Game Theory

Static Game under Complete Information

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Outline

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  博弈的标准型
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Decisions

A decision problem consists of three features

Actions (行动) all the alternatives from which the player can choose

Outcomes (后果) possible consequences that can result from any of the actions

Preferences (偏好) describe how the player ranks the set of possible outcomes, from most desired to least desired.

Example

You are taking a course for a grade. Your objective is some combination of learning the material and obtaining a good grade in the course, with higher grades being ***preferred*** *over lower ones. Your* ***set of possible actions*** *is deciding how hard to study. The* ***outcome*** *of your success is affected by the amount of work you choose to put into your course work.*

Anything missing?

Strategic Interactions (策略互动)

- You must surely know that grades are often set on a curve.
- Your grade relies on your success on the exam as an absolute measure of not only how much you got right but also how much **the other students** in the class got right.
- Each player is trying to guess what others are doing, and how to act accordingly.

Example

Troops A and B, are going to attack their enemy C. If they lunch the attach simultaneously, they win and each gets 1; If only one troop attacks, it results in a failure that gives -1. Keeping the status quo gives 0. If the two partners can reach to an "agreement"—attack at a particular time, then they will do it.

Components in Games of Complete Information

A game with complete information contains the following components: (in the example of "A & B attack C")

- **Actions** "attack" or "do not attack"
- **Outcomes** win (1)/fail (-1)/keep the status quo (0)
- **Combination of actions & outcomes** (1) attack & not $\to (-1, 0)$; (2) attack & attack $\to (1, 1)$; (3) not & attack $\to (0, -1)$; (4) not & not $\to (0, 0)$
- **Preferences** win $\succ$ keep the status quo $\succ$ fail

"Complete Information"

- Definition of "common knowledge":
  - everyone knows E
  - everyone knows that everyone knows E, and so on *ad infinitum*
- E.g., A & B attack C
  - A and B make a phone call.
  - A sends a message to notify B about the time of attack. The message might not be reached to B with a positive probability.
- Except for the lecture "information and knowledge," we stick with the "common knowledge" assumption.

Normal-form Representation (标准型)

- A normal-form game consists of three features:
  - A finite set of players, $N = \{1, 2, ..., n\}$;
  - A collection of sets of pure strategies, $\{S_1, S_2, ..., S_n\}$;
  - A set of payoff functions $\{v_1, v_2, ..., v_n\}$, each assigning a payoff value to each combination of chosen strategies.
- A pure strategy for player i is a deterministic plan of action. The set of all pure strategies for player i is denoted by S_i. A profile of pure strategies $s = (s_1, ..., s_n)$, $s_i \in S_i$ for all $i = 1, ..., n$, describes a particular combination of pure strategies chosen by all n players in the game.

Example (Prisoner's Dilemma)

Two suspects are caught. Each can choose to confess/fink (denoted by F), or to remain silence/mum (denoted by M). If both choose M, each get 2 years in jail. If one chooses M and the other chooses F, the one who chooses F gets 1 year and the one who chooses M gets 5. If both choose F, each get 4 years.

Normal-Form Representation of the Prisoner's Dilemma

- Players: $N = \{1, 2\}$;
- Strategy sets: $S_1 = \{M, F\}$ for player 1; and $S_2 = \{M, F\}$ for player 2.
- Payoffs: Let

$$v_i(\text{player 1's choice}, \text{player 2's choice})$$

be the payoff to player i depending on the choices of both players. The payoffs are

$$\begin{aligned} v_1(M, M) &= v_2(M, M) = -2 \\ v_1(F, F) &= v_2(F, F) = -4 \\ v_1(M, F) &= v_2(F, M) = -5 \\ v_1(F, M) &= v_2(M, F) = -1 \end{aligned}$$

Matrix Representation of the Prisoner's Dilemma

For a two-person finite (number of strategies in S_i is finite) game, we can use a "matrix" to summarize the components of the game.

- Player 1's actions are represented by rows.
- Player 2's actions are represented by columns.
- In each entry, write down player 1's payoff, player 2's payoff, generated by the corresponding actions, respectively.

Player 2 (Bob, B)

Player 1 (Alice, A)	B's action b_1	B's action b_2	…
A's action a_1	$v_A(a_1,b_1)$, $v_B(a_1,b_1)$	$v_A(a_1,b_2)$, $v_B(a_1,b_2)$	
A's action a_2	$v_A(a_2,b_1)$, $v_B(a_2,b_1)$	$v_A(a_2,b_2)$, $v_B(a_2,b_2)$	
…			

Prisoner's dilemma →

Suspect 2

Suspect 1	M	F
M	-2,-2	-5,-1
F	-1,-5	-4,-4

Dominance in Pure Strategies

- The Prisoner' s Dilemma: each of the two players has an action that is best (F) regardless of what his opponent chooses.

		Suspect 2 M	Suspect 2 F
Suspect 1	M	-2,-2	-5,-1
	F	-1,-5	-4,-4

- Definition: Let $s_i \in S_i$ and $s_i' \in S_i$ be possible strategies for player i. We say that s_i' is **strictly dominated** by s_i if for any possible combination of the other players' strategies, $s_{-i} \in S_{-i}$, player i's payoff from s_i' is strictly less than that from s_i. That is

$$v_i(s_i, s_{-i}) > v_i(s_i', s_{-i}) \text{ for all } s_{-i} \in S_{-i}$$

Dominant Strategy Equilibrium (占优策略均衡)

- $s_i \in S_i$ is a strictly dominant strategy for i if every other strategy of i is strictly dominated by it.

$$v_i(s_i, s_{-i}) > v_i(s_i', s_{-i}) \text{ for all } s_i' \in S_i,\ \ s_i \neq s_i' \ , s_{-i} \in S_{-i}$$

  - For suspect 1: $v_1(F, M) > v_1(M, M)$ and $v_1(F, F) > v_1(M, F)$
  - For suspect 2: $v_2(M, F) > v_2(M, M)$ and $v_2(F, F) > v_2(F, M)$
- The strategy profile $s^D \in S$ is a strict dominant strategy equilibrium if $s_i^D \in S_i$ is a strict dominant strategy for all $i \in N$.

IESDS

Iterated Elimination of Strictly Dominated Pure Strategies: **重复剔除严格劣策略**

- A rational player will never play a dominated strategy.
- If a rational player has a strictly dominant strategy then he will play it.

		Suspect 2	
		$\cancel{M}$	F
Suspect 1	$\cancel{M}$	$\cancel{-2}$,$\cancel{-2}$	$\cancel{-5}$,-1
	F	-1,$\cancel{-5}$	-4,-4

$\Rightarrow$

		Suspect 2
		F
Suspect 1	F	-4,-4

- For each player, M is strictly dominated by F.
  - Player 1 eliminates the first row
  - Player 2 eliminates the first column

  Leaving (F, F) survived from IESDS. And they both know that.

IESDS: Example

Assume player 1 can choose from $\{U_1, M_1, D_1\}$; player can choose from $\{L_2, C_2, R_2\}$. The matrix representation of the game is given by

	L_2	C_2	R_2
U_1	4,3	5,1	6,2
M_1	2,1	8,4	3,6
D_1	3,0	9,6	2,8

Notice that there is no strictly dominant/dominated strategy for player 1. Then check player 2: C_2 is strictly dominated by R_2:

	L_2	$\cancel{C_2}$	R_2
U_1	4,3	$5,\cancel{1}$	6,2
M_1	2,1	$8,\cancel{4}$	3,6
D_1	3,0	$9,\cancel{6}$	2,8

$\xrightarrow[\text{check 1's dominance}]{\text{2 deletes column } C_2}$

	L_2	R_2
U_1	4,3	6,2
$\cancel{M_1}$	$\cancel{2},1$	$\cancel{3},6$
$\cancel{D_1}$	$\cancel{3},0$	$\cancel{2},8$

$\xrightarrow[\text{1 deletes row } M_1 \text{ and } D_1]{M_1 \text{ and } D_1 \text{ are strictly dominated by } U_1}$

	L_2	R_2
U_1	4,3	6,2

$\xrightarrow[\text{2 deletes column } L_2]{R_2 \text{ is strictly dominated by } L_2}$

	L_2	$\cancel{R_2}$
U_1	4,3	$6,\cancel{2}$

$\xrightarrow{\text{The remaining outcome is}}$

	L_2
U_1	4,3

Battle of Sexes: No Dominated Strategy

- Alex prefers Opera over Football; Chris prefers Football over Opera
- Both prefer watching the same program to watching different programs

		Chris O	Chris F
Alex	O	2,1	0,0
Alex	F	0,0	1,2

- IESDS is not applicable to provide a solution. We need other solution concepts.

Best Response (最佳回应)

- The strategy $s_i \in S_i$ is player i's best response to the rival's strategy $s_{-i} \in S_{-i}$ if $v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i})\ \forall s_i' \in S_i$.
- If $\tilde{s}_i$ is a strictly dominated strategy for player i, then it cannot be a best response to any $s_{-i} \in S_{-i}$.
- If in a finite normal-form game s^* is a strictly dominant strategy equilibrium, or if it uniquely survives IESDS, then s_i^* is a best response to s_{-i}^* for all i.

Nash Equilibrium

Definition (Nash Equilibrium (Pure Strategy))

The pure-strategy profile $s^* = (s_1^*, ...s_n^*) \in S$ is a Nash equilibrium if s_i^* is a best response to s_{-i}^* for all $i \in N$.

If s^* is a Nash equilibrium, then nobody has an incentive to deviate:

$$v_i(s_i^*, s_{-i}^*) \geq v_i(s_i', s_{-i}^*),$$

for all $s_i' \in S$ and all $i \in N$.

John Nash

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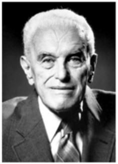


John C. Harsanyi
Prize share: 1/3

John F. Nash Jr.
Prize share: 1/3

Reinhard Selten
Prize share: 1/3

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 1994 was awarded jointly to John C. Harsanyi, John F. Nash Jr. and Reinhard Selten *"for their pioneering analysis of equilibria in the theory of non-cooperative games"*.

Using Best Responses to Find Nash Equilibrium

Step 1 Underlining Alex's best response, for each possible action taken by Chris.

		Chris O	Chris F
Alex	O	$\underline{2}$,1	0,0
	F	0,0	$\underline{1}$,2

Step 2 Underlining Chris's best response, for each possible action taken by Alex.

		Chris O	Chris F
Alex	O	$\underline{2}$,$\underline{1}$	0,0
	F	0,0	$\underline{1}$,$\underline{2}$

Step 3 The Nash equilibrium is (are) represented by the entry (entries) where the best responses between Alex and Chris "coincide."

		Chris O	Chris F
Alex	O	$\underline{2}$,$\underline{1}$	0,0
	F	0,0	$\underline{1}$,$\underline{2}$

There are two equilibria here: (O,O) and (F,F). Check: nobody has an incentive to deviate.

Example

Player 1 can choose $\{U, M, D\}$ while player 2 can choose $\{L, C, R\}$:

		Player 2 L	C	R
Player 1	U	7,7	4,2	1,8
	M	2,4	5,5	2,3
	D	8,1	3,2	0,0

Underlining player 1's best response (red)

Player 2

Player 1		L	C	R
	U	7,7	4,2	1,8
	M	2,4	$\underline{5}$,5	$\underline{2}$,3
	D	$\underline{8}$,1	3,2	0,0

Underlining player 2's best responses (blue)

Player 2

Player 1		L	C	R
	U	7,7	4,2	1,$\underline{8}$
	M	2,4	$\underline{5}$,$\underline{5}$	$\underline{2}$,3
	D	$\underline{8}$,1	3,$\underline{2}$	0,0

There exists a unique equilibrium: (M,C)

Example

Player 1 can choose $\{U, M, D\}$ while player 2 can choose $\{L, C, R\}$:

Player 2

Player 1		L	C	R
	U	4,3	5,1	6,2
	M	2,1	8,4	3,6
	D	3,0	9,6	2,8

Underlining the best responses:

Player 2

Player 1		L	C	R
	U	$\underline{4}$,$\underline{3}$	5,1	$\underline{6}$,2
	M	2,1	8,4	3,$\underline{6}$
	D	3,0	$\underline{9}$,6	2,$\underline{8}$

$\Rightarrow$ Unique equilibrium (U,L)

No Pure-Strategy-Equilibrium: Mixed Strategies

- The above equilibrium are called "pure-strategy equilibrium"
- If there's no pure-strategy-equilibrium, there may still exist "mixed-strategy-equilibrium"
- Example: Matching pennies
  Players 1 and 2 each put a penny on a table simultaneously. If the two pennies come up the same side (heads or tails) then player 1 gets both; otherwise player 2 does.

		Player 2 H	Player 2 T
Player 1	H	1,-1	-1,1
	T	-1,1	1,-1

Mixed Strategy (混合策略): Definition

- Let $S_i = \{s_{i1}, ..., s_{im}\}$ be player i's finite set of pure strategies. Define ΔS_i as the simplex of S_i, which is the set of all probability distributions over S_i. A mixed strategy for player i is an element $\sigma_i \in \Delta S_i$, so that $\sigma_i = \{\sigma_i(s_{i1}), ..., \sigma_i(s_{im})\}$ is a probability distribution over S_i, where σ_i is the probability that player i plays s_i.
  1. $\sigma_i(s_i) \geq 0$
  2. $\sum_{s_i \in S_i} \sigma_i(s_i) = 1.$
- Example: Matching Pennies
  - $\Delta S_i = \{\sigma_i(H), \sigma_i(T) | \sigma_i(H) \geq 0, \sigma_i(T) \geq 0, \sigma_i(H) + \sigma_i(T) = 1\}$
- The mixed-strategy profile $\sigma^* = \{\sigma_1^*, ..., \sigma_n^*\}$ is a Nash equilibrium if for each player $i = 1, 2, ..., n$, σ_i^* is a best response to σ_{-i}^*:

$$v_i(\sigma_i^*, \sigma_{-i}^*) \geq v_i(\sigma_i, \sigma_{-i}^*), \ \forall \sigma_i \in \Delta S_i$$

Player 1 chooses H with probability p (hence plays T with probability $1-p$); Player 2 chooses H with probability q (chooses T with probability $1-q$)

		Player 2	
		H (w.p. q)	T (w.p. $1-q$)
Player 1	H (w.p. p)	1, -1	-1, 1
	T (w.p. $1-p$)	-1, 1	1, -1

- For player 1:
  - The expected payoff of playing H: $1 \cdot q + (-1) \cdot (1-q) = 2q - 1$
  - Expected payoff of playing T: $(-1) \cdot q + 1 \cdot (1-q) = 1 - 2q$
- Player 2:
  - Playing H: $-p + 1 - p = 1 - 2p$;
  - Playing T: $p - (1-p) = 2p - 1$

- For player 1:
  - Playing H gives: $2q-1$
  - Playing T gives: $1-2q$

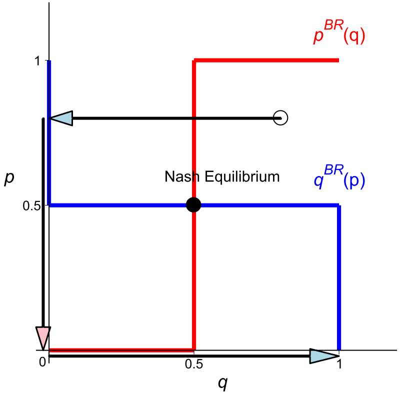
  $\Rightarrow H$ is preferred to L iff $2q-1>1-2q \Leftrightarrow q>\frac{1}{2}$
- Player 2:
  - Playing H gives: $1-2p$;
  - Playing T gives: $2p-1$

  $\Rightarrow H$ is preferred to L iff $1-2p>2p-1 \Leftrightarrow p<\frac{1}{2}$

The best-response correspondence:

$$p^{BR}(q)=\begin{cases} p=1 & \text{if } q>\frac{1}{2} \\ p=0 & \text{if } q<\frac{1}{2} \\ p\in[0,1] & \text{if } q=\frac{1}{2} \end{cases}, \quad q^{BR}(p)=\begin{cases} q=1 & \text{if } p<\frac{1}{2} \\ q=0 & \text{if } p>\frac{1}{2} \\ q\in[0,1] & \text{if } p=\frac{1}{2} \end{cases}$$

$$p^{BR}(q) = \begin{cases} p = 1 & \text{if } q > \frac{1}{2} \\ p = 0 & \text{if } q < \frac{1}{2} \\ p \in [0,1] & \text{if } q = \frac{1}{2} \end{cases}, \quad q^{BR}(p) = \begin{cases} q = 1 & \text{if } p < \frac{1}{2} \\ q = 0 & \text{if } p > \frac{1}{2} \\ q \in [0,1] & \text{if } p = \frac{1}{2} \end{cases}$$



Multiple Equilibrium: Pure and Mixed

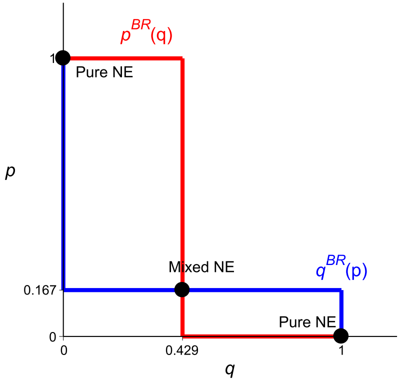
		Player 2	
		C (q)	R (1-q)
Player 1	M (p)	0, 0	$\underline{3}$, $\underline{5}$
	D (1-p)	$\underline{4}$, $\underline{4}$	0, 3

- Pure strategy equilibrium: (D, C) and (M, R)
- Mixed strategies?
  - Player 1:

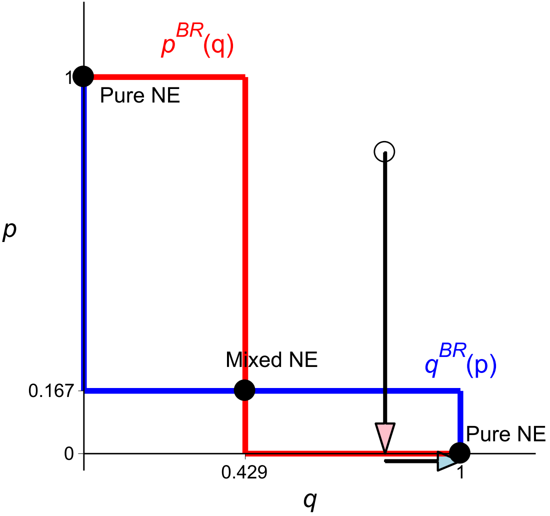
$$\begin{array}{l}\text{expected payoff of M: } 0 \cdot q + 3 \cdot (1-q) \\ \text{expected payoff of D: } 4 \cdot q + 0 \cdot (1-q)\end{array} \xrightarrow{\text{indifferent}} q = \frac{3}{7}$$

  - Player 2:

$$\begin{array}{l}\text{expected payoff of C: } 0 \cdot p + 4 \cdot (1-p) \\ \text{expected payoff of R: } 5 \cdot p + 3 \cdot (1-p)\end{array} \xrightarrow{\text{indifferent}} p = \frac{1}{6}$$



$$p^{BR}(q) = \begin{cases} p = 1 & \text{if } q < 3/7 \\ p = 0 & \text{if } q > 3/7 \\ p \in [0,1] & \text{if } q = 3/7 \end{cases}, \quad q^{BR}(p) = \begin{cases} q = 1 & \text{if } p < 1/6 \\ q = 0 & \text{if } p > 1/6 \\ q \in [0,1] & \text{if } p = 1/6 \end{cases}$$


p^BR(q)
1
Pure NE
p
Mixed NE
q^BR(p)
0.167
Pure NE
0
0.429
1
q

Rock-Paper-Scissors

- Two players play the “rock-paper-scissor” game. The matrix representation of the game is

Player 2

Player 1		R	P	S
	R	0,0	-1,<u>1</u>	<u>1</u>,-1
	P	<u>1</u>,-1	0,0	-1,<u>1</u>
	S	-1,<u>1</u>	<u>1</u>,-1	0,0

- Clearly, there is no pure-strategy equilibrium
- In order to find the Nash equilibrium (mixed strategy) of this game, we proceed in three steps.

		Player 2		
		R: $\sigma_2(R)$	P: $\sigma_2(P)$	S: $\sigma_2(S)$
Player 1	R: $\sigma_1(R)$	0, 0	-1, $\underline{1}$	$\underline{1}$, -1
	P: $\sigma_1(P)$	$\underline{1}$, -1	0, 0	-1, $\underline{1}$
	S: $\sigma_1(S)$	-1, $\underline{1}$	$\underline{1}$, -1	0, 0

- First, there is no NE if at least one player plays a pure strategy (no pure-strategy equilibrium)
- Second, there can be no NE in which at least one player mixes only between two pure strategies
  - Suppose player 2 only mixes between R and P, that is: $\sigma_2(R) + \sigma_2(P) = 1$ and $\sigma_2(S) = 0$
  - Then for player 1, R is a strictly dominated strategy
  - If player 1 drops R, then for player 2, playing S is always better than P, which contradicts with $\sigma_2(S) = 0$

			Player 2	
		R: $\sigma_2(R)$	P: $\sigma_2(P)$	S: $\sigma_2(S)$
Player 1	R: $\sigma_1(R)$	0, 0	-1, $\underline{1}$	$\underline{1}$, -1
	P: $\sigma_1(P)$	$\underline{1}$, -1	0, 0	-1, $\underline{1}$
	S: $\sigma_1(S)$	-1, $\underline{1}$	$\underline{1}$, -1	0, 0

- Third, each option must be played with positive probability.
  - For player $i \in \{1, 2\}$ $\sigma_i(R), \sigma_i(P), 1 - \sigma_i(R) - \sigma_i(P)$
  - $v_i(R, \sigma_j) = -\sigma_j(P) + 1 - \sigma_j(R) - \sigma_j(P)$
  - $v_i(P, \sigma_j) = \sigma_j(R) - [1 - \sigma_j(R) - \sigma_j(P)]$
  - $v_i(S, \sigma_j) = -\sigma_j(R) + \sigma_j(P)$
- The above three choices must be equally profitable; otherwise one of the options will be played with zero probability (which violates step 2)
- Therefore, by equating $v_i(R, \sigma_j) = v_i(P, \sigma_j) = v_i(S, \sigma_j)$, we obtain $\sigma_j^*(R) = \sigma_j^*(P) = \sigma_j^*(P) = \frac{1}{3}$

Public Goods

- In microeconomics, you have learned that "public goods" are those with non-rivalry in consumption and nonexclusive.
- Suppose Sam and Bob are roommates. They decides whether to buy an air-conditioner, i.e., a public good in the room.
  - The price of an air-conditioner is 3000
  - The air-conditioner brings 2000 for each person; without it, each person gets 0
  - If only one person buys it (hence the other person enjoys it for free), he pays 3000; If they jointly buy it, they split the cost, i.e., each pays 1500
  - Will they buy the air-conditioner?

Insufficient Provision of Public Goods

		Bob: Buy	Bob: Not
Sam	Buy	500,500	-1000,<u>2000</u>
	Not	<u>2000</u>,-1000	<u>0</u>,<u>0</u>

- "Not buy" is a dominant strategy for each player.
- "Individual rationality" results in an socially inefficient outcome:
  - "Welfare" of the two roommates is the sum of their utility minus the cost of the air-conditioner.
  - The free-market outcome (total surplus) is 0
  - The ideal outcome should be $2000 + 2000 - 3000 = 1000$
- $\Rightarrow$ "prisoner's dilemma"

Continuous Actions: The Tragedy of the Commons

- Public resources: items that are rival in consumption but non-exclusive, e.g., a pasture shared by local herders
  - Each herder wants to maximize his yield
  - The cost of that extra animal is shared by all the other herders.
- Total amount of pasture K
- Two persons, and each person chooses his/own size k_1 and k_2. The remaining pasture is of size $K - k_1 - k_2$
- The benefit of consuming an amount k_i gives $\ln k_i$; Each player also enjoys consuming the remainder of the pasture, giving each person a benefit $\ln(K - k_1 - k_2)$.
  - The payoff of person 1 is $v_1(k_1, k_2) = \ln k_1 + \ln(K - k_1 - k_2)$
  - The payoff of person 2 is $v_2(k_1, k_2) = \ln k_2 + \ln(K - k_1 - k_2)$

- Given k_2, player 1 solves

$$\max_{k_1} \ln k_1 + \ln(K - k_1 - k_2)$$

  - The first-order condition (FOC):

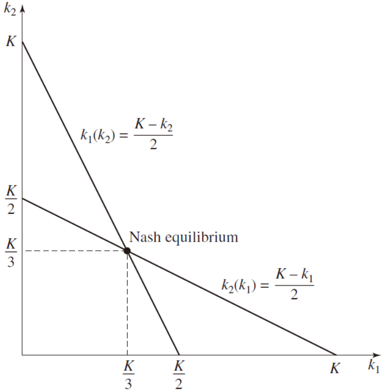
$$\frac{dv_1}{dk_1} = \frac{1}{k_1} - \frac{1}{K - k_1 - k_2} = 0 \Rightarrow k_1^{BR}(k_2) = \frac{K - k_2}{2}$$

  - The second-order derivative is $\frac{d^2 v_1}{dk_1^2} = -\frac{1}{k_1^2} - \frac{1}{(K-k_1-k_2)^2} < 0 \Rightarrow k_1^{BR}$ is a maximum. Hence player 1's **best response** is $k_1^{BR} = \frac{K-k_2}{2}$

- Similarly, player 2's best response is

$$\max_{k_2} v_2 \Rightarrow \frac{dv_2}{dk_2} = 0 \Rightarrow k_2^{BR}(k_1) = \frac{K - k_1}{2}$$

- The Nash equilibrium of the game is the "intersection" of the two best responses.



- At Nash equilibrium, no person has an incentive to deviate, i.e., the solution of two-unknowns & two-equations system

$$\begin{cases} k_1 = \frac{K - k_2}{2} \\ k_2 = \frac{K - k_1}{2} \end{cases} \Rightarrow k_1^* = k_2^* = \frac{K}{3}$$

The Nash Outcome v.s. Socially Optimal Outcome

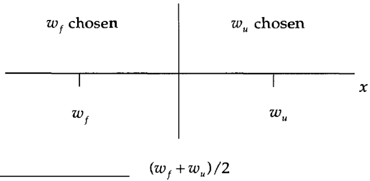
- Under Nash equilibrium, the payoff of each player is $\ln \frac{K}{3} + \ln \frac{K}{3} = 2\ln K - 2\ln 3$. Total surplus is $4\ln K - 4\ln 3$.
- Suppose a benevolent planner distributes the resources to maximize total surplus of the two persons:

$$\max_{k_1,k_2} \ln k_1 + \ln k_2 + 2\ln(K - k_1 - k_2)$$

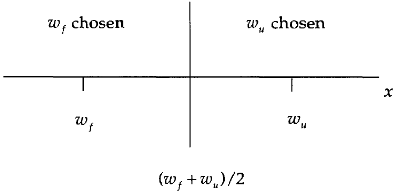
  - FOC k_1: $\frac{1}{k_1} = \frac{2}{K-k_1-k_2}$
  - FOC k_2: $\frac{1}{k_2} = \frac{2}{K-k_1-k_2}$
  - The socially efficient outcome is $k_1^o = k_2^o = \frac{K}{4} < k^* = \frac{K}{3}$
- Payoff of each is $\ln \frac{K}{4} + \ln \frac{K}{2} = 2\ln K - (\ln 4 + \ln 2)$
- Because $\ln(\cdot)$ is concave, hence $\ln\left(\frac{2+4}{2}\right) > \frac{\ln 2 + \ln 4}{2} \Rightarrow -2\ln 3 < -(\ln 4 + \ln 2)$

Final-Offer Arbitration[1]

- Many public-sector worker are forbidden to strike. Instead, wage disputes are settled by binding arbitration.
- The firm prefers a low wage; the union prefers a high wage.
- First, the firm and the union simultaneously make offers, denoted by w_f and w_u
- Second, the arbitrator chooses one of the two wages as the settlement.
- The arbitrator has an ideal settlement, denoted by x. After observing the offers from the two parties, the arbitrator chooses the offer that is closer to x:
  - choose w_f if $x < \frac{w_f + w_u}{2}$
  - choose w_u if $x > \frac{w_f + w_u}{2}$



[1] This example is optional



- The arbitrator knows x but the firm and union do not know x. The two parties believe that x is randomly distributed according to cumulative probability distribution denoted by $F(x)$, with associated probability density function $f(x) = F'(x)$.
- $\Pr(w_f \text{ is chosen}) = \Pr\left(x < \frac{w_f + w_u}{2}\right) = F\left(\frac{w_f + w_u}{2}\right)$
- $\Pr(w_u \text{ is chosen}) = \Pr\left(x > \frac{w_f + w_u}{2}\right) = 1 - F\left(\frac{w_f + w_u}{2}\right)$
- The expected wage settlement is

$$
\begin{aligned}
& w_f \Pr(w_f \text{ is chosen}) + w_u \Pr(w_u \text{ is chosen}) \\
=& w_f F\left(\frac{w_f + w_u}{2}\right) + w_u \left[1 - F\left(\frac{w_f + w_u}{2}\right)\right]
\end{aligned}
$$

Solve the Nash Equilibrium Offers

- The firm wishes to minimize the expected wage. Given w_u, the firm solves

$$\min_{w_f} w_f F\left(\frac{w_f + w_u}{2}\right) + w_u\left[1 - F\left(\frac{w_f + w_u}{2}\right)\right]$$

FOC: $(w_u - w_f)\frac{1}{2}f\left(\frac{w_f+w_u}{2}\right) = F\left(\frac{w_f+w_u}{2}\right)$

- The union wishes to maximize the expected wage, and solves

$$\max_{w_u} w_f F\left(\frac{w_f + w_u}{2}\right) + w_u\left[1 - F\left(\frac{w_f + w_u}{2}\right)\right]$$

FOC: $(w_u - w_f)\frac{1}{2}f\left(\frac{w_f+w_u}{2}\right) = 1 - F\left(\frac{w_f+w_u}{2}\right)$

- The two FOCs imply $F\left(\frac{w_f^* + w_u^*}{2}\right) = \frac{1}{2}$
- Plug $F\left(\frac{w_f^* + w_u^*}{2}\right) = \frac{1}{2}$ into one of the FOC, gives

$$w_u^* - w_f^* = \frac{1}{f\left(\frac{w_f^* + w_u^*}{2}\right)}$$

- Assume that x follows $N(\mu, \sigma^2)$

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

- Then $F\left(\frac{w_f^* + w_u^*}{2}\right) = \frac{1}{2} \Rightarrow \frac{w_f^* + w_u^*}{2} = \mu$
- And $w_u^* - w_f^* = \frac{1}{f\left(\frac{w_f^* + w_u^*}{2}\right)} = \frac{1}{f(\mu)} = \sqrt{2\pi}\sigma$
- Therefore, the Nash equilibrium is
  - $w_f^* = \mu - \sqrt{\frac{\pi}{2}}\sigma$
  - $w_u^* = \mu + \sqrt{\frac{\pi}{2}}\sigma$

Proof of $F\left(\frac{w_f^*+w_u^*}{2}\right)=\frac{1}{2} \Rightarrow \frac{w_f^*+w_u^*}{2}=\mu$ when x follows $N(\mu,\sigma^2)$.[2]

- It is equivalent to show that $F(\mu)=\int_{-\infty}^{\mu}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx=\frac{1}{2}$. Let $I=F(\mu)$ and show $I=\frac{1}{2}$.
- Change of variables: $t=\frac{x-\mu}{\sigma} \Rightarrow dt=\frac{1}{\sigma}dx$. Then $I=\int_{-\infty}^{0}\frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}dt$
- $I^2=\int_{-\infty}^{0}\int_{-\infty}^{0}\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dxdy$
- Polar coordinates: $x=r\cos\theta$, $y=r\sin\theta$, the integrand becomes $f(x,y)dxdy=f(r\cos\theta,r\sin\theta)rdrd\theta$
- When $x\in(-\infty,0]$ and $y\in(-\infty,0]$, the integral region becomes $\pi\leq\theta\leq\frac{3}{2}\pi$ and $0\leq r<+\infty$
- $I^2=\int_{\pi}^{\frac{3}{2}\pi}\left[\int_0^{+\infty}\frac{1}{2\pi}e^{-\frac{r^2}{2}}rdr\right]d\theta=\int_{\pi}^{\frac{3}{2}\pi}\left[-\frac{1}{2\pi}e^{-\frac{r^2}{2}}\Big|_0^{+\infty}\right]d\theta=\frac{1}{2\pi}\left(\frac{3}{2}\pi-\pi\right)=\frac{1}{4}$
- $I=F(\mu)=\frac{1}{2}$

□

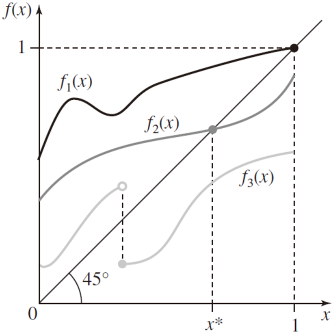
[2] Not required

Nash's Existence Theorem[3]

- Any n-player normal-form game with finite strategy sets S_i for all players has a Nash equilibrium in mixed strategies.
- The central idea of Nash's proof builds on the *fixed-point theorem*
- Brouwer's Fixed-Point Theorem: if $f(x)$ is a continuous function from the domain $[0,1]$ to itself then there exists at least one value $x^* \in [0,1]$ for which $f(x^*) = x^*$
  - Consider $G(x) = f(x) - x$, which is continuous in $[0,1]$
  - $G(0) = f(0) \geq 0$; $G(1) = f(1) - 1 \leq 0$
  - If $G(0) = 0$, then $x^* = 0$; if $G(1) = 0$, then $x^* = 1$
  - If $G(0) > 0$ and $G(1) < 0$, by Intermediate value theorem, there exists a x^* such that $G(x^*) = 0$, i.e., $f(x^*) = x^*$.

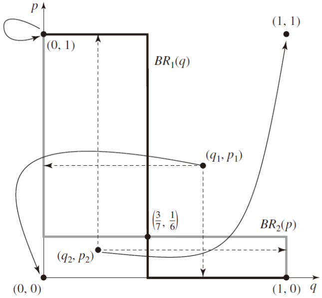
[3]Optional

Fixed-Point Theorem*



- Intuition: If you drop a world map randomly in the ground (in earth), there must be an overlapped point.

- There must be at least one mixed-strategy profile for which each player's strategy is itself a best response to this profile of strategies.
- Example: mapping $(q, p) \in [0, 1]^2$ to $(q^{BR}(p), p^{BR}(q))$: (q_1, p_1) is mapped onto $(0, 0)$; (q_2, p_2) is mapped onto $(1, 1)$.



There are 3 Nash equilibrium (fixed points): $(q, p) = (0, 1), (1, 0), \left(\frac{3}{7}, \frac{1}{6}\right)$, all of which are mapped onto themselves.