Intermediate Microeconomics

Production Theory

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Overview

- Production Technology
- Cost Minimization (成本最小化)
  - Conditional Input Demand (条件要素需求)
- Profit Maximization (利润最大化)
  - Input Demand (要素需求)

Production Function (生产函数)

- Assume that a firm uses two types of inputs, capital k and labor l, to produce q units of output according to the production function
$$q = f(k, l)$$
- Marginal product (边际产品) of a particular input:
$$\begin{array}{ll} \text{marginal product of capital} & MP_k = f'_k \\ \text{marginal product of labor} & MP_l = f'_l \end{array}$$
- Diminishing marginal productivity (边际产出递减):
$$\frac{\partial MP_k}{\partial k} = f''_{kk} < 0$$
$$\frac{\partial MP_l}{\partial l} = f''_{ll} < 0$$

Isoquant (等产量线) & RTS (边际技术替代率)

- An isoquant shows those combinations of k and l that can produce a given level of output:

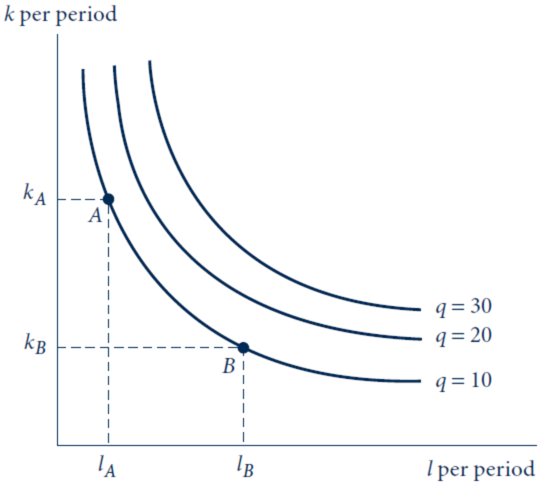
$$f(k,l) = q_0$$

  How to plot an isoquant: solve k as a function of l.

- The marginal rate of technical substitution (RTS) shows the rate at which labor can be substituted for capital while holding output constant along an isoquant:

$$RTS = -\left.\frac{dk}{dl}\right|_{q=q_0}$$

Isoquant (等产量线)



Diminishing RTS

- Fixing a particular output level q_0, total differentiate $f(k,l)=q_0$ gives

$$f'_k dk + f'_l dl = dq_0 = 0 \Leftrightarrow \frac{f'_l}{f'_k} = -\frac{dk}{dl} = RTS$$

- RTS is the ratio of the inputs' marginal productivities.
- Diminishing RTS: Fixing q_0, in an isoquant, $k(l)$ is a function of l. Then,

$$\begin{aligned}\frac{dRTS}{dl} = \frac{d\left(\frac{f'_l(k(l),l)}{f'_k(k(l),l)}\right)}{dl} &= \frac{f'_k\left(f''_{lk}\frac{dk}{dl}+f''_{ll}\right) - f'_l\left(f''_{kk}\frac{dk}{dl}+f''_{kl}\right)}{{f'_k}^2} \\ &= \frac{f'_k\left(-f''_{kl}\frac{f'_l}{f'_k}+f''_{ll}\right) - f'_l\left(-f''_{kk}\frac{f'_l}{f'_k}+f''_{kl}\right)}{{f'_k}^2} \\ &= \frac{{f'_l}^2 f''_{ll} - 2f'_k f'_l f''_{kl} + {f'_l}^2 f''_{kk}}{{f'_k}^3}\end{aligned}$$

- Diminishing RTS $\Leftrightarrow$ quasi-concavity of $f(k,l)$.

Example: Cobb-Douglas

$f(k,l) = k^a l^{1-a}$, where $0 < a < 1$

- MPK: $f'_k = a k^{a-1} l^{1-a}$
  - Diminishing MPK: $f''_{kk} = a(a-1)k^{a-2}l^{1-a} < 0$.
- MPL: $f'_l = (1-a)k^a l^{-a}$
  - Diminishing MPL: $f''_{ll} = -a(1-a)k^a l^{-a-1} < 0$.
- Isoquant: $q_0 = k^a l^{1-a} \Leftrightarrow k(l) = q_0^{\frac{1}{a}} l^{\frac{a-1}{a}}$
- RTS: $\frac{f'_l}{f'_k} = \frac{1-a}{a}\frac{k}{l} = \frac{1-a}{a} q_0^{\frac{1}{a}} l^{-\frac{1}{a}}$
- Diminishing RTS: $\frac{1-a}{a} q^{\frac{1}{a}} l^{-\frac{1}{a}}$ is decreasing in l.

Elasticity of Substitution (替代弹性)

For the production function $q = f(k, l)$, the elasticity of substitution measures the proportionate change in $\frac{k}{l}$ relative to the proportionate change in the RTS along an isoquant:

$$\sigma = \frac{\frac{\Delta(k/l)}{k/l}}{\frac{\Delta RTS}{RTS}} = \frac{d(k/l)}{dRTS}\frac{RTS}{k/l}.$$

Because $\frac{d\ln(x)}{dx} = \frac{1}{x}$, the elasticity of substitution can be expressed as

$$\sigma = \frac{\frac{d(k/l)}{k/l}}{\frac{dRTS}{RTS}} = \frac{d\ln(k/l)}{d\ln(RTS)}$$

Example: $f(k, l) = k^a l^{1-a}$, then $RTS = \frac{1-a}{a}\frac{k}{l}$. Then $\frac{k}{l} = \frac{a}{1-a}RTS \Rightarrow \ln\left(\frac{k}{l}\right) = \ln\frac{a}{1-a} + \ln(RTS) \Rightarrow d\ln\left(\frac{k}{l}\right) = d\ln(RTS) \Rightarrow \sigma = 1$.

CES Technology (替代弹性为常数的生产函数)

CES: constant elasticity of substitution.

$$q = f(k,l) = (k^\rho + l^\rho)^{\frac{\gamma}{\rho}}$$

$$RTS = \frac{f_l'}{f_k'} = \frac{\frac{\gamma}{\rho}(k^\rho + l^\rho)^{\frac{\gamma}{\rho}-1}\rho l^{\rho-1}}{\frac{\gamma}{\rho}(k^\rho + l^\rho)^{\frac{\gamma}{\rho}-1}\rho k^{\rho-1}} = \left(\frac{k}{l}\right)^{1-\rho}$$

$$\Rightarrow \ln(RTS) = (1-\rho)\ln\left(\frac{k}{l}\right)$$

$$\Rightarrow \sigma = \frac{d\ln(k/l)}{d\ln(RTS)} = \frac{1}{1-\rho}$$

Returns to Scale (规模报酬)

How output responds to increases in all inputs together? Given $q = f(k,l)$, if all inputs are multiplied by the same positive constant t, then we classify the returns to scale of the production function by

Effect on output	Returns to scale
$f(tk,tl) = tf(k,l)$	constant returns to scale
$f(tk,tl) < tf(k,l)$	decreasing returns to scale
$f(tk,tl) > tf(k,l)$	increasing returns to scale

For constant returns to scale, $f(k,l)$ is homogeneous of degree 1 (一次齐次):

$$f(tk,tl) = t^{\text{degree}} f(k,l) = t^1 f(k,l)$$

If a function is homogeneous of degree d: i.e.,

- $f(tk, tl) = t^d f(k, l)$

then its derivative is homogeneous of degree $d - 1$:

- $f_k(tk, tl) = t^{d-1} f_k(k, l)$

Proof: Differentiate the equation $f(tk, tl) = t^d f(k, l)$ with respect to k from both sides:

$$t f_k'(tk, tl) = t^d f_k'(k, l)$$
$$\Rightarrow f_k'(tk, tl) = t^{d-1} f_k'(k, l)$$

Therefore, if a production function exhibits constant returns to scale, the marginal productivity is homogeneous of degree 0.

Returns to Scale & Homogeneity of Partial Derivatives:

- Cobb-Douglas: $q = f(k,l) = k^a l^b$.
  $f(tk,tl) = (tk)^a (tl)^b = t^{a+b} k^a l^b = t^{a+b} f(k,l)$.
  - If $a+b=1$: constant returns to scale;
  - If $a+b>1$: increasing returns to scale;
  - If $a+b<1$: decreasing returns to scale.
  - Marginal product of capital: $f'_k(k,l) = ak^{a-1}l^b$; if k and l becomes tk and tl, then
    $f'_k(tk,tl) = a(tk)^{a-1}(tl)^b = t^{a+b-1}ak^{a-1}l^b = t^{a+b-1}f'_k(k,l)$.
- CES: $q = (k^\rho + l^\rho)^{\frac{\gamma}{\rho}}$.
  $f(tk,tl) = (t^\rho k^\rho + t^\rho l^\rho)^{\frac{\gamma}{\rho}} = [t^\rho(k^\rho + l^\rho)]^{\frac{\gamma}{\rho}} = t^\gamma f(k,l)$
  - $\gamma = 1$: CRS
  - $\gamma > 1$: IRS
  - $\gamma < 1$: DRS
  - MPK: $f'_k(k,l) = \frac{\gamma}{\rho}(k^\rho + l^\rho)^{\frac{\gamma}{\rho}-1}\rho k^{\rho-1} = \gamma\left(k^\rho + l^\rho\right)^{\frac{\gamma}{\rho}-1}k^{\rho-1}$.
    Verify $f'_k(tk,tl) = \gamma\left[(tk)^{rho} + (tl)^\rho\right]^{\frac{\gamma}{\rho}-1}(tk)^{\rho-1} = t^{\gamma-1}f'_k(k,l)$.

Cost Minimization

Assume that the input market is perfectly competitive: the per unit price of capital r, and the per unit price of labor w, are given.

- r: per unit cost of capital (e.g., interest rate);
- w: per unit cost of labor (e.g., wage)

Consider the following optimization problem: a firm with production technology $q = f(k,l)$ is going to minimize its total cost $rk + wl$ by producing q units of output. That is

$$\min_{k,l} rk + wl$$

$$s.t.\ q = f(k,l)$$

The Lagrangian for this constrained optimization is

$$\mathcal{L} = rk + wl + \lambda(q - f(k,l))$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial k} &= 0 \Rightarrow r = \lambda f'_k \\
\frac{\partial \mathcal{L}}{\partial l} &= 0 \Rightarrow w = \lambda f'_l \quad \Rightarrow \frac{f'_l}{f'_k} = RTS = \frac{w}{r} \\
\frac{\partial \mathcal{L}}{\partial \lambda} &= 0 \Rightarrow q = f(k,l)
\end{aligned}$$

- Recall that in consumer theory, expenditure minimization (EMP) solves

$$\min_{x,y} p_x x + p_y y$$
$$s.t.\ U(x,y) = u$$

  - The optimal choices (Hicksian demand) are $h_x(p_x, p_y, u)$ and $h_y(p_x, p_y, u)$.
  - Plug the solution h_x, h_y into the objective, the minimized expenditure $E(p_x, p_y, u)$ is a function of prices and u.
- Similarly, when a firm minimizes cost evaluated at a given level of output q:

$$\min_{k,l} rk + wl$$
$$s.t.\ f(k,l) = q$$

  - The optimal choices $k^*(r,w,q)$ and $l^*(r,w,q)$, as functions of input prices and output q, are called **contingent/conditional input demand** (**条件要素需求函数**).
  - Plug the solutoins k^* and l^* into the objective, we get **cost function** $C(r,w,q)$ (**成本函数**).

Example: Cobb-Douglas

The production function is $f(k,l) = k^a l^b$. The firm solves

$$\min_{k,l} rk + wl, \quad s.t. \ k^a l^b = q$$

- From the constraint, $l^b = qk^{-a} \Rightarrow l = k^{-\frac{a}{b}} q^{\frac{1}{b}}$.
- The objective becomes $rk + wk^{-\frac{a}{b}} q^{\frac{1}{b}}$. First order condition with respect to k gives the **input demand (conditional on q)** for k

$$r - \frac{a}{b} w k^{-\frac{a}{b}-1} q^{\frac{1}{b}} = 0 \Rightarrow k^*(r,w,q) = \left(\frac{a}{b}\frac{w}{r}\right)^{\frac{b}{a+b}} q^{\frac{1}{a+b}}$$

- Then the contingent input demand for l is $l^*(r,w,q) = \left(\frac{a}{b}\frac{w}{r}\right)^{-\frac{a}{a+b}} q^{\frac{1}{a+b}}$
- The **cost function** is given by $C(r,w,q) = rk^*(r,w,q) + wl^*(r,w,q) = \left[\left(\frac{a}{b}\right)^{\frac{b}{a+b}} + \left(\frac{a}{b}\right)^{\frac{-a}{a+b}}\right] r^{\frac{a}{a+b}} w^{\frac{b}{a+b}} q^{\frac{1}{a+b}}$

Example: CES

The production function is $(k^\rho + l^\rho)^{\frac{1}{\rho}}$. The firm minimizes $rk + wl$ under the constraint that $(k^\rho + l^\rho)^{\frac{1}{\rho}} = q \Leftrightarrow k^\rho + l^\rho = q^\rho$. The Lagrangian is $\mathcal{L} = rk + wl + \lambda(q^\rho - k^\rho - l^\rho)$.

- FOC wrt k gives $r = \lambda\rho k^{\rho-1} \Rightarrow r^{\frac{1}{\rho-1}} = (\lambda\rho)^{\frac{1}{\rho-1}} k \Rightarrow k^\rho = r^{\frac{\rho}{\rho-1}} (\lambda\rho)^{\frac{\rho}{1-\rho}}$;
- FOC wrt l gives $w = \lambda\rho l^{\rho-1} \Rightarrow l^\rho = w^{\frac{\rho}{\rho-1}} (\lambda\rho)^{\frac{\rho}{1-\rho}}$.
- The binding constraint implies that $q^\rho = (\lambda\rho)^{\frac{\rho}{1-\rho}} \left[r^{\frac{\rho}{\rho-1}} + w^{\frac{\rho}{\rho-1}}\right] \Rightarrow (\lambda\rho)^{\frac{\rho}{1-\rho}} = \frac{q^\rho}{r^{\frac{\rho}{\rho-1}} + w^{\frac{\rho}{\rho-1}}}$
- The conditional input demand for k is $k^*(r,w,q) = r^{\frac{1}{\rho-1}} q \left[r^{\frac{\rho}{\rho-1}} + w^{\frac{\rho}{\rho-1}}\right]^{-\frac{1}{\rho}}$ and the conditional input demand for l is $l^*(r,w,q) = w^{\frac{1}{\rho-1}} q \left[r^{\frac{\rho}{\rho-1}} + w^{\frac{\rho}{\rho-1}}\right]^{-\frac{1}{\rho}}$
- The cost function is $C(r,w,q) = rk^* + wl^* = \left[r^{\frac{\rho}{\rho-1}} + w^{\frac{\rho}{\rho-1}}\right]^{\frac{\rho-1}{\rho}} q$

Second-Order Conditions for Cost Minimization

- The first-order conditions of the Lagrangian is the necessary conditions. To ensure the minimum, we need second-order sufficiency.
- Recall that, the bordered Hessian for the Lagrangian $\mathcal{L} = rk + wl + \lambda(q - f(k,l))$ is

$$H_b = \begin{bmatrix} 0 & \mathcal{L}''_{\lambda k} & \mathcal{L}''_{\lambda l} \\ \mathcal{L}''_{\lambda k} & \mathcal{L}''_{kk} & \mathcal{L}''_{kl} \\ \mathcal{L}''_{\lambda l} & \mathcal{L}''_{lk} & \mathcal{L}''_{ll} \end{bmatrix} = \begin{bmatrix} 0 & -f'_k & -f'_l \\ -f'_k & -\lambda f''_{kk} & -\lambda f''_{kl} \\ -f'_l & -\lambda f''_{lk} & -\lambda f''_{ll} \end{bmatrix}$$

  - SOSC for maximum: $(-1)H_b$ is negative definite: the signs of the leading principal minors of H_b exhibit $- + - +$ starting from the second minor.
  - SOSC for minimum: $(-1)H_b$ is positive definite: all the leading principal minors of H_b is negative.

Input Price Changes and Conditional Input Demand

- At optimal choices, k^* and l^* depend on r, w and q.
- We would like to know: how will a change in r affect the choices on k^* and l^*.
- We need to obtain $\frac{\partial k^*}{\partial r}$ and $\frac{\partial k^*}{\partial w}$.
- The optimal choices k^* and l^*, are solved from the first-order conditions of the Lagrangian:

$$\begin{cases} \mathcal{L}'_\lambda = 0 \Rightarrow f\left(k^*, l^*\right) = q \\ \mathcal{L}'_k = 0 \Rightarrow r - \lambda f'_k(k^*, l^*) = 0 \\ \mathcal{L}'_l = 0 \Rightarrow w - \lambda f'_l(k^*, l^*) = 0 \end{cases}$$

- The solutions $k^*(r)$ and $l^*(r)$ are determined by the above three equations. Hence, we differentiate the above three equations with respect to r (the solution for λ is also a function of the same set of parameters):

$$\begin{cases} f'_k \frac{\partial k^*}{\partial r} + f'_l \frac{\partial l^*}{\partial r} = 0 \\ 1 - \lambda \left[f''_{kk} \frac{\partial k^*}{\partial r} + f''_{kl} \frac{\partial l^*}{\partial r} \right] - f'_k \frac{\partial \lambda}{\partial r} = 0 \\ 0 - \lambda \left[f''_{lk} \frac{\partial k^*}{\partial r} + f''_{ll} \frac{\partial l^*}{\partial r} \right] - f'_l \frac{\partial \lambda}{\partial r} = 0 \end{cases}$$

- Three unknowns: $\frac{\partial \lambda}{\partial r}$, $\frac{\partial k^*}{\partial r}$ and $\frac{\partial l^*}{\partial r}$ are solved by the three equations:

$$\begin{cases} f_k' \frac{\partial k^*}{\partial r} + f_l' \frac{\partial l^*}{\partial r} = 0 \\ 1 - \lambda \left[f_{kk}'' \frac{\partial k^*}{\partial r} + f_{kl}'' \frac{\partial l^*}{\partial r} \right] - f_k' \frac{\partial \lambda}{\partial r} = 0 \\ 0 - \lambda \left[f_{lk}'' \frac{\partial k^*}{\partial r} + f_{ll}'' \frac{\partial l^*}{\partial r} \right] - f_l' \frac{\partial \lambda}{\partial r} = 0 \end{cases}$$

- We can write the equations in the matrix form:

$$\underbrace{\begin{bmatrix} 0 & -f_k' & -f_l' \\ -f_k' & -\lambda f_{kk}'' & -\lambda f_{kl}'' \\ -f_l' & -\lambda f_{kl}'' & -\lambda f_{ll}'' \end{bmatrix}}_{=H_b} \begin{bmatrix} \frac{\partial \lambda}{\partial r} \\ \frac{\partial k^*}{\partial r} \\ \frac{\partial l^*}{\partial r} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

- Using the Cramer's rule, we can solve $\frac{\partial k^*}{\partial r}$

$$\frac{\partial k^*}{\partial r} = \frac{\begin{vmatrix} 0 & 0 & -f_l' \\ -f_k' & -1 & -\lambda f_{kl}'' \\ -f_l' & 0 & -\lambda f_{ll}'' \end{vmatrix}}{\det(H_b)} = \frac{{f_l'}^2}{\det(H_b)}$$

- For the cost minimization problem, $(-1)H_b$ is positive semidefinite, which implies that all the leading principal minors of H_b are negative, including $\det(H_b) < 0$. Hence, $\frac{\partial k^*}{\partial r} < 0$.

$$\underbrace{\begin{bmatrix} 0 & -f'_k & -f'_l \\ -f'_k & -\lambda f''_{kk} & -\lambda f''_{kl} \\ -f'_l & -\lambda f''_{kl} & -\lambda f''_{ll} \end{bmatrix}}_{=H_b} \begin{bmatrix} \frac{\partial \lambda}{\partial r} \\ \frac{\partial k^*}{\partial r} \\ \frac{\partial l^*}{\partial r} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

Similarly, using the Cramer's rule, we can solve $\frac{\partial l^*}{\partial r}$

$$\frac{\partial l^*}{\partial r} = \frac{\begin{vmatrix} 0 & -f'_k & 0 \\ -f'_k & -\lambda f''_{kk} & -1 \\ -f'_l & -\lambda f''_{kl} & 0 \end{vmatrix}}{\det(H_b)} = \frac{(-1)^5(-1)\left[-(-f'_k)(-f'_l)\right]}{\det(H_b)} > 0$$

Practice: show the cross-price effect, i.e., $\frac{\partial k^*}{\partial w} = \frac{\partial l^*}{\partial r}$

Cost Functions (成本函数)

- The cost function is given by (like expenditure function $E(p_x, p_y, u)$)
  $C(r, w, q) = rk^*(r, w, q) + wl^*(r, w, q)$
- Short run v.s. long run:
  - short run: some factors are fixed, e.g., unable to adjust capital $k = \bar{k}$;
  - long run: all inputs can be chosen.
- In short run: $C(r, w, q) = \underbrace{wl^*(r, w, q)}_{\text{variable: 可变}} + \underbrace{r\bar{k}}_{\text{fixed: 固定}}$
- Definitions:

$STC = wl^*(r, w, q) + r\bar{k}$	short-run total cost: 短期总成本
$SAC = STC/q$	short-run average cost: 短期平均成本
$SAVC = \frac{wl^*(r,w,q)}{q}$	short-run average variable cost: 短期平均可变成本
$SAFC = \frac{r\bar{k}}{q}$	short-run average fixed cost: 短期平均固定成本
$SMC = \frac{\partial STC}{\partial q}$	short-run marginal cost: 短期边际成本

Example: Short-run Cobb-Douglas Cost Function

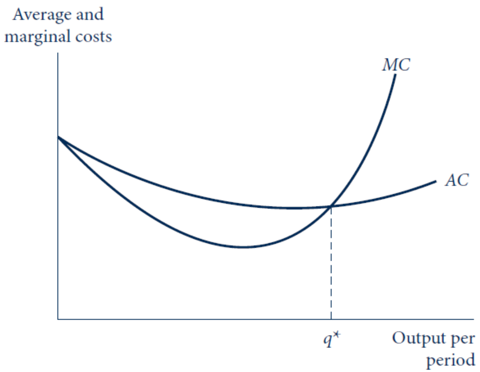
- Assume that in the short run, k is fixed at $\bar{k}$. Then

$$\min_{l} wl + r\bar{k}, \qquad s.t. \;\; l^a \bar{k}^{1-a} = q$$

- Since k is fixed, hence the the optimal labor choice (conditional on output q) is simply solved by the constraint: $l = q^{\frac{1}{a}} \bar{k}^{\frac{a-1}{a}}$.

$$\begin{aligned}
&C(r,w,q) = wq^{\frac{1}{a}}\bar{k}^{\frac{a-1}{a}} + r\bar{k} \\
&SAC = wq^{\frac{1}{a}-1}\bar{k}^{\frac{a-1}{a}} + \frac{r\bar{k}}{q} \\
&SAVC = wq^{\frac{1}{a}-1}\bar{k}^{\frac{a-1}{a}} \\
&SAFC = \frac{r\bar{k}}{q} \\
&SMC = \frac{\partial C}{\partial q} = \frac{w}{a}q^{\frac{1}{a}-1}\bar{k}^{\frac{a-1}{a}}
\end{aligned}$$

The Shape of Marginal Cost (MC) and Average Cost (AC)



- MC and the U-shaped AC intersect at the lowest point of AC
- $AC = \frac{C(q)}{q}$, $\frac{dAC}{dq} = \frac{qC'(q)-C(q)}{q^2} = \frac{MC}{q} - \frac{AC}{q}$. Evaluated at $\frac{dAC}{dq}\Big|_{q=q^*} = 0$, $MC(q^*) = AC(q^*)$.

Properties of Cost Function

More generally, let $\boldsymbol{p} = \{r, w, ...\}$ be the vector of input prices, $\boldsymbol{x} = [k, l, ...]^T$ be the vector of inputs (conditional demand), q be quantity of output. The cost function $C(\boldsymbol{p}, q) = \boldsymbol{p}\boldsymbol{x} = rk + wl + \cdots$ exhibits the following properties

Proposition (Properties of Cost Function)

1. $C(\boldsymbol{p}, q)$ *is non-decreasing in* $\boldsymbol{p}$*;*
2. $C(\boldsymbol{p}, q)$ *is homogeneous of degree 1 in* $\boldsymbol{p}$*;*
3. $C(\boldsymbol{p}, q)$ *is concave in* $\boldsymbol{p}$*.*

Monotonicity of Cost Function

We need to show that for $\boldsymbol{p}' > \boldsymbol{p}$, then $C(\boldsymbol{p}', q) > C(\boldsymbol{p}, q)$.

- Let $\boldsymbol{x}^*$ and $\boldsymbol{x}^{*\prime}$ be the cost-minimizing bundles associated with price levels $\boldsymbol{p}$ and $\boldsymbol{p}'$, respectively.
- If $\boldsymbol{x}^*$ is the solution for $\min_{\boldsymbol{x}} \boldsymbol{p}\boldsymbol{x}$, then $\boldsymbol{x}^{*\prime}$ is not optimal at price $\boldsymbol{p}$, i.e., $\boldsymbol{p}\boldsymbol{x}^* \leq \boldsymbol{p}\boldsymbol{x}^{*\prime}$.
- Similarly, if $\boldsymbol{x}^{*\prime}$ is the solution for $\min_{\boldsymbol{x}} \boldsymbol{p}'\boldsymbol{x}$, then for $\boldsymbol{p} < \boldsymbol{p}'$ and a fixed $\boldsymbol{x}^{*\prime}$, $\boldsymbol{p}\boldsymbol{x}^{*\prime} < \boldsymbol{p}'\boldsymbol{x}^{*\prime}$ holds.
- Combining $\boldsymbol{p}\boldsymbol{x}^* \leq \boldsymbol{p}\boldsymbol{x}^{*\prime}$ and $\boldsymbol{p}\boldsymbol{x}^{*\prime} < \boldsymbol{p}'\boldsymbol{x}^{*\prime}$, it implies $\boldsymbol{p}\boldsymbol{x}^* < \boldsymbol{p}'\boldsymbol{x}^{*\prime}$, where the left-hand side is cost function at price $\boldsymbol{p}$ and the right-hand side is the cost function at price $\boldsymbol{p}'$, i.e., $C(\boldsymbol{p}, q) < C(\boldsymbol{p}', q)$.

Homogeneity of Cost Function

We need to show that $C(t\boldsymbol{p}, q) = tC(\boldsymbol{p}, q)$.

- Claim: if $\boldsymbol{x}^*$ is the solution evaluated at price $\boldsymbol{p}$, then $\boldsymbol{x}^*$ is also the solution evaluated at price $t\boldsymbol{p}$.
- Suppose $\boldsymbol{x}^*$ is not the solution at price $t\boldsymbol{p}$: then let $\boldsymbol{x}^{*\prime}$ be the solution evaluated at price $t\boldsymbol{p}$. Hence, $\boldsymbol{x}^*$ does not minimize cost: $t\boldsymbol{p}\boldsymbol{x}^{*\prime} < t\boldsymbol{p}\boldsymbol{x}^*$.
- $t\boldsymbol{p}\boldsymbol{x}^{*\prime} < t\boldsymbol{p}\boldsymbol{x}^* \Rightarrow \boldsymbol{p}\boldsymbol{x}^{*\prime} < \boldsymbol{p}\boldsymbol{x}^*$, which contradicts with the fact that $\boldsymbol{x}^*$ is the cost-minimizing choice at price $\boldsymbol{p}$.

Concavity of Cost Function

For two different price levels $\boldsymbol{p}$ and $\boldsymbol{p}'$, we need to show that $C(\lambda \boldsymbol{p} + (1-\lambda)\boldsymbol{p}', q) \geq \lambda C(\boldsymbol{p}, q) + (1-\lambda)C(\boldsymbol{p}', q)$

- Let $\boldsymbol{p}'' = \lambda \boldsymbol{p} + (1-\lambda)\boldsymbol{p}'$ for shot, and $\boldsymbol{x}^{*\prime\prime}$ be the solution evaluated at price $\boldsymbol{p}''$.
- Let $\boldsymbol{x}^*$ be the solution evaluated at $\boldsymbol{p}$, and $\boldsymbol{x}^{*\prime}$ be the solution evaluated at $\boldsymbol{p}'$.
- $C(\boldsymbol{p}'', q) = \boldsymbol{p}''\boldsymbol{x}^{*\prime\prime} = \lambda \boldsymbol{p}\boldsymbol{x}^{*\prime\prime} + (1-\lambda)\boldsymbol{p}'\boldsymbol{x}^{*\prime\prime}$.
- Evaluated at price $\boldsymbol{p}$, the best choice is $\boldsymbol{x}^*$ instead of $\boldsymbol{x}^{*\prime\prime}$, hence $\lambda \boldsymbol{p}\boldsymbol{x}^{*\prime\prime} \geq \lambda \boldsymbol{p}\boldsymbol{x}^* = \lambda C(\boldsymbol{p}, q)$.
- Similarly, evaluated at price $\boldsymbol{p}'$, the best choice is $\boldsymbol{x}^{*\prime}$ rather than $\boldsymbol{x}^{*\prime\prime}$. Hence $(1-\lambda)\boldsymbol{p}'\boldsymbol{x}^{*\prime\prime} \geq (1-\lambda)C(\boldsymbol{p}', q)$
- Summing up, $C(\boldsymbol{p}'', q) \geq \lambda C(\boldsymbol{p}, q) + (1-\lambda)C(\boldsymbol{p}', q)$

The Envelop Theorem (包络定理)

Similar to Shephard's Lemma, we have

$$k^*(r,w,q) = \frac{\partial C(r,w,q)}{\partial r} \quad \text{and} \quad l^*(r,w,q) = \frac{\partial C(r,w,q)}{\partial w}$$

Proof:

- Differentiate cost function $C(r,w,q) = rk^*(r,w,q) + wl^*(r,w,q)$ with respect to r gives $k^* + r\frac{\partial k^*}{\partial r} + w\frac{\partial l^*}{\partial r}$. We need to show that the sum of the last two terms is zero.
- The solution k^* and l^* are obtained from three first-order conditions in Lagrangian.
- Differentiate the constraint $q = f(k^*, l^*)$ with respect to r gives $0 = f'_k\frac{\partial k^*}{\partial r} + f'_l\frac{\partial l^*}{\partial r}$.
- From the other two FOCs in Lagrangian: $r = \lambda f'_k$ and $w = \lambda f'_l$, then $r\frac{\partial k^*}{\partial r} + w\frac{\partial l^*}{\partial r} = \lambda f'_k\frac{\partial k^*}{\partial r} + \lambda f'_l\frac{\partial l^*}{\partial r} = \lambda\left(f'_k\frac{\partial k^*}{\partial r} + f'_l\frac{\partial l^*}{\partial r}\right) = 0$.
- Therefore, $\frac{\partial C}{\partial r} = k^*$.

Practice: show that $l^*(r,w,q) = \frac{\partial C(r,w,q)}{\partial w}$.

Profit Maximization (利润最大化)

Given the output price p (price-taker), a firm chooses k and l to maximize profit π:

$$\max_{k,l} pf(k,l) - rk - wl.$$

- FOC w.r.t. k: $pf'_k(k,l) = r$. Marginal contribution of capital = capital price.
- FOC w.r.t. l: $pf'_l(k,l) = w$. Marginal contribution of labor = labor price.
- Two unknowns, (k^*, l^*), are determined by above two equations. There are three parameters, (p, r, w). Hence the solutions $k^*(p, r, w)$ and $l^*(p, r, w)$, are functions of p, r, w.
- The solutions $k^*(p, r, w)$ and $l^*(p, r, w)$, are called (unconditional) input demand: (无条件) 要素需求.

Cost Minimization v.s. Profit Maximization

- Cost Minimization: given input prices r and w, the firm solves

$$\text{objective: } \min_{k,l} rk + wl$$

$$\text{constraint: } q = f(k,l)$$

  The solution $k^*(r,w,q)$ and $l^*(r,w,q)$ are functions of r and w. Conditional input demand on q.

- Profit Maximization: given output price p and input prices r and w, the firm solves the unconstrained problem:

$$\text{objective: } \max_{k,l} pf(k,l) - rk - wl$$

  no constraint

  The solution $k^*(p,r,w)$ and $l^*(p,r,w)$ are functions of prices p, r, w. There is no constraint. Therefore, the optimal choices are called input demand (without conditions).

Second Order Conditions for a Maximum

- The objective is $\pi = pf(k,l) - rk - wl$
- The first-order conditions:

$$\pi'_k = pf'_k - r = 0,\ \pi'_l = pf'_l - w = 0.$$

- Second-order condition: the Hessian matrix is negative semidefinite:

$$H = \begin{bmatrix} \pi''_{kk} & \pi''_{kl} \\ \pi''_{lk} & \pi''_{ll} \end{bmatrix} = \begin{bmatrix} pf''_{kk} & pf''_{kl} \\ pf''_{lk} & pf''_{ll} \end{bmatrix} = p\begin{bmatrix} f''_{kk} & f''_{kl} \\ f''_{lk} & f''_{ll} \end{bmatrix},$$

where $f''_{kk} \leq 0$ and $\det(H) \geq 0$.

The Effects of Price Changes on Input Demand

The first order conditions for $\max_{k,l} pf(k,l) - rk - wl$ are given by

$$pf_k'(k^*(p,r,w), l^*(p,r,w)) = r$$
$$pf_l'(k^*(p,r,w), l^*(p,r,w)) = w$$

We would like to see the effect of r on k^* and l^*. Differentiate the FOCs with respect to r:

前提是取了内点解

(FOC为0)

若取了非内点解呢?

$$pf_{kk}''\frac{\partial k^*}{\partial r} + pf_{kl}''\frac{\partial l^*}{\partial r} = 1$$
$$pf_{lk}''\frac{\partial k^*}{\partial r} + pf_{ll}''\frac{\partial l^*}{\partial r} = 0$$

The matrix representation is

$$p\begin{bmatrix} f_{kk}'' & f_{kl}'' \\ f_{kl}'' & f_{ll}'' \end{bmatrix}\begin{bmatrix} \frac{\partial k^*}{\partial r} \\ \frac{\partial l^*}{\partial r} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} f''_{kk} & f''_{kl} \\ f''_{kl} & f''_{ll} \end{bmatrix}}_{=H} \begin{bmatrix} \frac{\partial k^*}{\partial r} \\ \frac{\partial l^*}{\partial r} \end{bmatrix} = \frac{1}{p} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Using the Cramer's rule:

$$\Rightarrow \begin{array}{l} \dfrac{\partial k^*}{\partial r} = \dfrac{\begin{vmatrix} \frac{1}{p} & f''_{kl} \\ 0 & f''_{ll} \end{vmatrix}}{\det(H)} = \dfrac{\frac{1}{p} f''_{ll}}{\det(H)} \leq 0 \\ \text{sign}\left(\dfrac{\partial l^*}{\partial r}\right) = \dfrac{\begin{vmatrix} f''_{kk} & \frac{1}{p} \\ f''_{kl} & 0 \end{vmatrix}}{\det(H)} = \dfrac{-\frac{1}{p} f''_{kl}}{\det(H)} = -\text{sign}\left(f''_{kl}\right) \end{array}$$

You should compare $\frac{\partial l^*(p,r,w)}{\partial r}$ and $\frac{\partial l^*(r,w,q)}{\partial r}$.

Profit Function (利润函数)

Plug the input demand $k^*(p,r,w)$ and $l^*(p,r,w)$ into the objective π, we get profit function, i.e., the maximized profit:

$$\pi^* = \pi\left(p, k^*(p,r,w), l^*(p,r,w)\right)$$

Let $\boldsymbol{p} = [p, -r, -w]$ be the vector of prices, and let $\boldsymbol{q} = [f(k,l), k, l]^T$ be a vector of optimal choices, then profit function $\pi^*(\boldsymbol{p}) = \boldsymbol{pq} = pf - rk - wl$ has the following properties:

Proposition

1. *Non-decreasing in output prices; Non-increasing in input prices;*
2. *Homogeneous of degree 1 in* $\boldsymbol{p}$
3. *Convex in* $\boldsymbol{p}$

Properties of Profit Function: Monotonicity

We need to show that for
$\boldsymbol{p}' \geq \boldsymbol{p} \Rightarrow [p', -r', -w'] \geq [p, r, w] \Rightarrow p' \geq p$ and $[r', k'] \leq [r, k]$,
then $\pi(\boldsymbol{p}') \geq \pi(\boldsymbol{p})$

- If $\boldsymbol{q}$ is the solution evaluated at price level $\boldsymbol{p}$ and $\boldsymbol{q}'$ is the solution evaluated at price level $\boldsymbol{p}'$, then $\pi^*(\boldsymbol{p}) = \boldsymbol{p}\boldsymbol{q}$ and $\pi^*(\boldsymbol{p}') = \boldsymbol{p}'\boldsymbol{q}'$
- Fixing the price level $\boldsymbol{p}'$, then $\boldsymbol{q}'$ is the best choice hence $\boldsymbol{p}'\boldsymbol{q}' \geq \boldsymbol{p}'\boldsymbol{q}$.
- Fixing the choice $\boldsymbol{q}$, an exogenous increase in output price p, or a decrease in input price (r, k), gives a higher profit, i.e., $\boldsymbol{p}'\boldsymbol{q} \geq \boldsymbol{p}\boldsymbol{q}$.
- Combining the above two inequalities, $\boldsymbol{p}'\boldsymbol{q}' \geq \boldsymbol{p}\boldsymbol{q}$, or $\pi^*(\boldsymbol{p}') \geq \pi^*(\boldsymbol{p})$.

Properties of Profit Function: Homogeneity

We need to show that $\pi^*(t\boldsymbol{p}) = t\pi^*(\boldsymbol{p})$.

- Claim: if $\boldsymbol{q}$ is the optimal choice evaluated at price level $\boldsymbol{p}$, then it is also the best choice at price level $t\boldsymbol{p}$.
- Suppose not, and $\boldsymbol{q}'$ is the optimal choice at price level $t\boldsymbol{p}$. Then $t\boldsymbol{p}\boldsymbol{q}' \geq t\boldsymbol{p}\boldsymbol{q} \Rightarrow \boldsymbol{p}\boldsymbol{q}' \geq \boldsymbol{p}\boldsymbol{q}$, i.e., evaluated at price level $\boldsymbol{p}$, the choice $\boldsymbol{q}'$ is better than $\boldsymbol{q}$, which contradict with our assumption that $\boldsymbol{q}$ is the best choice at price $\boldsymbol{p}$.

Convexity of Profit Function

Let $\boldsymbol{p}'' = \lambda \boldsymbol{p} + (1-\lambda)\boldsymbol{p}'$, then we need to show that $\pi^*(\boldsymbol{p}'') \leq \lambda \pi^*(\boldsymbol{p}) + (1-\lambda)\pi^*(\boldsymbol{p}')$

- Let $\boldsymbol{q}$, $\boldsymbol{q}'$ and $\boldsymbol{q}''$ be the best choices evaluated at $\boldsymbol{p}$, $\boldsymbol{p}'$ and $\boldsymbol{p}''$, respectively.
- $\pi^*(\boldsymbol{p}'') = \boldsymbol{p}''\boldsymbol{q}'' = [\lambda \boldsymbol{p} + (1-\lambda)\boldsymbol{p}']\, \boldsymbol{q}'' = \lambda \boldsymbol{p}\boldsymbol{q}'' + (1-\lambda)\boldsymbol{p}'\boldsymbol{q}''$
- Evaluated at price level $\boldsymbol{p}$, $\boldsymbol{q}$ is the best choice (compared with $\boldsymbol{q}''$) hence $\lambda \boldsymbol{p}\boldsymbol{q}'' \leq \lambda \boldsymbol{p}\boldsymbol{q} = \lambda \pi^*(\boldsymbol{p})$.
- Evaluated at price level $\boldsymbol{p}'$, $\boldsymbol{q}'$ is the best choice (relative to $\boldsymbol{q}''$), hence $(1-\lambda)\boldsymbol{p}'\boldsymbol{q}'' \leq (1-\lambda)\boldsymbol{p}'\boldsymbol{q}' = (1-\lambda)\pi^*(\boldsymbol{p}')$.
- Summing the above two inequalities, we have $\pi^*(\boldsymbol{p}'') \leq \lambda \pi^*(\boldsymbol{p}) + (1-\lambda)\pi^*(\boldsymbol{p}')$.

Envelop Results

$$\frac{\partial \pi^*(p,r,w)}{\partial p} = q^*, \ \frac{\partial \pi^*(p,r,w)}{\partial r} = -k^*, \ \frac{\partial \pi^*(p,r,w)}{\partial w} = -l^*.$$

Proof of the first result:

- Differentiate the profit function $\pi^* = pf(k^*, l^*) - rk^* - wl^*$ with respect to p gives $\frac{\partial \pi^*}{\partial p} = q^* + p\left(f_k' \frac{\partial k^*}{\partial p} + f_l' \frac{\partial l^*}{\partial p}\right) - r\frac{\partial k^*}{\partial p} - w\frac{\partial l^*}{\partial p} =$ $q^* + (pf_k' - r)\frac{\partial k^*}{\partial p} + (pf_l' - w)\frac{\partial l^*}{\partial p}$.
- The FOCs of $\max_{k,l} pf(k,l) - rk - wl$ imply that $pf_k' = r$ and $pf_l' = w$, hence $\frac{\partial \pi^*}{\partial p} = q^* = f(k^*, l^*)$.

Proof of $\frac{\partial \pi^*}{\partial r} = -k^*$

- Differentiate $\pi^* = pf(k^*, l^*) - rk^* - wl^*$ with respect to r gives $\frac{\partial \pi^*}{\partial r} = p\left(f_k' \frac{\partial k^*}{\partial r} + f_l' \frac{\partial l^*}{\partial r}\right) - k^* - r\frac{\partial k^*}{\partial r} - w\frac{\partial l^*}{\partial r} = (pf_k' - r)\frac{\partial k^*}{\partial r} + (pf_l' - w)\frac{\partial l^*}{\partial r} - k^*$.
- The FOCs of $\max_{k,l} pf(k,l) - rk - wl$ imply that $pf_k' - r = 0$ and $pf_l' - w = 0$.
- Hence, $\frac{\partial \pi}{\partial r} = -k^*$

Example: Cobb-Douglas Technology

Assume that in the short run, given fixed $k = \bar{k}$, the firm solves

$$\max_{l} p\bar{k}^a l^b - r\bar{k} - wl$$

- FOC w.r.t. l: $bp\bar{k}l^{b-1} = w \Rightarrow l^*(p, r, w) = \left(\frac{w}{bp\bar{k}}\right)^{\frac{1}{b-1}}$
- The profit function is given by
  $\pi^* = p\bar{k}\left(\frac{w}{bp\bar{k}}\right)^{\frac{b}{b-1}} - r\bar{k} - w\left(\frac{w}{bp\bar{k}}\right)^{\frac{1}{b-1}}$.
- Verify $\frac{\partial \pi^*}{\partial p} = q^* = \bar{k}^a\left(\frac{w}{bp\bar{k}}\right)^{\frac{b}{b-1}}$