Game Theory

Auctions and Competitive Bidding

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Common Types of Auctions

- Open Auction
  - English Auction
    Bidders are all in a room and the price of the item goes up as long as someone is willing to bid it higher. Once the last increase is no longer challenged, the last bidder to increase the price wins the auction and pays that price for the item.
  - Dutch Auction
    Price starts at a prohibitively high value and the auctioneer gradually drops the price. Once a bidder shouts "buy," the auction ends and the bidder gets the good at the price at which he cried out.

- Sealed-bid auction
  - First-Price Sealed-Bid Auction
    Each bidder writes down his bid and place it in an envelope; the envelopes are opened simultaneously. The highest bidder wins and then pays a price equal to his own bid.
  - Second-Price Sealed-Bid Auction
    The difference (compared to the former type) is that although the highest bidder wins, he does not pay his bid but instead pays a price equal to the second-highest bid or the highest losing bid.

Question:

1. What is the optimal bidding strategy for each bidder (for a given auction rule)?
2. Which rule gives the highest revenue for the seller?

Class Game 1

Class Game 2

Example: Oil Tract Auction

Example

*Suppose two firms, A and B, are bidding for a tract of land that may have oil underground. The item is valued by v_A and v_B, respectively, for firm A and B. It is **common knowledge** that valuations are independently and uniformly distributed over $[0, 1]$. The firm who offers a higher price wins and the winner pays his/her biding price.*

What is the optimal bidding strategies?

- Let b_A and b_B be the bid. From the view of firm A, firm A wins the item provided that $b_A > b_B$, and wining the item gives $v_A - b_A$. Otherwise, losing the game gives 0 if $b_A < b_B$.
- Firm A chooses b_A to maximize

$$\begin{cases} v_A - b_A & \text{if} \quad b_A > b_B \\ \frac{v_A - b_A}{2} & \text{if} \quad b_A = b_B \\ 0 & \text{if} \quad b_A < b_B \end{cases}.$$

- To derive the explicit strategies, assume that each firm $(i = A, B)$ bids a fraction of its valuation, i.e.,

$$b_i = k_i v_i, \ 0 \leq k_i \leq 1, \ i = A, B.$$

- Firm A solves

$$\max_{b_A}(v_A - b_A)\,\mathrm{Pr}\,(b_A > b_B) + \frac{v_A - b_A}{2}\,\mathrm{Pr}(b_A = b_B) + 0 \cdot \mathrm{Pr}(b_A < b_B)$$

Note that

$$Pr(b_A > b_B) = \mathrm{Pr}(b_A > k_B v_B) = \mathrm{Pr}\left(v_B < \frac{b_A}{k_B}\right)$$

$$= \int_0^{\frac{b_A}{k_B}} f(v_B)dv_B = \int_0^{\frac{b_A}{k_B}} \frac{1}{1-0}dv_B = \frac{b_A}{k_B}.$$

- Rearranging, firm A chooses b_A to maximize

$$(v_A - b_A)\frac{b_A}{k_B}$$

F.O.C. w.r.t. b_A gives

$$b_A^* = \frac{v_A}{2}.$$

General Bidding Function $b(\cdot)$

Assume that each player 's bidding function $b(v)$ is strictly increasing and differentiable.

- Firm A's optimal bid solves $\max_{b_A}(v_A - b_A)\Pr(b_A > b(v_B))$
- Let $b^{-1}(\cdot)$ the inverse function (**反函数**). Then $b_A > b(v_B) \Rightarrow b^{-1}(b_A) > b^{-1}(b(v_B)) = v_B$.
- Valuations are uniformly distributed over $[0, 1]$:

$$\Pr(b_A > b(v_B)) = \Pr(v_B < b^{-1}(b_A)) = \int_0^{b^{-1}(b_A)} \frac{1}{1-0} dv = b^{-1}(b_A)$$

- Firm A's payoff becomes $(v_A - b_A)b^{-1}(b_A)$

- Firm A solves $\max_{b_A}(v_A - b_A)b^{-1}(b_A)$
- FOC b_A: $-b^{-1}(b_A) + (v_A - b_A)\frac{d}{db_A}b^{-1}(b_A) = 0$
- If the strategy $b(\cdot)$ is a symmetric Bayesian Nash equilibrium, then firm $i = A, B$'s strategy is $b(v_i)$.
- Replace b_A by $b(v_A)$ into the FOC:

$$
\begin{aligned}
& - b^{-1}(b(v_A)) + (v_A - b(v_A))\frac{d}{db_A}b^{-1}(b(v_A)) = 0 \\
\Leftrightarrow & - v_A + (v_A - b(v_A))\frac{1}{b'(v_A)} = 0 \\
\Leftrightarrow & b'(v_A)v_A + b(v_A) = v_A \\
\Leftrightarrow & \frac{db}{dv_A}v_A + b = v_A
\end{aligned}
$$

- Solve the differential equation $\frac{db}{dv_A}v_A + b = v_A$ where b is a function of v_A (“一阶线性非齐次微分方程”)
- Using the method of constant variation (常数变易法) to solve $y'(x)x + y = x$. First, solve $y'(x)x + y = 0$, or

$$\frac{dy}{dx}x = -y \Leftrightarrow \frac{dy}{y} = -\frac{dx}{x}$$

$$\Rightarrow \ln y = -\ln x + \ln C \Rightarrow y = \frac{C(x)}{x}$$

- Plug $y = \frac{C(x)}{x}$ into $\frac{dy}{dx}x + y = x$:

$$\frac{xC'(x) - C(x)}{x^2}x + \frac{C(x)}{x} = x \Rightarrow C'(x) = x$$

$$\Rightarrow C(x) = \frac{x^2}{2} + C \Rightarrow y(x)x = \frac{x^2}{2} + C$$

- Here, $b(v) = y(x)$.
- First, let $\frac{db}{dv_A} v_A + b = 0$

$$\frac{db}{dv_A} v_A = -b \Leftrightarrow \frac{db}{b} = -\frac{dv_A}{v_A}$$
$$\Rightarrow \ln b = -\ln v_A + \ln C \Rightarrow b = \frac{C(v_A)}{v_A}$$

- Plug $b = \frac{C(v_A)}{v_A}$ into $\frac{db}{dv_A} v_A + b = v_A$:

$$\frac{v_A C'(v_A) - C(v_A)}{v_A^2} v_A + \frac{C(v_A)}{v_A} = v_A \Rightarrow C'(v_A) = v_A$$
$$\Rightarrow C(v_A) = \frac{v_A^2}{2} + C \Rightarrow b(v_A) v_A = \frac{v_A^2}{2} + C$$

- $b(0) = 0 \Rightarrow C = 0 \Rightarrow b^*(v_A) = \frac{v_A}{2}$.

More (n) bidders

There are n symmetric bidders (the bidding function is assumed to be $b_i = kv_i$ and v_i is uniformly distributed over $[0, 1]$). From the view of bidder $i = 1, ..., n$, the probability of wining is

$$\begin{aligned}
\mathrm{Pr}(b_i \text{ is the highest}) &= \mathrm{Pr}(b_i > b_j,\ j \neq i,\ j = 1, ..., n) \\
&= \mathrm{Pr}(b_i > kv_j,\ j \neq i,\ j = 1, ..., n) \\
&= \mathrm{Pr}\left(v_j < \frac{b_i}{k},\ j \neq i,\ j = 1, ..., n\right). \\
&= \left(\frac{b_i}{k}\right)^{n-1}
\end{aligned}$$

The first-order condition of firm i's expected payoff, gives

$$\frac{d}{db_i}\left[(v_i - b_i)\left(\frac{b_i}{k}\right)^{n-1}\right] = 0 \Rightarrow \quad b_i^* = \frac{n-1}{n}v_i.$$

Second-price sealed-bid auction

Assumption (IPV: Independent Private Values)

Each person's willingness to pay depends only on his own type, and this in turn is private information. This differs from the common-values setting, in which the preferences of some players may depend on the types of other players.

The payoff function of bidder i:

$$\begin{cases} v_i - b_j, & b_i > b_j > \text{all the other bids} \\ 0, & b_i \text{ is not the highest bid} \end{cases}.$$

Proposition (Weakly Dominant Strategy)

In the second-price sealed-bid auction, each player has a weakly dominant strategy, which is to bid his true valuation. That is, $s_i(v_i) = v_i$ *for all* $i \in n$ *is a Bayesian Nash equilibrium in weakly dominant strategies.*

Proof: Suppose a bidder bids below his valuation: $b_i < v_i$, there are 3 possible outcomes

a. If $b_i = b^{[1]}$ is the highest bid, the bidder pays $b^{[2]} < b_i$. Bidding v_i whereby paying $b^{[2]}$ is as good as bidding $b^{[1]}$.
b. If another bidder bids $b_j = b^{[1]} > v_i$, then bidder i gets 0. Bidding his own valuation will not be worse off.
c. If the highest bidder j bids $b_i < b_j = b^{[1]} < v_i$, by deviating to bid $b_i = v_i$, bidder i will be better off because $v_i - b_j > 0$. Hence $b_i < v_i$ is not optimal.

English Auction

Proposition

In the button-auction model it is a weakly dominant strategy for each player to keep his button pressed as long as $p < v_i$ and to release it once $p = v_i$. This results in a Bayesian Nash equilibrium in weakly dominated strategies that is outcome-equivalent to the second-price sealed-bid auction.

Proof:

a. If the current price is $p < v_i$, bidder i (who is not the currently highest bidder) should hold his button, until $p = v_i$. Dropping out before $p = v_i$ loses the opportunity to win.
b. When $p = v_i$, bidder i should drop out.
c. When there are only two person in the game, release the button immediately after the other player drops out (pays the second-highest valuation).

First-Price Sealed-Bid and Dutch Auctions

Claim

In a first-price sealed-bid auction it is a dominated strategy for a player to bid his valuation.

Assumption

The higher a player's valuation, the higher is his bid, i.e., $s_i'(v_i) > 0$.

The monotonicity of bidding strategy ensures the existence of an invertible function $s_i^{-1}(\cdot)$:

$$\Pr(s_j(v_j) < b_i) = \Pr(v_j < s_j^{-1}(b_i)) = F_j\left(s_j^{-1}(b_i)\right).$$

The expected payoff of bidder i is

$$\pi_i = \Pr\left(b_i \text{ is the highest}\right)(v_i - b_i),$$

where (by IPV)

$$\begin{aligned}\Pr(b_i \text{ is the highest}) &= \underbrace{\Pr\left(v_{-i} < s_{-i}^{-1}(b_i)\right)}_{\# n-1} \\ &= \left[F\left(s^{-1}(b_i)\right)\right]^{n-1}\end{aligned}$$

The first-order condition gives (note that if $y = f(x)$ and $x = f^{-1}(y)$, then $\frac{dx}{dy} = \frac{df^{-1}(y)}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{f'(x)}$):

$$\begin{aligned}\frac{d\pi_i}{db_i} &= (n-1)F^{n-2}f\frac{ds^{-1}(b_i)}{db_i}(v_i - b_i) - F^{n-1} \\ &= (n-1)F^{n-2}f\frac{1}{s'(v_i)}(v_i - b_i) - F^{n-1} = 0.\end{aligned}$$

Symmetric Bayesian Nash Equilibrium, $b = s^*(v)$:

$$F^{n-1}s'(v) + (n-1)F^{n-2}fs(v) = (n-1)F^{n-2}fv. \quad .$$

Integrating both sides from $[\underline{v}, v]$, and using integration by parts ("分部积分": $\int udv = uv - \int vdu$)

$$\int_{\underline{v}}^{v} F^{n-1}ds(v) + \int_{\underline{v}}^{v} s(v)dF^{n-1} = \int_{\underline{v}}^{v} vdF^{n-1}$$

$$\Leftrightarrow F^{n-1}s(v)\Big|_{\underline{v}}^{v} - \int_{\underline{v}}^{v} s(v)dF^{n-1} + \int_{\underline{v}}^{v} s(v)dF^{n-1}$$

$$= vF^{n-1}\Big|_{\underline{v}}^{v} - \int_{\underline{v}}^{v} F^{n-1}dv$$

$$\Leftrightarrow F^{n-1}s(v) = vF^{n-1} - \int_{\underline{v}}^{v} F^{n-1}dv$$

$$\Leftrightarrow s(v) = v - \frac{\int_{\underline{v}}^{v} F^{n-1}dv}{F^{n-1}} < v.$$

Under uniform distribution with support $[0, 1]$,
$F(v) = \int_0^v \frac{1}{1-0} dt = v$, hence

$$s(v) = v - \frac{\int_0^v v^{n-1} dv}{v^{n-1}} = v - \frac{v}{n} = \left(\frac{n-1}{n}\right) v.$$

Recall that: $b_i = v_i$ for second-price sealed-bid auction (English); and $s(v_i) < v_i$ for first-price sealed-bid auction (Dutch), which format should be adopted by the auctioneer, in order to yield a higher expected revenue?

Claim

They are strategically equivalent.

Revenue Equivalence Theorem (William Vickrey, 1961; Myerson, 1981 and Riley and Samuelson, 1981)

Revenue Equivalence Theorem

Assume that valuations are independently and uniformly distributed over $[0,1]^n$. For the first-price sealed-bid auction, the seller receives the highest bid, $b_n^{[1]} = \max\{b_1, ..., b_n\}$. The expected value of the highest bid is given by

$$E(b_n^{[1]}).$$

To calculate $E(b_n^{[1]})$, we need to find the integral domain of $b_n^{[1]}$, and the density function $f_n^{[1]}(x)$.

- We have solved that $s(v) = \frac{n-1}{n} v$, hence the highest bid is no greater than $\frac{n-1}{n}$;
- The highest draw of $\{b_1, ..., b_n\}$, i.e., $b_n^{[1]}$ is called **first-order statistic** of the sample.

The CDF of the first-order statistic is

$$F_n^{[1]}(x) = \Pr(b_n^{[1]} \leq x) = \Pr(b_1 \leq x, ..., b_n \leq x)$$
$$= \left(\int_0^x \frac{1}{\frac{n-1}{n} - 0} dx \right)^n = \left(\frac{nx}{n-1} \right)^n .$$

The PDF is

$$f_n^{[1]}(x) = \frac{dF_n^{[1]}(x)}{dx} = \frac{d}{dx} \left[\left(\frac{nx}{n-1} \right)^n \right] = \frac{n}{x} \left(\frac{nx}{n-1} \right)^n .$$

The expected revenue of the highest bid is

$$E(b_n^{[1]}) = \int_0^{\frac{n-1}{n}} x f_n^{[1]}(x) dx = \frac{n-1}{n+1}.$$

For the second-price sealed-bid auction, the seller receives the second-highest bid $b_n^{[2]}$. We have shown that $b_i = v_i$ is a weakly dominant strategy. Therefore, the expected revenue of the seller is

$$E\left(\text{the second-highest value of } v\right).$$

To calculate the CDF and PDF of the **second-order statistic** of $v_n^{[2]}$, note that the event "the second-highest of the sample $\{v_1, ..., v_n\}$ is no greater than x" can be divided into two mutually exclusive subsets:

1. All v are no greater than x;
2. The highest v is greater than x, and the remaining v are no greater than x.

The CDF of the second-order statistic is

$$
\begin{aligned}
F_n^{[2]}(x) = & \Pr\left(\max\{v_1, ..., v_n\} \leq x\right) + \Pr\left(v_i > x,\ \ v_{-i} \leq x\right) \\
= & \Pi_{i=1}^{n} \Pr\left(v_i \leq x\right) + \sum_{i=1}^{n} \Pr(v_i > x) \Pi_{j=1}^{n-1} \Pr(v_j \leq x) \\
= & F(x)^n + \sum_{i=1}^{n} \left[1 - F(x)\right] F(x)^{n-1} \\
= & F(x)^n + n\left[1 - F(x)\right] F(x)^{n-1}
\end{aligned}
.
$$

In the second-price sealed-bid auction, biddings (i.e., valuations) are uniformly distributed over 0 and 1, i.e., $F(x) = \int_0^x \frac{1}{1-0} dv = x$. Therefore,

$$
F_n^{[2]}(x) = x^n + n(1-x)x^{n-1} = nx^{n-1} - (n-1)x^n.
$$

The PDF of the second-order statistic is

$$f_n^{[2]}(x) = n(n-1)x^{n-2} - n(n-1)x^{n-1}.$$

The expected revenue of the seller in the second-price sealed-bid auction is

$$E\left(v_n^{[2]}\right) = \int_0^1 x f_n^{[2]}(x)dx = \frac{n-1}{n+1}.$$

Winner's Curse

- In the above, we assume that the valuations of bidders are independent.
- Counter-example: house
  - private value of living
  - sell it at a later date
- The information of other people would enter into one's willingness to pay
- Common-values component

Winner's Curse: Example

- Two firms considering the purchase of a new oil field. It's common knowledge that the oil could be of
  - small, worth 10; $\Pr(v = 10) = \frac{1}{4}$
  - medium, worth 20; $\Pr(v = 20) = \frac{1}{2}$
  - large, worth 30; $\Pr(v = 30) = \frac{1}{4}$
- The oil is auctioned in a second-price sealed-bid auction
- Each of the two firms will perform a free exploration that provides some signal about the quantity of the oil
  - firm 1 receives a signal θ_1
  - firm 2 receives a signal θ_2
  - the signal outcomes are NOT independent

- The signal θ_i could be either H or L
  - if $v = 10$ then $\theta_1 = \theta_2 = L$
  - if $v = 20$ then:
    - either $(\theta_1, \theta_2) = (L, H)$
    - or $(\theta_1, \theta_2) = (H, L)$
  - if $v = 30$ then $\theta_1 = \theta_2 = H$
- prior

$$
\begin{array}{ccc|c|c|}
 & & \multicolumn{2}{c}{\theta_2} \\
 & & L & H \\
\cline{3-4}
\theta_1 & L & \frac{1}{4} & \frac{1}{4} \\
\cline{3-4}
 & H & \frac{1}{4} & \frac{1}{4} \\
\cline{3-4}
\end{array}
$$

- posterior: $\Pr(A|B) = \frac{\Pr(AB)}{\Pr(B)} = \frac{\Pr(AB)}{\Pr(BA_1)+\Pr(BA_2)+\cdots}$
  - e.g., upon observing $\theta_1 = L$, then firm 1 expects that the probability of $\theta_2 = L$ is $\Pr(\theta_2 = L|\theta_1 = L) = \frac{\Pr(\theta_1=\theta_2=L)}{\Pr(\theta_1=L)} = \frac{1/4}{\Pr(\theta_1=L;\theta_2=H)+\Pr(\theta_1=L;\theta_2=L)} = \frac{1/4}{1/4+1/4} = \frac{1}{2}$

- If firm 1 observes $\theta_1 = L$, he knows that
  - with probability $\frac{1}{2}$, $\theta_2 = L \Rightarrow v = 10$
  - with probability $\frac{1}{2}$, $\theta_2 = H \Rightarrow v = 20$
  - $E(v_1|\theta_1 = L) = \frac{1}{2} \cdot 10 + \frac{1}{2} \cdot 20 = 15$
- Similarly, upon observing $\theta_1 = H$, $E(v|\theta_1 = H) = \frac{1}{2} \cdot 20 + \frac{1}{2} \cdot 30 = 25$
- In a second-price sealed-bid auction, is it a Bayesian equilibrium for both bidders to submit truthful valuations?
  - Assume that player 2 submit his/her true expected valuation, then consider player 1's best response
- If $\theta_1 = L$, then with prob 0.5 player 2 finds $\theta_2 = L \Rightarrow$ player 2 bids $E(v_2|\theta_2 = L) = 15$, then they win with equal probability
  - In this case, both find L and the true value is 10
- With prob 0.5 player 2 finds $\theta_2 = H \Rightarrow$ player 2 bids $E(v_2|\theta_2 = H) = 25$, then player 1 loses.

- Therefore, if $\theta_1 = L$ and player 1 bids his/her true valuation $E(v_1|\theta_1 = L) = 15$, the expected payoff is

$$\underbrace{\frac{1}{2}\cdot\left[\underbrace{\frac{1}{2}(10-15)}_{\text{equal prob to win}}\right]}_{\text{oil is low}} + \underbrace{\frac{1}{2}\cdot 0}_{\text{oil is high and 1 loses}} = -1.25$$

- Therefore, bidding 15 is unprofitable $\Rightarrow$ bid should be less than 15 $\Rightarrow$ never win
- When one player wins the oil it is because his opponent's bid/signal is low $\Rightarrow$ "bad news"
- Winner's curse: a player wins when his/her signal is the most optimistic