Game Theory

Contract with Asymmetric Information

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Outline

- Adverse selection (逆向选择); Screening (甄别)
- Price discrimination (价格歧视/区别定价)
- Non-linear Pricing (非线性定价)
  - Non-linear two-part tariffs (“双” 两部收费)
- Incentive compatibility (激励相容)
- Downward-distortion (“低端” 扭曲) & “Zero at top” (对最高收入者实施 0 税率)
- Kuhn-Tucker Condition (库恩-塔克条件)
  - Constrained optimization (带有 (不等式) 约束的最优化)
  - Lagrangian (拉格朗日)
  - Complementary Slackness (互补-松弛)
- Spence-Mirrlees Single-Crossing (单交叉性)

Monopoly Pricing without Discrimination

Recall the monopoly's problem: The market demand is $Q(p)$ where $Q'(p) < 0$. A monopoly firm with constant marginal cost c, chooses a linear price p to maximize profit:

$$p^m = \arg\max_p (p - c)Q(p)$$

$$Q(p^m) + (p^m - c)Q'(p^m) = 0 \Rightarrow p^m - c = -\frac{Q(p^m)}{Q'(p^m)} > 0.$$

If the monopoly firm is operated by a benevolent planner who offers a linear price p^{FB} to maximize total surplus, then the **first-best** outcome is given by

$$p^{FB} = c < p^m.$$

Price Discrimination

The first-best outcome can be achieved, if, the monopoly firm is able to charge different prices to different consumers, i.e., **Perfect Price Discrimination**.

- Two consumers, type θ_L and θ_H, and $\theta_L < \theta_H$.
- Consumer i buys q_i units and pays a lump-sum transfer T_i to the seller.
- Utility: $\theta_i v(q_i) - T_i$. Outside option is zero.
  - Assume: $v'(q) > 0$ and $v''(q) < 0$
- Firm's profit (if both types are served): $T_H - cq_H + T_L - cq_L$.

First-Best Solutions

Suppose consumer types are perfectly observable: the firm knows who is type H and who is type L, and offer (T_H, q_H) to H and (T_L, q_L) to L.

- From the consumer side, accepting the offer if $\theta_i v(q_i) - T_i \geq 0$.
- For **each** consumer, the firm solves

$$\max_{T_i, q_i} T_i - cq_i$$
$$s.t.\ \theta_i v(q_i) - T_i \geq 0.$$

  The **participation** constraint should be **binding** (i.e., "$=$")
  - If "$>$", the firm can raise T_i to earn more.
  - If "$<$", the firm gets nothing.

  Sometimes the participation constraint is called **individual rationality**, i.e., IR.
- IR is binding: $T_i = \theta_i v(q_i)$.

First-Best Solutions

Plug the IR constraint into firm's objective:

$$\max_{q_i} \theta_i v(q_i) - cq_i$$
$$F.O.C. \Rightarrow \theta_i v'(q_i^{FB}) = c, i = H, L.$$

Because $\theta_L < \theta_H$ and $v'' < 0$:

$$\theta_L v'(q_L^{FB}) = c = \theta_H v'(q_H^{FB}) \Leftrightarrow v'(q_L^{FB}) > v'(q_H^{FB})$$
$$\Leftrightarrow q_L^{FB} < q_H^{FB}$$

Plug q_i^{FB} into the IR constraint, and by $v' > 0$:

$$T_H^{FB} = \theta_H v(q_H^{FB}) > \theta_L v(q_L^{FB}) = T_L^{FB}.$$

Features of Perfect Discrimination

- Different prices T_i for different consumers θ_i.
- Each consumer gets zero surplus.
- Marginal utility is equal to the marginal cost.
- A higher profit (compared with linear pricing), and a higher output.
- Total surplus is maximized.
  - The benevolent planner solves $\max_{T_H, T_L, q_H, q_L} \theta_H v(q_H) - T_H + \theta_L v(q_L) - T_L + T_H - cq_H + T_L - cq_L$
  - FOC implies $\theta_i v'(q_i) = c,\ i = H, L$.
  - Therefore, perfect discrimination = socially optimum [1]

However, such desirable outcome requires that the firm is able to distinguish θ_i.

[1] That's why we put "FB" (first-best) as superscripts.

Second-Best: Optimal Non-linear Pricing

- Now assume that buyers' types are θ_H and θ_L; but the seller does not know who is θ_H and who is θ_L.
- The seller offers two menus: (T_H, q_H) and (T_L, q_L). Each buyer chooses her/his menu **voluntarily**.
  The firm solves

$$\max_{T_i, q_i} (T_L - cq_L) + (T_H - cq_H).$$

Each buyer chooses $(T(q), q)$ to maximize utility:

$$q_i = \arg\max_{q} \theta_i v(q_i) - T(q_i)$$

All buyers participate:

$$\theta_i v(q_i) - T(q_i) \geq 0.$$

Revelation Principle: 激励相容 (IC) 与参与约束 (IR)

Among the four conditions:

$\theta_H v(q_H) - T_H \geq \theta_H v(q_L) - T_L$	H will not imitate L	IC_H
$\theta_L v(q_L) - T_L \geq \theta_L v(q_H) - T_H$	L will not imitate H	IC_L
$\theta_H v(q_H) - T_H \geq 0$	H participates	IR_H
$\theta_L v(q_L) - T_L \geq 0$	L participates	IR_L

- The first two are **incentive compatible** constraints: each type chooses the contract that is not intended to be designed for other types;
- The last two are **participation** constraints (IR: Individual Rationality).

How to use these constraints to solve the seller's problem?

Find the Binding "=" Constraints

$$
\begin{array}{lll}
\theta_H v(q_H) - T_H \geq \theta_H v(q_L) - T_L & \text{H will not imitate L} & IC_H \\
\theta_L v(q_L) - T_L \geq \theta_L v(q_H) - T_H & \text{L will not imitate H} & IC_L \\
\theta_H v(q_H) - T_H \geq 0 & \text{H participates} & IR_H \\
\theta_L v(q_L) - T_L \geq 0 & \text{L participates} & IR_L
\end{array}
$$

- $IR_L + IC_H \Rightarrow IR_H$ (IR_H is not binding):
  $\underbrace{\theta_H v(q_H) - T_H \geq \theta_H v(q_L) - T_L}_{IC_H} > \underbrace{\theta_L v(q_L) - T_L}_{IR_L} \geq 0.$
- Eliminate one of the two ICs.
  We claim that IC_H is binding and IC_L is redundant. And verify the claim after the solution is obtained.

Binding: $IC_H \& IR_L$; Redundant: $IR_H \& IC_L$

The remaining constraints are IC_H and IR_L:

$\theta_H v(q_H) - T_H \geq \theta_H v(q_L) - T_L$	H will not imitate L	IC_H
$\theta_L v(q_L) - T_L \geq 0$	L participates	IR_L

- IC_H will bind at equilibrium; otherwise, raising T_H is profitable;
- IR_L should bind at equilibrium; otherwise, raising T_L until it binds.

That is

$\theta_H v(q_H) - T_H = \theta_H v(q_L) - T_L$	H will not imitate L	IC_H
$\theta_L v(q_L) - T_L = 0$	L participates	IR_L

Solve the Relaxed Problem

For the binding constraints:

$$\begin{aligned} T_L &= \theta_L v(q_L) && IR_L \\ T_H &= \theta_H v(q_H) - \theta_H v(q_L) + \theta_L v(q_L) && IC_H \end{aligned}$$

The seller's problem becomes

$$\max_{T_i, q_i} (T_L - cq_L) + (T_H - cq_H)$$

$$\Rightarrow \max_{q_L, q_H} \theta_L v(q_L) - cq_L + \theta_H v(q_H) - \theta_H v(q_L) + \theta_L v(q_L) - cq_H$$

Optimal Second-Best Solution: Downward-Distortion

- The seller's objective

$$\max_{q_L, q_H} \theta_L v(q_L) - cq_L + \theta_H v(q_H) - \theta_H v(q_L) + \theta_L v(q_L) - cq_H$$

- F.O.C. w.r.p. q_H:

$$\theta_H v'(q_H^{SB}) = c \Leftrightarrow q_H^{SB} = q_H^{FB}$$

$\Rightarrow$ No distortion at the top.

- F.O.C. w.r.p. q_L:

$$\begin{aligned} &\theta_L v'(q_L^{SB}) - c - (\theta_H - \theta_L) v'(q_L^{SB}) = 0 \\ \Rightarrow &\theta_L v'(q_L^{SB}) = c + (\theta_H - \theta_L) v'(q_L) > c \\ \Rightarrow &\theta_L v'(q_L^{SB}) > c = \theta_L v'(q_L^{FB}) \Leftrightarrow q_L^{SB} < q_L^{FB} \end{aligned}$$

$\Rightarrow$ Downward distortion.

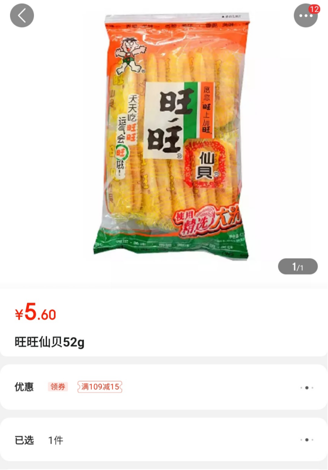
Evaluated at the solution, IR_L and IC_H are binding. For the remaining two constraints omitted, it can be verified that IC_L and IR_H are redundant:

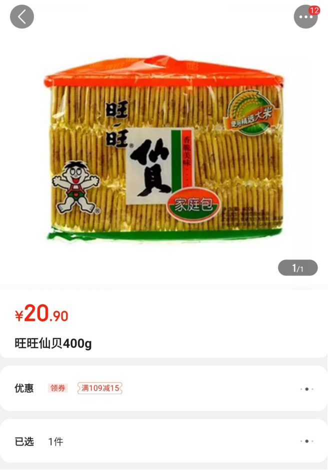
$\theta_H v(q_H) - T_H = \theta_H v(q_L) - T_L$	binding	IC_H
$\theta_L v(q_L) - T_L \geq \theta_L v(q_H) - T_H$	verify	IC_L
$\theta_H v(q_H) - T_H > 0$	implied by IR_L and IC_H	IR_H
$\theta_L v(q_L) - T_L = 0$	binding	IR_L

IC_L is redundant because evaluated at $T_L^{SB} = \theta_L v(q_L^{SB})$, the LHS of IC_L is zero. Using $T_H^{SB} = \theta_H v(q_H^{SB}) - \theta_H v(q_L^{SB}) + \theta_L v(q_L^{SB})$, the RHS of IC_L becomes
$(\theta_L - \theta_H) v(q_H^{SB}) + (\theta_H - \theta_L) v(q_L^{SB}) = (\theta_H - \theta_L) \left[v(q_L^{SB}) - v(q_H^{SB}) \right]$, which is negative by $q_L^{SB} < q_H^{SB}$.

Example: Downward-Distortion

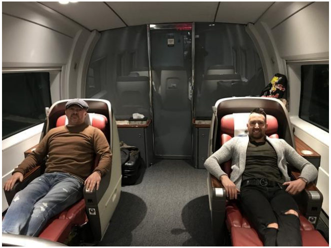
"量大从优"





Example: Downward-Distortion

二等座："我一直怀疑，设计车座的人是不是都以为我们长这样子!!"

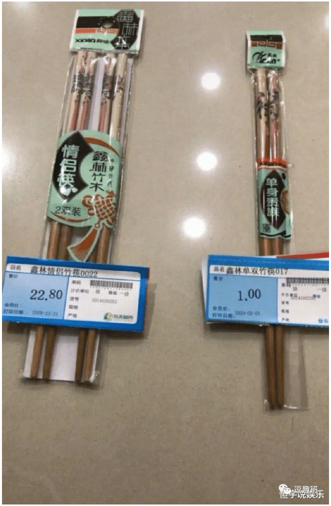


商务座

列车信息（以下余票信息仅供参考）			
2021-06-05（周六） G143次 北京南站（07:50开）—上海虹桥站（13:12到）			
商务座（￥1873.0）17张票	一等座（￥930.0）有票	二等座（￥553.0）有票	无座（￥553.0）无票
*显示的卧铺票价均为上铺票价，供您参考。具体票价以您确认支付时实际购买的铺别票价为准。			

假设腿一样长，但二等座设计的越舒服，θ_H 越有可能"伪装成" θ_L，从而违反 IC_H.

Counter-Example: Downward-Distortion?



Kuhn-Tucker Condition* (库恩-塔克条件)

Constrained Optimization with **Inequality Constraints**

- A general representation of "Lagrangian"
- The objective $\max_x f(x)$ ($f(x)$ is concave)
- Constraints: $g^i(x) \leq 0$, $i = 1, ..., m$, $x \geq 0$ (every $g^i(x)$ is convex)
- The Lagrangian: $\mathcal{L}(x, \lambda) = f(x) - \lambda g(x)$ where $\lambda \geq 0$.
- The optimal x^* satisfy
  1. $\frac{\partial \mathcal{L}}{\partial x_j} = f_j - \sum_{i=1}^m \lambda_i g_j^i \leq 0$; $\frac{\partial \mathcal{L}}{\partial x_j} x_j^* = 0$, $j = 1, ..., m$
  2. $g^i(x^*) \leq 0$ and $\lambda^* \geq 0$; $\lambda_i^* g^i(x^*) = 0$.
- "**Complementary Slackness**" (互补-松弛)
  - When $\lambda_i^* > 0$, then we say the constraint $g^i(x)$ is "binding"—that is, $g^i(x) = 0$.
  - Similarly, if $g^i(x) < 0$, then $\lambda_i^* = 0$.

Kuhn-Tucker: Decide Which Constraint is Binding

- The seller's objective: $T_H + T_L - cq_H - c_L$. Subjected to
  - IC_H: $\theta_H v(q_H) - T_H \geq \theta_H v(q_L) - T_L$, multiplier $\lambda_H \geq 0$
  - IC_L: $\theta_L v(q_L) - T_L \geq \theta_L v(q_H) - T_H$, multiplier $\lambda_L \geq 0$
  - IR_H: $\theta_H v(q_H) - T_H \geq 0$, μ_H, multiplier $\mu_H \geq 0$
  - IR_L: $\theta_L v(q_L) - T_L \geq 0$, μ_L, multiplier $\mu_L \geq 0$
- The Lagrangian is

$$
\begin{aligned}
\mathcal{L} = & T_H + T_L - cq_H - cq_L \\
& + \lambda_H \left[\theta_H v(q_H) - T_H - \theta_H v(q_L) + T_L\right] \\
& + \lambda_L \left[\theta_L v(q_L) - T_L - \theta_L v(q_H) + T_H\right] \\
& + \mu_H \left[\theta_H v(q_H) - T_H\right] \\
& + \mu_L \left[\theta_L v(q_L) - T_L\right]
\end{aligned}
$$

We have shown that IR_H is not binding, i.e.,
$\theta_H v(q_H) - T_H > 0 \Rightarrow \mu_H = 0$.

The Lagrangian becomes ($\mu_H = 0$)

$$\begin{aligned}
\mathcal{L} = & T_H + T_L - cq_H - cq_L \\
& + \lambda_H \left[\theta_H v(q_H) - T_H - \theta_H v(q_L) + T_L\right] \\
& + \lambda_L \left[\theta_L v(q_L) - T_L - \theta_L v(q_H) + T_H\right] \\
& + \mu_L \left[\theta_L v(q_L) - T_L\right]
\end{aligned}$$

- $\frac{\partial \mathcal{L}}{\partial q_H} = 0 \Rightarrow$
  $-c + (\lambda_H \theta_H - \lambda_L \theta_L) v'(q_H) = 0$
- $\frac{\partial \mathcal{L}}{\partial q_L} = 0 \Rightarrow$
  $-c + (-\lambda_H \theta_H + \lambda_L \theta_L + \mu_L \theta_L) v'(q_L) = 0$

Check "which IC is binding"

- $-c + (\lambda_H\theta_H - \lambda_L\theta_L)\, v'(q_H) = 0$
- $-c + (-\lambda_H\theta_H + \lambda_L\theta_L + \mu_L\theta_L)\, v'(q_L) = 0$
  - If both IC_H and IC_L are not binding $\Rightarrow \lambda_H = \lambda_L = 0$, the first equation implies $-c = 0$. A contradiction.
  - If both IC_H and IC_L are binding:
    $IC_H \overset{``='' }{\Longrightarrow} T_H - T_L = \theta_H\,(v(q_H) - v(q_L))$;
    $IC_L \overset{``='' }{\Longrightarrow} T_H - T_L = \theta_L\,(v(q_H) - v(q_L))$. A contradiction.
  - If IC_L is binding but IC_H is not $\Rightarrow \lambda_L > 0, \lambda_H = 0$, the first equation implies that $-c - \lambda_L\theta_L v'(q_H) = 0$. A contradiction.
  - The remaining possibility is $\lambda_H > 0$ and $\lambda_L = 0$, i.e., the first equation becomes $c = \lambda_H\theta_H v'(q_H)$; the second equation becomes $\mu_L\theta_L v'(q_L) = c + \lambda_H\theta_H v'(q_L)$.

Check whether "IR_L is biding" $\Rightarrow \mu_L > 0$

- If $\mu_L = 0$, then $\underbrace{\mu_L\theta_L v'(q_L)}_{=0} = \underbrace{c + \lambda_H\theta_H v'(q_L)}_{>0} \Rightarrow$ contradiction.

Therefore, $\lambda_H > 0$, $\lambda_L = 0$, $\mu_H = 0$ and $\mu_L > 0$. After eliminating IC_L and IR_H, the Lagrangian becomes

$$\begin{aligned}\mathcal{L} = & T_H + T_L - cq_H - cq_L \\ & +\lambda_H \left[\theta_H v(q_H) - T_H - \theta_H v(q_L) + T_L\right] \\ & +\mu_L \left[\theta_L v(q_L) - T_L\right]\end{aligned}$$

Because $\lambda_H > 0$ and $\mu_L > 0$, the constrained optimization can be transformed into an un-constrained optimization: when solving

$$\max_{q_H, q_L} T_H + T_L - cq_H - cq_L$$

T_H and T_L are replaced by using the two binding constraints, i.e.,

- $\mu_L > 0 \Rightarrow T_L = \theta_L v(q_L)$
- $\lambda_H > 0 \Rightarrow T_H = \theta_H v(q_H) - \theta_H v(q_L) + \theta_L v(q_L)$

More than Two Types*

Now there are n agents, whose valuations are ranked by

$$\theta_1 < ... < \theta_n.$$

Type $i = 1, ..., n$ occurs with prob β_i and $\sum_{i=1}^n \beta_i = 1$. The seller offers a non-linear schedule $\{(T_1, q_1), ..., (T_n, q_n)\}$ to solve

$$\max_{T_i, q_i} \sum_{i=1}^{n} \beta_i (T_i - c q_i)$$

subject to

$$\begin{array}{ll} \theta_i v(q_i) - T_i \geq \theta_i v(q_j) - T_j & IC_{ij}, \forall i, j, i \neq j \\ \theta_i v(q_i) - T_i \geq 0 & IR_i, \forall i \end{array}$$

Eliminate the Redundant Constraints

- The *IR* of the lowest type binds, and all the other *IR* are redundant:

$$\underbrace{\theta_1 v(q_1) - T_1 = 0}_{\text{suppose } IR_1 \text{ holds}} \Rightarrow$$

$$\underbrace{\theta_2 v(q_2) - T_2 \geq \theta_2 v(q_1) - T_1}_{IC_{21} \text{ holds}} > \theta_1 v(q_1) - T_1 = 0$$

$$\Rightarrow \underbrace{\theta_j v(q_j) - T_j > 0}_{IR_j \text{ holds}}, \forall j = 2, ..., n.$$

- How to eliminate the *IC*?

Spence-Mirrlees Single-Crossing Property

Assumption (Single-Crossing)

$\frac{\partial}{\partial \theta}\left[-\frac{\partial u/\partial q}{\partial u \partial T}\right] > 0$: *the "marginal rate of substitution" between commodity (q) in terms of numéraire (T) is increasing in types.*

Consider two arbitrary types, $\theta_i \neq \theta_j$, the *IC* are

$$\theta_i v(q_i) - T_i \geq \theta_i v(q_j) - T_j$$
$$\theta_j v(q_j) - T_j \geq \theta_j v(q_i) - T_i$$

$$\Rightarrow (\theta_i - \theta_j) v(q_i) \geq (\theta_i - \theta_j) v(q_j)$$
$$\Rightarrow \begin{cases} \theta_i > \theta_j \Rightarrow q_i > q_j \\ \theta_i < \theta_j \Rightarrow q_i < q_j \end{cases}$$

Hence, by single-crossing, $\theta_1 < \ldots < \theta_n$ is associated with $q_1 < \ldots < q_n$.

Local Downward Incentive Constraints (LDIC)

Consider three types, $\theta_1 < \theta_2 < \theta_3$. There are three "downward incentive" constraints:

$$\begin{array}{ll} \theta_3 v(q_3) - T_3 \geq \theta_3 v(q_2) - T_2 & \text{local downward } IC_{32} \\ \theta_2 v(q_2) - T_2 \geq \theta_2 v(q_1) - T_1 & \text{local downward } IC_{21} \\ \theta_3 v(q_3) - T_3 \geq \theta_3 v(q_1) - T_1 & \text{downward } IC_{31} \end{array}$$

By monotonicity $q_2(\theta_2) > q_1(\theta_1)$, $IC_{32} + IC_{21} \Rightarrow IC_{31}$:

$$\theta_3 v(q_3) - T_3 \geq \theta_3 v(q_2) - \theta_2 v(q_2) + \theta_2 v(q_1) - T_1 \quad IC_{32} + IC_{21}$$

$$\theta_3 v(q_3) - T_3 \geq \left[\underline{\theta_3 v(q_1)} - T_1\right] \underbrace{\underline{-\theta_3 v(q_1)} + \theta_3 v(q_2) - \theta_2 v(q_2) + \theta_2 v(q_1)}_{=(\theta_3 - \theta_2)[v(q_2) - v(q_1)] > 0}$$

LDICs binding

- All ICs are implied by binding LDICs.
- All LUICs are implied by binding LDICs (excise).
- The seller's problem reduces to

$$\begin{array}{ll}
\max_{q_i, T_i} \sum_{i=1}^{n} \beta_i (T_i - cq_i) & \\
\theta_1 v(q_1) - T_1 = 0 & IR_1 \\
\theta_i v(q_i) - T_i \geq \theta_i v(q_{i-1}) - T_{i-1} & IR_{i,i-1} (\# n - 1) \\
q_i > q_{i-1} & \text{monotonicity}
\end{array}$$

The Lagrangian is

$$\mathcal{L} = \sum_{i=1}^{n} \{\beta_i (T_i - cq_i) + \lambda_i [\theta_i v(q_i) - T_i - \theta_i v(q_{i-1}) + T_{i-1}]\}$$
$$+ \mu(\theta_1 v(q_1) - T_1)$$

$$\mathcal{L} = \sum_{i=1}^{n} \{\beta_i(T_i - cq_i) + \lambda_i [\theta_i v(q_i) - T_i - \theta_i v(q_{i-1}) + T_{i-1}]\}$$
$$+ \mu(\theta_1 v(q_1) - T_1)$$

- when $i = n$

$$\left.\begin{array}{l} \dfrac{\partial \mathcal{L}}{\partial q_n} = -\beta_n c + \lambda_n \theta_n v'(q_n) = 0 \\ \dfrac{\partial \mathcal{L}}{\partial T_n} = \beta_n - \lambda_n = 0 \end{array}\right. \Rightarrow \underbrace{\theta_n v'(q_n^{SB}) = c}_{\Leftrightarrow q_n^{SB} = q_n^{FB}}$$

- when $i < n$, note that in the *IC* constraint, there is a LDIC: $i + 1 \rightarrow i$

$$
\begin{aligned}
\mathcal{L} = & \sum_{i=1}^{n} \{\beta_i(T_i - cq_i) + \lambda_i [\theta_i v(q_i) - T_i - \theta_i v(q_{i-1}) + T_{i-1}]\} \\
& + \mu(\theta_1 v(q_1) - T_1) \\
= & \sum_{i=1}^{n} \beta_i(T_i - cq_i) + \lambda_n [\theta_n v(q_n) - T_n - \theta_n v(q_{n-1}) + T_{n-1}] \\
& + \lambda_{n-1} [\theta_{n-1} v(q_{n-1}) - T_{n-1} - \theta_{n-1} v(q_{n-2}) + T_{n-2}] + \lambda_{n-3}(...) \\
& + \mu(\theta_1 v(q_1) - T_1)
\end{aligned}
$$

when $i < n$,

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial q_i} &= -\beta_i c - \lambda_{i+1}\theta_{i+1} v'(q_i) + \lambda_i \theta_i v'(q_i) = 0 \\
\frac{\partial \mathcal{L}}{\partial T_i} &= \beta_i + \lambda_{i+1} - \lambda_i = 0.
\end{aligned}
$$

From

$$\frac{\partial \mathcal{L}}{\partial q_i} = -\beta_i c - \lambda_{i+1}\theta_{i+1}v'(q_i) + \lambda_i\theta_i v'(q_i) = 0$$

$$\frac{\partial \mathcal{L}}{\partial T_i} = \beta_i + \lambda_{i+1} - \lambda_i = 0$$

for $i = n-1$, for example, by $q_{n-1} < q_n$:

$$\lambda_{n-1}\theta_{n-1}v'(q_{n-1}) = \beta_{n-1}c + \lambda_n\theta_n \underbrace{v'(q_{n-1})}_{>v'(q_n)} > \beta_{n-1}c + \lambda_n c$$

$$\Leftrightarrow \theta_{n-1}v'(q_{n-1}) > \frac{\beta_{n-1} + \lambda_n}{\lambda_{n-1}} c = c$$

Similarly, for all $i < n$, $\theta_i v'(q_i^{SB}) > c = \theta_i(q_i^{FB}) \Rightarrow q_i^{SB} < q_i^{FB}$.

Continuum Types*

Assume that θ is distributed according to CDF $F(\cdot)$ and PDF $f(\cdot) = F'(\cdot)$ with support $[\theta_L, \theta_H]$.

- The firm solves $\max_{q(\theta), T(\theta)} \int_{\theta_L}^{\theta_H} (T(\theta) - cq(\theta))f(\theta)d\theta$
- For the consumers:
  - Each θ buys (IR): $\theta v(q(\theta)) - T(\theta) \geq 0$
  - Each θ will not buy the menu designed for others (IC): $\theta v(q(\theta)) - T(\theta) \geq \theta v(q(\tilde{\theta})) - T(\tilde{\theta}), \forall \theta, \tilde{\theta} \in [\theta_L, \theta_H]$
- Binding IR: $\theta_L v(q(\theta_L)) - T(\theta_L) = 0$;
- For IC: a rational agent with type θ chooses $(q(\hat{\theta}), T(\hat{\theta}))$:

$$\max_{\hat{\theta}} \theta v(q(\hat{\theta})) - T(\hat{\theta})$$
$$\Rightarrow \theta v'(q(\hat{\theta}))\frac{dq}{d\hat{\theta}} = T'(\hat{\theta})$$

The menu is "reasonable" provided that $\hat{\theta}$ "coincide" with θ.

Implementation: Mirrlees (1971)

- Agent θ chooses $(q(\hat{\theta}), T(\hat{\theta}))$: $\theta v'(q(\hat{\theta}))\frac{dq}{d\hat{\theta}} = T'(\hat{\theta})$;
- Agent θ chooses a contract that is intended to be designed for θ, i.e., evaluated at $\hat{\theta} = \theta$, the optimal choice of each agent is $\theta v'(q(\theta))\frac{dq}{d\theta} = T'(\theta)$.
- The indirect utility of agent θ is

$$V(\theta) = \theta v(q(\theta)) - T(\theta) = \max_{\hat{\theta}} \theta v(q(\hat{\theta})) - T(\hat{\theta})$$

- By Envelop Theorem: $\frac{dV}{d\theta} = v(q(\theta))$.
- Integrating from $[\theta_L, \theta]$:

$$\int_{\theta_L}^{\theta} \frac{dV}{d\theta} d\theta = V(\theta) - V(\theta_L) = \int_{\theta_L}^{\theta} v(q(\theta)) d\theta.$$

- By IR: $V(\theta_L) = 0 \Rightarrow V(\theta) = \int_{\theta_L}^{\theta} v(q(\theta))d\theta$.
- Indirect utility: $V(\theta) = \theta v(q(\theta)) - T(\theta)$
- The seller's profit becomes

$$\begin{aligned}
\pi &= \int_{\theta_L}^{\theta_H} [T(\theta) - cq(\theta)]\, f(\theta)d\theta \\
&= \int_{\theta_L}^{\theta_H} \left[\theta v(q(\theta)) - \int_{\theta_L}^{\theta} v(q(\theta))d\theta - cq(\theta)\right] f(\theta)d\theta \\
&= \int_{\theta_L}^{\theta_H} [\theta v(q(\theta)) - cq(\theta)]\, f(\theta)d\theta \\
&+ \quad \underbrace{\int_{\theta_L}^{\theta_H} \left[\int_{\theta_L}^{\theta} v(q(\theta))d\theta\right] d[1 - F(\theta)]}_{= \int_{\theta_L}^{\theta} v(q(\theta))d\theta [1-F(\theta)] \Big|_{\theta_L}^{\theta_H} - \int_{\theta_L}^{\theta_H} [1-F(\theta)] v(q(\theta))d\theta} \\
&= \int_{\theta_L}^{\theta_H} \{[\theta v(q(\theta)) - cq(\theta)]\, f(\theta) - [1 - F(\theta)]\, v(q(\theta))\}\, d\theta.
\end{aligned}$$

- The seller solves

$$\max_{q(\theta)} \int_{\theta_L}^{\theta_H} \left\{ \left[\theta v(q(\theta)) - cq(\theta) \right] f(\theta) - \left[1 - F(\theta) \right] v(q(\theta)) \right\} d\theta$$

$$\Rightarrow \left[\theta v'(q(\theta)) - c \right] f(\theta) = \left[1 - F(\theta) \right] v'(q(\theta))$$

$$\Rightarrow \left[\theta - \frac{1 - F(\theta)}{f(\theta)} \right] v'(q^{SB}(\theta)) = c.$$

- No distortion at top: when $\theta = \theta_H$,

$$\underbrace{\left(\theta_H - \frac{1 - F(\theta_H)}{f(\theta_H)} \right)}_{=\theta_H} v'(q(\theta_H)) = c \Rightarrow q^{SB}(\theta_H) = q^{FB}(\theta_H)$$

Second-Best Contract

Definition

$h(\theta) = \frac{f(\theta)}{1-F(\theta)}$ *hazard ratio;* $\theta - \frac{1-F(\theta)}{f(\theta)}$ *virtual type.*

Assumption

The hazard ratio and the virtual type is increasing in θ*.*

The second-best contract is

$$\underbrace{\left[\theta - \underbrace{\frac{1 - F(\theta)}{f(\theta)}}_{\text{inverse hazard ratio}}\right]}_{\text{virtual type}} \sqrt{\,}(q^{SB}(\theta)) = c$$

Second-Best v.s. Monopoly Linear Pricing

- Recall, the monopoly linear pricing: $T = pq$ and $\theta v(q) - T = \theta v(q) - pq \Rightarrow$:

$$\theta v'(q^m) = p^m$$

- The second-best non-linear pricing:

$$\left[\theta - \frac{1-F(\theta)}{f(\theta)}\right] v'(q^{SB}(\theta)) = c$$

$$\Rightarrow \underbrace{\frac{p-c}{p}}_{\text{price markup}} = \frac{1}{\theta} \underbrace{\frac{1-F(\theta)}{f(\theta)}}_{=\text{inverse hazard ratio}=1/h(\theta)}$$

- A higher markup (distortion) for the low-type to prevent adverse selection

$$\frac{d}{d\theta}\left(\frac{p-c}{p}\right) = -\frac{1}{\theta^2}\frac{1}{h(\theta)} + \frac{1}{\theta}\frac{d}{d\theta}\left[\frac{1}{h(\theta)}\right] < 0.$$

Example: Insurance

- Under complete information, people subject to diversifiable risk should receive complete insurance against the risk from a risk neutral insurance company. The result fails under asymmetric information
- Consider a risk-averse agent with $u(.)$ where $u'(.) > 0$ and $u''(.) < 0$. The agent's initial wealth is w. With probability θ the agent suffers a damage of d. Agent types could be θ_H or θ_L with probability β and $1 - \beta$, respectively.
- The net reimbursement in case of a damage is a; the insurance premium to be paid is n. The agent's expected utility is

$$U(a, n) = \theta \underbrace{u(w - d + a)}_{=u_a} + (1 - \theta) \underbrace{u(w - n)}_{=u_n}$$

Complete Information

- Agent's utility: $U(a, n) = \theta u(w - d + a) + (1 - \theta)u(w - n)$
- Reservation utility $R = \theta u(w - d) + (1 - \theta)u(w)$
- Observe that $R_H < R_L$:
  - $R_H - R_L = (\theta_H - \theta_L)u(w - d) - (\theta_H - \theta_L)u(w) < 0.$
- Under complete information, the insurance company solves

$$\max_{a,n} -\theta a + (1-\theta)n$$

$$s.t.\ \theta u(w - d + a) + (1 - \theta)u(w - n)$$

$$\geq \theta u(w - d) + (1 - \theta)u(w) = R$$

for each type.

- The Lagrangian:

$$
\begin{aligned}
&\mathcal{L} = -\theta a + (1-\theta)n \\
&+\lambda\left(\theta u(w-d+a) + (1-\theta)u(w-n) - R\right) \\
&a: \quad -\theta + \lambda\theta u'(w-d+a) = 0 \\
&n: \ 1-\theta-\lambda(1-\theta)u'(w-n) = 0 \\
&\lambda: \ \theta u(w-d+a) + (1-\theta)u(w-n) = R
\end{aligned}
$$

- The first two conditions:
  - $-d + a^* = -n^* \Rightarrow u_a^* = u_n^*$ for each type
- Combining the last equation: $\theta u_a^* + (1-\theta)u_n^* = R$ for each type
- Because $R_H < R_L$, we have $u_n^H < u_n^L$
- $u'(.) > 0$, then $u(w-n^H) < u(w-n^L) \Rightarrow n^H > n^L$.

Incomplete Information: Adverse Selection

- Because $R_H < R_L$, the high-risk agent is willing to take the insurance contract designed for the low-risk agent.
- The insurance company designs non-linear contracts: (a_H, n_H) and (a_L, n_L)
- The insurance company solves

$$\max_{a_H, n_H, a_L, n_L} (1-\beta)(-\theta_L a_L + (1-\theta_L)n_L) + \beta(-\theta_H a_H + (1-\theta_H)n_H)$$

  - IC_H: $\theta_H u(w-d+a_H) + (1-\theta_H)u(w-n_H) \geq \theta_H u(w-d+a_L) + (1-\theta_H)u(w-n_L)$
  - IC_L: $\theta_L u(w-d+a_L) + (1-\theta_L)u(w-n_L) \geq \theta_L u(w-d+a_H) + (1-\theta_L)u(w-n_H)$
  - IR_H: $\theta_H u(w-d+a_H) + (1-\theta_H)u(w-n_H) \geq R_H$
  - IR_L: $\theta_L u(w-d+a_L) + (1-\theta_L)u(w-n_L) \geq R_L$

- Denote $h = u^{-1}$ the inverse function of $u(.)$.
  - $u'(.) > 0 \Rightarrow h' = \frac{1}{u'(.)} > 0$
  - $u''(.) < 0 \Rightarrow h'' = -\frac{u''(.)}{(u'(.))^2} > 0$.
- $u_a = u(w - d - a) \Rightarrow w - d - a = h(u_a) \Rightarrow a = w - d - h(u_a)$
- $u_n = u(w - n) \Rightarrow w - n = h(u_n) \Rightarrow n = w - h(u_n)$
- The objective $\max_{a_H, n_H, a_L, n_L} (1-\beta)(-\theta_L a_L + (1-\theta_L)n_L) + \beta(-\theta_H a_H + (1-\theta_H)n_H)$ becomes
  - $\max_{u_a^H, u_n^H, u_a^L, u_n^L} (1-\beta)(-\theta_L d - \theta_L h(u_a^L) - (1-\theta_L)h(u_n^L) + w) + \beta(-\theta_H d - \theta_H h(u_a^H) - (1-\theta_H)h(u_n^H) + w)$
- The binding constraint is IC_H and IR_L
  - IC_H: $\theta_H u_a^H + (1-\theta_H)u_n^H \geq \theta_H u_a^L + (1-\theta_H)u_n^L$. Multiplier λ
  - IR_L: $\theta_L u_a^L + (1-\theta_L)u_n^L \geq R_L$. Multiplier μ

- The Lagrangian:

$$\begin{aligned}\mathcal{L} = &(1-\beta)\left[-\theta_L d - \theta_L h(u_a^L) - (1-\theta_L)h(u_n^L) + w\right] \\ &+ \beta\left[-\theta_H d - \theta_H h(u_a^H) - (1-\theta_H)h(u_n^H) + w\right] \\ &+ \lambda\left[\theta_H u_a^H + (1-\theta_H)u_n^H - \theta_H u_a^L - (1-\theta_H)u_n^L\right] \\ &+ \mu\left[\theta_L u_a^L + (1-\theta_L)u_n^L - R_L\right]\end{aligned}$$

- FOC u_a^L: $-\theta_L(1-\beta)h'(u_a^L) - \theta_H\lambda + \mu\theta_L = 0$
- FOC u_n^L: $-(1-\theta_L)(1-\beta)h'(u_n^L) - (1-\theta_H)\lambda + (1-\theta_L)\mu = 0$
- FOC u_a^H: $-\theta_H\beta h'(u_a^H) + \theta_H\lambda = 0.$
- FOC u_n^H: $-(1-\theta_H)\beta h'(u_n^H) + (1-\theta_H)\lambda = 0$

- From the FOCs of u_a^H and u_n^H:
  - $h'(u_a^H) = h'(u_n^H) \Rightarrow u_a^H = u_n^H = u^H$
  - No distortions for the high-risk agent
- FOC of $u_a^H \Rightarrow -\theta_H \beta h'(u_a^H) + \theta_H \lambda = 0$. Hence $\lambda > 0 \Rightarrow IC_H$ is binding:
  - $\theta_H u_a^H + (1-\theta_H) u_n^H = \theta_H u_a^L + (1-\theta_H) u_n^L$
- FOC of $u_a^L \Rightarrow \mu \theta_L = \theta_H \lambda + \theta_L (1-\beta) h'(u_a^L) > 0$. Hence $\mu > 0 \Rightarrow IR_L$ is binding
  - $\theta_L u_a^L + (1-\theta_L) u_n^L = R_L$
- Summing the binding IC_H and IR_L:
  - $u_a^H = u_n^H = R_L - (\theta_H - \theta_L)(u_n^L - u_a^L) = R_L - (\theta_H - \theta_L)\Delta u$
- Using $\Delta u = u_n^L - u_a^L$, from IR_L, we have
  - $u_a^L = R_L - (1-\theta_L)\Delta u$
  - $u_n^L = R_L + \theta_L \Delta u$
- Plug the red equations into the objective.

- The insurance company solves

$$\max_{\Delta u}(1-\beta)\left[-\theta_L d-\theta_L h(R_L-(1-\theta_L)\Delta u)\right]$$
$$+(1-\beta)\left[-(1-\theta_L)h(R_L+\theta_L\Delta u)+w\right]$$
$$+\beta\left[-\theta_H d-h(R_L-(\theta_H-\theta_L)\Delta u)+w\right]$$

- FOC Δu:

$$(1-\beta)\theta_L(1-\theta_L)h'(R_L-(1-\theta_L)\Delta u)$$
$$-(1-\beta)(1-\theta_L)\theta_L h'(R_L+\theta_L\Delta u)$$
$$+\beta(\theta_H-\theta_L)h'(R_L-(\theta_H-\theta_L)\Delta u)=0$$

- The second-best solution Δu^{SB} is implicitly determined by

$$\frac{\beta(\theta_H-\theta_L)}{(1-\beta)\theta_L(1-\theta_L)}h'\left(R_L-(\theta_H-\theta_L)\Delta u^{SB}\right)$$
$$=h'\left(R_L+\theta_L\Delta u^{SB}\right)-h'\left(R_L-(1-\theta_L)\Delta u^{SB}\right)$$

- Because $h'(.) = \frac{1}{u'(.)} > 0$, we know

$$h'\left(R_L + \theta_L \Delta u^{SB}\right) > h'\left(R_L - (1-\theta_L)\Delta u^{SB}\right)$$

- Because $h''(.) = -\frac{u''(.)}{(u'(.))^2} > 0$, we know

$$R_L + \theta_L \Delta u^{SB} > R_L - (1-\theta_L)\Delta u^{SB} \Rightarrow \Delta u^{SB} > 0$$

- That is, at second-best, $u_n^L > u_a^L$, or $u(w - d - a^{SB}) < u(w - n^{SB})$. Meanwhile, $u_a^H = u_n^H$.
- To reduce the incentives of the high-risk agent ot pretend being a low-risk one, the insurance company let the latter type bear some risk.

References

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