Macroeconomics A, EI056

Class 5

Intertemporal optimization: the Ramsey model

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What you will get from today class

- The standard intertemporal 'Ramsey' model of consumption (drawing on parts of Romer chapter 2).
  - SAVING
  - Consumption-investment **optimization**, the **Euler condition**.
- CONS Graphical presentation (phase diagram) of the solution.
  - INVEST **Control** variables and **state** variables. GIVEN EXOG. ENDOG; GIVEN K_t
- Overview of the steps and key ideas in computing an analytical solution.
  - Emphasis on understanding the logic and intuition.
- Extra slides at the end with a richer view of **investment** with adjustment costs (parts of Romer ch. 9).
- Technical appendix on Moodle also includes continuous time version, and matrix method.

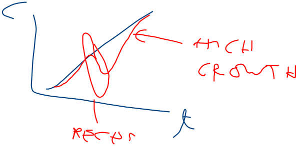
A question to start

$$\frac{C_{t+1}}{C_t} = 1 + \frac{C_{t+1} - C_t}{C_t}$$

FOLLOWED

A higher interest rate implies that households delay consumption. Periods of high interest rates should thus be associated with higher growth. This could imply that our view that high interest rates lead to recessions is wrong.

C_t



$i \Rightarrow \uparrow$ (SAVING)

DISCOUNT RATE $\rightarrow$ i vs DISCOUNT

Do you agree? Why or why not?

Focus of the model

- Understand the **intertemporal allocation** of resource, i.e. the consumption-saving decision.
- Dynamic relations between so-called **state** and **control** variables.
- Model is kept as simple as possible: labor is exogenous, no uncertainty or shocks (these are introduced next week).
- Pedagogical note.
  - Romer chapter is in continuous time (technical aspects can be hard to absorb).
  - Slides in **discrete** time, most standard in macro models (two key relations also presented in continuous time form for completeness).
  - Extra on Moodle: technical appendix with continuous time steps, excel illustration, Matlab code (basis for the analysis of more complex dynamic models,

HOUSEHOLD'S CHOICE

Intertemporal utility (1)

- Representative household. Population (and labor) is L_t growing at a rate n.
- Per-capita consumption is written as c_t. $\frac{C_t}{L_t}$
- Objective: maximize the intertemporal utility of consumption:

$$U_0 = \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} \frac{(c_t)^{1-\theta}}{1-\theta} \overset{?\,(1+n)^t L_0}{L_t} = \sum_{t=0}^{\infty} \left(\frac{1+n}{1+\rho} \right)^t \frac{(c_t)^{1-\theta}}{1-\theta} L_0$$

<1

- Let's deepen three points: time preference, specific consumption, and form of utility.

INPATIENCE

- $\rho > 0$: rate of **time preference**, the "interest rate" at which the household discounts the future.

4 ?.

  - Alternative notations: $\beta = 1/(1+\rho)$, or $e^{-\rho t}$ in Romer. Exact expression does not matter, as $e^{-\rho t} \simeq 1/(1+\rho)^t$ (when ρ is small).
  - Assume $\rho > n$ so $(1+n)/(1+\rho) < 1$, otherwise the sum is explosive.

Intertemporal utility (2)

- Utility of consumption: **concave function of per capita** consumption c_t, times population size.

$$U_0 = \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} \frac{(c_t)^{1-\theta}}{1-\theta} L_t = \sum_{t=0}^{\infty} \left(\frac{1+n}{1+\rho} \right)^t \frac{(c_t)^{1-\theta}}{1-\theta} L_0$$

- L_t is in as we put more weight on periods when there are more people.
  - L_t multiplies $(c_t)^{1-\theta}$: the utility of consumption is about what each persons consumes.
  - We do not use $(L_t c_t)^{1-\theta}$, as then the sheer size of the economy would affect how consumption is perceived.
  - $L_t (c_t)^{1-\theta}$ ensures that the economy does not behave differently when there are more people.

Utility of consumption

EPSTEIN-ZIN

- Mathematical form of utility: so-called constant relative risk aversion **CRRA utility**:

RRA = $-\frac{u''c}{u'} = \theta$

$$u(c_t) = \frac{(c_t)^{1-\theta}}{1-\theta}$$

- θ measures the **concavity** of the utility ($\theta = 1$ corresponds to log utility: $u(c_t) = \ln(c_t)$):

$$u'(c_t) = (c_t)^{-\theta} > 0 \quad ; \quad u''(c_t) = -\theta(c_t)^{-\theta-1} < 0 \quad ;$$

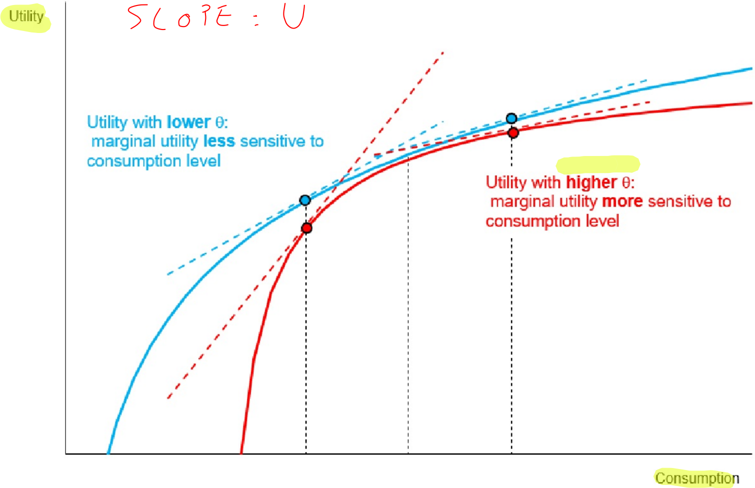
- With **higher** θ, the utility is more concave: **marginal utility is more sensitive** to the level of consumption.

Shape of the utility

- The higher θ, the more concave the utility: marginal utility more sensitive to the level of consumption.

U

SLOPE : U'



Budget constraint

- The household earns a wage w_t on labor, and supplies capital to firms, earning a rental rate r_t.
- The income is used to consume and accumulate capital (with depreciation δ):

$$w_t L_t + r_t K_t = C_t + \underbrace{\overbrace{K_{t+1} - K_t}^{\text{NET INVST}} + \delta K_t}_{\text{GROSS I}}$$

- Capital dynamics: relation between aggregate and per capita:

$$\frac{K_{t+1}}{L_{t+1}} \leftarrow \qquad k_{t+1} - k_t = \frac{1}{1+n}\left(\frac{K_{t+1} - K_t}{L_t} - k_t n\right)$$

- Budget constraint in per capita terms (as last week):

FLOW

$$w_t + r_t k_t = c_t + (1+n)(k_{t+1} - k_t) + (n + \delta) k_t$$

Optimization

- Household's optimization uses the **Lagrangian** (λ_t : shadow value of output):

$$\begin{aligned}\mathcal{L} &= L_0 \sum_{t=0}^{\infty} \left(\frac{1+n}{1+\rho}\right)^t \frac{(c_t)^{1-\theta}}{1-\theta} \\ &\quad + L_0 \sum_{t=0}^{\infty} \left(\frac{1+n}{1+\rho}\right)^t \lambda_t \left[\begin{array}{c} w_t + r_t k_t - c_t \\ -(1+n)(k_{t+1} - k_t) - (n+\delta) k_t \end{array} \right]\end{aligned}$$

- First-order condition with respect to c_t: marginal utility equal to the shadow value of output:

$$(c_t)^{-\theta} = \lambda_t$$

- First-order condition with respect to k_{t+1}: link of shadow values across time:

$$\lambda_t = \lambda_{t+1} \frac{1 + r_{t+1} - \delta}{1+\rho}$$

The Euler condition

- Combine both conditions to get a relation between the dynamics of consumption and the interest rate:

$\ell(1+\rho) \approx \rho$

growth rate

$$\frac{c_{t+1}}{c_t} = \left(\frac{1+r_{t+1}-\delta}{1+\rho}\right)^{\frac{1}{\theta}}$$

$$\ln\left(\frac{c_{t+1}}{c_t}\right) = \frac{1}{\theta}\left[\ln\left(1+r_{t+1}-\delta\right)-\ln\left(1+\rho\right)\right] \simeq \frac{1}{\theta}\left(r_{t+1}-(\delta+\rho)\right)$$

- c_{t+1}/c_t: relative demand for consumption across time.
- $r_{t+1}-(\delta+\rho)$: intertemporal price of consumption.
  - How much future consumption does one get by renouncing to one unit of consumption today. Includes the cost of waiting ρ.
- $1/\theta$: price elasticity of demand. More concave utility (higher θ) leads to a demand less sensitive to price.

FIRM'S CHOICE

Profit maximization

- Representative firm produces using capital and labor (for simplicity, abstract from productivity):

$$Y_t = F(K_t, L_t) \Rightarrow y_t = f(k_t)$$

- It maximizes profits: REAL

$$\Pi_t = F(K_t, L_t) - w_t L_t - r_t K_t = L_t [f(k_t) - w_t - r_t k_t]$$

- Equalize **marginal productivity** and **cost** for each factor:

$$r_t = f'(k_t) \qquad ; \qquad w_t = f(k_t) - k_t f'(k_t)$$

- Sum of factor payments adds up to GDP (because of constant returns to scale): $w_t + r_t k_t = f(k_t)$.

Country-level resource constraint

- GDP adds up to labor and capital payment, the household's constraint becomes:

$$y_t = f(k_t) = c_t + (1+n)(k_{t+1} - k_t) + (n+\delta) k_t$$

- Re-arrange to get the **dynamics of capital**:

$$k_{t+1} - k_t = \frac{1}{1+n}\left[f(k_t) - c_t - (n+\delta) k_t\right]$$

- **System in two equations**: the dynamics of capital and the **Euler** condition (real interest rate equal to the marginal product of capital):

$$\frac{c_{t+1}}{c_t} = \left(\frac{1 + f'(k_{t+1}) - \delta}{1+\rho}\right)^{\frac{1}{\theta}}$$

Continuous-time representation

- Utility of the household:

$$U_0 = L_0 \int_0^{\infty} e^{-(\rho - n)t} \frac{(c_t)^{1-\theta}}{1-\theta} dt$$

- Household's constraint: $c_t + \dot{k}_t + (n+\delta) k_t = w_t + r_t k_t$. Optimization leads to the Euler condition:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta} (r_t - (\delta + \rho)) = \frac{1}{\theta} \left(f'(k_t) - (\delta + \rho) \right)$$

- Firm's optimality conditions identical discrete time. Overall resource constraint is:

$$\dot{k}_t = f(k_t) - (n+\delta) k_t - c_t$$

- Dynamics system in two variables, c and k, and two equations, the Euler and the resource constraint.

SOLVING THE DYNAMIC SYSTEM

Dynamics of consumption

- Analysis in a phase diagram with consumption and capital.
- What combinations of consumption and capital does it take for a variable (c or k) to be constant?
  - What happens if we are not in such combinations?
- **Constant consumption** from Euler: capital stock is at the level $f'(k) = \delta + \rho$:

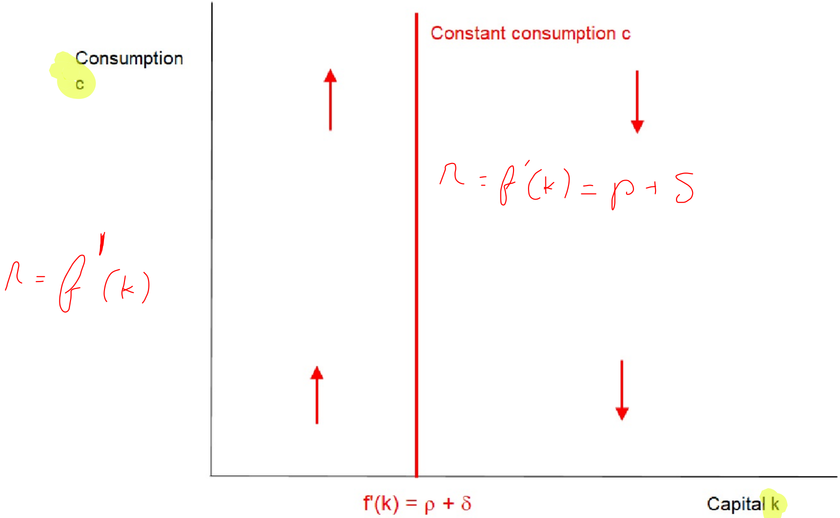
$$\dot{c}_t = 0 \Rightarrow f'(k_t) = \delta + \rho$$

← X c

- This gives a vertical line in a capital-consumption space (c_t does not enter the relation).
- Consumption increases if $f'(k_t) > \delta + \rho$, i.e. the real interest rate is high.
  - Corresponds to a low value of capital, hence to the left of the vertical line.

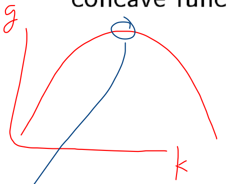
Diagram of consumption dynamics

- When capital is high, the real interest rate is low (saving is not rewarded), consumption is front loaded and then decreases.



Dynamics of capital

- Constant capital ($\dot{k}_t = 0$) implies (from the resource constraint) a concave function of consumption as a function of capital:

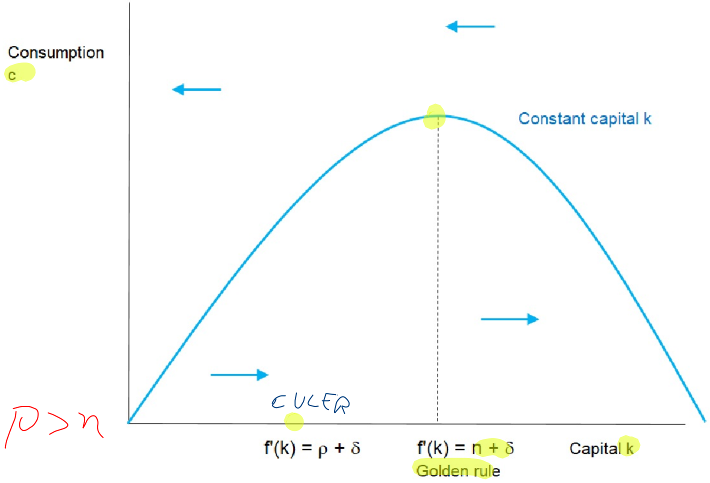


$$
\begin{aligned}
c_t &= g(k_t) = f(k_t) - (n+\delta)k_t \\
g'(k_t) &= f'(k_t) - (n+\delta) \\
g''(k_t) &= f''(k_t) < 0
\end{aligned}
$$

- Capital increases if $c_t < f(k_t) - (n+\delta)k_t$, as most resources are then available for investment.
- Maximum value of the function $g(k_t)$ corresponds to the **Golden rule** (as in the Solow model).
  - It is to the right of the vertical line: $f'\left(k_t^{golden}\right) = n + \delta < \rho + \delta$ (by the assumption $n < \rho$).

Diagram of capital dynamics

- When consumption is high, there is little output left for investment, and capital decreases.



Steady state and overall dynamics

- **Steady state**: capital and consumption are constant:

EULER →

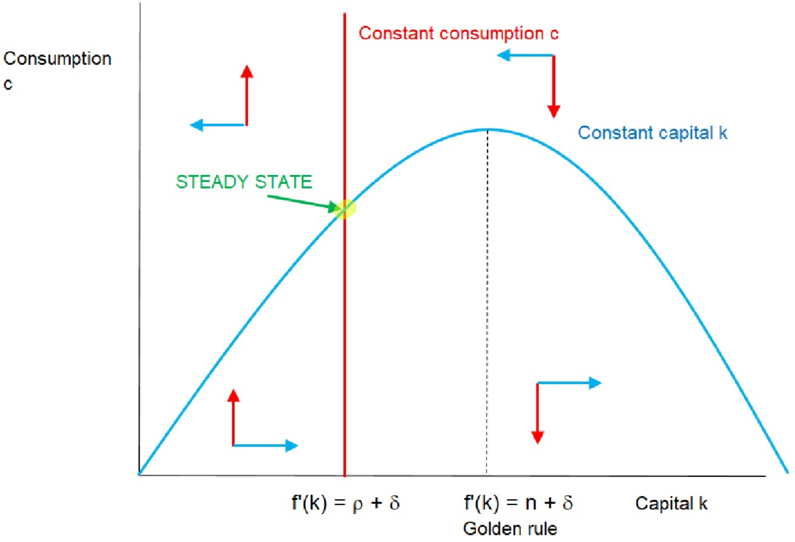
$$
\begin{aligned}
k^* &= \left(f'\right)^{-1}(\rho + \delta) \\
c^* &= f(k^*) - k^*(n + \delta)
\end{aligned}
$$

RESS CST

- Consumption per capita is not maximized.
  - Constant consumption implies $f'(k^*) = \rho + \delta > n + \delta$.
  - Intuition: the choice includes time preference. Prefer to consume a bit less today than the maximal consumption tomorrow.
- Dynamics of consumption and capital from a **phase diagram**.
  - From (nearly) any starting point the economy moves away from the steady state.
  - **Unless** it is exactly on a specific line that ensures convergence.
  - This line is the **saddle path**. The economy will always be on it.

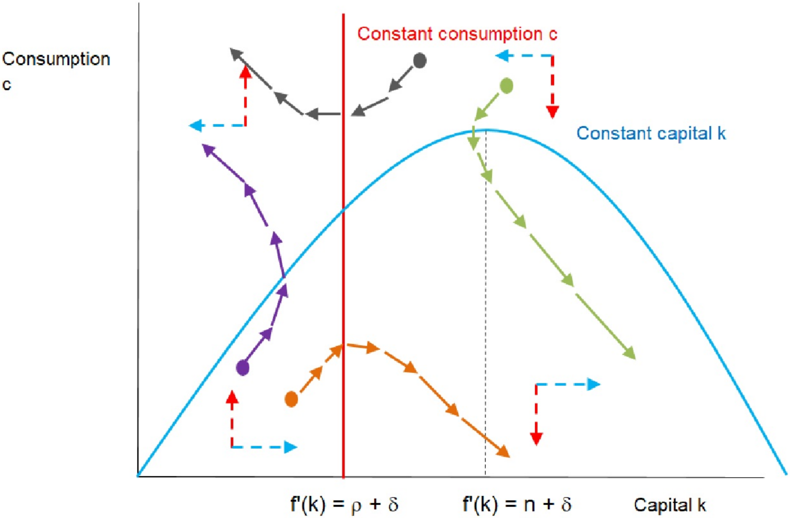
Overall dynamics

- Consumption is not maximized at the steady state.



Divergent paths

- From most starting points, we move away from the steady state.



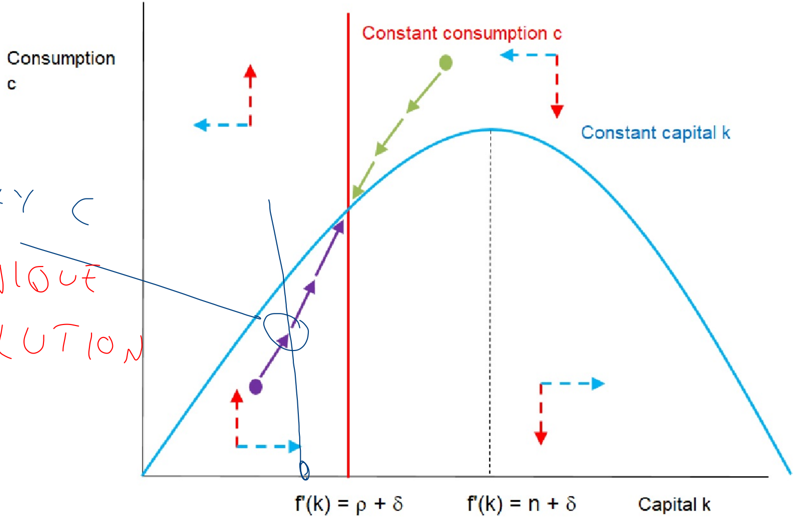
Saddle path

- Convergence is possible only when we start on a specific line.

ONLY C

UNIQUE

SOLUTION



State and control variables

- How do we know we are on the saddle path? Because the agent **chooses** to be on it.
  - Any other choice is irrational.
- Distinguish between **state** and **control** variables.
  - State variables given at the time of optimization: exogenous (productivity) or endogenous variables that are preset (capital).
  - Control variables freely chosen (subject to resource constraints).
- Given the state variables, the only **optimal choice** of control variable is the value of the saddle path.
  - Another choice would violate optimization (consumption or output goes to infinity).
  - This ensures a **unique solution**: for each value of k_t there is only one rational value of c_t.

ANALYTICAL SOLUTION

Solving by approximations

- No general closed-form solution. Use **log-linear approximations** around the steady state.
- Take a Cobb-Douglas production function: $y_t = (k_t)^\alpha$ where $\alpha < 1$.
- Euler condition and capital dynamics are:

$$\begin{aligned} \frac{c_{t+1}}{c_t} &= \left(\frac{1 + \alpha (k_{t+1})^{\alpha - 1} - \delta}{1 + \rho} \right)^{\frac{1}{\theta}} \\ c_t &= (k_t)^\alpha - (n + \delta) k_t - (1 + n)(k_{t+1} - k_t) \end{aligned}$$

- Solve for the steady state:

$$\begin{aligned} k^* &= \left(\frac{\alpha}{r^*} \right)^{\frac{1}{1-\alpha}} \quad ; \quad r^* = \rho + \delta \\ c^* &= \left[\frac{r^*}{\alpha} - (n + \delta) \right] \left(\frac{\alpha}{r^*} \right)^{\frac{1}{1-\alpha}} \end{aligned}$$

Linearization

- Denote $\hat{x}_t = (x_t - x^*) / x^*$, **percent deviations** from the steady-state.
- The system is approximated as:

$$\begin{aligned}
\hat{c}_{t+1} - \hat{c}_t &= -\frac{1}{\theta} \frac{r^*}{1 + r^* - \delta} (1 - \alpha) \hat{k}_{t+1} \\
\hat{k}_{t+1} &= \frac{1 + r^* - \delta}{1 + n} \hat{k}_t - \frac{1}{\alpha} \frac{r^* - \alpha (n + \delta)}{1 + n} \hat{c}_t
\end{aligned}$$

- Two ways to solve for the system:
  - Undetermined coefficients. Explicit solution, but feasible only for small models (in the technical short problems).
  - Blanchard-Kahn matrix computation (presented in technical appendix).

- Consumption and future capital (controls variables)are proportional to the current capital (state variable):

$\widehat{inv} = \eta \hat{k}_t$

$$\hat{c}_t = \eta_{ck} \hat{k}_t \qquad ; \qquad \hat{k}_{t+1} = \eta_{kk} \hat{k}_t$$

- η_{ck} and η_{kk}: **undetermined coefficients** that we need to solve for.
- Resource constraint gives η_{ck} as a linear function of η_{kk}:

$\eta_{kk} = f(\eta_{ck})$

$$\underbrace{\eta_{kk} \cancel{\hat{k}_t}}_{\hat{k}_{t+1}} = \frac{1 + r^* - \delta}{1 + n} \cancel{\hat{k}_t} - \frac{1}{\alpha} \frac{r^* - \alpha(n + \delta)}{1 + n} \underbrace{\eta_{ck} \cancel{\hat{k}_t}}_{\hat{c}_t}$$

- Euler condition gives a **quadratic** solution for η_{kk} (with two solutions).

$$\underbrace{\eta_{ck} \hat{k}_{t+1}}_{\hat{c}_{t+1}} - \underbrace{\eta_{ck} \hat{k}_t}_{\hat{c}_t} = -\frac{1}{\theta} \frac{r^*}{1 + r^* - \delta} (1 - \alpha) \hat{k}_{t+1}$$

$(\eta_{kk})^2$

$$\eta_{ck} \eta_{kk} \cancel{\hat{k}_t} - \eta_{ck} \cancel{\hat{k}_t} = -\frac{1}{\theta} \frac{r^*}{1 + r^* - \delta} (1 - \alpha) \eta_{kk} \cancel{\hat{k}_t}$$

Which solutions to choose?

QUADRATIC

- The quadratic solution is a polynomial in η_{kk}:

$$0 = Pol(\eta_{kk})$$

- Take the value of the polynomial for specific value of η_{kk}, for which we easily get the sign of $Pol(\eta_{kk})$:

$k_{t+1} = \eta_{kk} k_t$

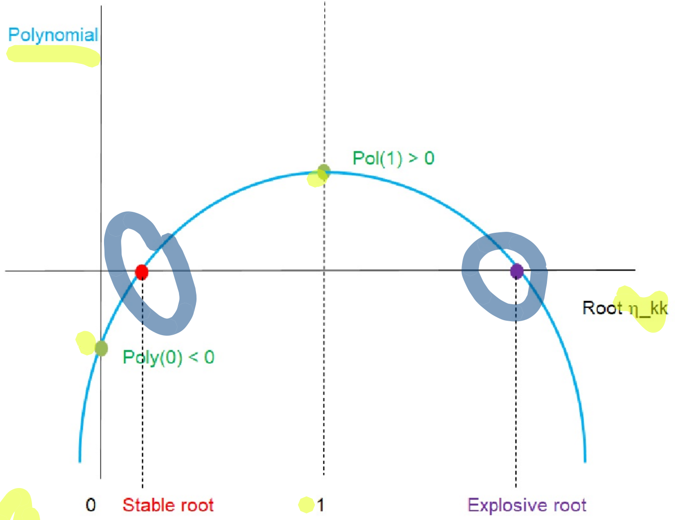
$\eta_{kk} \rightarrow -\infty$	implies	$Pol(\eta_{kk}) = -\infty < 0$
$\eta_{kk} = 0$	implies	$Pol(\eta_{kk}) < 0$
$\eta_{kk} = 1$	implies	$Pol(\eta_{kk}) > 0$
$\eta_{kk} \rightarrow +\infty$	implies	$Pol(\eta_{kk}) = -\infty < 0$

- First solution is $\eta_{kk} > 1$.
    - If capital is not at the steady state, capital and consumption have explosive paths.
    - Not on saddle path, hence not consistent with rationality.
- Correct solution is $0 < \eta_{kk} < 1$.
    - If capital is not at the steady state, we converge to it.
    - Corresponds to the saddle path, makes sense in economic terms.

Drawing the polynomial

- Allows to see where the roots lie.

POL(y_{kk})



Numerical illustration

- Calibration: $\alpha = 1/3$, $\theta = 1$, $n = 2\%$, $\rho = 3.5\%$, $\delta = 1.5\%$. It implies $r^* = 5\%$, $c^*/y^* = 77\%$, $\eta_{kk} = 0.948$, $\eta_{ck} = 0.59$.
- An exogenous drop of capital reduces consumption on impact, and then consumption and capital gradually increase back.

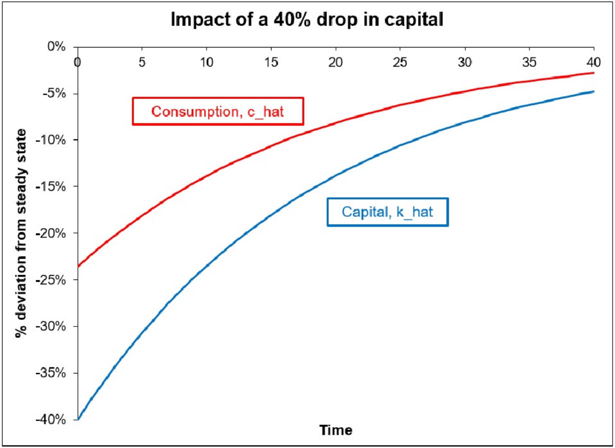
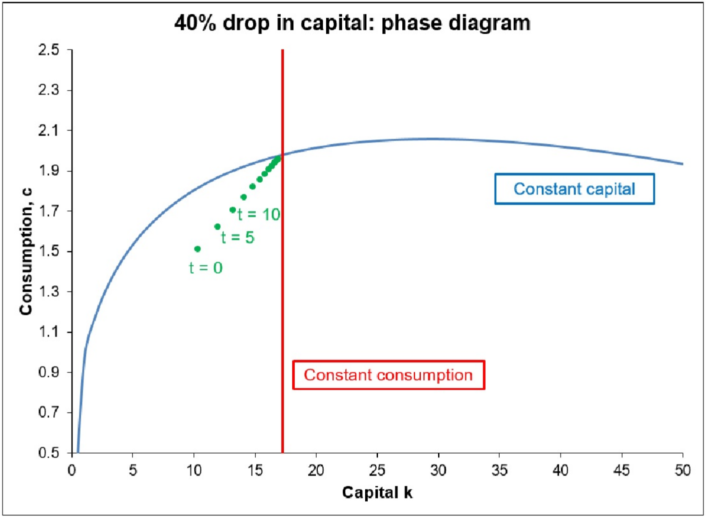


Illustration on the phase diagram

- Path of adjustment in the c, k space.



To sum up

- Macro dynamic models combine **decisions rules** (Euler) and **constraints**.
- Non-linear, so solve using **log linear approximations** around the steady state (locally valid solution).
- Express as **control** variables being functions of **state** variables ("state space" representation).
- Unique solution that is consistent with economic rationality.

ADDITIONAL SLIDES :

COSTLY CAPITAL ADJUSTMENT

Model with two goods

- The model has two goods:
  - **Consumption** good, consumed or set aside.
  - **Capital** good (input). Investment transform the consumption good set aside into the capital good (and conversely).
  - Think of capital as an asset, with a relative price (price of capital relative to consumption (real capital price).
- In the class the distinction is no material: the consumption side is immediately transformed into capital.
  - The two goods are identical, and therefore the relative price of capital to consumption is equal to 1.
- Alternative: transforming consumption goods into capital goods is costly, so they are not the same and the **relative price** can differ from unity.

Adjustment costs to capital

- For simplicity: focus on the firm's problem, set $n = \delta = 0$. We can present the solution graphically.
- The firm faces a constant wage w and a constant real interest rate r at which profits are discounted.
- Output is $f(k_t)$. Capital increases with investment, i_t:

$$k_{t+1} = i_t + k_t$$

- The firm faces an **adjustment cost** $C(i_t)$ when its investment differs from zero:

$$C(0) = 0 \qquad ; \qquad C'(0) = 0 \qquad ; \qquad C'' > 0$$

- The most standard specification is a **quadratic cost**:

$$C(i_t) = \frac{\phi}{2} \frac{(i_t)^2}{k_t}$$

Profits

- Firm's profit at time t are sales, minus the wage bill, minus the cost of investment including adjustment:

$$\Pi_t = f(k_t) - w_t - (i_t + C(i_t))$$

- Lagrangian for the firm's optimization:

$$\begin{aligned} \mathcal{L} &= \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} \left[f(k_t) - w_t - i_t - C(i_t) \right] \\ &\quad + \sum_{t=0}^{\infty} \frac{q_t}{(1+r)^t} \left[i_t - k_{t+1} + k_t \right] \end{aligned}$$

- q_s is the shadow value (in units of profits, i.e. consumption good) of an extra unit of capital. It can be understood as the **relative price** between the capital good and the consumption good.
- The firm chooses k and i to maximize the Lagrangian.

Optimal investment

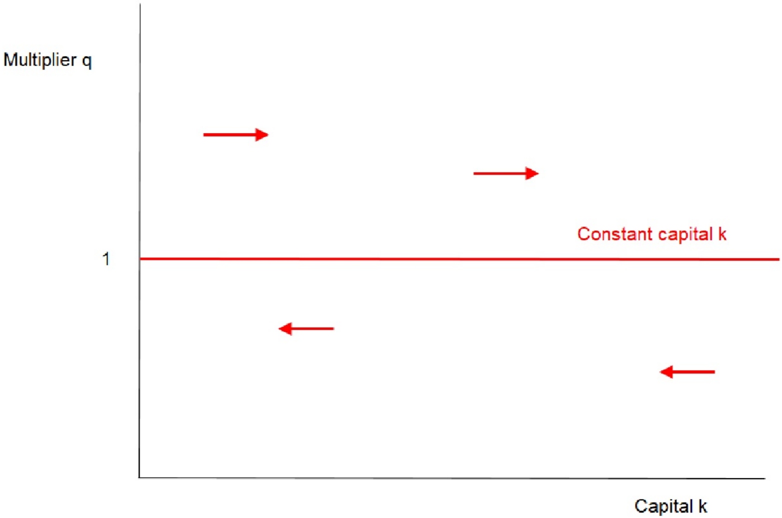
- First-order condition with respect to i_t:

$$q_t - 1 = C'(i_t) = C'(k_{t+1} - k_t)$$

- This gives a relation between k and q.
- Capital is constant $(k_{t+1} - k_t = 0)$ when $q = 1$.
- If $q > 1$ then investment is positive $(k_{t+1} - k_t > 0)$ as capital is valuable, and thus capital increases. q is called Tobin's Q.

Dynamics of capital quantity

- k increases when the relative price q is above 1.



Optimal capital stock

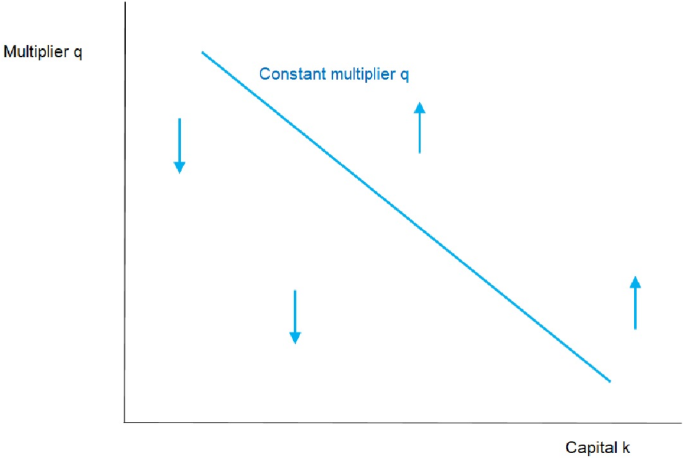
- First-order condition with respect to k_t:

$$rq_t = f'(k_{t+1}) + q_{t+1} - q_t$$

- Two investments are available: capital, or a bond paying the exogenous interest rate r.
  - Arbitrage: the firm is indifferent between the two. Otherwise, it invests only in the most profitable.
  - The return on capital (capital gain + marginal product) is equal to the interest rate times the value of capital.
- If q is constant ($q_{t+1} - q_t = 0$), we have a negative relation between q and k. A value of k_{t+1} above that relation implies a higher value of q_{t+1}, hence an increasing q.
- The steady state is given by the intersection of the two lines, and there is a unique saddle path leading to it.

Dynamics of capital price

- q increases when the capital quantity is high: a low k gives a low marginal return, so capital can deliver the overall return r (as the bond) only with a valuation gain (an increase in its price).



Phase diagram

- he steady state is given by the intersection of the two lines, and there is a unique saddle path leading to it.