EI060 Macroeconomics B
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PS1 Solutions

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1 Consumption Allocation

Problem Setup

The Home agent's consumption basket is given by

$$C_t = \left(\frac{C_{T,t}}{\gamma}\right)^{\gamma} \left(\frac{C_{N,t}}{1-\gamma}\right)^{1-\gamma},$$

where:

- $C_{T,t}$ is the quantity of the traded good (its price is normalized to 1),
- $C_{N,t}$ is the quantity of the domestic non-traded good (price $P_{N,t}$),
- γ is the expenditure share on the traded good.

The consumer minimizes total expenditure subject to attaining a given consumption level C_t. The problem is

$$\begin{aligned} \min_{C_{T,t}, C_{N,t}} & \ P_t C_t = C_{T,t} + P_{N,t} C_{N,t} \\ \text{s.t.} \quad & C_t = \left(\frac{C_{T,t}}{\gamma}\right)^{\gamma} \left(\frac{C_{N,t}}{1-\gamma}\right)^{1-\gamma}. \end{aligned}$$

Define the Lagrangian function:

$$\mathcal{L} = C_{T,t} + P_{N,t} C_{N,t} + \lambda \Big(C_t - \Big(\frac{C_{T,t}}{\gamma}\Big)^{\gamma} \Big(\frac{C_{N,t}}{1-\gamma}\Big)^{1-\gamma} \Big).$$

The FOCs with respect to $C_{T,t}$ and $C_{N,t}$ are:

$$\begin{aligned} \mathcal{L}_{C_{T,t}} &= 1 - \lambda\gamma \Big(\frac{C_{T,t}}{\gamma}\Big)^{\gamma-1} \frac{1}{\gamma} \Big(\frac{C_{N,t}}{1-\gamma}\Big)^{1-\gamma} = 0, \\ \mathcal{L}_{C_{N,t}} &= P_{N,t} - \lambda(1-\gamma) \Big(\frac{C_{T,t}}{\gamma}\Big)^{\gamma} \frac{1}{1-\gamma} \Big(\frac{C_{N,t}}{1-\gamma}\Big)^{-\gamma} = 0 \\ \Rightarrow \quad \frac{1}{P_{N,t}} &= \frac{\gamma}{1-\gamma} \frac{C_{N,t}}{C_{T,t}}. \end{aligned}$$

The dual (expenditure) minimization problem yields the unit cost function (composite price index) for the consumption bundle:

$$P_t C_t = \min \left\{ C_{T,t} + P_{N,t} C_{N,t} : \ C_t = \left(\frac{C_{T,t}}{\gamma} \right)^{\gamma} \left(\frac{C_{N,t}}{1-\gamma} \right)^{1-\gamma} \right\}.$$

So, we have:

$$\begin{aligned}
P_t C_t &= C_{T,t} + P_{N,t} C_{N,t} \\
&= C_{T,t} + \frac{1-\gamma}{\gamma} C_{T,t} \\
\Rightarrow C_{T,t} &= \gamma P_t C_t \\
\Rightarrow C_{N,t} &= \frac{1-\gamma}{\gamma} \frac{C_{T,t}}{P_{N,t}} \\
&= (1-\gamma) P_t C_t.
\end{aligned}$$

$$\begin{aligned}
\left(\frac{C_{T,t}}{\gamma} \right)^{\gamma} \left(\frac{C_{N,t}}{1-\gamma} \right)^{1-\gamma} &= C_t \\
\Rightarrow \quad (P_t C_t)^{\gamma} \left(\frac{P_t C_t}{P_N, t} \right)^{1-\gamma} &= C_t \\
\Rightarrow \quad P_t = (P_{N,t})^{1-\gamma}.
\end{aligned}$$

Analogously, for the Foreign agent, we have

$$\begin{aligned}
C^*_{T,t} &= \gamma P^*_t C^*_t \\
C^*_{N,t} &= (1-\gamma) P^*_t C^*_t \\
P^*_t &= (P^*_{N,t})^{1-\gamma}.
\end{aligned}$$

Economic Intuition

- The parameter γ reflects the expenditure share on the traded good.
- Since the traded good is the numéraire (price normalized to 1), its cost enters directly, while the cost of the non-traded good is weighted by its price $P_{N,t}$.
- The composite price index P_t is a weighted geometric mean of the individual prices. With the traded goods price equal to 1, we have $P_t = (P_{N,t})^{1-\gamma}$.
- The optimal consumption choices allocate expenditure in a way that equates the marginal rate of substitution to the ratio of prices.

2 Market Clearing

Under market clearing, the quantities of traded and non-traded goods produced must equal the quantities consumed. We have:

Non-Traded Goods Market

For Home, market clearing in non-traded goods is:

$$nC_{N,t} = A_{N,t}(L_{N,t})^{1-\alpha}.$$

Substituting $C_{N,t} = (1-\gamma)\frac{P_t C_t}{P_{N,t}}$ with $P_t = (P_{N,t})^{1-\gamma}$, we obtain:

$$\boxed{n(1-\gamma)(P_{N,t})^{-\gamma}C_t = A_{N,t}(L_{N,t})^{1-\alpha}.}$$

For Foreign, the market clearing condition is: $(1-n)C^*_{N,t} = A^*_{N,t}(L^*_{N,t})^{(1-\alpha)}$

$$\boxed{(1-n)(1-\gamma)(P^*_{N,t})^{-\gamma}C^*_t = A^*_{N,t}(L^*_{N,t})^{1-\alpha}.}$$

Traded Goods Market

Global market clearing for traded goods is:

$$nC_{T,t} + (1-n)C^*_{T,t} = A_{T,t}(n - L_{N,t})^{1-\alpha} + A^*_{T,t}\big((1-n) - L^*_{N,t}\big)^{1-\alpha}.$$

Substituting $C_{T,t} = \gamma P_t C_t$ with $P_t = (P_{N,t})^{1-\gamma}$ (and similarly for Foreign), we have:

$$\boxed{n\gamma(P_{N,t})^{1-\gamma}C_t + (1-n)\gamma(P^*_{N,t})^{1-\gamma}C^*_t = A_{T,t}(n - L_{N,t})^{1-\alpha} + A^*_{T,t}\big((1-n) - L^*_{N,t}\big)^{1-\alpha}.}$$

Intuition: Non-traded goods are produced and consumed domestically, while traded goods are balanced across countries.

3 Intertemporal Allocation

Home Optimization

The Home agent maximizes:

$$U_t = \sum_{s=0}^{\infty}(\beta_{H,t+s})^s \ln C_{t+s},$$

subject to the period-t budget constraint:

$$nP_tC_t + nB_{t+1} = A_{T,t}(n - L_{N,t})^{1-\alpha} + P_{N,t}A_{N,t}(L_{N,t})^{1-\alpha} + n(1+r_t)B_t.$$

Focusing on a single period (the infinite-horizon structure is standard), we write:

$$\mathcal{L}_t = \ln C_t + \beta_{H,t+1} \ln C_{t+1} - \lambda_t \Big[A_{T,t}(n - L_{N,t})^{1-\alpha} + P_{N,t} A_{N,t} (L_{N,t})^{1-\alpha} + n(1+r_t)B_t - nP_tC_t - nB_{t+1} \Big].$$

Take the FOCs:

$$\frac{\partial \mathcal{L}_t}{\partial C_t} = \frac{1}{C_t} - \lambda_t n P_t = 0 \quad \Rightarrow \quad \lambda_t = \frac{1}{nP_tC_t}$$

$$\frac{\partial \mathcal{L}_t}{\partial B_{t+1}} = -\lambda_t + \beta_{H,t+1}(1+r_{t+1})\lambda_{t+1} = 0.$$

Substitute the expressions for λ_t and λ_{t+1}:

$$\frac{1}{nP_tC_t} = \beta_{H,t+1}(1+r_{t+1}) \frac{1}{nP_{t+1}C_{t+1}}.$$

Cancel n and rearrange:

$$\frac{1}{C_t} = \beta_{H,t+1}(1+r_{t+1}) \frac{P_t}{P_{t+1}} \frac{1}{C_{t+1}}.$$

Set:

$$1 + r^C_{t+1} \equiv (1+r_{t+1}) \frac{P_t}{P_{t+1}},$$

so the Euler equation becomes:

$$\boxed{C_{t+1} = \beta_{H,t+1}(1+r^C_{t+1})C_t.}$$

From Question (1), we know that $P_t = (P_{N,t})^{(1-\gamma)}$, so, we have:

$$1 + r^C_{t+1} = (1+r_{t+1}) \frac{P_t}{P_{t+1}} = (1+r_{t+1}) \left(\frac{P_{N,t}}{P_{N,t+1}} \right)^{1-\gamma}.$$

Since

$$C_{T,t} = \gamma P_t C_t,$$

it follows that

$$C_{T,t+1} = \gamma P_{t+1} C_{t+1} = \gamma P_{t+1} \beta_{H,t+1}(1+r^C_{t+1})C_t.$$

As we note that

$$\beta_{H,t+1}(1+r^C_{t+1}) = \beta_{H,t+1}(1+r_{t+1}) \frac{P_t}{P_{t+1}},$$

so simplifying, we obtain:

$$\boxed{C_{T,t+1} = \beta_{H,t+1}(1+r_{t+1})C_{T,t}.}$$

Remark. The Foreign agent's optimization yields analogous Euler equations:

$$C^*_{t+1} = \beta_{F,t+1}(1+r^*_{C,t+1})C^*_t, \quad C^*_{T,t+1} = \beta_{F,t+1}(1+r_{t+1})C^*_{T,t},$$

with

$$1 + r^{*C}_{t+1} = (1+r_{t+1}) \frac{P^*_t}{P^*_{t+1}}.$$

Foreign Optimization

Similarly, for the Foreign country:

$$\boxed{C^*_{t+1} = \beta_{F,t+1}(1 + r^*_{C,t+1})C^*_t, \qquad C^*_{T,t+1} = \beta_{F,t+1}(1 + r_{t+1})C^*_{T,t},}$$

with

$$1 + r^{*C}_{t+1} = (1 + r_{t+1}) \left(\frac{P^*_{N,t}}{P^*_{N,t+1}} \right)^{1-\gamma}.$$

Intuition: Households equate the marginal utility cost of consuming today versus tomorrow, with intertemporal decisions affected by relative price changes.

4 Labor Allocation

The production functions are given by:

$$Y_{T,t} = A_{T,t}(n - L_{N,t})^{1-\alpha}, \quad Y_{N,t} = A_{N,t}(L_{N,t})^{1-\alpha}.$$

Compute the marginal product of labor in each sector:

$$\frac{\partial Y_{T,t}}{\partial (n - L_{N,t})} = (1 - \alpha)A_{T,t}(n - L_{N,t})^{-\alpha},$$

$$\frac{\partial Y_{N,t}}{\partial L_{N,t}} = (1 - \alpha)A_{N,t}(L_{N,t})^{-\alpha}.$$

The Home agent allocates labor so that the marginal value product in the traded sector equals the marginal value product (adjusted by the non-traded price) in the non-traded sector:

$$(1 - \alpha)A_{T,t}(n - L_{N,t})^{-\alpha} = P_{N,t}(1 - \alpha)A_{N,t}(L_{N,t})^{-\alpha}.$$

Cancel the common factor $1 - \alpha$ and rearrange:

$$\boxed{A_{T,t}(n - L_{N,t})^{-\alpha} = P_{N,t}A_{N,t}(L_{N,t})^{-\alpha}.}$$

The analogous condition for the Foreign country is:

$$\boxed{A^*_{T,t}\big((1 - n) - L^*_{N,t}\big)^{-\alpha} = P^*_{N,t}A^*_{N,t}(L^*_{N,t})^{-\alpha}.}$$

5 Resource Constraints and the Real Exchange Rate

Resource Constraints

Recall the Home budget constraint:

$$nP_tC_t + nB_{t+1} = A_{T,t}(n - L_{N,t})^{1-\alpha} + P_{N,t}A_{N,t}(L_{N,t})^{1-\alpha} + n(1 + r_t)B_t.$$

From Question 1, we have:

$$C_{N,t} = (1-\gamma)\frac{P_t}{P_{N,t}}C_t.$$

Since $P_t = (P_{N,t})^{1-\gamma}$, then:

$$C_{N,t} = (1-\gamma)(P_{N,t})^{-\gamma}C_t.$$

Given that non-traded goods are produced solely for domestic consumption, we also have the production identity (from Question 2):

$$n(1-\gamma)(P_{N,t})^{-\gamma}C_t = A_{N,t}(L_{N,t})^{1-\alpha}.$$

Thus, the expenditure on traded goods (which uses the consumption price index) plus net asset accumulation must equal traded output plus bond returns:

$$\boxed{n\gamma(P_{N,t})^{1-\gamma}C_t + nB_{t+1} = A_{T,t}(n-L_{N,t})^{1-\alpha} + n(1+r_t)B_t.}$$

Similarly, for Foreign we obtain:

$$\boxed{(1-n)\gamma(P^*_{N,t})^{1-\gamma}C^*_t - nB_{t+1} = A^*_{T,t}\big((1-n)-L^*_{N,t}\big)^{1-\alpha} - n(1+r_t)B_t.}$$

Then, we define the real exchange rate as the ratio of Foreign to Home consumption price indices:

$$Q_t \equiv \frac{P^*_t}{P_t}.$$

Since

$$P_t = (P_{N,t})^{1-\gamma} \quad \text{and} \quad P^*_t = (P^*_{N,t})^{1-\gamma},$$

we have:

$$\boxed{Q_t = \left(\frac{P^*_{N,t}}{P_{N,t}}\right)^{1-\gamma}.}$$

6 Steady State

In steady state, consumption is constant so that $C_{t+1} = C_t$. The Euler equation for the Home agent is

$$C_{t+1} = \beta_{H,t+1}(1+r^C_{t+1})C_t.$$

But by definition the real return in consumption units is

$$1+r^C_{t+1} = (1+r_{t+1})\left(\frac{P_t}{P_{t+1}}\right)^{1-\gamma}.$$

In steady state prices do not change ($P_t = P_{t+1}$) so that

$$1+r^C_{t+1} = 1+r_{t+1} \quad \Rightarrow \quad C_t = \beta_t(1+r_t)C_t.$$

Dividing by $C_t > 0$ and take $t = 0$, yields:

$$1 = \beta_0(1 + r_0).$$

Step 2. Price Normalization

By convention we normalize the steady state price of non-tradables to one:

$$P_{N,0} = 1, \quad \text{and similarly} \quad P^*_{N,0} = 1.$$

Then the consumption price indices become

$$P_0 = (P_{N,0})^{1-\gamma} = 1, \quad P^*_0 = 1.$$

Step 3. Labor Allocation in the Home Country

The Home intratemporal labor allocation condition is:

$$A_{T,0}(n - L_{N,0})^{-\alpha} = P_{N,0}A_{N,0}(L_{N,0})^{-\alpha}.$$

Since $P_{N,0} = 1$, this simplifies to:

$$A_{T,0}(n - L_{N,0})^{-\alpha} = A_{N,0}(L_{N,0})^{-\alpha}.$$

Rearrange by dividing both sides by $A_{T,0}$ and by $(L_{N,0})^{-\alpha}$:

$$\left(\frac{n - L_{N,0}}{L_{N,0}}\right)^{-\alpha} = \frac{A_{N,0}}{A_{T,0}}.$$

Taking the reciprocal and then the $1/\alpha$-th root,

$$\frac{n - L_{N,0}}{L_{N,0}} = \left(\frac{A_{T,0}}{A_{N,0}}\right)^{1/\alpha}.$$

Now, using the calibration

$$A_{N,0} = A_{T,0}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha},$$

we have

$$\frac{A_{T,0}}{A_{N,0}} = \left(\frac{\gamma}{1-\gamma}\right)^{\alpha}.$$

Then,

$$\frac{n - L_{N,0}}{L_{N,0}} = \left[\left(\frac{\gamma}{1-\gamma}\right)^{\alpha}\right]^{1/\alpha} = \frac{\gamma}{1-\gamma}.$$

Thus,

$$\boxed{L_{N,0} = n(1-\gamma).}$$

By symmetry, the corresponding condition for Foreign is:

$$A^*_{T,0}\Big((1-n) - L^*_{N,0}\Big)^{-\alpha} = P^*_{N,0}A^*_{N,0}(L^*_{N,0})^{-\alpha}.$$

Since $P^*_{N,0} = 1$, the same steps lead to:

$$\frac{(1-n) - L^*_{N,0}}{L^*_{N,0}} = \frac{\gamma}{1-\gamma},$$

so that

$$\boxed{L^*_{N,0} = (1-n)(1-\gamma).}$$

We derive the steady-state consumption from the market clearing conditions for non-traded and traded goods. The non-traded goods clearing condition is

$$nC_{N,0} = A_{N,0}(L_{N,0})^{1-\alpha}.$$

From Question 1 we have

$$C_{N,0} = (1-\gamma)\frac{P_0}{P_{N,0}}C_0.$$

Since $P_0 = 1$ and $P_{N,0} = 1$, it follows that

$$C_{N,0} = (1-\gamma)C_0.$$

Substitute into the clearing condition:

$$n(1-\gamma)C_0 = A_{N,0}(L_{N,0})^{1-\alpha}.$$

Recall that $L_{N,0} = n(1-\gamma)$, so

$$n(1-\gamma)C_0 = A_{N,0}\big[n(1-\gamma)\big]^{1-\alpha}.$$

Solve for C_0:

$$C_0^N = A_{N,0}\big[n(1-\gamma)\big]^{-\alpha}.$$

We then derive from the traded goods clearing condition:

$$
\begin{aligned}
& n\gamma(P_{N,t})^{1-\gamma}C_t + (1-n)\gamma(P^*_{N,t})^{1-\gamma}C^*_t = A_{T,t}(n-L_{N,t})^{1-\alpha} + A^*_{T,t}\big((1-n)-L^*_{N,t}\big)^{1-\alpha} \\
\Rightarrow\quad & n\gamma C_0 + \frac{\gamma}{1-\gamma}A^*_{N,0}(L^*_{N,0})^{1-\alpha} = A_{T,0}(n-L_{N,0})^{1-\alpha} + A^*_{T,0}(1-n-L^*_{N,0})^{1-\alpha} \\
\Rightarrow\quad & n\gamma C_0 + \frac{\gamma}{1-\gamma}A^*_{N,0}(1-n)^{1-\alpha}(1-\gamma)^{1-\alpha} = A_{T,0}(n-n(1-\gamma))^{1-\alpha} + \\
& A^*_{T,0}(1-n-(1-n)(1-\gamma))^{1-\alpha} \\
\Rightarrow\quad & n\gamma C_0 + \frac{\gamma}{1-\gamma}A^*_{N,0}(1-n)^{1-\alpha}(1-\gamma)^{1-\alpha} = A_{T,0}(n\gamma)^{1-\alpha} + A^*_{T,0}\Big[(1-n)\gamma\Big]^{1-\alpha} \\
\Rightarrow\quad & n\gamma C_0 + \frac{\gamma}{1-\gamma}A_{T,0}\Big(\frac{1-n}{n}\Big)^{\alpha}\Big(\frac{1-\gamma}{\gamma}\Big)^{\alpha}(1-n)^{1-\alpha}(1-\gamma)^{1-\alpha} = A_{T,0}(n\gamma)^{1-\alpha} + \\
& A_{T,0}\Big(\frac{1-n}{n}\Big)^{\alpha}\Big[(1-n)\gamma\Big]^{1-\alpha} \\
\Rightarrow\quad & n\gamma C_0 + A_{T,0}(1-n)n^{-\alpha}\gamma^{1-\alpha} = A_{T,0}(n\gamma)^{1-\alpha} + A_{T,0}(1-n)n^{-\alpha}\gamma^{1-\alpha} \\
\Rightarrow\quad & C^T_0 = A_{T,0}(n\gamma)^{-\alpha}.
\end{aligned}
$$

We take a weighted geometric mean with weights γ and $1-\gamma$. That is,

$$C_0 = (C^N_0)^{1-\gamma}\cdot(C^T_0)^{\gamma},$$

so that

$$C_0 = \left[A_{N,0}n^{-\alpha}(1-\gamma)^{-\alpha}\right]^{1-\gamma}\left[A_{T,0}n^{-\alpha}\gamma^{-\alpha}\right]^{\gamma}.$$

We obtain:

$$\boxed{C_0 = (A_{T,0})^{\gamma}(A_{N,0})^{1-\gamma}\left[n\gamma^{\gamma}(1-\gamma)^{1-\gamma}\right]^{-\alpha}.}$$

A similar derivation for Foreign (noting that the population is $1-n$) gives:

$$\boxed{C^*_0 = (A^*_{T,0})^{\gamma}(A^*_{N,0})^{1-\gamma}\left[(1-n)\gamma^{\gamma}(1-\gamma)^{1-\gamma}\right]^{-\alpha}.}$$

Because of the calibration (and the fact that the relative productivities satisfy

$$\frac{A_{N,0}}{A_{T,0}} = \left(\frac{1-\gamma}{\gamma}\right)^{\alpha} \quad\text{and}\quad \frac{A^*_{N,0}}{A^*_{T,0}} = \left(\frac{1-n}{n}\right)^{\alpha}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha}),$$

we can check that indeed

$$\frac{C_0}{C^*_0} = 1.$$

7 Log-Linear Approximation

A. Non-Traded Goods Market

The Home non-traded goods market clearing condition is:

$$n(1-\gamma)(P_{N,t})^{-\gamma}C_t = A_{N,t}(L_{N,t})^{1-\alpha}.$$

Taking logarithms, we have

$$\ln n + \ln(1-\gamma) - \gamma \ln P_{N,t} + \ln C_t = \ln A_{N,t} + (1-\alpha)\ln L_{N,t}.$$

Linearizing around the steady state (and noting that constants vanish in the difference), we obtain:

$$\boxed{-\gamma\widehat{P_{N,t}} + \widehat{C}_t = \widehat{A_{N,t}} + (1-\alpha)\widehat{L_{N,t}}.} \tag{7a}$$

B. Resource Constraint

The Home resource constraint is:

$$n\gamma(P_{N,t})^{1-\gamma}C_t + nB_{t+1} = A_{T,t}(n-L_{N,t})^{1-\alpha} + n(1+r_t)B_t.$$

Divide both sides by $n\gamma C_0$, we get:

$$\frac{(P_{N,t})^{1-\gamma}C_t}{C_0} + \widehat{B_{t+1}} = \frac{A_{T,t}(n-L_{N,t})^{1-\alpha}}{n\gamma C_0} + (1+r_t)\widehat{B_t}.$$

Taking logs and linearizing, we have:

$$(1-\gamma)\widehat{P_{N,t}} + \widehat{C}_t + \widehat{B_{t+1}} = \widehat{A_{T,t}} - \frac{(1-\alpha)(L_{N,t}-L_{N,0})}{n-L_{N,0}}\widehat{L_{N,t}} + \frac{1}{\beta_0}\widehat{B_t}.$$

Since in steady state $n - L_{N,0} = n\gamma$, and that $\widehat{L_{N,t}} = L_{N,t} - L_{N,0}$. Thus,

$$\boxed{(1-\gamma)\widehat{P_{N,t}} + \widehat{C}_t + \widehat{B_{t+1}} = \widehat{A_{T,t}} - (1-\alpha)\frac{1-\gamma}{\gamma}\widehat{L_{N,t}} + \frac{1}{\beta_0}\widehat{B_t}.} \tag{7c}$$

C. Euler Equation

Recall that:

$$C_{t+1} = C_t\beta_{H,t+1}(1+r_{t+1})\left(\frac{P_{N,t}}{P_{N,t+1}}\right)^{1-\gamma}$$

Taking logs and linearizing, we have:

$$\widehat{C_{t+1}} = \widehat{C}_t + \widehat{\beta_{H,t+1}} + \frac{r_{t+1}-r_0}{1+r_0} - (1-\gamma)(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}).$$

As $\beta_0(1+r_0) = 1$,

$$\begin{aligned}
\widehat{r_t} &= r_t - \frac{1-\beta_0}{\beta_0} \\
&= r_t - \frac{1-\frac{1}{1+r_0}}{\frac{1}{1+r_0}} \\
&= r_t - r_0
\end{aligned}$$

So, the Euler equation becomes:

$$\boxed{\widehat{C_{t+1}} = \widehat{C}_t + \widehat{\beta_{H,t+1}} + \beta_0 \widehat{r_{t+1}} - (1-\gamma)(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}).} \tag{7e}$$

D. Labor Allocation

The intratemporal condition is:

$$A_{T,t}(n - L_{N,t})^{-\alpha} = P_{N,t} A_{N,t} (L_{N,t})^{-\alpha}.$$

Taking logarithms:

$$\ln A_{T,t} - \alpha \ln(n - L_{N,t}) = \ln P_{N,t} + \ln A_{N,t} - \alpha \ln L_{N,t}.$$

Linearize around steady state:

$$\widehat{A_{T,t}} - \alpha \frac{L_{N,t} - L_{N,0}}{n - L_{N,0}} = \widehat{P_{N,t}} + \widehat{A_{N,t}} - \alpha \widehat{L_{N,t}}.$$

Using the fact that in steady state $n - L_{N,0} = n\gamma$:

$$\widehat{A_{T,t}} - \alpha \frac{\widehat{L_{N,t}} n(1-\gamma)}{n\gamma} = \widehat{P_{N,t}} + \widehat{A_{N,t}} - \alpha \widehat{L_{N,t}}.$$

$$\boxed{\widehat{A_{T,t}} + \frac{\alpha}{\gamma} \widehat{L_{N,t}} = \widehat{P_{N,t}} + \widehat{A_{N,t}}.} \tag{7g}$$

E. Real Exchange Rate

As we know:

$$1 + r^C_{t+1} = (1 + r_{t+1}) \left(\frac{P_{N,t}}{P_{N,t+1}} \right)^{1-\gamma},$$

take logs and linearize both sides, we have:

$$\frac{r^C_{t+1} - r^C_0}{1 + r^C_0} = \frac{r_{t+1} - r_0}{1 + r_0} + (1-\gamma)(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}).$$

$$\Rightarrow \beta_0 \widehat{r^C_{t+1}} = \beta_0 \widehat{r_{t+1}} + (1-\gamma)(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}).$$

$$\boxed{\widehat{r^C_{t+1}} = \widehat{r_{t+1}} + (1-\gamma)\frac{1}{\beta_0}(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}).} \tag{7i}$$

8 Worldwide Equilibrium

Define world aggregates as population-weighted averages (e.g., $\widehat{C^W_t} = n\widehat{C}_t + (1-n)\widehat{C^*_t}$). Then, from the above log-linearized equations one can show:

- **Non-Traded Goods Market:**

$$n \times (7a) + (1-n) \times (7b) \Rightarrow -\gamma \widehat{P_{N,t}^W} + \widehat{C_t^W} = \widehat{A_{N,t}^W} + (1-\alpha)\widehat{L_{N,t}^W}. \tag{8a}$$

- **Resource Constraint:**

$$n \times (7c) + (1-n) \times (7d) \Rightarrow (1-\gamma)\widehat{P_{N,t}^W} + \widehat{C_t^W} = \widehat{A_{T,t}^W} - (1-\alpha)\frac{1-\gamma}{\gamma}\widehat{L_{N,t}^W}. \tag{8b}$$

- **Euler Equation:**

$$n \times (7e) + (1-n) \times (7f) \Rightarrow \widehat{C_{t+1}^W} = \widehat{C_t^W} + (1-\gamma)(\widehat{P_{N,t}^W} - \widehat{P_{N,t+1}^W}) + \widehat{\beta_{H,t+1}^W} + \beta_0\widehat{r_{t+1}}. \tag{8c}$$

- **Labor Allocation:**

$$n \times (7g) + (1-n) \times (7h) \Rightarrow \widehat{A_{T,t}^W} + \frac{\alpha}{\gamma}\widehat{L_{N,t}^W} = \widehat{P_{N,t}^W} + \widehat{A_{N,t}^W}. \tag{8d}$$

- **Real Exchange Rate:**

$$n \times (7i) + (1-n) \times (7j) \Rightarrow \beta_0\widehat{r_{t+1}^{CW}} = (1-\gamma)(\widehat{P_{N,t}^W} - \widehat{P_{N,t+1}^W}) + \beta_0\widehat{r_{t+1}}. \tag{8e}$$

Let (8a)-(8d), we get:

$$(1-\gamma)\widehat{P_{N,t}^W} + \widehat{C_t^W} = \widehat{A_{T,t}^W} - (1-\alpha+\frac{\alpha}{\gamma})\widehat{L_{N,t}^W}.$$

Combine with (8b), we have:

$$\begin{aligned}
(1-\alpha+\frac{\alpha}{\gamma})\widehat{L_{N,t}^W} &= -(1-\alpha)\frac{1-\gamma}{\gamma}\widehat{L_{N,t}^W} \\
&= (1-\alpha)\widehat{L_{N,t}^W} - \frac{1-\alpha}{\gamma}\widehat{L_{N,t}^W} \\
\Rightarrow \frac{1}{\gamma}\widehat{L_{N,t}^W} &= 0.
\end{aligned}$$

Let (8b)-(8a), we have:

$$\begin{aligned}
\widehat{P_{N,t}^W} &= \widehat{A_{T,t}^W} - \widehat{A_{N,t}^W} - (1-\alpha)\Big(\frac{1-\gamma}{\gamma} + 1\Big)\widehat{L_{N,t}^W} \\
&= \widehat{A_{T,t}^W} - \widehat{A_{N,t}^W}.
\end{aligned}$$

Take $(1-\gamma) \times \text{(8a)} + \gamma\text{(8b)}$, we have:

$$(1-\gamma)\widehat{C_t^W} + \gamma\widehat{C_t^W} = \widehat{C_t^W} = (1-\gamma)\widehat{A_{N,t}^W} + \gamma\widehat{A_{T,t}^W}.$$

Finally, from (8c), we know that:

$$\begin{aligned}
\beta_0\widehat{r_{t+1}} + \widehat{\beta^W_{H,t+1}} &= \widehat{C^W_{t+1}} - \widehat{C^W_t} - (1-\gamma)\Big(\widehat{P^W_{N,t}} - \widehat{P^W_{N,t+1}}\Big) \\
&= \gamma\hat{A}^W_{t+1} + (1-\gamma)\widehat{A^W_{N,t+1}} - \gamma\widehat{A^W_{T,t+1}} - (1-\gamma)\widehat{A^W_{N,t+1}} \\
&- (1-\gamma)\Big(\widehat{A^W_{T,t}} - \widehat{A^W_{N,t}}\Big) + (1-\gamma)\Big(\widehat{A^W_{T,t+1}} - \widehat{A^W_{N,t+1}}\Big) \\
&= \widehat{A^W_{T,t+1}} - \widehat{A^W_{T,t}}.
\end{aligned}$$

Intuition: World aggregates respond only to symmetric shocks, with the real interest rate driven by global productivity changes and aggregate patience.

9 Cross-Country Differences

As we know that $Q_t = \left(\frac{P^*_{N,t}}{P_{N,t}}\right)^{1-\gamma}$, log-linearize the equation, we have:

$$\widehat{Q_t} = (1-\gamma)(\widehat{P^*_{N,t}}) - \widehat{P_{N,t}}.$$

Use (7a) - (7b), we get:

$$\begin{aligned}
\widehat{C}_t - \widehat{C^*_t} - \gamma(\widehat{P_{N,t}} - \widehat{P^*_{N,t}}) &= (\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) + (1-\alpha)(\widehat{L_{N,t}} - \widehat{L^*_{N,t}}) \\
\Rightarrow \widehat{C}_t - \widehat{C^*_t} + \frac{\gamma}{1-\gamma}\widehat{Q_t} &= (\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) + (1-\alpha)(\widehat{L_{N,t}} - \widehat{L^*_{N,t}}). \qquad (9a)
\end{aligned}$$

Use (7c) - (7d), we get:

$$\begin{aligned}
\text{LHS} &= (1-\gamma)(\widehat{P_{N,t}} - \widehat{P^*_{N,t}}) + \widehat{C}_t - \widehat{C^*_t} + \frac{\widehat{B_{t+1}}}{1-n} \\
\text{RHS} &= (\widehat{A_{T,t}} - \widehat{A^*_{T,t}}) - (1-\alpha)\frac{1-\gamma}{\gamma}(\widehat{L_{N,t}} - \widehat{L^*_{N,t}}) + \frac{1}{\beta_0}\frac{\widehat{B_t}}{1-n} \\
\text{As } \widehat{Q_t} &= (1-\gamma)(\widehat{P^*_{N,t}} - \widehat{P_{N,t}}), \text{ we have} \\
\text{LHS} &= -\widehat{Q_t} + (\widehat{C}_t - \widehat{C^*_t}) + \frac{\widehat{B_{t+1}}}{1-n} = \text{RHS} \qquad (9b)
\end{aligned}$$

Use (7e) - (7f), we get:

$$\begin{aligned}
(\widehat{C_{t+1}} - \widehat{C^*_{t+1}}) &= (\widehat{C}_t - \widehat{C^*_t}) + (1-\gamma)(\widehat{P_{N,t}} - \widehat{P^*_{N,t}}) + (1-\gamma)(\widehat{P^*_{N,t+1}} - \widehat{P_{N,t+1}}) + (\widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}}) \\
&= (\widehat{C}_t - \widehat{C^*_t}) - \widehat{Q_t} + \widehat{Q_{t+1}} + \widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}}. \qquad (9c)
\end{aligned}$$

Use (7g) - (7h), we get:

$$\begin{aligned}
(\widehat{A_{T,t}} - \widehat{A^*_{T,t}}) + \frac{\alpha}{\gamma}(\widehat{L_{N,t}} - \widehat{L^*_{N,t}}) &= (\widehat{P_{N,t}} - \widehat{P^*_{N,t}}) + (\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) \\
&= -\frac{1}{1-\gamma}\widehat{Q_t} + (\widehat{A_{N,t}} - \widehat{A^*_{N,t}}). \qquad (9d)
\end{aligned}$$

Use (7i) - (7j), we get:

$$\begin{aligned}\beta_0(\widehat{r^C_{t+1}} - \widehat{r^{C*}_{t+1}}) &= (1-\gamma)(\widehat{P_{N,t}} - \widehat{P^*_{N,t}}) + (1-\gamma)(\widehat{P^*_{N,t}} - \widehat{P_{N,t}}) \\ &= \widehat{Q_{t+1}} - \widehat{Q_t}. \end{aligned} \tag{9e}$$

Intuition: These equations link cross-country differences in consumption, labor, and the real exchange rate to differences in productivity and intertemporal preferences.

10 Long-Run Allocation (Period $t+1$)

Assume that from $t+1$ onward the economy reaches a new steady state with no further discount factor shocks ($\widehat{\beta_{H,t+2}} = \widehat{\beta_{F,t+2}} = 0$). In the steady state, the consumption growth rate is zero, the asset position is fixed and the real exchange rate is stable.
Using the labor allocation equation at $t+1$, we have:

$$\frac{\alpha}{\gamma}(\widehat{L_{N,t}} - \widehat{L^*_{N,t}}) = -\frac{1}{1-\gamma}\widehat{Q_{t+1}} + \Big[(\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) - (\widehat{A_{T,t}} - \widehat{A^*_{T,t}})\Big]. \tag{10.1}$$

Then, we use the market clearing condition for non-traded goods and the resource allocation constraints at $t+1$:

$$\frac{\gamma}{1-\gamma}\widehat{Q_{t+1}} + (\widehat{C_{t+1}} - \widehat{C^*_{t+1}}) = (\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) + (1-\alpha)(\widehat{L_{N,t+1}} - \widehat{L^*_{N,t+1}}) \tag{10.2}$$

$$-\widehat{Q_{t+1}} + (\widehat{C_{t+1}} - \widehat{C_{t+1}}^*) + \frac{\widehat{B_{t+2}}}{1-n} = (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) - (1-\alpha)\frac{1-\gamma}{\gamma}(\widehat{L_{N,t+1}} - \widehat{L^*_{N,t+1}}) + \frac{1}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}. \tag{10.3}$$

As $\widehat{B_{t+2}} = \widehat{B_{t+1}}$, using (10.2)-(10.3), we get:

$$\begin{aligned} \Big(\frac{1-\gamma}{\gamma}+1\Big)\widehat{Q_{t+1}} &= \Big[(\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}})\Big] \\ &\quad + (1-\alpha)\Big(1+\frac{1-\gamma}{\gamma}\Big)(\widehat{L_{N,t+1}} - \widehat{L^*_{N,t+1}}) - \frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\ \Rightarrow \quad \widehat{Q_{t+1}} &= (1-\gamma)\Big[(\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}})\Big] + (1-\alpha)\frac{1-\gamma}{\gamma}(\widehat{L_{N,t+1}} - \widehat{L^*_{N,t+1}}) \\ &\quad - (1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}. \end{aligned} \tag{10.4}$$

Replacing $\widehat{L_{N,t+1}} - \widehat{L^*_{N,t+1}}$ using (10.1), we get:

$$\begin{aligned}
\widehat{Q_{t+1}} &= (1-\gamma)\Big[(\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}})\Big] - \frac{1-\alpha}{\alpha}\widehat{Q_{t+1}} \\
&\quad + \frac{1-\alpha}{\alpha}(1-\gamma)\Big[(\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}})\Big] - (1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
\Rightarrow \quad \Big(1+\frac{1-\alpha}{\alpha}\Big)\widehat{Q_{t+1}} &= (1-\gamma)\Big(1+\frac{1-\alpha}{\alpha}\Big)\Big[(\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}})\Big] \\
&\quad - (1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
\Rightarrow \quad \widehat{Q_{t+1}} &= -(1-\gamma)\Big[(\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) - (\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}})\Big] - \alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}. \qquad (10a)
\end{aligned}$$

Comparing (10.4) and (10a), we have:

$$\begin{aligned}
(1-\alpha)\frac{1-\gamma}{\gamma}(\widehat{L_{N,t+1}} - \widehat{L^*_{N,t+1}}) - (1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} &= -\alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
\Rightarrow \quad \widehat{L_{N,t+1}} - \widehat{L^*_{N,t+1}} &= \gamma\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}. \qquad (10b)
\end{aligned}$$

Implementing (10a) and (10b) back into (10.2), we get:

$$\begin{aligned}
\widehat{C_{t+1}} - \widehat{C^*_{t+1}} &= \widehat{Q_{t+1}} + (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) - \frac{(1-\alpha)(1-\gamma)}{\gamma}\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} + \frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
&= -(1-\gamma)(\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) + (1-\gamma)(\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) + (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) \\
&\quad - \alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} - (1-\alpha)(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} + \frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\
&= \gamma(\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) + (1-\gamma)(\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}}) + \gamma\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \qquad (10c)
\end{aligned}$$

Interpretation:

- A positive $\widehat{B_{t+1}}$ (Home wealthier) implies higher relative consumption and a lower Q_{t+1} (Homes goods become relatively more expensive).
- Permanent productivity differences affect steady state consumption and prices directly.

11 Short-Run Allocation (Period t)

To simplify notation, we denote $\widehat{A_{N,t}} - \widehat{A^*_{N,t}} = \widetilde{A_{N,t}}$, $\widehat{A_{T,t}} - \widehat{A^*_{T,t}} = \widetilde{A_{T,t}}$, $\widehat{L_{N,t}} - \widehat{L^*_{N,t}} = \widetilde{L_{N,t}}$. Combining (9a) and (9d), we can get:

$$\begin{aligned}
\widetilde{C_t} + \frac{\gamma}{1-\gamma}\widehat{Q_t} - \gamma\widetilde{A_{T,t}} - \alpha\widetilde{L_{N,t}} &= \widetilde{A_{N,t}}(1-\gamma) + (1-\alpha)\widetilde{L_{N,t}} + \frac{\gamma}{1-\gamma}\widehat{Q_t} \\
\Rightarrow \widetilde{L_{N,t}} &= \widetilde{C_t} - \gamma\widetilde{A_{T,t}} - (1-\gamma)\widetilde{A_{N,t}} \qquad (11.1)
\end{aligned}$$

and that

$$(1-\alpha)\gamma\widetilde{A_{T,t}}+\alpha(1-\alpha)\widetilde{L_{N,t}}+\frac{\alpha\gamma}{1-\gamma}\widehat{Q_t}+\alpha\widetilde{C_t}=-\frac{\gamma(1-\alpha)}{1-\gamma}\widehat{Q_t}+(1-\alpha)\gamma\widetilde{A_{N,t}}+\alpha\widetilde{A_{N,t}}+\alpha(1-\alpha)\widetilde{L_{N,t}}$$

$$\Rightarrow \widehat{Q_t}=\Big(\frac{1-\gamma}{\gamma}\Big)[(1-\alpha)\gamma+\alpha]\widetilde{A_{N,t}}-(1-\alpha)(1-\gamma)\widetilde{A_{T,t}}-\frac{\alpha(1-\gamma)}{\gamma}\widetilde{C_t} \tag{11.2}$$

Implementing (11.1) and (11.2) back into (9b), we get:

$$\begin{aligned}
&-\Big(\frac{1-\gamma}{\gamma}\Big)[(1-\alpha)\gamma+\alpha]\widetilde{A_{N,t}}+(1-\alpha)(1-\gamma)\widetilde{A_{T,t}}+\frac{\alpha(1-\gamma)}{\gamma}\widetilde{C_t}+\widetilde{C_t}+\frac{\widehat{B_{t+1}}}{1-n}\\
&=\widetilde{A_{T,t}}-\frac{(1-\alpha)(1-\gamma)}{\gamma}\widetilde{C_t}+(1-\alpha)(1-\gamma)\widetilde{A_{T,t}}+\frac{(1-\alpha)(1-\gamma)^2}{\gamma}\widetilde{A_{N,t}}\\
&\Rightarrow \frac{\widehat{B_{t+1}}}{1-n}=\widetilde{A_{T,t}}+\frac{1-\gamma}{\gamma}\widetilde{A_{N,t}}-\frac{1}{\gamma}\widetilde{C_t}\\
&\Rightarrow \widetilde{C_t}=\gamma\widetilde{A_{T,t}}+(1-\gamma)\widetilde{A_{N,t}}-\gamma\frac{\widehat{B_{t+1}}}{1-n}
\end{aligned} \tag{11.3}$$

Bring (11.3) back to (11.2), we get:

$$\widehat{Q_t}=(1-\gamma)(\widetilde{A_{N,t}}-\widetilde{A_{T,t}})+\gamma\frac{\widehat{B_{t+1}}}{1-n} \tag{11.4}$$

Subtracting (11.4) from (10a), we get:

$$\widehat{Q_{t+1}}-\widehat{Q_t}=(1-\gamma)(\widetilde{A_{N,t+1}}-\widetilde{A_{N,t}})-(1-\gamma)(\widetilde{A_{T,t+1}}-\widetilde{A_{T,t}})-\alpha(1-\gamma)\frac{1}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \tag{11.5}$$

Subtracting (11.3) from (10c), we get:

$$\widetilde{C_{t+1}}-\widetilde{C_t}=\gamma(\widetilde{A_{T,t+1}}-\widetilde{A_{T,t}})+(1-\gamma)(\widetilde{A_{N,t+1}}-\widetilde{A_{N,t}})+\frac{\gamma}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \tag{11.6}$$

Bring (11.5) and (11.6) back to (9c), we get:

$$\begin{aligned}
&\gamma(\widetilde{A_{T,t+1}}-\widetilde{A_{T,t}})+(1-\gamma)(\widetilde{A_{N,t+1}}-\widetilde{A_{N,t}})+\frac{\gamma}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}\\
&=(1-\gamma)(\widetilde{A_{N,t+1}}-\widetilde{A_{N,t}})-(1-\gamma)(\widetilde{A_{T,t+1}}-\widetilde{A_{T,t}})-\alpha(1-\gamma)\frac{1}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}+\widehat{\beta_{H,t+1}}-\widehat{\beta_{F,t+1}}\\
&\Rightarrow \Big[\gamma+\alpha(1-\gamma)\Big]\frac{1}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n}=\widehat{\beta_{H,t+1}}-\widehat{\beta_{F,t+1}}-(\widetilde{A_{T,t+1}}-\widetilde{A_{T,t}})\\
&\Rightarrow \frac{\widehat{B_{t+1}}}{1-n}=\frac{\beta_0}{\gamma+\alpha(1-\gamma)}\Big[\widehat{\beta_{H,t+1}}-\widehat{\beta_{F,t+1}}-(\widehat{A_{T,t+1}}-\widehat{A^*_{T,t+1}})+(\widehat{A_{T,t}}-\widehat{A^*_{T,t}})\Big]
\end{aligned} \tag{11a}$$

Bring (11a) back to (11.3), we get:

$$\begin{aligned}\widehat{C_{t+1}} - \widehat{C}_t &= \gamma(\widehat{A_{T,t}} - \widehat{A^*_{T,t}}) + (1-\gamma)(\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) \\ &\quad - \frac{\gamma\beta_0}{\gamma+\alpha(1-\gamma)}\Big[\widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}} - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) + (\widehat{A_{T,t}} - \widehat{A^*_{T,t}})\Big]\end{aligned} \tag{11b}$$

Bring (11a) back to (11.4), we get:

$$\begin{aligned}\widehat{Q_t} &= -(1-\gamma)(\widehat{A_{T,t}} - \widehat{A^*_{T,t}}) + (1-\gamma)(\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) \\ &\quad + \frac{\alpha(1-\gamma)\beta_0}{\gamma+\alpha(1-\gamma)}\Big[\widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}} - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) + (\widehat{A_{T,t}} - \widehat{A^*_{T,t}})\Big]\end{aligned} \tag{11c}$$

Bring (11b) and (11c) back to (11.1), we get:

$$\widehat{L_{N,t}} - \widehat{L^*_{N,t}} = -\frac{\gamma\beta_0}{\gamma+\alpha(1-\gamma)}\Big[\widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}} - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) + (\widehat{A_{T,t}} - \widehat{A^*_{T,t}})\Big] \tag{11d}$$

By (9c), we know that:

$$\beta_0(\widehat{r^C_{t+1}} - \widehat{r^{C*}_{t+1}}) = \widehat{Q_{t+1}} - \widehat{Q_t}$$

Implementing (10.4) and (11c) back to the equation, we get:

$$\begin{aligned}\beta_0(\widehat{r^C_{t+1}} - \widehat{r^{C*}_{t+1}}) &= -(1-\gamma)\Big[(\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) - (\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}})\Big] - \alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\ &\quad - \Big\{-(1-\gamma)(\widehat{A_{T,t}} - \widehat{A^*_{T,t}}) + (1-\gamma)(\widehat{A_{N,t}} - \widehat{A^*_{N,t}}) + \alpha(1-\gamma)\frac{\widehat{B_{t+1}}}{1-n}\Big\}\end{aligned}$$

Bring (11a) back to the equation, we get:

$$\begin{aligned}\beta_0(\widehat{r^C_{t+1}} - \widehat{r^{C*}_{t+1}}) &= -(1-\gamma)\Big[(\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) - (\widehat{A_{N,t+1}} - \widehat{A^*_{N,t+1}})\Big] - \alpha(1-\gamma)\frac{1-\beta_0}{\beta_0}\frac{\widehat{B_{t+1}}}{1-n} \\ &\quad + (1-\gamma)\Big[(\widehat{A_{T,t}} - \widehat{A^*_{T,t}}) - (\widehat{A_{N,t}} - \widehat{A^*_{N,t}})\Big] \\ &\quad - \frac{\alpha(1-\gamma)}{\gamma+\alpha(1-\gamma)}\Big[\widehat{\beta_{H,t+1}} - \widehat{\beta_{F,t+1}} - (\widehat{A_{T,t+1}} - \widehat{A^*_{T,t+1}}) + (\widehat{A_{T,t}} - \widehat{A^*_{T,t}})\Big]\end{aligned} \tag{11e}$$

Interpretation:

- A temporary increase in Home patience (i.e. $\widehat{\beta_{H,t+1}} - \hat{\beta}_{F,t+1} > 0$) leads to $\widehat{B_{t+1}} > 0$ (Home runs a current account surplus), lower relative consumption, and a real exchange rate movement consistent with a trade surplus.

- Temporary traded or non-traded productivity shocks affect the current account and labor allocation differently.

- For permanent shocks ($\widehat{A_{T,t}} = \hat{A}_{T,t+1}$), intertemporal balance is restored with $\widehat{B_{t+1}} = 0$ and immediate adjustment to the new steady state.

12 Summary of Key Economic Insights

- **Consumption and Prices:** The structure of the consumption basket implies that a rise in the nontraded good price $P_{N,t}$ increases the overall consumption price P_t and shifts the consumption mix.
- **Market Clearing:** Non-traded goods are produced and consumed domestically, whereas traded goods are allocated internationally, linking domestic production choices to the real exchange rate.
- **Intertemporal Choices:** The Euler equations show that higher patience or higher real returns lead to deferred consumption. Cross-country differences in patience drive current account imbalances.
- **Labor Allocation:** Labor is reallocated across sectors until the marginal value products are equalized; productivity shifts affect both output composition and relative prices.
- **Steady State and Log-Linearization:** In a symmetric steady state, relative prices and allocations are balanced. Log-linearization permits analysis of small shocks and their propagation.
- **Short-Run vs. Long-Run Dynamics:** Temporary shocks generate current account imbalances and short-run reallocation, while permanent shocks adjust consumption and prices directly with no asset accumulation.
- **Wealth Effects:** A positive net asset position (Home wealthier) implies higher steady-state consumption and a relatively stronger (appreciated) currency.