Macroeconomics A; EI060

Technical appendix: real exchange rate and the terms-of-trade

Cédric Tille

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1 Traded and non-traded goods

1.1 Static consumption allocation

In a given period, the consumer's overall consumption C_t consists of a traded good C_t^T and a non traded good C_t^N:

$$C_t = \left[(\gamma)^\eta \left(C_t^T\right)^{1-\eta} + (1-\gamma)^\eta \left(C_t^N\right)^{1-\eta}\right]^{\frac{1}{1-\eta}}$$

The expenditure is, normalizing the price of the traded goods to 1, so P_t^N is the relative price of the non-traded good:

$$P_t C_t = C_t^T + P_t^N C_t^N$$

The consumer minimizes expenditure subject to a target level of the overall index. The Lagrangian is:

$$\mathcal{L}_t = C_t^T + P_t^N C_t^N + \lambda_t \left[C_t - \left[(\gamma)^\eta \left(C_t^T\right)^{1-\eta} + (1-\gamma)^\eta \left(C_t^N\right)^{1-\eta}\right]^{\frac{1}{1-\eta}}\right]$$

The first-order condition with respect to the consumption of traded and non-trade goods are:

$$\begin{aligned}
0 &= 1-\lambda_t\left[(\gamma)^{\eta}\left(C_t^T\right)^{1-\eta}+(1-\gamma)^{\eta}\left(C_t^N\right)^{1-\eta}\right]^{\frac{1}{1-\eta}-1}(\gamma)^{\eta}\left(C_t^T\right)^{-\eta} \\
&= 1-\lambda_t\left[(\gamma)^{\eta}\left(C_t^T\right)^{1-\eta}+(1-\gamma)^{\eta}\left(C_t^N\right)^{1-\eta}\right]^{\frac{\eta}{1-\eta}}(\gamma)^{\eta}\left(C_t^T\right)^{-\eta} \\
&= 1-\lambda_t\left(C_t\right)^{\eta}(\gamma)^{\eta}\left(C_t^T\right)^{-\eta} \\
0 &= P_t^N-\lambda_t\left[(\gamma)^{\eta}\left(C_t^T\right)^{1-\eta}+(1-\gamma)^{\eta}\left(C_t^N\right)^{1-\eta}\right]^{\frac{1}{1-\eta}-1}(1-\gamma)^{\eta}\left(C_t^N\right)^{-\eta} \\
&= P_t^N-\lambda_t\left(C_t\right)^{\eta}(1-\gamma)^{\eta}\left(C_t^N\right)^{-\eta}
\end{aligned}$$

Multiply these by C_t^T and C_t^N respectively, and add them up;

$$\begin{aligned}
0 &= C_t^T-\lambda_t\left(C_t\right)^{\eta}(\gamma)^{\eta}\left(C_t^T\right)^{1-\eta}+P_t^N C_t^N-\lambda_t\left(C_t\right)^{\eta}(1-\gamma)^{\eta}\left(C_t^N\right)^{1-\eta} \\
C_t^T+P_t^N C_t^N &= \lambda_t\left(C_t\right)^{\eta}(\gamma)^{\eta}\left(C_t^T\right)^{1-\eta}+\lambda_t\left(C_t\right)^{\eta}(1-\gamma)^{\eta}\left(C_t^N\right)^{1-\eta} \\
C_t^T+P_t^N C_t^N &= \lambda_t\left(C_t\right)^{\eta}\left[(\gamma)^{\eta}\left(C_t^T\right)^{1-\eta}+(1-\gamma)^{\eta}\left(C_t^N\right)^{1-\eta}\right] \\
C_t^T+P_t^N C_t^N &= \lambda_t\left(C_t\right)^{\eta}\left(C_t\right)^{1-\eta} \\
P_t C_t &= \lambda_t C_t \\
P_t &= \lambda_t
\end{aligned}$$

Using this, the optimality conditions are:

$$\begin{aligned}
0 &= 1-\lambda_t\left(C_t\right)^{\eta}(\gamma)^{\eta}\left(C_t^T\right)^{-\eta} \\
1 &= P_t\left(C_t\right)^{\eta}(\gamma)^{\eta}\left(C_t^T\right)^{-\eta} \\
\left(C_t^T\right)^{\eta} &= \left[\frac{1}{P_t}\right]^{-1}\left(C_t\right)^{\eta}(\gamma)^{\eta} \\
C_t^T &= \gamma\left[\frac{1}{P_t}\right]^{-\frac{1}{\eta}} C_t
\end{aligned}$$

and:

$$\begin{aligned}
0 &= P_t^N-\lambda_t\left(C_t\right)^{\eta}(1-\gamma)^{\eta}\left(C_t^N\right)^{-\eta} \\
P_t^N &= P_t\left(C_t\right)^{\eta}(1-\gamma)^{\eta}\left(C_t^N\right)^{-\eta} \\
\left(C_t^N\right)^{\eta} &= \left[\frac{P_t^N}{P_t}\right]^{-1}\left(C_t\right)^{\eta}(1-\gamma)^{\eta} \\
C_t^N &= (1-\gamma)\left[\frac{P_t^N}{P_t}\right]^{-\frac{1}{\eta}} C_t
\end{aligned}$$

The price index is obtained from the consumption index:

$$
\begin{aligned}
C_t &= \left[(\gamma)^{\eta}\left(C_t^T\right)^{1-\eta} + (1-\gamma)^{\eta}\left(C_t^N\right)^{1-\eta}\right]^{\frac{1}{1-\eta}} \\
C_t &= \left[(\gamma)^{\eta}\left(\gamma\left[\frac{1}{P_t}\right]^{-\frac{1}{\eta}} C_t\right)^{1-\eta} + (1-\gamma)^{\eta}\left((1-\gamma)\left[\frac{P_t^N}{P_t}\right]^{-\frac{1}{\eta}} C_t\right)^{1-\eta}\right]^{\frac{1}{1-\eta}} \\
C_t &= \left[\gamma\left(\left[\frac{1}{P_t}\right]^{-\frac{1}{\eta}}\right)^{1-\eta}(C_t)^{1-\eta} + (1-\gamma)\left(\left[\frac{P_t^N}{P_t}\right]^{-\frac{1}{\eta}}\right)^{1-\eta}(C_t)^{1-\eta}\right]^{\frac{1}{1-\eta}} \\
C_t &= C_t\left[\gamma\left[\frac{1}{P_t}\right]^{-\frac{1-\eta}{\eta}} + (1-\gamma)\left[\frac{P_t^N}{P_t}\right]^{-\frac{1-\eta}{\eta}}\right]^{\frac{1}{1-\eta}} \\
1 &= \left[\gamma\left[\frac{1}{P_t}\right]^{\frac{\eta-1}{\eta}} + (1-\gamma)\left[\frac{P_t^N}{P_t}\right]^{\frac{\eta-1}{\eta}}\right]^{\frac{1}{1-\eta}} \\
1 &= \gamma\left[\frac{1}{P_t}\right]^{\frac{\eta-1}{\eta}} + (1-\gamma)\left[\frac{P_t^N}{P_t}\right]^{\frac{\eta-1}{\eta}} \\
[P_t]^{\frac{\eta-1}{\eta}} &= \gamma + (1-\gamma)\left[P_t^N\right]^{\frac{\eta-1}{\eta}} \\
P_t &= \left[\gamma + (1-\gamma)\left[P_t^N\right]^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}
\end{aligned}
$$

1.2 Real exchange rate

Ratio of the price level abroad to the domestic price level. Assume that the rest of the world consumes only the traded good:

$$
\begin{aligned}
Q_t &= \frac{P_t^T}{P_t} \\
Q_t &= \frac{P_t^T}{\left[\gamma + (1-\gamma)\left[P_t^N\right]^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}}} \\
Q_t &= \left[\gamma + (1-\gamma)\left[P_t^N\right]^{\frac{\eta-1}{\eta}}\right]^{-\frac{\eta}{\eta-1}}
\end{aligned}
$$

An increase means foreign prices are higher. This is called a real depreciation for the domestic country (link with nominal exchange rate presented below).

An increase in the price of the non-traded good P_t^N reduces $Q_t$and is a real exchange rate appreciation for the country.

1.3 Intertemporal allocation

We consider a two period economy, where the agent maximizes the utility:

$$U_1 = \frac{(C_1)^{1-\sigma}}{1-\sigma} + \beta \frac{(C_2)^{1-\sigma}}{1-\sigma}$$

Output takes the form of endowments, and the agent can invest in a bond denominated in the traded good with a return r. The budget constraints, in terms of traded goods, are:

$$\begin{aligned} B_2 + P_1 C_1 &= Y_1^T + P_1^N Y_1^N \\ P_2 C_2 &= Y_2^T + P_2^N Y_2^N + (1+r) B_2 \end{aligned}$$

The intetemporal constraint is:

$$\begin{aligned} P_2 C_2 &= Y_2^T + P_2^N Y_2^N + (1+r) \left[Y_1^T + P_1^N Y_1^N - P_1 C_1 \right] \\ P_1 C_1 + \frac{P_2 C_2}{1+r} &= Y_1^T + P_1^N Y_1^N + \frac{Y_2^T + P_2^N Y_2^N}{1+r} \end{aligned}$$

The Lagrangian for the optimization is:

$$\mathcal{L}_t = \frac{(C_1)^{1-\sigma}}{1-\sigma} + \beta \frac{(C_2)^{1-\sigma}}{1-\sigma} + \lambda \left[Y_1^T + P_1^N Y_1^N + \frac{Y_2^T + P_2^N Y_2^N}{1+r} - P_1 C_1 - \frac{P_2 C_2}{1+r} \right]$$

The optimality conditions with respect to the two consumptions are:

$$\begin{aligned} 0 &= (C_1)^{-\sigma} - \lambda P_1 \\ 0 &= \beta (C_2)^{-\sigma} - \lambda \frac{P_2}{1+r} \end{aligned}$$

This gives the Euler condition:

$$\begin{aligned} (C_1)^{-\sigma} &= \lambda P_1 \\ (C_1)^{-\sigma} &= \frac{\beta (1+r)}{P_2} (C_2)^{-\sigma} P_1 \\ (C_1)^{-\sigma} &= \beta (1+r) \frac{P_1}{P_2} (C_2)^{-\sigma} \\ (C_1)^{-\sigma} &= \beta \left(1 + r^C\right) (C_2)^{-\sigma} \end{aligned}$$

Where $1+r$ is the real interest rate in terms of the traded good, and $1+r^C$ is the real interest rate in terms of the consumption basket. We can express the Euler in terms of the traded good consumption:

The solution for the consumption is thus:

$$\begin{aligned}
C_2 &= \left[\beta\left(1+r\right)\frac{P_1}{P_2}\right]^{\frac{1}{\sigma}} C_1 \\
\frac{1}{\gamma}\left[\frac{1}{P_2}\right]^{\frac{1}{\eta}} C_2^T &= \left[\beta\left(1+r\right)\frac{P_1}{P_2}\right]^{\frac{1}{\sigma}} \frac{1}{\gamma}\left[\frac{1}{P_1}\right]^{\frac{1}{\eta}} C_1^T \\
C_2^T &= \left[\beta\left(1+r\right)\frac{P_1}{P_2}\right]^{\frac{1}{\sigma}} \left[\frac{P_1}{P_2}\right]^{-\frac{1}{\eta}} C_1^T \\
C_2^T &= \left[\beta\left(1+r\right)\right]^{\frac{1}{\sigma}} \left[\frac{P_1}{P_2}\right]^{\frac{1}{\sigma}-\frac{1}{\eta}} C_1^T
\end{aligned}$$

Note that as the non-traded good is locally produced, $C_t^N = Y_t^N$, the intertemporal budget constraint simplifies to:

$$\begin{aligned}
P_1 C_1 + \frac{P_2 C_2}{1+r} &= Y_1^T + P_1^N Y_1^N + \frac{Y_2^T + P_2^N Y_2^N}{1+r} \\
\left[C_1^T + P_1^N C_1^N\right] + \frac{C_2^T + P_2^N C_2^N}{1+r} &= Y_1^T + P_1^N Y_1^N + \frac{Y_2^T + P_2^N Y_2^N}{1+r} \\
C_1^T + \frac{C_2^T}{1+r} &= Y_1^T + \frac{Y_2^T}{1+r}
\end{aligned}$$

The solution for the initial traded consumption is:

$$\begin{aligned}
C_1^T + \frac{C_2^T}{1+r} &= Y_1^T + \frac{Y_2^T}{1+r} \\
C_1^T + \frac{1}{1+r}\left[\beta\left(1+r\right)\right]^{\frac{1}{\sigma}} \left[\frac{P_1}{P_2}\right]^{\frac{1}{\sigma}-\frac{1}{\eta}} C_1^T &= Y_1^T + \frac{Y_2^T}{1+r} \\
C_1^T &= \frac{1}{1+\left(\beta\right)^{\frac{1}{\sigma}}\left(1+r\right)^{\frac{1-\sigma}{\sigma}}\left[\frac{P_2}{P_1}\right]^{\frac{1}{\eta}-\frac{1}{\sigma}}}\left[Y_1^T + \frac{Y_2^T}{1+r}\right]
\end{aligned}$$

he current account in the first period is the trade balance:

$$\begin{aligned}
CA_1 &= Y_1^T - C_1^T \\
CA_1 &= Y_1^T - \frac{1}{1+\left(\beta\right)^{\frac{1}{\sigma}}\left(1+r\right)^{\frac{1-\sigma}{\sigma}}\left[\frac{P_2}{P_1}\right]^{\frac{1}{\eta}-\frac{1}{\sigma}}}\left[Y_1^T + \frac{Y_2^T}{1+r}\right]
\end{aligned}$$

1.4 Simplified version

For simplicity, we set $\beta\left(1+r\right) = 1$. We also consider the case of $\eta = 1$. The consumption basket is then Cobb-Douglas:

$$C_t = \left(C_t^T\right)^{\gamma}\left(C_t^N\right)^{1-\gamma}$$

Following the same steps as above, we can show:

$$\begin{aligned}
C_t^T &= \gamma P_t C_t \\
P_t^N C_t^N &= (1-\gamma) P_t C_t \\
P_t &= \frac{1}{(\gamma)^{\gamma}(1-\gamma)^{1-\gamma}} \left(P_t^N\right)^{1-\gamma}
\end{aligned}$$

Combining the demand for traded and non-trade goods, we have:

$$\begin{aligned}
\frac{C_2^N}{C_1^N} &= \frac{P_1^N}{P_2^N}\frac{P_2 C_2}{P_1 C_1} \\
\frac{C_2^N}{C_1^N} &= \left(\frac{P_1}{P_2}\right)^{\frac{1}{1-\gamma}} \frac{P_2 C_2}{P_1 C_1} \\
\frac{Y_2^N}{Y_1^N} &= \left(\frac{P_1}{P_2}\right)^{\frac{1}{1-\gamma}} \frac{P_2 C_2}{P_1 C_1} \\
\frac{Y_2^N}{Y_1^N} &= \left(\frac{P_1}{P_2}\right)^{\frac{1}{1-\gamma}} \frac{P_2}{P_1 C_1} \left[\beta(1+r)\frac{P_1}{P_2}\right]^{\frac{1}{\sigma}} C_1 \\
\frac{Y_2^N}{Y_1^N} &= \left(\frac{P_1}{P_2}\right)^{\frac{1}{1-\gamma}} \left(\frac{P_1}{P_2}\right)^{\frac{1-\sigma}{\sigma}} \\
\frac{Y_2^N}{Y_1^N} &= \left(\frac{P_1}{P_2}\right)^{\frac{1}{1-\gamma}+\frac{1-\sigma}{\sigma}} \\
\frac{Y_2^N}{Y_1^N} &= \left(\frac{P_1}{P_2}\right)^{\frac{\sigma+1-\sigma-(1-\sigma)\gamma}{(1-\gamma)\sigma}} \\
\frac{Y_2^N}{Y_1^N} &= \left(\frac{P_1}{P_2}\right)^{\frac{1-\gamma+\sigma\gamma}{(1-\gamma)\sigma}} \\
\frac{P_1}{P_2} &= \left(\frac{Y_2^N}{Y_1^N}\right)^{\frac{(1-\gamma)\sigma}{1-\gamma+\sigma\gamma}} \\
\frac{P_2}{P_1} &= \left(\frac{Y_1^N}{Y_2^N}\right)^{\frac{(1-\gamma)\sigma}{1-\gamma+\sigma\gamma}}
\end{aligned}$$

The current account is then:

$$\begin{aligned}
CA_1 &= Y_1^T - \frac{1}{1+\beta\left[\frac{P_1}{P_2}\right]^{\frac{1-\sigma}{\sigma}}}\left[Y_1^T + \beta Y_2^T\right] \\
CA_1 &= Y_1^T - \frac{1}{1+\beta\left(\frac{Y_2^N}{Y_1^N}\right)^{\frac{(1-\gamma)(1-\sigma)}{1-\gamma+\sigma\gamma}}}\left[Y_1^T + \beta Y_2^T\right]
\end{aligned}$$

If endowments grow at a rate g in both sectors:

$$\frac{CA_1}{Y_1^T} = 1 - \frac{1}{1+\beta\left(\frac{Y_2^N}{Y_1^N}\right)^{\frac{(1-\gamma)(1-\sigma)}{1-\gamma+\sigma\gamma}}}\left[1+\beta\frac{Y_2^T}{Y_1^T}\right]$$

$$\frac{CA_1}{Y_1^T} = 1 - \frac{1+\beta g}{1+\beta\left(1+g\right)^{\frac{(1-\gamma)(1-\sigma)}{1-\gamma+\sigma\gamma}}}$$

2 The terms-of-trade

2.1 Consumption allocation

Instead of a unique traded good, we consider that there are tow different ones. One is produced at home (index H) and the other abroad (index F). For simplicity, we abstract from the non-traded good.

n a given period, the consumer's overall consumption of traded goodC_t is a basket of the two:

$$C_t = \left[(\theta)^{\nu}\left(C_t^H\right)^{1-\nu} + (1-\theta)^{\nu}\left(C_t^F\right)^{1-\nu}\right]^{\frac{1}{1-\nu}}$$

We normalize the foreign price to 1. The terms-of-trade are the ratio of the home good to the foreign one: $Q_t^{tot} = P_t^H/P_t^F = P_t^H$. The expenditure is:

$$P_t C_t = Q_t^{tot} C_t^H + C_t^F$$

The consumer minimizes expenditure subject to a target level of the overall index. The Lagrangian is:

$$\mathcal{L}_t = Q_t^{tot} C_t^H + C_t^F + \lambda_t\left[(\theta)^{\nu}\left(C_t^H\right)^{1-\nu} + (1-\theta)^{\nu}\left(C_t^F\right)^{1-\nu}\right]^{\frac{1}{1-\nu}}$$

The first-order condition with respect to the consumption of home and foreign goods are:

$$\begin{aligned}
0 &= Q_t^{tot} - \lambda_t\left[(\theta)^{\nu}\left(C_t^H\right)^{1-\nu} + (1-\theta)^{\nu}\left(C_t^F\right)^{1-\nu}\right]^{\frac{1}{1-\nu}-1}(\theta)^{\nu}\left(C_t^H\right)^{-\nu} \\
&= Q_t^{tot} - \lambda_t\left[(\theta)^{\nu}\left(C_t^H\right)^{1-\nu} + (1-\theta)^{\nu}\left(C_t^F\right)^{1-\nu}\right]^{\frac{\nu}{1-\nu}}(\theta)^{\nu}\left(C_t^H\right)^{-\nu} \\
&= Q_t^{tot} - \lambda_t\left(C_t\right)^{\nu}(\theta)^{\nu}\left(C_t^H\right)^{-\nu} \\
0 &= 1 - \lambda_t\left[(\theta)^{\nu}\left(C_t^H\right)^{1-\nu} + (1-\theta)^{\nu}\left(C_t^F\right)^{1-\nu}\right]^{\frac{1}{1-\nu}-1}(1-\theta)^{\nu}\left(C_t^F\right)^{-\nu} \\
&= 1 - \lambda_t\left(C_t\right)^{\nu}(1-\theta)^{\nu}\left(C_t^F\right)^{-\nu}
\end{aligned}$$

Multiply these by C_t^H and C_t^F respectively, and add them up;

$$\begin{aligned}
0 &= Q_t^{tot} C_t^H - \lambda_t (C_t)^{\nu} (\theta)^{\nu} \left(C_t^H\right)^{1-\nu} + C_t^F - \lambda_t (C_t)^{\nu} (1-\theta)^{\nu} \left(C_t^F\right)^{1-\nu} \\
Q_t^{tot} C_t^H + C_t^F &= \lambda_t (C_t)^{\nu} (\theta)^{\nu} \left(C_t^H\right)^{1-\nu} + \lambda_t (C_t)^{\nu} (1-\theta)^{\nu} \left(C_t^F\right)^{1-\nu} \\
Q_t^{tot} C_t^H + C_t^F &= \lambda_t (C_t)^{\nu} \left[(\theta)^{\nu} \left(C_t^H\right)^{1-\nu} + (1-\theta)^{\nu} \left(C_t^F\right)^{1-\nu}\right] \\
Q_t^{tot} C_t^H + C_t^F &= \lambda_t (C_t)^{\nu} (C_t)^{1-\nu} \\
P_t C_t &= \lambda_t C_t \\
P_t &= \lambda_t
\end{aligned}$$

Using this, the optimality conditions are:

$$\begin{aligned}
0 &= Q_t^{tot} - \lambda_t (C_t)^{\nu} (\theta)^{\nu} \left(C_t^H\right)^{-\nu} \\
Q_t^{tot} &= P_t (C_t)^{\nu} (\theta)^{\nu} \left(C_t^H\right)^{-\nu} \\
\left(C_t^H\right)^{\nu} &= \left[\frac{Q_t^{tot}}{P_t}\right]^{-1} (C_t)^{\nu} (\theta)^{\nu} \\
C_t^H &= \theta \left[\frac{Q_t^{tot}}{P_t}\right]^{-\frac{1}{\nu}} C_t
\end{aligned}$$

and:

$$\begin{aligned}
0 &= 1 - \lambda_t (C_t)^{\nu} (1-\theta)^{\nu} \left(C_t^F\right)^{-\nu} \\
1 &= P_t (C_t)^{\nu} (1-\theta)^{\nu} \left(C_t^F\right)^{-\nu} \\
\left(C_t^F\right)^{\nu} &= \left[\frac{1}{P_t}\right]^{-1} (C_t)^{\nu} (1-\theta)^{\nu} \\
C_t^F &= (1-\theta) \left[\frac{1}{P_t}\right]^{-\frac{1}{\nu}} C_t
\end{aligned}$$

The price index is obtained from the consumption index:

$$
\begin{aligned}
C_t &= \left[(\theta)^{\nu}\left(C_t^H\right)^{1-\nu}+(1-\theta)^{\nu}\left(C_t^F\right)^{1-\nu}\right]^{\frac{1}{1-\nu}} \\
C_t &= \left[(\theta)^{\nu}\left(\theta\left[\frac{Q_t^{tot}}{P_t}\right]^{-\frac{1}{\nu}}C_t\right)^{1-\nu}+(1-\theta)^{\nu}\left((1-\theta)\left[\frac{1}{P_t}\right]^{-\frac{1}{\nu}}C_t\right)^{1-\nu}\right]^{\frac{1}{1-\nu}} \\
C_t &= \left[\theta\left(\left[\frac{Q_t^{tot}}{P_t}\right]^{-\frac{1}{\nu}}\right)^{1-\nu}(C_t)^{1-\nu}+(1-\theta)\left(\left[\frac{1}{P_t}\right]^{-\frac{1}{\nu}}\right)^{1-\nu}(C_t)^{1-\nu}\right]^{\frac{1}{1-\nu}} \\
C_t &= C_t\left[\theta\left[\frac{Q_t^{tot}}{P_t}\right]^{-\frac{1-\nu}{\nu}}+(1-\theta)\left[\frac{1}{P_t}\right]^{-\frac{1-\nu}{\nu}}\right]^{\frac{1}{1-\nu}} \\
1 &= \left[\theta\left[\frac{Q_t^{tot}}{P_t}\right]^{\frac{\nu-1}{\nu}}+(1-\theta)\left[\frac{1}{P_t}\right]^{\frac{\nu-1}{\nu}}\right]^{\frac{1}{1-\nu}} \\
1 &= \theta\left[\frac{Q_t^{tot}}{P_t}\right]^{\frac{\nu-1}{\nu}}+(1-\theta)\left[\frac{1}{P_t}\right]^{\frac{\nu-1}{\nu}} \\
[P_t]^{\frac{\nu-1}{\nu}} &= \theta\left[Q_t^{tot}\right]^{\frac{\nu-1}{\nu}}+(1-\theta) \\
P_t &= \left[\theta\left[Q_t^{tot}\right]^{\frac{\nu-1}{\nu}}+(1-\theta)\right]^{\frac{\nu}{1-\nu}}
\end{aligned}
$$

2.2 Intertemporal constraint

We again consider a two period economy. Output of the home good takes the form of endowments, and the agent can invest in a bond denominated in the foreign traded good with a return r. The budget constraints, in terms of foreign traded good, are:

$$
\begin{aligned}
B_2 + P_1C_1 &= Q_1^{tot}Y_1^H \\
P_2C_2 &= Q_2^{tot}Y_2^H + (1+r)B_2
\end{aligned}
$$

The intetemporal constraint is:

$$
P_1C_1 + \frac{P_2C_2}{1+r} = Q_1^{tot}Y_1^H + \frac{Q_2^{tot}Y_2^H}{1+r}
$$

We consider a log utility of consumption::

$$
U_1 = ln\left(C_1\right) + \beta ln\left(C_2\right)
$$

The Lagrangian for the optimization is:

$$
\mathcal{L}_t = n\left(C_1\right) + \beta ln\left(C_2\right) + \lambda\left[Q_1^{tot}Y_1^H + \frac{Q_2^{tot}Y_2^H}{1+r} - P_1C_1 + \frac{P_2C_2}{1+r}\right]
$$

The optimality conditions with respect to the two consumptions are:

$$
\begin{aligned}
0 &= (C_1)^{-1} - \lambda P_1 \\
0 &= \beta (C_2)^{-1} - \lambda \frac{P_2}{1+r}
\end{aligned}
$$

This gives the Euler condition:

$$
\begin{aligned}
(C_1)^{-1} &= \lambda P_1 \\
(C_1)^{-1} &= \frac{\beta (1+r)}{P_2} (C_2)^{-1} P_1 \\
C_2 &= \beta (1+r) \frac{P_1}{P_2} C_1
\end{aligned}
$$

The consumption in the first period is:

$$
\begin{aligned}
P_1 C_1 + \frac{P_2 C_2}{1+r} &= Q_1^{tot} Y_1^H + \frac{Q_2^{tot} Y_2^H}{1+r} \\
P_1 C_1 + \frac{P_2}{1+r} \beta (1+r) \frac{P_1}{P_2} C_1 &= Q_1^{tot} Y_1^H + \frac{Q_2^{tot} Y_2^H}{1+r} \\
P_1 C_1 &= \frac{1}{1+\beta} \left[Q_1^{tot} Y_1^H + \frac{Q_2^{tot} Y_2^H}{1+r} \right]
\end{aligned}
$$

The current account in the first period is:

$$
\begin{aligned}
CA_1 &= Q_1^{tot} Y_1^H - P_1 C_1 \\
CA_1 &= Q_1^{tot} Y_1^H - \frac{1}{1+\beta} \left[Q_1^{tot} Y_1^H + \frac{Q_2^{tot} Y_2^H}{1+r} \right] \\
\frac{CA_1}{Q_1^{tot} Y_1^H} &= 1 - \frac{1}{1+\beta} \left[1 + \frac{1}{1+r} \frac{Q_2^{tot} Y_2^H}{Q_1^{tot} Y_1^H} \right] \\
\frac{CA_1}{Q_1^{tot} Y_1^H} &= \frac{1}{1+\beta} \left[\beta - \frac{1}{1+r} \frac{Q_2^{tot} Y_2^H}{Q_1^{tot} Y_1^H} \right]
\end{aligned}
$$

3 Combining the two levels

3.1 Consumption baskets and price indices

We nest the dimension of traded-nontraded goods, and different traded goods. For simplicity, we consider that the elasticities are $\eta = \nu = 1$. The consumption basket in the Home country is:

$$
C_t = \left(C_t^T\right)^{\gamma} \left(C_t^N\right)^{1-\gamma}
$$

The traded good basket is in turn:

$$
C_t^T = \left(C_t^H\right)^{\theta} \left(C_t^F\right)^{1-\theta}
$$

Prices are expressed in Home currency, and denoted by P_t^H, P_t^F,P_t^T,P_t^N and P_t.

The allocation between traded and non-traded goods minimizes the expenditure:

$$\mathcal{L}_t = P_t^T C_t^T + P_t^N C_t^N + \lambda_t \left[C_t - \left(C_t^T\right)^{\gamma} \left(C_t^N\right)^{1-\gamma} \right]$$

The optimal conditions are:

$$\begin{aligned} 0 &= P_t^T - \lambda_t \gamma \left(C_t^T\right)^{\gamma-1} \left(C_t^N\right)^{1-\gamma} \\ 0 &= P_t^N - \lambda_t (1-\gamma) \left(C_t^T\right)^{\gamma} \left(C_t^N\right)^{-\gamma} \end{aligned}$$

This implies:

$$\begin{aligned} P_t^T C_t^T + P_t^N C_t^N &= \lambda_t \left[\gamma \left(C_t^T\right)^{\gamma} \left(C_t^N\right)^{1-\gamma} + (1-\gamma) \left(C_t^T\right)^{\gamma} \left(C_t^N\right)^{1-\gamma} \right] \\ P_t C_t &= \lambda_t \left(C_t^T\right)^{\gamma} \left(C_t^N\right)^{1-\gamma} \\ P_t &= \lambda_t \end{aligned}$$

The demands and the price index are then:

$$\begin{aligned} C_t^T &= \gamma \frac{P_t C_t}{P_t^T} \\ C_t^N &= (1-\gamma) \frac{P_t C_t}{P_t^N} \\ P_t &= \frac{1}{(\gamma)^{\gamma} (1-\gamma)^{1-\gamma}} \left(P_t^T\right)^{\gamma} \left(P_t^N\right)^{1-\gamma} \end{aligned}$$

The allocation between the home and foreign traded goods minimizes the expenditure:

$$\mathcal{L}_t = P_t^H C_t^H + P_t^F C_t^F + \lambda_t \left[C_t^T - \left(C_t^H\right)^{\theta} \left(C_t^F\right)^{1-\theta} \right]$$

Following the same steps as above, this implies:

$$\begin{aligned} C_t^H &= \theta \frac{P_t^T C_t^T}{P_t^H} \\ C_t^F &= (1-\theta) \frac{P_t^T C_t^T}{P_t^F} \\ P_t^T &= \frac{1}{(\theta)^{\theta} (1-\theta)^{1-\theta}} \left(P_t^H\right)^{\theta} \left(P_t^F\right)^{1-\theta} \end{aligned}$$

Turning to the Foreign country, the consumption baskets are:

$$\begin{aligned} C_t^* &= \left(C_t^{T*}\right)^{\gamma} \left(C_t^{N*}\right)^{1-\gamma} \\ C_t^{T*} &= \left(C_t^{H*}\right)^{1-\theta} \left(C_t^{F*}\right)^{\theta} \end{aligned}$$

where $*$ denote consumptions in the foreign country. The weight θ applies to the domestic good, which is not the Foreign goods. The prices, expressed in Foreign currency, are denoted by P_t^{H*}, P_t^{F*},P_t^{T*},P_t^{N*} and P_t^*.

The analysis proceeds along the same steps as for the Home country, leading to:

$$\begin{aligned} C_t^{T*} &= \gamma \frac{P_t^* C_t^*}{P_t^{T*}} \\ C_t^{N*} &= (1-\gamma) \frac{P_t^* C_t^*}{P_t^{N*}} \\ P_t^* &= \frac{1}{(\gamma)^{\gamma} (1-\gamma)^{1-\gamma}} \left(P_t^{T*}\right)^{\gamma} \left(P_t^{N*}\right)^{1-\gamma} \end{aligned}$$

and:

$$\begin{aligned} C_t^{H*} &= (1-\theta) \frac{P_t^{T*} C_t^{T*}}{P_t^{H*}} \\ C_t^{F*} &= \theta \frac{P_t^{T*} C_t^{T*}}{P_t^{F*}} \\ P_t^{T*} &= \frac{1}{(\theta)^{\theta} (1-\theta)^{1-\theta}} \left(P_t^{F*}\right)^{\theta} \left(P_t^{H*}\right)^{1-\theta} \end{aligned}$$

3.2 Relative prices

The exchange rate between the Home and Foreign currencies is E_t. It is expressed in unit of Home currency for one unit of Foreign currency, with an increase representing a depreciation of the Home currency.

The real exchange rate, Home terms-of-trade, and Foreign terms-of-trade are:

$$\begin{aligned} Q_t^{rer} &= \frac{E_t P_t^*}{P_t} \\ Q_t^{tot} &= \frac{E_t P_t^{H*}}{P_t^F} \\ Q_t^{tot*} &= \frac{P_t^F}{E_t P_t^{H*}} = \frac{1}{Q_t^{tot}} \end{aligned}$$

Using the expressions for the various price indices, we get:

$$\begin{aligned} Q_t^{rer} &= \frac{E_t \left(P_t^{T*}\right)^{\gamma} \left(P_t^{N*}\right)^{1-\gamma}}{\left(P_t^T\right)^{\gamma} \left(P_t^N\right)^{1-\gamma}} \\ Q_t^{rer} &= \left(\frac{E_t P_t^{T*}}{P_t^T}\right)^{\gamma} \left(\frac{E_t P_t^{N*}}{P_t^N}\right)^{1-\gamma} \\ Q_t^{rer} &= \left(\frac{\left(E_t P_t^{F*}\right)^{\theta} \left(E_t P_t^{H*}\right)^{1-\theta}}{\left(P_t^H\right)^{\theta} \left(P_t^F\right)^{1-\theta}}\right)^{\gamma} \left(\frac{E_t P_t^{N*}}{P_t^N}\right)^{1-\gamma} \end{aligned}$$

Consider several cases. First, if the countries are of equal size and their consumption baskets are not biased towards domestic goods ($\theta = 0.5$):

$$Q_t^{rer} = \left(\left(\frac{E_t P_t^{H*}}{P_t^H}\right)^{0.5} \left(\frac{E_t P_t^{F*}}{P_t^F}\right)^{0.5}\right)^{\gamma} \left(\frac{E_t P_t^{N*}}{P_t^N}\right)^{1-\gamma}$$

If the law of one price holds, $E_t P_t^{H*} = P_t^H$ and $E_t P_t^{F*} = P_t^F$, the real exchange rate reflects the presence of non-traded goods:

$$Q_t^{rer} = \left(\frac{E_t P_t^{N*}}{P_t^N}\right)^{1-\gamma}$$

If the law of one price does not hold, that the exchange rate also reflects the gap between the prices of the same good across countries.

Second, if the countries have a bias towards domestic goods ($\theta > 0.5$) and the law of one price holds:

$$\begin{aligned}
Q_t^{rer} &= \left(\frac{\left(P_t^F\right)^\theta \left(P_t^H\right)^{1-\theta}}{\left(P_t^H\right)^\theta \left(P_t^F\right)^{1-\theta}}\right)^\gamma \left(\frac{E_t P_t^{N*}}{P_t^N}\right)^{1-\gamma} \\
Q_t^{rer} &= \left(\left(\frac{P_t^F}{P_t^H}\right)^{2\theta-1}\right)^\gamma \left(\frac{E_t P_t^{N*}}{P_t^N}\right)^{1-\gamma} \\
Q_t^{rer} &= \left(\left(\frac{1}{Q_t^{tot}}\right)^{2\theta-1}\right)^\gamma \left(\frac{E_t P_t^{N*}}{P_t^N}\right)^{1-\gamma}
\end{aligned}$$

4 Endogenous production

4.1 Technology and profit maximization

Instead of endowment, output of the traded and non-traded good is produced using labor and capital, with a productivity A:

$$\begin{aligned}
Y_t^T &= A_t^T \left(K_t^T\right)^{\alpha_T} \left(L_t^T\right)^{1-\alpha_T} \\
Y_t^N &= A_t^N \left(K_t^N\right)^{\alpha_N} \left(L_t^N\right)^{1-\alpha_N}
\end{aligned}$$

The costs of capital and labor are r and w. The price of the nontraded good is P_t^N, while that of the traded good is equal to 1. The profits in the traded and non-traded sectors are:

$$\begin{aligned}
\Pi_t^T &= A_t^T \left(K_t^T\right)^{\alpha_T} \left(L_t^T\right)^{1-\alpha_T} - rK_t^T - wL_t^T \\
\Pi_t^N &= P_t^N A_t^N \left(K_t^N\right)^{\alpha_N} \left(L_t^N\right)^{1-\alpha_N} - rK_t^N - wL_t^N
\end{aligned}$$

The conditions for profit maximization are:

$$\begin{aligned}
0 &= \alpha_T A_t^T \left(K_t^T\right)^{\alpha_T-1} \left(L_t^T\right)^{1-\alpha_T} - r \\
0 &= \left(1-\alpha_T\right) A_t^T \left(K_t^T\right)^{\alpha_T} \left(L_t^T\right)^{-\alpha_T} - w \\
0 &= P_t^N \alpha_N A_t^N \left(K_t^N\right)^{\alpha_N-1} \left(L_t^N\right)^{1-\alpha_N} - r \\
0 &= P_t^N \left(1-\alpha_N\right) A_t^N \left(K_t^N\right)^{\alpha_N} \left(L_t^N\right)^{-\alpha_N} - w
\end{aligned}$$

Using the notation $k = K/L$ we write:

$$\begin{aligned}
r &= \alpha_T A_t^T \left(k_t^T\right)^{\alpha_T - 1} \\
w &= \left(1-\alpha_T\right) A_t^T \left(k_t^T\right)^{\alpha_T} \\
r &= P_t^N \alpha_N A_t^N \left(k_t^N\right)^{\alpha_N - 1} \\
w &= P_t^N \left(1-\alpha_N\right) A_t^N \left(k_t^N\right)^{\alpha_N}
\end{aligned}$$

4.2 Wage and relative good price

k_t^T reflects the world real interest rate:

$$k_t^T = \left(\frac{\alpha_T A_t^T}{r}\right)^{\frac{1}{1-\alpha_T}}$$

and in turn determines the wage:

$$\begin{aligned}
w &= \left(1-\alpha_T\right) A_t^T \left(k_t^T\right)^{\alpha_T} \\
w &= \left(1-\alpha_T\right) A_t^T \left(\frac{\alpha_T A_t^T}{r}\right)^{\frac{\alpha_T}{1-\alpha_T}}
\end{aligned}$$

The optimality conditions in the non-traded sector give the price of the non-traded good:

$$\begin{aligned}
r &= P_t^N \alpha_N A_t^N \left(k_t^N\right)^{\alpha_N - 1} \\
r &= P_t^N \alpha_N A_t^N \left(\frac{w}{P_t^N \left(1-\alpha_N\right) A_t^N}\right)^{\frac{\alpha_N - 1}{\alpha_N}} \\
r &= P_t^N \alpha_N A_t^N \left(\frac{w}{P_t^N \left(1-\alpha_N\right) A_t^N}\right)^{\frac{\alpha_N - 1}{\alpha_N}} \\
r &= \alpha_N \left(\frac{w}{1-\alpha_N}\right)^{\frac{\alpha_N - 1}{\alpha_N}} P_t^N A_t^N \left(P_t^N A_t^N\right)^{\frac{1-\alpha_N}{\alpha_N}} \\
r &= \alpha_N \left(\frac{w}{1-\alpha_N}\right)^{\frac{\alpha_N - 1}{\alpha_N}} \left(P_t^N A_t^N\right)^{\frac{1}{\alpha_N}} \\
(r)^{\alpha_N} &= \left(\alpha_N\right)^{\alpha_N} \left(\frac{w}{1-\alpha_N}\right)^{\alpha_N - 1} P_t^N A_t^N \\
P_t^N &= \frac{1}{A_t^N} (r)^{\alpha_N} (w)^{1-\alpha_N} \left(\frac{1}{\alpha_N}\right)^{\alpha_N} \left(\frac{1}{1-\alpha_N}\right)^{1-\alpha_N}
\end{aligned}$$

Using the result for the wage, we have:

$$
\begin{aligned}
P_t^N &= \frac{1}{A_t^N}(r)^{\alpha_N}(w)^{1-\alpha_N}\left(\frac{1}{\alpha_N}\right)^{\alpha_N}\left(\frac{1}{1-\alpha_N}\right)^{1-\alpha_N} \\
P_t^N &= \frac{1}{A_t^N}(r)^{\alpha_N}\left((1-\alpha_T)A_t^T\left(\frac{\alpha_T A_t^T}{r}\right)^{\frac{\alpha_T}{1-\alpha_T}}\right)^{1-\alpha_N}\left(\frac{1}{\alpha_N}\right)^{\alpha_N}\left(\frac{1}{1-\alpha_N}\right)^{1-\alpha_N} \\
P_t^N &= \frac{\left(A_t^T\right)^{\frac{1-\alpha_N}{1-\alpha_T}}}{A_t^N}(r)^{\alpha_N-\frac{\alpha_T(1-\alpha_N)}{1-\alpha_T}}(\alpha_T)^{\frac{\alpha_T(1-\alpha_N)}{1-\alpha_T}}(\alpha_N)^{-\alpha_N}\left(\frac{1-\alpha_T}{1-\alpha_N}\right)^{1-\alpha_N} \\
P_t^N &= \frac{\left(A_t^T\right)^{\frac{1-\alpha_N}{1-\alpha_T}}}{A_t^N}(r)^{\frac{\alpha_N-\alpha_T}{1-\alpha_T}}(\alpha_T)^{\frac{\alpha_T(1-\alpha_N)}{1-\alpha_T}}(\alpha_N)^{-\alpha_N}\left(\frac{1-\alpha_T}{1-\alpha_N}\right)^{1-\alpha_N}
\end{aligned}
$$

The price of the non-traded goods reflects the different productivities in the two sectors, difference in labor shares, and the world real exchange rate. Taking logs, we have:

$$
ln\left(P_t^N\right) = \frac{1-\alpha_N}{1-\alpha_T}ln\left(A_t^T\right) - ln\left(A_t^N\right) + \frac{\alpha_N-\alpha_T}{1-\alpha_T}ln\left(r\right) + \phi
$$

In a two country model, we get a similar relation in the foreign county:

$$
ln\left(P_t^{N*}\right) = \frac{1-\alpha_N}{1-\alpha_T}ln\left(A_t^{T*}\right) - ln\left(A_t^{N*}\right) + \frac{\alpha_N-\alpha_T}{1-\alpha_T}ln\left(r\right) + \phi
$$

Hence the real exchange rate is proportional to:

$$
ln\left(\frac{P_t^{N*}}{P_t^N}\right) = \frac{1-\alpha_N}{1-\alpha_T}ln\left(\frac{A_t^{T*}}{A_t^T}\right) - ln\left(\frac{A_t^{N*}}{A_t^N}\right)
$$