Solution for Problem Set 1

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1 Consumption allocation

The expenditure is:

$$P_t C_t = C_t^T + P_t^N C_t^N \tag{1.1}$$

Therefore, the Lagrangian is:

$$\mathcal{L} = C_t^T + P_t^N C_t^N + \lambda \left[C_t - \frac{1}{\gamma^\gamma (1-\gamma)^{1-\gamma}} \left(C_t^T\right)^\gamma \left(C_t^N\right)^{1-\gamma} \right] \tag{1.2}$$

Taking FOCs:

$$\frac{\partial \mathcal{L}}{\partial C_t^T} = 1 - \lambda \frac{\gamma}{\gamma(1-\gamma)^{1-\gamma}} (C_t^T)^{\gamma-1} (C_t^N)^{1-\gamma} = 0. \tag{1.3}$$

$$\frac{\partial \mathcal{L}}{\partial C_t^N} = P_t^N - \lambda \frac{1-\gamma}{\gamma(1-\gamma)^{1-\gamma}} (C_t^T)^{\gamma} (C_t^N)^{-\gamma} = 0. \tag{1.4}$$

Hence,

$$1 = \lambda \frac{\gamma}{\gamma(1-\gamma)^{1-\gamma}} (C_t^T)^{\gamma-1} (C_t^N)^{1-\gamma} = \lambda \frac{\gamma C_t}{C_t^T} \tag{1.5}$$

$$P_t^N = \lambda \frac{1-\gamma}{\gamma(1-\gamma)^{1-\gamma}} (C_t^T)^{\gamma} (C_t^N)^{-\gamma} = \lambda \frac{(1-\gamma) C_t}{C_t^N} \tag{1.6}$$

Multiply (1.5)(1.6) by C_t^T and C_t^N respectively and add them up we get:

$$\begin{gathered} C_t^T + P_t^N C_t^N = P_t C_t = \gamma C_t^T \lambda + (1-\gamma) C_t^T \lambda = \lambda C_t \\ P_t = \lambda \end{gathered} \tag{1.7}$$

Substitute (1.7) to (1.5)(1.6) we have:

$$\gamma \frac{P_t C_t}{C_t^T} = 1 \quad ; \quad (1-\gamma) \frac{P_t C_t}{C_t^N} = P_t^N \tag{1.8}$$

Rearrange them we have:

$$C_t^T = \gamma P_t C_t \quad ; \quad C_t^N = (1-\gamma) \frac{P_t}{P_t^N} C_t \tag{1.9}$$

Plug (1.9) into the aggregator constraint we have:

$$
\begin{aligned}
C_t &= \frac{1}{\gamma^{\gamma}(1-\gamma)^{1-\gamma}} (\gamma P_t C_t)^{\gamma} \left((1-\gamma) \frac{P_t}{P_t^N} C_t \right)^{1-\gamma} \\
&= \frac{1}{\gamma^{\gamma}(1-\gamma)^{1-\gamma}} (\gamma^{\gamma} P_t^{\gamma} C_t^{\gamma}) \left((1-\gamma)^{1-\gamma} \left(\frac{P_t}{P_t^N} \right)^{1-\gamma} C_t^{1-\gamma} \right) \\
&= \frac{\gamma^{\gamma}(1-\gamma)^{1-\gamma} P_t^{\gamma+(1-\gamma)} C_t^{\gamma+(1-\gamma)}}{\gamma^{\gamma}(1-\gamma)^{1-\gamma}} \left(P_t^N \right)^{-(1-\gamma)} \\
&= P_t C_t \left(P_t^N \right)^{-(1-\gamma)}
\end{aligned}
\tag{1.10}
$$

Therefore:

$$
P_t \left(P_t^N \right)^{-(1-\gamma)} = 1 \quad \Longrightarrow \quad P_t = \left(P_t^N \right)^{1-\gamma} \tag{1.11}
$$

2 Market clearing

From (1.9) we know:

$$
\begin{gathered}
C_t^N = (1-\gamma) \frac{P_t}{P_t^N} C_t = (1-\gamma) \frac{\left(P_t^N \right)^{1-\gamma}}{P_t^N} C_t \\
C_t = (1-\gamma) \left(P_t^N \right)^{-\gamma} C_t
\end{gathered}
\tag{2.1}
$$

Therefore, total Home demand for n Home Households is:

$$
n C_t^N = n(1-\gamma) \left(P_t^N \right)^{-\gamma} C_t \tag{2.2}
$$

Market clearing for Home's non-tradable sector requires that supply equals demand:

$$
A_t^N \left(L_t^N \right)^{1-\alpha} = n (1-\gamma) \left(P_t^N \right)^{-\gamma} C_t \tag{2.3}
$$

Rearranging yields:

$$
n (1-\gamma) \left(P_t^N \right)^{-\gamma} C_t = A_t^N \left(L_t^N \right)^{1-\alpha} \tag{2.4}
$$

For foreign country, following the same steps we have:

$$
\begin{gathered}
C_t^{N*} = (1-\gamma) \left(P_t^{N*} \right)^{-\gamma} C_t^* \\
1 - n C_t^{N*} = 1 - n(1-\gamma) \left(P_t^{N*} \right)^{-\gamma} C_t^*
\end{gathered}
\tag{2.5}
$$

Therefore:

$$
(1-n)(1-\gamma) \left(P_t^{N*} \right)^{-\gamma} C_t^* = A_t^{N*} \left(L_t^{N*} \right)^{1-\alpha} \tag{2.6}
$$

For Home trade good market, combine (1.9) and (1.11) we have:

$$
C_t^T = \gamma P_t C_t = \gamma (P_t^N)^{(1-\gamma)} C_t \tag{2.7}
$$

Repeat the same steps above we get:

$$n\gamma(P_t^N)^{(1-\gamma)}C_t = A_t^T\left(n - L_t^N\right)^{1-\alpha} \tag{2.8}$$

For foreign country, similarly:

$$1 - n\gamma(P_t^{N*})^{(1-\gamma)}C_t^* = A_t^{T*}\left(1 - n - L_t^{N*}\right)^{1-\alpha} \tag{2.9}$$

Add them up we get:

$$n\gamma(P_t^N)^{(1-\gamma)}C_t + 1 - n\gamma(P_t^{N*})^{(1-\gamma)}C_t^* = A_t^T\left(n - L_t^N\right)^{1-\alpha} + A_t^{T*}\left(1 - n - L_t^{N*}\right)^{1-\alpha} \tag{2.10}$$

3 Intertemporal allocation

From question, we can set the Lagrangian as:

$$\begin{aligned}\mathcal{L} = \sum_{s=0}^{\infty}\Big\{&\left(\beta_{t+s}^H\right)^s \ln(C_{t+s}) \\ &+ \lambda_{t+s}\left[A_{t+s}^T\left(n - L_{t+s}^N\right)^{1-\alpha} + P_{t+s}^N A_{t+s}^N\left(L_{t+s}^N\right)^{1-\alpha} + n\left(1 + r_{t+s}\right)B_{t+s} - nP_{t+s}C_{t+s} - nB_{t+s+1}\right]\Big\}\end{aligned} \tag{3.1}$$

Taking FOCs:

$$\frac{\partial\mathcal{L}}{\partial C_{t+s}} = (\beta_{t+s}^H)^s\frac{1}{C_{t+s}} - n\lambda_{t+s}P_{t+s} = 0 \implies \lambda_{t+s} = \frac{(\beta_{t+s}^H)^s}{nP_{t+s}C_{t+s}} \tag{3.2}$$

$$\frac{\partial\mathcal{L}}{\partial B_{t+s+1}} = -n\lambda_{t+s} + \lambda_{t+s+1}\left[n\left(1 + r_{t+s+1}\right)\right] = 0 \implies \lambda_{t+s} = \left(1 + r_{t+s+1}\right)\lambda_{t+s+1} \tag{3.3}$$

Combine them together:

$$\lambda_{t+s} = \frac{(\beta_{t+s}^H)^s}{nP_{t+s}C_{t+s}} = \left(1 + r_{t+s+1}\right)\lambda_{t+s+1} = (1 + r_{t+s+1})\frac{(\beta_{t+s+1}^H)^{s+1}}{nP_{t+s+1}C_{t+s+1}} \tag{3.4}$$

Set $s = 0$ (for simplicity because the pattern is the same for all s) we get:

$$\frac{1}{P_tC_t} = (1 + r_{t+1})\frac{\beta_{t+1}^H}{P_{t+1}C_{t+1}} \tag{3.5}$$

Therefore:

$$C_{t+1} = \beta_{t+1}^H(1 + r_{t+1})\frac{P_tC_t}{P_{t+1}} = \beta_{t+1}^H(1 + r_{t+1}^C)C_t \tag{3.6}$$

Plug (1.11) here we get:

$$(1 + r_{t+1}^C) = (1 + r_{t+1})\frac{P_t}{P_{t+1}} = (1 + r_{t+1})\left(\frac{P_t^N}{P_{t+1}^N}\right)^{1-\gamma} \tag{3.7}$$

Plug (1.9) here we get:

$$C_{t+1} = \beta_{t+1}^{H}(1 + r_{t+1})\frac{P_t C_t}{P_{t+1}} \implies \frac{C_{t+1}^T}{\gamma P_{t+1}} = \beta_{t+1}^{H}(1 + r_{t+1})\frac{P_t}{P_{t+1}}\frac{C_t^T}{\gamma P_t}$$

$$C_{t+1}^T = \beta_{t+1}^{H}(1 + r_{t+1})C_t^T \tag{3.8}$$

4 Labor allocation

From Q2 we know that the Home agent sets L_t^N to maximize:

$$V(L_t^N) = A_t^T(n - L_t^N)^{1-\alpha} + P_t^N A_t^N (L_t^N)^{1-\alpha} \tag{4.1}$$

Taking FOC:

$$\frac{dV}{dL_t^N} = -A_t^T(1-\alpha)\left(n - L_t^N\right)^{-\alpha} + P_t^N A_t^N (1-\alpha)\left(L_t^N\right)^{-\alpha} = 0 \tag{4.2}$$

$$A_t^T\left(n - L_t^N\right)^{-\alpha} = P_t^N A_t^N \left(L_t^N\right)^{-\alpha} \tag{4.3}$$

5 Resource constraints

Home has n households. Each household's tradable-good spending has two parts:

1. tradable-good consumption: $C_t^T = \gamma(P_t^N)^{1-\gamma}C_t$. Across all n households, that is $n\gamma(P_t^N)^{1-\gamma}C_t$.
2. Bond purchases: B_{t+1} per household, total nB_{t+1}.

These must be financed by:

1. Home's tradable-goods output: $Y_t^T = A_t^T(n - L_t^N)^{1-\alpha}$.
2. Principal and interest on the bonds of the last period: $n(1 + r_t)B_t$.

Thus the Home resource constraint is:

$$n\gamma(P_t^N)^{1-\gamma}C_t + nB_{t+1} = A_t^T(n - L_t^N)^{1-\alpha} + n(1 + r_t)B_t \tag{5.1}$$

For foreign country, the logic remains the same but it pays the principal and interest on Home's bonds rather than gaining from them. Therefore, from the Foreign perspective, the resource constraint is:

$$1 - n\,\gamma\left(P_t^{N*}\right)^{1-\gamma}C_t^* - nB_{t+1} \;=\; A_t^{T*}\left(1 - n - L_t^{N*}\right)^{1-\alpha} - n\,(1 + r_t)\,B_t \tag{5.2}$$

For real exchange rate we have:

$$Q_t = \frac{P_t^*}{P_t} = \frac{(P_t^{N*})^{1-\gamma}}{(P_t^N)^{1-\gamma}} = \left(\frac{P_t^{N*}}{P_t^N}\right)^{1-\gamma} \tag{5.3}$$

6 Steady state

From the Home Euler equation:

$$C_{t+1} = \beta_{t+1}^H (1 + r_{t+1}) \left(\frac{P_t^N}{P_{t+1}^N} \right)^{1-\gamma} C_t$$

In steady state, all real variables are constant over time, so $C_{t+1} = C_t \equiv C_0$ and $P_{t+1}^N = P_t^N = P_0^N$. Also the discount factor $\beta_{t+1}^H \equiv \beta_0$. Hence:

$$C_0 = \beta_0 (1 + r_0) \left(\frac{P_0^N}{P_0^N} \right)^{1-\gamma} C_0 \implies 1 = \beta_0 (1 + r_0) \tag{6.1}$$

From the labor-allocation condition at Home and Foreign:

$$A_0^T \left(n - L_0^N \right)^{-\alpha} = P_0^N A_0^N \left(L_0^N \right)^{-\alpha}.$$

$$A_0^{T*} \left(1 - n - L_0^{N*} \right)^{-\alpha} = P_0^{N*} A_0^{N*} \left(L_0^{N*} \right)^{-\alpha}$$

Combine with the clearing of non-tradable good markets and resource constraints we get:

$$\frac{n\gamma (P_0^N)^{1-\gamma} C_0}{n - L_0^N} = \frac{n(1-\gamma)(P_0^N)^{-\gamma} C_0 P_0^N}{L_0^N} \quad \text{and} \quad \frac{(1-n)\gamma (P_0^{N*})^{1-\gamma} C_0^*}{1 - n - L_0^{N*}} = \frac{(1-n)(1-\gamma)(P_0^{N*})^{-\gamma} C_0^* P_0^{N*}}{L_0^{N*}}$$

Solving them we get:

$$\frac{\gamma}{n - L_0^N} = \frac{1-\gamma}{L_0^N} \implies \gamma L_0^N = (n - L_0^N)(1-\gamma) \implies L_0^N = n(1-\gamma) \tag{6.2}$$

$$\frac{\gamma}{1 - n - L_0^{N*}} = \frac{1-\gamma}{L_0^{N*}} \implies \gamma L_0^{N*} = (1 - n - L_0^{N*})(1-\gamma) \implies L_0^{N*} = (1-n)(1-\gamma) \tag{6.3}$$

From productivity levels: $A_0^N = A_0^T \left(\frac{1-\gamma}{\gamma} \right)^\alpha$, taking γ power of both sides we have:

$$(A_0^N)^\gamma = (A_0^T)^\gamma \left[\left(\frac{1-\gamma}{\gamma} \right)^\gamma \right]^\alpha = (A_0^T)^\gamma \left[(\gamma^\gamma (1-\gamma)^{-\gamma}) \right]^{-\alpha}$$

Therefore:

$$A_0^N = (A_0^N)^\gamma (A_0^N)^{1-\gamma} = (A_0^T)^\gamma (A_0^N)^{1-\gamma} \left[(\gamma^\gamma (1-\gamma)^{-\gamma}) \right]^{-\alpha} \tag{6.4}$$

For Home country consumption C_0, from clearing of non-tradable good markets equation we have:

$$n(1-\gamma) \left(P_0^N \right)^{-\gamma} C_0 = A_0^N \left(L_0^N \right)^{1-\alpha} \implies C_0 = A_0^N \left[n(1-\gamma) \right]^{-\alpha}$$

Plug (6.4) into it we get:

$$C_0 = (A_0^T)^\gamma (A_0^N)^{1-\gamma} \left[(\gamma^\gamma (1-\gamma)^{-\gamma}) \right]^{-\alpha} \left[n(1-\gamma) \right]^{-\alpha} = (A_0^T)^\gamma (A_0^N)^{1-\gamma} \left[n(\gamma)^\gamma (1-\gamma)^{1-\gamma} \right]^{-\alpha} \tag{6.5}$$

Repeat the same steps above for Foreign country consumption C_0^* we can easily get:

$$C_0^* = \left(A_0^{T*} \right)^\gamma \left(A_0^{N*} \right)^{1-\gamma} \left[(1-n)(\gamma)^\gamma (1-\gamma)^{1-\gamma} \right]^{-\alpha} \tag{6.6}$$

7 Log linear approximation

7.1 Clearing of the Non-tradable Good Markets

For Home country clearing condition equation, take natural logarithm of both sides we have:

$$\ln(n) + \ln(1-\gamma) - \gamma\ln(P_t^N) + \ln(C_t) = \ln(A_t^N) + (1-\alpha)\ln(L_t^N)$$

At steady state:

$$\ln(n) + \ln(1-\gamma) - \gamma\ln(P_0^N) + \ln(C_0) = \ln(A_0^N) + (1-\alpha)\ln(L_0^N)$$

Subtracting and simplifying:

$$\begin{aligned}
-\gamma\left[\ln(P_t^N) - \ln(P_0^N)\right] + \left[\ln(C_t) - \ln(C_0)\right] &= \left[\ln(A_t^N) - \ln(A_0^N)\right] + (1-\alpha)\left[\ln(L_t^N) - \ln(L_0^N)\right] \\
-\gamma\widehat{P_t^N} + \widehat{C_t} &= \widehat{A_t^N} + (1-\alpha)\widehat{L_t^N} \qquad (7.1)
\end{aligned}$$

For Foreign country, following the same steps we easily get:

$$-\gamma\widehat{P_t^{N*}} + \widehat{C_t^*} = \widehat{A_t^{N*}} + (1-\alpha)\widehat{L_t^{N*}} \qquad (7.2)$$

7.2 Resource Constraints

For Home country resource constraint equation, divide both sides by $n\gamma C_0$ we have:

$$\frac{(P_t^N)^{1-\gamma}C_t}{C_0} + \frac{B_{t+1}}{\gamma C_0} = \frac{A_t^T(n-L_t^N)^{1-\alpha}}{n\gamma C_0} + \frac{(1+r_t)B_t}{\gamma C_0}$$

For each term of the new equation:

$$\begin{aligned}
\frac{(P_t^N)^{1-\gamma}C_t}{C_0} &\approx \frac{\left(P_0^N e^{\widehat{P_t^N}}\right)^{1-\gamma}\left(C_0 e^{\widehat{C_t}}\right)}{C_0} = (P_0^N)^{1-\gamma} e^{(1-\gamma)\widehat{P_t^N}} e^{\widehat{C_t}} \approx (1 + (1-\gamma)\widehat{P_t^N})(1+\widehat{C_t}) \approx 1 + (1-\gamma)\widehat{P_t^N} + \widehat{C_t} \\
\frac{B_{t+1}}{\gamma C_0} &= \widehat{B_{t+1}} \\
\frac{A_t^T(n-L_t^N)^{1-\alpha}}{n\gamma C_0} &\approx \frac{A_0^T e^{\widehat{A_t^T}}(n-L_0^N)^{1-\alpha} e^{-(1-\alpha)\frac{L_0^N}{n-L_0^N}} L_t^N}{n\gamma C_0} \approx e^{\widehat{A_t^T}} e^{-(1-\alpha)\frac{L_0^N}{n-L_0^N}\widehat{L_t^N}} \\
&\approx \left(1+\widehat{A_t^T}\right)\left(1-(1-\alpha)\frac{L_0^N}{n-L_0^N}\widehat{L_t^N}\right) \approx 1 + \widehat{A_t^T} - (1-\alpha)\frac{1-\gamma}{\gamma}\widehat{L_t^N} \\
\frac{(1+r_t)B_t}{\gamma C_0} &= (\frac{1}{\beta_0} + \widehat{r_t})\widehat{B_t} \approx \frac{1}{\beta_0}\widehat{B_t} + \widehat{r_t}\widehat{B_t} \approx \frac{1}{\beta_0}\widehat{B_t}
\end{aligned}$$

Add them up we get:

$$(1-\gamma)\,\widehat{P_t^N} + \widehat{C_t} + \widehat{B_{t+1}} = \widehat{A_t^T} - (1-\alpha)\frac{1-\gamma}{\gamma}\widehat{L_t^N} + \frac{1}{\beta_0}\widehat{B_t} \qquad (7.3)$$

For Foreign country, following the similar steps (the only difference is that for foreign country, the bonds are its liabilities rather than assets, so the plus sign becomes a minus sign) we easily get :

$$(1-\gamma)\,\widehat{P_t^{N*}} + \widehat{C_t^*} - \frac{n}{1-n}\widehat{B_{t+1}} = \widehat{A_t^{T*}} - (1-\alpha)\frac{1-\gamma}{\gamma}\widehat{L_t^{N*}} - \frac{1}{\beta_0}\frac{n}{1-n}\widehat{B_t} \tag{7.4}$$

7.3 Euler Conditions

For Home country Euler equation, take natural logarithm of both sides we get:

$$\ln(C_{t+1}) = \ln(C_t) + \ln(\beta_{t+1}^H) + \ln(1+r_{t+1}) + (1-\gamma)\left[\ln(P_t^N) - \ln(P_{t+1}^N)\right]$$

At steady state:

$$\ln(C_0) = \ln(C_0) + \ln(\beta_0) + \ln(1+r_0) + (1-\gamma)[\ln(P_0^N) - \ln(P_0^N)]$$

Subtracting and simplifying:

$$\ln(C_{t+1}) - \ln(C_0) = \ln(C_t) - \ln(C_0) + \ln(\beta_{t+1}^H) - \ln(\beta_0) + \ln(1+r_{t+1}) - \ln(1+r_0) + (1-\gamma)[\ln(P_t^N) - \ln(P_0^N)] - [\ln(P_{t+1}^N) - \ln(P_0^N)]$$

$$\widehat{C_{t+1}} = \widehat{C_t} + \widehat{\beta_{t+1}^H} + [\ln(1+r_{t+1}) - \ln(1+r_0)] + (1-\gamma)[\widehat{P_t^N} - \widehat{P_{t+1}^N}]$$

$$\begin{aligned}\ln(1+r_{t+1}) - \ln(1+r_0) &\approx \ln(\frac{1}{\beta_0} + \widehat{r}_{t+1}) - \ln(\frac{1}{\beta_0}) \approx \ln(\frac{1}{\beta_0}(1+\beta_0\widehat{r}_{t+1})) - \ln(\frac{1}{\beta_0}) \\ &= \ln(\frac{1}{\beta_0}) + \ln(1+\beta_0\widehat{r}_{t+1}) - \ln(\frac{1}{\beta_0}) = \ln(1+\beta_0\widehat{r}_{t+1}) \approx \beta_0\widehat{r}_{t+1}\end{aligned}$$

Add them up we get:

$$\widehat{C_{t+1}} = \widehat{C_t} + \widehat{\beta_{t+1}^H} + \beta_0\widehat{r}_{t+1} + (1-\gamma)(\widehat{P_t^N} - \widehat{P_{t+1}^N}) \tag{7.5}$$

For Foreign country, following the same steps we easily get:

$$\widehat{C_{t+1}^*} = \widehat{C_t^*} + \widehat{\beta_{t+1}^F} + \beta_0\widehat{r}_{t+1} + (1-\gamma)(\widehat{P_t^{N*}} - \widehat{P_{t+1}^{N*}}) \tag{7.6}$$

7.4 Labor Allocations

For Home country labor allocation equation, take natural logarithm of both sides we get:

$$\ln(A_t^T) - \alpha\ln(n - L_t^N) = \ln(P_t^N) + \ln(A_t^N) - \alpha\ln(L_t^N)$$

At steady state:

$$\ln(A_0^T) - \alpha\ln(n - L_0^N) = \ln(P_0^N) + \ln(A_0^N) - \alpha\ln(L_0^N)$$

Subtracting and simplifying:

$$\ln(A_t^T) - \ln(A_0^T) - \alpha\left[\ln(n-L_t^N) - \ln(n-L_0^N)\right] = \ln(P_t^N) - \ln(P_0^N) + \ln(A_t^N) - \ln(A_0^N) - \alpha[\ln(L_t^N) - \ln(L_0^N)]$$

$$\widehat{A_t^T} - \alpha[\ln(n-L_t^N) - \ln(n-L_0^N)] = \widehat{P_t^N} + \widehat{A_t^N} - \alpha\widehat{L_t^N} \tag{7.7}$$

Taking the first-order Taylor expansion to the term $\ln\left(n - L_t^N\right)$ we get:

$$\ln\left(n - L_t^N\right) \approx \ln\left(n - L_0^N\right) - \frac{L_t^N - L_0^N}{n - L_0^N}$$
$$\ln\left(n - L_t^N\right) - \ln\left(n - L_0^N\right) \approx -\frac{L_t^N - L_0^N}{n - L_0^N} \approx -\frac{L_0^N}{n - L_0^N}\widehat{L_t^N} \tag{7.8}$$

Substituting (7.8) into (7.7) yields:

$$\widehat{A_t^T} - \alpha\left(-\frac{L_0^N}{n - L_0^N}\widehat{L_t^N}\right) = \widehat{P_t^N} + \widehat{A_t^N} - \alpha\widehat{L_t^N}$$
$$\widehat{A_t^T} + \alpha\frac{L_0^N}{n - L_0^N}\widehat{L_t^N} = \widehat{P_t^N} + \widehat{A_t^N} - \alpha\widehat{L_t^N}$$
$$\widehat{A_t^T} + \alpha\left(\frac{L_0^N}{n - L_0^N} + 1\right)\widehat{L_t^N} = \widehat{P_t^N} + \widehat{A_t^N}$$
$$\widehat{A_t^T} + \frac{\alpha}{\gamma}\widehat{L_t^N} = \widehat{P_t^N} + \widehat{A_t^N} \tag{7.9}$$

For Foreign country, following the same steps we easily get:

$$\widehat{A_t^{T*}} + \frac{\alpha}{\gamma}\widehat{L_t^{N*}} = \widehat{P_t^{N*}} + \widehat{A_t^{N*}} \tag{7.10}$$

7.5 Real Interest Rates

For Home country real interest equation, take natural logarithm of both sides we get:

$$\ln(1 + r_{t+1}^C) = \ln(1 + r_{t+1}) + (1 - \gamma)\left[\ln(P_t^N) - \ln(P_{t+1}^N)\right]$$

Take first-order approximation we get:

$$r_{t+1}^C = r_{t+1} + (1 - \gamma)[\ln(P_t^N) - \ln(P_{t+1}^N)]$$

At steady state:

$$r_0^C = r_0 + (1 - \gamma)\left[(\ln(P_t^N) - \ln(P_0^N)\right]$$

Subtracting and simplifying:

$$r_{t+1}^C - r_0^C = (r_{t+1} - r_0) + (1 - \gamma)\left[(\ln(P_t^N) - \ln(P_0^N)) - (\ln(P_{t+1}^N) - \ln(P_0^N))\right]$$
$$\widehat{r_{t+1}^C} = \widehat{r_{t+1}} + (1 - \gamma)(\widehat{P_t^N} - \widehat{P_{t+1}^N}) \tag{7.11}$$

For Foreign country, following the same steps we easily get:

$$\widehat{r_{t+1}^{C*}} = \widehat{r_{t+1}} + (1 - \gamma)(\widehat{P_t^{N*}} - \widehat{P_{t+1}^{N*}}) \tag{7.12}$$

8 Worldwide solution

Multiply (7.1) by n and (7.2) by $1-n$ respectively and add them up we easily get:

$$-\gamma\widehat{P^W_{N,t}} + \widehat{C^W_t} = \widehat{A^W_{N,t}} + (1-\alpha)\,\widehat{L^W_{N,t}} \tag{8.1}$$

Following the same steps we have:

$$(1-\gamma)\,\widehat{P^W_{N,t}} + \widehat{C^W_t} = \widehat{A^W_{T,t}} - (1-\alpha)\,\frac{1-\gamma}{\gamma}\widehat{L^W_{N,t}} \tag{8.2}$$

$$C^W_{t+1} = \widehat{C^W_t} + (1-\gamma)\left(\widehat{P^W_{N,t}} - \widehat{P^W_{N,t+1}}\right) + \widehat{\beta^W_{t+1}} + \beta_0\widehat{r_{t+1}} \tag{8.3}$$

$$\widehat{A^W_{T,t}} + \frac{\alpha}{\gamma}\widehat{L^W_{N,t}} = \widehat{P^W_{N,t}} + \widehat{A^W_{N,t}} \tag{8.4}$$

$$\beta_0\widehat{r^{CW}_{t+1}} = (1-\gamma)\left(\widehat{P^W_{N,t}} - \widehat{P^W_{N,t+1}}\right) + \beta_0\widehat{r_{t+1}} \tag{8.5}$$

Subtracting (8.2) from (8.1) and associating with (8.4) we have:

$$-\widehat{P^W_{N,t}} = \widehat{A^W_{N,t}} - \widehat{A^W_{T,t}} + \frac{2-\alpha-\gamma}{\gamma}\widehat{L^W_{N,t}} \quad ; \quad \widehat{A^W_{T,t}} + \frac{\alpha}{\gamma}\widehat{L^W_{N,t}} = \widehat{P^W_{N,t}} + \widehat{A^W_{N,t}}$$

$$\frac{\alpha}{\gamma}\widehat{L^W_{N,t}} = \frac{2-\alpha-\gamma}{\gamma}\widehat{L^W_{N,t}}$$

$$\frac{2-\gamma}{\gamma}\widehat{L^W_{N,t}} = 0$$

$$\widehat{L^W_{N,t}} = 0 \tag{8.6}$$

Therefore:

$$-\widehat{P^W_{N,t}} = \widehat{A^W_{N,t}} - \widehat{A^W_{T,t}} + \frac{2-\alpha-\gamma}{\gamma}\widehat{L^W_{N,t}} = \widehat{A^W_{N,t}} - \widehat{A^W_{T,t}}$$

$$\widehat{P^W_{N,t}} = \widehat{A^W_{T,t}} - \widehat{A^W_{N,t}} \tag{8.7}$$

Substituting (8.6) and (8.7) into (8.1) yields:

$$-\gamma\left(\widehat{A^W_{T,t}} - \widehat{A^W_{N,t}}\right) + \widehat{C^W_t} = \widehat{A^W_{N,t}}$$

$$\widehat{C^W_t} = -\gamma\widehat{A^W_{T,t}} + (1-\gamma)\widehat{A^W_{N,t}} \tag{8.8}$$

Substituting (8.7) and (8.8) into (8.3) yields:

$$\gamma\widehat{A^W_{T,t+1}} + (1-\gamma)\widehat{A^W_{N,t+1}} = \gamma\widehat{A^W_{T,t}} + (1-\gamma)\widehat{A^W_{N,t}} + (1-\gamma)\left[\widehat{A^W_{T,t}} - \widehat{A^W_{N,t}} - \widehat{A^W_{T,t+1}} + \widehat{A^W_{N,t+1}}\right] + \widehat{\beta^W_{t+1}} + \beta_0\widehat{r_{t+1}}$$

$$\beta_0\widehat{r_{t+1}} = -\widehat{\beta^W_{t+1}} + (\widehat{A^W_{T,t+1}} - \widehat{A^W_{T,t}}) \tag{8.9}$$

- **What Drives Relative Price:** If world tradable sector productivity $A^W_{T,t}$ rises more than non-tradable sector productivity $A^W_{N,t}$ then tradable goods become relatively cheaper to produce. The supply of tradable goods is more abundant, and their relative price falls, which means the relative price of non-tradable good rises. Hence $P^W_{N,t}$ is positive. Conversely, if non-tradable productivity increases faster than tradable, then non-tradable goods become cheaper, pushing $P^W_{N,t}$ down. Therefore, the driver of the relative price is the difference in T vs. N productivity at the world level:

$$\text{Relative Price Rises} \iff \Delta(T \text{ productivity}) > \Delta(N \text{ productivity})$$

- **What Drives Consumption:** The weights (γ and $1 - \gamma$) reflect the composition of the consumption basket between tradable and non-tradable goods. This shows that consumption growth depends directly on productivity growth in both sectors. Higher productivity in either sector leads to higher overall consumption. If T productivity jumps but N productivity is unchanged, consumption rises (but the relative price changes). If N productivity jumps, consumption also rises (and the relative price moves in the opposite direction). The sector that comprises a larger share of the consumption basket has a stronger influence on total consumption.

- **What Drives Real Interest Rate:** The real interest rate is driven by two key factors: Changes in productivity of the tradable sector and Time preference shocks.

  When tradable sector productivity is expected to rise in the future, the real interest rate increases. In other words, if the world expects more abundant future production of tradable goods, it is willing to lend only at a higher rate (there is more incentive to shift consumption to the future, so interest rates rise). This reflects the higher returns to investment in a more productive economy.

  A negative relationship exists between worldwide time preference shocks and the real interest rate. When consumers become more patient (higher β, lower desire to consume immediately), they're willing to save more at any given interest rate, pushing rates down. When consumers become more impatient (lower β, higher desire to consume immediately), they want to borrow more, pushing rates up.

9 Cross Country Differences

Subtracting (7.2) from (7.1) and simplifying yields:

$$\begin{aligned}
\widehat{C}_t - \widehat{C}^*_t - \gamma\left(\widehat{P^N_t} - \widehat{P^{N*}_t}\right) &= \widehat{A^N_t} + (1-\alpha)\widehat{L^N_t} - \widehat{A^{N*}_t} - (1-\alpha)\widehat{L^{N*}_t} \\
\widehat{C}_t - \widehat{C}^*_t + \frac{\gamma}{1-\gamma}\left[-(1-\gamma)\left(\widehat{P^N_t} - \widehat{P^{N*}_t}\right)\right] &= \widehat{A^N_t} - \widehat{A^{N*}_t} + (1-\alpha)\left(\widehat{L^N_t} - \widehat{L^{N*}_t}\right) \\
\widehat{C}_t - \widehat{C}^*_t + \frac{\gamma}{1-\gamma}\widehat{Q}_t &= \left(\widehat{A^N_t} - \widehat{A^{N*}_t}\right) + (1-\alpha)\left(\widehat{L^N_t} - \widehat{L^{N*}_t}\right) \qquad (9.1)
\end{aligned}$$

Similarly, subtracting (7.4) from (7.3), (7.6) from (7.5), (7.10) from (7.9) and (7.12) from (7.11) respectively and simplifying yields:

$$-\widehat{Q}_t + (\widehat{C}_t - \widehat{C}^*_t) + \frac{\widehat{B_{t+1}}}{1-n} = \left(\widehat{A^T_t} - \widehat{A^{T*}_t}\right) - (1-\alpha)\frac{1-\gamma}{\gamma}\left(\widehat{L^N_t} - \widehat{L^{N*}_t}\right) + \frac{1}{\beta_0}\frac{\widehat{B}_t}{1-n} \qquad (9.2)$$

$$\widehat{C_{t+1}} - \widehat{C^*_{t+1}} = \left(\widehat{C}_t - \widehat{C}^*_t\right) + \left(\widehat{Q_{t+1}} - \widehat{Q}_t\right) + \left(\widehat{\beta^H_{t+1}} - \widehat{\beta^F_{t+1}}\right) \qquad (9.3)$$

$$\left(\widehat{A^T_t} - \widehat{A^{T*}_t}\right) + \frac{\alpha}{\gamma}\left(\widehat{L^N_t} - \widehat{L^{N*}_t}\right) = \left(\widehat{A^N_t} - \widehat{A^{N*}_t}\right) - \frac{1}{1-\gamma}\widehat{Q}_t \qquad (9.4)$$

$$\beta_0\left(\widehat{r^C_{t+1}} - \widehat{r^{C*}_{t+1}}\right) = \left(\widehat{Q_{t+1}} - \widehat{Q_t}\right) \tag{9.5}$$

10 Long Run Allocation

In the new steady state, we simply know that consumption differences don't change, so: $\widehat{C_{t+2}} - \widehat{C^*_{t+2}} = \widehat{C_{t+1}} - \widehat{C^*_{t+1}}$, and bond holdings also remain constant: $\widehat{B_{t+2}} = \widehat{B_{t+1}}$. Therefore, for Euler equation (9.3) at period $t+2$ we have:

$$\left(\widehat{C_{t+2}} - \widehat{C^*_{t+2}}\right) = \left(\widehat{C_{t+1}} - \widehat{C^*_{t+1}}\right) + \left(\widehat{Q_{t+2}} - \widehat{Q_{t+1}}\right) + \left(\beta^H_{t+2} - \beta^F_{t+2}\right)$$

$$0 = \widehat{Q_{t+2}} - \widehat{Q_{t+1}}$$

This means $\widehat{Q_{t+2}} = \widehat{Q_{t+1}}$, i.e. the real exchange rate remains constant in the steady state. For the resource constraint difference equation (9.2) at period $t+1$ we have:

$$-\widehat{Q_{t+1}} + (\widehat{C_{t+1}} - \widehat{C^*_{t+1}}) + \frac{\widehat{B_{t+2}}}{1-n} = (\widehat{A^T_{t+1}} - \widehat{A^{T*}_{t+1}}) - (1-\alpha)\left(\frac{1-\gamma}{\gamma}\right)(\widehat{L^N_{t+1}} - \widehat{L^{N*}_{t+1}}) + \frac{1}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}$$

$$-\widehat{Q_{t+1}} + (\widehat{C_{t+1}} - \widehat{C^*_{t+1}}) = (\widehat{A^T_{t+1}} - \widehat{A^{T*}_{t+1}}) - (1-\alpha)\left(\frac{1-\gamma}{\gamma}\right)(\widehat{L^N_{t+1}} - \widehat{L^{N*}_{t+1}}) + \frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \tag{10.1}$$

For the market clearing condition equation (9.1) at period $t+1$ we have:

$$\widehat{C_{t+1}} - \widehat{C^*_{t+1}} = (\widehat{A^N_{t+1}} - \widehat{A^{N*}_{t+1)}}) + (1-\alpha)(\widehat{L^N_{t+1}} - \widehat{L^{N*}_{t+1}}) - \frac{\gamma}{1-\gamma}\widehat{Q_{t+1}} \tag{10.2}$$

Substituting (10.1) into (10.2) and simplifying we have:

$$-\frac{\widehat{Q_{t+1}}}{1-\gamma} + (\widehat{A^N_{t+1}} - \widehat{A^{N*}_{t+1}}) + (1-\alpha)(\widehat{L^N_{t+1}} - \widehat{L^{N*}_{t+1}}) = (\widehat{A^T_{t+1}} - \widehat{A^{T*}_{t+1}}) - (1-\alpha)\left(\frac{1-\gamma}{\gamma}\right)(\widehat{L^N_{t+1}} - \widehat{L^{N*}_{t+1}}) + \frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}$$

$$-\frac{\widehat{Q_{t+1}}}{1-\gamma} + (\widehat{A^N_{t+1}} - \widehat{A^{N*}_{t+1}}) + \frac{1-\alpha}{\gamma}(\widehat{L^N_{t+1}} - \widehat{L^{N*}_{t+1}}) = (\widehat{A^T_{t+1}} - \widehat{A^{T*}_{t+1}}) + \frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \tag{10.3}$$

For the labor allocation equation (9.4) at period $t+1$ we have:

$$(\widehat{A^N_{t+1}} - \widehat{A^{N*}_{t+1}}) = (\widehat{A^T_{t+1}} - \widehat{A^{T*}_{t+1}}) + \frac{\alpha}{\gamma}(\widehat{L^N_{t+1}} - \widehat{L^{N*}_{t+1}}) + \frac{1}{1-\gamma}\widehat{Q_{t+1}} \tag{10.4}$$

Substituting (10.4) into (10.3) and simplifying we get:

$$(\widehat{A^T_{t+1}} - \widehat{A^{T*}_{t+1}}) + \frac{\alpha}{\gamma}(\widehat{L^N_{t+1}} - \widehat{L^{N*}_{t+1}}) + \frac{1-\alpha}{\gamma}(\widehat{L^N_{t+1}} - \widehat{L^{N*}_{t+1}}) = (\widehat{A^T_{t+1}} - \widehat{A^{T*}_{t+1}}) + \frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}$$

$$\frac{1}{\gamma}\left(\widehat{L^N_{t+1}} - \widehat{L^{N*}_{t+1}}\right) = \frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}$$

$$\widehat{L^N_{t+1}} - \widehat{L^{N*}_{t+1}} = \gamma\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \tag{10.5}$$

Substituting (10.5) into (10.4) and simplifying we get:

$$
\begin{aligned}
(\widehat{A^T_{t+1}} - \widehat{A^{T*}_{t+1}}) + \alpha\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} &= -\frac{1}{1-\gamma}\widehat{Q_{t+1}} + (\widehat{A^N_{t+1}} - \widehat{A^{N*}_{t+1}}) \\
-\frac{1}{1-\gamma}\widehat{Q_{t+1}} &= (\widehat{A^T_{t+1}} - \widehat{A^{T*}_{t+1}}) - (\widehat{A^N_{t+1}} - \widehat{A^{N*}_{t+1}}) + \alpha\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
\widehat{Q_{t+1}} &= -(1-\gamma)\left[(\widehat{A^T_{t+1}} - \widehat{A^{T*}_{t+1}}) - (\widehat{A^N_{t+1}} - \widehat{A^{N*}_{t+1}})\right] - \alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \qquad (10.6)
\end{aligned}
$$

Substituting (10.6) into market clearing condition (10.2) and simplifying we get:

$$
\begin{aligned}
-\gamma\left[(\widehat{A^T_{t+1}} - \widehat{A^{T*}_{t+1}}) - (\widehat{A^N_{t+1}} - \widehat{A^{N*}_{t+1}})\right] - \gamma\alpha\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} + (\widehat{C_{t+1}} - \widehat{C^*_{t+1}}) &= (\widehat{A^N_{t+1}} - \widehat{A^{N*}_{t+1}}) + (1-\alpha)\gamma\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
\widehat{C_{t+1}} - \widehat{C^*_{t+1}} &= \gamma(\widehat{A^T_{t+1}} - \widehat{A^{T*}_{t+1}}) + (1-\gamma)(\widehat{A^N_{t+1}} - \widehat{A^{N*}_{t+1}}) + \gamma\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \qquad (10.7)
\end{aligned}
$$

- **Impact of $\widehat{B_{t+1}} > 0$:**

  1. Real Exchange Rate: Home's real exchange rate is more appreciated relative to Foreign since it starts the new steady state with a positive net asset position. This appreciation reflects the wealth effect - higher wealth leads to higher demand for non-tradable goods, increasing their relative price.
  2. Labor Allocation: Home devotes more labor to its non-tradable sector (less to tradable) than Foreign does.This reflects the wealth effect - increased wealth leads to higher demand for non-tradable goods, requiring more production and thus more labor in that sector. The Home country becomes more service-oriented relative to Foreign country.
  3. Consumption: Home consumes more than Foreign in the new steady state even if productivity keeps the same (the extra term $\gamma\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}$ in consumption differences). This directly reflects the wealth effect - the Home country can afford to consume more.

- **Long Run Impact of Productivity:**

  1. Real Exchange Rate: If Home has higher relative productivity in tradable goods compared to non-tradable goods than Foreign does, the Home real exchange rate appreciates (Balassa-Samuelson effect). This happens because higher productivity in tradable goods raises wages economy-wide, which increases costs and prices in the non-tradable sector where productivity hasn't risen as much.
  2. Labor Allocation: Productivity differences do not directly affect the relative labor allocation across countries. The allocation of labor to non-tradable goods is driven solely by the wealth effect. This suggests that productivity differences affect labor allocation only through their impact on wealth accumulation.
  3. Consumption: Productivity differences directly affect consumption differences. Higher productivity in either sector increases consumption because higher productivity in either sector allows a country to produce and consume more.

11 Short Run Allocation

Rearranging equation (9.4) we get:

$$\left(\widehat{A_t^T} - \widehat{A_t^{T*}}\right) + \frac{\alpha}{\gamma}\left(\widehat{L_t^N} - \widehat{L_t^{N*}}\right) = -\frac{1}{1-\gamma}\widehat{Q}_t + \left(\widehat{A_t^N} - \widehat{A_t^{N*}}\right)$$

$$\widehat{Q}_t = -(1-\gamma)\left[(\widehat{A_t^T} - \widehat{A_t^{T*}}) - (\widehat{A_t^N} - \widehat{A_t^{N*}}) + \frac{\alpha}{\gamma}(\widehat{L_t^N} - \widehat{L_t^{N*}})\right] \tag{11.1}$$

Substituting (11.1) into (9.1) and simplifying we get:

$$\frac{\gamma}{1-\gamma}\left\{-(1-\gamma)\left[(\widehat{A_t^T} - \widehat{A_t^{T*}}) - (\widehat{A_t^N} - \widehat{A_t^{N*}}) + \frac{\alpha}{\gamma}(\widehat{L_t^N} - \widehat{L_t^{N*}})\right]\right\} + (\widehat{C}_t - \widehat{C}_t^*) = (\widehat{A_t^N} - \widehat{A_t^{N*}}) + (1-\alpha)(\widehat{L_t^N} - \widehat{L_t^{N*}})$$

$$-\gamma\left[(\widehat{A_t^T} - \widehat{A_t^{T*}}) - (\widehat{A_t^N} - \widehat{A_t^{N*}}) + \frac{\alpha}{\gamma}(\widehat{L_t^N} - \widehat{L_t^{N*}})\right] + (\widehat{C}_t - \widehat{C}_t^*) = (\widehat{A_t^N} - \widehat{A_t^{N*}}) + (1-\alpha)(\widehat{L_t^N} - \widehat{L_t^{N*}})$$

$$(\widehat{L_t^N} - \widehat{L_t^{N*}}) = (\widehat{C}_t - \widehat{C}_t^*) - \gamma(\widehat{A_t^T} - \widehat{A_t^{T*}}) - (1-\gamma)(\widehat{A_t^N} - \widehat{A_t^{N*}}) \tag{11.2}$$

Substituting (11.2) into (9.1) and simplifying we get:

$$\frac{\gamma}{1-\gamma}\widehat{Q}_t = (\widehat{A_t^N} - \widehat{A_t^{N*}}) + (1-\alpha)\left[(\widehat{C}_t - \widehat{C}_t^*) - \gamma(\widehat{A_t^T} - \widehat{A_t^{T*}}) - (1-\gamma)(\widehat{A_t^N} - \widehat{A_t^{N*}})\right] - (\widehat{C}_t - \widehat{C}_t^*)$$

$$\frac{\gamma}{1-\gamma}\widehat{Q}_t = (\widehat{A_t^N} - \widehat{A_t^{N*}}) + (1-\alpha)(\widehat{C}_t - \widehat{C}_t^*) - (1-\alpha)\gamma(\widehat{A_t^T} - \widehat{A_t^{T*}}) - (1-\alpha)(1-\gamma)(\widehat{A_t^N} - \widehat{A_t^{N*}}) - (\widehat{C}_t - \widehat{C}_t^*)$$

$$\widehat{Q}_t = \frac{1-\gamma}{\gamma}\left[(\alpha + \gamma - \alpha\gamma)(\widehat{A_t^N} - \widehat{A_t^{N*}}) - (1-\alpha)\gamma(\widehat{A_t^T} - \widehat{A_t^{T*}}) - \alpha(\widehat{C}_t - \widehat{C}_t^*)\right] \tag{11.3}$$

Substituting (11.1), (11.2) and (11.3) into (9.2) and simplifying we get:

$$(1-\gamma)\left[(\widehat{A_t^T} - \widehat{A_t^{T*}}) - (\widehat{A_t^N} - \widehat{A_t^{N*}}) + \frac{\alpha}{\gamma}(\widehat{L_t^N} - \widehat{L_t^{N*}})\right] + (\widehat{C}_t - \widehat{C}_t^*) + \frac{\widehat{B_{t+1}}}{1-n} = (\widehat{A_t^T} - \widehat{A_t^{T*}}) - (1-\alpha)\frac{1-\gamma}{\gamma}(\widehat{L_t^N} - \widehat{L_t^{N*}})$$

$$(1-\gamma)\left[(\widehat{A_t^T} - \widehat{A_t^{T*}}) - (\widehat{A_t^N} - \widehat{A_t^{N*}})\right] + (\widehat{C}_t - \widehat{C}_t^*) + \frac{\widehat{B_{t+1}}}{1-n} = (\widehat{A_t^T} - \widehat{A_t^{T*}}) - \frac{1-\gamma}{\gamma}(\widehat{L_t^N} - \widehat{L_t^{N*}})$$

$$(1-\gamma)\left[(\widehat{A_t^T} - \widehat{A_t^{T*}}) - (\widehat{A_t^N} - \widehat{A_t^{N*}})\right] + (\widehat{C}_t - \widehat{C}_t^*) + \frac{\widehat{B_{t+1}}}{1-n}$$

$$= (\widehat{A_t^T} - \widehat{A_t^{T*}}) - \frac{1-\gamma}{\gamma}\left[(\widehat{C}_t - \widehat{C}_t^*) - \gamma(\widehat{A_t^T} - \widehat{A_t^{T*}}) - (1-\gamma)(\widehat{A_t^N} - \widehat{A_t^{N*}})\right]$$

$$\frac{\widehat{B_{t+1}}}{1-n} = \left[(2-\gamma)(\widehat{A_t^T} - \widehat{A_t^{T*}}) - \frac{1-\gamma}{\gamma}(\widehat{C}_t - \widehat{C}_t^*) + \frac{(1-\gamma)^2}{\gamma}(\widehat{A_t^N} - \widehat{A_t^{N*}})\right]$$

$$-\left[(1-\gamma)(\widehat{A_t^T} - \widehat{A_t^{T*}}) - (1-\gamma)(\widehat{A_t^N} - \widehat{A_t^{N*}}) + (\widehat{C}_t - \widehat{C}_t^*)\right]$$

$$\frac{\widehat{B_{t+1}}}{1-n} = (\widehat{A_t^T} - \widehat{A_t^{T*}}) - \frac{1}{\gamma}(\widehat{C}_t - \widehat{C}_t^*) + \frac{1-\gamma}{\gamma}(\widehat{A_t^N} - \widehat{A_t^{N*}})$$

$$\widehat{C}_t - \widehat{C}_t^* = \gamma\left(\widehat{A_t^T} - \widehat{A_t^{T*}}\right) + (1-\gamma)\left(\widehat{A_t^N} - \widehat{A_t^{N*}}\right) - \gamma\frac{\widehat{B_{t+1}}}{1-n} \tag{11.4}$$

Substituting (11.4) into (11.3) we easily get:

$$\widehat{Q}_t = (1-\gamma)\left[\left(\widehat{A_t^N} - \widehat{A_t^{N*}}\right) - \left(\widehat{A_t^T} - \widehat{A_t^{T*}}\right)\right] + \alpha(1-\gamma)\frac{\widehat{B_{t+1}}}{1-n} \tag{11.5}$$

Subtracting (10.7) from (11.4), (10.6) from (11.5) respectively and simplifying we have:

$$\left(\widehat{C_{t+1}}-\widehat{C^*_{t+1}}\right)-\left(\widehat{C_t}-\widehat{C^*_t}\right)=\gamma\left[\left(\widehat{A^T_{t+1}}-\widehat{A^{T*}_{t+1}}\right)-\left(\widehat{A^T_t}-\widehat{A^{T*}_t}\right)\right]+(1-\gamma)\left[\left(\widehat{A^N_{t+1}}-\widehat{A^{N*}_{t+1}}\right)-\left(\widehat{A^N_t}-\widehat{A^{N*}_t}\right)\right]+\frac{\gamma}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \tag{11.6}$$

$$\left(\widehat{Q_{t+1}}-\widehat{Q_t}\right)=(1-\gamma)\left[\left(\widehat{A^T_t}-\widehat{A^{T*}_t}\right)-\left(\widehat{A^N_t}-\widehat{A^{N*}_t}\right)-\left(\widehat{A^T_{t+1}}-\widehat{A^{T*}_{t+1}}\right)+\left(\widehat{A^N_{t+1}}-\widehat{A^{N*}_{t+1}}\right)\right]-\alpha(1-\gamma)\frac{1}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \tag{11.7}$$

Substituting (11.6) and (11.7) into (10.2) and simplifying we have:

$$\begin{aligned}
&\gamma\left[\left(\widehat{A^T_{t+1}}-\widehat{A^{T*}_{t+1}}\right)-\left(\widehat{A^T_t}-\widehat{A^{T*}_t}\right)\right]+(1-\gamma)\left[\left(\widehat{A^N_{t+1}}-\widehat{A^{N*}_{t+1}}\right)-\left(\widehat{A^N_t}-\widehat{A^{N*}_t}\right)\right]+\frac{\gamma}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}\\
&=(1-\gamma)\left[\left(\widehat{A^T_t}-\widehat{A^{T*}_t}\right)-\left(\widehat{A^N_t}-\widehat{A^{N*}_t}\right)-\left(\widehat{A^T_{t+1}}-\widehat{A^{T*}_{t+1}}\right)+\left(\widehat{A^N_{t+1}}-\widehat{A^{N*}_{t+1}}\right)\right]-\alpha(1-\gamma)\frac{1}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}+\left(\widehat{\beta^H_{t+1}}-\widehat{\beta^F_{t+1}}\right)\\
&\left(\widehat{A^T_{t+1}}-\widehat{A^{T*}_{t+1}}\right)-\left(\widehat{A^T_t}-\widehat{A^{T*}_t}\right)+\frac{1}{\beta_0}\left[\gamma+\alpha(1-\gamma)\right]\frac{\widehat{B_{t+1}}}{1-n}=\widehat{\beta^H_{t+1}}-\widehat{\beta^F_{t+1}}\\
&\frac{\widehat{B_{t+1}}}{1-n}=\frac{\beta_0}{\gamma+\alpha(1-\gamma)}\left[\left(\widehat{\beta^H_{t+1}}-\widehat{\beta^F_{t+1}}\right)-\left(\widehat{A^T_{t+1}}-\widehat{A^{T*}_{t+1}}\right)+\left(\widehat{A^T_t}-\widehat{A^{T*}_t}\right)\right]
\end{aligned} \tag{11.8}$$

Substituting (11.8) into (11.4) and (11.5) we easily have:

$$\widehat{C_t}-\widehat{C^*_t}=\gamma\left(\widehat{A^T_t}-\widehat{A^{T*}_t}\right)+(1-\gamma)\left(\widehat{A^N_t}-\widehat{A^{N*}_t}\right)-\frac{\gamma\beta_0}{\gamma+\alpha(1-\gamma)}\left[\left(\widehat{\beta^H_{t+1}}-\widehat{\beta^F_{t+1}}\right)-\left(\widehat{A^T_{t+1}}-\widehat{A^{T*}_{t+1}}\right)+\left(\widehat{A^T_t}-\widehat{A^{T*}_t}\right)\right] \tag{11.9}$$

$$\widehat{Q_t}=(1-\gamma)\left[\left(\widehat{A^N_t}-\widehat{A^{N*}_t}\right)-\left(\widehat{A^T_t}-\widehat{A^{T*}_t}\right)\right]+\frac{\alpha(1-\gamma)\beta_0}{\gamma+\alpha(1-\gamma)}\left[\left(\widehat{\beta^H_{t+1}}-\widehat{\beta^F_{t+1}}\right)-\left(\widehat{A^T_{t+1}}-\widehat{A^{T*}_{t+1}}\right)+\left(\widehat{A^T_t}-\widehat{A^{T*}_t}\right)\right] \tag{11.10}$$

Substituting (11.9) into (11.2) and simplifying we easily have:

$$\begin{aligned}
\widehat{L^N_t}-\widehat{L^{N*}_t}=&\left\{\gamma\left(\widehat{A^T_t}-\widehat{A^{T*}_t}\right)+(1-\gamma)\left(\widehat{A^N_t}-\widehat{A^{N*}_t}\right)-\frac{\gamma\beta_0}{\gamma+\alpha(1-\gamma)}\left[\left(\widehat{\beta^H_{t+1}}-\widehat{\beta^F_{t+1}}\right)-\left(\widehat{A^T_{t+1}}-\widehat{A^{T*}_{t+1}}\right)+\left(\widehat{A^T_t}-\widehat{A^{T*}_t}\right)\right]\right\}\\
&-\gamma(\widehat{A^T_t}-\widehat{A^{T*}_t})-(1-\gamma)(\widehat{A^N_t}-\widehat{A^{N*}_t})\\
\widehat{L^N_t}-\widehat{L^{N*}_t}=&-\frac{\gamma\beta_0}{\gamma+\alpha(1-\gamma)}\left[\left(\widehat{\beta^H_{t+1}}-\widehat{\beta^F_{t+1}}\right)-\left(\widehat{A^T_{t+1}}-\widehat{A^{T*}_{t+1}}\right)+\left(\widehat{A^T_t}-\widehat{A^{T*}_t}\right)\right]
\end{aligned} \tag{11.11}$$

Substituting (10.6) and (11.10) into (9.5) and simplifying we have:

$$\begin{aligned}
\beta_0\left(\widehat{r^C_{t+1}}-\widehat{r^{C*}_{t+1}}\right)=&-(1-\gamma)\left[(\widehat{A^T_{t+1}}-\widehat{A^{T*}_{t+1}})-(\widehat{A^N_{t+1}}-\widehat{A^{N*}_{t+1}})\right]-\alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}\\
&-\left\{(1-\gamma)\left[\left(\widehat{A^N_t}-\widehat{A^{N*}_t}\right)-\left(\widehat{A^T_t}-\widehat{A^{T*}_t}\right)\right]\right.\\
&\left.+\frac{\alpha(1-\gamma)\beta_0}{\gamma+\alpha(1-\gamma)}\left[\left(\widehat{\beta^H_{t+1}}-\widehat{\beta^F_{t+1}}\right)-\left(\widehat{A^T_{t+1}}-\widehat{A^{T*}_{t+1}}\right)+\left(\widehat{A^T_t}-\widehat{A^{T*}_t}\right)\right]\right\}\\
\beta_0\left(\widehat{r^C_{t+1}}-\widehat{r^{C*}_{t+1}}\right)=&-(1-\gamma)\left[\left(\widehat{A^T_{t+1}}-\widehat{A^{T*}_{t+1}}\right)-\left(\widehat{A^N_{t+1}}-\widehat{A^{N*}_{t+1}}\right)\right]+(1-\gamma)\left[\left(\widehat{A^T_t}-\widehat{A^{T*}_t}\right)-\left(\widehat{A^N_t}-\widehat{A^{N*}_t}\right)\right]\\
&-\frac{(1-\gamma)\alpha}{\gamma+\alpha(1-\gamma)}\left[\left(\widehat{\beta^H_{t+1}}-\widehat{\beta^F_{t+1}}\right)-\left(\widehat{A^T_{t+1}}-\widehat{A^{T*}_{t+1}}\right)+\left(\widehat{A^T_t}-\widehat{A^{T*}_t}\right)\right]
\end{aligned} \tag{11.12}$$

- **Impact of a Temporary Increase in Home Patience ($\beta^H_{t+1} - \beta^F_{t+1} > 0$):**

  1. Consumption: When Home households become more patient, Home's current consumption falls relative to Foreign's. However, future Home consumption will be higher, as Home accumulates foreign assets now, enabling higher future consumption.
  2. Labor Allocation: Labor shifts away from the non-tradable sector in Home. This makes sense because reduced demand for non-tradable goods means less labor is needed to produce them. Resources effectively shift toward the tradable sector, supporting the current account surplus.
  3. Real Exchange Rate: A temporary increase in Home patience will raise the real exchange rate, which means an appreciation of the Home real currency. This occurs because Home's reduced consumption decreases demand for all goods, but especially for non-tradable goods. The relative price of non-tradable goods falls in Home, causing the real exchange rate appreciation.
  4. Real Interest Rate: Real interest rates in Home fall relative to Foreign. The lower demand and higher savings in Home drives down the equilibrium interest rate.

- **Impact of a Temporary Shock in Home tradable Productivity ($\widehat{A^T_t} > 0$):**

  1. Consumption: Home consumption immediately rises relative to Foreign, as productivity gains boost income and wealth today. However, given the shock is temporary, households smooth consumption by saving part of the additional income.
  2. Labor Allocation: The temporary increase in tradable-sector productivity reduces the need for labor in the tradable sector. Therefore, labor shifts toward the non-tradable sector as higher Home income raises demand for non-tradable goods.
  3. Real Exchange Rate: Home real exchange rate depreciates because the supply of Home's tradable goods increases, reducing their relative prices.
  4. Real Interest Rate: Real interest rates in Home rise relative to Foreign because the higher returns to investment in Home due to increased productivity.

- **Impact of a Temporary Shock in Home Non-tradable Productivity ($\widehat{A^N_t} > 0$):**

  1. Consumption: Home consumption immediately rises relative to Foreign, as productivity gains boost income and wealth today. The increase is primarily in non-tradable goods.
  2. Labor Allocation: Labor allocation to the non-tradable sector may either decrease or increase, depending on substitution versus income effects. Direct productivity gains mean less labor needed per unit output in non-tradable goods. Yet higher real income may increase demand, potentially offsetting productivity gains.
  3. Real Exchange Rate: Home real exchange rate appreciates because non-tradable goods become cheaper, reducing their relative price.
  4. Real Interest Rate: Minimal change, since a temporary non-tradable productivity shock has limited effect on real returns, as the shock does not affect tradable-sector returns significantly.

- **Impact of Permanent Productivity Shocks:**

1. Consumption:Consumption significantly increases with permanent productivity gains (whether in tradable or non-tradable sectors), as households see their permanent income rise. Consumption smoothing leads to immediate consumption adjustment upward since there's less need to save for the future.
2. Labor Allocation: Permanent productivity gains in the tradable sector generally lead to higher incomes and increased demand for non-tradable goods, expanding employment in the non-tradable sector. Permanent productivity gains in the non-tradable sector may initially reduce labor needs there due to efficiency gains, but higher income and demand expansion typically offset it, keeping employment stable or higher.
3. Real Exchange Rate: Permanent tradable-sector productivity improvements lead to long-run real exchange rate appreciation (lower relative price of non-tradable goods over time due to wealth/income effect). Permanent non-tradable productivity gains strongly appreciate the real exchange rate immediately..
4. Real Interest Rate: Permanent productivity shocks raise the equilibrium real interest rate permanently because of the sustained higher returns to capital and higher marginal productivity, especially in tradable goods sector.