Toulouse School of Economics, 2012-2013
Macroeconomics 1 – Franck Portier

Problem set 8: Real Business Cycles - Solution

Problem I – A Simplified Real-Business-Cycle Model with Additive Technology Shocks

Consider an economy consisting of a constant population of infinitely-lived individuals. The representative individual maximizes the expected value of $\sum_{t=0}^{\infty} u(C_t)/(1+\rho)^t$, $\rho > 0$. The instantaneous utility function,$u(C_t)$, is $u(C_t) = C_t - \theta C_t^2$, $\theta > 0$. Assume that C is always in the range where $u'(C)$ is positive.

Output is linear in capital, plus an additive disturbance: $Y_t = AK_t + e_t$. There is no depreciation; thus $K_{t+1} = K_t + Y_t - C_t$, and the interest rate is A. Assume $A = \rho$. Finally, the disturbance follows a first-order autoregressive process: $e_t = \phi e_{t-1} + \varepsilon_t$, where $-1 < \phi < 1$ and where the ε_t's are mean-zero, i.i.d. shocks.

1 – Find the first-order condition (Euler equation) relating C_t and expectations of C_{t+1}.

The Lagrangian of the Social Planner problem is

$$\mathcal{L} = E_0 \sum_{t=0}^{\infty} \left(\frac{1}{1+\rho}\right)^t \left(C_t - \theta C_t^2 + \lambda_t \left((A+1)K_t + e_t - K_{t+1} - C_t\right)\right)$$

FOC with respect to C_t and K_{t+} are:

$$1 - 2\theta C_t = \lambda_t \tag{1}$$

$$\lambda_t = \frac{1}{1+\rho} E_t (1+1) \lambda_{t+1} \tag{2}$$

Substituting in (2) λ_t by its expression in (1), we obtain

$$1 - 2\theta C_t = \frac{1+A}{1+\rho} E_t \left(1 - 2\theta C_{t+1}\right)$$

Using $A = \rho$ and simplifying, one gets the Euler equation

$$C_t = E_t C_{t+1}$$

2 – Guess that consumption takes the form $C_t = \alpha + \beta K_t + \gamma e_t$. Given this guess, what is K_{t+1} as a function of K_t and e_t?

The resource constraint implies $K_{t+1} = (A+1)K_t - C_t + e_t$. Replacing C_t by its expression $\alpha + \beta K_t + \gamma e_t$, one gets

$$K_{t+1} = (A + 1 - \beta)K_t - \alpha + (1 - \gamma)e_t \tag{$\star$}$$

3 – What values must the parameters α, β, and γ have for the first-order condition in part (1) to be satisfied for all values of K_t and e_t?

Let's compute $E_t C_{t+1}$ using equation ($\star$):

$$\begin{aligned} E_t C_{t+1} &= \alpha + \beta E_t K_{t+1} + \gamma E_t e_{t+1} \\ &= \alpha(1-\beta) + \beta(A + 1 - \beta)K_t + (\beta(1-\gamma) + \gamma\phi)e_t \end{aligned}$$

The Euler equation implies $C_t = E_t C_{t+1}$. Knowing that $C_t = \alpha + \beta K_t + \gamma e_t$, one obtains the conitions

$$\begin{aligned} \alpha &= \alpha(1-\beta) \\ \beta &= \beta(A+1-\beta) \\ \gamma &= \beta(1-\gamma) + \gamma\phi \end{aligned}$$

from which we obtain

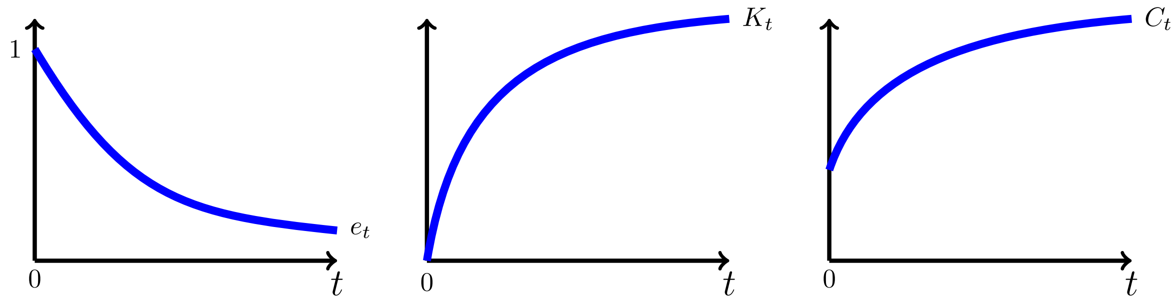
$$\begin{aligned} \alpha &= 0 \\ \beta &= A \\ \gamma &= \frac{A}{A+1-\phi} \end{aligned}$$

The equilibrium then writes

$$
\begin{aligned}
K_{t+1} &= K_t + \frac{1-\phi}{A+1-\phi} e_t \\
C_t &= AK_t + \frac{A}{A+1-\phi} e_t
\end{aligned}
$$

4 – What are the effects of a one-time shock to ε on the paths of Y, K, and C?

Figure 1: Impulse response to a one time shock ε



Problem III – An analytic model with log-linear depreciation

Consider a model economy populated with a representative household and a representative firm. The firm has a Cobb-Douglas technology:

$$Y_t = Z_t K_t^{\gamma} N_t^{1-\gamma} \tag{3}$$

where K_t is capital, N_t labor input, and Z_t a stochastic technological shock. All profits of the firm are distributed to the household. Capital evolves according to the log linear relation

$$K_{t+1} = AK_t^{1-\delta} I_t^{\delta} \qquad 0 < \delta \leq 1 \tag{4}$$

where δ is the rate of depreciation and where I_t is investment in period t.

The representative household works N_t, consumes C_t and invests I_t in period t. Preferences are given by:

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \left[\log C_t - V(N_t)\right] \tag{5}$$

where V is a convex function. Capital is accumulated by the household and rented to the firm.

Let κ denote the real rental rate of capital, P the price of the final good and W the nominal wage.

We first assume that $\delta = 1$ (full depreciation).

1 – Write down the budget constraint of the household and the profit function of the firm

2 – Derive FOCs of the utility and profit maximization

3 – Define a competitive equilibrium of this economy

4 – Solve the model and show that N_t is constant along an equilibrium path.

5 – Derive a $AR(1)$ process for log of output. Draw the Impulse Response Function of y to a unit shock to $\log Z = z$, assuming that z is *iid*. What are the determinants of the size and persistence of this IRF?

$\star$

We assume now that $\delta \in]0,1[$.

6 – What is the economic meaning of equation (2)?

7 – We still denote κ_t the real return in t on investment I_{t-1}:

$$\kappa_t = \frac{\partial Y_t}{\partial I_{t-1}} \tag{6}$$

and note that the real price, in terms of current output, of capital to be transmitted to the next period is not one as when $\delta = 1$. We denote it as q_t (like Tobin's q), and it is equal to:

$$q_t = \frac{\partial K_{t+1}/\partial K_t}{\partial K_{t+1}/\partial I_t} \tag{7}$$

8 – Comment the new budget constraint of the household

$$C_t + I_t = \frac{W_t}{P_t} N_t + \kappa_t (I_{t-1} + q_{t-1} K_{t-1}) \tag{8}$$

9 – Solve for the competitive equilibrium of this economy and compute $\frac{I_t}{C_t}$.

10 – Compute the constant level of N.

11 – Derive the new process for log of output. Draw output typical IRF and discuss of its shape as a function of parameters.

The first part of the exercise (full depreciation) has been solve in class. In each period the firm demands labor competitively so that the real wage is equal to the marginal productivity of labor:

$$\frac{W_t}{P_t} = \frac{\partial Y_t}{\partial N_t} = (1-\gamma)\frac{Y_t}{N_t} \tag{9}$$

Also the real return on capital is simply the marginal productivity of capital :

$$\kappa_t = \frac{\partial Y_t}{\partial K_t} = \gamma \frac{Y_t}{K_t} \tag{10}$$

The household maximizes the expected value of his discounted utility subject to the sequence of budget constraints. Denote by λ_t the marginal utility of real wealth in period t (i.e. the Lagrange multiplier associated with the corresponding budget constraint. Then the usual optimality conditions for the consumer's program yield:

$$\frac{1}{C_t} = \lambda_t \tag{11}$$

$$V'(N_t) = \lambda_t \frac{W_t}{P_t} \tag{12}$$

$$\lambda_t = \beta E_t(\lambda_{t+1}\kappa_{t+1}), \tag{13}$$

Combining (11), (13), the condition $Y_t = C_t + I_t$ and the definition of κ_t in (10), we obtain:

$$\frac{I_t}{C_t} = \beta\gamma + \beta\gamma E_t\left(\frac{I_{t+1}}{C_{t+1}}\right) \tag{14}$$

which, by solving forward, yields :

$$\frac{I_t}{C_t} = \frac{\beta\gamma}{1-\beta\gamma} \tag{15}$$

so that:

$$C_t = (1-\beta\gamma)Y_t \tag{16}$$

$$I_t = K_{t+1} = \beta\gamma Y_t \tag{17}$$

Now combining condition (12) with the expression of the real wage (9) and the value of consumption (16), we find that N_t is constant and equal to N, where N is given by:

$$NV'(N) = \frac{1-\gamma}{1-\beta\gamma} \tag{18}$$

Then formula (18) yields a Walrasian quantity of labor equal to:

$$N = \left[\frac{1-\gamma}{\xi\,(1-\beta\gamma)}\right]^{1/\nu} \tag{19}$$

To sum up, the dynamics is given by

$$N_t = N \tag{20}$$

$$Y_t = Z_t K_t^{\gamma} N^{1-\gamma} \tag{21}$$

$$\frac{W_t}{P_t} = (1-\gamma)\frac{Y_t}{N} \tag{22}$$

$$K_{t+1} = \beta\gamma Y_t \tag{23}$$

Consider now the case where capital evolves according to the log linear relation

$$K_{t+1} = A K_t^{1-\delta} I_t^{\delta} \qquad 0 < \delta \leq 1 \tag{24}$$

where δ is the rate of depreciation.
A first thing to note is that the price, in terms of current output, of capital to be transmitted to the next period is not one. We denote it as q_t (like Tobin's q), and it is equal to:

$$q_t = \frac{\partial K_{t+1}/\partial K_t}{\partial K_{t+1}/\partial I_t} = \frac{1-\delta}{\delta}\frac{I_t}{K_t} \tag{25}$$

We still call κ_t the real return in t on investment I_{t-1}. We have:

$$\kappa_t = \frac{\partial Y_t}{\partial I_{t-1}} = \frac{\partial Y_t}{\partial K_t}\frac{\partial K_t}{\partial I_{t-1}} = \gamma\delta\frac{Y_t}{I_{t-1}} \tag{26}$$

The household's budget constraint is now replaced by

$$C_t + I_t = \frac{W_t}{P_t}N_t + \kappa_t(I_{t-1} + q_{t-1}K_{t-1}) \tag{27}$$

The household maximizes his utility:

$$\sum_t \beta^t \left[\log C_t - V\left(L_t\right)\right] \tag{28}$$

subject to his budget constraints and the capital accumulation equations. The Lagrangean for this program is:

$$\sum_t \beta^t \left[\log C_t - V\left(L_t\right)\right] + \sum_t \beta^t \lambda_t \left[\frac{W_t}{P_t}N_t + \kappa_t(I_{t-1} + q_{t-1}K_{t-1}) - C_t - I_t\right] \tag{29}$$

$$+ \sum_t \beta^t \zeta_t \left(A K_{t-1}^{1-\delta} I_{t-1}^{\delta} - K_t\right)$$

The first order conditions for investment and capital yield:

$$\lambda_t = \beta E_t\left(\lambda_{t+1}\kappa_{t+1}\right) + \beta\delta E_t\left(\frac{\zeta_{t+1}K_{t+1}}{I_t}\right) \tag{30}$$

$$\zeta_t = \beta E_t\left(\lambda_{t+1}\kappa_{t+1}q_t\right) + \beta\left(1-\delta\right)E_t\left(\frac{\zeta_{t+1}K_{t+1}}{K_t}\right) \tag{31}$$

Comparing the two, and using the definition of q_t, we find:

$$\zeta_t = \frac{1-\delta}{\delta}\frac{I_t}{K_t}\lambda_t \tag{32}$$

Inserting this into either (30) or (31) we obtain:

$$\frac{I_t}{C_t} = \beta\gamma\delta E_t\left(\frac{Y_{t+1}}{C_{t+1}}\right) + \beta(1-\delta)E_t\left(\frac{I_{t+1}}{C_{t+1}}\right) \tag{33}$$

$$= \beta\gamma\delta + [\beta\gamma\delta + \beta(1-\delta)]E_t\left(\frac{I_{t+1}}{C_{t+1}}\right) \tag{34}$$

which solves as:

$$\frac{I_t}{C_t} = \frac{\beta\gamma\delta}{1-\beta(1-\delta+\gamma\delta)} \tag{35}$$

so that:

$$C_t = \frac{1-\beta(1-\delta+\gamma\delta)}{1-\beta+\beta\delta}Y_t \tag{36}$$

$$I_t = \frac{\beta\gamma\delta}{1-\beta+\beta\delta}Y_t \tag{37}$$

Finally, equations (9), (11), (12) and (36) yield a Walrasian quantity of labor N now given by :

$$N\ V'(N) = \frac{(1-\gamma)(1-\beta+\beta\delta)}{1-\beta+\beta\delta-\beta\gamma\delta} \tag{38}$$

Problem IV – Technological shocks, Preference shocks and the endogeneity of TFP

We consider here a variation of the simple analytical RBC model. We study a model economy $\mathcal{A}$ populated with a representative household and a representative firm. The firm has a Cobb-Douglas technology:

$$Y_t = e^{z_t}K_t^{\gamma}N_t^{1-\gamma} \tag{39}$$

where K_t is capital, N_t labor input, and e^{z_t} the stochastic Total Factor Productivity (TFP). All profits of the firm are distributed to the household. Capital evolves according to

$$K_{t+1} = I_t \tag{40}$$

where I_t is investment in period t.

The representative household works N_t and consumes C_t. Preferences are given by

$$U = E_0\sum_{t=0}^{\infty}\beta^t\left[\log C_t - e^{\chi_t}N_t\right] \tag{41}$$

where χ_t is a preference shock. Capital is accumulated by the household and rented to the firm.

Let κ_t denote the real rental rate of capital, P_t the price of the final good and W_t the nominal wage. It is assumed that γ and β are between 0 and 1.

1 – Write down the budget constraint of the household and the profit function of the firm

Budget constraint: $P_tC_t + P_tK_{t+1} = W_tN_t + \kappa_tK_t \quad (f)$
Profit: $\Pi_t = P_tY_t - W_tN_t - \kappa_tK_t = e^{z_t}K_t^{\gamma}N_t^{1-\gamma} - W_tN_t - \kappa_tK_t$

2 – Derive FOCs of the utility and profit maximization

Household Problem:

$$\max\mathcal{L} = E_0\sum_{t=0}^{\infty}\beta^t\left[\log C_t - e^{\chi_t}N_t - \lambda_t\left(P_tC_t + P_tK_{t+1} - W_tN_t - \kappa_tK_t\right)\right]$$

with K_0 given.
FOCs are:

w.r.t. C_t :	$1/C_t$	$=$	λ_t	(a)
w.r.t. N_t :	e^{χ_t}	$=$	λ_tW_t	(b)
w.r.t. K_{t+1}	λ_tP_t	$=$	$\beta E_t\lambda_{t+1}\kappa_{t+1}$	(c)

Firm problem:

$$\max \Pi_t = e^{z_t} K_t^{\gamma} N_t^{1-\gamma} - W_t N_t - \kappa_t K_t$$

FOCs are:

$$\begin{array}{llcll} \text{w.r.t. } N_t: & W_t/P_t & = & (1-\gamma)Y_t/N_t & (d) \\ \text{w.r.t. } K_t: & \kappa_t/P_t & = & \gamma Y_t/K_t & (e) \end{array}$$

3 − Define a competitive equilibrium of this economy

A competitive equilibrium is a sequence of prices $\{P_t, W_t, \kappa_t\}$ and quantities $\{C_t, K_{t+1}, N_{t+1}\}$ such that *(i)* quantities maximize profits and utility given prices (meaning that equations (a) to (f) hold) and *(ii)* markets clear, meaning that $Y_t = C_t + K_{t+1}$ (g) hold. Note that because of Walras law, one price can be chosen to 1 (for example P_t). The equilibrium will only determine *relative* prices.

4 − Solve the model and show that the equilibrium process of output is $y_t = z_t + \gamma y_{t-1} - (1-\gamma)\chi_t$ (1) (dropping constants)

Using (a) and (c), one gets $1/C_t = E_t \lambda_{t+1} P_{t+1} \kappa_{t+1}/P_{t+1}$. Using (e), one obtains $1/C_t = \beta\gamma E_t 1/C_{t+1} Y_{t+1}/K_{t+2}$. Using (f), one gets $1/C_t = \beta\gamma E_t 1/C_{t+1}(C_{t+1} + K_{t+2})/K_{t+2}$ which implies $K_{t+1}/C_t = \beta\gamma E_t(1 + K_{t+2}/C_{t+1})$.

Solving forward this equation, we obtain $K_{t+1}/C_t = \beta\gamma/(1-\beta\gamma)$. Using again (g), we obtain $K_{t+1} = \beta\gamma Y_t$ and $C_t = (1-\beta\gamma)Y_t$. Using (b), we obtain $e^{\chi_t} = \lambda_t P_t \times W_t/P_t = 1/C_t \times (1-\gamma)Y_t/N_t$ which gives $N_t = \frac{1-\gamma}{1-\beta\gamma} e^{-\chi_t}$.

Plugging the preceding results into the production function gives

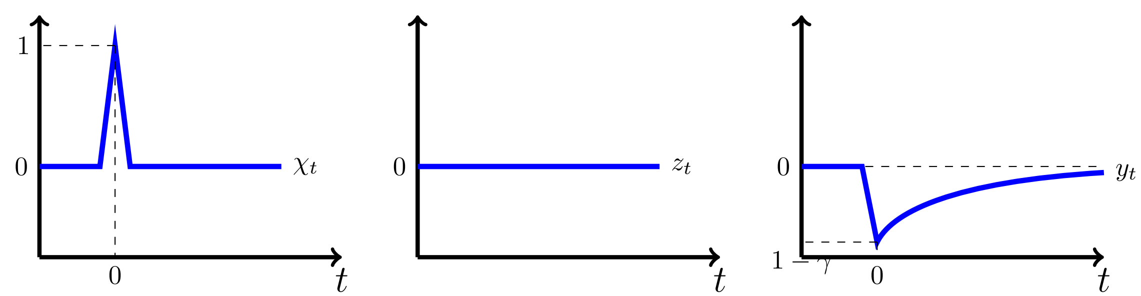
$$Y_t e^{z_t} K_t^{\gamma} N_t^{1-\gamma} = e^{z_t} \left(\beta\gamma Y_{t-1}\right)^{\gamma} \left(\frac{1-\gamma}{1-\beta\gamma} e^{-\chi_t}\right)^{1-\gamma}$$

Taking logs and dropping constants gives

$$y_t = z_t + \gamma y_{t-1} - (1-\gamma)\chi_t \tag{1}$$

5 − Assume $y_{-1} = 0$, $z_t = 0$ $\forall t$, $\chi_t = 0$ $\forall t$, except $\chi_0 = 1\%$. Draw the time path on χ_t, z_t and y_t. Explain why y_t is persistent.

Figure 2: Impulse response to a one time shock χ, economy $\mathcal{A}$



⋆

We now consider an economy $\mathcal{B}$, in which the TFP is not exogenous at the aggregate level, but given by $e^{z_t} = \overline{Y}_t^{\theta} e^{x_t}$, where x is the exogenous part of TFP and $\overline{Y}_t^{\theta}$ acts as an externality. More precisely, $\overline{Y}_t$ is taken as given by firms and households, but one has at the competitive equilibrium $\overline{Y}_t = Y_t$. I is assumed that θ is between 0 and 1.

6 – What is the economic interpretation of this externality?

This externality implies that inputs are individually (at the level of one firm) more productive when aggregate production is large. On can think of network externalities or thick market externalities (it is easier to do business in a boom).

7 – Solve for the competitive equilibrium and give the equilibrium process of output (again in logs). Comment

As $\overline{Y}$ enters as an externality, all individual decisions are kept the same, except that e^{z_t} has now a different expression. We can therefore directly get at the competitive equilibrium

$$Y_t e^{z_t} K_t^{\gamma} N_t^{1-\gamma} = e^{z_t} Y_t^{\theta} \left(\beta\gamma Y_{t-1}\right)^{\gamma} \left(\frac{1-\gamma}{1-\beta\gamma} e^{-\chi_t}\right)^{1-\gamma}$$

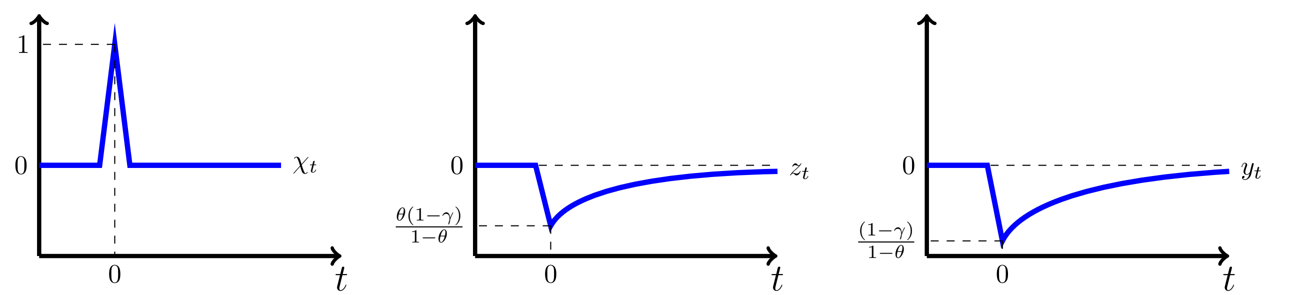
Solving for Y_t, taking logs and dropping constants, we obtain:

$$y_t = \frac{1}{1-\theta} x_t + \frac{\gamma}{1-\theta} y_{t-1} - \frac{1-\gamma}{1-\theta} \tag{2}$$

and $z_t = \theta y_t + x_t$

8 – Assume that an economist observes the economy $\mathcal{B}$, with externality, but thinks that he is observing economy $\mathcal{A}$, and is therefore using equation (1) to measure TFP. Draw the response of observed TFP z_t, of y_t and χ_t if $y_{-1} = 0$, $x_t = 0\ \forall t$, $\chi_t = 0\ \forall t$, except $\chi_0 = 1\%$.

Figure 3: Impulse response to a one time shock χ, economy $\mathcal{B}$ wrongly measured as an economy $\mathcal{A}$



9 – How to interpret the positive correlation between observed TFP and output? Could such an economist believe (wrongly) that technological shocks are driving part of the response of the economy? Discuss.

The correlation between TFP and output could lead the economist to attribute that fluctuation to technological shocks. In reality, the output movements are due to the preference shocks. Because the economist mismeasures productivity (z_t instead of x_t), the measure of TFP is *contaminated* by preference shocks.