Macroeconomics A; EI060

Short problems

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1 Infinite horizon optimization

Question: Consider that the consumer maximizes an intertemporal log utility:

$$U_t = \sum_{s=t}^{\infty} \beta^{s-t} ln\left(C_s\right)$$

Output is an endowment growing at a rate g, and the agent can save in a bond:

$$\begin{aligned} C_t + B_{t+1} &= (1+r)\,B_t + Y_t \\ CA_t &= rB_t + Y_t - C_t \end{aligned}$$

Show that:

$$C_{t+1} = \beta\,(1+r)\,C_t$$

Answer: We iterate the budget constraint forward, and use the transversality condition $\lim_{T\rightarrow\infty} B_{t+T+1}/\left(1+r\right)^{T+1} =$ 0:

$$\begin{aligned} B_t &= \frac{C_t - Y_t}{1+r} + \frac{B_{t+1}}{1+r} \\ B_t &= \frac{C_t - Y_t}{1+r} + \frac{1}{1+r}\left(\frac{C_{t+1} - Y_{t+1}}{1+r} + \frac{B_{t+2}}{1+r}\right) \\ B_t &= \sum_{s=t}^{t+T} \frac{C_s - Y_s}{\left(1+r\right)^{s-t+1}} + \frac{B_{t+T+1}}{\left(1+r\right)^{T+1}} \\ B_t &= \sum_{s=t}^{t+T} \frac{C_s - Y_s}{\left(1+r\right)^{s-t+1}} + \frac{B_{t+T+1}}{\left(1+r\right)^{T+1}} \\ \sum_{s=t}^{\infty} \frac{C_s}{\left(1+r\right)^{s-t}} &= (1+r)\,B_t + \sum_{s=t}^{\infty} \frac{Y_s}{\left(1+r\right)^{s-t}} \end{aligned}$$

The Lagrangian is then:

$$Z_t = \sum_{s=t}^{\infty} \beta^{s-t} ln\left(C_s\right) + \lambda_t \left[(1+r)\,B_t + \sum_{s=t}^{\infty} \frac{Y_s}{\left(1+r\right)^{s-t}} - \sum_{s=t}^{\infty} \frac{C_s}{\left(1+r\right)^{s-t}}\right]$$

The first order conditions with respect toC_t and C_{t+1} are:

$$0 = \frac{1}{C_t} - \lambda_t \qquad ; \qquad 0 = \beta \frac{1}{C_{t+1}} - \frac{\lambda_t}{1+r}$$

Combining them we get:

$$\begin{aligned} \frac{1}{C_t} &= \beta \frac{1+r}{C_{t+1}} = \lambda_t \\ C_{t+1} &= \beta (1+r) C_t \end{aligned}$$

So consumption grows at a rate $\beta (1+r) - 1$.

2 Consumption solution

Question: Assume that $g < r$. Show that consumption at time t is:

$$C_t = (1-\beta) \left[(1+r) B_t + \sum_{s=t}^{\infty} \frac{Y_s}{(1+r)^{s-t}} \right]$$

Answer: The Euler condition implies:

$$C_s = [\beta (1+r)]^{s-t} C_t$$

We start from the intertemporal constraint:

$$\begin{aligned} \sum_{s=t}^{\infty} \frac{C_s}{(1+r)^{s-t}} &= (1+r) B_t + \sum_{s=t}^{\infty} \frac{Y_s}{(1+r)^{s-t}} \\ C_t \sum_{s=t}^{\infty} \frac{[\beta (1+r)]^{s-t}}{(1+r)^{s-t}} &= (1+r) B_t + Y_t \sum_{s=t}^{\infty} \frac{(1+g)^{s-t}}{(1+r)^{s-t}} \\ C_t \sum_{y=0}^{\infty} (\beta)^y &= (1+r) B_t + \sum_{y=0}^{\infty} \left(\frac{1+g}{1+r} \right)^y \\ C_t \frac{1}{1-\beta} &= (1+r) B_t + Y_t \frac{1}{1 - \frac{1+g}{1+r}} \\ C_t &= (1-\beta) \left[(1+r) B_t + Y_t \frac{1+r}{r-g} \right] \\ C_t &= (1-\beta)(1+r) \left(B_t + Y_t \frac{1}{r-g} \right) \end{aligned}$$

3 Net exports and the current account

Question: Show that the trade balance and the current account are:

$$\begin{aligned} NX_t &= -(1-\beta)(1+r) B_t + \frac{\beta (1+r) - (1+g)}{r-g} Y_t \\ CA_t &= [\beta (1+r) - 1] B_t + \frac{\beta (1+r) - (1+g)}{r-g} Y_t \end{aligned}$$

Show that the debt dynamics are:

$$B_{t+1} = \beta(1+r)B_t + \frac{\beta(1+r)-(1+g)}{r-g}Y_t$$

Is the ratio of debt to GDP $\frac{B_t}{Y_t}$ constant? How about the ratio of other variables to GDP?

Answer: The net exports are output net of consumption:

$$\begin{aligned}
NX_t &= Y_t - C_t \\
NX_t &= -(1-\beta)(1+r)B_t + \left[1-(1-\beta)\frac{1+r}{r-g}\right]Y_t \\
NX_t &= -(1-\beta)(1+r)B_t + \left[1-\frac{1+r}{r-g}+\frac{1+r}{r-g}\beta\right]Y_t \\
NX_t &= -(1-\beta)(1+r)B_t + \frac{\beta(1+r)-(1+g)}{r-g}Y_t
\end{aligned}$$

The current account is the sum of the trade balance and primary income (interest on assets):

$$\begin{aligned}
CA_t &= rB_t + NX_t \\
CA_t &= [r-(1-\beta)(1+r)]B_t + \frac{\beta(1+r)-(1+g)}{r-g}Y_t \\
CA_t &= [\beta(1+r)-1]B_t + \frac{\beta(1+r)-(1+g)}{r-g}Y_t
\end{aligned}$$

From the budget constraint:

$$\begin{aligned}
B_{t+1} &= B_t + CA_t \\
B_{t+1} &= B_t + [\beta(1+r)-1]B_t + \frac{\beta(1+r)-(1+g)}{r-g}Y_t \\
B_{t+1} &= \beta(1+r)B_t + \frac{\beta(1+r)-(1+g)}{r-g}Y_t
\end{aligned}$$

In terms of ratio of debt to GDP, we get:

$$\begin{aligned}
B_{t+1} &= \beta(1+r)B_t + \frac{\beta(1+r)-(1+g)}{r-g}Y_t \\
\frac{B_{t+1}}{Y_{t+1}}\frac{Y_{t+1}}{Y_t} &= \beta(1+r)\frac{B_t}{Y_t} + \frac{\beta(1+r)-(1+g)}{r-g} \\
\frac{B_{t+1}}{Y_{t+1}}(1+g) &= \beta(1+r)\frac{B_t}{Y_t} + \frac{\beta(1+r)-(1+g)}{r-g} \\
\frac{B_{t+1}}{Y_{t+1}}(1+g) &= \beta(1+r)\frac{B_t}{Y_t} + \frac{\beta(1+r)-(1+g)}{r-g} \\
\frac{B_{t+1}}{Y_{t+1}} &= \frac{\beta(1+r)}{1+g}\frac{B_t}{Y_t} + \frac{\beta(1+r)-(1+g)}{(r-g)(1+g)} \\
\frac{B_{t+1}}{Y_{t+1}} - \frac{B_t}{Y_t} &= \frac{\beta(1+r)-(1+g)}{1+g}\frac{B_t}{Y_t} + \frac{\beta(1+r)-(1+g)}{(r-g)(1+g)} \\
\frac{B_{t+1}}{Y_{t+1}} - \frac{B_t}{Y_t} &= \frac{\beta(1+r)-(1+g)}{1+g}\left(\frac{B_t}{Y_t}+\frac{1}{r-g}\right)
\end{aligned}$$

Unless the initial debt ratio is exactly equal to $\frac{B_t}{Y_t} = -\frac{1}{r-g}$, the ratio is not constant.

The other variables are then not constant ratios of GDP in general:

$$
\begin{aligned}
\frac{C_t}{Y_t} &= (1-\beta)(1+r)\left(\frac{B_t}{Y_t} + \frac{1}{r-g}\right) \\
\frac{NX_t}{Y_t} &= -(1-\beta)(1+r)\frac{B_t}{Y_t} + \frac{\beta(1+r)-(1+g)}{r-g} \\
\frac{CA_t}{Y_t} &= [\beta(1+r)-1]\frac{B_t}{Y_t} + \frac{\beta(1+r)-(1+g)}{r-g}
\end{aligned}
$$

4 Constant consumption

Question: Consider that $\beta(1+r) = 1$.

What are the values of consumption C_t, net exports NX_t, the current account CA_t, in periods t and $t+1$?

What are the debt dynamics $B_{t+1} - B_t$?

Are variables constant, in terms of ratio to GDP?

Answer: We first recall the general expressions:

$$
\begin{aligned}
C_t &= (1-\beta)(1+r)\left(B_t + Y_t\frac{1}{r-g}\right) \\
NX_t &= -(1-\beta)(1+r)B_t + \frac{\beta(1+r)-(1+g)}{r-g}Y_t \\
CA_t &= [\beta(1+r)-1]B_t + \frac{\beta(1+r)-(1+g)}{r-g}Y_t \\
B_{t+1} &= \beta(1+r)B_t + \frac{\beta(1+r)-(1+g)}{r-g}Y_t
\end{aligned}
$$

If $\beta(1+r) = 1$, consumption, net exports, and the current account in period t are:

$$
\begin{aligned}
C_t &= \frac{1-\beta}{\beta}\left(B_t + \frac{\beta}{1-\beta(1+g)}Y_t\right) \\
NX_t &= -\frac{1-\beta}{\beta}B_t - \frac{\beta g}{1-\beta(1+g)}Y_t \\
CA_t &= -\frac{\beta g}{1-\beta(1+g)}Y_t
\end{aligned}
$$

The debt dynamics show that assets decrease through time, ultimately turning into growing liabilities:

$$
B_{t+1} = B_t - \frac{\beta g}{1-\beta(1+g)}Y_t
$$

Using the debt dynamics, we see that consumption constant:

$$
\begin{aligned}
C_{t+1} &= \frac{1-\beta}{\beta}\left(B_{t+1} + \frac{\beta}{1-\beta(1+g)}Y_{t+1}\right) \\
C_{t+1} &= \frac{1-\beta}{\beta}\left(B_t - \frac{\beta g}{1-\beta(1+g)}Y_t + \frac{\beta(1+g)}{1-\beta(1+g)}Y_t\right)
\end{aligned}
$$

$$
\begin{aligned}
C_{t+1} &= \frac{1-\beta}{\beta}\left(B_t + \frac{\beta}{1-\beta(1+g)}Y_t\right) \\
C_{t+1} &= C_t
\end{aligned}
$$

As output is growing, the ratio of consumption to output decreases, and converges to zero in the long run.

The net exports are growing through time:

$$
\begin{aligned}
NX_{t+1} &= -\frac{1-\beta}{\beta}B_{t+1} - \frac{\beta g}{1-\beta(1+g)}Y_{t+1} \\
NX_{t+1} &= -\frac{1-\beta}{\beta}B_t + \frac{\beta g}{1-\beta(1+g)}\left[\frac{1-\beta}{\beta}Y_t - Y_{t+1}\right] \\
NX_{t+1} &= -\frac{1-\beta}{\beta}B_t + \frac{\beta g}{1-\beta(1+g)}\left[\frac{1-\beta}{\beta} - (1+g)\right]Y_t \\
NX_{t+1} &= NX_t + \frac{\beta g}{1-\beta(1+g)}Y_t + \frac{\beta g}{1-\beta(1+g)}\left[\frac{1-\beta}{\beta} - (1+g)\right]Y_t \\
NX_{t+1} &= NX_t + \frac{\beta g}{1-\beta(1+g)}\left[\frac{1}{\beta} - (1+g)\right]Y_t \\
NX_{t+1} &= NX_t + gY_t
\end{aligned}
$$

In terms of ratio to GDP, the trade balance also grows, and converges to 100% as time goes to infinity:

$$
\begin{aligned}
\frac{NX_{t+1}}{Y_{t+1}}(1+g) &= \frac{NX_t}{Y_t} + g \\
\frac{NX_{t+1}}{Y_{t+1}} &= \frac{g + \frac{NX_t}{Y_t}}{1+g} \\
\frac{NX_{t+1}}{Y_{t+1}} &= \frac{NX_t}{Y_t} - \frac{1+g}{1+g}\frac{NX_t}{Y_t} + \frac{g + \frac{NX_t}{Y_t}}{1+g} \\
\frac{NX_{t+1}}{Y_{t+1}} &= \frac{NX_t}{Y_t} + \frac{g}{1+g}\left(1 - \frac{NX_t}{Y_t}\right)
\end{aligned}
$$

The current account grows, but is stable relative to GDP:

$$
\begin{aligned}
CA_{t+1} &= -\frac{\beta g}{1-\beta(1+g)}Y_{t+1} \\
CA_{t+1} &= -\frac{\beta g(1+g)}{1-\beta(1+g)}Y_t \\
CA_{t+1} &= (1+g)\,CA_t \\
\frac{CA_{t+1}}{Y_{t+1}}(1+g) &= (1+g)\frac{CA_t}{Y_t} \\
\frac{CA_{t+1}}{Y_{t+1}} &= \frac{CA_t}{Y_t}
\end{aligned}
$$

The dynamics of the ratio of debt to GDP are:

$$
\begin{aligned}
B_{t+1} &= B_t - \frac{\beta g}{1-\beta(1+g)}Y_t \\
\frac{B_{t+1}}{Y_{t+1}}(1+g) &= \frac{B_t}{Y_t} - \frac{\beta g}{1-\beta(1+g)}
\end{aligned}
$$

$$\left(\frac{B_{t+1}}{Y_{t+1}} - \frac{B_t}{Y_t}\right)(1+g) = -g\frac{B_t}{Y_t} - \frac{\beta g}{1-\beta(1+g)}$$

$$\left(\frac{B_{t+1}}{Y_{t+1}} - \frac{B_t}{Y_t}\right)(1+g) = -\frac{g}{1+g}\left[\frac{B_t}{Y_t} + \frac{\beta}{1-\beta(1+g)}\right]$$

Ultimately, it stabilizes at $-\frac{\beta}{1-\beta(1+g)}$ which is g times the constant ratio of current account to GDP.

Consider for simplicity that the economy as not assets initially ($B_t = 0$). As output is growing and consumption is smoothed, the economy starts with a trade deficit ($NX_t < 0$) and a current account deficit ($CA_t < 0$), hence accumulates debt.

As output is growing, it will at some point exceed consumption, and the economy then have a trade surplus. It will however by then have a large debt, and the interest payments are such that it always runs a current account deficit, growing in line with GDP. Ultimately, output goes entirely into exports, and these large exports are enough to pay part of the debt interest and keep the ratio of debt to GDP constant.

5 Autarky

Question: If the economy is in autarky, consumption is always equal to output.

Show that the real interest rate is:

$$1 + r^A = \frac{1+g}{\beta}$$

Answer: The Euler condition implies:

$$C_{t+1} = \beta(1+r)C_t$$

As consumption is equal to output:

$$\begin{aligned} Y_{t+1} &= \beta\left(1+r^A\right)Y_t \\ 1+g &= \beta\left(1+r^A\right) \\ 1+r^A &= \frac{1+g}{\beta} \end{aligned}$$

6 Open economy, autarky interest rate

Question: Consider that the economy is open, and the world interest rate is equal to the autarky interest rate $1 + r^A$.

What are the values of consumption C_t, net exports NX_t, the current account CA_t, in periods t and $t+1$?

What are the debt dynamics $B_{t+1} - B_t$?

Are variables constant, in terms of ratio to GDP?

Answer: We first recall the general expressions:

$$C_t = (1-\beta)(1+r)\left(B_t + Y_t\frac{1}{r-g}\right)$$

$$
\begin{aligned}
NX_t &= -(1-\beta)(1+r)B_t + \frac{\beta(1+r)-(1+g)}{r-g}Y_t \\
CA_t &= [\beta(1+r)-1]B_t + \frac{\beta(1+r)-(1+g)}{r-g}Y_t \\
B_{t+1} &= \beta(1+r)B_t + \frac{\beta(1+r)-(1+g)}{r-g}Y_t
\end{aligned}
$$

If $1+r^A = \frac{1+g}{\beta}$, consumption, net exports, and the current account in period t are:

$$
\begin{aligned}
C_t &= \frac{1-\beta}{\beta}(1+g)B_t + Y_t \\
NX_t &= -\frac{1-\beta}{\beta}(1+g)B_t \\
CA_t &= gB_t \\
B_{t+1} &= (1+g)B_t
\end{aligned}
$$

The debt dynamics show that asset (or liabilities) grow at the rate of GDP, and never change sign:

$$
B_{t+1} = (1+g)B_t
$$

As assets and output both grow at the same rate, all other variables grow at that rate. For instance, consumption is:

$$
\begin{aligned}
C_{t+1} &= \frac{1-\beta}{\beta}(1+g)B_{t+1} + Y_{t+1} \\
C_{t+1} &= \frac{1-\beta}{\beta}(1+g)(1+g)B_t + (1+g)Y_t \\
C_{t+1} &= (1+g)\left[\frac{1-\beta}{\beta}(1+g)B_t + Y_t\right] \\
C_{t+1} &= (1+g)C_t
\end{aligned}
$$

With all variables, including output, growing at the same rate, all ratio to GDP are constant. The economy is thus immediately in a steady state, as the real interest rate reflects the growth rate of output, and thus leads consumption to track it. By contrast, when $r^A > r$, consumption grows at a rate slower than output, and ultimately we get in the situation where the ratio of consumption to output is zero, and (nearly) all output goes towards net exports.