Macroeconomics A: Review Session III

Discretion vs. Commitment

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Outline

1. **Monetary Policy in an AS-AD Setting**
   - Expanding the Class Notes

2. **Understanding Phase Diagrams**
   - Direction Fields
   - Phase Diagrams

Table of contents

Natural Level of Output

Last week we saw two terms:

- The output target Y^* in the central bank Taylor rule
- The natural level of output Y^n

The **natural level of output** is different from **potential output**

- Potential output: all available resources are used efficiently; there is perfect competition
- Natural output: output given flexible prices and wages but imperfect competition (markups)
- See a more formal definition here

The **output gap** is the difference between the actual output of an economy and its potential output

- The Taylor rule uses potential output
- The AS curve is determined by natural output

Taylor Rule

The original Taylor rule is a simple numerical formula that relates the FOMC's target for the federal funds rate to the state of the economy

$$i = \pi + 0.5(\pi - \pi^*) + 0.5(Y - Y^*) + 2$$

- i = the federal funds rate
- π = the rate of inflation ($\pi^* = 2$)
- $Y - Y^*$ = output gap

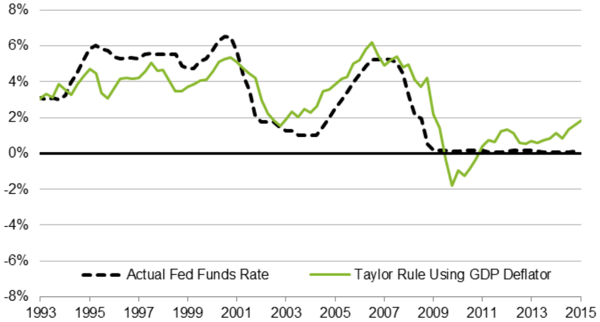
More generally, a 'Taylor rule' is a formula describing how the central bank sets the policy rate

- The policy rate determines the rate at which banks can borrow
- Banks make a profit on the spread—i.e. the difference between the return on lending and the cost of borrowing money
- Therefore, the policy rate also affects the cost of borrowing for households and firms (albeit indirectly) and thereby output

Taylor Rule Predicts Policy Well

The original Taylor rule performs surprisingly well

Figure: Taylor Rule vs. Fed Funds Rate, 1993-2015



- Note that the FOMC does not mechanically follow this rule!

Taylor Principle

The Taylor principle asserts that if the policy rate moves more than one-to-one to changes in inflation, inflation will stabilize around target

- Implies the central bank can set the rate of inflation if it wants

Last week, we saw that surprise inflation can raise output above its natural level

$$Y_{AS} = Y^n + \eta\,(\pi - \pi^e) \quad \text{where} \quad \eta > 0 \tag{1}$$

This creates a temptation for the central bank to raise output by creating surprise inflation

- This may work, but what if households begin adjusting expectations to incorporate 'surprises' ?

Central Bank Loss Function

To figure out how central bank behavior and household expectations interact, we need to describe the central bank's goals

Two cases with quadratic loss

- Discretion: central bank pursues its objective function $V(Y, \pi)$
- Commitment: central bank keeps inflation on target $\pi = \pi^*$

We can specify the loss function of the central bank as

$$V = \frac{1}{2}\left[\lambda(Y - Y^*)^2 + \pi^2\right] \quad \text{where} \quad Y^* = Y^n + k$$

Plugging in equation 1 for Y and the definition of Y^*

$$V = \frac{1}{2}\left[\lambda(\eta\,(\pi - \pi^e) - k)^2 + \pi^2\right]$$

Inflation Expectations under Discretion

Minimizing with respect to π gives

$$\frac{\partial V}{\partial \pi} = 0 \implies \lambda\eta(\eta\,(\pi - \pi^e) - k) + \pi = 0$$

Rearranging terms, we have the inflation the central bank would choose

$$\pi = \frac{\lambda\eta^2\pi^e + \lambda\eta k}{1 + \lambda\eta^2}$$

Households know the central bank will choose this and set $\pi^e = \pi$

Solving gives $\pi^e = \lambda\eta k$

With this rate of expected inflation, the loss function becomes

$$V_d = \frac{1}{2}\left[\lambda k^2 + (\lambda\eta k)^2\right]$$

Inflation Expectations under Commitment

Central bank sets $\pi = \pi^*$ and inflation expectations are $\pi^e = \pi^*$

$$V_c = \frac{1}{2}\left[\lambda(-k)^2 + (\pi^*)^2\right]$$

Minimizing the loss function gives the optimal choice for π^*

$$\frac{\partial V_c}{\partial \pi^*} = 0 \implies \pi^* = 0$$

Accordingly

$$V_c = \frac{1}{2}\lambda k^2$$

This implies the loss is always greater under discretion

$$V_d > V_c$$

Introducing Shocks

The aggregate supply equation can include stochastic shocks e

$$Y_{AS} = Y^n + \eta(\pi - \pi^e) + e \tag{2}$$

The quadratic loss function becomes

$$\mathbb{E}[V] = \mathbb{E}\left[\frac{1}{2}\left[\lambda(\eta(\pi - \pi^e) + e - k)^2 + \pi^2\right]\right]$$

Note that $\mathbb{E}[e] = 0$ and that $\mathbb{E}[e^2] = \sigma^2 > 0$

Under commitment, we still get that $\pi = \pi^e = 0$

$$\begin{aligned}\mathbb{E}[V_c] &= \mathbb{E}\left[\frac{1}{2}\left[\lambda(e-k)^2\right]\right] = \frac{1}{2}\left[\lambda\sigma^2 - 2\lambda\mathbb{E}[e]k + \lambda k^2\right] \\ &= \frac{1}{2}\left[\lambda\sigma^2 + \lambda k^2\right]\end{aligned}$$

Solving the Discretion Case with Shocks

Minimizing V_d with respect to π gives

$$\frac{\partial V_d}{\partial \pi} \implies \lambda\eta(\eta\,(\pi - \pi^e) + e - k) + \pi = 0$$

Rearranging terms, we have the inflation the central bank would choose

$$\pi = \frac{\lambda\eta^2\pi^e + \lambda\eta k - \lambda\eta e}{1 + \lambda\eta^2}$$

$$\pi^e = \mathbb{E}\left[\frac{\lambda\eta^2\pi^e + \lambda\eta k - \lambda\eta e}{1 + \lambda\eta^2}\right] = \lambda\eta k$$

Plugging this back into π gives

$$\pi = \frac{\lambda\eta^2\lambda\eta k + \lambda\eta k - \lambda\eta e}{1 + \lambda\eta^2} = \lambda\eta k - \frac{\lambda\eta e}{1 + \lambda\eta^2}$$

Exercise: Find $\mathbb{E}[V_d]$ and compare to $\mathbb{E}[V_c]$

Solving the Discretion Case with Shocks

Exercise: Find $\mathbb{E}[V_d]$ and compare to $\mathbb{E}[V_c]$

$$\mathbb{E}[V_d] = \mathbb{E}\left[\frac{1}{2}\left[\lambda\left(\eta\left(-\frac{\lambda\eta e}{1+\lambda\eta^2}\right)+e-k\right)^2+\left(\lambda\eta k-\frac{\lambda\eta e}{1+\lambda\eta^2}\right)^2\right]\right]$$

All cross-products with e are zero in expectation so that

$$\mathbb{E}[V_d] = \frac{1}{2}\left[\lambda\mathbb{E}\left[\left(\frac{e}{1+\lambda\eta^2}\right)^2\right]+\lambda k^2+(\lambda\eta k)^2+\mathbb{E}\left[\left(\frac{\lambda\eta e}{1+\lambda\eta^2}\right)^2\right]\right]$$

Further simplifying

$$\begin{aligned}\mathbb{E}[V_d] &= \frac{1}{2}\left[\lambda(1+\lambda\eta^2)\frac{\sigma^2}{(1+\lambda\eta^2)^2}+\lambda(1+\lambda\eta^2)k^2\right] \\ &= \frac{1}{2}\left[\frac{\lambda}{1+\lambda\eta^2}\sigma^2+\lambda(1+\lambda\eta^2)k^2\right]\end{aligned}$$

Answer: $\partial V_d < \partial V_c$ given $\partial\sigma^2$ and $\partial V_c < \partial V_d$ given ∂k^2

Table of contents

What Are Differential Equations?

- Differential equations express the rate of change of the current state as a function of the current state (motion or 'growth')

$$\frac{dx(t)}{dt} \equiv \dot{x}(t) = f(t, x(t))$$

- Initial (and/or terminal conditions) are needed to situate the path of an object (say a cannonball or economy)

$$x(0) = x_0 \quad \text{and} \quad x(T) = x_T$$

- Example: Suppose that GDP (y) grows at some constant rate

$$\frac{dy}{dt} = \dot{y} = gy(t)$$

- We will later see that $y(t) = y_0 e^{gt}$

Basics

- Broadly speaking, a differential equation is any equation which involves a derivative

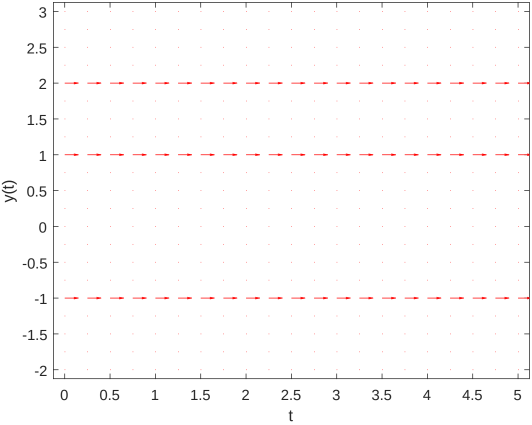
$$y' = (y^2 - y - 2)(1 - y)^2$$

- The equation is 'autonomous' since depends only on y(t) and not the independent value of t
- Not sure how to solve this yet, but we can find the roots for $y' = 0$

$$0 = (y^2 - y - 2)(1 - y)^2 = (y - 2)(y + 1)(1 - y)^2$$

- The roots are $y = \{-1, 1, 2\}$
- The slope of the tangent lines are zero at these values
- We can plot a direction field accordingly

Direction Field at $y' = 0$



- The graph is now divided into four regions
- We now want to see how y evolves over time within each region

What Happens at $y' \leq -1$?

- Solving for $y = -2$

$$y' = (-2-2)(-2+1)(1+2)^2 = -4 \times -1 \times 3^2 = 36$$

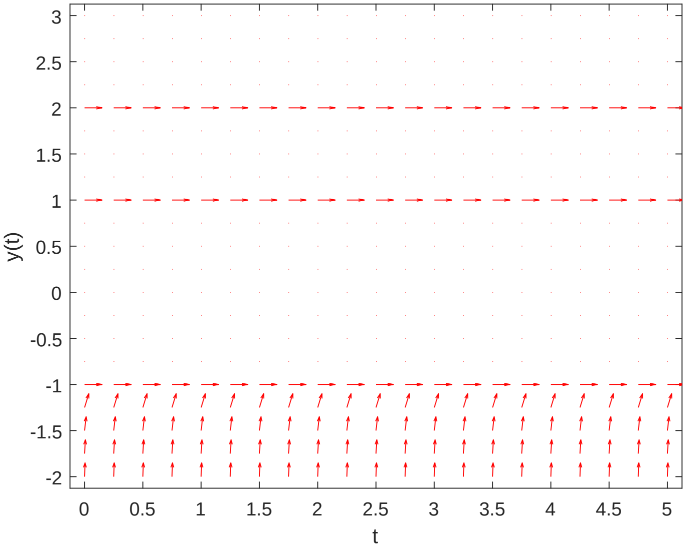
- Indicates the slope is extremely steep
- At y = -1.1, we get

$$y' = (-1.1-2)(-1.1+1)(1+1.1))^2 = 1.3671$$

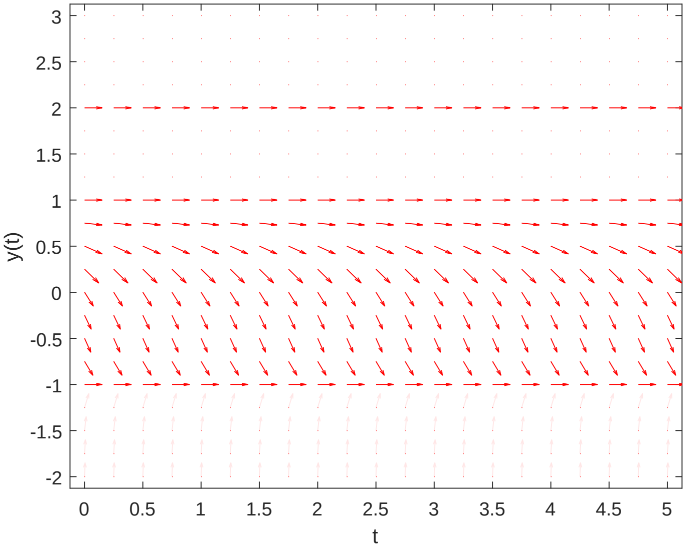
- As $y \to -1^-$, the slope is still positive but less steep
- We can test this across the four regions we identified earlier

Matlab code for the following figures is here

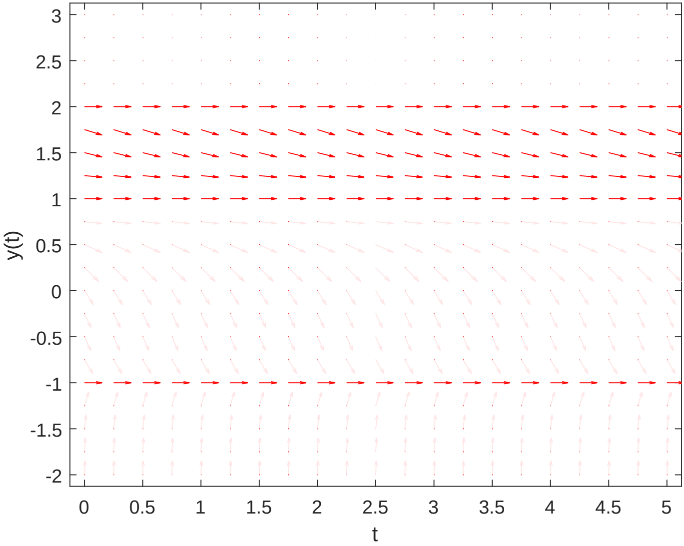
Direction Field for $y < -1$



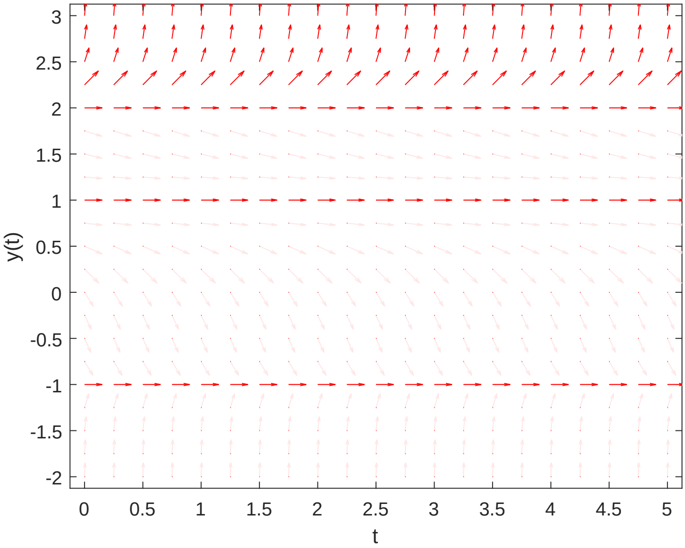
Direction Field for $-1 < y < 1$



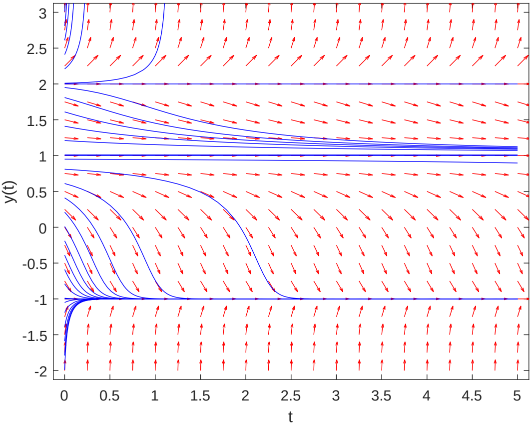
Direction Field for $1 < y < 2$



Direction Field for $y > 2$



Integral Curves



- Each curve gives the path y' will take given a certain initial condition
- Here, the initial values are evenly spaced across the interval $(-2, 3)$

Wrapping Up

- From what we have seen, we can characterize the behavior of y based on $y(0)$ as $t \to \infty$

Value of $y(0)$	$y(t)$ as $t \to \infty$
$y(0) < 1$	$y \to -1$
$1 \leq y(0) < 2$	$y \to 1$
$y(0) = 2$	$y \to 2$
$y(0) > 2$	$y \to \infty$

- Stability: for each root y_k we have $\widetilde{y}_k \in (y_k - \epsilon, y_k + \epsilon)$ where $\epsilon > 0$
  - $y_k = k$ is asymptotically stable if $\lim_{t \to \infty} f(t, \widetilde{y}_k) = y_k$
  - $y_k = k$ is asymptotically unstable if $\lim_{t \to \infty} f(t, \widetilde{y}_k) \neq y_k$
  - $y_k = k$ is asymptotically semi-stable if it is neither asymptotically stable nor unstable
- Question: which root(s) is stable, unstable, and/or semi-stable?

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- Question: which root(s) is stable, unstable, and/or semi-stable?

stable: $y_k = -1$; semi-stable: $y_k = 1$; unstable: $y_k = 2$;

Fixed Points and Systems of Equations

- A fixed point for a system is where $g(x^*) = x^*$
- For differential equations, the fixed point is given by $\dot{x}(t) = 0$, which requires that $f(x^*) = 0$
- We have already seen a one-dimensional system, but what about two or more dimensions?
- Take that $x_i > 0$

$$\dot{x}_1 = x_1^{\alpha} - x_2 \qquad \dot{x}_2 = b + x_1^{-1} - x_2 \qquad 0 < \alpha < 1 < b$$

- If a fixed point exists, it must satisfy

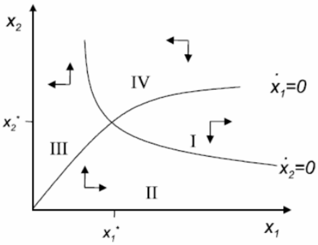
$$(x_1^*)^{\alpha} = x_2^*; \quad x_2^* = b + (x_1^*)^{-1} \implies (x_1^*)^{\alpha} = b + (x_1^*)^{-1} \tag{3}$$

- The LHS (RHS) of eq. 3 is monotonically increasing (decreasing)
- This implies one and only one intersection

Phase Diagram

- The intersection of the two lines gives $\mathbf{x}^* = (x_1^*, x_2^*)$
- This represents the steady state of the system where $\dot{x}_1 = \dot{x}_2 = 0$

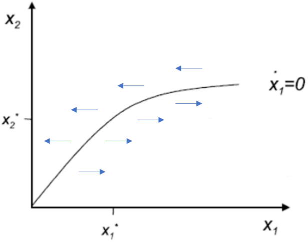
Figure: Phase Diagram



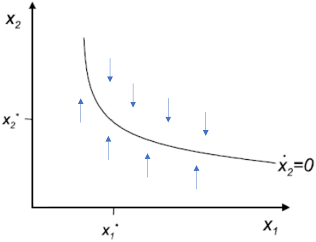
- What happens when $(x_1, x_2) \neq (x_1^*, x_2^*)$?
- It is helpful to plot the relation between x_1 and x_2 separately

Breaking Down the Phase Diagram

- There are four regions and four sets of arrows
- (a) x_1 increases whenever $x_2 < x_1^{\alpha}$, otherwise decreases
- (b) x_2 increases whenever $x_2 < x_1^{-1} + b$, otherwise decreases



(a)



(b)

Saddle Path Stable System

- We like systems where there is only one path to the steady state
- This accommodates ‘purposeful’ behavior over time