Winter Bootcamp

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This week

Winter bootcamp from Monday 10/02 to Friday 14/02, only mornings (9h-12h).

Room S8.

We will cover integration and STATA. You will need these notions for Micro II and Econometrics II.

No pass/fail for this week, you already got the 6 credits in September.

No problem sets but exercises to work on during the class.

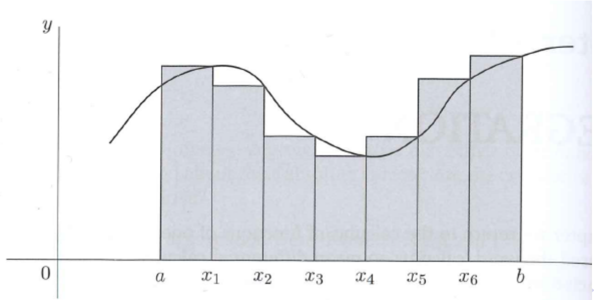
Integration

Areas and integrals

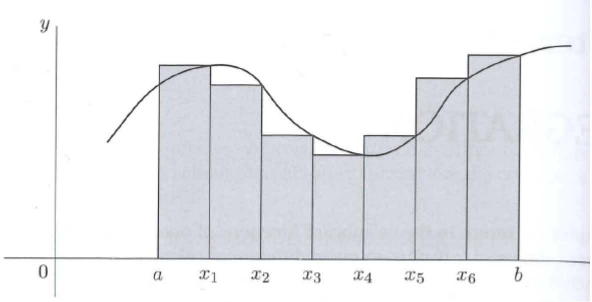
Let a and b be real numbers such that $a < b$, and let f be a continuous function.

Central problem : find the area bounded by the curve $y = f(x)$, the vertical lines $x = a$ and $x = b$, and the x-axis.

The area can first be approximated by the sum of areas of rectangles.



Areas and integrals



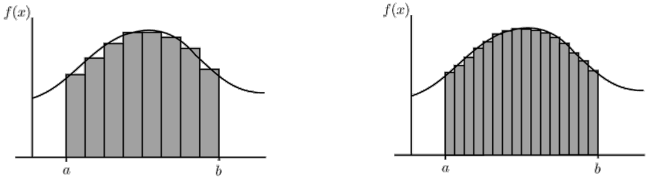
$$(x_1 - x_0)f(x_1) + \cdots + (x_7 - x_6)f(x_7) = \sum_{i=1}^{7}(x_i - x_{i-1})f(x_i)$$

with $x_0 = a$ and $x_7 = b$.

Areas and integrals

The area would then be better approximated by the sum of the areas of a very large number of very narrow rectangles.

(a) 8 sub-intervals (b) 16 sub-intervals



If the sum of the areas of the rectangles tends to a limit as the length of the sub-intervals tends to 0, this limit is the required area.

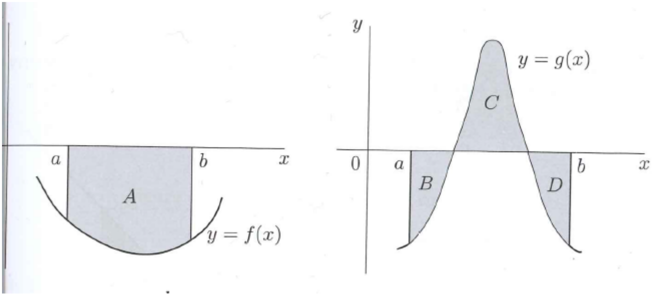
It is written :

$$\int_a^b f(x)dx$$

and is called the **integral** of $f(x)$ wrt x from $x = a$ to $x = b$

Areas and integrals

By convention, areas below the x-axis are negative.



$$\int_a^b f(x)dx = -A \quad \int_a^b g(x)dx = -B + C - D$$

Areas and integrals

We also define

(1) If $a < b$, $\int_b^a f(x)dx = -\int_a^b f(x)dx$

(2) $\int_a^a f(x)dx = 0$

Areas and integrals

Properties of integrals

(1) If k is constant, $\int_a^b k\, dx = k(b-a)$

(2) $\int_a^b x\, dx = \frac{1}{2}(b^2 - a^2)$

(3) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

Exercise 1 prove properties (1) to (3) using graphs.

Exercises

Exercise 2 using properties (1)-(3), represent graphically $f(x)$ and calculate $\int_0^3 f(x)dx$ where

$$f(x) = \begin{cases} 2 & \text{if } x \leq 2, \\ x & \text{if } x > 2 \end{cases}$$

Exercise 3 using properties (1)-(3), represent graphically $g(x)$ and calculate $\int_0^3 g(x)dx$ where

$$g(x) = \begin{cases} 1 & \text{if } x \leq 2, \\ x & \text{if } x > 2 \end{cases}$$

Given a real number a and a continuous function f, we can define another function F by letting

$$F(t) = \int_a^t f(x)dx \quad \text{for all } t$$

It turns out that this function F is differentiable and

$$F'(t) = f(t) \quad \text{for all } t$$

Fundamental theorem of calculus : if you integrate a function and then differentiate the result, you retrieve the function you started with.

$\rightarrow$ **Integration is the reverse of differentiation.**

Proposition : if the function g has a continuous derivative,

$$\int_a^b g'(x)dx = g(b) - g(a)$$

Areas and integrals

Integrating derivatives and primitives

Given a continuous function f, we define a **primitive** of f to be a function g such that

$$g'(x) = f(x)$$

Common method of integration

To find $\int_a^b f(x)dx$, we have to find **a** primitive g of f.

Then $\int_a^b f(x)dx = g(b) - g(a)$.

Exercises

Example : $\frac{d}{dx}(kx^2) = 2kx$ for any constant k, kx^2 is a primitive of $2kx$.

In particular, $x^2/2$ is a primitive of x. It follows that

$$\int_a^b x dx = \frac{1}{2}(b^2 - a^2)$$

Exercise 4 Write down a primitive for $8x^3$ and evaluate $\int_1^2 8x^3 dx$

Exercise 5 Write down a primitive for $12x^5$ and evaluate $\int_2^3 12x^5 dx$

Rules of integration

Calculating indefinite integrals

Proposition : If $g(x)$ is a primitive of $f(x)$, then the set of all primitives of $f(x)$ consists of the functions of the form $g(x) + C$, where C is constant.

Given a function $f(x)$, the set of all its primitives is called the **indefinite integral** and denoted by

$$\int f(x)dx = g(x) + C$$

where C is constant of integration.

For example,

$$\int 6x^2 dx = 2x^3 + C$$

Rules of integration

Calculating indefinite integrals

Rule 1 $\int x^{\alpha} dx = \frac{1}{\alpha+1} x^{\alpha+1} + C$ if $\alpha \neq -1$

Rule 2 $\int \frac{1}{x} dx = \ln x + C$, provided $x > 0$

Rule 3 $\int e^{x} dx = e^{x} + C$

Rule 4 $\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$

Rule 5 If k is a constant, $\int k f(x) dx = k \int f(x) dx$

Extending the rules

Rule 1' $\int (x-a)^{\alpha} dx = \frac{1}{\alpha+1}(x-a)^{\alpha+1} + C$ if $\alpha \neq -1$

Rule 2' $\int \frac{1}{x-a} dx = \ln(x-a) + C$, provided $x > 0$

Rule 3' $\int e^{ax} dx = e^{ax}/a + C$

Rules of integration

Calculating indefinite integrals

Example

$$\begin{aligned}\int (9x^2+8)dx &= 9\int x^2 dx + \int 8dx \quad \text{by Rules 4 and 5} \\ &= 9(\frac{1}{3}x^3 + C_1) + (8x + C_2) \quad \text{by Rule 1}\end{aligned}$$

where C_1 and C_2 are arbitrary constants.

Setting $C = 9C_1 + C_2$, we see that

$$\int (9x^2+8)dx = 3x^3 + 8x + C$$

Exercises

Exercise 6 Solve for the following indefinite integrals

(a) $\int (9x^2 + 8)dx$

(b) $\int (6x + \frac{4}{x^3} - 9e^x)dx$

(c) $\int (6(x-1)^3 + 8e^{-4x})dx$

(d) $\int (x^{0.8} - 2)(x^{-0.6} - 3)dx$
Hint : develop the expression inside the integral.

Rules of integration

Calculating definite integrals

In contrast with the indefinite integrals discussed before, integrals in the form

$$\int_a^b f(x)dx$$

are known as **definite integrals**.

$f(x)$ is called the **integrand**.

a, b are **limits of integration**.

$[a, b]$ is the **range of integration**

Rules of integration

Calculating definite integrals

(i) Calculate the indefinite integral $\int f(x)dx = g(x) + C$

(ii) Calculate the definite integral as the difference of the indefinite integral values at the two limits of integration.

$$\Big[g(x) + C\Big]_a^b = (g(b) + C) - (g(a) + C)$$

(iii) Notice that the Cs cancel out. The definite integral becomes

$$\int_a^b f(x)dx = g(b) - g(a)$$

Exercises

Example

$$\int_1^3 (9x^2+8)dx = \Big[3x^3+8x\Big]_1^3 = (81+24)-(3+8) = 94$$

Exercise 7 Solve for the following definite integrals

(a) $\int_1^3 (9x^2+8)dx$

(b) $\int_{10}^{11} \frac{2}{x}dx$

(c) $\int_{-1}^{1} \frac{2x+3}{x+2}dx$

Exercises

Exercise 8 Find the following definite integrals :

$$(a) \quad \int_1^2 x^4 dx \quad (b) \quad \int_0^3 e^{w/4} dw$$

Exercise 9 Find the following indefinite integrals :

$$(c) \quad \int (2x^3 + 3x - 1) dx \quad (d) \quad \int (2e^{5t} + 5e^{-5t} - 5t) dt$$

Exercise 10 Find the following integrals by first simplifying the integrand :

$$(e) \quad \int (x^{3/4} - 6)/x \, dx \quad (x > 0) \quad (f) \quad \int (e^{2x} + e^{-2x})(e^{3x} + e^{-3x}) dx$$

Methods of Integration

Integration by parts

We showed that if the function g has a continuous derivative,

$$\int_a^b g'(x)dx = g(b) - g(a)$$

Suppose that $g(x)$ can be expressed as a product of two functions, say $g(x) = p(x)q(x)$.

Using the product rule,

$$\int_a^b \Big(p'(x)q(x) + p(x)q'(x)\Big)dx = p(b)q(b) - p(a)q(a)$$

Rearranging,

$$\int_a^b p'(x)q(x)dx = \Big[p(x)q(x)\Big]_a^b - \int_a^b p(x)q'(x)dx$$

The technique is known as **integration by parts**.

Methodology :

Suppose the integrand $f(x)$ can be written as a product of two functions.

Denote the first function by $p'(x)$, where $p(x)$ is an easily-found primitive, and the second function by $q(x)$, where $q(x)$ becomes simpler to integrate when differentiated.

Then $\int_a^b f(x)dx$ is equal to the LHS of the below equation and is evaluated as the RHS.

$$\int_a^b p'(x)q(x)dx = \Big[p(x)q(x)\Big]_a^b - \int_a^b p(x)q'(x)dx$$

The indefinite-integrate version is

$$\int p'(x)q(x)dx = p(x)q(x) - \int p(x)q'(x)dx$$

Example : $\int x^3 \ln x \, dx$

Note that an obvious primitive of x^3 is $x^4/4$, while $\ln x$ simplifies on differentiation to $1/x$.

Therefore, $p(x) = x^4/4$ and $q(x) = \ln x$

$$\begin{aligned}\int x^3 \ln x \, dx &= \frac{x^4}{4} \ln x - \int \frac{x^4}{4} \frac{1}{x} dx \\ &= \frac{1}{4}\left(x^4 \ln x - \int x^3 dx\right) \\ &= \left(\frac{4 \ln x - 1}{16}\right) x^4 + C\end{aligned}$$

Exercises

Exercise 11 Use integration by parts to determine

(a) $\int_0^2 x^2 e^{x/2} dx$

(b) $\int 4x(x+1)^3 dx$

(c) $\int \frac{2x}{(x-8)^3} dx$

Methods of Integration

Integration by substitution

Let f and g be functions. Let F be a primitive for f; thus F' and f are the same function. By the composite function rule,

$$\frac{d}{dx}F(g(x)) = f(g(x))g'(x)$$

Integrating from $x = a$ to $x = b$ we have

$$\int_a^b \frac{d}{dx}F(g(x)) = F(g(b)) - F(g(a)) = \int_a^b f(g(x))g'(x)dx$$

Setting $t = g(x)$, with the consequence that $dt/dx = g'(x)$ and therefore

$$dt = g'(x)dx$$

Setting $g(a) = r$ and $g(b) = s$, we have

$$F(s) - F(r) = \int_r^s f(t)dt$$

Methods of Integration

Integration by substitution

Hence,

$$\int_a^b f(g(x))g'(x)dx = \int_r^s f(t)dt, \quad \text{where } r = g(a) \text{ and } s = g(b)$$

The technique is known as **integration by substitution**.

Important : Note that we are integrating not wrt to x but wrt t, where $t = g(x)$. Therefore the limits of integration have to change too !

$$x = a \to t = g(a) = r$$
$$x = b \to t = g(b) = s$$

Methodology :

Suppose we have a situation where the integrand can be written in the form $f(g(x))g'(x)$ and f is easy to integrate.

Then, we may integrate with respect to t rather than x.

$$\int_a^b f(g(x))g'(x)dx = \int_r^s f(t)dt, \quad \text{where } r = g(a) \text{ and } s = g(b)$$

The indefinite-integral form is

$$\int f(g(x))g'(x)dx = \int f(t)dt, \quad \text{where } t = g(x)$$

Important : we have to convert the answer into an expression of x rather than t using $t = g(x)$.

Example :

$$\int \frac{\ln x}{x} dx$$

Since the integrand is $(\ln x) d(\ln x)$, we may set $t = \ln x$, $f(t) = t$

$$\int \frac{\ln x}{x} dx = \int t dt = \frac{1}{2} t^2 + C = \frac{1}{2} (\ln x)^2 + C$$

where C is an arbitrary constant.

Exercises

Exercise 12 Solve the following integrals using integration by substitution

(a) $\int (x+4)^5 dx$

(b) $\int (ax+b)^n dx$

(c) $\int_0^X xe^{-x^2/2} dx$

Methodology :

Suppose another situation, we wish to calculate $\int f(x)dx$, where $f(x)$ involves some complication.

Then a suitable substitution of the form $x = g(t)$ may enable us to simplify the integration.

$$\int_a^b f(x)dx = \int_u^v f(g(t))g'(t)dt, \quad \text{where } g(u) = a \text{ and } g(v) = b$$

Important : calculate using the proper limits, $g(u) = a$ and $g(v) = b$.

Methods of Integration

Integration by substitution

The indefinite-integral form is

$$\int f(x)dx = \int f(g(t))g'(t)dt, \quad \text{where } x = g(t)$$

Important : we have to convert the answer into an expression of x rather than t using $x = g(t)$.

Exercises

Exercise 13 Evaluate $\int_1^2 x\sqrt{x-1}dx$ using integration by substitution.

Exercises

Exercise 14 Use integration by parts to find :

$$(a) \quad \int x\sqrt{x+1}dx, \quad (b) \quad \int_0^3 (x+3)(x+2)^{-4}dx$$

Exercise 15 If a is a constant, then $\frac{d}{dx}(x+a) = 1$. Use this fact and integration by parts to find

$$\int \ln(x+a)dx$$

Exercise 16 Use integration by substitution to find :

$$(c) \quad \int x^3 \exp(x^4+1)dx \quad ,(d) \quad \int \frac{6x}{x^2+1}dx$$

Infinite integrals

It's easy to see that for any positive number X,

$$\int_1^X x^{-2} dx = 1 - X^{-1}$$

As $X \to \infty$, $1/X \to 0$. Hence,

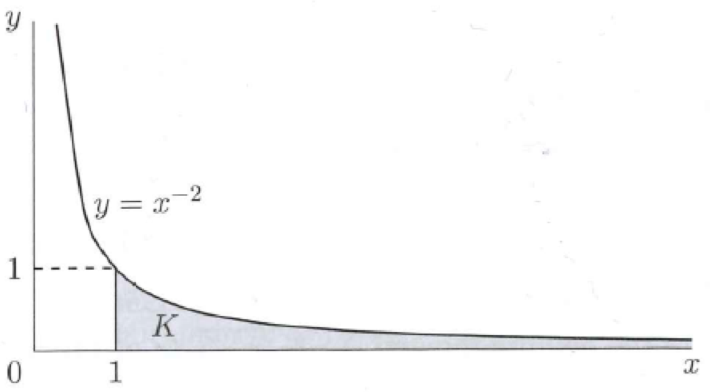
$$\lim_{X \to \infty} \int_1^X x^{-2} dx = 1$$

We write more concisely as

$$\int_1^\infty x^{-2} dx = 1$$

This is called an **infinite integral** (integration is taking place over an infinite interval).

Infinite integrals



$K = 1$ while K extends infinitely far to the right.

Infinite integrals

Suppose we have a continuous function $f(x)$ defined for $x \geq a$ and suppose

$$\int_a^X f(x)dx = L$$

as $X \to \infty$.

If L is finite, we say that the infinite integral **exists** or **converges**.

If L is infinite, we say that the infinite integral **does not exist** or **diverge**.

Two-sided infinite integrals

If $f(x)$ is a continuous function for all $x \leq a$, and if the definite integral $\int_Y^a f(x)dx$ approaches a finite limit as $Y \to -\infty$, we denote the limit by

$$\int_{-\infty}^{a} f(x)dx$$

If the integrals $\int_{-\infty}^{a} f(x)dx$ and $\int_a^{\infty} f(x)dx$ both exist, we denote their sum by

$$\int_{-\infty}^{\infty} f(x)dx$$

Exercises

Exercise 17

We have shown in **Exercise 13**

$$\int_0^X xe^{-x^2/2}dx = 1 - e^{-X^2/2}$$

Calculate

$$\int_{-\infty}^{\infty} xe^{-x^2/2}dx = \int_0^{\infty} xe^{-x^2/2}dx + \int_{-\infty}^{0} xe^{-x^2/2}dx$$

Improper integrals

An **improper integral** is an integral where the integrand is not defined at one of the limits of integration.

Example :

$$I = \int_0^1 x^{-1/2} dx$$

is an improper integral since $x^{-1/2} \to \infty$ as $x \to 0$.

Exercises

Exercise 18

(a) Find $\int_1^X x^{-3/2} dx$. Hence find $\int_1^\infty x^{-3/2} dx$.

(b) Show that $\int_1^\infty x^{-1} dx$ diverges.

(c) Try to formulate a general result concerning the existence or otherwise of $\int_1^\infty x^{-\alpha} dx$ where α is a constant.

Exercise 19

(a) Find $\int_\delta^1 x^{-1/3} dx$. Hence find $\int_0^1 x^{-1/3} dx$.

(b) Show that $\int_0^1 x^{-1} dx$ diverges.

(c) Try to formulate a general result concerning the existence or otherwise of $\int_0^1 x^{-\alpha} dx$ where α is a constant.

Differentiation under the integral sign

Suppose $f(x, y)$ is a function of two variables and a, b are constants.

We can define

$$I(y) = \int_a^b f(x, y) dx$$

Suppose f is a smooth function, $I'(y)$ exists and is given by

$$I'(y) = \int_a^b \frac{\partial}{\partial y} f(x, y) dx$$

Differentiation under the integral sign

To summarise

$$\frac{d}{dy}\Big(\int_a^b f(x,y)dx\Big) = \int_a^b \frac{\partial}{\partial y} f(x,y)dx$$

This is known as **differentiation under the integral sign**.

Differentiation under the integral sign

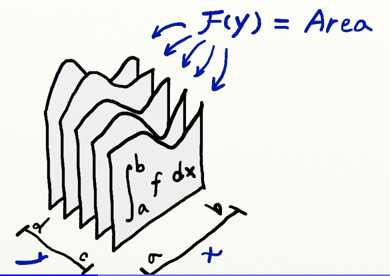
Intuition behind

To understand some formulas from calculus intuitively, we could try to interpret the notions in discrete ways.

For $y \in [c, d]$, the values

$$F(y) = \int_a^b f(x, y)dx$$

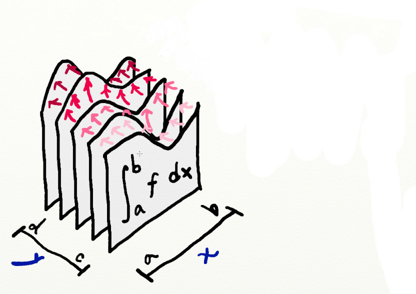
denote the areas of the slices (integrals along the x-axis specified by y)



Differentiation under the integral sign

Intuition behind

To measure the difference between each two adjacent areas (discrete analog of $F'(y)$), one tried to draw all the little changes in height between the slices.



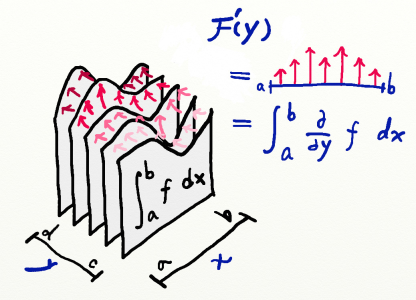
Pink arrows are grouped by their colors. Each group of pink arrows indicates the the rate of change of area between each two adjacent slices.

Differentiation under the integral sign

Intuition behind

On the other hand, every single pink arrow can also be regarded as the partial derivative $\frac{\partial}{\partial y} f$ at each point (x, y).

Thus, summing up a group of partial derivatives means measuring the difference between two areas.



Differentiation under the integral sign

The general case

We generalise the previous equation by allowing a and b to depend on y.

We first define a function of three variables

$$J(a, b, y) = \int_a^b f(x, y)dx$$

By the **fundamental theorem of calculus**, we know that

$$\frac{\partial J}{\partial b} = f(b, y)$$

Moreover,

$$\frac{\partial J}{\partial y} = \int_a^b \frac{\partial f}{\partial y} dx$$

by **differentiation under the integral sign**.

Differentiation under the integral sign

The general case

Finally, we use the fact that

$$J(a, b, y) = -\int_b^a f(x, y)dx$$

Hence,

$$\frac{\partial J}{\partial a} = -f(a, y)$$

To summarise

$$\frac{\partial J}{\partial a} = -f(a, y) \quad \frac{\partial J}{\partial b} = f(b, y) \quad \frac{\partial J}{\partial y} = \int_a^b \frac{\partial f}{\partial y}dx$$

Differentiation under the integral sign

The general case

Let $I(y)$ be defined as before but now a and b depends on y : $a = p(y)$, $b = q(y)$. Then

$$I(y) = J(p(y), q(y), y)$$

By the rules of **total differentiation**

$$I'(y) = \frac{\partial J}{\partial a} p'(y) + \frac{\partial J}{\partial b} q'(y) + \frac{\partial J}{\partial y}$$

Differentiation under the integral sign

The general case

Replacing with the expressions we obtained above

$$\frac{d}{dy}\Big(\int_{p(y)}^{q(y)} f(x,y)dx\Big) = f(q(y),y)q'(y) - f(p(y),y)p'(y) + \int_{p(y)}^{q(y)} \frac{\partial}{\partial y} f(x,y)dx$$

which is known as the **Leibniz's formula**.

Differentiation under the integral sign

Example :

Find $\frac{dI}{dx}$ if

$$I = \int_{x}^{2x} \frac{e^{xt}}{t} dt$$

Applying **Leibniz's formula**, we get

$$\int_{x}^{2x} \frac{te^{xt}}{t} dt + \frac{e^{x.2x}}{2x}.2 - \frac{e^{x.x}}{x}.1$$

Simplifying

$$\begin{aligned} \frac{1}{x}\left(e^{2x^2} - e^{x^2}\right) + \left[\frac{e^{xt}}{x}\right]_{x}^{2x} \\ = \frac{2}{x}\left(e^{2x^2} - e^{x^2}\right) \end{aligned}$$

Exercises

Exercise 20

Differentiate the following expressions with respect to y :

(a) $\int_1^5 \frac{f(x)}{x+y} dx$

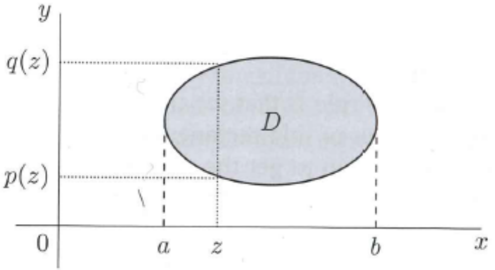
(b) $\int_1^{e^y} f(xy) dx$

(c) $\int_y^{y^2} \frac{1}{2} t^2 y dt$

Double integrals

Just as a integral of a function of one variable represents an **area in the plane**, a double integral of a function of two variables represents a **volume in three-dimensional space**.

Let D be a region in the xy-plane that is 'well-behaved', i.e. consists of one piece, with no hole, and can be contained within a rectangle.



Double integrals

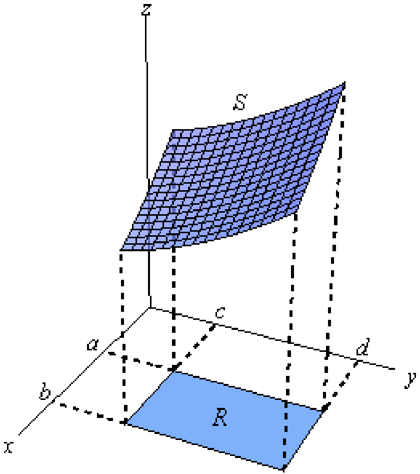
Let $f(x, y)$ be a continuous function of two variables.

If $f(x, y) \geq 0$ for all x and y, so that $z = f(x, y)$ represents a region in xyz-space, then

$$\int\int_D f(x, y) dx\, dy$$

is the volume over D under the surface $z = f(x, y)$

Double integrals



Double integrals

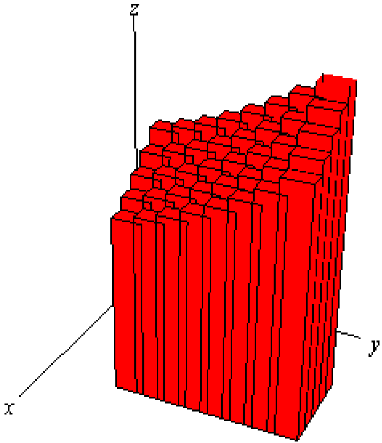
As with single integrals, double integrals may be regarded as limits of sums.

Suppose we approximate the region D by a large collection of small non-overlapping rectangles, $Q_1, \ldots, Q_N$.

For $k = 1, \ldots, N$, let a_k be the area of Q_k and let (x_k, y_k) be a point in Q_k. Then

$$\sum_{k=1}^{N} a_k f(x_k, y_k) \approx \int\int_D f(x, y) dx\, dy$$

Double integrals



Double integrals can be computed by a method known as **repeated integrals**.

It consists of the calculation of two definite integrals of functions of one variable, one after the other.

Double integrals

Repeated integrals

Method A Suppose there are real numbers a and b, and functions p and q, such that

$$D = \{(x, y) \in \mathbb{R}^2 \, : \, a \leq x \leq b \quad \text{and } p(x) \leq y \leq q(x)\}$$

Then,

$$\int \int_D f(x, y) dx \, dy = \int_a^b \Big[\int_{p(x)}^{q(x)} f(x, y) dy \Big] dx$$

Intuition :
Think region D being approximated by a large number of thin rectangular strips, each parallel to the y-axis.

We integrate over each strip to get the term in square brackets, and then put the pieces together by integrating wrt x.

Method B Similar to Method A but the roles of x and y are reversed : divide D into thin parallel to the x-axis rather than the y-axis.

Suppose there are real numbers α and β, and functions ϕ and ψ, such that

$$D = \{(x, y) \in \mathbb{R}^2 \ : \ \alpha \leq y \leq \beta \quad \text{and } \phi(y) \leq x \leq \psi(y)\}$$

Then,

$$\int \int_D f(x, y) dx\, dy = \int_\alpha^\beta \Big[\int_{\psi(y)}^{\phi(y)} f(x, y) dx \Big] dy$$

Double integrals

Repeated integrals

The integral inside the bracket is called the **inner integral**.

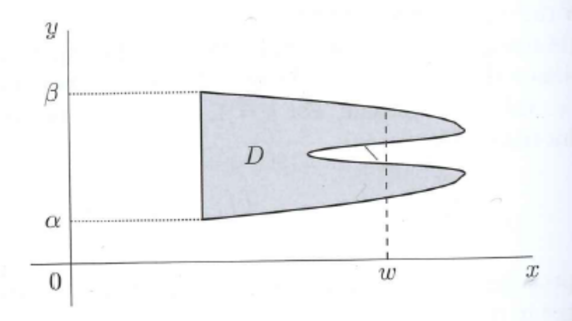
Calculation of the inner integral under Method A is performed treating x as constant.

Under Method B we calculate the inner integral treating y as constant.

Which method is better ? It varies from case to case.

Double integrals

Repeated integrals



There are values of x, for instance w, such that the set of values of y for which $(x, y) \in D$ is not one interval but two.

In this case, $\int\int_D f(x,y)dx\,dy$ can be calculated by Method B but not Method A.

Double integrals
The simplest case

Let,

$$f(x, y) = g(x)h(y), \quad D = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b \text{ and } u \leq y \leq v\}$$

where a, b, u and v are constants.

Method A tells us

$$\int \int_D f(x, y)dx\, dy = \int_a^b \left[\int_u^v g(x)h(y)dy \right] dx$$

Double integrals

The simplest case

Let $J = \int_u^v h(y)dy$. Since $g(x)$ does not depend on y, the inner integral is equal to $Jg(x)$.

Since J is a constant, it follows that

$$\int\int_D g(x)h(y)dx\,dy = J\int_a^b g(x)dx$$

Hence, by definition of J,

$$\int\int_D g(x)h(y)dx\,dy = \left(\int_u^v h(y)dy\right)\left(\int_a^b g(x)dx\right)$$

Method B yields the same result.

Exercises

Exercise 21 Let $D = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 2 \text{ and } 1 \leq y \leq 3\}$ and $f(x, y) = x^2y^3$. Calculate

$$\int \int_D f(x, y) dx\, dy$$

Double integrals

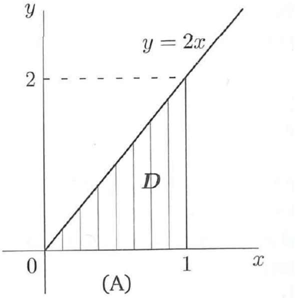
Harder examples

We wish to evaluate

$$I = \int\int_D (x+y)dx\, dy$$

with

$$D = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 2x\}$$



Double integrals

Harder examples

Using Method A

$$I = \int_0^1 \left[\int_0^{2x} (x+y)dy \right] dx = \int_0^1 \left[xy + \frac{1}{2}y^2 \right]_{y=0}^{y=2x} dx$$

Since $x(2x) + \frac{1}{2}(2x)^2 = 4x^2$,

$$I = 4\int_0^1 x^2 dx = 4/3$$

Exercises

Exercise 22 evaluate the previous integral using Method B. Define carefully D.

Exercise 23 Let D be the region in the positive quadrant bounded by the y-axis, the line $y = 1$ and the curve $y = x^3$. We wish to evaluate

$$I = \int\int_D (x^2 + y^2) dx\, dy$$

Define and depict region D and evaluate I using (a) Method A and (b) Method B.

Exercises

Exercise 24 Let E be the region in the positive quadrant bounded by the x-axis and the straight lines $x = 2$ and $y = x$. We wish to evaluate

$$I = \int \int_E \exp(x^2) dx\, dy$$

Define and depict region E and evaluate I by Method A.