# Game Theory

## Dynamic Game of Incomplete Information (Signaling Game)

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# Outline

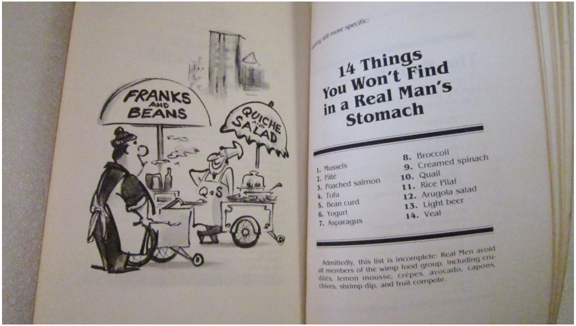
- Signaling Game (信号博弈)
- Perfect Bayesian Nash Equilibrium (PBNE) 完美贝叶斯纳什均衡
    - Separating Equilibrium (分离均衡)
    - Pooling Equilibrium (混同均衡)
- Equilibrium refinement (排除混同均衡的精炼):
    - Intuition Criterion (Cho & Kreps, 1987) 直觉标准 (required)
    - "Universal Divinity ($D_1$-Criterion)" (Banks & Sobel, 1987) "普世神性" (not covered)
- Example: Job Market Signaling (学历是一种信号)

# Dynamic Game with Incomplete Information

- In games of incomplete information there is at least one player who is uninformed about the type of another player.
- In some circumstances, it will be to the benefit of players to reveal their types to their opponents.
  1. Nature chooses a type for player 1 that player 2 does not know, but cares about.
  2. Player 1 has a rich action set in the sense that there are at least as many actions as there are types, and each action imposes a different cost on each type
  3. Player 1 chooses an action first, and player 2 then responds after observing player 1's choice
  4. Given player 2's belief about player 1;s strategy, player 2 updates his belief after observing player 1's choice. Player 2 then makes his choice as a best response to his updated beliefs.

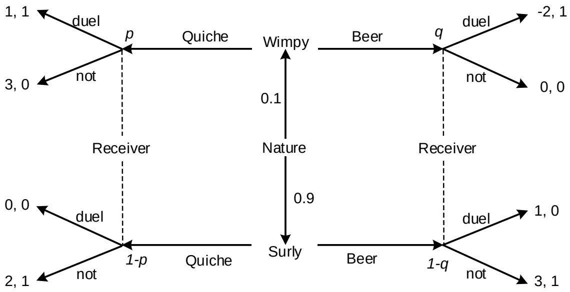
# Example: Beer and Quiche

"Real Men Don't Eat Quiche" is a book by Bruce Feirstein (1982), satirizing stereotypes of masculinity.



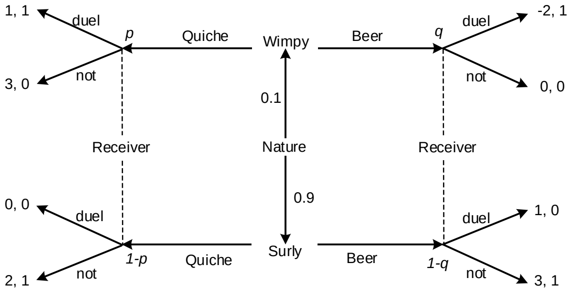

- "wimpy" (weak) type
- "surly" (strong) type

1. Nature draws the types of player 1, “wimpy” or “surly”
    - Assume that the “wimpy” type prefers quiche for breakfast
    - The “surly” type prefers beer for breakfast
2. Player 1 choose quiche or beer (send a signal)
3. Player 2 (receiver), observes the signal (quiche or beer), but does not know the type of player 1
    - Player 2 forms his/her own belief about the type of the sender, based on the signal (quiche/beer) observed.
4. The receiver decides to “duel” or “not”
    - Player 2 prefers to duel with the wimpy type, but not the surly type

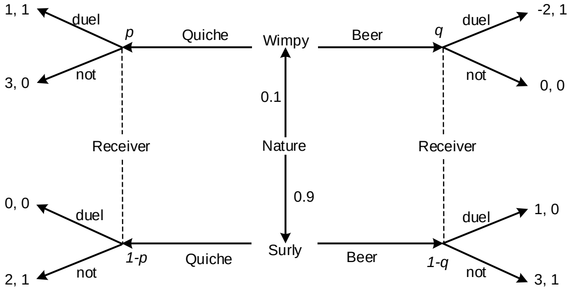



1. Nature draws a type $t_1 = w$ or $t_2 = s$ for the sender (player 1). The prior probability distribution is $\Pr(w) = 0.1$ and $\Pr(s) = 0.9$
2. The sender knows his/her own type and then chooses a message $Q$ or $B$;
3. The receiver observes the message (but not type) and then chooses an action $d$ or $n$
4. Payoffs are realized.

# Beliefs

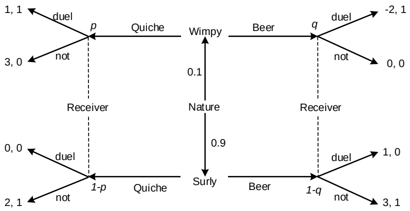



- Prior: $\Pr(w) = 0.1$ and $\Pr(s) = 0.9$
- Posterior: upon receiving the signal, player 2 revises his/her beliefs:
  - if $B$ is observed, the probability that $B$ is sent by type $w$ is $q = \Pr(w|B) = \frac{\Pr(w \text{ sends } B)}{\Pr(B)} = \frac{\Pr(w \text{ sends } B)}{\Pr(w \text{ sends } B)+\Pr(s \text{ sends } B)}$; and $1 - q = \Pr(s|B) = \frac{\Pr(s \text{ sends } B)}{\Pr(s \text{ sends } B)+\Pr(w \text{ sends } B)} = 1 - \Pr(w|B) = 1 - q$



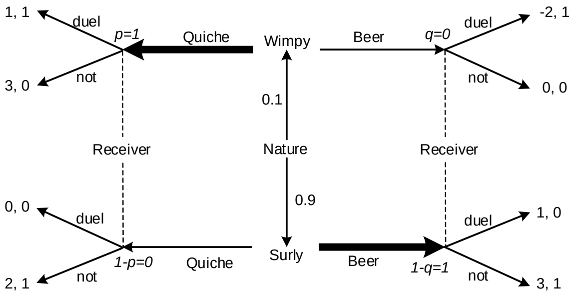

- Prior: $\Pr(w) = 0.1$ and $\Pr(s) = 0.9$
- Posterior: upon receiving the signal, player 2 revises his/her beliefs:
  - if $Q$ is observed, the probability that $Q$ is sent by type $w$ is $p = \Pr(w|Q) = \frac{\Pr(w \text{ sends } Q)}{\Pr(Q)} = \frac{\Pr(w \text{ sends } Q)}{\Pr(w \text{ sends } Q)+\Pr(s \text{ sends } Q)}$; and $1 - p = \Pr(s|Q) = \frac{\Pr(s \text{ sends } Q)}{\Pr(s \text{ sends } Q)+\Pr(w \text{ sends } Q)} = 1 - \Pr(w|Q) = 1 - p$

# Pooling (混同) & Separating (分离)



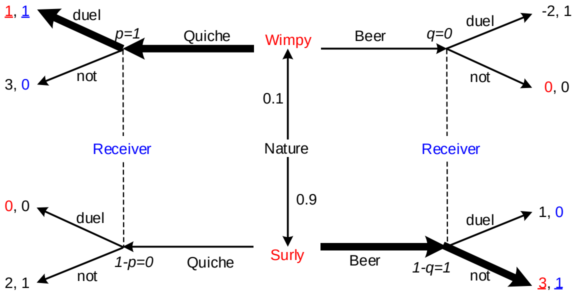

- Sender's strategies can be categorized into 4 cases:
  - (i) $w$ chooses $Q$ and $s$ chooses $B$ (separating)
  - (ii) $w$ chooses $B$ and $s$ chooses $Q$ (separating)
  - (iii) Both $w$ and $s$ send $Q$ (pooling on $Q$)
  - (iv) Both $w$ and $s$ send $B$ (pooling on $B$)
- Separating: each type chooses a different action $\Rightarrow$ revealing his/her type to the receiver
- Pooling: both types choose the same action $\Rightarrow$ revealing no information to receiver

# Case (i): $w$ sends $Q$ and $s$ sends $B$



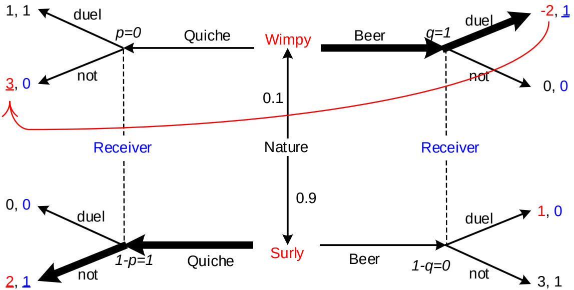

- $p = \Pr(w|Q) = \frac{\Pr(w \cap Q)}{\Pr(w \cap Q) + \Pr(s \cap Q)} = \frac{\Pr(w)\Pr(Q|w)}{\Pr(w)\Pr(Q|w) + \Pr(s)\Pr(Q|s)} = \frac{\Pr(w) \cdot 1}{\Pr(w) \cdot 1 + (1 - \Pr(w)) \cdot 0} = 1$
- Here, $\Pr(Q|w) = 1$ and $\Pr(Q|s) = 0$
- Posterior $p = 1$. Similarly, $1 - p = \Pr(s|Q) = 0$; $q = \Pr(w|B) = 0$; $1 - q = \Pr(s|B) = 1$

# Case (i): $w$ sends $Q$ and $s$ sends $B$



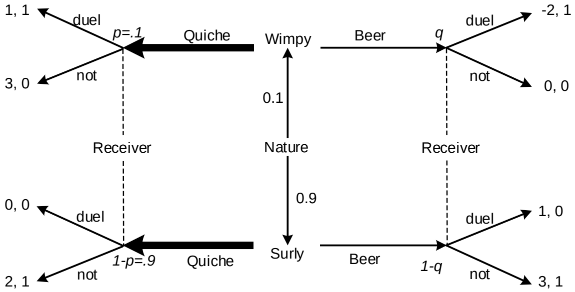

- $p = \Pr(w|Q) = 1$ and $1 - q = \Pr(s|B) = 1 \Rightarrow q = 0$
- Receiver: $d$ is a best response to $Q$; and $n$ is a best response to $B$
- Check whether the sender has an incentive to deviate:
  - For type $w$, deviate from $Q$ to $B$ is not profitable $(1 > 0)$
  - For type $s$, deviate from $B$ to $Q$ is not profitable $(3 > 0)$
- $\{(Q, B), (d, n), p = 1, q = 0\}$ is a separating equilibrium.

# Case (ii): $w$ sends $B$ and $s$ sends $Q$



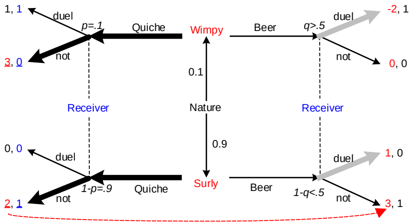

- $q = \Pr(w|B) = 1 \Rightarrow 1 - q = 0$ and $1 - p = \Pr(s|Q) = 1 \Rightarrow p = 0$
- Receiver: $n$ is a best response to $Q$; and $d$ is a best response to $B$
- Sender: type $w$ has an incentive deviate to $Q$ (do not need to pretend to be surly because player 2 does not duel with the one who eats quiche)
- $\{(B, Q), (n, d)\}$ is not an equilibrium

# Case (iii): Pooling on $Q$




- The signal $Q$ reveals no information:
$\Pr(w|Q) = \frac{\Pr(w \cap Q)}{\Pr(w \cap Q) + \Pr(s \cap Q)} = \frac{\Pr(w)\Pr(Q|w)}{\Pr(w)\Pr(Q|w) + \Pr(s)\Pr(Q|s)} = \frac{\Pr(w)\cdot 1}{\Pr(w)\cdot 1 + (1 - \Pr(w))\cdot 1} = \Pr(w)$
- Here, $\Pr(Q|w) = \Pr(Q|s) = 1$.
- Posterior $p = \Pr(w) = 0.1 =$ prior (no updates)

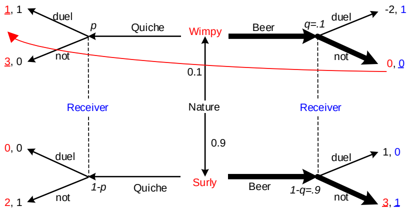
# Case (iii): Pooling on $Q$




- Pooling on $Q \Rightarrow p = \Pr(w|Q) = \Pr(w) = 0.1$
- Receiver: $E(d) = p \cdot 1 + (1-p) \cdot 0 = 0.1$; $E(n) = p \cdot 0 + (1-p) \cdot 1 = 0.9 \Rightarrow n$ is a best response to $Q$
- **Define posterior beliefs off-equilibrium path (that is not reached) at $B$:**

  pooling on $Q$ is an equilibrium provided that neither type will deviate to $B$.
  - $w$ will definitely not deviate to $B$
  - $s$ potentially benefits from deviating to $B$ (he likes beer) if $n$ is a best response to $B$. To avoid deviation, $E(n) = q \cdot 0 + (1-q) \cdot 1 = 1 - q$ at $B$ should be no greater than $E(d) = q \cdot 1 + (1-q) \cdot 0 = q$ at $B$: $E(n) < E(d) \Leftrightarrow 1 - q < q \Leftrightarrow q > \frac{1}{2}$
- $\{(Q, Q), (n, d), p = 0.1, q > \frac{1}{2}\}$ is a pooling equilibrium

# Case (iv): Pooling on $B$




- Pooling on $B \Rightarrow q = \Pr(w|B) = \Pr(w) = 0.1$
- Receiver: $E(d) = q \cdot 1 + (1-q) \cdot 0 = 0.1$;
  $E(n) = q \cdot 0 + (1-q) \cdot 1 = 0.9 \Rightarrow n$ is a best response to $B$
- **Define posterior beliefs off-equilibrium path (that is not reached) at** $Q$:
  - Notice that type $w$ will definitely deviate to $Q$ regardless of the responses of player 2.
- Pooling on $B$ is not an equilibrium

# Definitions: (On/Off) Equilibrium Path

- Let $\sigma^* = (\sigma_1^*, ...\sigma_n^*)$ be a Bayesian Nash equilibrium profile of strategies in a game of incomplete information. We say that an information set is **on the equilibrium path** if given $\sigma^*$ and given the distribution of types, it is reached with positive probability. We say that an information set is **off equilibrium path** if given $\sigma^*$ and the distribution of types, it is reached with zero probability.
  - E.g., case (i) & (ii): both information sets of the receiver are reached (separating)
  - E.g., case (iii): pooling on $Q$ (on the equilibrium path); information set at $B$ is not reached

# Definitions: Beliefs

- A system of Beliefs of an extensive-form game assigns a probability distribution over decision nodes to **every** information set.
  - Sender: nature assigns priors $\Pr(t_1 = w) = 0.1$, $\Pr(t_2 = s) = 0.9$
  - $q = \Pr(w|B)$
  - $1 - q = \Pr(s|B)$
  - $p = \Pr(w|Q)$
  - $1 - p = \Pr(s|Q)$

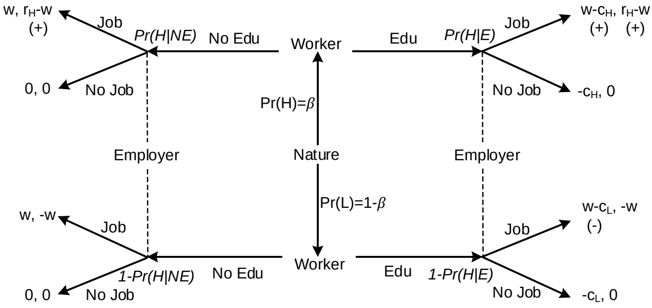
## A Bayesian Nash equilibrium profile (BNE) constitutes a perfect Bayesian equilibrium (PBNE) if they satisfy:

1. Every player has a well-defined belief over where he/she is in each of his/her information set, i.e., the game has a system of beliefs
2. All information sets beliefs that are on the equilibrium path be consistent with Bayes' rule
   - e.g., case (i): $p = 1$, $q = 0$; case (iii): $p = 0.1$
3. At information sets that are off the equilibrium path any belief can be assigned to which Bayes' rule does not apply
   - e.g., case (iii): $q$ could be any number $\in [0, 1]$
4. Sequentially rational: in every information set (including the ones that are not reached) players will play a best response to their beliefs.
   - e.g., case (iii): $(nd)$ are BR to $(QB)$ even though $B$ is not reached $\Leftarrow d$ as a BR to $B$ according to the off-equilibrium belief $q$ (not according to Bayes' rule) to support the equilibrium

# Job Market Signaling (Micheal Spence, 1973)

- A worker has two productivity levels, $t_1 = r_H$ or $t_2 = r_L$. Assume $r_H > r_L = 0$.
- An employer's (prior) belief of a worker's productivity is $\Pr(H) = \beta$ and $\Pr(L) = 1 - \beta$.
- Each worker can choose to get educated, incurring a cost $c_H$ and $c_L$, where $c_H < c_L$.
    - Assumption: Education is orthogonal to productivity.
- The employer chooses to whether offering a job with wage $w$, or no job. The employer gets $r_i - w$ (or 0). Assume $r_H > w$, $c_L > w > c_H$.

1. "Nature" moves first: the work's type is privately known;
2. The worker moves second: choose whether $E$ (education) or $NE$ (no education);
3. The employer moves last: offering a job $J$ or no job $NJ$.

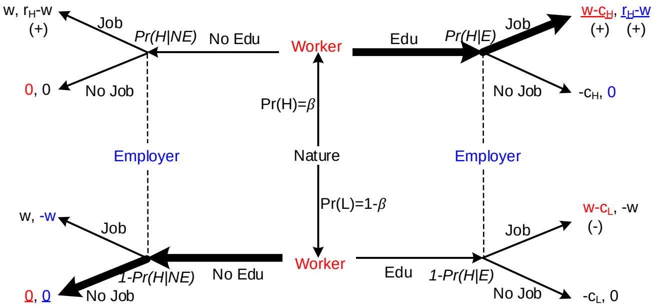



1. *H* sends *E* and *L* sends *NE*
2. *H* sends *NE* and *L* sends *E*
3. Both *H* and *L* send *E*
4. Both *H* and *L* send *NE*

Define PBNE:

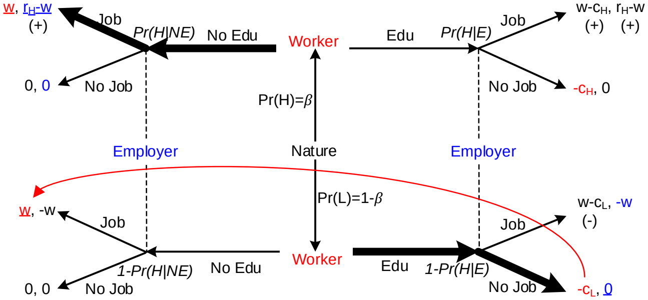
$\{(H\text{'s choice}, L\text{'s choice}), (\text{reaction to } E, \text{reaction to } NE), \Pr(H|E), \Pr(H|NE)\}$

# (i) Separating: $H$ sends $E$ and $L$ sends $NE$



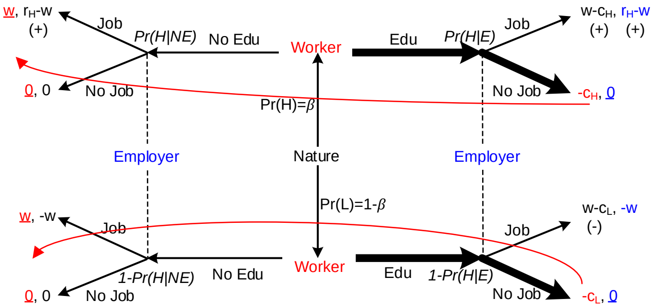

- $\Pr(H|E) = \frac{\Pr(H)\Pr(E|H)}{\Pr(H)\Pr(E|H)+\Pr(L)\Pr(E|L)} = \frac{\beta\cdot 1}{\beta\cdot 1+(1-\beta)\cdot 0} = 1$, $\Pr(L|NE) = 1 - \Pr(H|NE) = 1$
- Employer: $J$ is BR to $E$; $NJ$ is BR to $NE$.
- Worker: Type $H$ won't deviate to $NE \Leftarrow w - c_H > 0$; Type $L$ won't deviate to $E \Leftarrow 0 > w - c_L$
- Separating PBNE: $\{(E, NE), (J, NJ), \Pr(H|E) = 1, \Pr(H|NE) = 0\}$ is PBNE.

# (ii) Separating: *H* sends *NE* and *L* sends *E*



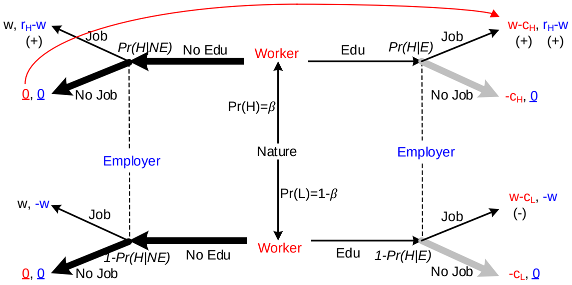

- $\Pr(H|NE) = 1$, $\Pr(L|E) = 1$.
- Employer: *J* is BR to *NE*; *NJ* is BR to *E*.
- Worker: $w > -c_H \Rightarrow$ type *H* won't deviate to *E*; $-c_L < w \Rightarrow$ type *L* deviates to *NE* (mimics the H type)
- Not PBNE.

# (iii): Pooling on $E$ (Assume $\beta r_H < w$)




- $\Pr(H|E) = \frac{\Pr(H)\Pr(E|H)}{\Pr(H)\Pr(E|H)+\Pr(L)\Pr(E|L)} = \frac{\Pr(H)\cdot 1}{\Pr(H)\cdot 1+(1-\Pr(H))\cdot 1} = \Pr(H) = \beta$
- Employer: Upon receiving $E$:
  $E(J) = \beta \cdot (r_H - w) + (1-\beta)\cdot(-w) = \beta r_H - w < 0$; $E(NJ) = 0$. $NJ$ is BR to $E \Rightarrow$
- Worker: Both types will deviate to $NE$ ($-c_H < 0 < w$; $-c_L < 0 < w$).
- Not PBNE

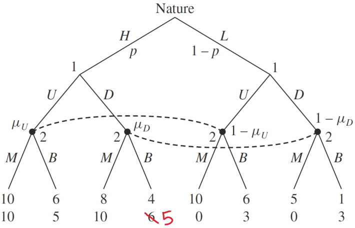
# (iv): Pooling on *NE* (Assume $\beta r_H < w$)



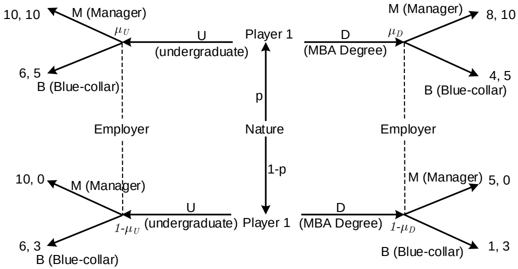

- $\Pr(H|NE) = \Pr(H) = \beta$
- Employer: Upon receiving *NE*, $E(J) = \beta(r_H - w) + (1-\beta)(-w) < 0$, $E(NJ) = 0 \Rightarrow$ *NJ* is BR to *NE*.
- Type *L* will definitely not deviate to *E* $(0 > w - c_L > -c_L)$.
- Type *H* might deviate to *E*. To avoid so, find the employer's BR to **off-equilibrium belief at path** *E*:
  - To support a pooling equilibrium on *NE*, the boss should play *NJ* as a BR to *E*.
  - At path *E*: $E(NJ) = 0$; $E(J) = \Pr(H|E)(r_H - w) + (1 - \Pr(H|E))(-w)$.
    $E(NJ) \geq E(J) \Leftrightarrow \Pr(H|E) \leq \frac{w}{r_H}$
- Pooling PBNE: $\{(NE, NE), (NJ, NJ), \Pr(H|NE) = \beta, \Pr(H|E) \leq \frac{w}{r_H}\}$

# 教材勘误

Textbook

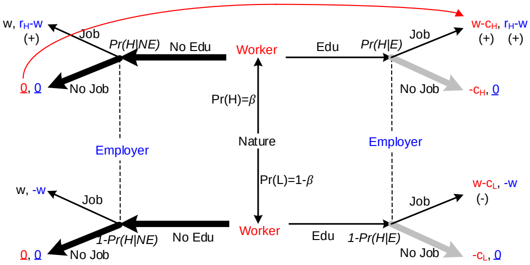



Correction

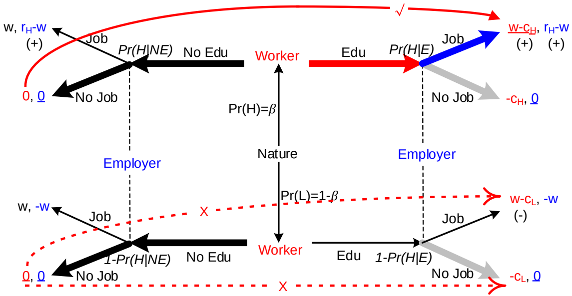


# Equilibrium Refinement: Intuition Criterion

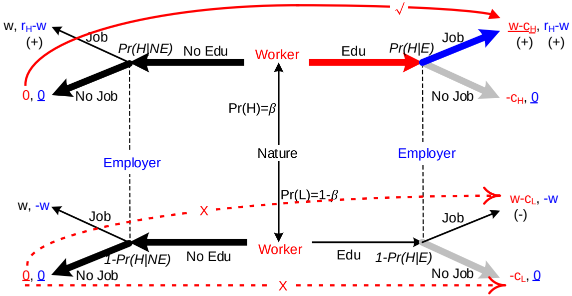
- The "pooling equilibrium" has unappealing feature in the sense that no information is revealed
- Cho and Kreps (1987)'s "intuition criterion" eliminates some equilibria that seem impossible
- Recall the pooling equilibrium (on $(NE, NE), (NJ, NJ)$) in the job market signaling game:




  - Both type send $NE$ and $NJ$ is BR to $NE$

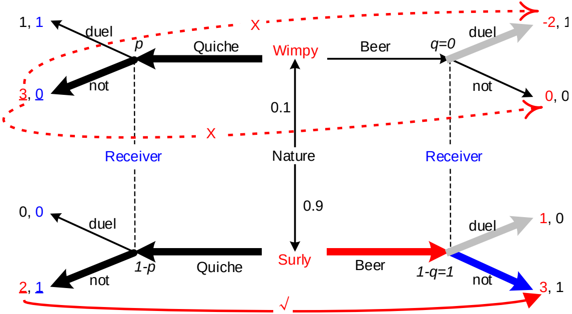



- Consider the following speech make by the worker who claims himself/herself to be a $H$ type:
  - "*I am type H. To convince you I am going to deviate and choose E. If you believe me and put me into a job, I will get* $w - c_H$ *instead of 0. The reason you should believe me is that if I were an L type who chose E and you were to assign a job, then I would get* $w - c_L$ *instead of 0. Therefore you should believe me when I tell you that I am an H type because no L type in his/her right mind would do what I am about to do.*"




- What should the employer think? The argument **makes sense** because if the candidate were an $L$ type then there is no way in which he/she could gain from this move.
  - "*Let me see which type can gain from this deviation. If neither can or if both can, I will keep my off-the-equilibrium-path beliefs as before. But if only one type of worker can benefit and the other type can only lose, then I should update my beliefs accordingly and act upon these new, more "sophisticated" beliefs.*"
- Update $\Pr(H|E) \leq \frac{w}{r_H}$ to $\Pr(H|E) = 1$ in such situation.

# Refinement according to Cho & Kreps (1987)




- Starting from $\{(Q, Q), (n, d), p = 0.1, q > 0.5\}$
- The surly type claims: "*See me choose Beer should convince you that I am the surly type: choosing Beer could not possibly have improved the lot of the wimpy type; and if choosing Beer will convince you that I am the surly type then doing so will improve my lot.*"
- Update $1 - q = \Pr(s|B)$ to $1 - q = 1 \Rightarrow q = 0$

# Job Market Signaling: Continuous Signals

- The type $i$ worker can choose to get $e$ years of education, incurring a cost $c_i(e) = \theta_i e$, where $\theta_H < \theta_L$. The marginal cost of the high-productivity work is relatively lower. Education is orthogonal to productivity.
- Upon observing $e$,
  - Separating equilibrium: for informative signals, the firm offers $w_H$ and $w_L$ for different levels of $e$;
  - Pooling equilibrium: for uninformative signals, the firm offers an expected wage

$$w(e) = r_H \Pr(H|e) + r_L \Pr(L|e)$$

- When the worker learns about his/her type, he/she chooses $e$ to maximize $w(e) - \theta_i e$, i.e., $e$ should be a best response for $w(e)$.

# Separating Equilibrium

Observe that $\theta_H < \theta_L$, i.e., the high-productivity worker has a relatively lower marginal cost of obtaining an additional year of education.

- If different $e$ give different signals, then firm offers different wages for different signals: $w_H = \underbrace{\Pr(H|e_H)}_{=1} r_H + \underbrace{\Pr(L|e_H)}_{=0} r_L = r_H$ and $w_L = r_L$. Then $r_H - \theta_H e_H$ is the payoff of the type-$H$-worker. Similarly, $r_L - \theta_L e_L$ is the payoff of the type-$L$-worker.
- If types are distinguishable, there's no need to obtain any education for the low-type worker, i.e., $e_L = 0$.
- At such separating equilibrium, by choosing $e_H$, H will not deviate to $e_L = 0$ if $\underbrace{r_H - \theta_H e_H}_{\text{H chooses } e_H} \geq \underbrace{r_L - \theta_H \cdot 0}_{\text{H chooses } e_L} \Rightarrow e_H \leq \frac{r_H - r_L}{\theta_H}$
- Similarly, $\underbrace{r_L - \theta_L \cdot 0}_{\text{L chooses } e_L} \geq \underbrace{r_H - \theta_L e_H}_{\text{L chooses } e_H} \Rightarrow e_H \geq \frac{r_H - r_L}{\theta_L}$.

# Pooling Equilibrium

Another possible outcome is that both types choose $e_H = e_L = e^p$ and firm offers a flat wage: $w(e^p) = \Pr(H|e^p)r_H + \Pr(L|e^p)r_L$, where $\Pr(H|e^p) = \frac{\Pr(\text{H chooses } e^p)}{\Pr(\text{H chooses } e^p)+\Pr(\text{L chooses } e^p)} = \frac{\Pr(e^p|H)\Pr(H)}{\Pr(e^p|H)\Pr(H)+\Pr(e^p|L)\Pr(L)} = \frac{1\cdot\beta}{1\cdot\beta+1\cdot(1-\beta)} = \beta$, $\Pr(L|e^p) = 1-\beta$.

- The flat wage is $w = \beta r_H + (1-\beta)r_L$
- Notice that $e_H = e_L = e^p = 0$ could be an (pooling) equilibrium:
  - For both type $i = H, L$, $w - \theta_i \cdot 0 \geq w - \theta_i\tilde{e}$ (for $\tilde{e} > 0$)
- For a pooling equilibrium at $e^p$, each type does not deviate (to an otherwise separating equilibrium):
  - For H: $w - \theta_H e^p \geq r_H - \theta_H e_H \Rightarrow \theta_H e^p \leq \theta_H e_H - (1-\beta)(r_H - r_L) \leq \beta(r_H - r_L)$;
  - For L: $w - \theta_L e^p \geq r_L - \theta_L \cdot 0 \Rightarrow \theta_L e^p \leq \beta(r_H - r_L)$.
  - $\theta_H < \theta_L \Rightarrow e^p \leq \frac{\beta(r_H - r_L)}{\theta_L}$