Macroeconomics A; EI060

Short problems

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Class of March 19, 2025

1 Maximum debt under default risk

Question: Consider a two period model. The country starts period 2 with a debt d_1 and gets a stochastic output y_2 uniformly distributed over the interval $\left[0, y_2^H\right]$.

The probability that $y_2 < \alpha$ is $\frac{\alpha}{y_2^H}$.

If the country does not default it repays $(1+r^s)\,d_1$. In case of default, the borrower loses ϕy_2 and the lender gets nothing. The probability of default is:

$$
\begin{aligned}
\pi &= Prob\left[y_2 < \frac{(1+r^s)\,d_1}{\phi}\right] \\
\pi &= \frac{(1+r^s)\,d_1}{\phi y_2^H}
\end{aligned}
$$

The lender is risk neutral and requires an expected return $1+r$. Show that:

$$
1+r^s = \frac{1+r}{1-(1+r^s)\left(\frac{d_1}{\phi y_2^H}\right)}
$$

What is the probability of default and r^sif $d_1 \leq 0$?

Show that the maximum debt that the economy can borrow is:

$$
d_1^{high} = \frac{\phi y_2^H}{4\,(1+r)}
$$

This is subtle, so proceed as follows:

1. The expression above has a left-hand side $1+r^s$ and a more complex right hand side, call it a function $G\,(1+r^s)$.
2. Show that $G' > 0$ and $G'' > 0$, $G\,(0) = 1+r$. At $1+r^s = 0$, how do the left and right-hand side compare (which is larger)?
3. The maximmum debt is when the two side of the equations are tangent, i.e. have the same level and slope. The equality of slopes gives $(1+r^s) = 2\,(1+r)$, and the equality of levels gives d_1^{high}.

2 Endogenous interest rate

Question: Take the non-linear expression in $1+r^s$:

$$1+r^s = \frac{1+r}{1-(1+r^s)\left(\frac{d_1}{\phi y_2^H}\right)}$$

Show that it is a quadratic polynomial:

$$0 = \frac{d_1}{\phi y_2^H}(1+r^s)^2 - (1+r^s) + (1+r)$$

Show that the solution is:

$$1+r^s = 2(1+r)\frac{d_1^{high}}{d_1}\left(1-\sqrt{1-\frac{d_1}{d_1^{high}}}\right)$$

What is the interest rate if $d_1 = d_1^{high}$?

Note: there is another solution with a higher interest rate, but it is unstable.

3 Intertemporal allocation

Question: The borrower maximizes a linear utility, where $r < \delta$:

$$U = c_1 + \frac{1}{1+\delta}Ec_2$$

The budget constraints are:

$$\begin{aligned} c_1 &= y_1 + d_1 \\ c_2^{ND} &= y_2 - (1+r^s)\,d_1 \\ c_2^D &= (1-\phi)\,y_2 \end{aligned}$$

Some useful properties that you can take as given:

$$\begin{aligned} \pi E(y_2 \mid D) + (1-\pi)\,E(y_2 \mid ND) &= E(y_2) = \frac{y_2^H}{2} \\ E(y_2 \mid D) &= \frac{\pi y_2^H}{2} \end{aligned}$$

Show that the utility is:

$$U = Y_1 + \frac{\delta - r}{1+\delta}d_1 + \frac{1}{1+\delta}E(y_2) - \frac{1}{1+\delta}\pi\phi E(y_2 \mid D)$$

Using the expected values of output, show that:

$$U = y_1 + \frac{\delta - r}{1+\delta}d_1 + \frac{1}{1+\delta}\frac{y_2^H}{2}\left(1-\phi\pi^2\right)$$

Show that the optimal debt level is such that (recall that π is a function of the debt):

$$\delta - r = \phi\pi y_2^H \frac{\partial \pi}{\partial d_1}$$

4 Marginal impact of debt on default

Question: The probability of default, and the relation between the risk-free rate and risky rate are:

$$\begin{aligned}\pi &= \frac{(1+r^s)\,d_1}{\phi y_2^H}\\ 1+r &= (1-\pi)(1+r^s)\end{aligned}$$

Differentiate these two relations with respect to the risky interest rate, the probability of default, and the debt/GDP ratio to get:

$$\begin{aligned}d\pi &= d(1+r^s)\left(\frac{d_1}{\phi y_2^H}\right)+(1+r^s)\,d\left(\frac{d_1}{\phi y_2^H}\right)\\ (1+r^s)\,d\pi &= (1-\pi)\,d(1+r^s)\end{aligned}$$

Combine these to show:

$$\frac{\partial\pi}{\partial d_1} = \frac{(1+r^s)}{1-\frac{1+r^s}{1-\pi}\frac{d_1}{\phi y_2^H}}\frac{1}{\phi y_2^H}$$

5 Overall solution

Question: The system is given by the probability of default, the relation between the risk-free rate and the risky rate, and the optimality condition above:

$$\begin{aligned}\pi &= \frac{(1+r^s)\,d_1}{\phi y_2^H}\\ 1+r &= (1-\pi)(1+r^s)\\ \delta-r &= \phi\pi y_2^H\frac{\partial\pi}{\partial d_1}\end{aligned}$$

Using our results, show that:

$$\begin{aligned}\pi &= \frac{\delta-r}{1+2\delta-r}\\ 1+r^s &= \frac{1+r}{1+\delta}(1+2\delta-r)\\ \frac{d_1}{\phi y_2^H} &= \frac{(1+\delta)(\delta-r)}{(1+r)(1+2\delta-r)^2}\end{aligned}$$