PS1 Solutions

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Solution 1 (Gains from Trade)**.**
Sorry professor, we are really not sure about our logic and proof of this question, so we put two answers here, and really expect to learn about the proof of this question. **Sufficient Condition:** $\mathbf{p}^2 \leq \mathbf{p}^1$ (with at Least One Strict Inequality)

- **Definition:** The inequality $\mathbf{p}^2 \leq \mathbf{p}^1$ means that each component (price of each good) in $\mathbf{p}^2$ is less than or equal to the corresponding component in $\mathbf{p}^1$, with at least one component being strictly lower.
- **Welfare Gain:** With the same income, the consumer can purchase at least the original bundle (and likely an improved bundle), thereby strictly increasing (or at least not decreasing) utility.

Thus, $\mathbf{p}^2 \leq \mathbf{p}^1$ **guarantees** that Australia's representative consumer gains. **The Rationale for Requiring** $\mathbf{p}^2 \leq \mathbf{p}^1$

- **Unambiguous Improvement:** Demonstrating that every price relevant to Australia does not increase (with at least one strictly decreasing) leaves no ambiguity: the consumer's cost of living falls.
- **Simplification:** Without this uniform price decrease, some goods may become cheaper while others become more expensive, complicating the direct welfare comparison.

Sufficient versus Necessary Condition

- **Sufficient:** The condition $\mathbf{p}^2 \leq \mathbf{p}^1$ is sufficient to ensure a welfare gain.
- **Not Strictly Necessary:** Even if some components of $\mathbf{p}^2$ exceed those of $\mathbf{p}^1$, Australia's consumer might still be better off if the goods that become cheaper are those that the consumer values most. In such cases, the outcome depends on the specific consumption patterns and preferences.

Role of China's Trade Balance

- China's trade deficit with Britain and surplus with Australia reshuffle global supply and demand.

- This adjustment can lower the prices of goods consumed by Australia while potentially raising the prices of other goods.
- If the net effect is that the overall cost of Australia's consumption basket declines, then the representative consumer benefits.

Conclusion

- **Guaranteed Gain:** To unambiguously show that Australia's representative consumer is better off after China's entry, one typically assumes $\mathbf{p}^2 \leq \mathbf{p}^1$. Under this assumption, the consumer's real purchasing power increases.
- **General Case:** Without this assumption, the outcome depends on which prices rise or fall relative to Australian consumption preferences. Nonetheless, $\mathbf{p}^2 \leq \mathbf{p}^1$ remains a sufficient condition for a welfare gain.

Proof 2

Initially, Australia (A) trades freely with Britain (B). Both are large countries. The vector of equilibrium prices in the initial free-trade equilibrium is $\mathbf{p}_1$. Now it becomes possible for Australia to trade (freely) with China (C). The vector of equilibrium prices in the three-country world is $\mathbf{p}_2$. China runs a trade deficit with Britain balanced by a surplus with Australia. Let us define the following notation:

- $e^A(\mathbf{p}, u)$: Australia's expenditure function at price vector $\mathbf{p}$ and utility level u
- $r^A(\mathbf{p})$: Australia's revenue (GDP) function at price vector $\mathbf{p}$
- m_1^A: Australia's trade balance in the initial two-country equilibrium
- m_2^A: Australia's trade balance in the new three-country equilibrium
- $\mathbf{x}^A(\mathbf{p}, u)$: Australia's net export vector at price vector $\mathbf{p}$ and utility level u

In the initial equilibrium, Australia's budget constraint is:

$$e^A(\mathbf{p}_1, u_1) = r^A(\mathbf{p}_1) + m_1^A \tag{1}$$

After China enters the trading system, Australia's budget constraint becomes:

$$e^A(\mathbf{p}_2, u_2) = r^A(\mathbf{p}_2) + m_2^A \tag{2}$$

To determine whether Australia gains, we need to compare u_2 with u_1. Using a first-order Taylor expansion of the expenditure function around u_1:

$$e^A(\mathbf{p}_2, u_2) \approx e^A(\mathbf{p}_2, u_1) + \frac{\partial e^A(\mathbf{p}_2, u_1)}{\partial u}(u_2 - u_1) \tag{3}$$

where $\frac{\partial e^A(\mathbf{p}_2,u_1)}{\partial u} > 0$ represents the marginal cost of utility. From equations (2) and (3), we get:

$$\frac{\partial e^A(\mathbf{p}_2,u_1)}{\partial u}(u_2-u_1) = r^A(\mathbf{p}_2) - e^A(\mathbf{p}_2,u_1) + m_2^A - (r^A(\mathbf{p}_1) - e^A(\mathbf{p}_1,u_1) + m_1^A) \tag{5}$$

Simplifying:

$$\frac{\partial e^A(\mathbf{p}_2,u_1)}{\partial u}(u_2-u_1) = [r^A(\mathbf{p}_2) - r^A(\mathbf{p}_1)] - [e^A(\mathbf{p}_2,u_1) - e^A(\mathbf{p}_1,u_1)] + [m_2^A - m_1^A] \tag{6}$$

According to Shephard's Lemma:

$$\nabla_p e^A(\mathbf{p},u) = \mathbf{c}^A(\mathbf{p},u) \tag{7}$$

where $\mathbf{c}^A(\mathbf{p},u)$ is the Hicksian demand function.
According to Hotelling's Lemma:

$$\nabla_p r^A(\mathbf{p}) = \mathbf{y}^A(\mathbf{p}) \tag{8}$$

where $\mathbf{y}^A(\mathbf{p})$ is the supply function.
Using the mean value theorem, we can write:

$$e^A(\mathbf{p}_2,u_1) - e^A(\mathbf{p}_1,u_1) = \nabla_p e^A(\tilde{\mathbf{p}},u_1)\cdot(\mathbf{p}_2-\mathbf{p}_1) = \mathbf{c}^A(\tilde{\mathbf{p}},u_1)\cdot(\mathbf{p}_2-\mathbf{p}_1) \tag{9}$$

Similarly:

$$r^A(\mathbf{p}_2) - r^A(\mathbf{p}_1) = \nabla_p r^A(\hat{\mathbf{p}})\cdot(\mathbf{p}_2-\mathbf{p}_1) = \mathbf{y}^A(\hat{\mathbf{p}})\cdot(\mathbf{p}_2-\mathbf{p}_1) \tag{10}$$

where $\tilde{\mathbf{p}}$ and $\hat{\mathbf{p}}$ are some intermediate price vectors between $\mathbf{p}_1$ and $\mathbf{p}_2$.
The net export vector can be expressed as:

$$\mathbf{x}^A(\mathbf{p},u) = \mathbf{y}^A(\mathbf{p}) - \mathbf{c}^A(\mathbf{p},u) \tag{11}$$

Assuming the price changes are relatively close that $\tilde{\mathbf{p}} \approx \hat{\mathbf{p}} = \mathbf{p}^*$ (some intermediate price vector), we can approximately express:

$$[r^A(\mathbf{p}_2) - r^A(\mathbf{p}_1)] - [e^A(\mathbf{p}_2,u_1) - e^A(\mathbf{p}_1,u_1)] \approx \mathbf{x}^A(\mathbf{p}^*,u_1)\cdot(\mathbf{p}_2-\mathbf{p}_1) \tag{12}$$

Substituting this result into equation (6):

$$\frac{\partial e^A(\mathbf{p}_2,u_1)}{\partial u}(u_2-u_1) \approx \mathbf{x}^A(\mathbf{p}^*,u_1)\cdot(\mathbf{p}_2-\mathbf{p}_1) + [m_2^A - m_1^A] \tag{13}$$

Since $\frac{\partial e^A(\mathbf{p}_2,u_1)}{\partial u} > 0$, a sufficient condition for Australia's welfare to improve ($u_2 > u_1$) is:

$$\mathbf{x}^A(\mathbf{p}^*,u_1)\cdot(\mathbf{p}_2-\mathbf{p}_1) + [m_2^A - m_1^A] > 0 \tag{14}$$

Given that China runs a trade surplus with Australia, this implies $m_2^A < m_1^A$, meaning $[m_2^A - m_1^A] < 0$.
Therefore, to ensure welfare improvement, the following must hold:

$$\mathbf{x}^A(\mathbf{p}^*, u_1) \cdot (\mathbf{p}_2 - \mathbf{p}_1) > [m_1^A - m_2^A] \tag{15}$$

This condition states that the terms of trade improvement effect due to price changes must exceed the negative impact of trade balance deterioration. Specifically, if:

1. The prices of goods that Australia exports to Britain increase
2. The prices of goods that Australia imports from China decrease
3. The terms of trade improvement effect from these price changes exceeds the impact of trade balance deterioration

then the representative consumer in Australia will experience a welfare gain.

Solution 2 (Ricardian Trade and Technological Progress)**.**

1. For absolute advantage, we compare unit labor productivity directly:
   - Clothing
     - Home: produces z unit per hour.
     - Foreign: produces 1 unit per hour.

     Home has an absolute advantage in clothing if $z > 1$; otherwise, Foreign does.
   - Food
     - Home: produces z unit per hour.
     - Foreign: produces 4 unit per hour.

     Home has an absolute advantage in food if $z > 4$; otherwise, Foreign does.

   Thus, comparing the absolute advantage, we have the following table:

	Clothing	Food
$z > 4$	Home	Home
$1 < z < 4$	Home	Foreign
$z < 1$	Foreign	Foreign

2. For comparative advantage, we compare the opportunity cost of producing one unit of one good in terms of the other good:
   - Home: The opportunity cost of 1 unit of Clothing over 1 unit of Food is: $\frac{a_C}{a_F} = 1$.

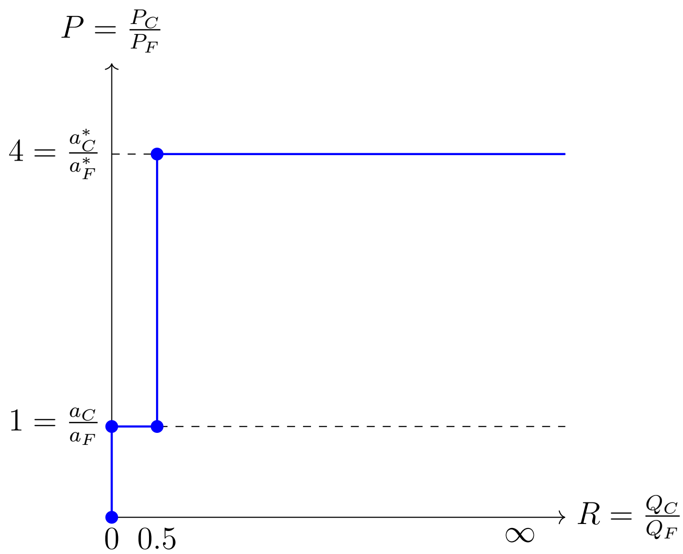
- Foreign: The opportunity cost of 1 unit of Clothing over 1 unit of Food is: $\frac{a_C^*}{a_F^*} = 4$.

Thus, Home has a comparative advantage in Clothing, and Foreign has a comparative advantage in Food, regardless of z.

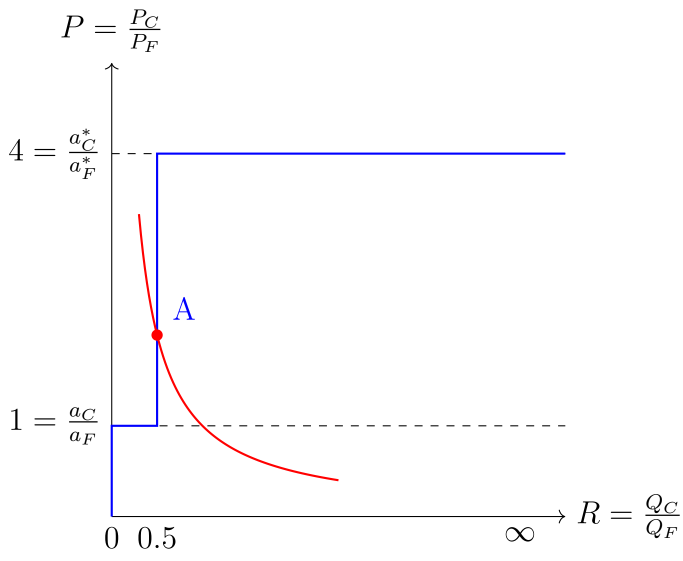
3. If $z = 2$, the relative price of Clothing is $P = \frac{P_C}{P_F} = \frac{a_C}{a_F} = 1$, $P^* = \frac{a_C^*}{a_F^*} = 4$. $Q_C = \frac{L}{a_C} = 2000 = Q_F$, $Q_C^* = \frac{L^*}{a_C^*} = 1000$, $Q_F^* = \frac{L^*}{a_F^*} = 4000$.

   (a) Draw the world relative supply of clothing.

   - When $P < 1$, both Home and Foreign produce only Food, giving $R = \frac{Q_C+Q_C^*}{Q_F+Q_F^*} = 0$;
   - When $P = 1$, Home can vary production between $(\text{Clothing}, \text{Food}) = \left[(2000, 0), (0, 2000)\right]$ and Foreign produces only Food, giving $R \in [0, 0.5]$;
   - When $1 < P < 4$, Home produces only Clothing and Foreign produces only Food, giving $R = 0.5$;
   - When $P = 4$, Home produces only Clothing and Foreign can vary production between $(\text{Clothing}, \text{Food}) = \left[(0, 4000), (1000, 0)\right]$, giving $R \in [0.5, \infty)$;
   - When $P > 4$, both Home and Foreign produce only Clothing, giving $R = \infty$.



   (b) Under the Cobb-Douglas utility function $U(C, F) = CF$, we can tell that consumers will spend the same expenditure on both goods. So, worldwide, $P_C Q_C = P_F Q_F$, which gives $\frac{P_C}{P_F} = \frac{Q_F}{Q_C} = \frac{1}{R} = 2$.



(c) In Home, a worker produces $z = 2$ units of Clothing per hour, hence the value of one hour's output is: $w = 2P_C$; While in foreign, a worker produces 4 units of Food per hour, having a value of $p^* = 4P_F$. Given that $\frac{P_C}{P_F} = 2$, we know it that $\frac{w}{w^*} = 1$.

4. As analyzed before, the change of z won't affect the world relative price. We assume that after the change both countries remain completely specialized in their comparative-advantage goods.

(a) Home produces $1000z$ units of Clothing, and Foreign produces 4000 unnits of food. The world relative supply of Clothing is $R = \frac{1000z}{4000} = \frac{z}{4}$.
For Home, a worker's nominal income is $w = zP_C$, and for Foreign, $w^* = 4P_F$. Our Cobb-Douglas utility with equal share tells us that $P = \frac{P_C}{P_F} = \frac{Q_F}{Q_C} = \frac{1}{R}$, thus $P = \frac{4}{z}$.
Thus the free-trade relative price is $\frac{P_C}{P_F} = \frac{4}{z}$, the wage ratio is:

$$\frac{w}{w^*} = \frac{zP_C}{4P_F} = 1.$$

(b) If z increases, the relative price $P = \frac{4}{z}$ decreases, clothing gets cheaper relative to food on world markets. To assess welfare, consider that consumers have the utility function $U(C, F) = C \cdot F$. The price index for such a utility function is given by

$$P = 2\sqrt{p_C\, p_F} = 2\sqrt{\frac{4}{z} \cdot 1} = \frac{4}{\sqrt{z}}.$$

Thus, the real wage (or real income) in each country is

$$\frac{w}{P} = \frac{4}{4/\sqrt{z}} = \sqrt{z}.$$

Intuitively, Home exports clothing; so Home workers' real wage measured in units of clothing rises directly with z. Meanwhile Foreign exports food; as clothing becomes cheaper, Foreign's real wage in terms of clothing goes up as well.

Solution 3 (Two-by-Two-by-Two with Fixed Coefficients)**.**

1. - **Relative Factor Abundance:** $RFA_A = \frac{K_A}{L_A} = \frac{420}{460} \approx 1.095$, $RFA_G = \frac{K_G}{L_G} = \frac{900}{600} = 1.5$. Thus Germany is relatively capital abundant and Austria is relatively labor abundant.
   - **Relative Factor Intensity of Goods:** $RFIG_B = \frac{a_{KB}}{a_{LB}} = 3$, $RFIG_S = \frac{a_{KS}}{a_{LS}} = 0.5$. Thus Buns are capital intensive, while Sausages are labor intensive.
   - **Comparative Advantage:** By the Heckscher-Ohlin theorem, the relatively capital-abundant country, Germany, will have a comparative advantage in the capital-intensive good, Buns, and the relatively labor-abundant country, Austria, in the labor-intensive good, Sausages.
   - **Autarkic Relative Price:** In autarky, each country's relative price reflects its "shadow" cost. Because factor prices adjust differently in each country (with the excess factor receiving a zero "price"), we expect:
     - Austria: As labor is in excess, the wage is set to 0, hence $P_A = \frac{P_{BA}}{P_{SA}} = \frac{a_{KB}}{a_{KS}} = 3$;
     - Germany: As capital is in excess, the rental rate is set to 0, hence $P_G = \frac{P_{BG}}{P_{SG}} = \frac{a_{LB}}{a_{LS}} = \frac{1}{2}$.

     Thus, the autarkic price of Buns is higher in Austria than in Germany. Under trade we expect Germany to export Buns and Austria to export Sausages.
   - **Free Trade:** Under trade we expect Germany to export Buns and Austria to export Sausages.

2. We separate the two countries' production functions and factor endowments: $B = Q_{BA} + Q_{BG}$, $S = Q_{SA} + Q_{SG}$.

   (a) For Austria, the full employment conditions are:

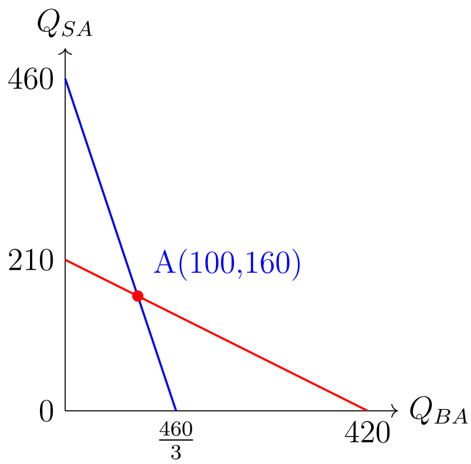
   - **Labor:**

   $$L_A = a_{LB}Q_{BA} + a_{LS}Q_{SA} \Rightarrow Q_{BA} + 2Q_{SA} = 420$$

   - **Capital:**

   $$K_A = a_{KB}Q_{BA} + a_{KS}Q_{SA} \Rightarrow 3Q_{BA} + Q_{SA} = 460$$

   Both constraints hold with equality, we can have: $(Q_{BA}, Q_{SA}) = (100, 160)$.



(b) For Germany, the full employment conditions are:

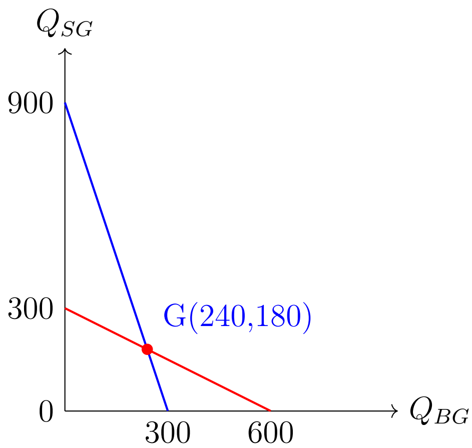
- **Labor:**

$$L_G = a_{LB}Q_{BG} + a_{LS}Q_{SG} \Rightarrow Q_{BG} + 2Q_{SG} = 600$$

- **Capital:**

$$K_G = a_{KB}Q_{BG} + a_{KS}Q_{SG} \Rightarrow 3Q_{BG} + Q_{SG} = 900$$

Solve the equations, we have: $(Q_{BG}, Q_{SG}) = (240, 180)$.



3. Consumers have Leontief preferences (they want to consume 1 Bun and 1 Sausage per hotdog). Because the consumption ratio is fixed at 1, autarkic equilibrium production must satisfy $B = S$.

   (a) We first find the ebalanced production point in both countries:

   - Austria:

$$\text{Labor: } B + 2B \leq 420$$
$$\text{Capital: } 3B + B \leq 460$$

The binding constraint is capital, so the maximum balanced production in Austria is $B = S = 115$.

- Germany:

$$\text{Labor: } B + 2B \leq 600$$
$$\text{Capital: } 3B + B \leq 900$$

The binding constraint is labor, so the maximum balanced production in Germany is $B = S = 200$.

The autarkic relative price is set by the zero-profit conditions.

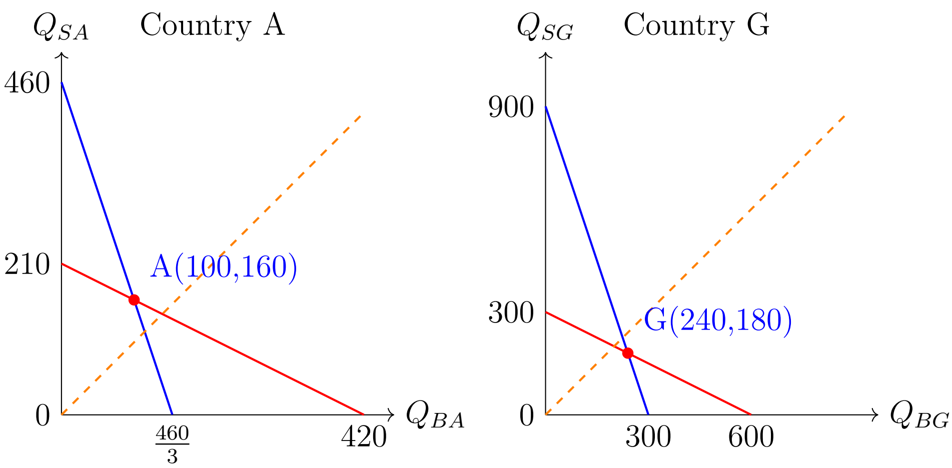
For Austria, as balanced production $(115, 115)$ lies in the interior of the production possibilities frontier, labor is in excess. (Thus $W = 0$, and prices are determined solely by the rental rate R.)

In Germany, capital is in excess, so $R = 0$.

The zero-profit conditions then are:

$$P_{BA} = W + 3R = 3R$$
$$P_{SA} = 2W + R = R$$
$$P_{BG} = W$$
$$P_{SG} = 2W$$

These relative prices are in line with our prediction from part (1). For Austria $\frac{P_{BA}}{P_{SA}} = 3 > \frac{1}{2} = \frac{P_{BG}}{P_{SG}}$, and it exports sausages, so this is in line with the Heckscher-Ohlin theorem.



(b) In Austria, labor is in excess, hence $W = 0$. Total national income is $R \cdot K_A = 460R$. Each hotdog costs $P_{BA} + P_{SA} = 4R$, so each owner of a unit of capital can buy $\frac{R}{4R} = \frac{1}{4}$ hotdogs.

In Germany, capital is in excess, so $R = 0$. Total income is $W \cdot L_G = 600W$. Each hotdog costs $P_{BG} + P_{SG} = 3W$, so each worker can buy $\frac{W}{3W} = \frac{1}{3}$ hotdogs.

4. (a) The production of both countries will be at the kink point where both factors are fully employed: In Austria: $(Q_B, Q_S) = (100, 160)$, and in Germany: $(Q_B, Q_S) = (240, 180)$.

   For the market clearing condition, in Austria, we have: $p_B C_B + p_S C_S = p_B Q_B + p_S Q_S$, as we are computing with regard to hotdogs which is a bundle of 1 Bun and 1 Sausage, and the condition that $\frac{p_B}{p_S} = 2$, $2C_B + C_S = 360$ and $C_B = C_S$, thus $C_B = C_S = 120$, Austria consumes 120 hotdogs, giving that it export 40 sausages and import 20 buns.

   As for Germany, we have: $p_B C_B + p_S C_S = p_B Q_B + p_S Q_S$, and the condition that $\frac{p_B}{p_S} = 2$, $2C_B + C_S = 660$ and $C_B = C_S$, thus $C_B = C_S = 220$, Germany consumes 180 hotdogs, giving that it export 20 buns and import 40 sausages.

   This is a Free-Trade equilibrium.

   (b) Under perfect competition, the zero-profit condition requires that the price of each good equals its unit cost. Let W denote the wage rate and R the rental rate of capital. Then, for each good we have:

$$P_B = W + 3R,$$
$$P_S = 2W + R.$$

   Substituting the given prices:

$$2 = W + 3R, \quad (1)$$
$$1 = 2W + R. \quad (2)$$

   Thus we have the free-trade factor prices in monetary units:

$$W = \frac{1}{5}, \quad R = \frac{3}{5}.$$

   Since a hotdog is a bundle of 1 Bun and 1 Sausage, its price is:

$$P_H = P_B + P_S = 2 + 1 = 3.$$

   Thus, the real wage (in hotdog units) is:

$$\tilde{W} = \frac{W}{P_H} = \frac{1/5}{3} = \frac{1}{15},$$

   and the real rental rate is:

$$\tilde{R} = \frac{R}{P_H} = \frac{3/5}{3} = \frac{1}{5}.$$

   This result confirms that under free trade the factor prices are equalized across countries, in line with the **factor price equalization theorem**.

(c)
- In **Austria**, compared to autarky, workers (the relatively abundant factor) see an increase in their real income while capital owners (the relatively scarce factor) experience a decline in their real returns.
- In **Germany**, the opposite occurs: workers (the relatively scarce factor) suffer a decrease in real income, while capital owners (the relatively abundant factor) benefit from higher returns.

This pattern of distribution is in accordance with the Stolper-Samuelson theorem, which states that free trade will benefit the factor that is relatively abundant in a country and hurt the factor that is relatively scarce.

- **Aggregate Effects:** Free trade leads to an efficient reallocation of resources, where each country produces a mix of both goods such that the markets for goods, labor, and capital clear. The unified factor prices under free trade ensure that overall, the economy attains a higher level of output and welfare. Specifically, the gains from trade come from:
  - Enhanced production efficiency by specializing according to comparative advantage.
  - Lower production costs, which result in a lower aggregate price level and higher real incomes.
- **Distributional Conflict:** Even though the overall welfare of an economy increases under free trade, the adjustment process generates income redistribution within each country. In Austria, where capital is relatively scarce, free trade raises workers' wages and lowers returns to capital, while in Germany, where labor is relatively scarce, free trade lowers workers' wages and increases returns to capital. This divergence in the impact on different groups within a country explains why:
  - Some groups (e.g., capital owners in Austria or workers in Germany) may oppose free trade because their relative income declines.
  - Despite these distributional conflicts, the aggregate gains are positive as both countries experience improved overall resource allocation and increased national welfare.

Thus, while free trade improves efficiency and increases total welfare (aggregate gains), it also intensifies income disparities between factors of production. These distributional conflicts are a central reason why domestic politics and interest group pressures can lead to divergent attitudes toward free trade policies.

Solution 4 (Two-by-Two-by-Two)**.**

1. Each country produces two goods using:

- **Good 1:** $x_1 = L_1$, only labor used,
- **Good 2:** $x_2 = K_2$, only capital used.

In any competitive equilibrium, the price of each good equals unit cost: $p_1 = w_J$, $p_2 = r_J$. As the household's utility is an 'equal-exponent' cobb-Douglas function, the expenditure on each good is the same, giving $p_1 c_1 = p_2 c_2$. Thus $\frac{p_1}{p_2} = \frac{c_2}{c_1}$. In autarky equilibrium, comsumptions equals output, thus $c_{1J} = x_{1J} = L_J$ and $c_{2J} = x_{2J} = K_J$. With $Y_J = w_J L_J + r_J K_J$, we know that $w_J L_J = r_J K_J$.

Thus in autarky, the relative price of Good 1 is $p_1/p_2 = w_J/r_J = K_J/L_J$.

2. Now if two countries integrate, then both countries have the same wage and rental rate: $p_1 = w$ and $p_2 = r$. Households, with the same preferences, spend half their income on each good, where income is $Y = w(L_H + L_F) + r(K_H + K_F)$. Based on similar analysis, we know that $\frac{w}{r} = \frac{K_H + K_F}{L_H + L_F} = 1$. Determined by world endowments and preferences.

3. Now suppose that goods trade is free but factors are immobile internationally. In that case each country's factor market outcome remains as in autarky, so we have:

   - **Home:** $\frac{p_{1H}}{p_{2H}} = \frac{w_H}{r_H} = \frac{K_H}{L_H} = 3$,
   - **Foreign:** $\frac{p_{1F}}{p_{2F}} = \frac{w_F}{r_F} = \frac{K_F}{L_F} = \frac{1}{3}$.

   In this trade equilibrium, factor price equalization (FPE) does not occur because factor markets remain separated. Each country's wages and rental rates continue to be determined by its own factor endowment.

4. Because factor prices in a competitive industry must equal the unit cost of production, the zero-profit conditions now read: In country H, if producing good 1, $p_1 = \frac{w_H}{9}$; if producing good 2, $p_2 = w_F$. As before, for good 2, $p_2 = r$ holds in either country.

   FPE requires identical production technologies (or, more precisely, identical "unit cost functions") so that the goods-market equilibrium forces the same factor prices everywhere. Here, even if the same world goods prices prevail, the "effective" unit cost for good 1 differs across countries.

   Thus, even under free goods trade the domestic factor prices in will remain different. In other words, the equilibrium will not exhibit factor price equalization.

5. W.l.o.g, we normalize $p_1 = 1$ and $p_2 = P$.

   - Country H: When producing good 1, $1 = \frac{w_H}{9} \Rightarrow w_H = 9$. When producing good 2, $P = r_H$.
   - Country F: When producing good 1, $1 = w_F$. When producing good 2, $P = r_F$.

Notice that for good 1 the effective cost in H is: $\frac{w_H}{9} = 1$, and in F is $w_F = 1$.

For country H:

$$\frac{w_H}{r_H} = \frac{K_H}{L_H} = 3$$

so $r_H = 3$.

For country F:

$$\frac{w_F}{r_F} = \frac{K_F}{L_F} = \frac{1}{3}$$

so $r_F = 3$. Notice that for good 2, the zero-profit conditions gives $r_J = P$, hence we have $P = 3$. Assume that in each country the production of goods follows the Cobb–Douglas cost functions derived earlier. For a given country with wage w and rental rate r, the unit cost functions are

$$c_1(w, r) = w^{\frac{2}{3}} r^{\frac{1}{3}}, \quad c_2(w, r) = w^{\frac{1}{3}} r^{\frac{2}{3}}.$$

Let us normalize the rental rate to 1 in each country (since only the ratio matters). Then, in country H, setting $r^H = 1$ and using $\frac{w^H}{1} = 3$, we have $w^H = 3$. Thus,

$$c_{1,H} = 3^{\frac{2}{3}} \quad \text{and} \quad c_{2,H} = 3^{\frac{1}{3}}.$$

Similarly, in country F, with $r^F = 1$ and $w^F = \frac{1}{3}$,

$$c_{1,F} = \left(\frac{1}{3}\right)^{\frac{2}{3}} \quad \text{and} \quad c_{2,F} = \left(\frac{1}{3}\right)^{\frac{1}{3}}.$$

The opportunity cost of producing one unit of good 1 (measured in units of good 2) in each country is then given by the ratio of the unit costs:

$$\mathrm{OC}_1^H = \frac{c_{1,H}}{c_{2,H}} = \frac{3^{\frac{2}{3}}}{3^{\frac{1}{3}}} = 3^{\frac{1}{3}},$$

$$\mathrm{OC}_1^F = \frac{c_{1,F}}{c_{2,F}} = \frac{3^{-\frac{2}{3}}}{3^{-\frac{1}{3}}} = 3^{-\frac{1}{3}}.$$

Since $3^{\frac{1}{3}} > 3^{-\frac{1}{3}}$, country H has a higher opportunity cost of producing good 1 than country F. As country H is capital abundant, it will mainly produce capital-abundant good 2, and country F will mainly produce labor-abundant good 1.

6. As we have the Cobb-Douglas production function, we use its property to have:

$$\frac{wL_1}{rK_1} = \frac{1-\alpha}{\alpha} \Rightarrow \frac{L_1}{K_1} = \frac{2r}{w}$$

and the same for good 2:

$$\frac{L_2}{K_2} = \frac{r}{2w}.$$

Substitute $L = \frac{2r}{w} K$ into the production constraint:

$$A_1 \, K^{\frac{1}{3}} \left(\frac{2r}{w} \, K\right)^{\frac{2}{3}} = A_1 \, \left(\frac{2r}{w}\right)^{\frac{2}{3}} K = 1,$$

so that

$$K^* = \frac{1}{A_1 \left(\frac{2r}{w}\right)^{\frac{2}{3}}}.$$

Then,

$$L^* = \frac{2r}{w} \, K^* = \frac{1}{A_1} \left(\frac{2r}{w}\right)^{\frac{1}{3}}.$$

The minimum total cost is

$$TC = wL^* + rK^* = w \left[\frac{1}{A_1} \left(\frac{2r}{w}\right)^{\frac{1}{3}}\right] + r \left[\frac{1}{A_1} \left(\frac{2r}{w}\right)^{-\frac{2}{3}}\right].$$

Simplify each term:

$$w \left(\frac{2r}{w}\right)^{\frac{1}{3}} = 2^{\frac{1}{3}} w^{\frac{2}{3}} r^{\frac{1}{3}},$$

$$r \left(\frac{2r}{w}\right)^{-\frac{2}{3}} = \frac{w^{\frac{2}{3}} r^{\frac{1}{3}}}{2^{\frac{2}{3}}}.$$

Thus,

$$TC = \frac{1}{A_1} \left[2^{\frac{1}{3}} + \frac{1}{2^{\frac{2}{3}}}\right] w^{\frac{2}{3}} r^{\frac{1}{3}}.$$

Note that

$$2^{\frac{1}{3}} + \frac{1}{2^{\frac{2}{3}}} = \frac{2^{\frac{1}{3}} \cdot 2^{\frac{2}{3}} + 1}{2^{\frac{2}{3}}} = \frac{2+1}{2^{\frac{2}{3}}} = \frac{3}{2^{\frac{2}{3}}}.$$

Since $A_1 = \frac{3}{2^{\frac{2}{3}}}$, we obtain

$$TC = \frac{3}{A_1 2^{\frac{2}{3}}} \, w^{\frac{2}{3}} r^{\frac{1}{3}} = w^{\frac{2}{3}} r^{\frac{1}{3}}.$$

Thus, the unit cost function is

$$c_1(w, r) = w^{\frac{2}{3}} r^{\frac{1}{3}}.$$

Similar for good 2, its unit cost function is

$$c_2(w, r) = w^{\frac{1}{3}} r^{\frac{2}{3}}.$$

Under perfect competition, product prices equal unit costs. Let the price of good 2 be the numeraire:

$$p_2 = c_2(w, r) = w^{\frac{1}{3}} r^{\frac{2}{3}} = 1.$$

Then, the price of good 1 is

$$p_1 = c_1(w, r) = w^{\frac{2}{3}} r^{\frac{1}{3}}.$$

The relative price is given by

$$P = \frac{p_1}{p_2} = \frac{w^{\frac{2}{3}} r^{\frac{1}{3}}}{w^{\frac{1}{3}} r^{\frac{2}{3}}} = \left(\frac{w}{r}\right)^{\frac{1}{3}}.$$

Thus, we have

$$\frac{w}{r} = P^3.$$

This relationship defines the *FPE set* in the (w, r)-space; all factor price pairs satisfying $w = P^3 r$ lie on this line. Using the normalization $p_2 = 1$, the cost condition for Department 2 is:

$$w^{\frac{1}{3}} r^{\frac{2}{3}} = 1.$$

Raising both sides to the power of 3:

$$w\, r^2 = 1 \quad \Rightarrow \quad r = \frac{1}{\sqrt{w}}.$$

Substitute into $\frac{w}{r} = P^3$:

$$\frac{w}{1/\sqrt{w}} = w^{\frac{3}{2}} = P^3,$$

so that

$$w^{\frac{3}{2}} = P^3 \quad \Rightarrow \quad w = P^2,$$

and consequently,

$$r = \frac{1}{P}.$$

Thus, the equilibrium factor prices are uniquely determined as:

$$w^* = P^2, \quad r^* = \frac{1}{P},$$

and the relative product price is

$$p_1 = w^{\frac{2}{3}} r^{\frac{1}{3}} = (P^2)^{\frac{2}{3}} \left(\frac{1}{P}\right)^{\frac{1}{3}} = P^{\frac{4}{3}} P^{-\frac{1}{3}} = P.$$

In the integrated economy, total resources are allocated between the two sectors:

$$K_1 + K_2 = 1, \quad L_1 + L_2 = 1.$$

Cost minimization in each sector yields optimal input ratios. For Department 1:

$$\frac{L_1}{K_1} = \frac{2r}{w} = \frac{2}{P^3},$$

and for good 2:

$$\frac{L_2}{K_2} = \frac{r}{2w} = \frac{1}{2P^3}.$$

These determine the internal allocations in each sector and, together with the overall resource constraints, lead to a unique integrated equilibrium. The derived relationship

$$\frac{w}{r} = P^3$$

defines the FPE set in the (w, r)-plane. In a fully integrated economy with complete factor mobility, the competitive equilibrium requires that all sectors share the same factor prices. Thus, the unique equilibrium factor price pair (w^*, r^*) must lie on the FPE set. In our case, the equilibrium

$$w^* = P^2, \quad r^* = \frac{1}{P}$$

satisfies

$$\frac{w^*}{r^*} = P^3.$$

Therefore, the integrated equilibrium *does* achieve factor price equalization.

7. Although country H is absolutely more productive in producing good 1 (with 1/4 unit of labor producing 9/4 units), its labor is very scarce relative to its capital. Mathematically, while the *absolute productivity* in good 1 is high in H, its *relative factor price* is

$$\frac{w^H}{r^H} = 3,$$

which implies a high cost of labor. Consequently, the *opportunity cost* of producing good 1 (a labor–intensive good) is higher in H:

$$\mathrm{OC}_1^H = 3^{\frac{1}{3}},$$

compared to the opportunity cost in F:

$$\mathrm{OC}_1^F = 3^{-\frac{1}{3}}.$$

Thus, country F, which is labor abundant (and has a lower wage–rental ratio of 1/3), has a comparative advantage in producing good 1. Conversely, country H is relatively capital abundant and, with a lower unit cost in good 2 (since $c_{2,H} = 3^{\frac{1}{3}}$ is lower than $c_{1,H}$), it specializes in producing good 2.

- **Wage-Rental Ratios:** In trade equilibrium, we have

$$\frac{w^H}{r^H} = 3 \quad \text{(in country } H\text{)} \quad \text{and} \quad \frac{w^F}{r^F} = \frac{1}{3} \quad \text{(in country } F\text{)}.$$

- **Opportunity Costs:** The opportunity cost of producing good 1 is

$$\mathrm{OC}_1^H = 3^{\frac{1}{3}} \quad \text{in } H, \qquad \mathrm{OC}_1^F = 3^{-\frac{1}{3}} \quad \text{in } F.$$

  Since $3^{\frac{1}{3}} > 3^{-\frac{1}{3}}$, country F has the lower opportunity cost for good 1.

- **Pattern of Trade:** Therefore, even though country H has a high absolute productivity in good 1, its high wage–rental ratio makes its opportunity cost for good 1 high. As a result, under free trade:
  - Country F specializes in the production of the labor–intensive good 1 and exports it.
  - Country H specializes in the production of the capital–intensive good 2 and exports it.

Solution 5 (Gravity with Multilateral Resistance)**.**
Let the income of country j denoted by Y_j, and the price of a good from country i in country j be p_{ij}. For consumers, their UMP is:

$$\max_{x_{kj}} \Big[\sum_k \alpha_{kj}^{\frac{1}{\sigma}} x_{kj}^{\frac{\sigma-1}{\sigma}}\Big]^{\frac{\sigma}{\sigma-1}}$$
$$\text{s.t. } \sum_k x_{kj} p_{kj} \leq Y_j$$

Define the Lagrangian:

$$\mathcal{L} = \Big[\sum_k \alpha_{kj}^{\frac{1}{\sigma}} x_{kj}^{\frac{\sigma-1}{\sigma}}\Big]^{\frac{\sigma}{\sigma-1}} - \lambda\Big(\sum_k x_{kj} p_{kj} - Y_j\Big)$$

The FOCs are:

$$\frac{\partial \mathcal{L}}{\partial x_{kj}} = \Big[\sum_k \alpha_{kj}^{\frac{1}{\sigma}} x_{kj}^{\frac{\sigma-1}{\sigma}}\Big]^{\frac{1}{\sigma-1}} \alpha_{kj}^{\frac{1}{\sigma}} x_{kj}^{-\frac{1}{\sigma}} - \lambda p_{kj} = 0 \tag{1}$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_k x_{kj} p_{kj} - Y_j = 0 \tag{2}$$

From (1), we know that for two countries k and k', we have:

$$\frac{p_{kj}}{p_{k'j}} = \frac{\alpha_{kj}^{\frac{1}{\sigma}} x_{kj}^{-\frac{1}{\sigma}}}{\alpha_{k'j}^{\frac{1}{\sigma}} x_{k'j}^{-\frac{1}{\sigma}}} \Leftrightarrow \frac{\alpha_{kj}}{\alpha_{k'j}} = \frac{p_{kj}^{\sigma}}{p_{k'j}^{\sigma}} \frac{x_{kj}}{x_{k'j}}$$

Rearranging and multiplying both sides by $p_{k'j}$ yields:

$$x_{k'j}p_{k'j} = \frac{1}{\alpha_{kj}} x_{kj} p_{kj}^{\sigma} \alpha_{k'j} p_{k'j}^{1-\sigma}$$

Summing for all countries gives:

$$\sum_{k'} x_{k'j}p_{k'j} = \frac{1}{\alpha_{kj}} x_{kj} p_{kj}^{\sigma} \sum_{k'} \alpha_{k'j} p_{k'j}^{1-\sigma} \Leftrightarrow Y_j = \frac{1}{\alpha_{kj}} x_{kj} p_{kj}^{\sigma} P_j^{1-\sigma} \tag{3}$$

where $P_j = \left[\sum_k \alpha_{kj} p_{kj}^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$ is the Dixit-Stiglitz price index.
Rearrange (3) gives the CES demand function:

$$x_{kj} = \alpha_{kj} p_{kj}^{-\sigma} Y_j P_j^{\sigma-1} \tag{4}$$

Noting that the value of total trade is simply equal to the price times quantity, which gives $X_{kj} = p_{kj}x_{kj}$, we have:

$$X_{kj} = \alpha_{kj} p_{kj}^{1-\sigma} Y_j P_j^{\sigma-1} \tag{5}$$

As we suppose that the market for each country/good is perfectly competitive, the price of a good is simply the marginal cost. Let w_i be the wage of a worker in country i.
Because we assume that one unit of labor produces one unit of the local output, the wage in country i is simply the price of the good produced in country i: $p_i = w_i$.
So, the price of j consuming 1 unit of country i's good is:

$$p_{ij} = \tau_{ij} w_i \Rightarrow \tau_{ij} = \frac{p_{ij}}{p_i}. \tag{6}$$

1. Let λ_{ij} be the share of total spending in country j that is devoted to goods imported from country i. Implementing (5), (6) and the price index, We have:

$$\begin{aligned}
\lambda_{ij} = \frac{X_{ij}}{Y_j} &= \alpha_{ij} p_{ij}^{1-\sigma} P_j^{\sigma-1} \\
&= \alpha_{ij} \Big(\tau_{ij} w_i\Big)^{1-\sigma} \left[\sum_k \alpha_{kj} \Big(\tau_{kj} w_k\Big)^{1-\sigma}\right]^{\sigma-1} \\
&= \frac{\alpha_{ij} \tau_{ij}^{1-\sigma} w_i^{1-\sigma}}{\sum_k \alpha_{kj} \Big(\tau_{kj} w_k\Big)^{1-\sigma}}
\end{aligned} \tag{*}$$

2. As market clearing requires income is equal to payments to, the Y we previously defined is the same as the X_i(defined as GDP in country i).

Income in a country is also equal to its total sales:

$$\begin{aligned} X_i &= \sum_j X_{ij} = \sum_j \alpha_{ij}\tau_{ij}^{1-\sigma} w_i^{1-\sigma} X_j P_j^{\sigma-1} \\ \Rightarrow w_i^{1-\sigma} &= X_i / \sum_j \alpha_{ij}\tau_{ij}^{1-\sigma} X_j P_j^{\sigma-1} \end{aligned}$$

Replace this equation back into (5), we have:

$$\begin{aligned} X_{ij} &= \alpha_{ij}\tau_{ij}^{1-\sigma} w_i^{1-\sigma} X_j P_j^{\sigma-1} \\ &= \alpha_{ij}\tau_{ij}^{1-\sigma} \frac{X_i}{\sum_k \alpha_{ik}\tau_{ik}^{1-\sigma} X_k P_k^{\sigma-1}} X_j P_j^{\sigma-1} \\ &= X_i X_j \left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma} \frac{\alpha_{ij}}{\sum_k \alpha_{ik}\left(\frac{\tau_{ik}}{P_k}\right)^{1-\sigma} X_k} \end{aligned} \tag{**}$$

3. Under CES preferences, utility of the representative agent is the real wage. Thus in our notation, we have: $X_j = U_j P_j$. The per capita welfare W_i can be written as:

$$W_j = \frac{U_j}{L_j} = \frac{X_j}{P_j L_j} = \frac{w_j L_j}{P_j L_j} = \frac{w_j}{P_j} \tag{7}$$

We assume that $\tau_{jj} = 1$, and by choosing $i = j$, (*) implies:

$$\lambda_{jj} = \alpha_{jj} w_j^{1-\sigma} P_j^{\sigma-1} \Rightarrow P_j = \left(\lambda_{jj}\alpha_{jj}^{-1} w_j^{\sigma-1}\right)^{\frac{1}{\sigma-1}} = \lambda_{jj}^{\frac{1}{\sigma-1}} \alpha_{jj}^{-\frac{1}{\sigma-1}} w_j$$

Replacing this equation into (7), we have:

$$W_j = \frac{w_j}{\lambda_{jj}^{\frac{1}{\sigma-1}} \alpha_{jj}^{-\frac{1}{\sigma-1}} w_j} = \lambda_{jj}^{-\frac{1}{\sigma-1}} \alpha_{jj}^{\frac{1}{\sigma-1}} \tag{***}$$