Microeconomic Foundations I:
Choice and Competitive Markets

Student's Guide

Chapter 11: Classic Demand Theory

Summary of the Chapter

This chapter completes the development of the theory of the consumer in parallel with the theory of the firm developed in Chapter 9. The key question, answered (more or less) by the Integrability Theorem (which is never formally stated, hence the ”more or less”) is, When is a (parametric) family of demand functions the Marshallian demand of a utility-maximizing consumer? But the path that takes us to this climax (and the denouement that follows) is long and winding:

1. We begin with *Roy's Identity* and the *Slutsky Equation*, two important "identities" from the theory of consumer demand. Simple derivations are provided, and intuition for the two results are given, relating them to the notion of compensated demand, wherein one tries to isolate the *substitution* and *income* effects of a change in a price to (a) the level of indirect utility (Roy's Identity) and (b) the quantities consumed (the Slutsky Equation).

2. Section 11.2 concerns differentiability of indirect utility, both in prices and in income. Included here is a more robust derivation of Roy's Identity.

3. Section 11.3 does for the indirect utility function what we did last chapter for the expenditure function: Which utility functions give rise to the same indirect utility? (How) Can you invert indirect utility to get the utility function (or, *a* utility function) that generates it? What are necessary and sufficient conditions on an indirect utility to be "legitimate," meaning, generated by a continuous (quasi-concave, nondecreasing, and locally insatiable) utility function?

4. In Section 11.4, the Implicit-Function Theorem is used to generate conditions under which Marshallian demand (if it is single-valued) is (locally) differentiable in prices and income.
5. Section 11.5 provides the climax, the Integrability Theorem. Or, rather, the section discusses the Integrability Theorem, describing what it says (roughly) and how it works (again, roughly), but without giving all the fine mathematical details.
6. The Slutsky Equation (and symmetry of the so-called Slutsky matrix) is used to motivate a discussion of complementary and substitute goods in Section 11.6.
7. Afriat's Theorem, based on revealed preference, gives one set of answers to the question, When do demand data arise from a utility-maximizing consumer? The Integrability Theorem gives a seemingly very different answer to what is, at its core, the same question. We conclude in Section 11.7 by asking, How are these connected? As with the Integrability Theorem, we discuss the connection instead of developing it in all its details.

Solutions to Starred Problems

■ 11.1. It is clear that at the optimal solution, x_1 and x_2 must both be strictly positive, since otherwise $U(x_1, x_2, x_3)$ will be $-\infty$. Their bangs for the buck are $1/x_1$ for commodity one and $3/2x_2$ for commodity 2. The bang for the buck of the third commodity is $1/3(x_3 + 10)$, which is $1/30$ at $x_3 = 0$. Thus consumption of the third commodity only begins when the bangs for the buck of the other two commodities fall to $1/30$, which is where

$$\frac{1}{x_1} = \frac{1}{30} \text{ or } x_1 = 30 \quad \text{and} \quad \frac{3}{2x_2} = \frac{1}{30} \text{ or } x_2 = 45,$$

at a total cost of 120. Hence:

- For $y \leq 120$, only commodities 1 and 2 are purchased, and equal bangs for the buck means

  $$x_1 = \frac{2x_2}{3},$$

  hence the budget equation is

  $$x_1 + 2x_2 = \frac{2x_2}{3} + 2x_2 = \frac{8x_2}{3} = y,$$

  hence $x_2 = 3y/8$ and $x_1 = y/4$. This gives indirect utility $\ln(y/4)+3\ln(3y/8)+\ln(10) = \ln(y)-\ln(4)+3\ln(y)+3\ln(3/8)+\ln(10) = 4\ln(y)+k$, for k some constant, and $\partial \nu/\partial y =$

$4/y$. The multiplier λ is, of course, the bang for the buck of the two commodities that are consumed, which is $1/(y/4) = 4/y$ for commodity 1 and $3/[2(3y/8)] = 4/y$ for commodity 2.

- For $y > 120$, all three commodities are purchased, and equal bangs for the buck gives

$$x_1 = \frac{2x_2}{3} = 3(x_3 + 10),$$

hence the budget equation is

$$x_1 + 2x_2 + 3x_3 = 3(x_3 + 10) + 9(x_3 + 10) + 3x_3 = 15x_3 + 120 = y,$$

or

$$x_3 = \frac{y - 120}{15}, x_1 = \frac{y + 30}{5}, \text{ and } x_2 = \frac{3(y + 30)}{10}.$$

Indirect utility is

$$\nu((1, 2, 3), y) = \ln(y + 30) + 3\ln(y + 30) + \ln(y + 30) + K,$$

where I've factored out a constant K, or $5\ln(y + 30) + K$, so that $\partial\nu/\partial y = 5/(y + 30)$. The multiplier is the bang for the buck of the three commodities: For commodity 1, this is $1/x_1 = 1/[(y + 30)/5] = 5/(y + 30)$. For commodity 2, it is $3/(2x_2) = 3/[2(3(y + 30)/10)] = 5/(y + 30)$. And for commodity 3, it is $1/[3(x_3 + 10) = 1/[3(y + 30)/15] = 5/(y + 30)$.

■ 11.4. Suppose that u is continuous and locally insatiable. By Corollary 11.6, the indirect utility function generated by u is identical to the indirect utility function generated by "its" nondecreasing and quasi-concave "equivalent," $\hat{u}$. Since this half of the proof involves demonstrating properties of ν, we can assume w.l.o.g. that u is also nondecreasing and quasi-concave. Fix $x^0 \in R^k_+$, and suppose $v \geq 0$ and $\epsilon > 0$ satisfy $\nu(p, p \cdot x^0) \geq v + \epsilon$ for all $p \in R^k_{++}$. Then according to Proposition 11.7, $u(x^0) \geq v + \epsilon$. Because u is continuous, there exists $\delta < 1$ (but close to 1) such that $u(\delta x^0) \geq v + \epsilon/2$. But then, for any strictly positive p, since δx^0 is feasible for the CP at prices p and income $y = p \cdot \delta x^0 = \delta p \cdot x^0$, we have $\nu(p, \delta p \cdot x^0) \geq u(\delta x^0) \geq v + \epsilon/2 > v$.

Conversely, suppose that ν has the property (11.7), and we construct u from ν via (11.5) and (11.6). We already know that u is upper semi-continuous, so to verify that u is continuous, we need only show that it is lower semi-continuous. Suppose, then, by way of contradiction that for some sequence $\{x^n\}$ with limit x^0, $\lim_n u(x^n) < u(x^0)$. Without loss of generality, we can assume that $u(x^n)$ is nondecreasing in n. Let $v' =$

$\lim_n u(x^n)$, and let $\epsilon = (u(x^0) - v')/2 > 0$ and $v = v' + \epsilon$. Then it is clear that $\nu(p, p \cdot x^0) \geq u(x^0) = v + \epsilon$ for all strictly positive p, so by property (11.7), for some $\delta < 1$, $\nu(p, \delta p \cdot x^0) = \nu(p, p \cdot \delta x^0) > v$ for all p. By the same argument as used in the proof of Proposition 10.17, for large-enough n, $x^n \geq \delta x^0$. And since ν is nondecreasing in its last argument, this implies that for all large-enough n, $\nu(p, p \cdot x^n) > v$ for all p. But then looking at (11.5) and (11.6), this implies that for all large n, $u(x^n) > v$, which is the desired contradiction.

■ 11.6. There are (at least) two ways to attack this problem and, despite what the problem statement says, in one of them you don't (necessarily) have to worry about first-order conditions for the EMP.

Of course, we must know first of all that Hicksian demand is a function and not a correspondence, in which case Hicksian demand will be the "same" as Marshallian demand. Writing $h(p, v)$ for Hicksian demand and $d(p, y)$ for Marshallian demand, this gives us the first line of attack: We have

$$h(p, v) = d(p, e(p, v)),$$

and so if d is continuously differentiable and e is continuously differentiable (in both p and v), then the chain-rule establishes the differentiability of h. Proposition 11.10 gives sufficient conditions for differentiability of d, and we (almost!) know what it takes for e to be differentiable; I add the parenthetical *almost!* because we only really proved differentiability in p. But following our proof of the differentiability of ν in y, it isn't hard to obtain differentiability of e in v. (If you did Problem 2, you will have done this.) So that's one line of attack.

Rather than finish that line of attack, I'll take a direct approach, mimicking the proof of Proposition 11.10. The first step is to discuss the solution of the EMP using calculus. We need to assume that u is (at least) continuously differentiable. The problem is to

$$\text{minimize } p \cdot x, \quad \text{subject to } u(x) \geq v, x \geq 0.$$

We will want to know that the first-order/complementary-slackness conditions are necessary at any optimal solution, so we need to know that, at any solution, the constraint qualification holds. We know that if u is differentiable, it is continuous, and we know that if u is continuous, then the constraint $u(x) \geq v$ must hold with equality. (See Proposition 10.2(d).) So that constraint is binding. Now if $v = u(0)$, then we are stuck: The solution is $x = 0$, all the nonnegativity constraints bind, and the constraint qualification fails miserably. So we must restrict attention to $v > u(0)$. (You may not recall, but back in Chapter 3, when we said that the constaint qualification held for the CP, it was only for the case $y > 0$.)

But if $v > u(0)$, then at any solution, at least one x_i must be strictly positive. (That is, for any x that meets the constraint $u(x) \geq v$, some $x_i > 0$.) This means that the constraint qualification will hold as long as $\partial u/\partial x_i > 0$ for any of the is that have $x_i > 0$. To ensure this is so, we need to add an assumption (it isn't true in general), and I'll simply assume that u is strictly increasing and, moreover, has strictly positive partial derivatives at all nonzero arguments. Then the constraint qualification holds (if you aren't sure about this, go to Appendix 5 to recall what this means and verify that it is so), and the first-order/complementary-slackness conditions are indeed necessary.

And what are they? They are

$$\begin{aligned}
&p_i - \hat{\lambda}\frac{\partial u}{\partial x_i} - \hat{\mu}_i = 0,\ i = 1, \ldots, k \\
&\hat{\lambda} \geq 0, u(x) \geq v, \hat{\lambda}(u(x) - v) = 0, \\
&\hat{\mu}_i \geq 0, x_i \geq 0, \hat{\mu}_i x_i = 0,\ i = 1, \ldots, k
\end{aligned}.$$

We can rewrite this by eliminating the multipliers on the nonnegativity constraints and noting that we know the constraint $u(x) \geq v$ must hold with equality; we have

$$\begin{aligned}
&u(x) = v; \quad x_i \geq 0, i = 1, \ldots, k; \quad \hat{\lambda} \geq 0 \\
&p_i = \hat{\lambda}\frac{\partial u}{\partial x_i} \text{ if } x_i > 0; \quad p_i \geq \hat{\lambda}\frac{\partial u}{\partial x_i} \text{ if } x_i = 0.
\end{aligned}$$

It is worth staring at these optimality conditions before moving on to the rest of the problem: We have the utility and feasibility constraints in the first line (and the constraint on the mulitplier; given our assumption on marginal utilities, this isn't much of a constraint given what must be true on the second line). As for the second line, let $\lambda = 1/\hat{\lambda}$, and rewrite the equality and inequality with λ on the left-hand side and $(\partial u/\partial x_i)/p_i$ on the right-hand side. (We know that $\hat{\lambda} > 0$ because some good is consumed at a positive level, and its price and marginal utility are both strictly positive.) This should look very familiar: The bangs-for-the-buck of goods that are strictly positive must all be equal, and they must (all, equally) exceed the bang-for-the-buck of any good that is up against the nonnegativity constraint. Where have we seen that before? In the first-order/complementary-slackness conditions of the CP, of course. This has the same first-order conditions as the CP, which in retrospect shouldn't come as a surprise: We know that solutions to the CP are solutions to the EMP (and vice versa) for the appropriate change of variables. So the marginal conditions on the commodities should be the same.

And, just to do a bit more on this, go back to the equalities and inequalities in line 2, and keep $\hat{\lambda}$ on the right-hand side, putting $p_i/(\partial u/\partial x_i)$ on the left-hand side. To give $p_i/(\partial u/\partial x_i)$ a name, it is the *buck-for-the-bang*, or how much expenditure can be saved per unit of utility lost, on the margin, by decreasing the consumption of good i. The first-order conditions are that, for goods that are at strictly positive levels, their bucks-for-the-bang must all be equal, and they must be no greater than the bucks-for-the-bang of goods that are up against the nonnegativity constraint. Logic similar to the logic of Chapter 3 applies: If the buck-for-the-bang of good i is less than that of good j, the thing to do is to increase the consumption of good i while decreasing that of good j in a ratio that keeps utility constant; this should lead to less expenditure. Which works, unless you can't decrease the consumption of good j, because it is at level 0, from which no decreases are feasible.

Now to finish the assigned problem. We're going to mimic the proof of Proposition 11.10 and apply the Implicit-Function Theorem to the equalities in the first-order, complementary slackness conditions, in a neighborhood in which the EMP has unique solutions and the set of binding constraints does not change. We know that the utility-level constraint will be binding, and some subset of the nonnegativity constraints will bind—for commodity indices i that are strictly positive, we have the equations

$$p_i = \hat{\lambda}\frac{\partial u}{\partial x_i}.$$

Renumber the variables so that these indices are 1 through n: We are looking at the function

$$G(p, v, \hat{\lambda}, x) = \begin{cases} u(x) - v \\ p_1 - \hat{\lambda}u_1(x) \\ \quad \cdots \\ p_n - \hat{\lambda}u_n(x) \end{cases}$$

where u_i is short hand for $\partial u/\partial x_i$. Solutions of the first-order/complementary-slackness conditions (locally) are solutions to $G(p, v, \hat{\lambda}, x) = 0$, and we want to implicitly define functions $\hat{\lambda}(p, v)$ and $x(p, v)$ that satisfy

$$G(p, v, \hat{\lambda}(p, v), x(p, v)) = 0.$$

(In the proof of Proposition 11.10, I put hats on the implicitly defined functions, which is unfortunate given my use of a hat here in $\hat{\lambda}$. But I trust that if you are sophisticated enough to be following this discussion, this won't bother you.)

We'll need that u is twice-continuously differentiable to apply the Implicit-Function Theorem, of course, and we need to know that the matrix of partial derivatives in the eliminated variables in nonsingular. The partial derivative in $\hat{\lambda}$ is $(0, u_1, \ldots, u_n)$ and in x_i is $u_i, \hat{\lambda}u_{1i}, \ldots, \hat{\lambda}u_{ni}$. Multiply all the rows except the first by $1/\hat{\lambda}$ and then multiply the first column by $\hat{\lambda}$, and out pops the same bordered Hessian as in Section 11.4; non-singularity of the bordered Hessian is just what we need here, precisely as in that section.

To sum up, we get differentiability of Hicksian demand in neighborhoods of points in which the binding constraints in the EMP do not change, if u is twice-continuously differentiable, u is strictly increasing (and with strictly positive first derivatives), and the bordered Hessian from Proposition 11.10 is non-singular.