Macroeconomics A; EI056

Technical appendix: The Cagan model

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1 Main relation

The money demand is given by:

$$m_t - p_t = -\gamma \pi_{t+1}^e = -\gamma \left(p_{t+1}^e - p_t \right) \tag{1}$$

We solve for the price level by iterating (1) forward:

$$\begin{aligned}
m_t - (1+\gamma) p_t &= -\gamma p_{t+1}^e \\
(1+\gamma) p_t &= m_t + \gamma p_{t+1}^e \\
p_t &= \frac{1}{1+\gamma} m_t + \frac{\gamma}{1+\gamma} p_{t+1}^e \\
p_t &= \frac{1}{1+\gamma} m_t + \frac{\gamma}{1+\gamma} \left(\frac{1}{1+\gamma} m_{t+1}^e + \frac{\gamma}{1+\gamma} p_{t+2}^e \right) \\
p_t &= \frac{1}{1+\gamma} \sum_{s=0}^{\infty} \left(\frac{\gamma}{1+\gamma} \right)^s m_{t+s}^e \qquad (2)
\end{aligned}$$

Note that

$$\begin{aligned}
p_{t-1} &= \frac{1}{1+\gamma} \sum_{s=0}^{\infty} \left(\frac{\gamma}{1+\gamma} \right)^s m_{t+s-1}^e \\
p_{t-1} &= \frac{1}{1+\gamma} m_{t-1}^e + \frac{1}{1+\gamma} \sum_{s=1}^{\infty} \left(\frac{\gamma}{1+\gamma} \right)^s m_{t+s-1}^e \\
p_{t-1} &= \frac{1}{1+\gamma} m_{t-1}^e + \frac{1}{1+\gamma} \frac{\gamma}{1+\gamma} \sum_{s=0}^{\infty} \left(\frac{\gamma}{1+\gamma} \right)^s m_{t+s}^e \\
p_{t-1} &= \frac{1}{1+\gamma} m_{t-1}^e + \frac{\gamma}{1+\gamma} p_t \qquad (3)
\end{aligned}$$

2 A steady growth rate of money

Consider that money growth at a constant rate:

$$m_{t+1} = m_t + \mu$$

Take (1) for t and $t-1$:

$$\begin{aligned}
(m_t - p_t) - (m_{t-1} - p_{t-1}) &= -\gamma \pi^e_{t+1} + \gamma \pi^e_t \\
(m_t - m_{t-1}) - (p_t - p_{t-1}) &= -\gamma \pi^e_{t+1} + \gamma \pi^e_t \\
(m_t - m_{t-1}) - \pi_t &= -\gamma \pi^e_{t+1} + \gamma \pi^e_t \\
\pi_t &= \gamma \left(\pi^e_{t+1} - \pi^e_t\right) + (m_t - m_{t-1}) \qquad (4)
\end{aligned}$$

Along a steady path expectations are equal to actual outcomes. (4) then implies:

$$\begin{aligned}
\pi_t &= \gamma \left(\pi^e_{t+1} - \pi^e_t\right) + (m_t - m_{t-1}) \\
\pi_t &= \gamma \left(\pi_{t+1} - \pi_t\right) + \mu \\
\pi_t &= \frac{\gamma}{1+\gamma}\pi_{t+1} + \frac{1}{1+\gamma}\mu \\
\pi_t &= \frac{\gamma}{1+\gamma}\left(\frac{\gamma}{1+\gamma}\pi_{t+2} + \frac{1}{1+\gamma}\mu\right) + \frac{1}{1+\gamma}\mu \\
\pi_t &= \frac{1}{1+\gamma}\mu \sum_{s=0}^{\infty}\left(\frac{\gamma}{1+\gamma}\right)^s \\
\pi_t &= \frac{1}{1+\gamma}\mu \frac{1}{1-\frac{\gamma}{1+\gamma}} \\
\pi_t &= \mu
\end{aligned}$$

(3) then implies:

$$\begin{aligned}
p_t &= \frac{1}{1+\gamma} m_t + \frac{\gamma}{1+\gamma} p_{t+1} \\
p_t &= \frac{1}{1+\gamma} m_t + \frac{\gamma}{1+\gamma}\left(\pi_{t+1} + p_t\right) \\
p_t &= m_t + \gamma \pi_{t+1} \\
p_t &= m_t + \gamma \mu \qquad (5)
\end{aligned}$$

3 A disinflation episode

3.1 Adaptive expectations, no pre-announcement

Consider a stabilization episode. Until T money grows at a high rate: $m_T = m_{T-1} + \mu_0$. Starting at $T+1$ money grows at a slower rate: $m_{T+1} = m_T + \mu_1$, where $\mu_1 < \mu_0$.

Start with a situation where inflation expectations are adaptive:

$$\pi_{t+1}^e - \pi_t^e = \delta\left(\pi_t - \pi_t^e\right) \tag{6}$$

(6) implies that (4) becomes:

$$\begin{aligned}
\pi_t &= \gamma\left(\pi_{t+1}^e - \pi_t^e\right) + (m_t - m_{t-1}) \\
\pi_t &= \gamma\delta\left(\pi_t - \pi_t^e\right) + (m_t - m_{t-1}) \\
\pi_t &= -\frac{\gamma\delta}{1-\gamma\delta}\pi_t^e + \frac{1}{1-\gamma\delta}\left(m_t - m_{t-1}\right) \qquad (7)
\end{aligned}$$

(6) then gives the dynamics of expectations:

$$\begin{aligned}
\pi_{t+1}^e - \pi_t^e &= \delta\left(\pi_t - \pi_t^e\right) \\
\pi_{t+1}^e - \pi_t^e &= \delta\left(-\frac{\gamma\delta}{1-\gamma\delta}\pi_t^e + \frac{1}{1-\gamma\delta}\left(m_t - m_{t-1}\right) - \pi_t^e\right) \\
\pi_{t+1}^e - \pi_t^e &= -\frac{\delta}{1-\gamma\delta}\pi_t^e + \frac{\delta}{1-\gamma\delta}\left(m_t - m_{t-1}\right) \\
\pi_{t+1}^e &= \frac{1-(1+\gamma)\delta}{1-\gamma\delta}\pi_t^e + \frac{\delta}{1-\gamma\delta}\left(m_t - m_{t-1}\right) \qquad (8)
\end{aligned}$$

The economy is initially at a steady path allocation, with expectations equal to actual outcome, and money growing at a rate μ_0. We therefore have:

$$\pi_{T+1}^e = \mu_0$$

The subsequent dynamics of expected inflation are given by (8):

$$\begin{aligned}
\pi_{T+2}^e &= \frac{\delta}{1-\gamma\delta}\mu_1 + \frac{1-(1+\gamma)\delta}{1-\gamma\delta}\pi_{T+1}^e \\
&= \mu_1 + \frac{1-(1+\gamma)\delta}{1-\gamma\delta}\left(\mu_0 - \mu_1\right) \\
\pi_{T+3}^e &= \frac{\delta}{1-\gamma\delta}\mu_1 + \frac{1-(1+\gamma)\delta}{1-\gamma\delta}\pi_{T+2}^e \\
&= \mu_1 + \left(\frac{1-(1+\gamma)\delta}{1-\gamma\delta}\right)^2\left(\mu_0 - \mu_1\right) \\
\pi_{T+s}^e &= \mu_1 + \left(\frac{1-(1+\gamma)\delta}{1-\gamma\delta}\right)^{s-1}\left(\mu_0 - \mu_1\right)
\end{aligned}$$

Actual inflation is given by (7):

$$
\begin{aligned}
\pi_T &= \mu_0 \\
\pi_{T+1} &= \frac{1}{1-\gamma\delta}\mu_1 - \frac{\gamma\delta}{1-\gamma\delta}\pi^e_{T+1} \\
&= \mu_1 - \frac{\gamma\delta}{1-\gamma\delta}\left(\mu_0-\mu_1\right) \\
\pi_{T+2} &= \frac{1}{1-\gamma\delta}\mu_1 - \frac{\gamma\delta}{1-\gamma\delta}\pi^e_{T+2} \\
&= \mu_1 - \frac{\gamma\delta}{1-\gamma\delta}\frac{1-(1+\gamma)\delta}{1-\gamma\delta}\left(\mu_0-\mu_1\right) \\
\pi_{T+3} &= \mu_1 - \frac{\gamma\delta}{1-\gamma\delta}\left(\frac{1-(1+\gamma)\delta}{1-\gamma\delta}\right)^2\left(\mu_0-\mu_1\right) \\
\pi_{T+s} &= \mu_1 - \frac{\gamma\delta}{1-\gamma\delta}\left(\frac{1-(1+\gamma)\delta}{1-\gamma\delta}\right)^{s-1}\left(\mu_0-\mu_1\right)
\end{aligned}
$$

3.2 Rational expectations, no pre-announcement

Under rational expectations, $\pi^e_t = \pi_t$ at all times. (4) then implies:

$$
\begin{aligned}
\pi_T &= \mu_0 \\
\pi_{T+1} &= \gamma\left(\pi_{T+2}-\pi_{T+1}\right)+\mu_1 \\
&= \mu_1
\end{aligned}
$$

where the last step follows from iterating forward.

3.3 Rational expectations, with pre-announcement

Finally, consider a situation where change is announced at time S by the government. Recall that the money supply is:

$$
\begin{aligned}
m_t &= m_{t-1}+\mu_0 & \qquad t &< T+1 \\
m_t &= m_{t-1}+\mu_1 & \qquad t &\geq T+1
\end{aligned}
$$

Between the announcement and the change, the price level is given by:

$$
\begin{aligned}
p_t &= \frac{1}{1+\eta}\sum_{s=t}^{\infty}\left(\frac{\eta}{1+\eta}\right)^{s-t}E_t m_s \\
&= \frac{1}{1+\eta}\sum_{s=t}^{T}\left(\frac{\eta}{1+\eta}\right)^{s-t}E_t m_s + \frac{1}{1+\eta}\sum_{s=T+1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^{s-t}E_t m_s \qquad (9)
\end{aligned}
$$

Start with the last term in (9):

$$
\begin{aligned}
&\frac{1}{1+\eta}\sum_{s=T+1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^{s-t}E_t m_s\\
=&\frac{1}{1+\eta}\left(\frac{\eta}{1+\eta}\right)^{T+1-t}\sum_{s=T+1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^{s-T-1}E_t m_s\\
=&\frac{1}{1+\eta}\left(\frac{\eta}{1+\eta}\right)^{T+1-t}\sum_{s=T+1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^{s-T-1}\left[m_{T+1}+\mu_1\left(s-T-1\right)\right]\\
=&\left(\frac{\eta}{1+\eta}\right)^{T+1-t}\left[\begin{array}{c} m_{T+1}\frac{1}{1+\eta}\sum_{s=T+1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^{s-T-1} \\ +\mu_1\frac{1}{1+\eta}\sum_{s=T+1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^{s-T-1}\left(s-T-1\right)\end{array}\right]\\
=&\left(\frac{\eta}{1+\eta}\right)^{T+1-t}\left[m_{T+1}\frac{1}{1+\eta}\frac{1}{1-\frac{\eta}{1+\eta}}+\mu_1\frac{1}{1+\eta}\eta\left(1+\eta\right)\right]\\
=&\left(\frac{\eta}{1+\eta}\right)^{T+1-t}\left(m_{T+1}+\eta\mu_1\right)
\end{aligned}
$$

The first term in (9) is:

$$
\begin{aligned}
&\frac{1}{1+\eta}\sum_{s=t}^{T}\left(\frac{\eta}{1+\eta}\right)^{s-t}E_t m_s\\
=&\frac{1}{1+\eta}\sum_{s=t}^{T}\left(\frac{\eta}{1+\eta}\right)^{s-t}\left[m_t+\mu_0\left(s-t\right)\right]\\
=&\frac{1}{1+\eta}\sum_{s=t}^{\infty}\left(\frac{\eta}{1+\eta}\right)^{s-t}\left[m_t+\mu_0\left(s-t\right)\right]-\frac{1}{1+\eta}\sum_{s=T+1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^{s-t}\left[m_t+\mu_0\left(s-t\right)\right]\\
=&m_t+\eta\mu_0-\left(\frac{\eta}{1+\eta}\right)^{T+1-t}\frac{1}{1+\eta}\sum_{s=T+1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^{s-T-1}\left[\begin{array}{c}\left(m_t+\mu_0\left(T+1-t\right)\right)\\ +\mu_0\left(s-T-1\right)\end{array}\right]\\
=&m_t+\eta\mu_0-\left(m_t+\mu_0\left(T+1-t\right)\right)\left(\frac{\eta}{1+\eta}\right)^{T+1-t}\frac{1}{1+\eta}\sum_{s=T+1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^{s-T-1}\\
&-\mu_0\left(\frac{\eta}{1+\eta}\right)^{T+1-t}\frac{1}{1+\eta}\sum_{s=T+1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^{s-T-1}\left(s-T-1\right)\\
=&m_t+\eta\mu_0-\left(m_t+\mu_0\left(T+1-t\right)+\eta\mu_0\right)\left(\frac{\eta}{1+\eta}\right)^{T+1-t}
\end{aligned}
$$

(9) is then:

$$\begin{aligned}
p_t &= m_t + \eta\mu_0 - \left(m_t + \mu_0\left(T+1-t\right) + \eta\mu_0\right)\left(\frac{\eta}{1+\eta}\right)^{T+1-t} \\
&\quad + \left(\frac{\eta}{1+\eta}\right)^{T+1-t}\left(m_{T+1} + \eta\mu_1\right) \\
&= m_t + \eta\mu_0 - \left(m_t + \mu_0\left(T+1-t\right) + \eta\mu_0\right)\left(\frac{\eta}{1+\eta}\right)^{T+1-t} \\
&\quad + \left(\frac{\eta}{1+\eta}\right)^{T+1-t}\left(m_T + \mu_1 + \eta\mu_1\right) \\
&= m_t + \eta\mu_0 - \left(m_t + \mu_0\left(T+1-t\right) + \eta\mu_0\right)\left(\frac{\eta}{1+\eta}\right)^{T+1-t} \\
&\quad + \left(\frac{\eta}{1+\eta}\right)^{T+1-t}\left(m_t + \mu_0\left(T-t\right) + \mu_1 + \eta\mu_1\right) \\
&= m_t + \eta\mu_0 + \left(1+\eta\right)\left(\mu_1 - \mu_0\right)\left(\frac{\eta}{1+\eta}\right)^{T+1-t}
\end{aligned}$$

To sum up, we have:

$$\begin{aligned}
p_t &= m_t + \eta\mu_0 && t < S \\
p_t &= m_t + \eta\mu_0 + \left(1+\eta\right)\left(\mu_1 - \mu_0\right)\left(\frac{\eta}{1+\eta}\right)^{T+1-t} && S \leq t < T+1 \\
p_t &= m_t + \eta\mu_1 && t \geq T+1
\end{aligned}$$