PS1 Solutions

Jingle fu, Flammarion Akafack, Chaitanya Venkateswaran

Question 1. *Sandra pays income tax according to the schedule*

$$T(X) \;=\; \begin{cases} 0 & if X < E \\ t(X-E) & if X \geq E \end{cases}$$

where X is her pre-tax income; E and t are positive constants with $t < 1$. Sandra is also eligible for an income-related transfer

$$B(X) \;=\; \begin{cases} s(P-X) & if X < P \\ 0 & if X \geq P \end{cases}$$

where P and s are constants such that $P > 0$ and $t < s < 1$. Sandra's disposable income Y is therefore equal to $F(X)$, where

$$F(X) = X - T(X) + B(X)$$

Sketch the graph $Y = F(X)$ in each of the following case:

(i) $E > P$;

(ii) $E < P$, $s + t < 1$;

(iii) $E < P$, $s + t > 1$.

Solution 1.

(i) For $E > P$, the function can be separated into 3 parts:

	$[0, P)$	$[P, E)$	$[E, +\infty)$
$T(X)$	0	0	$t(X-E)$
$B(X)$	$s(P-X)$	0	0
Y	$X + s(P-X)$	X	$X - t(X-E)$

(ii) For $E < P$, the function can be separated into 3 parts:

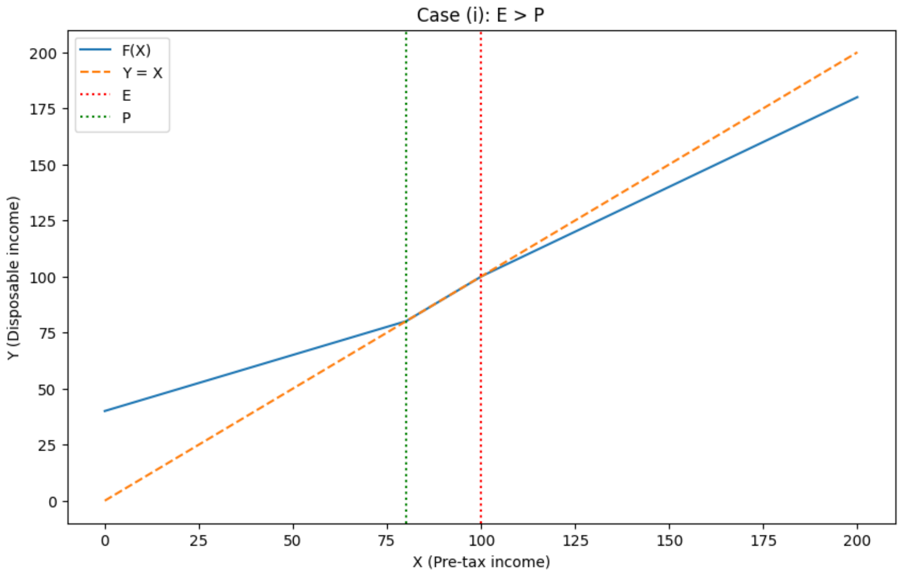

Figure 1: Case(i)

	$[0, E)$	$[E, P)$	$[P, +\infty)$
$T(X)$	0	$t(X-E)$	$t(X-E)$
$B(X)$	$s(P-X)$	$s(P-X)$	0
Y	$X+s(P-X)$	$X-t(X-E)+s(P-X)=(1-t-s)X+tE+sP$	$X-t(X-E)$

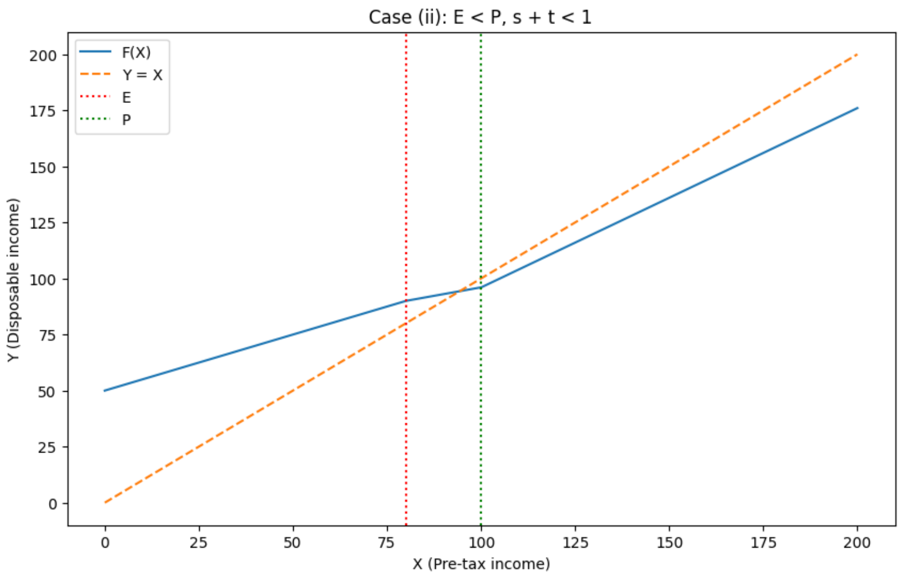

Figure 2: Case(ii)

(iii) We have the same piecewise function. Only for the interval $[E, P)$, the slope of the function is negative.

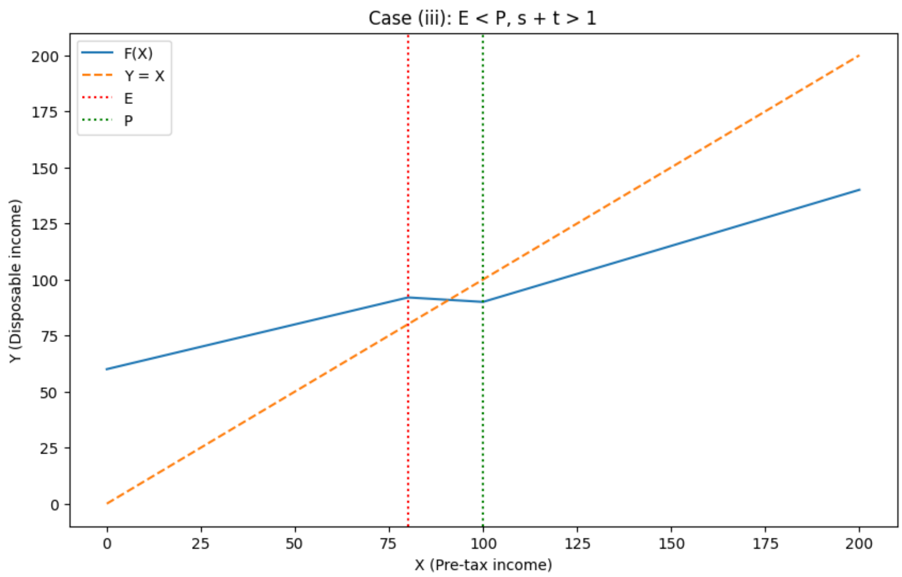

Figure 3: Case(iii)

Question 2. *Given the function*

$$y = \frac{x^2}{x^2 - x + 1}$$

(i) Use the quotient rule to calculate $\frac{dy}{dx}$

(ii) Obtain the same result by writing $y = u^{-1}$*, where* $u = 1 - x^{-1} + x^{-2}$*, and using the composite function rule.*

Solution 2.

(i) The quotient rule states that for a function $y = \frac{f(x)}{g(x)}$*, the derivative is given by:*

$$\begin{aligned} y' = \frac{dy}{dx} &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \\ \Leftrightarrow y' &= \frac{2x(x^2 - x + 1) - (2x - 1)(x^2)}{(x^2 - x + 1)^2} \\ \Leftrightarrow y' &= \frac{-x^2 + 2x}{(x^2 - x + 1)^2} \end{aligned}$$

By simplifying the numerator, the derivative can also be written as: $y' = \frac{x(-x+2)}{(x^2-x+1)^2}$

(ii) Since $y = u^{-1}$ *and* $u = 1 - x^{-1} + x^{-2}$*, the function* y *can be written as follow :*

$$y = u^{-1} = \frac{1}{u}$$

And the function u can be rewritten and transformed as:

$$u = 1 - \frac{1}{x} + \frac{1}{x^2} = \frac{x^2 - x + 1}{x^2}$$

By using the composite function rule, we can find the derivative ($\frac{dy}{dx}$):

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Thus,

$$\frac{du}{dy} = -u^{-2} = -\frac{1}{u^2}$$

and the derivative $u' = \frac{du}{dx}$ is given by:

$$\begin{aligned} u' = \frac{du}{dx} &= \frac{(2x-1)x^2 - (2x)(x^2 - x + 1)}{x^4} \\ &= \frac{x^2 - 2x}{x^4} = \frac{x-2}{x^3} \end{aligned}$$

So,

$$\frac{dx}{dy} = -\frac{(x^2 - x + 1)^2}{x^4} \cdot \frac{x^3}{x-2} = \frac{x(2-x)}{(x^2 - x + 1)^2}$$

in conclusion we obtain the same derivative or result.

Question 3. *A data set consists of n observations on k variables $x_1, x_2, ...x_k$; the i^{th} observation is denoted $(x_{1i}, x_{2i}, \cdots, x_{ki})$. Let $\mathbf{X}$ be the $n \times k$ matrix whose i^{th} row is the i^{th} observation. Calculate the matrix $\mathbf{X}'\mathbf{X}$, expressing its entries in $\sum$-notation, and verify that it is symmetric.*

Solution 3.

$$
\begin{aligned}
\mathbf{X}'\mathbf{X} &= \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k1} & x_{k2} & \cdots & x_{kn} \end{bmatrix} \cdot \begin{bmatrix} x_{11} & x_{21} & \cdots & x_{k1} \\ x_{12} & x_{22} & \cdots & x_{k2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \cdots & x_{kn} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{m=1}^{n} x_{1m}x_{1m} & \sum_{m=1}^{n} x_{1m}x_{2m} & \cdots & \sum_{m=1}^{n} x_{1m}x_{km} \\ \sum_{m=1}^{n} x_{2m}x_{1m} & \sum_{m=1}^{n} x_{2m}x_{2m} & \cdots & \sum_{m=1}^{n} x_{2m}x_{km} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{m=1}^{n} x_{km}x_{1m} & \sum_{m=1}^{n} x_{km}x_{2m} & \cdots & \sum_{m=1}^{n} x_{km}x_{km} \end{bmatrix}
\end{aligned}
$$

Thus, we know that

$$(\mathbf{X}'\mathbf{X})_{ij} = \sum_{m=1}^{n} x_{im}x_{jm}.$$

Since i and j are not fixed, we switch them and it is obvious that $(\mathbf{X}'\mathbf{X})_{ij} = (\mathbf{X}'\mathbf{X})_{ji}$.

Question 4. *Let $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ be invertible matrices of the same order. Simplify the expressions*

$$(\mathbf{I}+\mathbf{A})\mathbf{A}^{-1}(\mathbf{I}-\mathbf{A}), \mathbf{A}(3\mathbf{A}^{-1}+4\mathbf{B}^{-1})\mathbf{B}, (\mathbf{A}\mathbf{B}^{-1}\mathbf{C})^{-1}$$

Solution 4.

$$(\mathbf{I}+\mathbf{A})\mathbf{A}^{-1}(\mathbf{I}-\mathbf{A}) = (\mathbf{I}+\mathbf{A})(\mathbf{A}^{-1}\mathbf{I}-\mathbf{I}) = \mathbf{I}\mathbf{A}^{-1} - \mathbf{I} + \mathbf{A}\mathbf{A}^{-1} - \mathbf{A} = \mathbf{A}^{-1} - \mathbf{A}$$

$$\mathbf{A}(3\mathbf{A}^{-1}+4\mathbf{B}^{-1})\mathbf{B} = (3\mathbf{I}+4\mathbf{A}\mathbf{B}^{-1})\mathbf{B} = 3\mathbf{B}+4\mathbf{A}$$

$$(\mathbf{A}\mathbf{B}^{-1}\mathbf{C})^{-1} = \mathbf{C}^{-1}\mathbf{B}\mathbf{A}^{-1}$$

Question 5. *Given a quadratic function*

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{b}^T\mathbf{x} + \mathbf{c}$$

where $\mathbf{x} \in \mathbb{R}^{3\times 1}$ and $\mathbf{b} \in \mathbb{R}^{3\times 1}$ are column vectors, $\mathbf{A} \in \mathbb{R}^{3\times 3}$ is a symmetric matrix, $c \in \mathbb{R}$ is a scalar.

(i) Derive the gradient of $f(\mathbf{x})$ with respect to $\mathbf{x}$.

(ii) Derive the Hessian of $f(\mathbf{x})$ with respect to $\mathbf{x}$.

Solution 5.

(i) Since we know that $A \in \mathbb{R}^{3\times 3}$ is a symmetric matrix ($A = A^\top$), the gradient of f with respect to x is given by:

$$f' = \frac{\partial f}{\partial x} = Ax + b$$

(ii) The second-order partial derivative of f with respect to x and x' is given by:

$$\frac{\partial^2 f}{\partial x \partial x'} = \frac{\partial}{\partial x'}\left(\frac{\partial f}{\partial x}\right) = (Ax + b)' = A$$

Question 6. *Suppose the function from our In-class Exercise*

$$y = -x^4 + 4x^3 - 6x^2 + 8x + 3$$

is defined only for $x \geq 0$. Derive the coordinates of the local minimum point and the global maximum point. What happens if the function is defined only for (i) $x \geq 1$, (ii) $x \geq 2$, (iii) $x \geq 3$?

Solution 6.

$y^{'} = -4x^3 + 12x^2 - 12x + 8 = -4(x-2)(x^2 - x + 1)$

$y^{''} = -12x^2 + 24x - 12 = -12(x-1)^2 \leq 0$

Let $y^{'} = 0$, we have $x = 2$. Since $y^{''} \leq 0$, we know that $y^{'}$ is decreasing,

	$[0, 2)$	2	$(2, +\infty)$
$y^{'}$	> 0	0	< 0

Hence, y is increasing from $[0, 2)$ and decreasing from $(2, +\infty)$. Thus $x = 2, y = 11$ is a global maximum point, and a boundary local minimum point at $(0, 3)$.

(i) If $x \geq 1$, derivative conditions don't change, the global maximum point is $(2, 11)$ and a boundary local minimum point at $(1, 8)$.

(ii) If $x \geq 2$, derivative conditions change to:

	2	$(2, +\infty)$
$y^{'}$	0	< 0
$y^{''}$	< 0	< 0

the global maximum point is $(2, 11)$ and no local minimum point.

(iii) If $x \geq 3$, derivative conditions change to:

	$[3, +\infty)$
y'	< 0

y continuous decreasing with no point's derivative equals to 0, thus in this case, the function has no local minimum point. The boundary global maximum point is $(3, 0)$.

Question 7. *Which of the following functions of x are convex? Which are concave?*
(i) $(2x-1)^6$;
(ii) $\sqrt{1+x^2}$;
(iii) $x^5 - x$.

Solution 7.

The identification of the convexity and concavity of each function involves considering the following property: A function is convex if its second derivative is non-negative for all x (i.e., $f''(x) \geq 0$). Conversely, a function is concave if its second derivative is non-positive for all x (i.e., $f''(x) \leq 0$). Now, let's consider each function:

(i) $f(x) = (2x-1)^6$

-First derivative $f'(x)$***:***

Given that : $(g(f(x))' = g'(f(x)) \cdot f'(x)$

$$\Leftrightarrow f'(x) = 6(2x-1)^5 \cdot 2 = 12(2x-1)^5$$

-Second derivative $f''(x)$***:***

$$f''(x) = 0 \cdot (2x-1)^5 + 12 \cdot 5 \cdot (2x-1)^4 \cdot 2 = 120(2x-1)^4 \geq 0$$

In conclusion, since $f''(x) \geq 0$ ***for all*** x***, so f(x) is convex.***

(ii) $f(x) = \sqrt{1+x^2}$

-First derivative $f'(x)$***:*** *Given that* $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$*, then*

$$f'(x) = \frac{2x}{2\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}}$$

-Second derivative $f''(x)$***:***

Since

$$\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

then we have:

$$\begin{aligned}f'(x) &= \frac{1\cdot(\sqrt{1+x^2}) - \frac{x}{\sqrt{1+x^2}}\cdot x}{\left(\sqrt{1+x^2}\right)^2} \\ &= \frac{1+x^2-x^2}{(1+x^2)^{1/2}\cdot(1+x^2)} \\ &= \frac{1}{(1+x^2)^{3/2}} \geq 0\end{aligned}$$

In conclusion, since $f''(x) \geq 0$ ***for all*** x***, so f(x) is convex.***

(iii) $f(x) = (x^5 - x)$

-First derivative $f'(x)$ ***is as follow :***

$$f'(x) = 5x^4 - 1$$

-Second derivative $f''(x)$ ***in turn is given by:***

$$f''(x) = 20x^3$$

In conclusion, in this specific case, the function $f(x)$ ***is convex for all*** $x > 0$ ***and concave for all*** $x < 0$***. If*** $x = 0$***, the function has an inflection point.***

Question 8. *Compute the first and second derivatives of each of the following functions:*

(i) $e^{x^2\cdot 3x-2}$*;*

(ii) $\ln{(x^4+2)}^2$*;*

(iii) $\frac{x}{\ln x}$

Solution 8.

(i) If we regard the question as $y = e^{x^2\cdot 3x-2}$*, we will have* $y' = 9x^2e^{3x^3-2}$*,* $y'' = (81x^4 + 18x)e^{3x^3-2}$.

If, (is it possible that Joy you get a typo here?) $y = e^{x^2\cdot(3x-2)}$*, then we will have* $y' = (9x^2 - 4x)e^{x^2\cdot(3x-2)}$*, and* $y'' = (9x^2-4x)^2e^{x^2\cdot(3x-2)} + (18x-4)e^{x^2\cdot(3x-2)} = (81x^4 - 72x^3 + 16x^2 - 18x - 4)e^{x^2\cdot(3x-2)}$.

(ii) $y' = \frac{8x^3}{(x^4+2)}$*,* $y'' = \frac{24x^2(x^4+2)-32x^6}{(x^4+2)^2} = \frac{8x^2(6-x^4)}{(x^4+2)^2}$

(iii) $y' = \frac{\ln x - 1}{\ln^2 x}$, $y'' = \frac{\frac{1}{x}\ln x - (\ln x - 1)\frac{2}{x}\ln x}{\ln^4 x} = \frac{2-\ln x}{x\ln^3 x}$

Question 9. *Let*

$$f(x) = x^2 \ln x$$

(i) Find the linear and quadratic approximations to $f(x)$ for values of x close to 2. Construct a numerical table similar to our In-class Exercise for $x = 1.80, 1.95, 2.02, 2.10$ and 2.25.

(ii) Find third-order approximations to $f(x)$ for values of x close to 1 and for values of x close to 2 . Use these approximations to extend the numerical tables of Example 1 and part (a) of this exercise by an additional row.

Solution 9.

(i) $f' = x + 2x\ln x$, $f(2) = 4\ln 2$, $f'(2) = 2 + 4\ln 2$, *linear approx of the function is*

$$f(x) = 4\ln 2 + (2 + 4\ln 2)(x-2)$$

quadratic approximation of the function is

$$f(x) = 4\ln 2 + (2 + 4\ln 2)(x-2) + \frac{3 + 2\ln 2}{2}(x-2)^2$$

x	Exact Value	Linear Approx	Quadratic Approx
1.80	1.904429	1.818071	1.905797
1.95	2.539421	2.533959	2.539442
2.02	2.868919	2.868040	2.868918
2.10	3.271944	3.249848	3.271779
2.25	4.105334	3.965736	4.102808

Table 1: Comparison of E.V. and Linear and Quad Approx

(ii)

$$f(x) = 4\ln 2 + (2 + 4\ln 2)(x-2) + \frac{3 + 2\ln 2}{2}(x-2)^2 + \frac{1}{6}(x-2)^3$$

x	Exact Value	Linear Approx	Quadratic Approx	Cubic Approx
1.80	1.904429	1.818071	1.905797	1.904464
1.95	2.539421	2.533959	2.539442	2.539421
2.02	2.868919	2.868040	2.868918	2.868919
2.10	3.271944	3.249848	3.271779	3.271946
2.25	4.105334	3.965736	4.102808	4.105412

Table 2: Comparison of E.V. and Approximations at $x = 2$

x	Exact Value	Linear Approx	Quadratic Approx	Cubic Approx
0.80	-0.142812	-0.20	-0.14000	-0.142667
0.95	-0.046292	-0.05	-0.04625	-0.046292
1.02	0.020603	0.02	0.02060	0.020603
1.10	0.115325	0.10	0.11500	0.115333
1.25	0.348662	0.25	0.34375	0.348958

Table 3: Comparison of E.V. and Approximations at $x = 1$

Question 10. *Consider the standard consumption Euler equation that emerges from household optimization problems with CRRA utility:*

$$\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = \beta(1 + r_t)$$

Log-linearize the above equation.

Solution 10.

$$\sigma(\ln c_{t+1} - \ln c_t) = \ln \beta + \ln (1 + r_t)$$

Thus,

$$\sigma(\ln c^* + \frac{c_{t+1} - c^*}{c^*} - \ln c^* - \frac{c_t - c^*}{c^*}) = \ln \beta + \ln (1 + r^*) + \frac{r_t - r^*}{1 + r^*}$$

$$\Rightarrow \sigma(\hat{c}_{t+1} - \hat{c}_t) = \frac{r_t - r^*}{1 + r^*} = (1 - \beta)\hat{r}_t$$

where $\hat{r}_t = \frac{r_t - r^*}{r^*}$, $\hat{c}_{t+1} = \frac{c_{t+1} - c^*}{c^*}$,$\hat{c}_t = \frac{c_t - c^*}{c^*}$