Measurement Equations

- Useful to think of ***solution*** to DSGE in ***state space***:

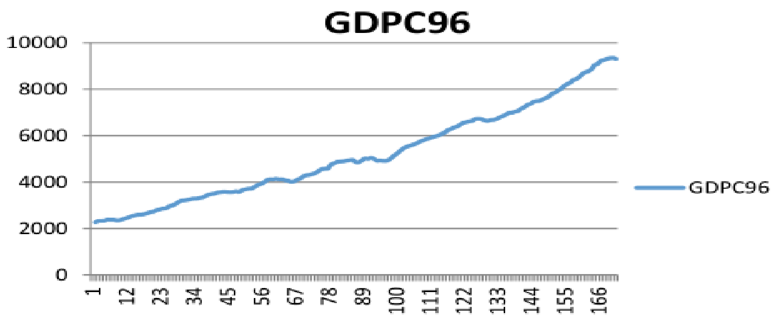
$$x_t = g\left(x_{t-1}, \varepsilon_t^{struct}\right)$$

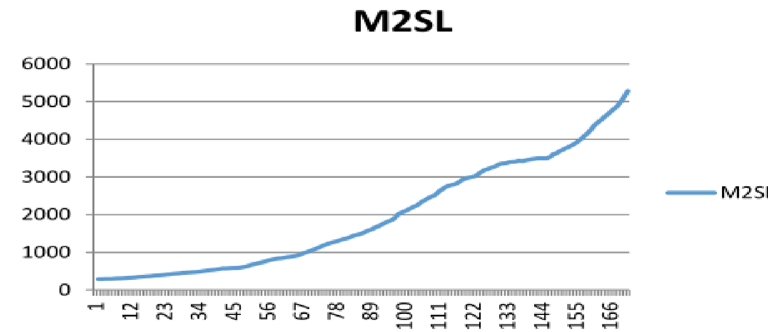
$$y_t^{obs} = h\left(x_t, \varepsilon_t^{obs}\right)$$

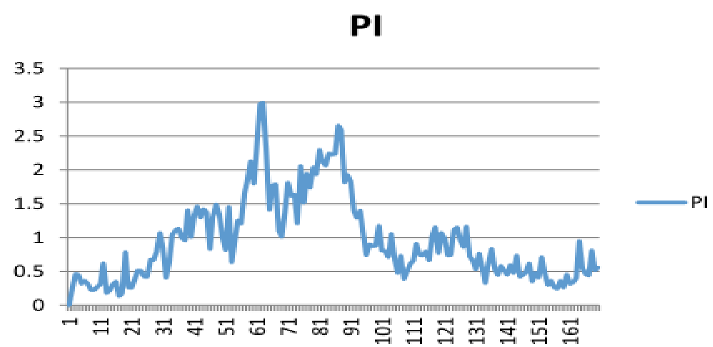
  - x_t = ***state*** (= model) variables
  - ε_t^{struct} = structural shocks
  - y_t^{obs} =***observed*** variables
  - ε_t^{obs} = measurement error
- First = ***transition equation*** (= DSGE model)
- Second = ***observation equation*** (describes how ***observed*** variables ***map*** into ***state*** variables, potentially including ***measurement error***)

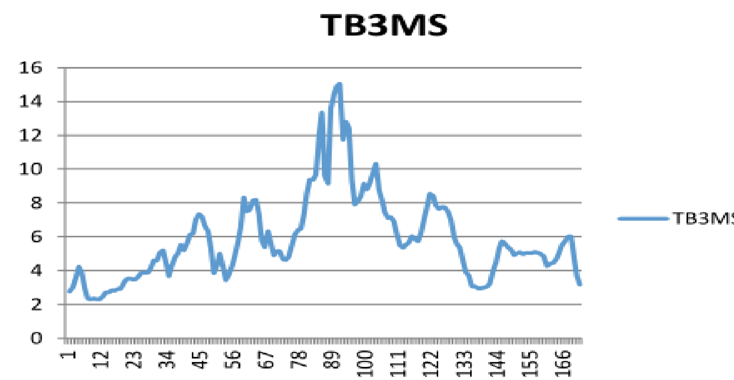
- In ***estimation*** phase, Dynare ***automatically*** computes ***mapping*** from state variables x_t into observables y_t^{obs} ***provided*** it is told how observed data is related to other model variables
- ➔unless observed variables y_t^{obs} correspond ***exactly*** to an ***actual model variable***, must add ***separate equations*** detailing how y_t^{obs} is linked to model variables
- These are ***measurement equations***
- ➔ providing ***Dynare*** with information on which are the observed variables ➔ ***varobs*** command

- ***Most crucial issue*** when specifying observation equations is that ***model*** (= state) variables like output y_t are typically assumed to be ***stationary***
- Whereas ***empirically observed*** macro aggregate variables typically have a growth ***trend***



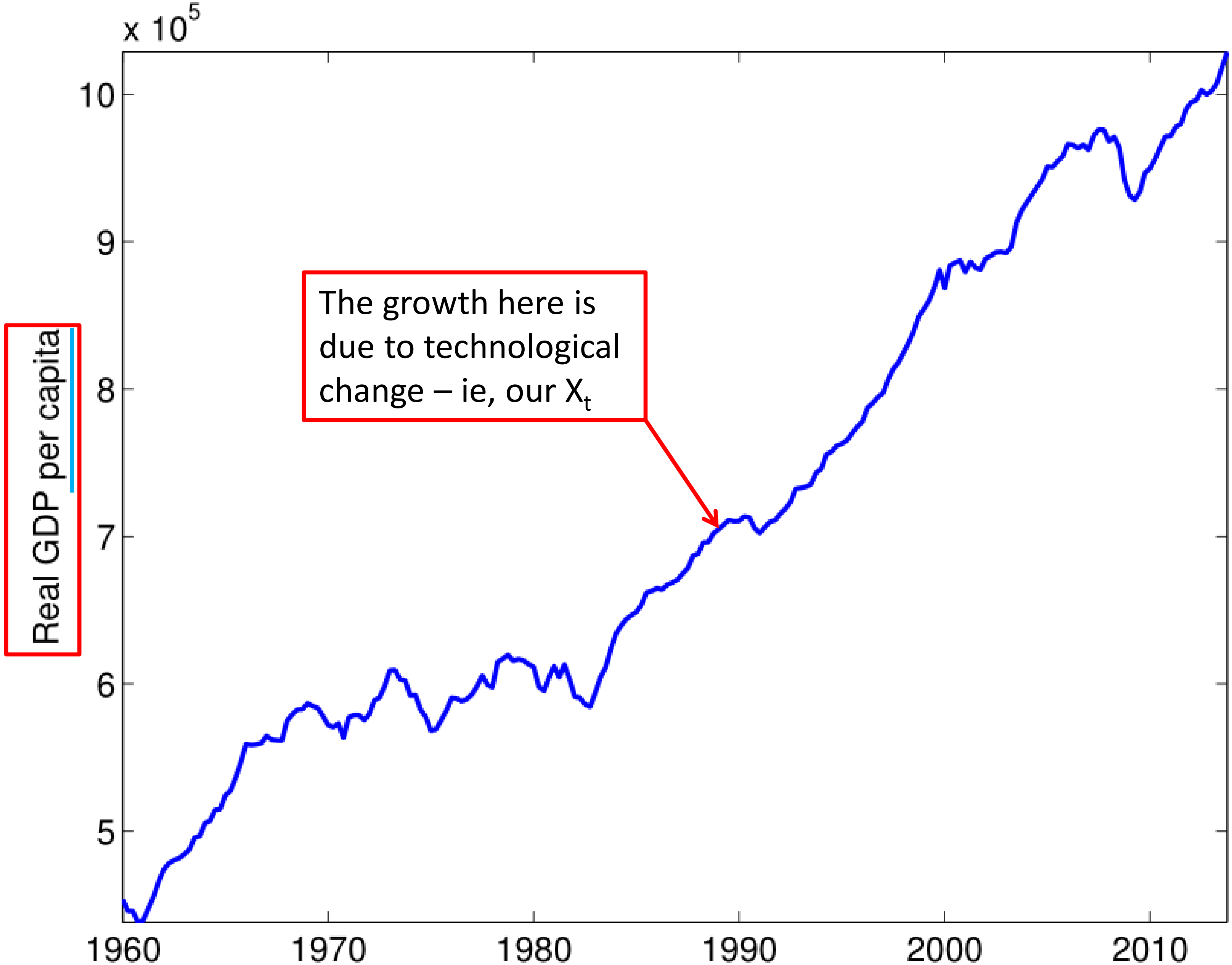






- Reason for model variables being ***stationary*** in ***Dynare*** is that DSGE models are solved using ***perturbation*** techniques, i.e. using a ***Taylor approximation*** around an approximation point, typically (deterministic) ***steady state***
- This obviously ***requires*** model to ***actually have*** a well-defined steady state
- However, ***actual*** economies ***don't*** tend to ***converge*** back to a steady state, but ***grow*** over time
- How to deal with this?

- Typically, economists conceptualize this as a “steady state” in ***intensive*** form (***technology-weighted per capita***) and then model using intensive form variables
- ***Per capita*** variables then grow along a BGP ➔ output, consumption, and investment ***per capita*** grow at rate of ***technology*** growth X_t
- while ***total*** output, consumption, and investment grow at rate of ***population plus technology*** growth



x 10^5
10
9
8
7
6
5
1960
1970
1980
1990
2000
2010
Real GDP per capita
The growth here is due to technological change – ie, our X_t

- How to get around the problem of ***modeling*** an economy in (***stationary***) ***intensive form*** and only ***observing*** actual ***growing*** variables?
- ***Answer***: Enter data ***made stationary*** or ***transformed*** into intensive form
- ***Common ways*** of getting trend out of trending variables like output are:
  - One-sided HP-filter (Stock and Watson 1999)
  - Linear-(quadratic) trend (see e.g. King and Rebelo 1999)
  - First-difference filter (see e.g. Smets and Wouters 2007)
- ∃ others (eg 2-sided HP and Band Pass) ➔ NO !!

- More specifically: If model is ***log-linearised***, then typically model aggregate variables will have form

$$\hat{y}_t \equiv \log(y_t) - \log(\bar{y})$$

- where
  - y_t = generic variable in ***intensive form***
  - hats ➔ variables in ***percentage deviations*** (log diffs)
  - bars ➔ ***steady state*** values of y_t
- $\hat{y}_t$ are variables used and entered into ***Dynare model***

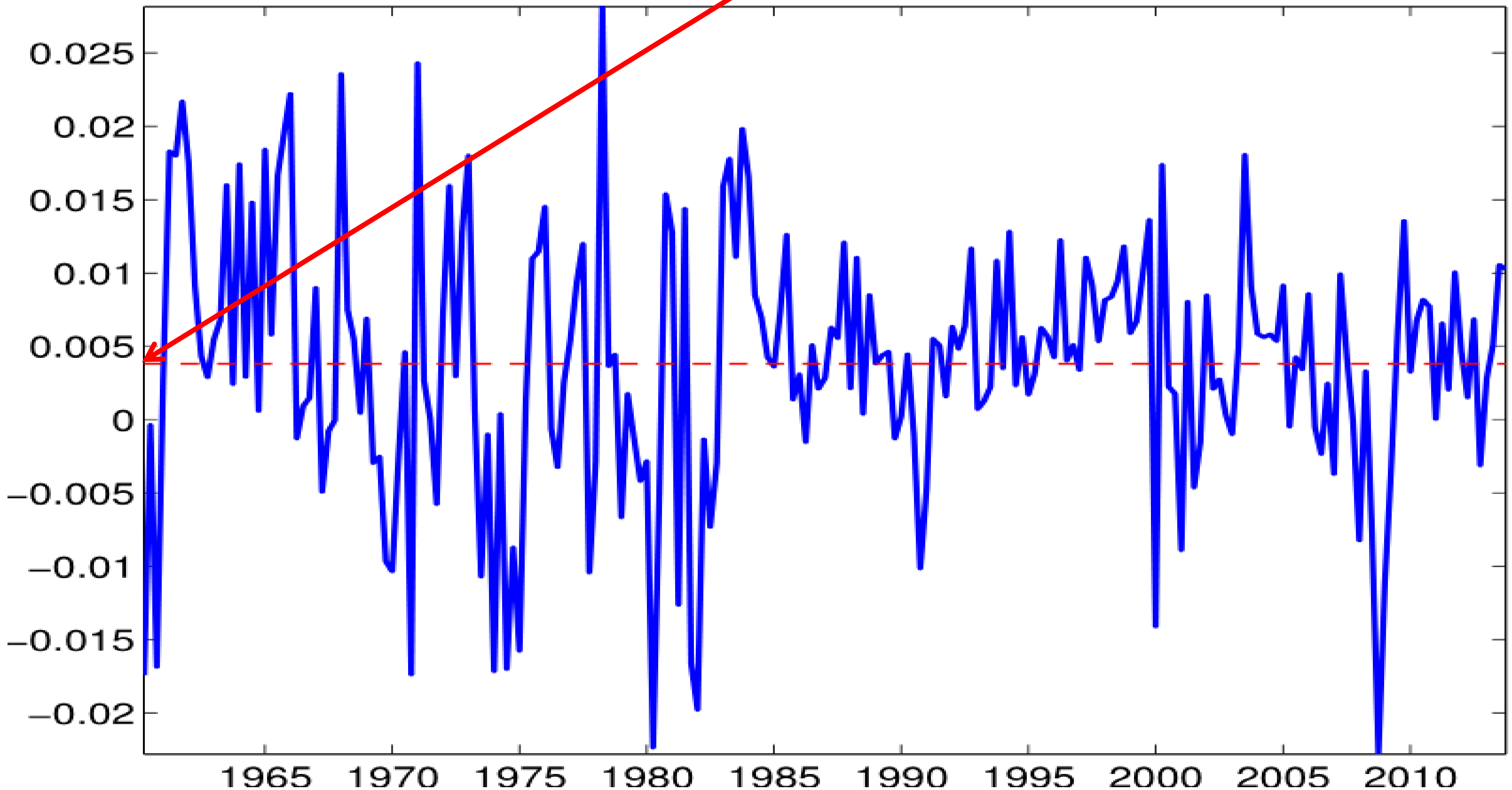
- ***Do not be confused by two steady-state concepts!***
- ***Steady-state*** of ***original model*** variable y_t (eg. technology-weighted real per capita output) is entered as a ***parameter*** (eg, y_ss)
- ***Steady-state*** of ***Dynare model*** variable $\hat{y}_t$ in this framework is just (by definition) ***zero*** because

$$\hat{y}_t \equiv \log(y_t) - \log(\bar{y})$$

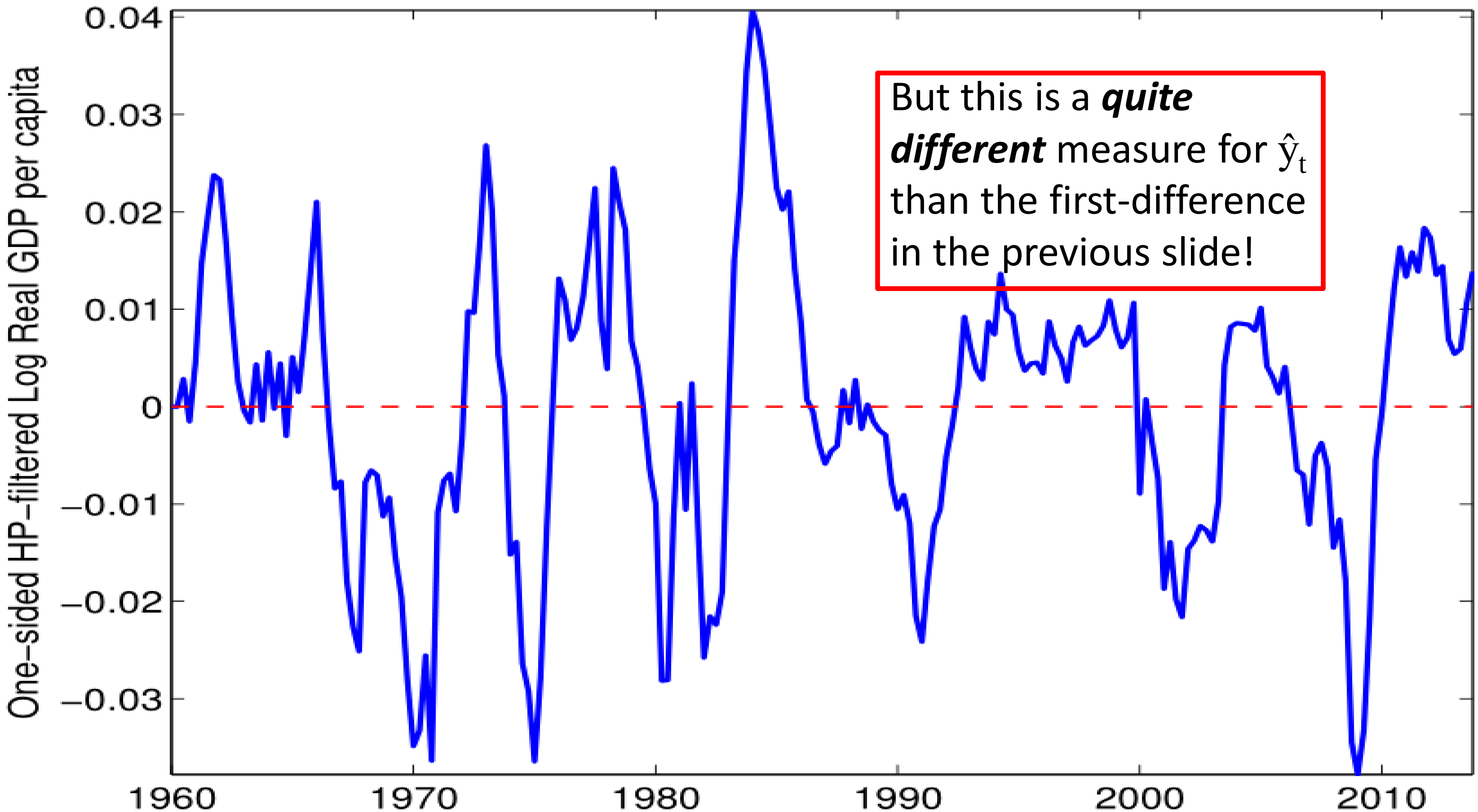
- ➔ for a log-linearised model, steady-state in Dynare will be shown in output as 0

- Now, consider an aggregate like output
- How do we obtain technology-weighted per capita real output variable?
- In practice, take following approach:
  1. download both ***real*** GDP (***seasonally adjusted !***) and Population data
  2. construct real ***per capita*** GDP
  3. de-trend ***log*** of real per capita GDP using a filter ➔ cyclical component of variable $\hat{y} = \log(y) - \log(\bar{y})$
  4. if de-trended variable still has a ***non-zero*** mean, subtract that mean
- ***OK*** because ∃ natural equivalence between model concept of ***deviations from steady state*** and ***cyclical fluctuation*** around a trend in data

- For example, if real seasonally-adjusted US GDP per capita is detrended using a ***dlog*** filter, the mean is very small but still ***non-zero*** (hence step 4)
- $\hat{y}$ = diff(log(dataQ(:,1)/4))



- The “1-sided HP filter” gives a ***zero mean*** directly
- ŷ = log(dataQ(:,1)/4)-one_sided_hp_filter(log(dataQ(:,1)/4));



- What about inflation?
- In the underlying model, typically the inflation variable used is ***gross inflation*** $\Pi_t \equiv P_t/P_{t-1}$
- If that model is log-linearized, the $\hat{\Pi}_t$ used in Dynare model will be ***percentage deviation*** of gross inflation from its steady state (which is approximately ***net*** inflation)
- And this is just the definition of inflation in general use, and found in databases
- A word of caution: DSGE models typically use ***quarter-to-quarter gross inflation*** but datasets typically contain ***year-to-year net inflation*** ➔ need to divide by 4

- For nominal interest rate R_t, treatment is similar to that for inflation: DSGE model uses quarterly gross interest rate
- If model is log-linearized, $\hat{R}_t$ used in Dynare model will be ***percentage deviation*** of quarterly gross interest rate from its steady state (which is approximately ***net*** quarterly interest rate)
- And this is just the definition of the interest rate in general use, and found in databases
- As for inflation, however, DSGE models typically use ***quarter-to-quarter gross interest rate*** but datasets typically contain ***annualised net interest rate*** → need to divide by 4

- To summarise: simplest model transforms variables like Y (real GDP) as follows:
- $Y \rightarrow \ln(Y) \rightarrow \ln(Y) - \ln(Y_{SS}) \rightarrow y$
- This in turn implies that
- $dy = y - y(-1)$

$$= \ln(Y) - \ln(Y_{SS}) - [\ln(Y(-1)) - \ln(Y_{SS})\ ]$$

$$= \ln(Y) - \ln(Y(-1))$$

$$= d\ln(Y)$$

$$\approx \%\Delta Y$$

Steady-state is constant, by definition, so Y_SS(-1)=Y_SS

- So in model's measurement equations, ***dy*** matches ***percentage change in Y***

- Further transformations are frequently also used:

1. Measure in ***per capita*** terms (eg, SW):

- $Y/N \rightarrow \ln(Y/N) \rightarrow \ln(Y/N) - \ln(Y_{SS}/N_{SS}) \rightarrow y$
- This in turn implies that
- $dy = y - y(-1) = \ln(Y/N) - \ln(Y_{SS}/N_{SS}) - [\ln(Y(-1)/N(-1)) - \ln(Y_{SS}/N_{SS})]$

$$= \ln(Y/N) - \ln(Y(-1)/N(-1))$$

$$= d\ln(Y/N)$$

$$\approx \%\Delta Y - \%\Delta N$$

- So in model's measurement equations, ***dy*** now matches ***percentage change in Y adjusted by population growth rate***

- Another transformation frequently used (eg, Ireland):

2. Measure in ***technology-weighted*** terms:

- $Y/A \rightarrow \ln(Y/A) \rightarrow \ln(Y/A) - \ln(Y_{SS}/A_{SS}) \rightarrow y$
- This in turn implies that
- $dy = y - y(-1) = \ln(Y/A) - \ln(Y_{SS}/A_{SS}) -$

$$[\ln(Y(-1)/A(-1)) - \ln(Y_{SS}/A_{SS})\ ]$$

$$= \ln(Y/A) - \ln(Y(-1)/A(-1))$$

$$= d\ln(Y/A)$$

$$\approx \%\Delta Y - \%\Delta A$$

- So in model's measurement equations, ***dy*** now matches ***percentage change in Y adjusted by technology growth rate***

- Finally, combined transformation:

3. Measure in ***per capita technology-weighted*** terms:

- $Y/AN \Rightarrow \ln(Y/AN) \Rightarrow \ln(Y/AN) - \ln(Y_{SS}/A_{SS}\,N_{SS}) \Rightarrow y$
- This in turn implies that
- $dy = y - y(-1) = \ln(Y/AN) - \ln(Y_{SS}/A_{SS}\,N_{SS}) -$

$$[\ln(Y(-1)/A(-1)N(-1)) - \ln(Y_{SS}/A_{SS}\,N_{SS})\,]$$

$$= \ln(Y/AN) - \ln(Y(-1)/A(-1)N(-1))$$

$$= d\ln(Y/AN)$$

$$\approx \%\Delta Y - \%\Delta A - \%\Delta N$$

- So in model's measurement equations, ***dy*** now matches ***percentage change in Y adjusted by both technology and population growth rates***

- Practical example: famous SW model
- SW define aggregate variables in ***per-capita*** terms
- Thus, their measurement equations should be of type: dy = y-y(-1) = %ΔY - %ΔN
- dy = yobs – gr_pop ➔ yobs = dy + gr_pop
- *where* yobs = %ΔY and gr_pop = %ΔN
- They actually write: yobs = y-y(-1) + ctrend;
- Why?
- Essentially, they assume that population growth rate is a constant, measured by ctrend ("common trend") – and ***estimated*** in model

An Example

- Start with a Baseline RBC Model, specified already in ***per-capita*** terms (remember Ireland …)

$$\max_{\{C_t, I_t, K_t, B_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t \log(C_t)$$

subject to

$$C_t + I_t + \frac{B_t}{P_t} = Y_t + \frac{B_{t-1} R_{t-1}}{P_t}$$

$$Y_t = A_t K_{t-1}^{\alpha} (X_t h_t)^{1-\alpha} = A_t K_{t-1}^{\alpha} X_t^{1-\alpha}$$

$$K_t = (1-\delta) K_{t-1} + I_t$$

$$\frac{R_t}{\bar{R}} = \left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\phi}$$

- A_t is Total Factor Productivity (TFP)
  X_t is labour augmenting technology growth

- Optimising ➔ FOCs

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \left[\alpha A_{t+1} K_t^{\alpha-1} X_{t+1}^{1-\alpha} + (1-\delta)\right]$$

$$\frac{1}{C_t P_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{P_{t+1}}$$

$$C_t + K_t - (1-\delta) K_{t-1} + \frac{B_t}{P_t} = A_t K_{t-1}^{\alpha} X_t^{1-\alpha} + \frac{B_{t-1} R_{t-1}}{P_t}$$

$$\frac{R_t}{\bar{R}} = \left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\phi}$$

- ***Nominal price P_t*** is ***not*** uniquely determined ➔ prices and nominal variables have to be ***rewritten*** in terms of ***inflation Π_t*** and ***real*** variables

- Here, this ➔

$$\frac{1}{C_t} = \beta E_t \frac{1}{C_{t+1}} \frac{R_t}{\Pi_{t+1}}$$

$$C_t + K_t - (1-\delta)K_{t-1} + \frac{B_t}{P_t} = A_t K_{t-1}^{\alpha} X_t^{1-\alpha} + \frac{B_{t-1}}{P_{t-1}} R_{t-1} \frac{1}{\Pi_t}$$

- Assuming that real bonds are in zero net supply ($B_t/P_t = 0$) ➔ budget constraint

$$C_t + K_t - (1-\delta)K_{t-1} = A_t K_{t-1}^{\alpha} X_t^{1-\alpha}$$

- Assume a law of motion for TFP: $A_t = \exp(z_t)$
- where $z_t = \rho z_{t-1} + \varepsilon_t,\ \varepsilon_t \sim \mathcal{N}(0, \sigma^2)$
- Also assume zero labour-augmenting technology growth (ie, $X_t \equiv 1$) ➔ mod-file

• ➔ *Dynare model*

```
model;
1/c=beta*(1/c(+1))*(alpha*A(+1)*k^(alpha-1)+(1-delta));
1/c=beta*(1/c(+1))*(R/Pi(+1));
A*k(-1)^alpha=c+k-(1-delta)*k(-1);
y=A*k(-1)^alpha;
R/Rbar=(Pi/Pibar)^phi_pi;
A=exp(z);
z=rhoz*z(-1)+eps_z;
end;
```

Note: no X_t here since assumed = 1

• ➔ log-linearised ***Dynare model***

```
model(linear);
#k_ss=((1/beta-(1-delta))/alpha)^(1/(alpha-1));
#y_ss=k_ss^alpha;
#c_ss=y_ss-delta*k_ss;

(-1/c_ss)*c=(-1/c_ss)*c(+1)+
    beta*(1/c_ss)*alpha*k_ss^(alpha-1)*(A(+1)+(alpha-1)*k);
-c=-c(+1)+R-Pi(+1);
y_ss*y=c_ss*c+k_ss*k-(1-delta)*k_ss*k(-1);
y=A+alpha*k(-1);
R=phi_pi*Pi;
A=rhoA*A(-1)+eps_A;
end;
```

These are the steady-state values as ***parameters***

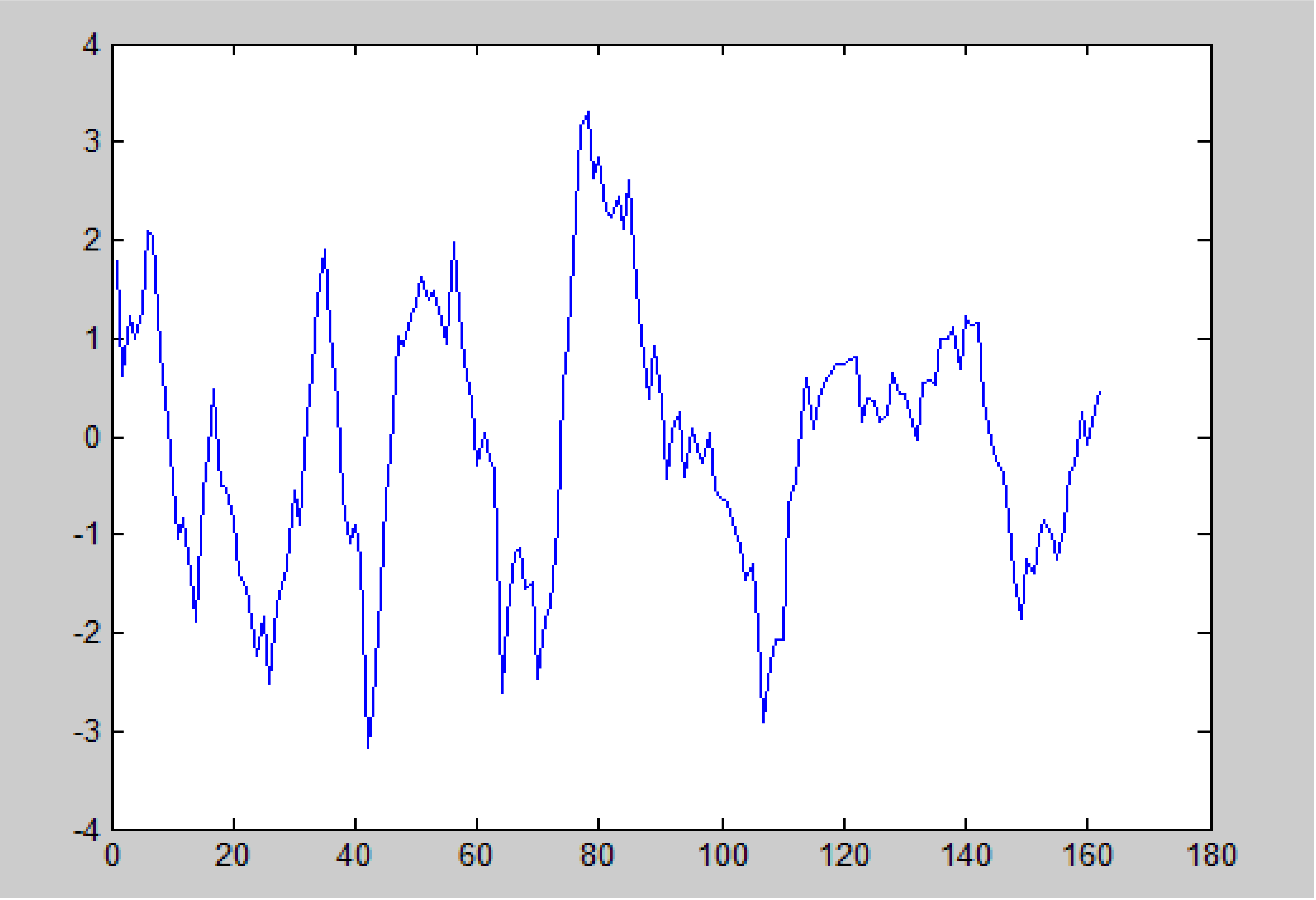
Here, the variables should be understood to have hats, eg y here stands for $\hat{y}$ in the model where $\hat{y} = \log(y) - \log(\bar{y})$

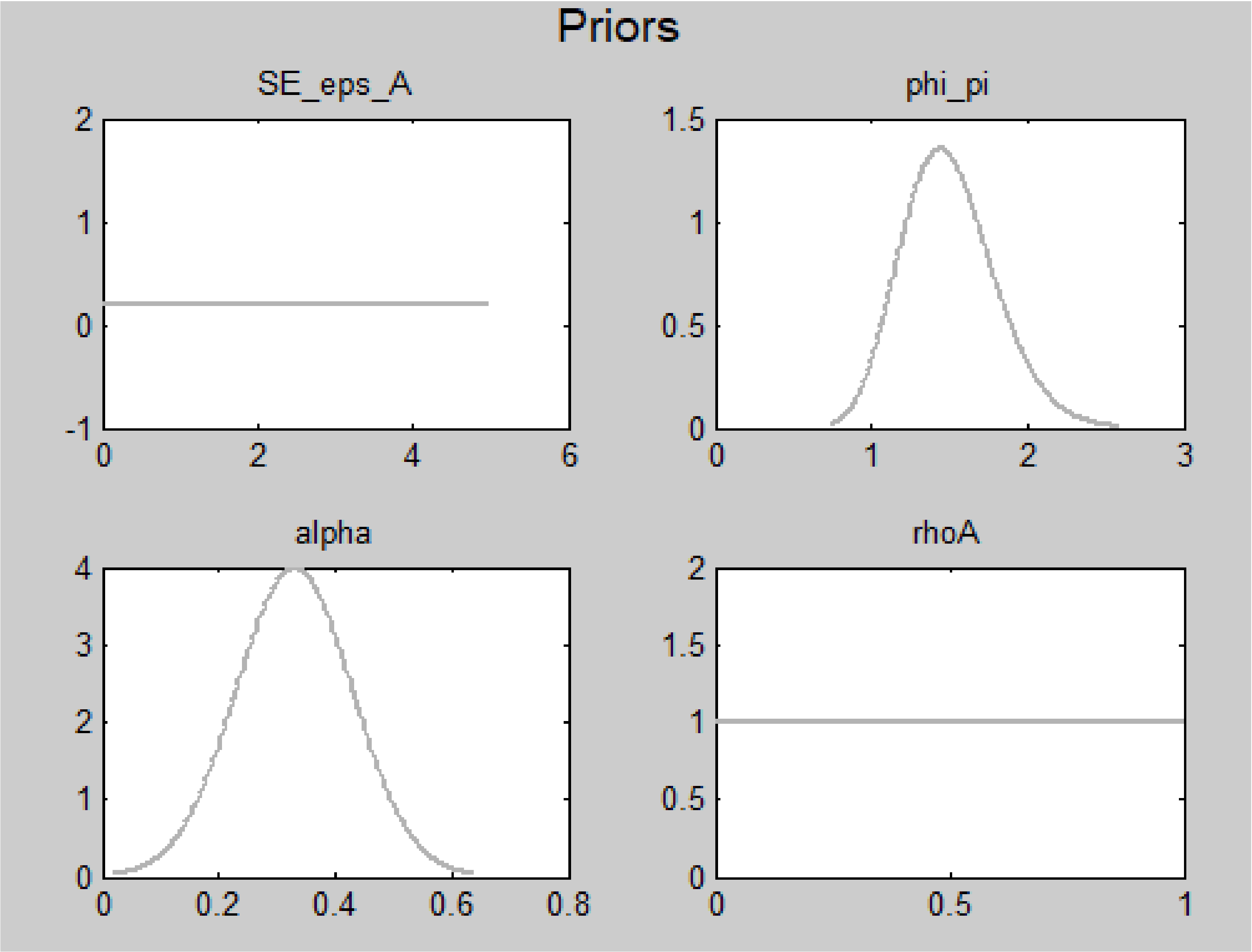
- As an example, ***use SW's data*** in our simple model
- Since model has only ***one shock***, can use only one observed variable
- ***Select*** consumption
- ***Transform*** nominal consumption data by
  - deflating by GDP deflator → real consumption
  - dividing by population → real per capita consumption
  - taking logs
- ***De-trend*** transformed consumption using some filter (eg, 1-sided HP) – see ***swdata1_hp1b.m***

```
%   lp - linear trend
%   qp - quadratic trend
%   hpfilter - smoothed trend
%   diff - first difference

%%preliminaries
begindate = '01.03.1947'; % beginning of the sample
enddate = '01.12.2004';     % end of the sample
trendtype = 6;              % 1=linear, 2=quadratic, 3=hp 2-sided, 4 = hp 1-sided, 5=dlog, 6 = alternative hp 1-sided
lambda = 1600;              % hp smoothing parameter

%% loading the data from file
[dataQ,datadate,raw] = xlsread('data/SW_usmodel_data.xls');
% Real Log per Capita: consumption(13) investment(14) output(15) hours(16)   inflation(17)  real wage(18)    interest rate(19)
% NOTE: dataQ does NOT contain the date column, so its column 13 corresponds to column 14 of SW_usmodel_data

%% different detrending methods
if trendtype==1;
   cgap = lp(dataQ(:,13)); % SW data is already in logs
elseif trendtype==2;
   cgap = qp(dataQ(:,13)); % SW data is already in logs
elseif trendtype==3;
   cgap = (dataQ(:,7))-hpfilter(dataQ(:,13), lambda);  % SW data is already in logs
elseif trendtype==4;
   cgap = (dataQ(:,7))-one_sided_hp_filter_serial(dataQ(:,13));  % SW data is already in logs
elseif trendtype==5;
   cgap = diff(dataQ(:,13)); % SW data is already in logs
elseif trendtype==6;
   cgap = (dataQ(:,13))-one_sided_hp_filter(dataQ(:,13));  % SW data is already in logs
else
   disp('Error! Need to specify detrending method for cgap')
end

% defining variables for Dynare model
c_obs=cgap;
```

- Estimate model using detrended transformed consumption data:
- //measurement equations
- c_obs=c_hat;
- end;
- …
- varobs c_obs;
- estimation(Tex,datafile=swdata1_hp1b, mode_compute=4,first_obs=71,mode_check);

- c_obs looks like this with 1-sided HP filter:




Priors
SE_eps_A
2
1
0
-1
0
2
4
6
phi_pi
1.5
1
0.5
0
0
1
2
3
alpha
4
3
2
1
0
0
0.2
0.4
0.6
0.8
rhoA
2
1.5
1
0.5
0
0
0.5
1

- Results:

RESULTS FROM POSTERIOR ESTIMATION

parameters

	prior mean	mode	s.d.	prior	pstdev
phi_pi	1.500	1.4399	0.2939	gamm	0.3000
alpha	0.330	0.1581	0.1649	norm	0.1000
rhoA	0.500	0.5116	0.3758	unif	0.2887

standard deviation of shocks

	prior mean	mode	s.d.	prior	pstdev
eps_A	2.500	4.2191	4.1221	unif	1.4434

Log data density [Laplace approximation] is -143.198528.

Statistically insignificant

- Recall that in their model SW actually use a ***first-difference*** filter and enter data into model via ***measurement equations*** like:
- yobs=y-y(-1)+ctrend;
- What happens if we do the same in our estimated model?

- Results:

RESULTS FROM POSTERIOR ESTIMATION

parameters

	prior mean	mode	s.d.	prior	pstdev
phi_pi	1.500	1.4400	0.2825	gamm	0.3000
alpha	0.330	0.5342	0.0698	norm	0.1000
rhoA	0.500	0.9988	0.0023	unif	0.2887
ctrend	0.400	0.3346	0.0975	norm	0.1000

standard deviation of shocks

	prior mean	mode	s.d.	prior	pstdev
eps_A	2.500	2.5439	0.1511	unif	1.4434

Log data density [Laplace approximation] is -391.749917.

Note that now all parameters are **statistically significant**