ECON 39: Undergraduate International Trade
Problem Set #1 Solution
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Preliminaries

This week, we are learning about the Ricardian model of trade. A word of warning: the following problem set is hard. I am purposely keeping things abstract so that when it comes time for the first midterm, you will be more than adequately prepared!

Questions

1. Suppose there is a single country (call it country 1) inhabited by L_1 identical workers. Workers have one unit of time with which to produce and can choose to produce either product A or product B (or split her time between the two). A worker can make $\frac{1}{\alpha_1^A}$ units of product A in a unit of time or $\frac{1}{\alpha_1^B}$ units of product B in a unit of time.

   (a) Define the production possibility set and the production possibility frontier for country 1.

   - *The production possibility set is every bundle of produce that country* 1 *can produce:*

   $$\left\{Q_1^A, Q_1^B | \alpha_1^A Q_1^A + \alpha_1^B Q_1^B \leq L_1\right\}$$

   .

   - *The production possibility frontier is the most (say) of* Q_1^B *that can be produced given that country* 1 *is producing* Q_1^A *units of good* 1*:*

   $$\begin{aligned} \alpha_1^A Q_1^A + \alpha_1^B Q_1^B = L_1 &\iff \\ Q_1^B &= \frac{L_1}{\alpha_1^B} - \frac{\alpha_1^A}{\alpha_1^B} Q_1^A \end{aligned}$$

   (b) Suppose that each worker has preferences $U_1\left(C_1^A, C_1^B\right)$, where C_1^A is the quantity consumed of product A and C_1^B is the quantity consumed of product B. Let Q_1^A and Q_1^B denote the quantity produced of products A and B, respectively.

   i. List the exogenous model parameters.
      - *The exogenous model parameters are productivities* α_1^A *and* α_1^B*, population* L_1*, and preferences* $U_1(\cdot, \cdot)$*.*

   ii. List the endogenous model outcomes.
      - *The endogenous model outcomes are equilibrium relative prices* $\frac{p^A}{p^B}$*, quantities produced (per worker)* $\left\{Q_1^A, Q_1^B\right\}$*, and equilibrium quantities consumed (per worker)* $\left\{C_1^A, C_1^B\right\}$*. Note that we could without loss of generality normalize the price of one of the goods to one; I will not do that to show you the scale of prices does not matter.*

   iii. Define the equilibrium.
      - *For any productivities* α_1^A *and* α_1^B*, population* L_1*, and preferences* $U_1(\cdot, \cdot)$*, equilibrium is a set of relative prices* $\frac{p^A}{p^B}$*, quantities produced* $\left\{Q_1^A, Q_1^B\right\}$*, and equilibrium quantities consumed* $\left\{C_1^A, C_1^B\right\}$ *such that (1) given relative prices, producers choose quantities produced to maximize profits; (2) given prices and incomes, consumers choose quantities consumed to maximize utility; and (3) markets clear, i.e. the quantity produced is equal to the quantity consumed of each good.*

   (c) Suppose that $U_1\left(C_1^A, C_1^B\right) = \beta_1 \ln\left(C_1^A\right) + \beta_2 \ln\left(C_1^B\right)$, where $\beta_1 > 0$ and $\beta_2 > 0$. Find the equilibrium.

- *Let us begin with producer profit maximization. Since we have no concept of firms (yet), this is equivalent to a worker choosing how much of her time to allocate to the production of each good to maximize her income. Let θ denote the fraction of time a worker allocates to the production of good A and $(1-\theta)$ denote the fraction of time a worker allocates to the production of good B. The workers problem is:*

$$\max_{\theta\in[0,1]} \theta\frac{p^A}{\alpha_1^A} + (1-\theta)\frac{p^B}{\alpha_1^B}$$

  Note that another way of writing this problem is:

$$\max_{\theta\in[0,1]} p^A Q_1^A + p^B Q_1^B \ \ s.t. \ \ Q_1^A\alpha_1^A + Q_1^B\alpha_1^B \leq L_1$$

  (why are the two equivalent?). Assuming that both goods are produced (why is this okay?), the solution to either maximization problem is:

$$w_1 = \frac{p^A}{\alpha_1^A} = \frac{p^B}{\alpha_1^B}, \tag{1}$$

  where w_1 is the wage of the worker in country 1. This equation makes the worker indifferent between both goods being produced.
- *Now let us move to the consumer utility maximization problem. Each worker maximizes:*

$$\max_{C_1^A, C_1^B} \beta_1 \ln\left(C_1^A\right) + \beta_2 \ln\left(C_1^B\right) \ \ s.t. \ \ p^A C_1^A + p^B C_2^A \leq w_1$$

  The associated Lagrangian is:

$$\mathcal{L} : \beta_1 \ln\left(C_1^A\right) + \beta_2 \ln\left(C_1^B\right) - \lambda\left(p^A C_1^A + p^B C_2^A - w_1\right)$$

  The first order conditions are:

$$\begin{aligned} \frac{\partial L}{\partial C_1^A} &= 0 population L_1, \iff \\ \frac{\beta_1}{C_1^A} &= \lambda p^A \iff \\ \beta_1 &= \lambda p^A C_1^A \end{aligned}$$

  and:

$$\begin{aligned} \frac{\partial L}{\partial C_1^B} &= 0 \iff \\ \frac{\beta_2}{C_1^B} &= \lambda p^B \iff \\ \beta_2 &= \lambda p^B C_1^B \end{aligned}$$

  Combining the two first order conditions (to cancel out the λ) yields:

$$\frac{p^A C_1^A}{\beta_1} = \frac{p^B C_1^B}{\beta_2}$$

Now we substitute this expression into the budget constraint:

$$\begin{aligned}
p^A C_1^A + p^B C_2^A &= w_1 \iff \\
p^A C_1^A + \left(\frac{\beta_2}{\beta_1} p^A C_1^A\right) &= w_1 \iff \\
\left(1 + \frac{\beta_2}{\beta_1}\right) p^A C_1^A &= w_1 \iff \\
\left(\frac{\beta_1 + \beta_2}{\beta_1}\right) p^A C_1^A &= w_1 \iff \\
p^A C_1^A &= \frac{\beta_1}{\beta_1 + \beta_2} w_1 \qquad (2)
\end{aligned}$$

and, similarly,

$$\begin{aligned}
p^B C_1^B &= \frac{\beta_2}{\beta_1} \left(p^A C_1^A\right) \iff \\
p^B C_1^B &= \frac{\beta_2}{\beta_1 + \beta_2} w_1, \qquad (3)
\end{aligned}$$

As we said in class, the Cobb-Douglas preferences imply consumers spend a constant fraction of their income on each good. (Why are these preferences also Cobb-Douglas? How can we see that without loss of generality we can assume $\beta_2 = 1 - \beta_1$?)

- *Finally, let us solve for prices and quantities. We get prices from the second equality in equations (1):*

$$\begin{aligned}
\frac{p^A}{\alpha_1^A} &= \frac{p^B}{\alpha_1^B} \iff \\
\frac{p^A}{p^B} &= \frac{\alpha_1^A}{\alpha_1^B}.
\end{aligned}$$

Combining equations (1) and (2) yields equilibrium consumption of good A:

$$\begin{aligned}
p^A C_1^A &= \frac{\beta_1}{\beta_1 + \beta_2} w_1 \iff \\
p^A C_1^A &= \frac{\beta_1}{\beta_1 + \beta_2} \left(\frac{p^A}{\alpha_1^A}\right) \iff \\
C_1^A &= \frac{\beta_1}{\beta_1 + \beta_2} \times \frac{1}{\alpha_1^A} \qquad (4)
\end{aligned}$$

Combining equations (1) and (3) yields equilibrium consumption of good B:

$$\begin{aligned}
p^B C_1^B &= \frac{\beta_2}{\beta_1 + \beta_2} w_1 \iff \\
p^B C_1^B &= \frac{\beta_2}{\beta_1 + \beta_2} \left(\frac{p^B}{\alpha_1^B}\right) \iff \\
C_1^B &= \frac{\beta_2}{\beta_1 + \beta_2} \times \frac{1}{\alpha_1^B} \qquad (5)
\end{aligned}$$

This gives us the quantity consumed of both goods as a function of exogenous model parameters. Moreover, since markets clear from equilibrium #3, the equilibrium quantity consumed is also equal to the equilibrium quantity produced. As a final check to make sure we have the correct *equilibrium, we can check to make sure the solution is on the production possibility frontier:*

$$Q_1^B = \frac{L_1}{\alpha_1^B} - \frac{\alpha_1^A}{\alpha_1^B} Q_1^A \iff$$
$$\left(\frac{\beta_2}{\beta_1+\beta_2} \times \frac{1}{\alpha_1^B} \times L_1\right) = \frac{L_1}{\alpha_1^B} - \frac{\alpha_1^A}{\alpha_1^B}\left(\frac{\beta_1}{\beta_1+\beta_2} \times \frac{1}{\alpha_1^A} \times L_1\right) \iff$$
$$\left(\frac{\beta_2}{\beta_1+\beta_2} \times \frac{1}{\alpha_1^B}\right) + \left(\frac{\beta_1}{\beta_1+\beta_2} \times \frac{1}{\alpha_1^A}\right)\frac{\alpha_1^A}{\alpha_1^B} = \frac{1}{\alpha_1^B} \iff$$
$$\frac{\beta_2}{\beta_1+\beta_2} + \frac{\beta_1}{\beta_1+\beta_2} = 1,$$

which is true. (Note that we had to be careful, as the quantities we solved for above were per worker, whereas the PPF was as the economy as a whole). Note that we never solved separately for p_1 and p_2, only their ratio, but were still able to solve for all quantities. This is because the level of prices do not matter.

2. Now suppose that there is an additional country (call it country 2) inhabited by L_2 identical workers. As in country 1, workers have one unit of time with which to produce and can choose to produce either product A or product B (or split her time between the two). However, the productivity of workers in country 2 is different: a country 2 worker can make $\frac{1}{\alpha_2^A}$ units of product A in a unit of time or $\frac{1}{\alpha_2^B}$ units of product B in a unit of time.

   (a) Suppose that workers in both countries have identical preferences $U\left(C_i^A, C_i^B\right)$, where $i \in \{1,2\}$ (that is, i is the name of the country).

      i. List the exogenous model parameters.

         - *The exogenous model parameters are productivities $\{\alpha_c^g\}$, populations $\{L_c\}$, and preferences $U_c(\cdot,\cdot)$ for all $g \in \{A,B\}$ and $c \in \{1,2\}$.*

      ii. List the endogenous model outcomes.

         - *The endogenous model outcomes are equilibrium relative prices $\frac{p^A}{p^B}$, quantities produced (per worker) $\{Q_c^g\}$, and equilibrium quantities consumed (per worker) $\{C_c^g\}$ for all $g \in \{A,B\}$ and $c \in \{1,2\}$.*

      iii. Define the equilibrium.

         - *For any productivities productivities $\{\alpha_c^g\}$, populations $\{L_c\}$, and preferences $U_c(\cdot,\cdot)$ for all $g \in \{A,B\}$ and $c \in \{1,2\}$, equilibrium is a set of relative prices $\frac{p^A}{p^B}$, quantities produced (per worker) $\{Q_c^g\}$, and equilibrium quantities consumed (per worker) $\{C_c^g\}$ for all $g \in \{A,B\}$ and $c \in \{1,2\}$ such that (1) given relative prices,* ***in each country*** *producers choose quantities produced to maximize profits; (2) given prices and incomes, consumers* ***in each country*** *choose quantities consumed to maximize utility; and (3) markets clear, i.e. the quantity produced of each good* ***in the world*** *is equal to the quantity consumed of each good* ***in the world****.*

   (b) Suppose that $\alpha_1^A > \alpha_2^A$ and $\alpha_1^B > \alpha_2^B$ (i.e. that country 2 is more efficient at producing both goods). In equilibrium, will it be the case that $C_1^A > Q_1^A$, $C_1^A < Q_1^A$, $C_1^A = Q_1^A$, or not enough information?

      - *This is not enough information, as it is the relative productivities of each country that determines the pattern of trade.*

   (c) Suppose that $\frac{\alpha_1^A}{\alpha_1^B} > \frac{\alpha_2^A}{\alpha_2^B}$. In equilibrium, will it be the case that $C_1^A > Q_1^A$, $C_1^A < Q_1^A$, $C_1^A = Q_1^A$, or not enough information?

      - *In this case, country 2 is relatively more productive at producing good A versus good B than country 1. As a result, country 2 will specialize in the production of good A and country 1 will specialize in the production of good B. This implies that $C_1^A > Q_1^A$, i.e. country 1 will import good A.*

(d) Continue to assume that $\frac{\alpha_1^A}{\alpha_1^B} > \frac{\alpha_B^A}{\alpha_B^B}$. In addition, suppose that $U\left(C_i^A, C_i^B\right) = \beta_1 \ln\left(C_i^A\right) + \beta_2 \ln\left(C_i^B\right)$, where $\beta_1 > 0$ and $\beta_2 > 0$ and $i \in \{1, 2\}$.

i. *Suppose that in equilibrium, one country completely specializes in the production of a good, while the other produces some of each good. Which country will be specializing in what good? What will the equilibrium relative price be? Find the equilibrium consumption and production of both goods in both countries.*

- *Country 1 has a comparative advantage in good B and country 2 has a comparative advantage in good A, so country 1 will specialize (and export) good B and country 2 will specialize (and export) good A. If country 1 completely specializes in the production of good B and country 2 workers produce both goods A and B, then the world relative price has to ensure workers in country 2 are willing to produce both goods, i.e.* $\frac{p^A}{\alpha_2^A} = \frac{p^B}{\alpha_2^B} \iff \frac{p^A}{p^B} = \frac{\alpha_2^A}{\alpha_2^B}$. *Conversely, if country 2 completely specializes in the production of good A and country 1 workers produce both goods, then the world relative price has to ensure workers in country 1 are willing to produce both goods, i.e.* $\frac{p^A}{\alpha_1^A} = \frac{p^B}{\alpha_1^B} \iff \frac{p^A}{p^B} = \frac{\alpha_1^A}{\alpha_1^B}$.
- Let us consider the case where country 1 completely specializes in the production of good B and country 2 produces both goods A and B (the other case is similar). Let us normalize the price of good B to one, so that $\frac{p^A}{p^B} = p = \frac{\alpha_2^A}{\alpha_2^B}$. Given the patterns of specialization, the income in country 1 is $Y_1 = \frac{L_1}{\alpha_1^B}$ and the income in country 2 is $Y_2 = \frac{L_2}{\alpha_2^B}$. Given the Cobb-Douglas preferences, country 1 will consume $C_1^A = \frac{\beta_1}{\beta_1+\beta_2}\frac{Y_1}{p} = \frac{\beta_1}{\beta_1+\beta_2}\frac{L_1}{\alpha_1^B}\frac{\alpha_2^B}{\alpha_2^A}$ and $C_1^B = \frac{\beta_2}{\beta_1+\beta_2}Y_1 = \frac{\beta_2}{\beta_1+\beta_2}\frac{L_1}{\alpha_1^B}$. Similarly, country 2 will consume $C_2^A = \frac{\beta_1}{\beta_1+\beta_2}\frac{Y_2}{p} = \frac{\beta_1}{\beta_1+\beta_2}\frac{L_2}{\alpha_2^A}$ and $C_2^B = \frac{\beta_2}{\beta_1+\beta_2}Y_2 = \frac{\beta_2}{\beta_1+\beta_2}\frac{L_2}{\alpha_2^B}$. This implies that the total world demand for good A is $C_1^A + C_2^A = \frac{\beta_1}{\beta_1+\beta_2}\frac{L_1}{\alpha_1^B}\frac{\alpha_2^B}{\alpha_2^A} + \frac{\beta_1}{\beta_1+\beta_2}\frac{L_2}{\alpha_2^A} = \frac{\beta_1}{\beta_1+\beta_2}\frac{1}{\alpha_2^A}\left(L_1\frac{\alpha_2^B}{\alpha_1^B} + L_2\right)$. Since country 2 is the only country producing good A, it must be that $Q_2^A = \frac{\beta_1}{\beta_1+\beta_2}\frac{1}{\alpha_2^A}\left(L_1\frac{\alpha_2^B}{\alpha_1^B} + L_2\right)$ (and $Q_1^A = 0$). Country 1 only produces good B, so $Q_1^B = \frac{L_1}{\alpha_1^B}$. For markets to clear, it must be that $Q_2^B = C_1^B + C_2^B - Q_1^B = \frac{\beta_2}{\beta_1+\beta_2}\frac{L_1}{\alpha_1^B} + \frac{\beta_2}{\beta_1+\beta_2}\frac{L_2}{\alpha_2^B} - \frac{L_1}{\alpha_1^B} = \frac{\beta_2}{\beta_1+\beta_2}\frac{L_2}{\alpha_2^B} - \frac{\beta_1}{\beta_1+\beta_2}\frac{L_1}{\alpha_1^B}$.

ii. *Suppose that in equilibrium, both countries completely specialize in the production of a single good. How much of each good will be produced in the world? Find the equilibrium relative price and the equilibrium consumption of both goods in both countries.*

- *If both countries completely specialize (country 1 only produces good B and country 2 only produces good A), then we can immediately figure out the total world production of each good:*

$$Q_{world}^A = \frac{L_2}{\alpha_2^A} \text{ and } Q_{world}^B = \frac{L_1}{\alpha_1^B}.$$

- *Moreover, we can figure out the total income of each country as a function of the relative price. Let us normalize the price of good A to 1, so that the price of good B is $\frac{p^B}{p^A}$.*

$$Y_1 = w_1 \times L_1 = \frac{L_1}{\alpha_1^B}$$
$$Y_2 = w_2 \times L_2 = \frac{p^B}{p^A} \times \frac{L_2}{\alpha_2^A}$$

- *Since the preferences are the same as the previous question, we know that the total quantity consumed in each country is:*

$$C_c^A = \frac{\beta_1}{\beta_1+\beta_2} \times w_c \times L_c = \frac{\beta_1}{\beta_1+\beta_2} \times Y_c$$
$$C_c^B = \frac{\beta_2}{\beta_1+\beta_2} \times \frac{w_c \times L_c}{p^B/p^A} = \frac{\beta_2}{\beta_1+\beta_2} \times \frac{Y_c}{p^B/p^A}$$

for $c \in \{1,2\}$. (The fact that the consumption of both goods scales proportionally with income is because Cobb-Douglas preferences are "homothetic.")

- *This means that the total world consumption of each good can be written as:*

$$C^A_{world} = C^A_1 + C^A_2 = \frac{\beta_1}{\beta_1+\beta_2} \times (Y_1 + Y_2)$$
$$C^B_{world} = C^B_1 + C^B_2 = \frac{\beta_2}{\beta_1+\beta_2} \times (\frac{Y_1+Y_2}{p^B/p^A})$$

Substituting in the expressions for income yields:

$$C^A_{world} = C^A_1 + C^A_2 = \frac{\beta_1}{\beta_1+\beta_2} \times (\frac{L_1}{\alpha^B_1} + \frac{p^B}{p^A} \times \frac{L_2}{\alpha^A_2})$$
$$C^B_{world} = C^B_1 + C^B_2 = \frac{\beta_2}{\beta_1+\beta_2} \times (\frac{L_1}{\alpha^B_1} / \frac{p^B}{p^A} + \frac{L_2}{\alpha^A_2})$$

- *From equilibrium condition #3, we can then calculate the relative world prices from either good (we'll choose good A):*

$$\begin{aligned} Q^A_{world} &= C^A_{world} \iff \\ \frac{L_2}{\alpha^A_2} &= \frac{\beta_1}{\beta_1+\beta_2} \times (\frac{L_1}{\alpha^B_1} + \frac{p^B}{p^A} \times \frac{L_2}{\alpha^B_2}) \iff \\ \frac{p^B}{p^A} &= \left(\frac{\beta_1+\beta_2}{\beta_1}\right) \frac{L_2}{\alpha^A_2} - \frac{L_1}{\alpha^B_1} \times \frac{\alpha^2_B}{L_2} \end{aligned} \tag{6}$$

- *With relative prices, we can then calculate the consumption in both countries:*

$$C^A_1 = \frac{\beta_1}{\beta_1+\beta_2} \times \frac{L_1}{\alpha^B_1};\ C^B_1 = \frac{\beta_2}{\beta_1+\beta_2} \times \frac{L_1}{\alpha^B_1}$$
$$C^A_2 = \frac{\beta_1}{\beta_1+\beta_2} \times \frac{p^B}{p^A} \times \frac{L_2}{\alpha^B_2};\ C^B_2 = \frac{\beta_2}{\beta_1+\beta_2} \times \frac{p^B}{p^A} \times \frac{L_2}{\alpha^B_2},$$

where $\frac{p^B}{p^A}$ is defined in equation (6) (and is sort of messy).