PS1 Solutions

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Solution 1 (Gains from Trade).
Assuming the consumption bundle of Australia(A) before trading with China is $\mathbf{c_1}$ and after is $\mathbf{c_2}$.
Before China's involvement, according to the Weak Axiom of Revealed Preference, we have $\mathbf{p_1 c_1} \leq \mathbf{p_1 c_2}$.
After China's entering, we have $\mathbf{p_2 c_2} \leq \mathbf{p_2 c_1}$.

Solution 2 (Ricardian Trade and Technological Progress).

1. For absolute advantage, we compare unit labor productivity directly:

    - Clothing
        - Home: produces z unit per hour.
        - Foreign: produces 1 unit per hour.

      Home has an absolute advantage in clothing if $z > 1$; otherwise, Foreign does.
    - Food
        - Home: produces z unit per hour.
        - Foreign: produces 4 unit per hour.

      Home has an absolute advantage in food if $z > 4$; otherwise, Foreign does.

    Thus, comparing the absolute advantage, we have the following table:

	Clothing	Food
$z > 4$	Home	Home
$1 < z < 4$	Home	Foreign
$z < 1$	Foreign	Foreign

2. For comparative advantage, we compare the opportunity cost of producing one unit of one good in terms of the other good:

    - Home: The opportunity cost of 1 unit of Clothing over 1 unit of Food is: $\frac{a_C}{a_F} = 1$.

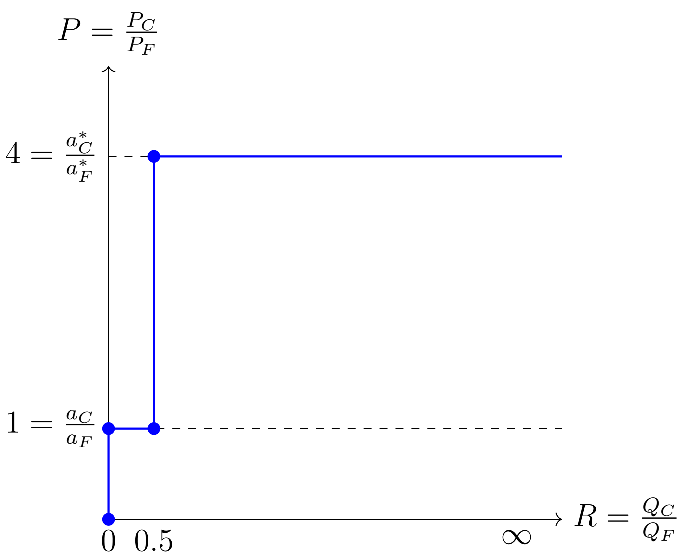
- Foreign: The opportunity cost of 1 unit of Clothing over 1 unit of Food is: $\frac{a_C^*}{a_F^*} = 4$.

Thus, Home has a comparative advantage in Clothing, and Foreign has a comparative advantage in Food, regardless of z.

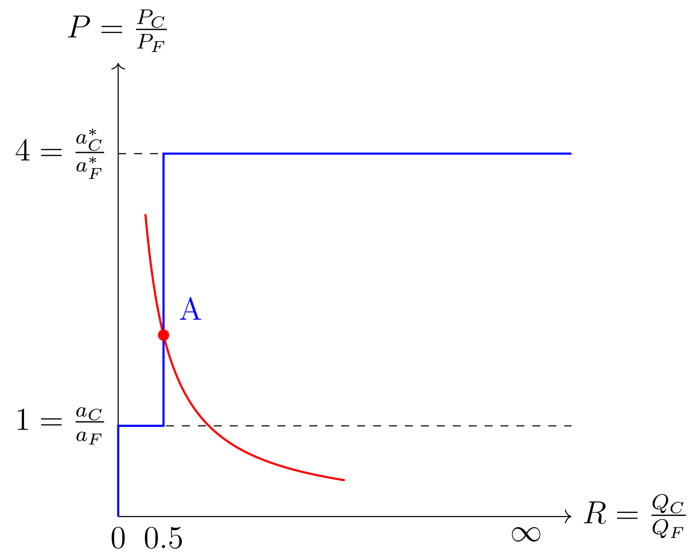
3. If $z = 2$, the relative price of Clothing is $P = \frac{P_C}{P_F} = \frac{a_C}{a_F} = 1$, $P^* = \frac{a_C^*}{a_F^*} = 4$. $Q_C = \frac{L}{a_C} = 2000 = Q_F$, $Q_C^* = \frac{L^*}{a_C^*} = 1000$, $Q_F^* = \frac{L^*}{a_F^*} = 4000$.

   (a) Draw the world relative supply of clothing.

   - When $P < 1$, both Home and Foreign produce only Food, giving $R = \frac{Q_C + Q_C^*}{Q_F + Q_F^*} = 0$;
   - When $P = 1$, Home can vary production between $(\text{Clothing}, \text{Food}) = \Big[(2000, 0), (0, 2000)\Big]$ and Foreign produces only Food, giving $R \in [0, 0.5]$;
   - When $1 < P < 4$, Home produces only Clothing and Foreign produces only Food, giving $R = 0.5$;
   - When $P = 4$, Home produces only Clothing and Foreign can vary production between $(\text{Clothing}, \text{Food}) = \Big[(0, 4000), (1000, 0)\Big]$, giving $R \in [0.5, \infty)$;
   - When $P > 4$, both Home and Foreign produce only Clothing, giving $R = \infty$.



   (b) Under the Cobb-Douglas utility function $U(C, F) = CF$, we can tell that consumers will spend the same expenditure on both goods. So, worldwide, $P_C Q_C = P_F Q_F$, which gives $\frac{P_C}{P_F} = \frac{Q_F}{Q_C} = \frac{1}{R} = 2$.



(c) In Home, a worker produces $z = 2$ units of Clothing per hour, hence the value of one hour's output is: $w = 2P_C$; While in foreign, a worker produces 4 units of Food per hour, having a value of $p^* = 4P_F$. Given that $\frac{P_C}{P_F} = 2$, we know it that $\frac{w}{w^*} = 1$.

4. As analyzed before, the change of z won't affect the world relative price. We assume that after the change both countries remain completely specialized in their comparative-advantage goods.

(a) Home produces $1000z$ units of Clothing, and Foreign produces 4000 unnits of food. The world relative supply of Clothing is $R = \frac{1000z}{4000} = \frac{z}{4}$.

For Home, a worker's nominal income is $w = zP_C$, and for Foreign, $w^* = 4P_F$. Our Cobb-Douglas utility with equal share tells us that $P = \frac{P_C}{P_F} = \frac{Q_F}{Q_C} = \frac{1}{R}$, thus $P = \frac{4}{z}$.

Thus the free-trade relative price is $\frac{P_C}{P_F} = \frac{4}{z}$, the wage ratio is:

$$\frac{w}{w^*} = \frac{zP_C}{4P_F} = 1.$$

(b) If z increases, the relative price $P = \frac{4}{z}$ decreases, since the

Solution 3 (Two-by-Two-by-Two with Fixed Coefficients)**.**

1. - **Relative Factor Abundance:** $RFA_A = \frac{K_A}{L_A} = \frac{420}{460} \approx 1.095$, $RFA_G = \frac{K_G}{L_G} = \frac{900}{600} = 1.5$. Thus Germany is relatively capital abundant and Austria is relatively labor abundant.
   - **Relative Factor Intensity of Goods:** $RFIG_B = \frac{a_{KB}}{a_{LB}} = 3$, $RFIG_S = \frac{a_{KS}}{a_{LS}} = 0.5$. Thus Buns are capital intensive, while Sausages are labor intensive.

- **Comparative Advantage:** By the Heckscher-Ohlin theorem, the relatively capital-abundant country, Germany, will have a comparative advantage in the capital-intensive good, Buns, and the relatively labor-abundant country, Austria, in the labor-intensive good, Sausages.

- **Autarkic Relative Price:** In autarky, each country's relative price reflects its "shadow" cost. Because factor prices adjust differently in each country (with the excess factor receiving a zero "price"), we expect:
  - Austria: As labor is in excess, the wage is set to 0, hence $P_A = \frac{P_{BA}}{P_{SA}} = \frac{a_{KB}}{a_{KS}} = 3$;
  - Germany: As capital is in excess, the rental rate is set to 0, hence $P_G = \frac{P_{BG}}{P_{SG}} = \frac{a_{LB}}{a_{LS}} = \frac{1}{2}$.

  Thus, the autarkic price of Buns is higher in Austria than in Germany. Under trade we expect Germany to export Buns and Austria to export Sausages.

- **Free Trade:** Under trade we expect Germany to export Buns and Austria to export Sausages.

2. We separate the two countries' production functions and factor endowments: $B = Q_{BA} + Q_{BG}$, $S = Q_{SA} + Q_{SG}$.

   (a) For Austria, the full employment conditions are:

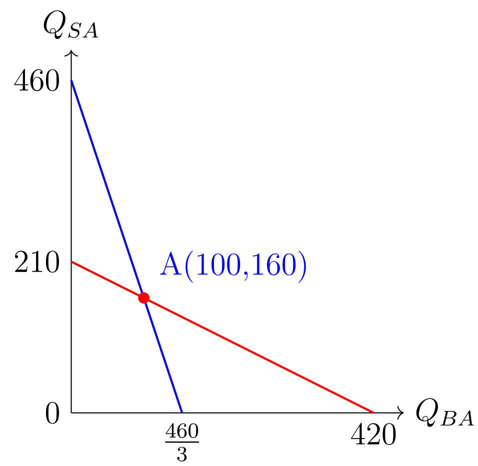
   - **Labor:**

   $$L_A = a_{LB}Q_{BA} + a_{LS}Q_{SA} \Rightarrow Q_{BA} + 2Q_{SA} = 420$$

   - **Capital:**

   $$K_A = a_{KB}Q_{BA} + a_{KS}Q_{SA} \Rightarrow 3Q_{BA} + Q_{SA} = 460$$

   Both constraints hold with equality, we can have: $(Q_{BA}, Q_{SA}) = (100, 160)$.



   (b) For Germany, the full employment conditions are:

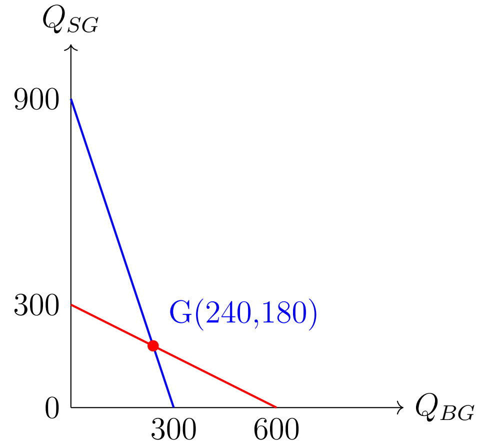
- **Labor:**

$$L_G = a_{LB}Q_{BG} + a_{LS}Q_{SG} \Rightarrow Q_{BG} + 2Q_{SG} = 600$$

- **Capital:**

$$K_G = a_{KB}Q_{BG} + a_{KS}Q_{SG} \Rightarrow 3Q_{BG} + Q_{SG} = 900$$

Solve the equations, we have: $(Q_{BG}, Q_{SG}) = (240, 180)$.



3. Consumers have Leontief preferences (they want to consume 1 Bun and 1 Sausage per hotdog). Because the consumption ratio is fixed at 1, autarkic equilibrium production must satisfy $B = S$.

   (a) We first find the ebalanced production point in both countries:

   - Austria:

$$\text{Labor: } B + 2B \leq 420$$
$$\text{Capital: } 3B + B \leq 460$$

     The binding constraint is capital, so the maximum balanced production in Austria is $B = S = 115$.

   - Germany:

$$\text{Labor: } B + 2B \leq 600$$
$$\text{Capital: } 3B + B \leq 900$$

     The binding constraint is labor, so the maximum balanced production in Germany is $B = S = 200$.

   The autarkic relative price is set by the zero-profit conditions.

   For Austria, as balanced production $(115, 115)$ lies in the interior of the production possibilities frontier, labor is in excess. (Thus $W = 0$, and prices are determined solely by the rental rate R.)

In Germany, capital is in excess, so $R = 0$.

The zero-profit conditions then are:

$$\begin{aligned} P_{BA} &= W + 3R = 3R \\ P_{SA} &= 2W + R = R \\ P_{BG} &= W \\ P_{SG} &= 2W \end{aligned}$$

These relative prices are in line with our prediction from part (1).

(b) In Austria, labor is in excess, hence $W = 0$. Total national income is $R \cdot K_A = 460R$. Each hotdog costs $P_{BA} + P_{SA} = 4R$, so each owner of a unit of capital can buy $\frac{R}{4R} = \frac{1}{4}$ hotdogs.

In Germany, capital is in excess, so $R = 0$. Total income is $W \cdot L_G = 600W$. Each hotdog costs $P_{BG} + P_{SG} = 3W$, so each worker can buy $\frac{W}{3W} = \frac{1}{3}$ hotdogs.

4. (a)

Solution 4 (Two-by-Two-by-Two)**.**

1.

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Solution 5 (Gravity with Multilateral Resistance)**.**

Let the income of country j denoted by Y_j, and the price of a good from country i in country j be p_{ij}. For consumers, their UMP is:

$$\begin{aligned} &\max_{x_{kj}} \left[\sum_k \alpha_{kj}^{\frac{1}{\sigma}} x_{kj}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} \\ &\text{s.t. } \sum_k x_{kj} p_{kj} \leq Y_j \end{aligned}$$

Define the Lagrangian:

$$\mathcal{L} = \left[\sum_k \alpha_{kj}^{\frac{1}{\sigma}} x_{kj}^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}} - \lambda \Big(\sum_k x_{kj} p_{kj} - Y_j \Big)$$

The FOCs are:

$$\frac{\partial \mathcal{L}}{\partial x_{kj}} = \left[\sum_k \alpha_{kj}^{\frac{1}{\sigma}} x_{kj}^{\frac{\sigma-1}{\sigma}}\right]^{\frac{1}{\sigma-1}} \alpha_{kj}^{\frac{1}{\sigma}} x_{kj}^{-\frac{1}{\sigma}} - \lambda p_{kj} = 0 \tag{1}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \sum_k x_{kj} p_{kj} - Y_j = 0 \tag{2}$$

From (1), we know that for two countries k and k', we have:

$$\frac{p_{kj}}{p_{k'j}} = \frac{\alpha_{kj}^{\frac{1}{\sigma}} x_{kj}^{-\frac{1}{\sigma}}}{\alpha_{k'j}^{\frac{1}{\sigma}} x_{k'j}^{-\frac{1}{\sigma}}} \Leftrightarrow \frac{\alpha_{kj}}{\alpha_{k'j}} = \frac{p_{kj}^{\sigma}}{p_{k'j}^{\sigma}} \frac{x_{kj}}{x_{k'j}}$$

Rearranging and multiplying both sides by $p_{k'j}$ yields:

$$x_{k'j} p_{k'j} = \frac{1}{\alpha_{kj}} x_{kj} p_{kj}^{\sigma} \alpha_{k'j} p_{k'j}^{1-\sigma}$$

Summing for all countries gives:

$$\sum_{k'} x_{k'j} p_{k'j} = \frac{1}{\alpha_{kj}} x_{kj} p_{kj}^{\sigma} \sum_{k'} \alpha_{k'j} p_{k'j}^{1-\sigma} \Leftrightarrow Y_j = \frac{1}{\alpha_{kj}} x_{kj} p_{kj}^{\sigma} P_j^{1-\sigma} \tag{3}$$

where $P_j = \left[\sum_k \alpha_{kj} p_{kj}^{1-\sigma}\right]^{\frac{1}{1-\sigma}}$ is the Dixit-Stiglitz price index.
Rearrange (3) gives the CES demand function:

$$x_{kj} = \alpha_{kj} p_{kj}^{-\sigma} Y_j P_j^{\sigma-1} \tag{4}$$

Noting that the value of total trade is simply equal to the price times quantity, which gives $X_{kj} = p_{kj} x_{kj}$, we have:

$$X_{kj} = \alpha_{kj} p_{kj}^{1-\sigma} Y_j P_j^{\sigma-1} \tag{5}$$

As we suppose that the market for each country/good is perfectly competitive, the price of a good is simply the marginal cost. Let w_i be the wage of a worker in country i.
Because we assume that one unit of labor produces one unit of the local output, the wage in country i is simply the price of the good produced in country i: $p_i = w_i$.
So, the price of j consuming 1 unit of country i's good is:

$$p_{ij} = \tau_{ij} w_i \Rightarrow \tau_{ij} = \frac{p_{ij}}{p_i}. \tag{6}$$

1. Let λ_{ij} be the share of total spending in country j that is devoted to goods imported

from country i. Implementing (5), (6) and the price index, We have:

$$\begin{aligned}
\lambda_{ij} &= \frac{X_{ij}}{Y_j} = \alpha_{ij} p_{ij}^{1-\sigma} P_j^{\sigma-1} \\
&= \alpha_{ij}\left(\tau_{ij} w_i\right)^{1-\sigma}\left[\sum_k \alpha_{kj}\left(\tau_{kj} w_k\right)^{1-\sigma}\right]^{\sigma-1} \\
&= \frac{\alpha_{ij}\tau_{ij}^{1-\sigma} w_i^{1-\sigma}}{\sum_k \alpha_{kj}\left(\tau_{kj} w_k\right)^{1-\sigma}}
\end{aligned} \tag{*}$$

2. As market clearing requires income is equal to payments to, the Y we previously defined is the same as the X_i(defined as GDP in country i).

   Income in a country is also equal to its total sales:

$$\begin{aligned}
X_i &= \sum_j X_{ij} = \sum_j \alpha_{ij}\tau_{ij}^{1-\sigma} w_i^{1-\sigma} X_j P_j^{\sigma-1} \\
&\Rightarrow w_i^{1-\sigma} = X_i / \sum_j \alpha_{ij}\tau_{ij}^{1-\sigma} X_j P_j^{\sigma-1}
\end{aligned}$$

   Replace this equation back into (5), we have:

$$\begin{aligned}
X_{ij} &= \alpha_{ij}\tau_{ij}^{1-\sigma} w_i^{1-\sigma} X_j P_j^{\sigma-1} \\
&= \alpha_{ij}\tau_{ij}^{1-\sigma}\frac{X_i}{\sum_k \alpha_{ik}\tau_{ik}^{1-\sigma} X_k P_k^{\sigma-1}} X_j P_j^{\sigma-1} \\
&= X_i X_j \left(\frac{\tau_{ij}}{P_j}\right)^{1-\sigma}\frac{\alpha_{ij}}{\sum_k \alpha_{ik}\left(\frac{\tau_{ik}}{P_k}\right)^{1-\sigma} X_k}
\end{aligned} \tag{**}$$

3. Under CES preferences, utility of the representative agent is the real wage. Thus in our notation, we have: $X_j = U_j P_j$. The per capita welfare W_i can be written as:

$$W_j = \frac{U_j}{L_j} = \frac{X_j}{P_j L_j} = \frac{w_j L_j}{P_j L_j} = \frac{w_j}{P_j} \tag{7}$$

   We assume that $\tau_{jj} = 1$, and by choosing $i = j$, (*) implies:

$$\lambda_{jj} = \alpha_{jj} w_j^{1-\sigma} P_j^{\sigma-1} \Rightarrow P_j = \left(\lambda_{jj}\alpha_{jj}^{-1} w_j^{\sigma-1}\right)^{\frac{1}{\sigma-1}} = \lambda_{jj}^{\frac{1}{\sigma-1}}\alpha_{jj}^{-\frac{1}{\sigma-1}} w_j$$

   Replacing this equation into (7), we have:

$$W_j = \frac{w_j}{\lambda_{jj}^{\frac{1}{\sigma-1}}\alpha_{jj}^{-\frac{1}{\sigma-1}} w_j} = \lambda_{jj}^{-\frac{1}{\sigma-1}}\alpha_{jj}^{\frac{1}{\sigma-1}} \tag{***}$$