Exercise 2

(a)

$t=1$

$$c_1^H + p_1 c_1^F = y_1^H - B_2$$

$t=2$

$$c_2^H + p_2 c_2^F = y_2^H + (1+r) B_2$$

$$\hookrightarrow B_2 = \frac{c_2^H + p_2 c_2^F - y_2^H}{1+r}$$

Combining we get the intertemporal budget constraint:

$$c_1^H + p_1 c_1^F = y_1^H - \frac{c_2^H + p_2 c_2^F - y_2^H}{1+r}$$

(b) We can write the maximization problem as

$$\mathcal{L} = \ln(c_1^H) + \alpha \ln(c_1^F) + \beta(\ln(c_2^H) + \alpha\ln(c_2^F))$$

$$-\lambda_1 (c_1^H + p_1 c_1^F - y_2^H + (1+r) B_2)$$

$$-\beta\lambda_2 (c_2^H + p_2 c_2^F - y_2^H - (1+r) B_2)$$

$$\frac{\partial \mathcal{L}}{\partial c_1^H} = 0 \Rightarrow \frac{1}{c_1^H} = \lambda_1 \qquad \frac{\partial \mathcal{L}}{\partial c_1^F} = 0 \Rightarrow \frac{\alpha}{c_1^F} = \lambda_1 p_1$$

$$\frac{\partial \mathcal{L}}{\partial c_2^H} = \frac{\beta 1}{c_2^H} = \beta\lambda_2 \qquad \frac{\partial \mathcal{L}}{\partial c_2^F} \Rightarrow \frac{\alpha}{c_2^F} = \lambda_2 p_2$$

$$\frac{\partial \mathcal{L}}{\partial B_2} = \lambda_1 = \beta(1+r)\lambda_2$$

Combining, we have

$$p_1 = \frac{\alpha C_1^H}{C_1^F} \qquad p_2 = \frac{\alpha C_2^H}{C_2^F}$$

$$\frac{1}{C_1^H} = \beta(1+r)\frac{1}{C_2^H} \Rightarrow \frac{C_2^H}{C_1^H} = \beta(1+r)$$

$$\frac{1}{C_1^F} = \frac{p_2}{p_1}\beta(1+r)\frac{1}{C_1^F} \Rightarrow \frac{C_2^F}{C_1^F} = \frac{p_1}{p_2}\beta(1+r)$$

$$p_1 = p_2 = p$$

$$\Rightarrow \frac{\cancel{\alpha} C_1^H}{C_1^F} = \frac{\cancel{\alpha} C_2^H}{C_2^F}$$

$$\frac{C_2^H}{C_1^H} = \frac{C_2^F}{C_1^F} = \beta(1+r)$$

(c)

when $\beta = \frac{1}{1+r}$

$$\frac{c_2^H}{c_1^H} = \frac{c_2^F}{c_1^F} = 1$$

$$\Longrightarrow c_1^H = c_2^H = c^H$$

$$c_1^F = c_2^F = c^F$$

Let's take the intertemporal budget constraint

$$c^H + p c^F = y_1^H - \frac{c^H + p c^F - y_2^H}{1+r}$$

$$W = \frac{(2+r)(c^H + p c^F)}{1+r}$$

$$= \frac{(2+r)(c^H + \alpha c^H)}{1+r}$$

$$= \frac{(2+r)(1+\alpha)}{1+r} c^H$$

$$c^H = \frac{(1+r)W}{(2+r)(1+\alpha)}$$

$$c^F = \frac{\alpha}{p} \frac{(1+r)W}{(2+r)(1+\alpha)}$$

d) If $Y_1^H = Y_2^H$ then

$$W = \frac{(2+r)}{1+r} Y^H$$

We can use this in the budget constraint

$$B_2 = Y^H - C^H - pC^F$$

$$= Y^H - \frac{Y^H}{1+\alpha} - \alpha\frac{Y^H}{1-\alpha}$$

$$= 0$$

$$B_2 = 0 \Rightarrow CA_1 = 0$$

$$TB_1 = Y^H - C^H - pC^F = \underline{0}$$

$$TB_2 = Y^H - C^H - pC^F = -(1+r)B_2 = \underline{0}$$

(e) $X_t = Y^H - C^H$

$$M_t = C^F$$

$$\frac{X+M}{Y^H} = \frac{Y^H + C^F - C^H}{Y^H}$$

$$\frac{X+M}{Y^H} = 1 + \frac{\alpha}{\rho}\frac{1}{1+\alpha} - \frac{1}{1+\alpha}$$

$$= \frac{\alpha}{1+\alpha} + \frac{\alpha}{1+\alpha}\rho$$

$$= \frac{(1+\rho)\alpha}{1+\alpha}$$ ✓

(f) If $Y_2^H > Y_1^H$ and $Y_2^H = (1+g)Y_1^H$

$$W = Y_1^H + \frac{1+g}{1+r}Y_1^H$$

$$= \frac{2+r+g}{1+r}Y_1^H$$

$$B_2 = CA_1 = Y_1^H - C_1^H - pC_1^F$$

$$= Y_1^H - \frac{1+r}{(1+\alpha)(2+r)}W - \rho\frac{\alpha}{\rho}\frac{1+r}{(1+\alpha)(2+r)}W$$

$$= Y_1^H - \frac{1+r}{2+r}W$$

$$= Y_1^H - \frac{2+r+g}{2+r}Y_1^H$$

$$= \frac{-g}{2+r}Y_1^H < 0$$