Macroeconomics A; EI060

Short problems

Cédric Tille

Class of March 12, 2025

1 Euler under uncertainty

Question: he consumer maximizes the following utility:

$$u(C_1) + \beta E u(C_2)$$

where k denotes the state of nature of probability $\pi(k)$. He can invest in a bond giving an interest rate $r = (1/\beta - 1)$. Output is an endowment, which can vary in period 2. The budget constraints are:

$$\begin{aligned} C_1 + B_2 &= Y_1 \\ C_2(k) &= \frac{1}{\beta} B_2 + Y_2(k) \end{aligned}$$

Show that:

$$u'(C_1) = E[u'(C_2)]$$

Answer: Writing things in terms of state of nature, the Lagrangian is:

$$\begin{aligned} \mathcal{L} = & \; u(C_1) + \beta \sum_k \pi(k) u(C_2(k)) \\ & + \lambda_1 [Y_1 - C_1 - B_2] \\ & + \sum_k \lambda_2(k) \left[\frac{1}{\beta} B_2 + Y_2(k) - C_2(k) \right] \end{aligned}$$

The optimality conditions for C_1, $C_2(k)$ and B_2 are:

$$\begin{aligned} 0 &= u'(C_1) - \lambda_1 \\ 0 &= \beta \pi(k) u'(C_2(k)) - \lambda_2(k) \\ 0 &= -\lambda_1 + \frac{1}{\beta} \sum_k \lambda_2(k) \end{aligned}$$

Combining, we get the Euler condition:

$$\begin{aligned}
\lambda_1 &= \frac{1}{\beta}\sum_k \lambda_2(k) \\
u'(C_1) &= \sum_k \pi(k)\, u'(C_2(k)) \\
u'(C_1) &= \sum_k \pi(k)\, u'(C_2(k)) \\
u'(C_1) &= E\left[u'(C_2)\right]
\end{aligned}$$

2 Approximation of budget constraint

Question: We approximate the system around an allocation where $C_2(k) = C_1 = Y_2(k) = Y_1 = \overline{C}$

Define the approximation, scaled by the steady state value. For instance:

$$\widehat{C}_1 = \frac{C_1 - \overline{C}}{\overline{C}}$$

Show that the budget constraint in period 2 in state k can be written as the linear approximation:

$$\widehat{C}_2(k) = \frac{1}{\beta}\left(\widehat{Y}_1 - \widehat{C}_1\right) + \widehat{Y}_2(k)$$

Answer: The budget constraint is:

$$\begin{aligned}
C_2(k) &= \frac{1}{\beta}B_2 + Y_2(k) \\
C_2(k) &= \frac{1}{\beta}(Y_1 - C_1) + Y_2(k)
\end{aligned}$$

We approximate it in a linear way as:

$$\begin{aligned}
C_2(k) &= \frac{1}{\beta}(Y_1 - C_1) + Y_2(k) \\
\overline{C} + \frac{C_2(k) - \overline{C}}{\overline{C}} &= \frac{1}{\beta}\left(\overline{C} + \frac{Y_1 - \overline{C}}{\overline{C}} - \overline{C} + \frac{C_1 - \overline{C}}{\overline{C}}\right) + \overline{C} + \frac{Y_2(k) - \overline{C}}{\overline{C}} \\
\frac{C_2(k) - \overline{C}}{\overline{C}} &= \frac{1}{\beta}\left(\frac{Y_1 - \overline{C}}{\overline{C}} + \frac{C_1 - \overline{C}}{\overline{C}}\right) + \frac{Y_2(k) - \overline{C}}{\overline{C}} \\
\widehat{C}_2(k) &= \frac{1}{\beta}\left(\widehat{Y}_1 - \widehat{C}_1\right) + \widehat{Y}_2(k)
\end{aligned}$$

3 Quadratic approximation of Euler condition

Question: Recal that a general function of consumption can be expanded as such up to a quadratic term:

$$f(C) = f\left(\overline{C}\right) + f'\left(\overline{C}\right)\left(C - \overline{C}\right) + \frac{1}{2}f''\left(\overline{C}\right)\left(C - \overline{C}\right)^2$$

Show that:

$$\begin{aligned}
u'(C_1) &= u'(\overline{C}) + u''(\overline{C})\,\overline{C}\widehat{C}_1 + \frac{1}{2}u'''(\overline{C})(\overline{C})^2\left(\widehat{C}_1\right)^2 \\
u'(C_2(k)) &= u'(\overline{C}) + u''(\overline{C})\,\overline{C}\widehat{C}_2(k) + \frac{1}{2}u'''(\overline{C})(\overline{C})^2\left(\widehat{C}_2(k)\right)^2
\end{aligned}$$

Show that the quadratic approximation of the Euler is:

$$\widehat{C}_1 = E\left[\widehat{C}_2\right] + \frac{1}{2}\frac{u'''(\overline{C})}{u''(\overline{C})}\overline{C}\left[-\left(\widehat{C}_1\right)^2 + E\left[\left(\widehat{C}_2\right)^2\right]\right]$$

Answer: Using the formula, we write:

$$\begin{aligned}
u'(C_1) &= u'(\overline{C}) + u''(\overline{C})(C_1 - \overline{C}) + \frac{1}{2}u'''(\overline{C})(C_1 - \overline{C})^2 \\
u'(C_1) &= u'(\overline{C}) + u''(\overline{C})\,\overline{C}\frac{C_1 - \overline{C}}{\overline{C}} + \frac{1}{2}u'''(\overline{C})(\overline{C})^2\left(\frac{C_1 - \overline{C}}{\overline{C}}\right)^2 \\
u'(C_1) &= u'(\overline{C}) + u''(\overline{C})\,\overline{C}\widehat{C}_1 + \frac{1}{2}u'''(\overline{C})(\overline{C})^2\left(\widehat{C}_1\right)^2
\end{aligned}$$

Similarly:

$$\begin{aligned}
u'(C_2(k)) &= u'(\overline{C}) + u''(\overline{C})(C_2(k) - \overline{C}) + \frac{1}{2}u'''(\overline{C})(C_2(k) - \overline{C})^2 \\
u'(C_2(k)) &= u'(\overline{C}) + u''(\overline{C})\,\overline{C}\frac{C_2(k) - \overline{C}}{\overline{C}} + \frac{1}{2}u'''(\overline{C})(\overline{C})^2\left(\frac{C_2(k) - \overline{C}}{\overline{C}}\right)^2 \\
u'(C_2(k)) &= u'(\overline{C}) + u''(\overline{C})\,\overline{C}\widehat{C}_2(k) + \frac{1}{2}u'''(\overline{C})(\overline{C})^2\left(\widehat{C}_2(k)\right)^2
\end{aligned}$$

Combining these, the Euler condition is:

$$\begin{aligned}
0 &= u'(C_1) - \sum_k \pi(k)\,u'(C_2(k)) \\
0 &= u'(\overline{C}) + u''(\overline{C})\,\overline{C}\widehat{C}_1 + \frac{1}{2}u'''(\overline{C})(\overline{C})^2\left(\widehat{C}_1\right)^2 \\
&\quad - \sum_k \pi(k)\left[u'(\overline{C}) + u''(\overline{C})\,\overline{C}\widehat{C}_2(k) + \frac{1}{2}u'''(\overline{C})(\overline{C})^2\left(\widehat{C}_2(k)\right)^2\right] \\
0 &= u'(\overline{C}) + u''(\overline{C})\,\overline{C}\widehat{C}_1 + \frac{1}{2}u'''(\overline{C})(\overline{C})^2\left(\widehat{C}_1\right)^2 \\
&\quad - u'(\overline{C})\sum_k \pi(k) - u''(\overline{C})\,\overline{C}\sum_k \pi(k)\,\widehat{C}_2(k) \\
&\quad - \frac{1}{2}u'''(\overline{C})(\overline{C})^2\sum_k \pi(k)\left(\widehat{C}_2(k)\right)^2 \\
0 &= u''(\overline{C})\,\overline{C}\widehat{C}_1 + \frac{1}{2}u'''(\overline{C})(\overline{C})^2\left(\widehat{C}_1\right)^2 \\
&\quad - u''(\overline{C})\,\overline{C}E\left[\widehat{C}_2\right] - \frac{1}{2}u'''(\overline{C})(\overline{C})^2 E\left[\left(\widehat{C}_2\right)^2\right] \\
0 &= \widehat{C}_1 - E\left[\widehat{C}_2\right] + \frac{1}{2}\frac{u'''(\overline{C})}{u''(\overline{C})}\overline{C}\left[\left(\widehat{C}_1\right)^2 - E\left[\left(\widehat{C}_2\right)^2\right]\right]
\end{aligned}$$

$$\widehat{C}_1 = E\left[\widehat{C}_2\right] + \frac{1}{2}\frac{u'''\left(\overline{C}\right)}{u''\left(\overline{C}\right)}\overline{C}\left[-\left(\widehat{C}_1\right)^2 + E\left[\left(\widehat{C}_2\right)^2\right]\right]$$

4 First-order solution

Question: We can represent the deviation of variable from the reference point as composed to several orders. Focus on the first two:

$$\widehat{C} = \widehat{C}_{O(1)} + \widehat{C}_{O(2)}$$

$\widehat{C}_{O(1)}$ is proportional to the innovations of shocks, while $\widehat{C}_{O(2)}$ is proportional to the squares innovations of shocks (which in expected terms are their variance).

When taking the quadratic approximation of the function $f(C)$, we can split it between its first and second order components:

$$\begin{aligned}
f(C) - f\left(\overline{C}\right) &= f'\left(\overline{C}\right)\overline{C}\widehat{C} + \frac{1}{2}f''\left(\overline{C}\right)\left(\overline{C}\right)^2\left(\widehat{C}\right)^2 \\
\left[f(C) - f\left(\overline{C}\right)\right]_{O(1)} &= f'\left(\overline{C}\right)\overline{C}\widehat{C}_{O(1)} \\
\left[f(C) - f\left(\overline{C}\right)\right]_{O(2)} &= f'\left(\overline{C}\right)\overline{C}\widehat{C}_{O(2)} + \frac{1}{2}f''\left(\overline{C}\right)\left(\overline{C}\right)^2\left(\widehat{C}_{O(1)}\right)^2
\end{aligned}$$

Note that $\left(\widehat{C}_{O(1)}\right)^2$ is of order 2.

We consider that output in the first period is constant. Output in period 2 can fluctuate, but the expected value of these shocks is zero.

Show that:

$$\widehat{C}_{1,O(1)} = E\left[\widehat{C}_{2,O(1)}\right] = 0$$

Does the uncertainty of output in period 2 has an effect?

Answer: The linear approximation of the budget constraint of period 2 only has first-order elements:

$$\widehat{C}_{2,O(1)}(k) = \frac{1}{\beta}\left(\widehat{Y}_{1,O(1)} - \widehat{C}_{1,O(1)}\right) + \widehat{Y}_{2,O(1)}(k)$$

As output of the first period never changes, we have: $\widehat{Y}_{1,O(1)} = 0$.

The firs-order component of the Euler condition only takes its linear terms:

$$\widehat{C}_{1,O(1)} = E\left[\widehat{C}_{2,O(1)}\right]$$

We write this as:

$$\begin{aligned}
\widehat{C}_{1,O(1)} &= E\left[\widehat{C}_{2,O(1)}\right] \\
\widehat{C}_{1,O(1)} &= E\left[\frac{1}{\beta}\left(\widehat{Y}_{1,O(1)} - \widehat{C}_{1,O(1)}\right) + \widehat{Y}_{2,O(1)}\right] \\
\widehat{C}_{1,O(1)} &= E\left[-\frac{1}{\beta}\widehat{C}_{1,O(1)} + \widehat{Y}_{2,O(1)}\right] \\
\frac{1+\beta}{\beta}\widehat{C}_{1,O(1)} &= E\left[\widehat{Y}_{2,O(1)}\right]
\end{aligned}$$

As the second period output shocks cancel out in expected terms, we have $E\left[\widehat{Y}_{2,O(1)}\right] = 0$ hence:

$$\begin{aligned}
\widehat{C}_{1,O(1)} &= \frac{\beta}{1+\beta} E\left[\widehat{Y}_{2,O(1)}\right] \\
\widehat{C}_{1,O(1)} &= 0
\end{aligned}$$

Uncertainty does not have an effect. This is not surprising: uncertainty is about the variance of output in period 2, which is a second-order dimension that is not captured in our analysis focusing on the first-order dimension.

5 Precautionary savings

Question: We now turn to the second-order dimension of the Euler condition. Show that:

$$\widehat{C}_{1,O(2)} = E\left[\widehat{C}_{2,O(2)}\right] - \left(\frac{1}{2}\frac{u'''\left(\overline{C}\right)}{-u''\left(\overline{C}\right)}\overline{C}\right) E\left[\left(\widehat{Y}_{2,O(1)}\left(k\right)\right)^2\right]$$

How does the volatility of output matters for initial consumption?

What are the characteristics of the utility function that are needed to have an effect? Is it the case for the following two utilities (assume a is small):

$$\begin{aligned}
u\left(C\right) &= C - \frac{a}{2}C^2 \\
u\left(C\right) &= \frac{\left(C\right)^{1-\sigma}}{1-\sigma}
\end{aligned}$$

Answer: The second-ordwer consumptions enter through the linear components, while the quadratic components reflect the first-order terms:

$$\widehat{C}_{1,O(2)} = E\left[\widehat{C}_{2,O(2)}\right] + \frac{1}{2}\frac{u'''\left(\overline{C}\right)}{u''\left(\overline{C}\right)}\overline{C}\left[-\left(\widehat{C}_{1,O(1)}\right)^2 + E\left[\left(\widehat{C}_{2,O(1)}\right)^2\right]\right]$$

Using our results in first-order terms. we write:

$$\begin{aligned}
\widehat{C}_{1,O(2)} &= E\left[\widehat{C}_{2,O(2)}\right] + \frac{1}{2}\frac{u'''\left(\overline{C}\right)}{u''\left(\overline{C}\right)}\overline{C}\left[-\left(\widehat{C}_{1,O(1)}\right)^2 + E\left[\left(\frac{1}{\beta}\left(\widehat{Y}_{1,O(1)} - \widehat{C}_{1,O(1)}\right) + \widehat{Y}_{2,O(1)}\left(k\right)\right)^2\right]\right] \\
\widehat{C}_{1,O(2)} &= E\left[\widehat{C}_{2,O(2)}\right] + \frac{1}{2}\frac{u'''\left(\overline{C}\right)}{u''\left(\overline{C}\right)}\overline{C}E\left[\left(\widehat{Y}_{2,O(1)}\left(k\right)\right)^2\right] \\
\widehat{C}_{1,O(2)} &= E\left[\widehat{C}_{2,O(2)}\right] - \frac{1}{2}\frac{u'''\left(\overline{C}\right)}{-u''\left(\overline{C}\right)}\overline{C}E\left[\left(\widehat{Y}_{2,O(1)}\left(k\right)\right)^2\right]
\end{aligned}$$

With decreasing marginal utility, we have $u''\left(\overline{C}\right) < 0$. If the marginal utility is convex ($u'''\left(\overline{C}\right) > 0$), a higher volatility of output $E\left[\left(\widehat{Y}_{2,O(1)}\left(k\right)\right)^2\right]$ leads to a reduction of initial consumption $\widehat{C}_{1,O(2)}$, i.e more initial savings. These savings are called precautionary as they reflect the variance of output. Being proportional to the variance, they are second-order in our labeling.

The effect depends on $u'''\left(\overline{C}\right)$. This is absent with a linear-quadratic utility:

$$u\left(C\right) = C - \frac{a}{2}C^2$$

$$
\begin{aligned}
u'(C) &= 1 - aC > 0 \\
u''(C) &= -a < 0 \\
u'''(C) &= 0
\end{aligned}
$$

It is present however with a CRRA utility:

$$
\begin{aligned}
u(C) &= \frac{(C)^{1-\sigma}}{1-\sigma} \\
u'(C) &= (C)^{-\sigma} > 0 \\
u''(C) &= -\sigma (C)^{-\sigma-1} < 0 \\
u'''(C) &= \sigma (1+\sigma) (C)^{-\sigma-2} > 0
\end{aligned}
$$