PS4 Solutions

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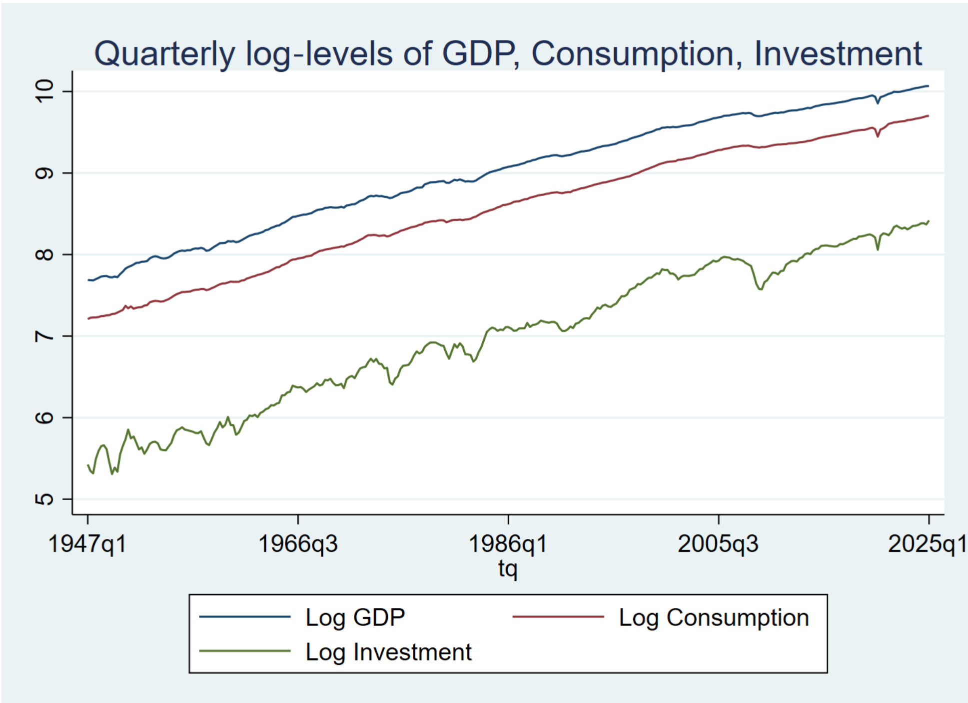
1 Problem 1

Solution (a).




```
import delimited using "GDP.csv", varnames(1) clear
gen date = date(observation_date, "YMD")
format date %td
gen tq = qofd(date)
format tq %tq
rename gdpc1 GDP
keep tq GDP
save gdp_temp, replace

import delimited using "RPCE.csv", varnames(1) clear
gen date = date(observation_date, "YMD")
format date %td
gen tq = qofd(date)
format tq %tq
```

```
rename pcecc96 PCE
keep tq PCE
save pce_temp, replace

import delimited using "GPDI.csv", varnames(1) clear
gen date = date(observation_date, "YMD")
format date %td
gen tq = qofd(date)
format tq %tq
rename gpdic1 GPDI
keep tq GPDI

merge m:1 tq using gdp_temp
drop _merge
merge m:1 tq using pce_temp
drop _merge

tsset tq, quarterly
gen lnGDP  = ln(GDP)
gen lnPCE  = ln(PCE)
gen lnGPDI = ln(GPDI)

set scheme s2color

tsline lnGDP lnPCE lnGPDI, ///
    legend(order(1 "Log GDP" 2 "Log Consumption" 3 "Log Investment")) ///
    title("Quarterly log-levels of GDP, Consumption, Investment")

graph export "a.png", as(png) replace

```

Solution (b).

Let $\mathbf{X} = (\mathbf{1}, \mathbf{t})$ be the $N \times 2$ design matrix, with rows $(1, t)$, and $\mathbf{y} = (y_1, \dots, y_N)'$. Then the OLS estimator is

$$\begin{pmatrix} \hat\beta_1 \\ \hat\beta_2 \end{pmatrix} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \frac{1}{S_{tt}} \begin{pmatrix} \sum_t t^2 & -\sum_t t \\ -\sum_t t & \sum_t 1 \end{pmatrix} \begin{pmatrix} \sum_t y_t \\ \sum_t t\, y_t \end{pmatrix},$$

where

$$\bar t = \frac{1}{N}\sum_{t=1}^{N} t, \quad \bar y = \frac{1}{N}\sum_{t=1}^{N} y_t, \quad S_{tt} = \sum_{t=1}^{N} (t - \bar t)^2.$$

Equivalently,

$$\hat\beta_2 = \frac{\sum_{t=1}^{N} (t - \bar t)(y_t - \bar y)}{\sum_{t=1}^{N} (t - \bar t)^2}, \quad \hat\beta_1 = \bar y - \hat\beta_2\, \bar t.$$

The fitted values and residuals are

$$\hat{y}_t = \hat{\beta}_1 + \hat{\beta}_2 t, \qquad \hat{u}_t = y_t - \hat{y}_t.$$

y_t is the log of three variables: GDP, PCE, and GPDI. The estimated models are as below:

Window	Series	$\hat{\beta}_1$	$\hat{\beta}_2$ (per qtr)	Annualized Growth (%)
1965 Q1–2006 Q4	ln GDP	7.8509	0.0077707	3.11
	ln PCE	7.3017	0.0083520	3.34
	ln GPDI	5.4989	0.0100216	4.01
2007 Q1–2019 Q4	ln GDP	8.4997	0.0048888	1.96
	ln PCE	8.0947	0.0049259	1.97
	ln GPDI	5.2006	0.0104206	4.17
2007 Q1–2022 Q2	ln GDP	8.4867	0.0049376	1.98
	ln PCE	8.0394	0.0051370	2.05
	ln GPDI	5.3432	0.0098678	3.95

Table 1: Trend regression estimates and annualized growth rates

```
gen t = _n

* 1) 1965Q1-2006Q4
reg lnGDP t if inrange(tq, tq(1965q1), tq(2006q4))
predict uhat_GDP_65_06, resid
display "GDP 1965-06 annualized % = " _b[t]*4*100
reg lnPCE t if inrange(tq, tq(1965q1), tq(2006q4))
predict uhat_PCE_65_06, resid
display "PCE 1965-06 annualized % = " _b[t]*4*100
reg lnGPDI t if inrange(tq, tq(1965q1), tq(2006q4))
predict uhat_GPDI_65_06, resid
display "GPDI 1965-06 annualized % = " _b[t]*4*100

* 2) 2007Q1-2019Q4
reg lnGDP t if inrange(tq, tq(2007q1), tq(2019q4))
predict uhat_GDP_07_19, resid
display "GDP 2007-19 annualized % = " _b[t]*4*100

reg lnPCE t if inrange(tq, tq(2007q1), tq(2019q4))
predict uhat_PCE_07_19, resid
display "PCE 2007-19 annualized % = " _b[t]*4*100

reg lnGPDI t if inrange(tq, tq(2007q1), tq(2019q4))
predict uhat_GPDI_07_19, resid
display "GPDI 2007-19 annualized % = " _b[t]*4*100

* 3) 2007Q1-2022Q2
```

```
reg lnGDP t if inrange(tq, tq(2007q1), tq(2022q2))
predict uhat_GDP_07_22, resid
display "GDP 2007-22 annualized % = " _b[t]*4*100

reg lnPCE t if inrange(tq, tq(2007q1), tq(2022q2))
predict uhat_PCE_07_22, resid
display "PCE 2007-22 annualized % = " _b[t]*4*100

reg lnGPDI t if inrange(tq, tq(2007q1), tq(2022q2))
predict uhat_GPDI_07_22, resid
display "GPDI 2007-22 annualized % = " _b[t]*4*100

```

Solution (c).
$\hat{\beta}_2$ measures the *average quarterly* change in $\ln y_t$.
The *approximate quarterly growth rate* is $\hat{\beta}_2 \times 100\%$.
The *exact annualized growth rate* is

$$\left(e^{4\hat{\beta}_2} - 1\right) \times 100\% \approx 4\hat{\beta}_2 \times 100\%,$$

- **1965–2006:** GDP grows at about 0.78% per quarter $\Rightarrow$ 3.11% per year; Consumption: 0.84% qtr $\Rightarrow$ 3.34% yr; Investment: 1.00% qtr $\Rightarrow$ 4.01% yr.
- **2007–2019:** GDP/Consumption trend roughly halves to 0.49% qtr $\Rightarrow$ 1.96% yr. Investment trend remains high, 1.04% qtr $\Rightarrow$ 4.17% yr.
- **2007–2022:** GDP/Consumption trend stays near 2.05% annualized; Investment trend slightly eases to 3.95%.

GDP and PCE grow at a similar rate, but investment grow faster, at about twice the rate of GDP and PCE.

Solution (d).
The (kth) sample autocorrelation of $\{\hat{u}_t\}$ is

$$\hat{\rho}(k) = \frac{\sum_{t=k+1}^{N} \hat{u}_t\, \hat{u}_{t-k}}{\sum_{t=1}^{N} \hat{u}_t^2}, \qquad k = 0, 1, 2, \ldots.$$

We compute $\hat{\rho}(k)$ for $k = 1, \ldots, K$ (here $K = 8$) in each subsample to assess the persistence of residual cycles.

```
corrgram uhat_GDP_65_06 if inrange(tq, tq(1965q1), tq(2006q4)), lags(8)
corrgram uhat_PCE_65_06 if inrange(tq, tq(1965q1), tq(2006q4)), lags(8)
corrgram uhat_GPDI_65_06 if inrange(tq, tq(1965q1), tq(2006q4)), lags(8)
```

Table 2: Sample autocorrelations (AC) of detrended residuals by lag

Window	Series	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6	Lag 7	Lag 8
1965 Q1–2006 Q4	GDP	0.9207	0.8038	0.6640	0.5231	0.3799	0.2457	0.1218	0.0211
	PCE	0.9463	0.8724	0.7853	0.6864	0.5883	0.4880	0.3892	0.2972
	GPDI	0.9143	0.8018	0.6805	0.5437	0.4146	0.3171	0.2192	0.1401
2007 Q1–2019 Q4	GDP	0.8952	0.7492	0.5887	0.4183	0.2782	0.1379	0.0236	-0.0304
	PCE	0.9183	0.8167	0.6955	0.5650	0.4382	0.3085	0.2057	0.1304
	GPDI	0.8844	0.7045	0.4978	0.2961	0.1250	-0.0311	-0.1723	-0.2628
2007 Q1–2022 Q2	GDP	0.6975	0.5012	0.3325	0.1832	0.0808	-0.0320	-0.0779	-0.0736
	PCE	0.7061	0.5167	0.3387	0.1636	0.0639	-0.0153	-0.0416	-0.0518
	GPDI	0.8437	0.6556	0.4650	0.2928	0.1289	-0.0343	-0.1699	-0.2362

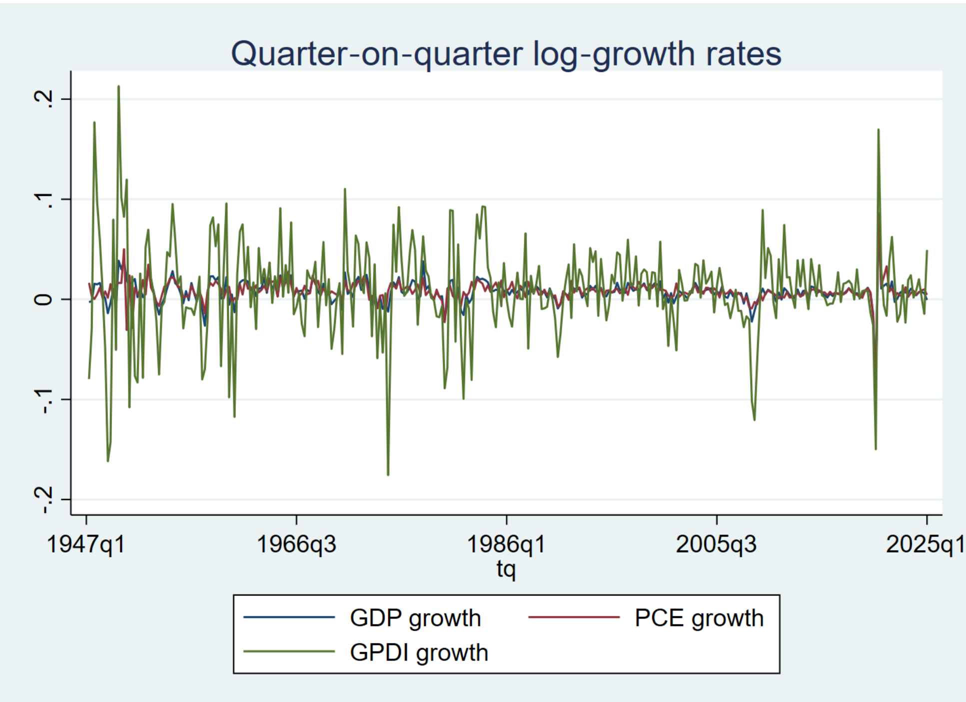
```

corrgram uhat_GDP_07_19 if inrange(tq, tq(2007q1), tq(2019q4)), lags(8)
corrgram uhat_PCE_07_19 if inrange(tq, tq(2007q1), tq(2019q4)), lags(8)
corrgram uhat_GPDI_07_19 if inrange(tq, tq(2007q1), tq(2019q4)), lags(8)

corrgram uhat_GDP_07_22 if inrange(tq, tq(2007q1), tq(2022q2)), lags(8)
corrgram uhat_PCE_07_22 if inrange(tq, tq(2007q1), tq(2022q2)), lags(8)
corrgram uhat_GPDI_07_22 if inrange(tq, tq(2007q1), tq(2022q2)), lags(8)

```

Solution (e).




```
gen dgdp  = D.lnGDP
gen dpce  = D.lnPCE
```

```
gen dgpdi = D.lnGPDI

tsline dgdp dpce dgpdi, ///
    legend(order(1 "GDP growth" 2 "Cons growth" 3 "Inv growth")) ///
    title("Quarter-on-quarter log-growth rates")

graph export "e.png", as(png) replace

```

Solution (f).
For each series $g_t \in \{\Delta y_t, \Delta c_t, \Delta i_t\}$ in a given subsample of length M, we compute

$$\bar{g} = \frac{1}{M}\sum_{t=1}^{M} g_t, \qquad s_g = \sqrt{\frac{1}{M-1}\sum_{t=1}^{M}\left(g_t - \bar{g}\right)^2}.$$

These compare the *average realized growth* $\bar{g}$ with the *trend-based* quarterly slope $\hat{\beta}_2$, and measure cyclical volatility via s_g.

Window	Series	Obs	Mean of $\Delta \ln y$	Std. of $\Delta \ln y$	$\hat{\beta}_2$ (per qtr)
1965 Q2–2006 Q4	GDP	167	0.0080837	0.0083304	0.0077707
	PCE	167	0.0087461	0.0068057	0.0083520
	GPDI	167	0.0104904	0.0394865	0.0100216
2007 Q2–2019 Q4	GDP	51	0.0045828	0.0061137	0.0048888
	PCE	51	0.0045793	0.0043225	0.0049259
	GPDI	51	0.0058344	0.0346509	0.0104206
2007 Q2–2022 Q2	GDP	61	0.0045452	0.0159656	0.0049376
	PCE	61	0.0050521	0.0175900	0.0051370
	GPDI	61	0.0064772	0.0444316	0.0098678

Comparison to part (b):

- *1965 Q2-2006 Q4:* The sample means of $\Delta \ln y$ for GDP (0.00799), PCE (0.00867) and GPDI (0.01001) are almost identical to the estimated quarterly trend slopes $\hat{\beta}_2$ (0.0077707, 0.0083520, 0.0100216). This confirms that, over the long sample, the linear-trend regression accurately captures the average growth rate.

- *2007 Q2-2019 Q4:* GDP and PCE mean growth rates ($\approx$0.00458) lie slightly below their $\hat{\beta}_2$ ($\approx$0.00489 and 0.00493), reflecting that downturns (2008-09 crisis) pull the sample average below the fitted trend. For GPDI, the mean (0.00583) is markedly below its trend slope (0.01042), since investment experienced large negative shocks that the OLS trend, minimizing squared errors, spreads more evenly across the sample.

- *2007 Q2-2022 Q2:* Including the COVID-19 shock further widens the gap: GDP and PCE means (0.00455, 0.00505) remain below their slopes (0.00494, 0.00514), and investment's mean (0.00648) stays well under its slope (0.00987). This again shows that severe cyclical downturns pull down the simple average growth below the fitted linear trend.

```
* 1965Q2-2006Q4
sum dgdp dpce dgpdi if inrange(tq, tq(1965q2), tq(2006q4))

* 2007Q2-2019Q4
sum dgdp dpce dgpdi if inrange(tq, tq(2007q2), tq(2019q4))

* 2007Q2-2022Q2
sum dgdp dpce dgpdi if inrange(tq, tq(2007q2), tq(2022q2))

```

Solution (g).

Window	Series	Mean of $\Delta \ln y$	Std. dev. of $\Delta \ln y$
1965 Q1–1983 Q4	GDP	0.0087338	0.0117128
	PCE	0.0089202	0.0100335
	GPDI	0.0104315	0.0602435
1984 Q1–2006 Q4	GDP	0.0080725	0.0051354
	PCE	0.0086507	0.0048490
	GPDI	0.0107001	0.0267834

- *Mean growth rates:* GDPs average growth rises slightly from 0.7886% to 0.8073%; PCE falls marginally from 0.8686% to 0.8651%; GPDI increases from 0.9159% to 1.0700%. Overall, the mean growth rates remain essentially unchanged in magnitude.
- *Volatility:* All three series exhibit a dramatic reduction in standard deviation.
  - GDP's σ falls from 1.10% to 0.51%.
  - PCE's σ falls from 0.86% to 0.48%.
  - GPDI's σ falls from 5.05% to 2.68%.

  This confirms the *Great Moderation*after 1984, while average growth remained stable, business-cycle volatility was substantially dampened.

```
* 1965Q2-1983Q4
sum dgdp dpce dgpdi if inrange(tq, tq(1947q2), tq(1983q4))

* 1984Q1-2006Q4
sum dgdp dpce dgpdi if inrange(tq, tq(1984q1), tq(2006q4))

```

2 Problem 2

Solution (a).
Weak Stationarity: A stochastic process $\{y_t\}$ is weakly stationary if (i) its mean is the same at every time period, $\mu_t = \mu \quad \forall t$; and (ii) every autocovariance $\gamma_{h,t} = \mathrm{Cov}(y_t, y_{t-h})$ depends only on the displacement h, not on the time period t, or equivalently, $\gamma_{h,t} = \gamma_h, \quad \forall t$
Strict Stationarity: A process is strictly stationary if for any values $h_1, \dots, h_k$, the joint distribution of $(y_t, y_{t-h_1}, \dots, y_{t-h_k})$ depends only on the intervals separating the dates (displacements) $h_1, \dots, h_k$ and not on the date t itself, i.e.

$$f_{Y_t, Y_{t-h_1}, \dots, Y_{t-h_k}} = f_{Y_\tau, Y_{\tau-h_1}, \dots, Y_{\tau-h_k}} \quad \forall \tau.$$

Weak Stationarity

A process $\{y_t\}$ is weakly stationary if

$$\mathbb{E}[y_t] = \mu \quad \forall t, \quad \mathrm{Cov}(y_t, y_{t-k}) = \gamma_k \text{ depends only on } k.$$

Here

$$\mathbb{E}[y_t] = \sum_{j=0}^{3} \psi_j \, \mathbb{E}[u_{t-j}] = 0,$$

so the mean is constant. Next,

$$\mathrm{Cov}(y_t, y_{t-k}) = \mathbb{E}\big[y_t \, y_{t-k}\big] = \sum_{j=0}^{3} \sum_{i=0}^{3} \psi_j \psi_i \, \mathbb{E}\big[u_{t-j} u_{t-k-i}\big].$$

Since $\mathbb{E}[u_s u_r] = 0$ for $s \neq r$, only terms with $t - j = t - k - i$ survive, so $\mathrm{Cov}(y_t, y_{t-k})$ depends on k alone. Hence $\{y_t\}$ is weakly stationary.

Strict Stationarity

If $\{u_t\}$ is strictly stationary (e.g. i.i.d.), then any finite-order linear filter $\sum_{j=0}^{3} \psi_j \, u_{t-j}$ yields a strictly stationary $\{y_t\}$. Thus under i.i.d. innovations, y_t is strictly stationary.

Solution (b).
Define $\gamma_k = \mathrm{Cov}(y_t, y_{t-k})$. With ψ_j as above,

$$\gamma_k = \sum_{j=0}^{3} \sum_{i=0}^{3} \psi_j \psi_i \, E[u_{t-j} u_{t-k-i}] = \sum_{j=0}^{3} \psi_j \, \psi_{j-k},$$

interpreting $\psi_m = 0$ for $m > 3$. From the question, we know that:

$$\sigma^2 = \mathbb{V}[u_t] = \mathbb{E}[u_t^2] - \mathbb{E}[u_t]^2 = \mathbb{E}[u_t^2] = 1$$

Thus:

$$\begin{aligned}
\gamma_0 &= \sum_{j=0}^{3} \psi_j^2 = 1^2 + (-2.4)^2 + 0.8^2 + (-0.4)^2 = 7.56, \\
\gamma_1 &= \psi_0\psi_1 + \psi_1\psi_2 + \psi_2\psi_3 = (1 \cdot (-2.4) + (-2.4) \cdot 0.8 + 0.8 \cdot (-0.4)) = -4.64, \\
\gamma_2 &= \psi_0\psi_2 + \psi_1\psi_3 = (1 \cdot 0.8 + (-2.4) \cdot (-0.4)) = 1.76, \\
\gamma_3 &= \psi_0\psi_3 = (1 \cdot (-0.4)) = -0.4, \\
\gamma_k &= 0 \quad \text{for } |k| \geq 4.
\end{aligned}$$

Solution (c).
Consider

$$S_T = \sum_{t=1}^{T} y_t, \qquad V_T = \mathbb{V}[S_T] = \sum_{i=1}^{T}\sum_{j=1}^{T} \gamma_{i-j}.$$

Re-index $h = j - i$:

$$V_T = \sum_{h=-(T-1)}^{T-1} (T - |h|)\, \gamma_h.$$

Hence

$$\mathbb{V}\left[\tfrac{1}{\sqrt{T}} S_T\right] = \frac{V_T}{T} = \gamma_0 + 2\sum_{h=1}^{T-1}\Big(1 - \tfrac{h}{T}\Big)\gamma_h.$$

As $T \to \infty$, $\frac{h}{T} \to 0$ for fixed h, and $\gamma_h = 0$ for $h \geq 4$, so

$$\lim_{T\to\infty} \mathbb{V}\left[\tfrac{1}{\sqrt{T}} S_T\right] = \gamma_0 + 2(\gamma_1 + \gamma_2 + \gamma_3) = 7.56 + 2(-4.64 + 1.76 - 0.4) = 1.$$

Solution (d).
Assume $\{u_t\}$ is i.i.d. with $\mathbb{E}[u_t] = 0$, $\mathbb{V}[u_t] = \sigma^2$.

Strict Stationarity

$x_t = f(u_t, u_{t-4})$ depends on two i.i.d. draws; hence its joint distributions do not change with shifts in t. So $\{x_t\}$ is strictly stationary.

Ergodicity

An i.i.d. sequence is ergodic and any measurable function of it remains ergodic. Thus $\{x_t\}$ is ergodic.

White-Noise Properties

- $\mathbb{E}[x_t] = \mathbb{E}[u_t]\mathbb{E}[u_{t-4}] = 0$.
- For $k \neq 0$,

$$\mathrm{Cov}(x_t, x_{t-k}) = \mathbb{E}[u_t u_{t-4} u_{t-k} u_{t-k-4}] = 0,$$

  since among the four factors at least one is independent with zero mean.

Hence $\{x_t\}$ is zero-mean uncorrelated white noise, but not i.i.d. (since x_t and x_{t+4} share u_t).

Solution (e).
We have:

$$\mathbb{E}\left[\tfrac{1}{T}\sum_{t=1}^{T} x_t\right] = \tfrac{1}{T}\sum_{t=1}^{T}\mathbb{E}[x_t] = 0,$$

and

$$\mathbb{V}\left[\tfrac{1}{\sqrt{T}}\sum_{t=1}^{T} x_t\right] = \sum_{h=-(T-1)}^{T-1}(T - |h|)\gamma(h)$$

where $\gamma(h) = \mathrm{Cov}(x_t, x_{t-h})$. We know that $\gamma(h) = 0$ for $h \neq 0$ and $\gamma(0) = \mathbb{E}[u_t^2 u_{t-4}^2] = \sigma^4$. Thus:

$$\mathbb{V}\left[\tfrac{1}{\sqrt{T}}S_T\right] = \frac{1}{T}T\gamma(0) = \sigma^4.$$