 3p_x + 5p_y \Rightarrow x_A = y_A = \frac{3p_x+5p_y}{p_x+p_y}$.
- Bob's budget: $p_x x_B + p_y y_B = 5p_x + 3p_y \Rightarrow x_B = y_B = \frac{5p_x+3p_y}{p_x+p_y}$.
- For any prices, $x_A^* + x_B^* = 8$ and $y_A^* + y_B^* = 8$ hold.