Macroeconomics A

Lecture 3 - Business Cycle Facts and RBC Model

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Business Cycles

The questions:

- Why do we have aggregate fluctuations (booms & recessions)?
- What can we do against them?

To answer these questions, we need to construct models that are accurate in both their qualitative and quantitative predictions.
$\Rightarrow$ very ambitious!

Undergraduate macro

Economy divided into three markets: labor market, money market, goods market.

- IS: $Y = \underbrace{A(Y, r, T, Y'^e, r'^e, T'^e)}_{\text{Demand for Goods}} + G$
- LM: $M = P \cdot L(Y, i)$
- PC: $\pi = \pi^e + \underbrace{f(Y, z)}_{\text{Deviation of Output from MR level}}$

where Y'^e is the expected output in the *next* period and $r = i - \pi^e$ is the real expected interest rate

Assumptions / implications of this model

- in the short run the price level, P is given $\Rightarrow$
  hence IS & LM determines output
  demand matters & many factors affect demand
- in the medium run $\pi^e = \pi$, so $f(Y, z) = 0 \Rightarrow$ output at the natural level
- money affects output in the short run, not in the medium run
- money growth affects inflation and the nominal rate one-for-one in the medium run
- a fiscal expansion increases output in the short run, may decrease it in the medium run
- expectations matter: an anticipated fiscal contraction can be expansionary
  effect of money depends on i, π^e, i', π'^e

Strength and weakness of these models

Strengths

- provides a simple way of thinking about *general equilibrium* (complicated)
- implicit micro-foundations: life cycle, q theory,...
- time-tested shortcuts (but often not based on empirical evidence)

Weaknesses:

- Static, not dynamic
- not really made for quantitative analysis (quantitative attempts very short-lived)
- need explicit micro-foundations: hard to do welfare analysis without that (Lucas critique: curves may not be invariant to policy)

Dynamic Stochastic General Equilibrium (DSGE) models

D some things don't make sense in static models (for example: investment)
relatively short-term analysis

S shocks hit the economy, and force it off the balanced growth path (BGP) $\Rightarrow$
fluctuations do not mean dis-equilibrium, this is the reaction of the economy to an outside shock

G E this is macro

- the models are based on
  - perfectly/monopolistically competitive markets
  - optimizing agents

  $\Rightarrow$ the economy is in equilibrium
- **micro-founded** models

There are different schools of thought...

SUBSTANTIVE DIFFERENCES

METHODOLOGICAL DIFFERENCES

	supply shocks	demand shocks rigidity	
simple models/ qualitative insights	$A \cdot L^{\alpha} K^{1-\alpha}$ growth model	SALT WATER contributed what is important RIGIDITY	→ aim: see the mechanisms very clearly
complex models quantitative matching	FRESH WATER contributed the METHODOLOGY	now there is more harmony in macro	→ transparency is the cost

↓

we are going to start here, because it is easier

Path to get there:

1. real business cycle (RBC) models
2. introduce money
3. introduce nominal rigidities - New Keynesian (NK) models

Some predictions in the end are not very different from the IS-LM. However, better sense

- of the role of distortions
- of optimal policy

Big disadvantage: usually need the computer to solve the models

Business Cycle Facts I.

We'll study properties of quarterly **detrended macro time series**:

$$x_t = log(X_t) - log(X_t^*)$$

is the percentage deviation of variable X from its trend, X^*

how is the trend defined?

- first linear
- now more sophisticated filters: Baxter-King ("Bandpass") filter, Hodrick-Prescott ("HP") filter

Linear: just run OLS regression and use residuals. Bandpass/HP filter: programs are available

Business Cycle Facts II.

look at the highest correlation with GDP

$$\rho(x_t, y_{t+k}) \quad k = -6, -5, ..., 0, ...5, 6$$

- if $\rho > 0$, then x is pro-cyclical
- if $\rho < 0$, then x is counter-cyclical
- if $k < 0$, then x lags behind output
- if $k > 0$, then x leads output

$\rho(x_t, x_{t+k})$ is called the *autocorrelation function* (as a function of k) of the stochastic process x_t.

Business Cycle Facts III.

Series	Standard deviation	Correlation with GDP	Lag
y	1.66	1	0
c	1.26	0.9	0
i	4.97	0.9	0
g	2.49	0.15	-6
hours	1.61	0.9	-1
n	1.39	0.8	-1
tfp	2.29	0.9	-1
$\frac{w}{P}$	0.64	0.16	0

- everything quite pro-cyclical
- except: government spending $\rightarrow$ does not seem to support the statement that the cause if BC is gov spending
- real wage is mildly pro-cyclical $\rightarrow$ big problem in many models aggregation bias: average wage evolves differently than the wage of a continuously employed worker

Source: Stock and Watson (1999)

Business Cycle Facts IV.

Standard deviations (unit of measure is percentage deviation from trend) $\Rightarrow$ they are comparable

- GDP is more volatile than consumption
- investment is much more volatile than GDP
- government spending is pretty volatile
- working hours is almost exactly as volatile as GDP
- vast majority of the volatility of working hours is explained by the employment volatility
  $\rightarrow$ weird, because it should be cheaper to adjust the working hours of employees than to hire/fire people
- TFP is very volatile
  $\rightarrow$ school of thoughts disagree whether this is a cause or consequence of business cycles

Some other facts (numbers not here)

- There is significant heterogeneity in output volatilities across sectors and in wages across worker groups
- Equity returns are large and volatile relative to risk-free returns ($\Rightarrow$ macro-finance puzzles)
- Most macro aggregates displayed decreased volatility from the 1980's onwards (the "Great Moderation")
  - ... until the financial crisis in 2008
- The capital stock is rather smooth compared to other series (but: hard to measure, changes in measurement!)
- The trade balance is highly volatile and procyclical
- All major macro aggregates display significant serial correlation. $\rightarrow$ Is this because shocks are autocorrelated, or because of a propagation mechanism that endogenously smoothes the response of macro variables to a shock?

Basic model

Start with the most basic model, has to contain

- uncertainty - productivity shocks
- consumption/saving choice

$\Rightarrow$ Ramsey model, but stochastic, due to technological shocks

- add a labor/leisure choice as well

Many limitations: infinite horizon, no heterogeneity, no money
Good starting point: analyze the effect of shocks, propagation mechanisms, consumption smoothing

Production function

$$Y_t = K_t^{\alpha} (A_t L_t)^{1-\alpha} \quad 0 < \alpha < 1$$

goal: match the data
in the data the capital share is acyclical: $\Rightarrow$ Cobb-Douglas

Technological progress

$$A_t = A_t^* \widetilde{A}_t$$

- deterministic component

$$A_t^* = G^t \overline{A}$$

  long-run non-stochastic log-linear trend, $G > 1$
- shock process is an autoregressive process of order 1

$$\ln(\widetilde{A}_t) = \rho \ln(\widetilde{A}_{t-1}) + \varepsilon_{A,t}$$

  $E(\varepsilon_{A,t}) = 0$ and is iid (independent and identically distributed across time)

Capital accumulation

$$K_{t+1} = (1-\delta)K_t + Y_t - C_t$$

Household objective function

$$U = E_t \sum_{i=0}^{\infty} \beta^i U(C_{t+i}, 1 - L_{t+i})$$

C_t is consumption and L_t is the fraction of time spent working $(0 < L_t < 1)$
$U_1, U_2 > 0$ and $U_{11}, U_{22} < 0$

Solving this problem

Just like last week: household maximizes utility subject to budget constraint:

$$\max E_t \sum_{i=0}^{\infty} \beta^i U(C_{t+i}, 1 - L_{t+i})$$

$$\begin{aligned} \text{s.t. } S_{t+i} + C_{t+i} &= \widetilde{R}_{t+i} K_{t+i} + W_{t+i} L_{t+i} \\ K_{t+i+1} &= (1-\delta) K_{t+i} + S_{t+i} \\ K_{t+i} &\geq 0;\ K_0 > 0 \end{aligned}$$

Lagrangian:

$$\begin{aligned} \mathcal{L} =& E_t \sum_{i=0}^{\infty} \beta^i U(C_{t+i}, 1 - L_{t+i}) - \\ & E_t \sum_{i=0}^{\infty} \beta^i \lambda_{t+i} (K_{t+i+1} - (1-\delta) K_{t+i} - \widetilde{R}_{t+i} K_{t+i} - W_{t+i} L_{t+i} + C_{t+i}) \end{aligned}$$

Our exercise here:

Imagine the following:

- the economy is initially on its non-stochastic BGP
- there is a sudden realization of $\varepsilon_{A,t} \neq 0$
- this moves the economy away from its non-stochastic BGP
- over time the economy moves back to its BGP
- interpret the economy's deviation from the BGP as a business cycle
- does it look like what we see in the data?

Household behavior I.

inter-temporal FOC

$$U_1(C_t, 1 - L_t) = \beta E_t \left(R_{t+1} U_1(C_{t+1}, 1 - L_{t+1}) \right) \tag{1}$$

where $R_{t+1} \equiv 1 + \tilde{R}_{t+1} - \delta$.

Interpretation (should be familiar)

- decrease consumption by ε, so decrease utility by $U_1(C_t, 1 - L_t)\varepsilon$
- save and get R_{t+1} next period, so an increase in expected utility of $E_t \left(R_{t+1} U_1(C_{t+1}, 1 - L_{t+1}) \right) \varepsilon$

Household behavior II.

intra-temporal FOC

$$U_1(C_t, 1 - L_t)W_t = U_2(C_t, 1 - L_t) \tag{2}$$

Interpretation

- increase work by ε, so decrease in utility by $U_2(C_t, 1 - L_t)\varepsilon$
- extra wage: $W_t\varepsilon \rightarrow$ increase consumption $\rightarrow$ increase in utility by $U_1(C_t, 1 - L_t)W_t\varepsilon$

$$U_1(C_t, 1 - L_t)W_t = U_2(C_t, 1 - L_t)$$

if W_t constant, then if $(1 - L_t) \downarrow \Rightarrow U_2(C_t, 1 - L_t) \uparrow$
$\Rightarrow$ for intra-temporal FOC to hold need:
$U_1(C_t, 1 - L_t)W_t \uparrow \Rightarrow C_t \downarrow$
$\Leftrightarrow$ work more $\Rightarrow$ consume less from the intra-temporal FOC

Barro and King (1984) insight: to have both consumption and labor pro-cyclical (as in the data) need:

* highly pro-cyclical wage and/or
* high substitutability between leisure and consumption

quantitative question: is the observed pro-cyclicality of the wage enough given the large pro-cyclicality of labor?

Balanced growth restrictions

To progress further we need to specify the utility function of the household.

What can the utility function look like?

Imposing balanced growth path restrictions might help.
What do they mean? And are they reasonable?

in the steady state:

- L, labor is constant $\Rightarrow$ production side: we need labor augmenting technological progress
- C and W are growing at rate G, which is the rate of technological progress

can identify a set of utility functions that allow a BGP when techn is labor augmenting (King, Plosser, Rebelo (1988, JME))

two cases that are often used (where utility is separable in leisure and consumption):

1. $U(C, 1-L) = \ln C + b \ln(1-L)$
   $\rightarrow$ use this today
2. $U(C, 1-L) = \ln C + \theta \frac{(1-L)^{1-\gamma}}{1-\gamma}$
   $\rightarrow$ most used New-Keynesian specification, look at it later on

Using the specification $U(C, 1-L) = \ln(C) + v(1-L)$ we get the following:

- the intra-temporal foc becomes:

$$\frac{W_t}{C_t} = v'(1 - L_t)$$

  equalize the marginal utility of leisure to the wage times the marginal value of capital (i.e. λ_t, and therefore the marginal utility of consumption)

- while the inter-temporal foc becomes:

$$1 = E_t\left(\beta R_{t+1}\frac{C_t}{C_{t+1}}\right)$$

  this is the usual condition for consumption

What are the effects of a positive technological shock? It increases current and future R and W.

- consumption
  income effect: people feel richer $\Rightarrow$ consumption up
  substitution effect: saving is worth more $\Rightarrow$ consumption down
  net effect is probably consumption up

- leisure
  income effect: people feel richer $\rightarrow$ they want to enjoy more leisure $\rightarrow$ leisure up
  substitution effect: higher wage $\Rightarrow$ leisure down
  net effect depends on the relative strength of the two forces

* *transitory shock* $\rightarrow$ smaller wealth effect and stronger substitution effect
* *permanent shock* $\rightarrow$ it is possible that consumption goes up and employment goes down

Employment effects another way

combine inter- and intra-temporal conditions and assume that $v(1-L) = b\ln(1-L)$

- the intra-temporal condition is:

$$\frac{W_t}{C_t} = \frac{b}{1-L_t}$$

- using this in the inter-temporal condition we get:

$$1 = E_t\left(\beta R_{t+1}\frac{W_t}{W_{t+1}}\frac{1-L_t}{1-L_{t+1}}\right)$$

* *transitory shock* $\rightarrow W_t \uparrow$ but not $W_{t+1} \Rightarrow (1-L_t)/(1-L_{t+1}) \downarrow$
$\rightarrow$ employment increases today
* *permanent shock* $\rightarrow W_t/W_{t+1}$ pretty much constant $\Rightarrow$ $(1-L_t)/(1-L_{t+1})$ constant as well $\rightarrow$ employment does not change

A very special case

$$U(C_t) = \log C_t$$

$$Z_t F(K_t, 1) = Z_t K_t^{\alpha}$$

$$\delta = 1$$

- full depreciation $\Rightarrow$ almost like a two-period model
- separable log-utility:

$$U(C_t, 1 - L_t) = \ln(C_t) + b \ln(1 - L_t)$$

The two FOCs become:

$$\frac{1}{C_t} = \beta E_t \left(\frac{R_{t+1}}{C_{t+1}} \right) \tag{3}$$

and

$$\frac{W_t}{C_t} = \frac{b}{1 - L_t} \tag{4}$$

Suppose that the solution is $C_t = (1 - s_t)Y_t$; we have $R_t = \alpha \left(\frac{A_t L_t}{K_t}\right)^{1-\alpha} + (1 - 1)$, and $K_{t+1} = s_t Y_t$, we can manipulate (3) to get:

$$\frac{s_t}{1 - s_t} = \alpha \beta E_t \left(\frac{1}{1 - s_{t+1}}\right)$$

which implies that the optimal saving rate is constant: $s_t = \alpha\beta$, therefore

$$\frac{C_t}{Y_t} = 1 - \alpha\beta$$

Using that $\frac{C_t}{Y_t} = 1 - \alpha\beta$ and that $W_t = (1-\alpha)\frac{Y_t}{L_t}$ in (4) gives

$$\frac{(1-\alpha)\frac{Y_t}{L_t}}{(1-\alpha\beta)Y_t} = \frac{(1-\alpha)}{(1-\alpha\beta)L_t} = \frac{b}{1-L_t}$$

rearranging for L_t we get:

$$L_t = \frac{1-\alpha}{b(1-\alpha\beta)+1-\alpha}$$

1. $\Rightarrow$ **constant** working hours
2. very pro-cyclical wage

not very good news - far from where we want to be

Intuition:

- re-write (4) as: $L_t = 1 - \frac{bC_t}{W_t}$
- can think of C_t as capturing the income effects
  (remember: $1/C_t$ is the marginal value of wealth)
  C_t moves around a lot in cycles $\Rightarrow$ the income effect is very large $\Rightarrow$ people feel much richer $\Rightarrow$ they want to consume more and enjoy more leisure
- and of W_t as capturing the substitution effects
  higher W_t $\Rightarrow$ people want to work harder to take advantage of the higher wages
- here they exactly cancel each other out

in models where C_t is less variable, maybe substitution effect can dominate $\Rightarrow$ labor up in booms

Summary so far:

- consumption is too pro-cyclical
  too pro-cyclical, moves one-for-one with Y_t
- investment $I_t = s_t Y_t = \alpha\beta Y_t$ also pro-cyclical but not enough
  investment in the data is super pro-cyclical, with constant saving rate it is only pro-cyclical
  we **need a pro-cyclical saving rate** to match the data:

$$Var(\ln I_t) = Var(\ln s_t) + Var(\ln Y_t) + 2Cov(\ln s_t, \ln Y_t)$$

  for this a pro-cyclical interest rate is needed

  effects of R on saving?
  $r \uparrow \Rightarrow$ inter-temporal substitution effect, $C_t \downarrow$
  $\Rightarrow$ income effect, $C_t \uparrow$
  (in this model they cancel out)
- W_t is too pro-cyclical

Output dynamics

- constant saving rate and labor supply $\Rightarrow$ like the Solow model in discrete time: $y_t^* = \frac{Y_t^*}{A_t^* L^*} = \left(\frac{\alpha\beta}{G}\right)^{\frac{\alpha}{1-\alpha}}$, rearranged:

$$Y_t^* = \left(\frac{\alpha\beta}{G}\right)^{\frac{\alpha}{1-\alpha}} A_t^* L^*$$

- What we are interested in is $\ln \widetilde{Y}_t = \ln Y_t - \ln Y_t^*$. Assume that initially we are in the steady state, $\ln \widetilde{Y}_0 = 0$. Then we can express

$$\ln \widetilde{Y}_t = (1-\alpha)\left(\alpha^t \ln \widetilde{A}_0 + \alpha^{t-1} \ln \widetilde{A}_1 + \ldots + \alpha \ln \widetilde{A}_{t-1} + \ln \widetilde{A}_t\right)$$

One way to show this (normalize $L^* = 1$ for simplicity):

$$\begin{aligned}
Y_1 &= K_1^{\alpha} A_1^{1-\alpha} = (sY_0)^{\alpha} A_1^{1-\alpha} = (sY_0^*)^{\alpha} A_1^{1-\alpha} \\
&= \left(s \left(\frac{s}{G} \right)^{\frac{\alpha}{1-\alpha}} A_0^* \right)^{\alpha} A_1^{1-\alpha} \\
&= \left(\frac{s}{G} \right)^{\alpha} G^{\alpha} \left(\frac{s}{G} \right)^{\frac{\alpha^2}{1-\alpha}} (A_0^*)^{\alpha} A_1^{1-\alpha} \\
&= \left(\frac{s}{G} \right)^{\frac{\alpha(1-\alpha)}{1-\alpha} + \frac{\alpha^2}{1-\alpha}} G^{\alpha} (A_0^*)^{\alpha} A_1^{1-\alpha} \\
&= \left(\frac{s}{G} \right)^{\frac{\alpha}{1-\alpha}} (A_1^*)^{\alpha} A_1^{1-\alpha} = \left(\frac{s}{G} \right)^{\frac{\alpha}{1-\alpha}} (A_1^*)^{\alpha} \left(A_1^* \widetilde{A}_1 \right)^{1-\alpha} \\
&= \left(\frac{s}{G} \right)^{\frac{\alpha}{1-\alpha}} A_1^* \widetilde{A}_1^{1-\alpha} = Y_1^* \widetilde{A}_1^{1-\alpha}
\end{aligned}$$

Now move to Y_2 and keep going.

The economy's reaction to a shock

experiment: there is one shock at time 0 and then never again

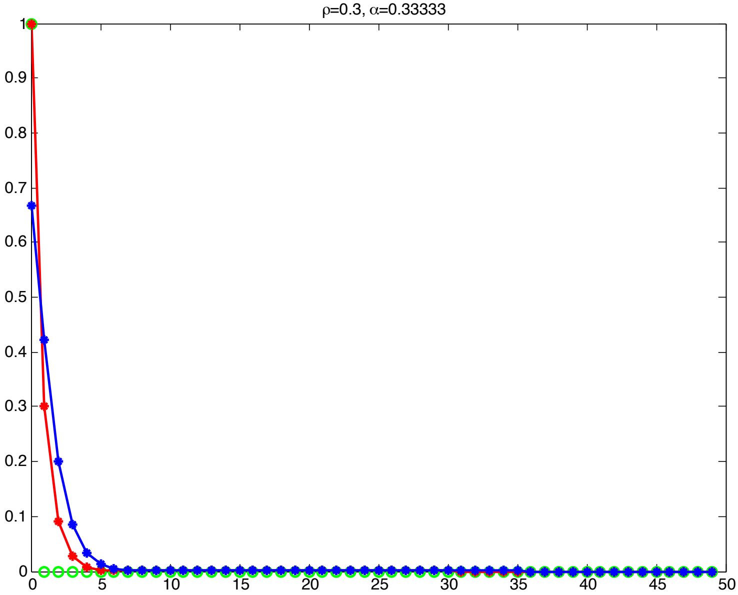
$$\ln \widetilde{A}_t = \rho^t \varepsilon_{A,0}$$

what is the economy's log deviation from trend? what is the economy's **impulse response function**?
use formula from before:

$$\begin{aligned} \ln \widetilde{Y}_t &= (1-\alpha)\left(\alpha^t \ln \widetilde{A}_0 + \alpha^{t-1} \ln \widetilde{A}_1 + ... + \alpha \ln \widetilde{A}_{t-1} + \ln \widetilde{A}_t\right) \\ &= (1-\alpha)\left(\alpha^t + \alpha^{t-1}\rho + ... + \alpha\rho^{t-1} + \rho^t\right)\varepsilon_{A,0} \end{aligned}$$

in the long run output goes back to the BGP level:

$$\lim_{t\to\infty} \ln \widetilde{Y}_t = 0$$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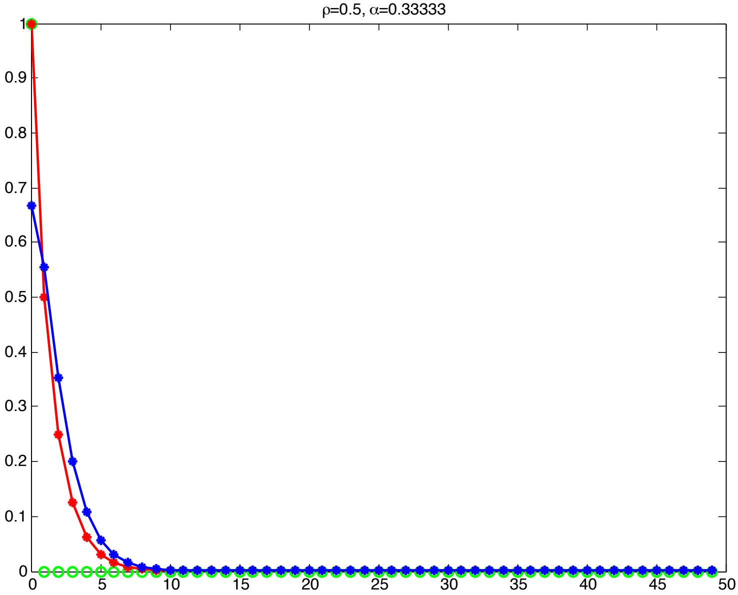

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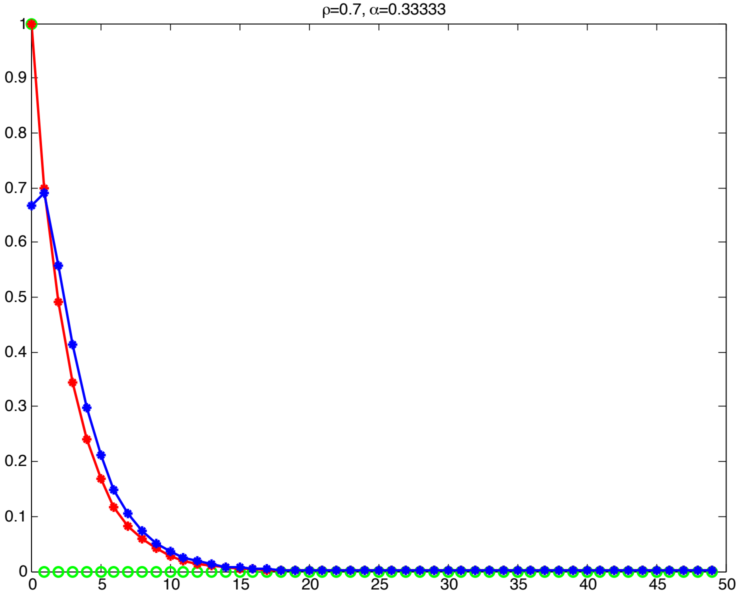
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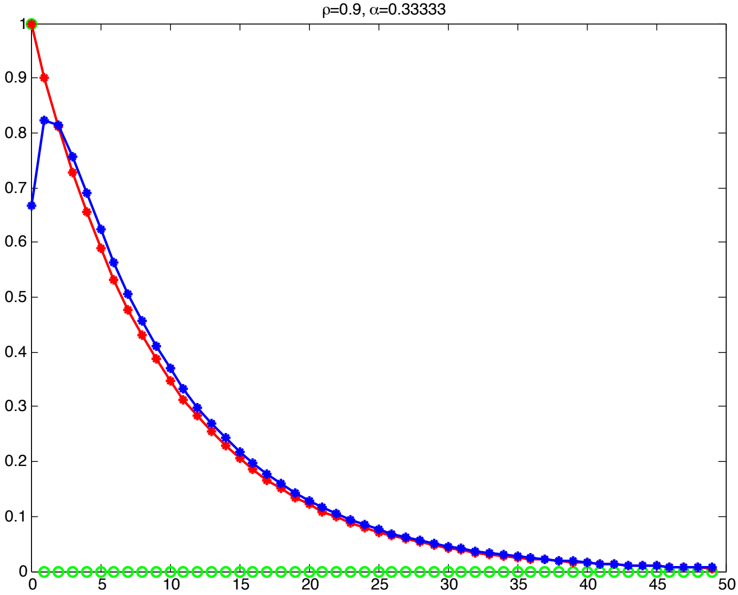
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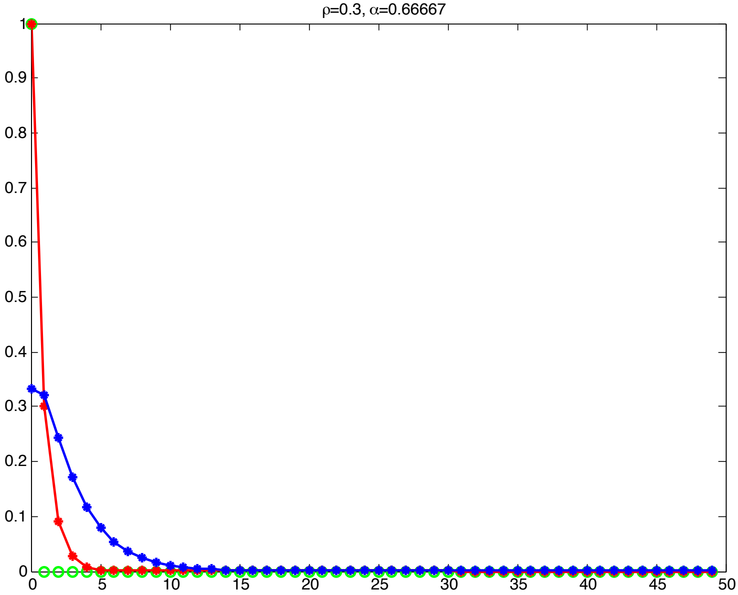

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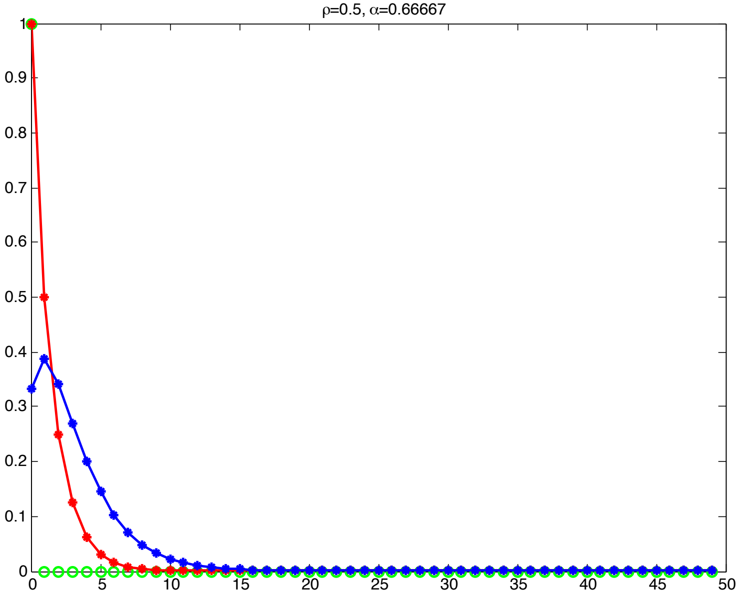
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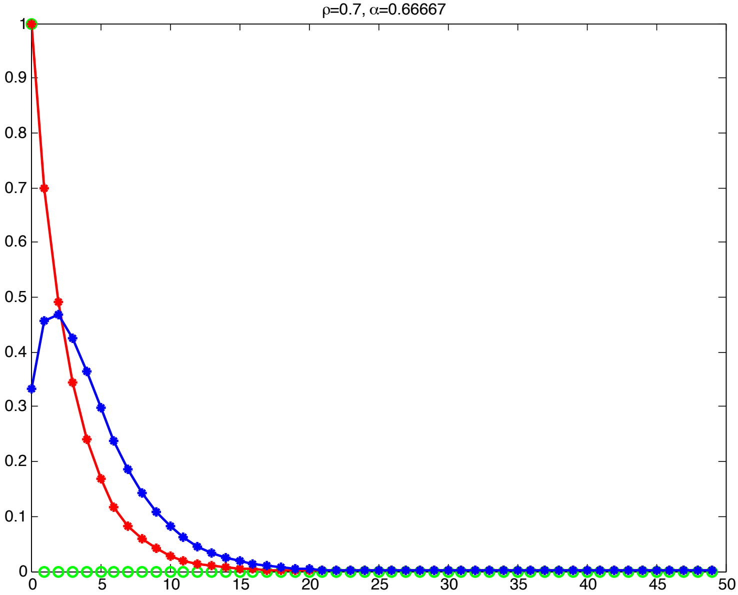
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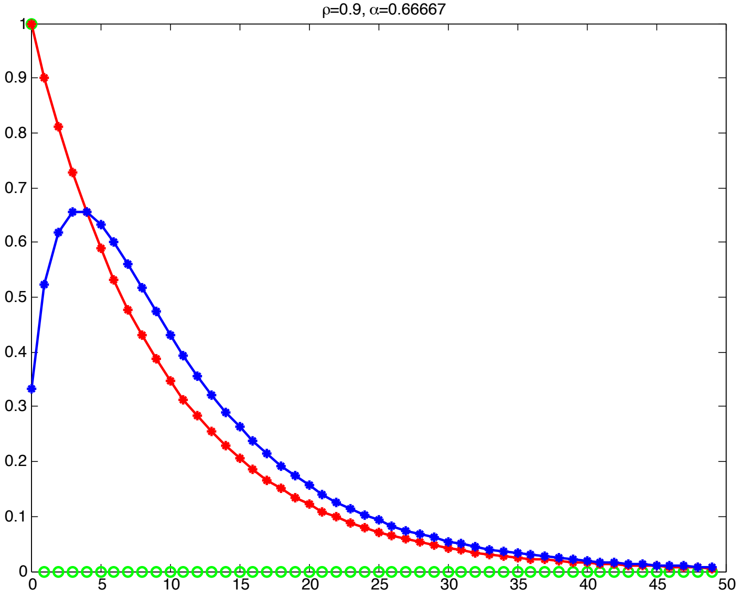

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The graphs show

- hump shape in the impulse response of Y_t
  $\rightarrow$ this is due to the increase in total investment, since the saving rate is constant, and output increases
  so there is **amplification**
  this comes in part from α, but not only
  output today is only higher because A is higher, output tomorrow is higher because K is higher (due to α) and A is higher (due to ρ)
- output stays above trend longer than the shock (which is just one period) $\rightarrow$ so there is **persistence**
  two sources:
  1. exogenous persistence, $\rho > 0$, this is the intrinsic persistence of the technology shock
     this determines mostly the persistence and it is chosen by us
  2. endogenous persistence, $\alpha > 0$, which works through investment

Conclusions so far

special case does not look good

- not enough persistence: very quick return to the BGP unless very high ρ
- not enough amplification
- need a more general model
- and possibly need to look for other sources of shocks

need to:

- get consumption to be less pro-cyclical
  and investment to be more pro-cyclical
- get labor effort to respond
- get more endogenous persistence

A more general model

Two differences:
not full depreciation: $0 < \delta < 1$

household objective function:

$$U = E_t \sum_{i=0}^{\infty} \beta^i U(C_{t+i}, 1 - L_{t+i})$$

where the utility function is

$$U(C_t, 1 - L_t) = \ln(C_t) + \theta \frac{(1 - L_t)^{1-\gamma_l}}{1 - \gamma_l}$$

log in consumption, CES in leisure
$\gamma_l = 1 \Rightarrow$ log in leisure as well

Firm behavior

$$R_t = \alpha \left(\frac{A_t L_t}{K_t} \right)^{1-\alpha} + (1 - \delta) \tag{5}$$

the gross interest rate has to equal the gross marginal product of capital

$$W_t = (1 - \alpha) A_t^{1-\alpha} \left(\frac{K_t}{L_t} \right)^{\alpha} \tag{6}$$

the wage rate has to equal the marginal product of labor

Household behavior

inter-temporal FOC

$$\frac{1}{C_t} = \beta E_t \frac{R_{t+1}}{C_{t+1}} \tag{7}$$

intra-temporal FOC

$$\frac{W_t}{C_t} = \theta(1 - L_t)^{-\gamma_l} \tag{8}$$

Strategy

- goal: solve for the dynamic path of Y_t, C_t, L_t, K_t etc for any realization of shock, $\varepsilon_{A,t}$
  find the impulse response functions
- we do not know how to do this in closed form (as opposed to before, when we did)
- instead:
  1. solve for the non-stochastic BGP
  2. rewrite the model in terms of log-deviations from the non-stochastic BGP
  3. study this alternative model, which is log-linear, and an approximation of an original model around the non-stochastic BGP

1. The non-stochastic BGP

- Y^*, K^*, C^*, W^* all grow at rate G (the non-stochastic growth rate of technology)
- R^*, L^* are constant (why?)

Gross rental rate from the inter-temporal foc:

$$\frac{1}{C_t} = \beta E_t \left(\frac{R_{t+1}}{C_{t+1}} \right)$$

non-stochastic & the growth rate of C_t

$$R^* = \frac{G}{\beta}$$

Capital per efficiency units of labor

$$R_t = \alpha \left(\frac{A_t L_t}{K_t} \right)^{1-\alpha} + (1 - \delta)$$

combine with the previous to get

$$\frac{K_t^*}{A_t^* L^*} = \left(\frac{\alpha}{\frac{G}{\beta} - (1 - \delta)} \right)^{\frac{1}{1-\alpha}}$$

Output per efficiency units of labor

$$Y_t = (A_t L_t)^{1-\alpha} K_t^{\alpha}$$

rearrange and combine with the previous to get:

$$\frac{Y_t^*}{A_t^* L^*} = \left(\frac{\alpha}{\frac{G}{\beta} - (1 - \delta)} \right)^{\frac{\alpha}{1-\alpha}}$$

Consumption relative to capital combine the capital accumulation equation: $K_{t+1} = (1-\delta)K_t + Y_t - C_t$ with $K^*_{t+1} = GK^*_t$ to get:

$$\frac{C^*_t}{K^*_t} = 1 - \delta - G + \frac{Y^*_t}{K^*_t}$$

where $\frac{Y^*_t}{K^*_t}$ we know from the last two equations.

Wage use the wage rate, $W_t = (1-\alpha)Y_t/L_t$

$$\frac{W^*_t}{C^*_t} = \frac{1-\alpha}{L^*}\frac{Y^*_t}{C^*_t}$$

Labour supply use the intra-temporal foc $W_t/C_t = \theta(1-L_t)^{-\gamma_l}$ and combine with the previous:

$$\frac{1-\alpha}{L^*}\frac{Y^*_t}{C^*_t} = \theta(1-L^*)^{-\gamma_l}$$

Note: $\frac{Y^*_t}{C^*_t}$ we know because we know $\frac{Y^*_t}{K^*_t}$ and $\frac{C^*_t}{K^*_t}$, hence this last eqn gives us L^* and then the previous pins down W^*, ...

2. The log-linearized model

small letters denote the log-deviations from the non-stochastic BGP: $c_t = \ln C_t - \ln C_t^*$.

Have to log-linearize all the equations that describe the model.

1. The production function

$$Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$$

re-write as

$$Y_t^* e^{y_t} = K_t^{*\alpha} e^{\alpha k_t}(A_t^* L_t^*)^{1-\alpha} e^{(a_t + l_t)(1-\alpha)}$$

and use production function in steady state to get

$$y_t = \alpha k_t + (1-\alpha)(a_t + l_t)$$

2. Capital accumulation

$$K_{t+1} = (1-\delta)K_t + Y_t - C_t$$

re-write as

$$K_{t+1}^* e^{k_{t+1}} = (1-\delta)K_t^* e^{k_t} + Y_t^* e^{y_t} - C_t^* e^{c_t}$$

Now use the fact that for x small enough $e^x \approx 1 + x$ to get:

$$K_{t+1}^* + K_{t+1}^* k_{t+1} \approx (1-\delta)K_t^* + (1-\delta)K_t^* k_t + Y_t^* + Y_t^* y_t - C_t^* - C_t^* c_t$$

which simplifies to

$$\frac{K_{t+1}^*}{K_t^*} k_{t+1} \approx (1-\delta)k_t + \frac{Y_t^*}{K_t^*} y_t - \frac{C_t^*}{K_t^*} c_t$$

3. The rental rate

$$R_t = \alpha \left(\frac{A_t L_t}{K_t} \right)^{1-\alpha} + (1-\delta)$$

re-write as

$$R_t^* e^{r_t} = \alpha \left(\frac{A_t^* L^*}{K_t^*} \right)^{1-\alpha} e^{(1-\alpha)(a_t + l_t - k_t)} + (1-\delta)$$

this is approximately

$$R_t^* r_t \approx \alpha(1-\alpha) \left(\frac{A_t^* L^*}{K_t^*} \right)^{1-\alpha} (a_t + l_t - k_t)$$

4. The wage rate

$$W_t = (1-\alpha)A_t^{1-\alpha}\left(\frac{K_t}{L_t}\right)^{\alpha}$$

re-write as

$$W_t^* e^{w_t} = (1-\alpha){A_t^*}^{1-\alpha} e^{(1-\alpha)a_t}\left(\frac{K_t^*}{L_t^*}\right)^{\alpha} e^{\alpha(k_t - l_t)}$$

which simplifies to

$$w_t = (1-\alpha)a_t + \alpha(k_t - l_t)$$

5. the intra-temporal FOC

$$\frac{W_t}{C_t} = \theta(1 - L_t)^{-\gamma_l}$$

re-write and approximate:

$$\frac{W_t^* e^{w_t}}{C_t^* e^{c_t}} = \theta(1 - L^* e^{l_t})^{-\gamma_l} \Rightarrow \frac{W_t^* + W_t^* w_t}{C_t^* + C_t^* c_t} \approx \theta(1 - L^* - L^* l_t)^{-\gamma_l}$$

we do a first order taylor approximation around $c_t = w_t = l_t = 0$:

$$\begin{aligned} \frac{W_t^*}{C_t^*} + \frac{W_t^*}{C_t^*} w_t - \frac{W_t^*}{C_t^*} c_t \approx& \theta(1 - L^*)^{-\gamma_l} + \theta\gamma_l(1 - L^*)^{-\gamma_l - 1} L^* l_t \\ \frac{W_t^*}{C_t^*}(w_t - c_t) \approx& \theta\gamma_l(1 - L^*)^{-\gamma_l - 1} L^* l_t \end{aligned}$$

which simplifies to

$$w_t - c_t \approx \gamma_l \frac{L^*}{1 - L^*} l_t$$

6. The inter-temporal FOC

$$\frac{1}{C_t} = \beta E_t \left(\frac{R_{t+1}}{C_{t+1}} \right)$$

re-write as

$$\frac{1}{C_t^* e^{c_t}} = \beta E_t \left(\frac{R^* e^{r_{t+1}}}{C_{t+1}^* e^{c_{t+1}}} \right)$$

which is

$$\frac{1}{e^{c_t}} = E_t \left(\frac{e^{r_{t+1}}}{e^{c_{t+1}}} \right) \Rightarrow \frac{1}{1+c_t} \approx E_t \left(\frac{1+r_{t+1}}{1+c_{t+1}} \right)$$

finally, do a first order Taylor approximation

$$-c_t \approx E_t(r_{t+1} - c_{t+1})$$

A 'recipe' for log-linearizations

- $g(X_t) = 0$ is what you want to log-linearize
- then $g(X_t^*) = 0$ holds
- and you can rewrite: $g(X_t) = g(X_t^* e^{x_t}) = 0$ where $x_t = \ln X_t - \ln X_t^*$
- look at what you have, can you use $g(X_t^*) = 0$ to get something linear in x_t?
  - if yes, stop, you have an exact expression
  - if not, continue
- approximate everywhere e^{x_t} with $1 + x$
- look at what you have, try again to use $g(X_t^*) = 0$ to get something linear in x_t.
  - if yes, stop
  - if not, continue
- take a first order Taylor approximation around $x_t = 0$, and use $g(X_t^*) = 0$, this always works

Summary of the log-linear model

1. the production function
   $y_t = \alpha k_t + (1-\alpha)(a_t + l_t)$
2. the resource constraint
   $\frac{K^*_{t+1}}{K^*_t} k_{t+1} \approx (1-\delta)k_t + \frac{Y^*_t}{K^*_t} y_t - \frac{C^*_t}{K^*_t} c_t$
3. interest rate
   $R^*_t r_t \approx \alpha(1-\alpha)\left(\frac{A^*_t L^*}{K^*_t}\right)^{1-\alpha}(a_t + l_t - k_t)$
4. wage
   $w_t = (1-\alpha)a_t + \alpha(k_t - l_t)$
5. labor/leisure choice
   $w_t - c_t \approx \gamma_l \frac{L^*}{1-L^*} l_t$
6. Euler equation
   $-c_t \approx E_t(r_{t+1} - c_{t+1})$
7. the shock process
   $a_t = \rho a_{t-1} + \varepsilon_{A,t}$
   where $a_t = \ln(\widetilde{A}_t)$

- we have a system of 7 first order linear stochastic difference equations, in the 7 variables, $y_t, k_t, l_t, c_t, r_t, w_t$ and a_t

  note that we can express all the steady state variables, or the ratios that we need, in terms of parameters of the model: $\alpha, \beta, \gamma_l, \delta, \rho, G, \theta$

- we want to solve for the path of the endogenous variables as functions of the model's parameters and the path of the exogenous shock process, ε_t

  i.e. we want to get the impulse response functions

Solving the log-linear model

there are two ways of solving the model:

1. the easy way
   give the system of linear first-order stochastic equations to Matlab (or Octave) and have Matlab solve it for you

2. the hard way
   solve the equations by hand - method of undetermined coefficients

Calibration

- the time series properties of the model depends on the model parameters
- calibration is a method of picking numbers for these parameters
- many ways of doing it
  - pick parameters one-by-one to match certain moments in the data
  - jointly calibrate parameters in order to minimize the distance from model and data moments
- today: Campbell's way

- G

  in order to match the average annual growth rate in the US, which has been 2%
  if we interpret this as steady state growth $\Rightarrow$ this is also the annual growth rate of A^*
  transformed into quarterly growth rate: $G = 1.005$

- β

  in order to match the annual real interest rate in the US, which has been roughly 6%
  again, take this as an estimate of the non-stochastic BGP
  quarterly this means: $R^* = 1.015$
  in the steady state we have: $R^* = G/\beta \Rightarrow$ gives us β
  notice: we never use β on its own, just as G/β, hence we can forget about β and use $R^* = 1.015$

- α

  in order to match the labor share over long-run US history, which has been 2/3
  hence we pick $\alpha = 1/3$

- δ

  consensus view (based on NIPA statistics) is that the annual depreciation rate is around 10%
  quarterly this means: $\delta = 0.025$

- θ

  hhs spend about 1/3 of their time on market activities
  again interpreting this as the steady state implies $L^* = 1/3$
  remember that in the steady state L^* is pinned down by the intra-temporal foc: $\frac{1-\alpha}{L^*}\frac{Y_t^*}{C_t^*} = \theta(1 - L^*)^{-\gamma_l}$
  conditional on γ and $L^* = 1/3$ and Y_t^*/C_t^* implied by the other parameters gives us θ
  but θ does not appear anywhere else except in pinning down L^* $\Rightarrow$ forget about θ and use $L^* = 1/3$

- we do not make choices for γ_l and ρ instead look at how the model behaves for different values
  - ρ persistence of the technology shock, $\rho \in [0, 1)$, the closer to 1 the more persistent the shock is
  - γ_l controls the elasticity of labor supply to wage changes, $\gamma_l \in [0, \infty]$, the closer to zero the more elastic is the labor supply

Re-writing the log-linearized model for Dynare

The package that does this for you is called "Dynare" (available for Matlab, Octave, and Julia)

You have classify all variables into three groups:

- exogenous variables
  in this example ε_t
- endogenous variables
  - state variables
    endogenous variables known at the beginning of time t
    in this example k_t
    (usual notation in matlab: write k_t as k_{t-1})
  - control variables
    endogenous variables to be determined at time t
    here y_t, c_t, w_t, r_t, l_t

You have to give matlab the variables, the parameters and the equations. Key limitation: finitely many equations, finite-dimensional state variables!

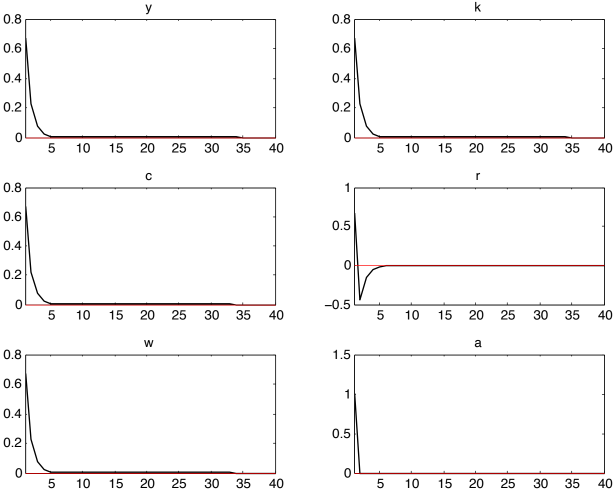
Software

You'll need

- A programming language/environment. For our purposes, I recommend MATLAB (Student version 35 eur), GNU Octave (free), or Julia (experimental, free)
  - MATLAB: www.mathworks.com
  - GNU Octave: www.gnu.org/software/octave
  - Julia: www.julialang.org
- Code for solving RBC/DSGE models:
  - Dynare: www.cepremap.fr/en/modelling/dynare/

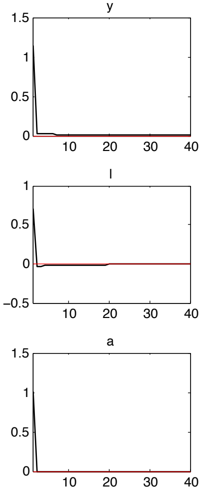
The combination of Matlab + Dynare is the default, and is the most stable

An old acquaintance



$\delta = 1, \gamma_l = 1, \rho = 0$

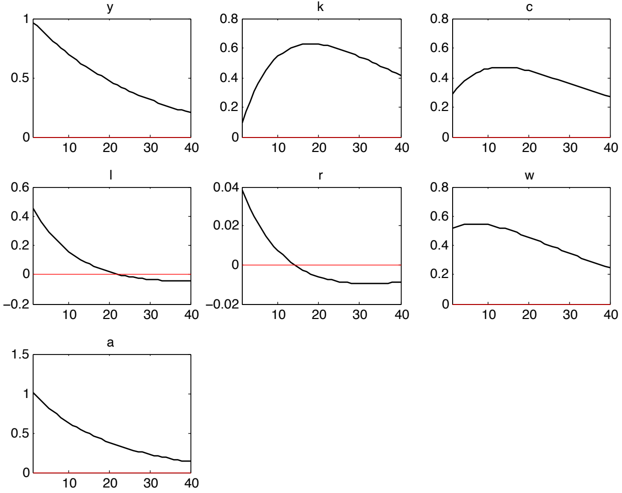
An old acquaintance + incomplete depreciation



$\delta = 0.025, \gamma_l = 1, \rho = 0$

l responds, but not enough (lack of amplification); insufficient persistence of Y; consumption response too small; wage too procyclical

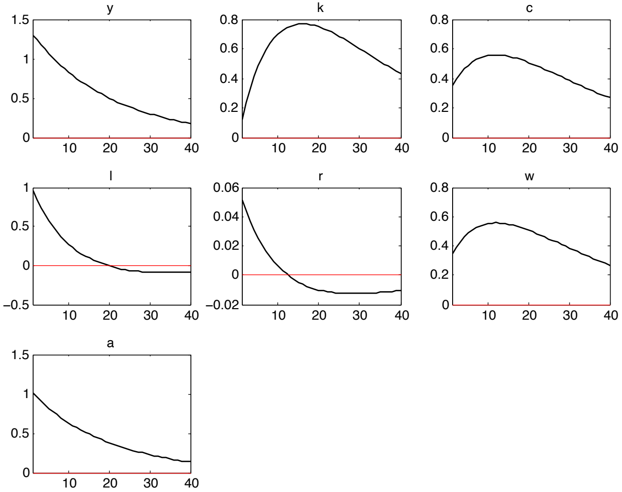
An old acquaintance + incomplete depr + persistent shock



$\delta = 0.025, \gamma_l = 1, \rho = 0.95$

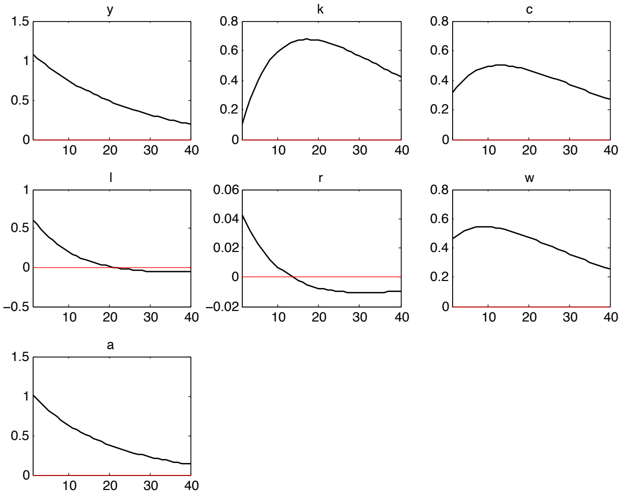
Still not much amplification (l response). Output inherits shock's persistence. Consumption response not strong enough. Wage even more procyclical.

Very elastic labor supply



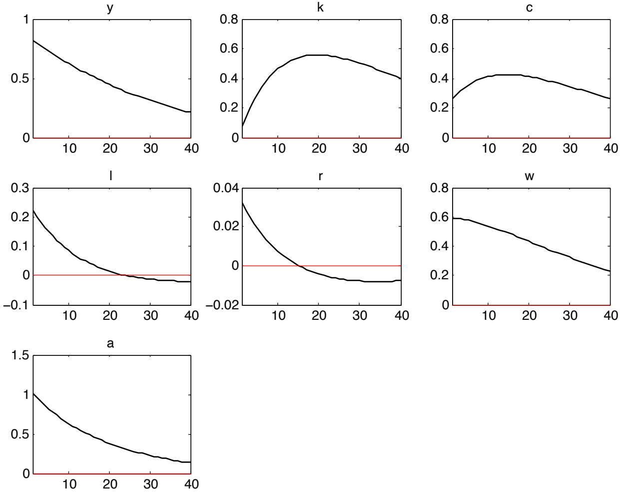
$\delta = 0.025, \gamma_l = 0, \rho = 0.95$ Desired labor supply response. Otherwise no change.

Elastic labor supply



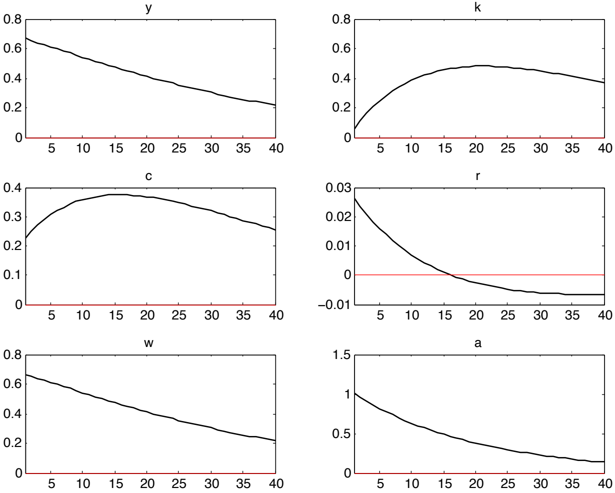
$\delta = 0.025, \gamma_l = 0.5, \rho = 0.95$

Less elastic labor supply



$\delta = 0.025, \gamma_l = 1.5, \rho = 0.95$

Very inelastic labor supply



$\delta = 0.025, \gamma_l = 1000000000, \rho = 0.95$

Summary - What basic RBC models cannot do

endogenous persistence

- we were expecting persistence to come from capital accumulation
- capital accumulation goes up for a while, but quantitatively not enough
- we also wanted a hump-shape response in Y (that's what VARs tell us), again this would have come from capital accumulation
- $\rightarrow$ output dynamics are essentially the same as the assumed dynamics of the shocks

"Capital accumulation has an important effect on the dynamics of the economy only when the underlying technology shock is persistent. The stochastic growth model is unable to generate persistent effects from transitory shocks." (Campbell, 1994)

amplification problem

- we were expecting this to come from the labor supply
- only in extreme cases is the labor supply and/or capital accumulation response strong enough to generate multiplier like responses
- i.e. in which y goes up or falls more than one-for-one with the change in a
- this means that slower than normal, but still positive technological growth can cause output to grow slower, but not to fall
- for output to fall, technology must decline
- for strongly pro-cyclical labor supply we need strongly pro-cyclical wages
  possible solutions: composition bias, implicit contracts, indivisible labor

RBCs and the Solow residual

- take $Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$, plug in data for $Y_t, K_t, L_t \rightarrow A_t$
- detrend $A_t \rightarrow a_t$, the log deviation from trend, this is the **Solow residual**
- which is very pro-cyclical and very persistent, it also implies that technological change is sometimes negative
- use the log-linearized equations of the model to generate a predicted path for $y_t, k_t, l_t, c_t, r_t, w_t$
- if your model is right, then these should be really close to the actually observed data
- AND THEY ARE
- some people think this is victory

- but is the Solow residual an accurate measure of technology?
- can all the TFP changes be interpreted as technological shocks on this very short frequency data?
- what about
  - labor hoarding?
  - capacity utilization?
- suppose firms don't fire people and don't throw away capital if in that quarter for some reason output was lower $\rightarrow$ temporary observed changes in output without observable capital/labor change
  $\Rightarrow$ high correlation between output and TFP, but here what we observe is $y_t \rightarrow a_t$ and we are trying to say that an exogenous $a_t \rightarrow y_t$

Basu and Fernald (1997) and Basu, Fernald and Kimball (2001) fix this problem of measurement:

- control for capacity utilization
  instead of $K_t \rightarrow u_t K_t$ in the production function
  u_t is utilization, approximated by electricity utilization
- control for labor hoarding
  instead of $L_t \rightarrow h_t L_t$
  h_t approximated by the number of accidents
- after controlling for all these, the new TFP process is
  1. much less volatile
  2. negatively correlated with factor inputs

Issue 1: new TFP process much less volatile
why is this a problem?
if you plug these back into our equations, the model predictions look pretty pathetic

King and Rebelo (1999) present a response to this problem

- main point: same corrections that make the empirical Solow residual smaller, also make the required magnitude of the theoretical shocks smaller
- they change the model by introducing variable capacity/labor utilization $\rightarrow$ with zero marginal cost you can use more/less labor and capital
- $\Rightarrow$ small productivity shocks have large amplification: big output effect through capacity utilization and labor hoarding
- though still have to be positively correlated with inputs

Issue 2: negative correlation with inputs
why is this a problem?
$Corr(A_t, L_t) < 0 \Leftrightarrow$ positive productivity shock $\Rightarrow$ labor input goes down
this is not consistent with these RBC models

1. from previous macro models this was an obvious conclusion: technological shocks & demand does not change $\Rightarrow$ supply same amount with less workers, due to productivity rise
   everything was demand driven – firms supply what people demand, and productivity shocks do not affect the demand
2. if this is true in the data, plus observed employment and output is positively correlated, then **there must be other shocks** as well

A very cheap way of fixing the labor supply problem

- Key problem: estimates for worker's measured labor supply elasticity is very low
- Possible reasons: can only choose full-time/part-time/non-employment

"Indivisible labor" setup:
Change RBC model so that

- there is a unit mass of workers
- time endowment is 1 unit
- worker i either works 1 unit or 0 units, that is, labor is indivisible

Hence, h_t workers are employed, $1 - h_t$ workers are unemployed.

Indivisible labor

Framework introduces heterogeneity across agents

- In principle, consumption and saving would depend on employment spells through income and wealth
- Let's get rid of this: assume asset markets are such that c_t is the same for all households
- Being employed is a pure lottery:
  - probability of being employed is h_t
  - employement status is not persistent
  - households choose *probability* of being employed (agents are ex ante identical $\Rightarrow$ all choose the same h_t)

Household's problem

Assume period utility function

$$U(C_t, L_t) - \theta \frac{L^{1-\gamma}}{1-\gamma}$$

- Ex ante workers are the same $\Rightarrow$ representative household maximizes expectation of the (PDV of the) above (over C and h in every period)
- Hence, max expectation of period utility function:

$$max_{C_t, h_t} U(C_t) - \left(h_t \theta \frac{1^{1-\gamma}}{1-\gamma} + (1-h_t)\theta \frac{0^{1-\gamma}}{1-\gamma} \right)$$

or

$$max_{C_t, h_t} U(C_t) - h_t \frac{\theta}{1-\gamma}$$

- That is, we have linear utility of work and high elasticity of labor supply
- Do we have an interior solution for h_t? Yes, h and C are linked via the BC and labor demand.

FOC and Results

- Micro-elasticity of labor supply: zero
- Macro-elasticity of labor supply: large
- References: Rogerson 1988 J Mon Econ, Hansen 1985 J Mon Econ

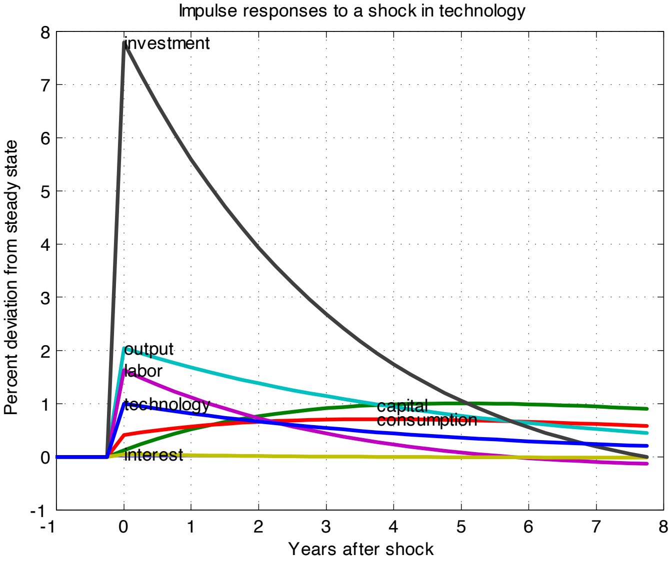


Figure 11.2: Implulse Response for the Hansen Model, Technology Shock

(Source: Krueger, Quantitative Macroeconomics)