Macroeconomics A; EI060

Short problems

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1 Consumption allocation

Question: Consider a two period model where the consumers maximizes a log utility:

$$U_1 = ln\left(C_1\right) + \beta ln\left(C_2\right)$$

The consumption basket is given by:

$$\begin{aligned} C_t &= \left(C_t^T\right)^{\gamma} \left(C_t^N\right)^{1-\gamma} \\ C_t^T &= \left(C_t^H\right)^{\theta} \left(C_t^F\right)^{1-\theta} \end{aligned}$$

Take the Foreign traded good to have a price of 1. The price of the Home traded good is P_t^H and that of the non-traded good P_t^N .

Show that:

$$\begin{aligned} P_t^H C_t^H &= \gamma\theta P_t C_t \\ C_t^F &= \gamma\left(1-\theta\right) P_t C_t \\ P_t^N C_t^N &= \left(1-\gamma\right) P_t C_t \end{aligned}$$

and:

$$P_t = \frac{1}{\left(\gamma\theta\right)^{\gamma\theta}\left(\gamma\left(1-\theta\right)\right)^{\gamma(1-\theta)}\left(1-\gamma\right)^{1-\gamma}} \left(P_t^H\right)^{\gamma\theta} \left(P_t^N\right)^{1-\gamma}$$

Answer: The consumption basket is:

$$C_t = \left(C_t^H\right)^{\gamma\theta} \left(C_t^F\right)^{\gamma(1-\theta)} \left(C_t^N\right)^{1-\gamma}$$

The expenditure is:

$$P_t^H C_t^H + C_t^F + P_t^N C_t^N$$

We minimize the expenditure, subject to a target value for C_t. The Lagrangian is:

$$\mathcal{L}_t = P_t^H C_t^H + C_t^F + P_t^N C_t^N + \lambda_t \left[C_t - \left(C_t^H\right)^{\gamma\theta} \left(C_t^F\right)^{\gamma(1-\theta)} \left(C_t^N\right)^{1-\gamma}\right]$$

The optimality conditions are:

$$\begin{aligned}
0 &= P_t^H - \lambda_t \left(C_t^H\right)^{\gamma\theta-1} \left(C_t^F\right)^{\gamma(1-\theta)} \left(C_t^N\right)^{1-\gamma} \gamma\theta \\
0 &= 1 - \lambda_t \left(C_t^H\right)^{\gamma\theta} \left(C_t^F\right)^{\gamma(1-\theta)-1} \left(C_t^N\right)^{1-\gamma} \gamma(1-\theta) \\
0 &= P_t^N - \lambda_t \left(C_t^H\right)^{\gamma\theta} \left(C_t^F\right)^{\gamma(1-\theta)} \left(C_t^N\right)^{-\gamma} (1-\gamma)
\end{aligned}$$

Multiply by C_t^H, C_t^F and C_t^Nrespectively, and add up:

$$\begin{aligned}
P_t^H C_t^H + C_t^F + P_t^N C_t^N &= \lambda_t \left(C_t^H\right)^{\gamma\theta} \left(C_t^F\right)^{\gamma(1-\theta)} \left(C_t^N\right)^{1-\gamma} \left[\gamma\theta + \gamma(1-\theta) + (1-\gamma)\right] \\
P_t C_t &= \lambda_t \left(C_t^H\right)^{\gamma\theta} \left(C_t^F\right)^{\gamma(1-\theta)-1} \left(C_t^N\right)^{1-\gamma} \\
P_t C_t &= \lambda_t C_t
\end{aligned}$$

Using this, the first-order conditions give the demands:

$$\begin{aligned}
P_t^H C_t^H &= \gamma\theta P_t C_t \\
C_t^F &= \gamma(1-\theta) P_t C_t \\
P_t^N C_t^N &= (1-\gamma) P_t C_t
\end{aligned}$$

The price index is:

$$\begin{aligned}
C_t &= \left(C_t^H\right)^{\gamma\theta} \left(C_t^F\right)^{\gamma(1-\theta)} \left(C_t^N\right)^{1-\gamma} \\
C_t &= \left(\gamma\theta \frac{P_t C_t}{P_t^H}\right)^{\gamma\theta} \left(\gamma(1-\theta) P_t C_t\right)^{\gamma(1-\theta)} \left((1-\gamma)\frac{P_t C_t}{P_t^N}\right)^{1-\gamma} \\
1 &= \left(\gamma\theta \frac{P_t}{P_t^H}\right)^{\gamma\theta} \left(\gamma(1-\theta) P_t\right)^{\gamma(1-\theta)} \left((1-\gamma)\frac{P_t}{P_t^N}\right)^{1-\gamma} \\
P_t &= \frac{1}{(\gamma\theta)^{\gamma\theta} \left(\gamma(1-\theta)\right)^{\gamma(1-\theta)} (1-\gamma)^{1-\gamma}} \left(P_t^H\right)^{\gamma\theta} \left(P_t^N\right)^{1-\gamma}
\end{aligned}$$

2 Intertemporal allocation

Question: Outputs are endowments, and the consumer can save in a bond denominated in the Foreign traded good, with a return r. We asssume $\beta(1+r) = 1$.

Show that:

$$\begin{aligned}
C_2 &= \left(1 + r^C\right) C_1 \\
1 + r^C &= \left(\frac{P_1^H}{P_2^H}\right)^{\gamma\theta} \left(\frac{P_1^N}{P_2^N}\right)^{1-\gamma}
\end{aligned}$$

Answer: The budget constraints are:

$$\begin{aligned}
B_2 + P_1 C_1 &= P_1^H Y_1^H + P_1^N Y_1^N \\
P_2 C_2 &= P_2^H Y_2^H + P_2^N Y_2^N + (1+r) B_2
\end{aligned}$$

The Lagrangian is then:

$$\mathcal{L}_t = ln(C_1) + \beta ln(C_2) + \lambda \left[P_1^H Y_1^H + P_1^N Y_1^N + \frac{P_2^H Y_2^H + P_2^N Y_2^N}{1+r} - P_1 C_1 - \frac{P_2 C_2}{1+r}\right]$$

The optimality conditions are:

$$
\begin{aligned}
0 &= (C_1)^{-1} - \lambda P_1 \\
0 &= \beta (C_2)^{-1} - \lambda \frac{P_2}{1+r}
\end{aligned}
$$

This gives the Euler condition:

$$
\begin{aligned}
C_2 &= \beta (1+r) \frac{P_1}{P_2} C_1 \\
C_2 &= \frac{P_1}{P_2} C_1
\end{aligned}
$$

Using the expression of the price indices, we have:

$$
\frac{P_1}{P_2} = \left(\frac{P_1^H}{P_2^H}\right)^{\gamma\theta} \left(\frac{P_1^N}{P_2^N}\right)^{1-\gamma}
$$

The intetemporal constraint is:

3 Real exchange rate

Question: The consumption of non-traded good is equal to its endowment each period. Show that:

$$
\frac{P_2}{P_1} = \left(\frac{Y_1^N}{Y_2^N}\right)^{1-\gamma} \left(\frac{P_1^H}{P_2^H}\right)^{\gamma\theta}
$$

Answer: We write the ratio of expenditures as follows:

$$
\begin{aligned}
\frac{P_2^N C_2^N}{P_1^N C_1^N} &= \frac{(1-\gamma) P_2 C_2}{(1-\gamma) P_1 C_1} \\
\frac{C_2^N}{C_1^N} &= \frac{P_1^N}{P_2^N} \frac{P_2 C_2}{P_1 C_1} \\
\frac{C_2^N}{C_1^N} &= \left(\frac{P_1}{P_2}\right)^{\frac{1}{1-\gamma}} \left(\frac{P_1^H}{P_2^H}\right)^{\frac{\gamma\theta}{1-\gamma}} \frac{P_2 C_2}{P_1 C_1} \\
\frac{Y_2^N}{Y_1^N} &= \left(\frac{P_1}{P_2}\right)^{\frac{1}{1-\gamma}} \left(\frac{P_1^H}{P_2^H}\right)^{\frac{\gamma\theta}{1-\gamma}} \\
\frac{P_2}{P_1} &= \left(\frac{Y_1^N}{Y_2^N}\right)^{1-\gamma} \left(\frac{P_1^H}{P_2^H}\right)^{\gamma\theta}
\end{aligned}
$$

4 Current account

Question: The value of the spending on traded goods in a period is: $P_1^H C_1^H + C_1^F$. Show that:

$$
P_1^H C_1^H + C_1^F + \beta \left(P_2^H C_2^H + C_2^F\right) = P_1^H Y_1^H + \beta P_2^H Y_2^H
$$

Using the allocation of consumption, and the Euler condition, show that:

$$P_1^H C_1^H + C_1^F = \frac{1}{1+\beta}\left[P_1^H Y_1^H + \beta P_2^H Y_2^H\right]$$

Show that the current account is:

$$\frac{CA_1}{P_1^H Y_1^H} = \frac{\beta}{1+\beta}\left(1 - \frac{P_2^H Y_2^H}{P_1^H Y_1^H}\right)$$

What is the impact of the dynamics of the non-traded endowment?

Answer: The intertemporal budget constraint is written as follows:

$$\begin{aligned} & P_1^H C_1^H + C_1^F + P_1^N Y_1^N + \frac{P_2^H C_2^H + C_2^F + P_2^N Y_2^N}{1+r} \\ = \; & P_1^H Y_1^H + P_1^N Y_1^N + \frac{P_2^H Y_2^H + P_2^N Y_2^N}{1+r} \end{aligned}$$

As consumption of non-traded goods is equal to its endowment, we get:

$$P_1^H C_1^H + C_1^F + \beta\left(P_2^H C_2^H + C_2^F\right) = P_1^H Y_1^H + \beta P_2^H Y_2^H$$

The allocation of demand implies that the value of traded consumption is:

$$P_1^H C_1^H + C_1^F = \gamma P_1 C_1$$

The Euler implies a constant overall spending, $C_2 = \frac{P_1}{P_2} C_1$, hence a constant spending on traded goods. Therefore:

$$P_1^H C_1^H + C_1^F = \frac{1}{1+\beta}\left[P_1^H Y_1^H + \beta P_2^H Y_2^H\right]$$

The current account is then:

$$\begin{aligned} CA_1 &= P_1^H Y_1^H - \left(P_1^H C_1^H + C_1^F\right) \\ CA_1 &= P_1^H Y_1^H - \frac{1}{1+\beta}\left[P_1^H Y_1^H + \beta P_2^H Y_2^H\right] \\ \frac{CA_1}{P_1^H Y_1^H} &= 1 - \frac{1}{1+\beta}\left[1 + \beta \frac{P_2^H Y_2^H}{P_1^H Y_1^H}\right] \\ \frac{CA_1}{P_1^H Y_1^H} &= \frac{\beta}{1+\beta} - \frac{\beta}{1+\beta}\frac{P_2^H Y_2^H}{P_1^H Y_1^H} \\ \frac{CA_1}{P_1^H Y_1^H} &= \frac{\beta}{1+\beta}\left(1 - \frac{P_2^H Y_2^H}{P_1^H Y_1^H}\right) \end{aligned}$$

The non-traded endowment plays no role. This is because the effects that we saw in class through the $\frac{1}{\eta} - \frac{1}{\sigma}$ term cancels out as $\sigma = 1$ (log utility) and $\eta = 1$ (Cobb-Douglas index).