Macroeconomics A; EI056

Short problems

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1 Adverse selection

1.1 Payoff of borrower

Question: Consider a borrower getting a loan from a lender of an amount L and interest rate r_l.

The borrower invests in a project that delivers $R' + x$ with 50% probability and $R' - x$ with 50% probability.

We assume that $R' - x < (1 + r_l) L$, so in case of bad payoff the borrower just defaults and pays nothing.

What is the expected return for the borrower? How does it depend on the risk x? What is the intuition?

Answer: If the project works, the borrower pays back the loan and keeps the difference. Otherwise, he defaults and get nothing.

The expected payoff is then:

$$E\pi^B(x) = \frac{1}{2}\left(R' + x - (1 + r_l) L\right)$$

This is clearly increasing in x. Intuitively, a higher x (more risk) makes the bad payoff really low (but the borrower does not care about that) and the high payoff really high. In case of success the borrower thus get a higher $R' + x - (1 + r_l) L$, which is not offset by a worse situation in case of failure as there is then default.

1.2 Payoff of lender

Question: The lender gets the loan paid back, or in case of default gets the return on the project $R' - x$.

What is the expected return for the lender? How does it depend on the risk x? What is the intuition?

Answer: The expected payoff for the lender is:

$$E\pi^L(x) = \frac{1}{2}(1 + r_l) L + \frac{1}{2}(R' - x)$$

This is clearly decreasing in x. Intuitively, a higher x (more risk) implies that the lender gets only a little amount in case of default, but does not get more in case of success, as the borrower then simply repays the loan.

1.3 Heterogeneous borrowers

Question: The population is made of 50% of risky borrowers whose projects have $x = x_b$ and 50% of safe borrowers whose projects have $x = x_g < x_b$.

The lender must charge the same interest rate to all borrowers, as he cannot tell who is risky and who is not.

Show that $r_l > r_l^* = \frac{R'+x_g}{L} - 1$ if the safe borrowers do not take a loan.

What is the expected payoff for the risky borrower if $r_l = r_l^*$?

Answer: The expected payoff for the safe borrower is:

$$E\pi^B(x_g) = \frac{1}{2}\left(R' + x_g - (1+r_l)L\right)$$

This is positive only if:

$$\begin{aligned}
0 &< R' + x_g - (1+r_l)L \\
(1+r_l)L &< R' + x_g \\
1+r_l &< \frac{R'+x_g}{L} \\
r_l &< \frac{R'+x_g}{L} - 1
\end{aligned}$$

Of the interest rate is higher, the safe borrower does not take a loan.

he expected payoff for the Risky borrower is:

$$\begin{aligned}
E\pi^B(x_b) &= \frac{1}{2}\left(R' + x_b - (1+r_l^*)L\right) \\
E\pi^B(x_b) &= \frac{1}{2}\left(R' + x_g - x_g + x_b - (1+r_l^*)L\right) \\
E\pi^B(x_b) &= \frac{1}{2}\left(R' + x_g - (1+r_l^*)L\right) + \frac{1}{2}(x_b - x_g) \\
E\pi^B(x_b) &= \frac{1}{2}(x_b - x_g) > 0
\end{aligned}$$

It is thus work borrowing for the risky borrower.

1.4 Expected payoff of lender with heterogeneous borrowers

Question: Show that if all potential borrowers take a loan, the expected payoff of the lender:

Show that if only risky borrowers take a loan, the expected payoff of the lender:

What happens to the lender's expected payoff if the interest rate moves from slightly below r_l^* to slightly above r_l^*?

Answer: The lender's expected payoff for individual risky and safe borrowers are:

$$\begin{aligned} E\pi^L(x_g) &= \frac{1}{2}(1+r_l)L + \frac{1}{2}(R'-x_g) \\ E\pi^L(x_b) &= \frac{1}{2}(1+r_l)L + \frac{1}{2}(R'-x_b) \end{aligned}$$

When both types of borrowers are present, the lender gets:

$$\begin{aligned} E\pi^{L,b\&g} &= \frac{1}{2}E\pi^L(x_g) + \frac{1}{2}E\pi^L(x_b) \\ E\pi^{L,b\&g} &= \frac{1}{4}(1+r_l)L + \frac{1}{4}(R'-x_g) + \frac{1}{4}(1+r_l)L + \frac{1}{4}(R'-x_b) \\ E\pi^{L,b\&g} &= \frac{1}{2}(1+r_l)L + \frac{1}{4}(R'-x_b) + \frac{1}{4}(x_b-x_g) + \frac{1}{4}(1+r_l)L + \frac{1}{4}(R'-x_b) \\ E\pi^{L,b\&g} &= \frac{1}{2}(1+r_l)L + \frac{1}{2}(R'-x_b) + \frac{1}{4}(x_b-x_g) \end{aligned}$$

When only risky borrowers are present, the lender gets:

$$\begin{aligned} E\pi^{L,b} &= E\pi^L(x_b) \\ E\pi^{L,b} &= \frac{1}{2}(1+r_l)L + \frac{1}{2}(R'-x_b) \end{aligned}$$

When the interest rate is just below r_l^* both borrowers are present. When the interest rate gets slightly higher than r_l^*, the safe borrowers leave. The change of expected lender's payoff from moving from just below r_l^* to just above r_l^* is:

$$\begin{aligned} E\pi^{L,b} - E\pi^{L,b\&g} &= \left[\frac{1}{2}(1+r_l^*)L + \frac{1}{2}(R'-x_b)\right] - \left[\frac{1}{2}(1+r_l^*)L + \frac{1}{2}(R'-x_b) + \frac{1}{4}(x_b-x_g)\right] \\ E\pi^{L,b} - E\pi^{L,b\&g} &= -\frac{1}{4}(x_b-x_g) < 0 \end{aligned}$$

There is thus a discrete drop in profits for the small movement in interest rate, as it induces the best borrowers to leave.

2 Bank and risk sharing

2.1 Utility under autarky

Question: Consider a two-period model with a unit mass of small agents. Each agent gets 1 unit of a good in period 0. They can consume the unit in period 1, or keep it until period 2 and get $R > 1$ units.

In period 1, each agent learns its type. If impatient, which happens with probability t she get utility only from consumption in period 1. With probability $1-t$ she get utility only from consumption in period 2. Utility is thus:

$$\begin{aligned} &\frac{1}{1-\sigma}(c_1)^{1-\sigma} \text{with probability } t \\ &\frac{1}{1-\sigma}(c_2)^{1-\sigma} \text{with probability } 1-t \end{aligned}$$

What is the expected utility for an agent who does not transact with anyone else?

Answer: Under autarky, the agent consumes $c_1 = 1$ if impatient, and $c_2 = R$ if patient.

The expected utility is:

$$\begin{aligned} & t\frac{1}{1-\sigma}\left(c_1\right)^{1-\sigma}+(1-t)\frac{1}{1-\sigma}\left(c_2\right)^{1-\sigma} \\ = & t\frac{1}{1-\sigma}+(1-t)\frac{1}{1-\sigma}(R)^{1-\sigma} \end{aligned}$$

2.2 Allocation under pooling

Question: Consider that there is a bank. All agents deposit their unit of endowment.

In period 1, an agent can come to the bank and ask for c_1^* units of consumption. Alternatively, she can come to the bank and ask for c_2^* units of consumption.

The budget constraints of the bank are:

$$tc_1^*+s=1 \qquad ; \qquad sR=(1-t)\,c_2^*$$

where s is the amount kept from period 1 to 2, and t is the proportion of agents coming to the bank in the first period.

Show that a bank maximizing welfare chooses:

$$c_1^*=\frac{1}{1-(1-t)\left[1-(R)^{\frac{1-\sigma}{\sigma}}\right]} \qquad ; \qquad c_2^*=\frac{1}{1+t\left[(R)^{\frac{\sigma-1}{\sigma}}-1\right]}R$$

How does this compare to the consumption under autarky (assume $\sigma>1$)?

Answer: The bank maximizes the expected utility subject to the constraint that

$$(1-t)\,c_2^*=R\left(1-tc_1^*\right)$$

The objective is thus:

$$\begin{aligned} & t\frac{1}{1-\sigma}\left(c_1^*\right)^{1-\sigma}+(1-t)\frac{1}{1-\sigma}\left(c_2^*\right)^{1-\sigma} \\ = & t\frac{1}{1-\sigma}\left(c_1^*\right)^{1-\sigma}+(1-t)\frac{1}{1-\sigma}\left(R\frac{1-tc_1^*}{1-t}\right)^{1-\sigma} \end{aligned}$$

The first-order condition is:

$$
\begin{aligned}
0 &= t\left(c_1^*\right)^{-\sigma} + (1-t)\left(R\frac{1-tc_1^*}{1-t}\right)^{-\sigma} R\frac{-t}{1-t} \\
t\left(c_1^*\right)^{-\sigma} &= tR\left(R\frac{1-tc_1^*}{1-t}\right)^{-\sigma} \\
\left(c_1^*\right)^{-\sigma} &= (R)^{1-\sigma}\left(\frac{1-tc_1^*}{1-t}\right)^{-\sigma} \\
\left(c_1^*\right)^{-1} &= (R)^{\frac{1-\sigma}{\sigma}}\left(\frac{1-tc_1^*}{1-t}\right)^{-1} \\
\frac{1-tc_1^*}{1-t} &= (R)^{\frac{1-\sigma}{\sigma}} c_1^* \\
1-tc_1^* &= (1-t)(R)^{\frac{1-\sigma}{\sigma}} c_1^* \\
1 &= \left[t + (1-t)(R)^{\frac{1-\sigma}{\sigma}}\right] c_1^* \\
1 &= \left[t - 1 + 1 + (1-t)(R)^{\frac{1-\sigma}{\sigma}}\right] c_1^* \\
1 &= \left[1 - (1-t) + (1-t)(R)^{\frac{1-\sigma}{\sigma}}\right] c_1^* \\
1 &= \left[1 - (1-t)\left[1 - (R)^{\frac{1-\sigma}{\sigma}}\right]\right] c_1^* \\
c_1^* &= \frac{1}{1 - (1-t)\left[1 - (R)^{\frac{1-\sigma}{\sigma}}\right]}
\end{aligned}
$$

As $\sigma > 1$, we have: $(R)^{\frac{\sigma-1}{\sigma}} > 1$ and $(R)^{\frac{1-\sigma}{\sigma}} = 1/(R)^{\frac{1-\sigma}{\sigma}} < 1$. Therefore $1 - (R)^{\frac{1-\sigma}{\sigma}} > 1$, so the denominator is smaller than 1. Consumption is larger than under autarky for impatient agents: $c_1^* > 1$.

Consumption in the second period is then:

$$\begin{aligned}
c_2^* &= R\frac{1-tc_1^*}{1-t} \\
c_2^* &= R\frac{1}{1-t} - R\frac{t}{1-t}\frac{1}{1-(1-t)\left[1-(R)^{\frac{1-\sigma}{\sigma}}\right]} \\
c_2^* &= \left[\frac{1}{1-t} - \frac{t}{1-t}\frac{1}{1-(1-t)\left[1-(R)^{\frac{1-\sigma}{\sigma}}\right]}\right]R \\
c_2^* &= \left[1-(1-t)\left[1-(R)^{\frac{1-\sigma}{\sigma}}\right]-t\right]\frac{1}{1-t}\frac{1}{1-(1-t)\left[1-(R)^{\frac{1-\sigma}{\sigma}}\right]}R \\
c_2^* &= \left[(1-t)-(1-t)\left[1-(R)^{\frac{1-\sigma}{\sigma}}\right]\right]\frac{1}{1-t}\frac{1}{1-(1-t)\left[1-(R)^{\frac{1-\sigma}{\sigma}}\right]}R \\
c_2^* &= \left[1-\left[1-(R)^{\frac{1-\sigma}{\sigma}}\right]\right]\frac{1}{1-(1-t)\left[1-(R)^{\frac{1-\sigma}{\sigma}}\right]}R \\
c_2^* &= (R)^{\frac{1-\sigma}{\sigma}}\frac{1}{1-(1-t)\left[1-(R)^{\frac{1-\sigma}{\sigma}}\right]}R \\
c_2^* &= \frac{1}{(R)^{\frac{\sigma-1}{\sigma}}-(1-t)\left[(R)^{\frac{\sigma-1}{\sigma}}-(R)^{\frac{\sigma-1}{\sigma}}(R)^{\frac{1-\sigma}{\sigma}}\right]}R \\
c_2^* &= \frac{1}{(R)^{\frac{\sigma-1}{\sigma}}-(1-t)\left[(R)^{\frac{\sigma-1}{\sigma}}-1\right]}R \\
c_2^* &= \frac{1}{(R)^{\frac{\sigma-1}{\sigma}}-\left[(R)^{\frac{\sigma-1}{\sigma}}-1\right]+t\left[(R)^{\frac{\sigma-1}{\sigma}}-1\right]}R \\
c_2^* &= \frac{1}{1+t\left[(R)^{\frac{\sigma-1}{\sigma}}-1\right]}R
\end{aligned}$$

As $(R)^{\frac{\sigma-1}{\sigma}} > 1$ the denominator is larger than 1, and consumption is smaller than under autarky for patient agents: $c_2^* < R$.

2.3 Interpretation

Question: What are the values of c_1^* and c_2^* when $R = 1$?
What are the derivatives of c_1^* and c_2^* with respect to R (evaluate them at $R = 1$)?
Show that $c_2^* > c_1^*$. Hint: think about how an increase in R starting at $R = 1$ affects $c_2^* - c_1^*$.

Answer: If $R = 1$ we have $c_1^* = c_2^* = 1$.

The derivative of c_1^* is:

$$\begin{aligned}
\frac{\partial c_1^*}{\partial R} &= -\frac{1}{\left(1-(1-t)\left[1-(R)^{\frac{1-\sigma}{\sigma}}\right]\right)^2}\left(-(1-t)\left[-\frac{1-\sigma}{\sigma}(R)^{\frac{1-\sigma}{\sigma}-1}\right]\right) \\
\frac{\partial c_1^*}{\partial R} &= -\frac{1}{\left(1-(1-t)\left[1-(R)^{\frac{1-\sigma}{\sigma}}\right]\right)^2}(1-t)\frac{1-\sigma}{\sigma}(R)^{\frac{1-\sigma}{\sigma}-1} \\
\frac{\partial c_1^*}{\partial R} &= \frac{1}{\left(1-(1-t)\left[1-(R)^{\frac{1-\sigma}{\sigma}}\right]\right)^2}(1-t)\frac{\sigma-1}{\sigma}(R)^{\frac{1-\sigma}{\sigma}-1} \\
\frac{\partial c_1^*}{\partial R} &= (1-t)\frac{\sigma-1}{\sigma}>0
\end{aligned}$$

The derivative of c_2^* is:

$$\begin{aligned}
\frac{\partial c_2^*}{\partial R} &= \frac{\left(1+t\left[(R)^{\frac{\sigma-1}{\sigma}}-1\right]\right)-\frac{\sigma-1}{\sigma}Rt(R)^{\frac{\sigma-1}{\sigma}-1}}{\left(1+t\left[(R)^{\frac{\sigma-1}{\sigma}}-1\right]\right)^2} \\
\frac{\partial c_2^*}{\partial R} &= \frac{1-t+t(R)^{\frac{\sigma-1}{\sigma}}-\frac{\sigma-1}{\sigma}t(R)^{\frac{\sigma-1}{\sigma}}}{\left(1+t\left[(R)^{\frac{\sigma-1}{\sigma}}-1\right]\right)^2} \\
\frac{\partial c_2^*}{\partial R} &= \frac{1-t+\frac{1}{\sigma}t(R)^{\frac{\sigma-1}{\sigma}}}{\left(1+t\left[(R)^{\frac{\sigma-1}{\sigma}}-1\right]\right)^2} \\
\frac{\partial c_2^*}{\partial R} &= 1-t+\frac{1}{\sigma}t>0
\end{aligned}$$

Comparing them we get:

$$\begin{aligned}
\frac{\partial\left(c_2^*-c_1^*\right)}{\partial R} &= 1-t+\frac{1}{\sigma}t-(1-t)\frac{\sigma-1}{\sigma} \\
\frac{\partial\left(c_2^*-c_1^*\right)}{\partial R} &= 1-t+\frac{1}{\sigma}t-(1-t)+(1-t)\frac{1}{\sigma} \\
\frac{\partial\left(c_2^*-c_1^*\right)}{\partial R} &= \frac{1}{\sigma}t+(1-t)\frac{1}{\sigma} \\
\frac{\partial\left(c_2^*-c_1^*\right)}{\partial R} &= \frac{1}{\sigma}>0
\end{aligned}$$

An increase in R raises both consumptions, but does so more for consumption in the second period. We therefore get that for $R>1$ consumption is higher in the second period $c_2^*>c_1^*$.

2.4 Interpretation

Question: Intuitively, why is the consumption different under pooling that under autarky?
Would an impatient agent ever lie about who she truly is?
Is it optimal for a patient agent to claim to be impatient?

Answer: The pooling allocation acts as an insurance, making consumptions less extreme:

$$1<c_1^*<c_2^*<R$$

The bank acts as an insturance mechanism.

No impatient agent would ever lie. Claiming to be patient, and getting consumption in period 2 is pointless, as the agent does not care about consuming then.

A patient agent can wait and get c_2^*. She can also claim to be impatient, get c_1^* in period 1, and wait until period 2 to consume. But as $c_2^* > c_1^*$ this is not a good idea.