PS5 Solutions

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We use the following notation:

$$\Phi_1 = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}, \quad \Phi_\varepsilon = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}, \quad \Sigma_u := \mathbb{V}[u_t] = \mathbb{E}[u_t u_t'] = \Phi_\varepsilon \Phi_\varepsilon'.$$

Solution (a).
The VAR(1) is weakly stationary iff all eigenvalues of Φ_1 lie strictly inside the unit circle,

$$\rho(\Phi_1) = \max_i \left|\lambda_i(\Phi_1)\right| < 1.$$

Solution (b).
Assuming y_t is stationary, $\mathbb{E}[y_t] = \mathbb{E}[\Phi_1 y_{t-1} + \Phi_\varepsilon \varepsilon_t] = \Phi_1 \mathbb{E}[y_{t-1}]$, as $\mathbb{E}[y_t] = \mu, \forall t$, $\mu = 0$.

$$\begin{aligned} \Gamma_{yy}(0) = \mathbb{E}[y_t y_t'] &= \mathbb{E}\left[(\Phi_1 y_{t-1} + u_t)(\Phi_1 y_{t-1} + u_t)'\right] \\ &= \mathbb{E}\left[\Phi_1 y_{t-1} y_{t-1}' \Phi_1' + \Phi_1 y_{t-1} u_t' + u_t y_{t-1}' \Phi_1' + u_t u_t'\right] \\ &= \Phi_1 \mathbb{E}[y_{t-1} y_{t-1}'] \Phi_1' + \Phi_1 \mathbb{E}[y_{t-1} u_t'] + \mathbb{E}[u_t y_{t-1}'] \Phi_1' + \mathbb{E}[u_t u_t'] \\ &= \Phi_1 \Gamma_{yy}(0) \Phi_1' + \mathbb{E}[u_t u_t'] \\ &= \Phi_1 \Gamma_{yy}(0) \Phi_1' + \Sigma_u \end{aligned}$$

Since u_t contains contemporaneous shocks ε_t which are independent of past y_{t-1}, $\mathbb{E}[y_{t-s} u_t'] = 0$ for $s \geq 0$, $\mathbb{E}[y_{t-1} u_t'] = 0$ and $\mathbb{E}[u_t y_{t-1}'] = 0$. As $\Sigma_u' = (\Phi_\varepsilon \Phi_\varepsilon')' = \Phi_\varepsilon \Phi_\varepsilon' = \Sigma_u$ is Hermitian, this is a discrete Lyapunov equation, which can be solved for $\Gamma_{yy}(0)$ using the vectorization operator:

$$\operatorname{vec}(\Gamma_{yy}(0)) = (I_{k^2} - \Phi_1 \otimes \Phi_1)^{-1} \operatorname{vec}(\Sigma_u)$$

where $k = 2$ is the dimension of y_t (so $k^2 = 4$), and $\otimes$ is the Kronecker product.

$$\begin{aligned} \Gamma_{yy}(1) = \mathbb{E}[y_t y_{t-1}'] &= \mathbb{E}[(\Phi_1 y_{t-1} + u_t) y_{t-1}'] \\ &= \Phi_1 \mathbb{E}[y_{t-1} y_{t-1}'] + \mathbb{E}[u_t y_{t-1}'] \end{aligned}$$

Again, $\mathbb{E}[u_t y_{t-1}'] = 0$. So, $\Gamma_{yy}(1) = \Phi_1 \Gamma_{yy}(0)$.

Solution (c).

Starting from $y_t = \Phi_1 y_{t-1} + \Phi_\varepsilon \varepsilon_t$. By repeated substitution, we can write y_t in its MA(∞) representation (assuming stationarity):

$$y_t = \sum_{j=0}^{\infty} \Phi_1^j \Phi_\varepsilon \varepsilon_{t-j} + \lim_{k\to\infty} \Phi_1^k y_{t-k}.$$

(Here $\Phi_1^0 = I_k$, where $k = 2$). As we assume that y_t is stationary, we know that Φ_1 has all the eigenvalues in the unit circle, and the last term vanishes as $k \to \infty$. Define $e_1 = (1, 0)'$, $e_2 = (0, 1)'$. A one-unit structural shock at t affects y_{t+h} by

$$\Psi(h) := \frac{\partial y_{t+h}}{\partial \varepsilon_t} = \Phi_1^h \Phi_\varepsilon, \quad h = 0, 1, \dots$$

Impact of a labour-supply shock three periods ago on log wages

$$\frac{\partial w_t}{\partial \varepsilon_{b,t-3}} = e_1' \Psi(3)\, e_2 = e_1' \Phi_1^3 \Phi_\varepsilon e_2 = \left(\Phi_1^3\right)_{1\bullet} b_{12}.$$

where $\left(\Phi_1^3\right)_{1\bullet}$ denotes the first row of Φ_1^3.

Solution (d).
From the reduced-form VAR, we can consistently estimate Φ_1 and $\Sigma_u = \Phi_\varepsilon \Phi_\varepsilon'$.

$$\Sigma_u = \begin{bmatrix} b_{11}^2 + b_{12}^2 & b_{11}b_{21} + b_{12}b_{22} \\ \cdot & b_{21}^2 + b_{22}^2 \end{bmatrix},$$

provides three distinct equations for the four unknowns in Φ_ε. If Φ_ε is a solution, then for any $k \times k$ (here we have $k = 2$) orthogonal matrix P (such that $PP' = I_k$), $\Phi_\varepsilon^* = \Phi_\varepsilon P$ is also a solution because $\Phi_\varepsilon^*(\Phi_\varepsilon^*)' = (\Phi_\varepsilon P)(\Phi_\varepsilon P)' = \Phi_\varepsilon PP'\Phi_\varepsilon' = \Phi_\varepsilon I_k \Phi_\varepsilon' = \Phi_\varepsilon \Phi_\varepsilon' = \Sigma_u$. The identification problem is to find restrictions to pin down P.

Solution (e).
As $u_t = (u_{w,t}, u_{h,t})' = \Phi_\varepsilon \varepsilon_t$, we know:

$$\begin{aligned} u_{w,t} &= b_{11}\varepsilon_{a,t} + b_{12}\varepsilon_{b,t} \\ u_{h,t} &= b_{21}\varepsilon_{a,t} + b_{22}\varepsilon_{b,t} \end{aligned}$$

and the asusmption gives that $u_{h,t}$ is only affected by $\varepsilon_{b,t}$, so $b_{21} = 0$.
Since we need only 1 restriction for 2×2 matrix, this is exactly enough for identification (up to sign normalizations). With $b_{21} = 0$, Φ_ε becomes upper triangular:

$$\Phi_\varepsilon = \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix}$$

Then $\Sigma_u = \Phi_\varepsilon \Phi_\varepsilon' = \begin{bmatrix} b_{11} & b_{12} \\ 0 & b_{22} \end{bmatrix} \begin{bmatrix} b_{11} & 0 \\ b_{12} & b_{22} \end{bmatrix} = \begin{bmatrix} b_{11}^2 + b_{12}^2 & b_{12}b_{22} \\ b_{12}b_{22} & b_{22}^2 \end{bmatrix}$.

Let $\Sigma_u = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$ (where $\sigma_{12} = \sigma_{21}$).

1. From $\sigma_{22} = b_{22}^2$, we get $b_{22} = \sqrt{\sigma_{22}}$ (by convention, positive).
2. From $\sigma_{12} = b_{12}b_{22}$, we get $b_{12} = \sigma_{12}/b_{22}$ (assuming $b_{22} \neq 0$).
3. From $\sigma_{11} = b_{11}^2 + b_{12}^2$, we get $b_{11} = \sqrt{\sigma_{11} - b_{12}^2}$ (by convention, positive, and assuming $\sigma_{11} - b_{12}^2 \geq 0$).

This uniquely identifies Φ_ε (given sign normalizations for diagonal elements). This procedure is equivalent to finding an upper Cholesky factor of Σ_u.

Solution (f).

1. Labor supply shock ($\varepsilon_{b,t}$) moves wages (w_t) and hours (h_t) in opposite directions upon impact: $\frac{\partial w_t}{\partial \varepsilon_{b,t}} = b_{12}$ and $\frac{\partial h_t}{\partial \varepsilon_{b,t}} = b_{22}$. So, $b_{12} \cdot b_{22} < 0$.
2. Demand shock ($\varepsilon_{a,t}$) moves wages (w_t) and hours (h_t) in the same direction upon impact: $\frac{\partial w_t}{\partial \varepsilon_{a,t}} = b_{11}$ and $\frac{\partial h_t}{\partial \varepsilon_{a,t}} = b_{21}$. So, $b_{11} \cdot b_{21} > 0$.

These are inequality restrictions. They do not typically lead to point identification. Let $\Phi_{\varepsilon,0}$ be any matrix such that $\Phi_{\varepsilon,0}\Phi_{\varepsilon,0}' = \Sigma_u$ (e.g., from a Cholesky decomposition of Σ_u). Then any other valid matrix is $\Phi_\varepsilon = \Phi_{\varepsilon,0}P$, where P is an orthogonal matrix.

For $k = 2$, P can be a rotation matrix $P(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ (or $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, depending on convention). The sign restrictions define a set of admissible rotation angles θ. If this set is not a singleton (or two points corresponding to P and $-P$ after sign normalizations), then Φ_ε is not uniquely identified. Generally, sign restrictions lead to set identification, meaning there is a range of θ values (and thus a set of Φ_ε matrices) consistent with the restrictions. So, this is not enough to uniquely identify Φ_ε, the model is only *set-identified.*

Solution (g). NO question (g).

Solution (h).
From the structural model, $\frac{\partial y_t}{\partial \varepsilon_t} = \Phi_\varepsilon$. This gives us:

$$\frac{\partial w_t}{\partial \varepsilon_{a,t}} = b_{11}, \quad \frac{\partial w_t}{\partial \varepsilon_{b,t}} = b_{12}$$
$$\frac{\partial h_t}{\partial \varepsilon_{a,t}} = b_{21}, \quad \frac{\partial h_t}{\partial \varepsilon_{b,t}} = b_{22}$$

The assumption that labor demand depends only on technology shocks means: $\frac{\partial \varphi^D}{\partial \varepsilon_{b,t}} = 0$. However, this doesn't mean that hours don't respond to preference shocks at all. Since hours are determined in equilibrium where demand equals supply, a preference shock will affect wages, which will then affect labor demand.
To formalize this, we can use the total derivative of the labor demand function with respect to the preference shock:

$$\frac{\partial h_t}{\partial \varepsilon_{b,t}} = \frac{\partial \varphi^D}{\partial \varepsilon_{b,t}} + \frac{\partial \varphi^D}{\partial w_t} \frac{\partial w_t}{\partial \varepsilon_{b,t}}$$

Since $\frac{\partial \varphi^D}{\partial \varepsilon_{b,t}} = 0$, we have:

$$\frac{\partial h_t}{\partial \varepsilon_{b,t}} = \frac{\partial \varphi^D}{\partial w_t} \frac{\partial w_t}{\partial \varepsilon_{b,t}} \Rightarrow b_{22} = \frac{\partial \varphi^D}{\partial w_t} b_{12}$$

From $\Sigma_u = \Phi_\varepsilon \Phi_\varepsilon'$, we have three restrictions, and our labor market restriction gives us a relationship between b_{12} and b_{22}, but with an unknown parameter $\frac{\partial \varphi^D}{\partial w_t}$.
Therefore, the given restriction alone is not sufficient to uniquely identify Φ_ε. This is why in the following question, we give a certain α as this parameter to help identify Φ_ε.

Solution (i).

Let L be the lower Cholesky factor of Σ_u, such that $LL' = \Sigma_u$. $L = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix}$, where

$$\sigma_{11} = l_{11}^2, \quad \sigma_{12} = l_{11} l_{21}, \quad \sigma_{22} = l_{21}^2 + l_{22}^2$$

Any Φ_ε such that $\Phi_\varepsilon \Phi_\varepsilon' = \Sigma_u$ can be written as $\Phi_\varepsilon = LP(\theta)$ for some orthogonal matrix $P(\theta)$. For $k = 2$, a common rotation matrix is $P(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

$$\Phi_\varepsilon = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} l_{11}\cos\theta & -l_{11}\sin\theta \\ l_{21}\cos\theta + l_{22}\sin\theta & -l_{21}\sin\theta + l_{22}\cos\theta \end{bmatrix}$$

So, $b_{12} = -l_{11}\sin\theta$ and $b_{22} = -l_{21}\sin\theta + l_{22}\cos\theta$.
The restriction $b_{12} = (\alpha - 1)b_{22}$ becomes:

$$-l_{11}\sin\theta = (\alpha - 1)[-l_{21}\sin\theta + l_{22}\cos\theta]$$

$$[(\alpha - 1)l_{21} - l_{11}]\sin\theta = (\alpha - 1)l_{22}\cos\theta$$

If $(\alpha - 1)l_{22} \neq 0$ and the coefficient of $\sin\theta$ is not zero (and $\cos\theta \neq 0$ to avoid division by zero for $\tan\theta$):

$$\tan\theta = \frac{(\alpha - 1)l_{22}}{(\alpha - 1)l_{21} - l_{11}}$$

As Cholesky decomposition is unique (up to sign normalizations), l_{11}, l_{21}, l_{22} are uniquely determined, hence $\tan\theta$ is uniquely determined. Hence this equation determines θ up to a multiple of π. For example, if θ_0 is a solution, then $\theta_0 + \pi$ is also a solution. Adding π to θ changes $P(\theta)$ to $-P(\theta)$, which flips the sign of all elements in Φ_ε. This means Φ_ε is identified up to an overall sign change.