Macroeconomics A; EI056

Technical appendix: The Overlapping Generations model

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1 Optimization infinite horizon

1.1 Household

Consider the utility used in the class of last week:

$$U_0 = \sum_{s=0}^{\infty} \frac{1}{(1+\rho)^s} \frac{(C_s)^{1-\theta}}{1-\theta} L_s$$

where C is per capital consumption. The household can invest in capital K and a government bond B. Both pay the same rate of return. The budget constraint is:

$$W_t L_t + r_t K_t + r_t B_t - L_t T_t = L_t C_t + K_{t+1} - K_t + B_{t+1} - B_t$$

where T_t are taxes per capita.

The technology is Cobb-Douglas:

$$W_t L_t + r_t K_t = (K_t)^{\alpha} (A_t L_t)^{1-\alpha}$$

Productivity A grows at a rate g and labor L at a rate n (in the class we consider $g = n = 0$ for simplicity). We express the budget constraint is scaled terms:

$$\begin{aligned}
W_t L_t + r_t K_t + r_t B_t - L_t T_t &= L_t C_t + K_{t+1} - K_t + B_{t+1} - B_t \\
\frac{W_t L_t}{A_t L_t} + r_t \frac{K_t}{A_t L_t} + r_t \frac{B_t}{A_t L_t} - \frac{L_t T_t}{A_t L_t} &= \frac{L_t C_t}{A_t L_t} + \frac{K_{t+1}}{A_{t+1} L_{t+1}} \frac{A_{t+1} L_{t+1}}{A_t L_t} - \frac{K_t}{A_t L_t} \\
&\quad + \frac{B_{t+1}}{A_{t+1} L_{t+1}} \frac{A_{t+1} L_{t+1}}{A_t L_t} - \frac{B_t}{A_t L_t} \\
\frac{W_t}{A_t} + r_t k_t + r_t b_t - \frac{T_t}{A_t} &= \frac{C_t}{A_t} + k_{t+1} (1+n)(1+g) - k_t \\
&\quad + b_{t+1} (1+n)(1+g) - b_t
\end{aligned}$$

$$w_t - \tau_t + r_t (k_t + b_t) = c_t + (k_{t+1} + b_{t+1})(1+n)(1+g) - (k_t + b_t)$$

This constraint can be iterated forward to give the intertemporal constraint:

$$\begin{aligned}
w_t - \tau_t + r_t (k_t + b_t) &= c_t + (k_{t+1} + b_{t+1})(1+n)(1+g) - (k_t + b_t) \\
(1+r_t)(k_t + b_t) &= c_t - (w_t - \tau_t) + (k_{t+1} + b_{t+1})(1+n)(1+g) \\
(k_t + b_t) &= \frac{(1+n)(1+g)}{(1+r_t)} \frac{c_t - (w_t - \tau_t)}{(1+n)(1+g)} + \frac{(1+n)(1+g)}{(1+r_t)} (k_{t+1} + b_{t+1})
\end{aligned}$$

Iterate forward:

$$\begin{aligned}
(k_t + b_t) &= \frac{(1+n)(1+g)}{(1+r_t)} \frac{c_t - (w_t - \tau_t)}{(1+n)(1+g)} + \frac{(1+n)(1+g)}{(1+r_t)} (k_{t+1} + b_{t+1}) \\
(k_t + b_t) &= \frac{(1+n)(1+g)}{(1+r_t)} \frac{c_t - (w_t - \tau_t)}{(1+n)(1+g)} \\
&\quad + \frac{(1+n)(1+g)}{(1+r_t)} \frac{(1+n)(1+g)}{(1+r_{t+1})} \frac{c_{t+1} - (w_{t+1} - \tau_{t+1})}{(1+n)(1+g)} \\
&\quad + \frac{(1+n)(1+g)}{(1+r_t)} \frac{(1+n)(1+g)}{(1+r_{t+1})} (k_{t+2} + b_{t+2})
\end{aligned}$$

Define the compound discount factor:

$$R_{t,t+s} = \frac{(1+n)(1+g)}{1+r_t} \frac{(1+n)(1+g)}{1+r_{t+1}} \cdots \frac{(1+n)(1+g)}{1+r_{t+s}} = \prod_{i=0}^{s} \frac{(1+n)(1+g)}{1+r_{t+i}}$$

We then write:

$$(k_t + b_t) = R_{t,t} \frac{c_t - (w_t - \tau_t)}{(1+n)(1+g)} + R_{t,t+1} \frac{c_{t+1} - (w_{t+1} - \tau_{t+1})}{(1+n)(1+g)} + R_{t,t+1} (k_{t+2} + b_{t+2})$$

Iterating to infinity we get:

$$(k_t + b_t) = \sum_{s=0}^{\infty} R_{t,t+s} \frac{c_{t+s} - (w_{t+s} - \tau_{t+s})}{(1+n)(1+g)} + \lim_{k \to \infty} R_{t,t+k} (k_{t+k+1} + b_{t+k+1})$$

The transversality condition implies that the last term is zero, hence:

$$\sum_{s=0}^{\infty} R_{t,t+s} \frac{c_{t+s}}{(1+n)(1+g)} = (k_t + b_t) + \sum_{s=0}^{\infty} R_{t,t+s} \frac{w_{t+s} - \tau_{t+s}}{(1+n)(1+g)} \tag{1}$$

1.2 Government and Ricardian equivalence

The government spends G units per capita, and funds itself by taxes and debt:

$$B_{t+1} = L_t G_t - L_t T_t + (1+r_t) B_t$$

We scale this as follows:

$$\begin{aligned}
B_{t+1} &= L_tG_t - L_tT_t + (1+r_t)\,B_t \\
\frac{B_{t+1}}{A_{t+1}L_{t+1}}\frac{A_{t+1}L_{t+1}}{A_tL_t} &= \frac{L_tG_t}{A_tL_t} - \frac{L_tT_t}{A_tL_t} + (1+r_t)\,\frac{B_t}{A_tL_t} \\
b_{t+1}\,(1+n)\,(1+g) &= \frac{G_t}{A_t} - \frac{T_t}{A_t} + (1+r_t)\,b_t \\
b_{t+1}\,(1+n)\,(1+g) &= g_t - \tau_t + (1+r_t)\,b_t
\end{aligned}$$

Iterating forward we write:

$$\begin{aligned}
(1+r_t)\,b_t &= (\tau_t - g_t) + (1+n)\,(1+g)\,b_{t+1} \\
b_t &= \frac{(1+n)\,(1+g)}{(1+r_t)}\frac{(\tau_t - g_t)}{(1+n)\,(1+g)} + \frac{(1+n)\,(1+g)}{(1+r_t)}b_{t+1} \\
b_t &= \frac{(1+n)\,(1+g)}{(1+r_t)}\frac{(\tau_t - g_t)}{(1+n)\,(1+g)} \\
&\quad + \frac{(1+n)\,(1+g)}{(1+r_t)}\left[\frac{(1+n)\,(1+g)}{(1+r_{t+1})}\frac{\tau_{t+1} - g_{t+1}}{(1+n)\,(1+g)} + \frac{(1+n)\,(1+g)}{(1+r_{t+1})}b_{t+2}\right]
\end{aligned}$$

Going to infinity and using the transversality condition:

$$\begin{aligned}
b_t &= \sum_{s=0}^{\infty} R_{t,t+s}\frac{\tau_{t+s} - g_{t+s}}{(1+n)\,(1+g)} + \lim_{k\to\infty} R_{t,t+k}b_{t+k+1} \\
b_t + \sum_{s=0}^{\infty} R_{t,t+s}\frac{g_{t+s}}{(1+n)\,(1+g)} &= \sum_{s=0}^{\infty} R_{t,t+s}\frac{\tau_{t+s}}{(1+n)\,(1+g)} \qquad (2)
\end{aligned}$$

Combining (1) and (2) shows that private demand is only affected by government spending g and not by the timing of the taxes τ:

$$\begin{aligned}
\sum_{s=0}^{\infty} R_{t,t+s}\frac{c_{t+s}}{(1+n)\,(1+g)} &= (k_t + b_t) + \sum_{s=0}^{\infty} R_{t,t+s}\frac{w_{t+s} - \tau_{t+s}}{(1+n)\,(1+g)} \\
\sum_{s=0}^{\infty} R_{t,t+s}\frac{c_{t+s}}{(1+n)\,(1+g)} &= k_t + \sum_{s=0}^{\infty} R_{t,t+s}\frac{w_{t+s}}{(1+n)\,(1+g)} + b_t - \sum_{s=0}^{\infty} R_{t,t+s}\frac{\tau_{t+s}}{(1+n)\,(1+g)} \\
\sum_{s=0}^{\infty} R_{t,t+s}\frac{c_{t+s}}{(1+n)\,(1+g)} &= k_t + \sum_{s=0}^{\infty} R_{t,t+s}\frac{w_{t+s}}{(1+n)\,(1+g)} + \sum_{s=0}^{\infty} R_{t,t+s}\frac{\tau_{t+s}}{(1+n)\,(1+g)} \\
&\quad - \sum_{s=0}^{\infty} R_{t,t+s}\frac{g_{t+s}}{(1+n)\,(1+g)} - \sum_{s=0}^{\infty} R_{t,t+s}\frac{\tau_{t+s}}{(1+n)\,(1+g)} \\
\sum_{s=0}^{\infty} R_{t,t+s}\frac{c_{t+s}}{(1+n)\,(1+g)} &= k_t + \sum_{s=0}^{\infty} R_{t,t+s}\frac{w_{t+s} - g_{t+s}}{(1+n)\,(1+g)} \qquad (3)
\end{aligned}$$

1.3 Optimal consumption

For brevity, consider a log utility of consumption:

$$
\begin{aligned}
U_0 &= \sum_{s=0}^{\infty} \frac{1}{(1+\rho)^s} \ln(C_s) L_s \\
U_0 &= \sum_{s=0}^{\infty} \frac{1}{(1+\rho)^s} \ln\left(c_s A_0 (1+g)^s\right) (1+n)^s L_0 \\
U_0 &= L_0 \sum_{s=0}^{\infty} \left(\frac{1+n}{1+\rho}\right)^s \ln(c_s) + L_0 \sum_{s=0}^{\infty} \left(\frac{1+n}{1+\rho}\right)^s \ln\left(A_0 (1+g)^s\right)
\end{aligned}
$$

We can ignore the last term as it is not affected by any economic variable. The maximization is subject to:

$$
w_t - \tau_t + r_t (k_t + b_t) = c_t + (k_{t+1} + b_{t+1})(1+n)(1+g) - (k_t + b_t)
$$

The Lagrangian is:

$$
\begin{aligned}
\mathcal{L} &= L_0 \sum_{s=0}^{\infty} \left(\frac{1+n}{1+\rho}\right)^s \ln(c_s) \\
&\quad + L_0 \sum_{d=0}^{\infty} \left(\frac{1+n}{1+\rho}\right)^s \lambda_s \begin{bmatrix} w_{t+s} - \tau_{t+s} + r_{t+s}(k_{t+s} + b_{t+s}) - c_{t+s} \\ -(k_{t+s+1} + b_{t+s+1})(1+n)(1+g) + (k_{t+s} + b_{t+s}) \end{bmatrix}
\end{aligned}
$$

There is one resource constraint for each period t, with associated multiplier λ_t. The first order conditions with respect to consumption is:

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial c_t} \\
0 &= L_0 \left(\frac{1+n}{1+\rho}\right)^t \left((c_t)^{-1} - \lambda_t\right) \\
(c_t)^{-1} &= \lambda_t
\end{aligned}
$$

The first order conditions with respect to assets is (omit L_0 for brevity) :

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial (k_{t+1} + b_{t+1})} \\
0 &= \frac{\partial}{\partial (k_{t+1} + b_{t+1})} \begin{bmatrix} \left(\frac{1+n}{1+\rho}\right)^t \ln(c_t) \\ + \left(\frac{1+n}{1+\rho}\right)^t \lambda_t \begin{bmatrix} w_t - \tau_t + r_t (k_t + b_t) \\ -(k_{t+1} + b_{t+1})(1+n)(1+g) \\ -c_t + (k_t + b_t) \end{bmatrix} \\ + \left(\frac{1+n}{1+\rho}\right)^{t+1} \ln(c_{t+1}) \\ + \left(\frac{1+n}{1+\rho}\right)^{t+1} \lambda_{t+1} \begin{bmatrix} w_{t+1} - \tau_{t+1} + r_{t+1}(k_{t+1} + b_{t+1}) \\ -(k_{t+2} + b_{t+2})(1+n)(1+g) \\ -c_{t+1} + (k_{t+1} + b_{t+1}) \end{bmatrix} \\ + \ldots \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
0 &= -\left(\frac{1+n}{1+\rho}\right)^t \lambda_t (1+n)(1+g) + \left(\frac{1+n}{1+\rho}\right)^{t+1} \lambda_{t+1} [r_{t+1}+1] \\
\lambda_t (1+n)(1+g) &= \frac{1+n}{1+\rho} \lambda_{t+1} (1+r_{t+1}) \\
\lambda_t &= \frac{1+r_{t+1}}{1+\rho} \lambda_{t+1}
\end{aligned}
$$

Combining the two conditions, we get:

$$
\begin{aligned}
\frac{1}{c_t} &= \frac{1+r_{t+1}}{1+\rho} \frac{1}{c_{t+1}} \\
\frac{c_{t+1}}{c_t} &= \frac{1+r_{t+1}}{1+\rho}
\end{aligned}
$$

For simplicity, consider the case where the interest rate is constant at r, which implies:

$$
R_{t,t+s} = \left(\frac{(1+n)(1+g)}{1+r}\right)^s
$$

Iterating the Euler condition forward, we get:

$$
\begin{aligned}
c_{t+1} &= \frac{1+r}{1+\rho} c_t \\
c_{t+s} &= \left(\frac{1+r}{1+\rho}\right)^s c_t
\end{aligned}
$$

Using this into (3)

$$
\begin{aligned}
\sum_{s=0}^{\infty} R_{t,t+s} \frac{c_{t+s}}{(1+n)(1+g)} &= k_t + \sum_{s=0}^{\infty} R_{t,t+s} \frac{w_{t+s} - g_{t+s}}{(1+n)(1+g)} \\
c_t \sum_{s=0}^{\infty} \frac{1}{(1+n)(1+g)} \left(\frac{(1+n)(1+g)}{1+r} \frac{1+r}{1+\rho}\right)^s &= k_t + \sum_{s=0}^{\infty} R_{t,t+s} \frac{w_{t+s} - g_{t+s}}{(1+n)(1+g)} \\
c_t \sum_{s=0}^{\infty} \frac{1}{(1+n)(1+g)} \left(\frac{(1+n)(1+g)}{1+\rho}\right)^s &= k_t + \sum_{s=0}^{\infty} R_{t,t+s} \frac{w_{t+s} - g_{t+s}}{(1+n)(1+g)}
\end{aligned}
$$

Assuming that $(1+n)(1+g) < 1+\rho$ we have:

$$
\begin{aligned}
c_t \frac{1}{(1+n)(1+g)} \frac{1}{1 - \frac{(1+n)(1+g)}{1+\rho}} &= k_t + \sum_{s=0}^{\infty} R_{t,t+s} \frac{w_{t+s} - g_{t+s}}{(1+n)(1+g)} \\
c_t \frac{1+\rho}{(1+n)(1+g)} \frac{1}{1+\rho-(1+n)(1+g)} &= k_t + \sum_{s=0}^{\infty} R_{t,t+s} \frac{w_{t+s} - g_{t+s}}{(1+n)(1+g)}
\end{aligned}
$$

Hence consumption is a constant share of current and future income:

$$
c_t = (1+n)(1+g) \frac{1+\rho-(1+n)(1+g)}{1+\rho} \left[k_t + \sum_{s=0}^{\infty} R_{t,t+s} \frac{w_{t+s} - g_{t+s}}{(1+n)(1+g)}\right]
$$

2 Overlapping generation model

2.1 Household's optimization

Agents live for two periods. An agent borne at time t maximizes a utility over consumption when young at time t, $C_{1,t}$, and when old at time $t+1$, $C_{2,t+1}$:

$$U_t = \frac{(C_{1,t})^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{(C_{2,t+1})^{1-\theta}}{1-\theta}$$

where $\rho > -1$ ($\rho > 0$ makes more intuitive sense). There are L_t young agents at time t, with population growing at an exogenous rate n. An agent born at time t supplies one unit of time when young, getting a wage W_t, and can save resources into old age earning a rate of return r_{t+1}. The agent faces a tax $T_{1,t}$ when young and $T_{2,t+1}$ when old. The budget constraints are ($S_{1,t}$ denotes savings by the young agent):

$$\begin{aligned} C_{1,t} + S_{1,t} &= W_t - T_{1,t} \\ C_{2,t+1} &= (1+r_{t+1}) S_{1,t} - T_{2,t+1} \end{aligned}$$

Combining these we get the intertemporal budget constraint:

$$C_{1,t} + \frac{C_{2,t+1}}{1+r_{t+1}} = W_t - \left(T_{1,t} + \frac{T_{2,t+1}}{1+r_{t+1}}\right)$$

The Lagrangian solved by the young agent at time t is:

$$\mathcal{L}_t = \frac{(C_{1,t})^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{(C_{2,t+1})^{1-\theta}}{1-\theta} + \lambda_t \left[W_t - \left(T_{1,t} + \frac{T_{2,t+1}}{1+r_{t+1}}\right) - C_{1,t} - \frac{C_{2,t+1}}{1+r_{t+1}}\right]$$

The first-order condition with respect to consumption when young is:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}_t}{\partial C_{1,t}} \\ 0 &= (C_{1,t})^{-\theta} - \lambda_t \\ (C_{1,t})^{-\theta} &= \lambda_t \end{aligned}$$

The first-order condition with respect to consumption old is:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}_t}{\partial C_{2,t+1}} \\ 0 &= \frac{1}{1+\rho}(C_{2,t+1})^{-\theta} - \lambda_t \frac{1}{1+r_{t+1}} \\ \frac{1+r_{t+1}}{1+\rho}(C_{2,t+1})^{-\theta} &= \lambda_t \end{aligned}$$

Combining the two conditions gives the Euler equation:

$$\begin{aligned}
\frac{1+r_{t+1}}{1+\rho}\left(C_{2,t+1}\right)^{-\theta} &= \left(C_{1,t}\right)^{-\theta} \\
\left(\frac{C_{2,t+1}}{C_{1,t}}\right)^{\theta} &= \frac{1+r_{t+1}}{1+\rho} \\
\frac{C_{2,t+1}}{C_{1,t}} &= \left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}}
\end{aligned}$$

To derive the solution for consumption, we use the Euler condition and the budget constraint:

$$\begin{aligned}
C_{1,t}+\frac{C_{2,t+1}}{1+r_{t+1}} &= W_t-\left(T_{1,t}+\frac{T_{2,t+1}}{1+r_{t+1}}\right) \\
C_{1,t}+\frac{C_{1,t}}{1+r_{t+1}}\left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}} &= W_t-\left(T_{1,t}+\frac{T_{2,t+1}}{1+r_{t+1}}\right) \\
C_{1,t}\left(1+\left(1+r_{t+1}\right)^{\frac{1-\theta}{\theta}}\left(1+\rho\right)^{-\frac{1}{\theta}}\right) &= W_t-\left(T_{1,t}+\frac{T_{2,t+1}}{1+r_{t+1}}\right) \\
C_{1,t} &= \frac{1}{1+\left(1+r_{t+1}\right)^{\frac{1-\theta}{\theta}}\left(1+\rho\right)^{-\frac{1}{\theta}}}\left[W_t-\left(T_{1,t}+\frac{T_{2,t+1}}{1+r_{t+1}}\right)\right] \\
C_{1,t} &= \frac{\left(1+\rho\right)^{\frac{1}{\theta}}}{\left(1+\rho\right)^{\frac{1}{\theta}}+\left(1+r_{t+1}\right)^{\frac{1-\theta}{\theta}}}\left[W_t-\left(T_{1,t}+\frac{T_{2,t+1}}{1+r_{t+1}}\right)\right] \quad (4)
\end{aligned}$$

The savings are then:

$$\begin{aligned}
S_{1,t} &= W_t-T_{1,t}-C_{1,t} \\
S_{1,t} &= W_t-T_{1,t}-\frac{\left(1+\rho\right)^{\frac{1}{\theta}}}{\left(1+\rho\right)^{\frac{1}{\theta}}+\left(1+r_{t+1}\right)^{\frac{1-\theta}{\theta}}}\left[W_t-T_{1,t}-\frac{T_{2,t+1}}{1+r_{t+1}}\right] \\
S_{1,t} &= \left(1-\frac{\left(1+\rho\right)^{\frac{1}{\theta}}}{\left(1+\rho\right)^{\frac{1}{\theta}}+\left(1+r_{t+1}\right)^{\frac{1-\theta}{\theta}}}\right)\left(W_t-T_{1,t}\right)+\frac{\left(1+\rho\right)^{\frac{1}{\theta}}}{\left(1+\rho\right)^{\frac{1}{\theta}}+\left(1+r_{t+1}\right)^{\frac{1-\theta}{\theta}}}\frac{T_{2,t+1}}{1+r_{t+1}} \\
S_{1,t} &= \frac{\left(1+r_{t+1}\right)^{\frac{1-\theta}{\theta}}}{\left(1+\rho\right)^{\frac{1}{\theta}}+\left(1+r_{t+1}\right)^{\frac{1-\theta}{\theta}}}\left(W_t-T_{1,t}\right)+\frac{\left(1+\rho\right)^{\frac{1}{\theta}}}{\left(1+\rho\right)^{\frac{1}{\theta}}+\left(1+r_{t+1}\right)^{\frac{1-\theta}{\theta}}}\frac{T_{2,t+1}}{1+r_{t+1}}
\end{aligned}$$

The consumption when old is:

$$\begin{aligned}
C_{2,t+1} &= \left(\frac{1+r_{t+1}}{1+\rho}\right)^{\frac{1}{\theta}}C_{1,t} \\
C_{2,t+1} &= \frac{\left(1+r_{t+1}\right)^{\frac{1}{\theta}}}{\left(1+\rho\right)^{\frac{1}{\theta}}+\left(1+r_{t+1}\right)^{\frac{1-\theta}{\theta}}}\left[W_t-\left(T_{1,t}+\frac{T_{2,t+1}}{1+r_{t+1}}\right)\right] \quad (5)
\end{aligned}$$

2.2 Firms

The firm produces output using a constant returns to scale technology, which we assume to be Cobb-Douglas:

$$Y_t = F\left(K_t, A_tL_t\right) = \left(K_t\right)^{\alpha}\left(A_tL_t\right)^{1-\alpha}$$

The wage is equal to the marginal product of labor:

$$W_t = (1-\alpha)(K_t)^{\alpha}(A_t)^{1-\alpha}(L_t)^{-\alpha} \tag{6}$$

The marginal product of capital is equal to the real interest rate:

$$r_t = \alpha (K_t)^{\alpha-1}(A_t L_t)^{1-\alpha} \tag{7}$$

We assume that productivity A_t grows at a rate g.

2.3 Capital accumulation and market clearing

The young agents can invest in capital K and government bonds B. The capital and government bonds available at time $t+1$ is held by the old agents, who bought it using their savings at time t:

$$\begin{aligned} K_{t+1} + B_{t+1} &= S_{1,t} L_t \\ K_{t+1} + B_{t+1} &= \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}}(W_t - T_{1,t}) L_t \\ &\quad + \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} \frac{T_{2,t+1}}{1+r_{t+1}} L_t \end{aligned} \tag{8}$$

There is no government spending, so the budget constraint is:

$$-B_{t+1} = L_t T_{1,t} + L_{t-1} T_{2,t} - (1+r_t) B_t \tag{9}$$

The clearing of the good market implies that output is consumed and invested:

$$Y_t = L_t C_{1,t} + L_{t-1} C_{2,t} + K_{t+1} - K_t \tag{10}$$

2.4 Scaling the equations

As was done with the Solow model, we scale variables by effective labor $A_t L_t$ (with some exceptions) to get a stationary system (this was not needed in class as we considered constant A and L). The consumption levels (4) and (5) are already in per-capita terms, so we only scale them by productivity A_t. The consumption of a young agent (4) is:

$$\begin{aligned} C_{1,t} &= \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} \left[W_t - \left(T_{1,t} + \frac{T_{2,t+1}}{1+r_{t+1}} \right) \right] \\ \frac{C_{1,t}}{A_t} &= \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} \left[\frac{W_t}{A_t} - \left(\frac{T_{1,t}}{A_t} + \frac{A_{t+1}}{A_t} \frac{1}{1+r_{t+1}} \frac{T_{2,t+1}}{A_{t+1}} \right) \right] \\ c_{1,t} &= \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} \left[w_t - \left(\tau_{1,t} + \frac{1+g}{1+r_{t+1}} \tau_{2,t+1} \right) \right] \end{aligned} \tag{11}$$

where the wage and taxes are also scaled only by productivity. The consumption of an old agent (5) is:

$$
\begin{aligned}
C_{2,t+1} &= \frac{(1+r_{t+1})^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} \left[W_t - \left(T_{1,t} + \frac{T_{2,t+1}}{1+r_{t+1}} \right) \right] \\
\frac{C_{2,t+1}}{A_{t+1}} &= \frac{(1+r_{t+1})^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} \left[\frac{W_t}{A_t}\frac{A_t}{A_{t+1}} - \left(\frac{T_{1,t}}{A_t}\frac{A_t}{A_{t+1}} + \frac{1}{1+r_{t+1}}\frac{T_{2,t+1}}{A_{t+1}} \right) \right] \\
c_{2,t+1} &= \frac{(1+r_{t+1})^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} \frac{1}{1+g} \left[w_t - \left(\tau_{1,t} + \frac{1+g}{1+r_{t+1}} \tau_{2,t+1} \right) \right] \qquad (12)
\end{aligned}
$$

The wage (6) is scaled as ($k_t = K_t/(A_t L_t)$):

$$
\begin{aligned}
\frac{W_t}{A_t} &= (1-\alpha)(K_t)^{\alpha} \frac{(A_t)^{1-\alpha}}{A_t} (L_t)^{-\alpha} \\
w_t &= (1-\alpha) \left(\frac{K_t}{A_t L_t} \right)^{\alpha} \\
w_t &= (1-\alpha)(k_t)^{\alpha} \qquad (13)
\end{aligned}
$$

The interest rate (7) is scaled as:

$$
\begin{aligned}
r_t &= \alpha (K_t)^{\alpha-1} (A_t L_t)^{1-\alpha} \\
r_t &= \alpha (k_t)^{\alpha-1} \qquad (14)
\end{aligned}
$$

The dynamics of assets (8) is scaled as:

$$
\begin{aligned}
\frac{K_{t+1}}{A_{t+1}L_{t+1}} + \frac{B_{t+1}}{A_{t+1}L_{t+1}} &= \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} (W_t - T_{1,t}) \frac{L_t}{A_{t+1}L_{t+1}} \\
&\quad + \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} \frac{T_{2,t+1}}{1+r_{t+1}} \frac{L_t}{A_{t+1}L_{t+1}} \\
k_{t+1} + b_{t+1} &= \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} \left(\frac{A_t w_t - T_{1,t}}{A_{t+1}} \right) \frac{1}{1+n} \\
&\quad + \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} \frac{T_{2,t+1}}{1+r_{t+1}} \frac{1}{A_{t+1}} \frac{1}{1+n} \\
k_{t+1} + b_{t+1} &= \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} \frac{w_t - \tau_{1,t}}{(1+g)(1+n)} \\
&\quad + \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} \frac{\tau_{2,t+1}}{1+r_{t+1}} \frac{1}{1+n} \qquad (15)
\end{aligned}
$$

The government budget constraint (9) becomes:

$$
-B_{t+1} = L_t T_{1,t} + L_{t-1} T_{2,t} - (1+r_t) B_t
$$

$$
\begin{aligned}
-\frac{B_{t+1}}{A_{t+1}L_{t+1}} &= \frac{A_t L_t}{A_{t+1}L_{t+1}}\left(\frac{L_t T_{1,t}}{A_t L_t}+\frac{L_{t-1}T_{2,t}}{A_t L_t}-(1+r_t)\frac{B_t}{A_t L_t}\right)\\
b_{t+1} &= -\frac{1}{(1+g)(1+n)}\left(\tau_{1,t}+\frac{1}{1+n}\tau_{2,t}-(1+r_t)\,b_t\right) \qquad (16)
\end{aligned}
$$

The clearing of the good market (10) is:

$$
\begin{aligned}
Y_t &= L_t C_{1,t}+L_{t-1}C_{2,t}+K_{t+1}-K_t\\
\frac{Y_t}{A_t L_t} &= \frac{L_t C_{1,t}}{A_t L_t}+\frac{L_{t-1}C_{2,t}}{A_t L_t}+\frac{K_{t+1}}{A_{t+1}L_{t+1}}\frac{A_{t+1}L_{t+1}}{A_t L_t}-\frac{K_t}{A_t L_t}\\
y_t &= c_{1,t}+\frac{1}{1+n}c_{2,t}+(1+g)(1+n)\,k_{t+1}-k_t\\
(k_t)^\alpha &= c_{1,t}+\frac{1}{1+n}c_{2,t}+(1+g)(1+n)\,k_{t+1}-k_t \qquad (17)
\end{aligned}
$$

As the wage and real interest rate are functions of capital, (15)-(16) give a highly non-linear relation between k_{t+1} and k_t:

$$
\begin{aligned}
k_{t+1} &= \frac{\left(1+\alpha\,(k_{t+1})^{\alpha-1}\right)^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+\left(1+\alpha\,(k_{t+1})^{\alpha-1}\right)^{\frac{1-\theta}{\theta}}}\frac{(1-\alpha)(k_t)^\alpha-\tau_{1,t}}{(1+g)(1+n)}\\
&\quad+\frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+\left(1+\alpha\,(k_{t+1})^{\alpha-1}\right)^{\frac{1-\theta}{\theta}}}\frac{\tau_{2,t+1}}{1+\alpha\,(k_{t+1})^{\alpha-1}}\frac{1}{1+n}\\
&\quad+\frac{1}{(1+g)(1+n)}\left(\tau_{1,t}+\frac{1}{1+n}\tau_{2,t}-(1+r_t)\,b_t\right)
\end{aligned}
$$

3 A simpler version of the model

3.1 General equations

Consider a log utility of consumption ($\theta = 1$). The model is characterized by (11)-(17) which become:

$$
\begin{aligned}
c_{1,t} &= \frac{1+\rho}{2+\rho}\left[w_t - \left(\tau_{1,t} + \frac{1+g}{1+r_{t+1}}\tau_{2,t+1}\right)\right] \\
c_{2,t+1} &= \frac{1+r_{t+1}}{2+\rho}\frac{1}{1+g}\left[w_t - \left(\tau_{1,t} + \frac{1+g}{1+r_{t+1}}\tau_{2,t+1}\right)\right] \\
w_t &= (1-\alpha)(k_t)^{\alpha} \\
r_t &= \alpha (k_t)^{\alpha-1} \\
k_{t+1} + b_{t+1} &= \frac{1}{2+\rho}\frac{w_t - \tau_{1,t}}{(1+g)(1+n)} + \frac{1+\rho}{2+\rho}\frac{\tau_{2,t+1}}{1+r_{t+1}}\frac{1}{1+n} \\
b_{t+1} &= -\frac{1}{(1+g)(1+n)}\left(\tau_{1,t} + \frac{1}{1+n}\tau_{2,t} - (1+r_t)\, b_t\right) \\
(k_t)^{\alpha} &= c_{1,t} + \frac{1}{1+n}c_{2,t} + (1+g)(1+n)\, k_{t+1} - k_t
\end{aligned}
$$

The utility of the agent borne at time t is:

$$
\begin{aligned}
U_t &= \ln(C_{1,t}) + \frac{1}{1+\rho}\ln(C_{2,t+1}) \\
U_t &= \ln(c_{1,t}) + \frac{1}{1+\rho}\ln(c_{2,t+1}) + \ln(A_t) + \frac{1}{1+\rho}\ln(A_{t+1})
\end{aligned}
$$

The scaled utility is:

$$
V_t = U_t - \ln(A_t) - \frac{1}{1+\rho}\ln(A_{t+1}) = \ln(c_{1,t}) + \frac{1}{1+\rho}\ln(c_{2,t+1})
$$

3.2 Steady state

In general the model is highly non-linear. We thus first solve for a steady state where there are not taxes and no government debt ($\tau_{1,t} = \tau_{2,t+1} = b_t = 0$), where asterisks denote steady-state values. The system becomes:

$$
\begin{aligned}
c_1^* &= \frac{1+\rho}{2+\rho}w^* \\
c_2^* &= \frac{1+r^*}{2+\rho}\frac{1}{1+g}w^* \\
w^* &= (1-\alpha)(k^*)^{\alpha} \\
r^* &= \alpha (k^*)^{\alpha-1} \\
k^* &= \frac{1}{2+\rho}\frac{w^*}{(1+g)(1+n)}
\end{aligned}
$$

The last equation and the wage imply:

$$k^* = \left[\frac{1}{2+\rho}\frac{(1-\alpha)}{(1+g)(1+n)}\right]^{\frac{1}{1-\alpha}}$$

The values for the other variables follow easily. In particular, the real interest rate is:

$$r^* = \frac{\alpha}{1-\alpha}(2+\rho)(1+g)(1+n)$$

The scaled utility is:

$$V_t^* = \ln\left(\frac{1+\rho}{2+\rho}w^*\right) + \frac{1}{1+\rho}\ln\left(\frac{1+r^*}{2+\rho}\frac{1}{1+g}w^*\right)$$

3.3 Linear approximations

We now take linear approximations of the various variables around the steady state, denoting deviations by hatted values (specifically: $\hat{c}_{1,t} = (c_{1,t} - c_1^*)/c_1^*$, $\hat{c}_{2,t} = (c_{2,t} - c_2^*)/c_2^*$, $\hat{k}_t = (k_t - k^*)/k^*$, and $\hat{w}_t = (w_t - w^*)/w^*$). We also define: $\hat{r}_{t+1} = (r_{t+1} - r^*)/(1+r^*)$, $\hat{\tau}_{1,t} = \tau_{1,t}/w^*$, $\hat{\tau}_{2,t} = \tau_{2,t}/w^*$, and $\hat{b}_t = b_t/k^*$. The system of equations is expanded as:

$$\hat{c}_{1,t} = \hat{w}_t - \hat{\tau}_{1,t} - \frac{1+g}{1+r^*}\hat{\tau}_{2,t+1} \tag{18}$$

$$\hat{c}_{2,t+1} = \hat{r}_{t+1} + \hat{w}_t - \hat{\tau}_{1,t} - \frac{1+g}{1+r^*}\hat{\tau}_{2,t+1} \tag{19}$$

$$\hat{w}_t = \alpha\hat{k}_t \tag{20}$$

$$\hat{r}_t = \frac{r^*}{1+r^*}(\alpha-1)\hat{k}_t \tag{21}$$

$$\hat{k}_{t+1} + \hat{b}_{t+1} = \hat{w}_t - \hat{\tau}_{1,t} + (1+\rho)\frac{1+g}{1+r^*}\hat{\tau}_{2,t+1} \tag{22}$$

$$\hat{b}_{t+1} = -(2+\rho)\left(\hat{\tau}_{1,t} + \frac{1}{1+n}\hat{\tau}_{2,t}\right) + \frac{1+r^*}{(1+g)(1+n)}\hat{b}_t \tag{23}$$

$$\frac{1+r^*}{(1+g)(1+n)}\hat{k}_t = (1+\rho)\hat{c}_{1,t} + \frac{1+r^*}{(1+g)(1+n)}\hat{c}_{2,t} + \hat{k}_{t+1} \tag{24}$$

(24) is actually redundant.

The scaled utility is:

$$\begin{aligned}
\hat{v}_t &= V_t - V^* \\
\hat{v}_t &= \hat{c}_{1,t} + \frac{1}{1+\rho}\hat{c}_{2,t+1} \\
\hat{v}_t &= \frac{2+\rho}{1+\rho}\left[\hat{w}_t - \hat{\tau}_{1,t} - \frac{1+g}{1+r^*}\hat{\tau}_{2,t+1}\right] + \frac{1}{1+\rho}\hat{r}_{t+1}
\end{aligned}$$

3.4 The impact of taxes

Consider that we start at the steady state with zero taxes. At time t the government unexpectedly introduces a transfer to both households: $\hat{\tau}_{1,t} = \hat{\tau}_{2,t} = \hat{\tau}_t < 0$. It also announces that from $t+1$ onwards, taxes are increased on all agents to keep the debt constant ($\hat{\tau}_{1,t+1} = \hat{\tau}_{2,t+1} = \hat{\tau}_{t+1}$). The debt dynamics (23) are thus:

$$\hat{b}_{t+2} = -\left(2+\rho\right)\frac{2+n}{1+n}\hat{\tau}_{t+1} + \frac{1+r^*}{\left(1+g\right)\left(1+n\right)}\hat{b}_{t+1}$$

As debt is kept constant, $\hat{b}_{t+2} = \hat{b}_{t+1}$, (23) gives the value of future taxes:

$$\begin{aligned}
\hat{b}_{t+1} &= -\left(2+\rho\right)\frac{2+n}{1+n}\hat{\tau}_{t+1} + \frac{1+r^*}{\left(1+g\right)\left(1+n\right)}\hat{b}_{t+1} \\
\left(2+\rho\right)\frac{2+n}{1+n}\hat{\tau}_{t+1} &= \left[\frac{1+r^*}{\left(1+g\right)\left(1+n\right)} - 1\right]\hat{b}_{t+1} \\
\hat{\tau}_{t+1} &= \frac{1+n}{2+n}\frac{1}{2+\rho}\left[\frac{1+r^*}{\left(1+g\right)\left(1+n\right)} - 1\right]\hat{b}_{t+1}
\end{aligned}$$

All variables can be computed as functions of $\hat{\tau}_t$ and $\hat{\tau}_{t+1}$ using our solution. Specifically, given $\hat{b}_t$ we compute $\hat{b}_{t+1}$ using(23). Then given $\hat{k}_t$ we compute $\hat{w}_t$ and $\hat{r}_t$ using (20)-(21). We then compute $\hat{k}_{t+1}$ using (22). Finally, we compute $\hat{c}_{1,t}$ and $\hat{c}_{2,t}$ using (18)-(19).

A specific aspect arises if the tax policy was not expected at time t. In that case, the capital stock is unchanged at period t. In addition, the consumption of old agents in period t simply reflects the transfer:

$$\begin{aligned}
dc_{2,t} &= -d\tau_t \\
c_2^*\hat{c}_{2,t} &= -w^*\hat{\tau}_t \\
\hat{c}_{2,t} &= -\frac{\left(2+\rho\right)\left(1+g\right)}{1+r^*}\hat{\tau}_t
\end{aligned}$$

4 Viewing constraints as shadow interest rates

In the OLG, agents maxmize utility subject to the budget constraint:

$$\begin{aligned} U_t &= \frac{(c_{1,t})^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{(c_{2,t+1})^{1-\theta}}{1-\theta} \\ c_{1,t} + s_{1,t} &= w_t - \tau_{1,t} \qquad ; \qquad c_{2,t+1} = (1+r_{t+1})\, s_{1,t} - \tau_{2,t+1} \end{aligned}$$

We introduce two constraints, that may or may not be binding. The first one states that the agent cannot borrow more than an amount s^{low}, so savings have to exceed $-s^{low}$:

$$s_{1,t} \geq -s^{low}$$

The second states that the agent cannot save more than an amount s^{high},:

$$s_{1,t} \leq s^{high}$$

The Lagrangian with the two flow budget constraints and the two limiting constraints on savings is:

$$\begin{aligned} \mathcal{L}_t = &\frac{(c_{1,t})^{1-\theta}}{1-\theta} + \frac{1}{1+\rho}\frac{(c_{2,t+1})^{1-\theta}}{1-\theta} \\ &+\lambda_{1,t}\left[w_t - \tau_{1,t} - c_{1,t} - s_{1,t}\right] \\ &+\lambda_{2,t}\left[(1+r_{t+1})\, s_{1,t} - \tau_{2,t+1} - c_{2,t+1}\right] \\ &+\mu^{low}\left[s_{1,t} + s^{low}\right] \\ &-\mu^{high}\left[s_{1,t} - s^{high}\right] \end{aligned}$$

$\mu^{low} > 0$ is the utility value of relaxing the borrowing constraint (if binding) and $\mu^{high} > 0$ is the utility value of relaxing the lending constraint (if binding)

The first-order conditions with respect to $c_{1,t}$, $c_{2,t+1}$, and $s_{1,t}$ are:

$$\begin{aligned} (c_{1,t})^{-\theta} &= \lambda_{1,t} \\ \frac{1}{1+\rho}(c_{2,t+1})^{-\theta} &= \lambda_{2,t} \\ \lambda_{1,t} &= \lambda_{2,t}(1+r_{t+1}) + \mu^{low} - \mu^{high} \end{aligned}$$

Combining we get:

$$\begin{aligned} (c_{1,t})^{-\theta} &= \frac{1+r_{t+1}}{1+\rho}(c_{2,t+1})^{-\theta} + \mu^{low} - \mu^{high} \\ (c_{1,t})^{-\theta} &= \frac{1+r_{t+1} + \mu^{low}(1+\rho)(c_{2,t+1})^{\theta} - \mu^{high}(1+\rho)(c_{2,t+1})^{\theta}}{1+\rho}(c_{2,t+1})^{-\theta} \end{aligned}$$

If the constraints are not biding ($\mu^{low} = \mu^{high} = 0$) we get the Euler condition at the usual market interest rate:

$$\left(c_{1,t}\right)^{-\theta} = \frac{1+r_{t+1}}{1+\rho}\left(c_{2,t+1}\right)^{-\theta}$$

If the borrowing constraint is binding ($\mu^{low} > 0$ and $\mu^{high} = 0$) we get:

$$\left(c_{1,t}\right)^{-\theta} = \frac{1+r_{t+1}+\mu^{low}\left(1+\rho\right)\left(c_{2,t+1}\right)^{\theta}}{1+\rho}\left(c_{2,t+1}\right)^{-\theta}$$

We can therefore understand this as a situation where the agent effectively faces an interest rate $r_{t+1}+\mu^{low}\left(1+\rho\right)\left(c_{2,t+1}\right)^{\theta}$ that is larger than r_{t+1}, so she can borrow but at a rate that is higher than the market interest rate. If the agent would like to borrow but cannot at all ($s^{low} = 0$), she effectively faces an infinite interest rate.

If the saving constraint is binding ($\mu^{low} = 0$ and $\mu^{high} > 0$) we get:

$$\left(c_{1,t}\right)^{-\theta} = \frac{1+r_{t+1}-\mu^{high}\left(1+\rho\right)\left(c_{2,t+1}\right)^{\theta}}{1+\rho}\left(c_{2,t+1}\right)^{-\theta}$$

We can therefore understand this as a situation where the agent effectively faces an interest rate $r_{t+1}-\mu^{high}\left(1+\rho\right)\left(c_{2,t+1}\right)^{\theta}$ that is smaller than r_{t+1}, so she can lend but at a rate that is lower than the market interest rate. If the agent would like to save but cannot at all ($s^{high} = 0$), she effectively faces an interest rate of -100%.

5 Integrating OLG and infinite lives

5.1 Bequests

Consider the case were agents live for one period, but also cares for their children:

$$U_t = u(C_t) + \beta U_{t+1} = \sum_{s=t}^{\infty} \beta^{s-t} u(C_s)$$

The budget constraints includes a bequest:

$$\begin{aligned} C_t + H_{t+1} &= (1+r) H_t + Y_t - T_t \\ \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} C_s &= (1+r) H_t + \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} (Y_s - T_s) \end{aligned}$$

which is the infinitively lived model (up to the value of β).

5.2 The Weil model

5.2.1 Building blocks

Families are infinitively lived. In each period new families are born with no assets. The interest rate is constant at r. The welfare of family of vintage v and its budget constraint is:

$$\begin{aligned} U_t^v &= \sum_{s=t}^{\infty} \beta^{s-t} \ln(c_s^v) \\ \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} c_s^v &= (1+r) b_t^v + \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} (y_s^v - \tau_s^v) \qquad ; \qquad b_v^v = 0 \end{aligned}$$

The size of the family does not grow, but the population grows at rate n from new individuals ($N_0 = 1$). The size of the population and new vintages are:

time	population	new vintage
0	1	1
1	$1+n$	n
2	$(1+n)^2$	$(1+n)^2 - (1+n) = n(1+n)$
3	$(1+n)^3$	$(1+n)^3 - (1+n)^2 = n(1+n)^2$

The dynamics of a family's consumption lead to the Euler equation:

$$c_{s+1}^v = \beta(1+r) c_s^v$$

Combining with the budget constraint, we get:

$$c_t^v = (1-\beta)\left[(1+r) b_t^v + \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} (y_s^v - \tau_s^v)\right]$$

The overall per capita consumption is computed by aggregating the consumption of the various families:

$$c_t = \frac{c_t^0 + nc_t^1 + n(1+n)c_t^2 + n(1+n)^2 c_t^3 + ... + n(1+n)^{t-1} c_t^t}{(1+n)^t} = \frac{c_t^0 + n\sum_{v=1}^t (1+n)^{v-1} c_t^v}{(1+n)^t}$$

We can write the per-capita consumption as:

$$c_t = (1-\beta)\left[(1+r)b_t + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(y_s - \tau_s)\right]$$

The corresponding relation in the infinite lived representative agent setup (denoted by Repr) is:

$$c_t^{\mathrm{Repr}} = (1-\beta)\left[(1+r)b_t + \sum_{s=t}^{\infty}\left(\frac{1+n}{1+r}\right)^{s-t}(y_s - \tau_s)\right]$$

Future income is discounted by less in the representative agent setup.

The budget constraint of a particular family is:

$$b_{t+1}^v = (1+r)b_t^v + (y_t^v - \tau_t^v) - c_t^v$$

We take a weighted sum of this relation, with the weights reflecting the size of the various families:

$$\frac{b_{t+1}^0 + nb_{t+1}^1 + n(1+n)b_{t+1}^2 + ... + n(1+n)^{t-1} b_{t+1}^t}{(1+n)^t} = (1+r)b_t + (y_t - \tau_t) - c_t$$

where b_t, y_t and τ_t are weighted averages of b_t^v, y_t^v and τ_t^v across the families that are constructed along similar lines as c_t.

Notice that as $b_{t+1}^{t+1} = 0$ (the newborns have no assets), hence we write

$$\begin{aligned}
& \frac{b_{t+1}^0 + nb_{t+1}^1 + n(1+n)b_{t+1}^2 + ... + n(1+n)^{t-1} b_{t+1}^t}{(1+n)^t} \\
= \quad & (1+n)\frac{b_{t+1}^0 + nb_{t+1}^1 + n(1+n)b_{t+1}^2 + ... + n(1+n)^{t-1} b_{t+1}^t + n(1+n)^t b_{t+1}^{t+1}}{(1+n)^{t+1}} \\
= \quad & (1+n)b_{t+1}
\end{aligned}$$

The aggregate budget constraint is then:

$$\begin{aligned}
b_{t+1} &= \frac{(1+r)b_t + (y_t - \tau_t) - c_t}{1+n} \\
&= \frac{\beta(1+r)}{1+n}b_t + \frac{(y_t - \tau_t) - (1-\beta)\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(y_s - \tau_s)}{1+n}
\end{aligned}$$

Notice that the form of the budget constraint is the same as in the representative agent setup:

$$b_{t+1}^{\mathrm{Repr}} = \frac{(1+r)b_t + (y_t - \tau_t) - c_t}{1+n}$$

Consider the simple case with constant output and no government:

$$b_{t+1} = \frac{\beta(1+r)}{1+n} b_t + \frac{\beta(1+r)-1}{r(1+n)} y$$

So we can have a steady state even if $\beta(1+r) > 1$, which is not possible in the representative agent model. The steady state is:

$$\frac{b}{y} = \frac{\beta(1+r)-1}{1+n-\beta(1+r)} \frac{1}{r}$$

Note that if $\beta(1+r) < 1$ the economy is a steady-state debtor.

5.2.2 Output fluctuations

If $\beta(1+r) < 1$ a permanent rise in output leads to an increase in steady-state debt as the agents are impatient. In case of a one-period increase in output at time t (initially at the steady state) we get:

$$b_{t+1} = \frac{\beta(1+r)}{1+n} b + \frac{\beta(1+r)-1}{r(1+n)} y + \frac{\beta}{1+n} dy_t = b + \frac{\beta}{1+n} dy_t$$

So there is a transitory effect, but then things gradually go back to the steady state as new agents 'dilute' the effect (this would not be the case in the infinitively lived case).

5.2.3 Government debt

For simplicity, set $\beta(1+r) = 1$:

$$b_{t+1} = \frac{1}{1+n} b_t$$

As long as $n > 0$ there is a well defined steady state. This is because agents today to not consider future unborn generations in their planning.

The government budget constraint is:

$$b^G_{t+1} = \frac{(1+r) b^G_t + \tau_t - g_t}{1+n}$$

The government starts with zero debt. It then issues debt of $\bar{d}$, leaving the debt steady at this level thereafter. Future taxes are then:

$$\tau_s = \bar{d}(r-n) + g_s$$

The per-capita aggregate consumption is:

$$\begin{aligned}
c_t &= (1-\beta)\left[(1+r)\left(b_t+\bar{d}\right) + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(y_s-\tau_s)\right] \\
&= (1-\beta)\left[(1+r)\left(b_t+\bar{d}\right) + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}\left(y_s-\bar{d}(r-n)-g_s\right)\right] \\
&= (1-\beta)\left[(1+r)b_t + \bar{d}\left[(1+r)-(r-n)\frac{1+r}{r}\right] + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(y_s-g_s)\right]
\end{aligned}$$

$$= \quad (1-\beta)\left[(1+r)\,b_t + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(y_s - g_s)\right] + (1-\beta)(1+r)\,\frac{n}{r}\bar{d}$$

So debt has an impact as long as $n > 0$. Extra debt boosts consumption as some of the gift today is paid for by taxes on unborn agents in the future.

By contrast there is no effect in the representative agent setup:

$$\begin{aligned}
c_t^{\mathrm{Repr}} &= (1-\beta)\left[(1+r)\left(b_t + \bar{d}\right) + \sum_{s=t}^{\infty}\left(\frac{1+n}{1+r}\right)^{s-t}(y_s - \tau_s)\right] \\
&= (1-\beta)\left[(1+r)\left(b_t + \bar{d}\right) + \sum_{s=t}^{\infty}\left(\frac{1+n}{1+r}\right)^{s-t}\left(y_s - \bar{d}\,(r-n) - g_s\right)\right] \\
&= (1-\beta)\left[(1+r)\left(b_t + \bar{d}\right) - \bar{d}\,(1+r) + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(y_s - g_s)\right] \\
&= (1-\beta)\left[(1+r)\,b_t + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}(y_s - g_s)\right]
\end{aligned}$$