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The properties of the propensity score

Before we start, recall the definition of the Propensity Score as:

$$p(X_i) = Pr(D_i = 1 \mid X_i) = E(D_i \mid X_i) \tag{1}$$

where D_i is a dummy treatment indicator and X_i a set of observable control variables.

Theorem 1 (The Balancing Property)**.**

$$D_i \perp X_i \mid p(X_i)$$

In words, the distributions of the treatment status D_i and the observable control variables X_i are orthogonal to each other, once conditioning on the propensity score $p(X_i)$.

Proof. Given that D_i is a binary variable, its distribution is fully summarized by its mean and Theorem 1 is equivalent to the following statement:

$$E[D_i \mid X_i, p(X_i)] = E[D_i \mid p(X_i)] \tag{2}$$

In words, once conditioning on $p(X_i)$, it is irrelevant whether the mean of D_i is computed further conditioning on X_i or not. We proceed with the proof by showing that both the term on the left and on the right hand sides of equation 2 are equal to the propensity score itself and, thus, are also equal to each other.

Let us start with the left hand side:

$$E[D_i \mid X_i, p(X_i)] = E[D_i \mid X_i] = p(X_i) \tag{3}$$

where the first equality comes trivially from the fact that $p(X_i)$ is simply a function of X_i, so that, conditioning on X_i, knowledge of $p(X_i)$ is irrelevant. The second equality comes directly from the definition of the propensity score 1.

Now, let us look at the right hand side of equation 2. In order to show that $E[D_i \mid p(X_i)] = p(X_i)$ we need to apply the *Law of Iterated Expectations*, which is reported here for convenience:

$$\textbf{Law of Iterated Expectations:}\ E_A(A) = E_B[E_{A|B}(A \mid B)] \tag{4}$$

where A and B are random numbers and where the subscripts to the expectation operators indicate the distributions over which the expected value is computed.

We are going to apply this exact same law to $E[D_i \mid p(X_i)]$, where, for analogy with equation 4, A is defined as $A = D_i|p(X_i)$ and $B = X_i|p(X_i)$. Then, direct application of the Law of Iterated Expectations with such definitions to the right hand side of equation 2 leads to the following:

$$\begin{aligned} E_{D|p(X)}[D_i \mid p(X_i)] &= E_{X|p(X)}\left\{E_{D|X,p(X)}[D_i \mid X_i, p(X_i)] \mid p(X_i)\right\} \\ &= E[p(X_i) \mid p(X_i)] = p(X_i) \end{aligned} \tag{5}$$

The first equality comes from the Law of Iterated Expectations, while the second uses equation 3 from the first part of this proof and the third equality holds trivially. The combination of equations 3 and 5 proves the theorem. □

Theorem 2 (Unconfoundedness)**.**

$$\begin{array}{c} y_i^0 \perp D_i \mid X_i \\ \Downarrow \\ y_i^0 \perp D_i \mid p(X_i) \end{array}$$

In words, conditional independence of y_i^0 given X_i, which is the hypothesis of this theorem, implies conditional independence of y_i^0 given the propensity score $p(X_i)$.[1]

Proof. Once again, we use the fact that D_i is a dummy variable to restate theorem 2 as:

$$E[D_i \mid y_i^0, X_i] = E[D_i \mid X_i] \tag{6}$$

$$\Downarrow$$

$$E[D_i \mid y_i^0, p(X_i)] = E[D_i \mid p(X_i)] \tag{7}$$

The proof proceeds by showing that both the left and the right hand sides of equation 7 are equal to the propensity score itself and, hence, they are also

[1]The theorem holds equally conditioning on y_i^1 instead of y_i^0, or both.

equal to each other. In doing so we will make use of the assumption of the theorem (equation 6).

Notice that in the proof of the Balancing Property we have already shown that the right hand side of equation 7 is equal to the propensity score (see equation 5):

$$E[D_i \mid p(X_i)] = p(X_i) \tag{8}$$

We still need to prove that $E[D_i \mid y_i^0, p(X_i)] = p(X_i)$. We do it by using the Law of Iterated Expectations (equation 4), where now $A = D_i|y_i^0, p(X_i)$ and $B = X_i|y_i^0, p(X_i)$. Hence:

$$\begin{aligned} E[D_i \mid y_i^0, p(X_i)] &= E_{X|y_i^0,p(X)}\left\{E_{D|y^0,X,p(X)}[D_i \mid y_i^0, X_i, p(X_i)] \mid y_i^0, p(X_i)\right\} \\ &= E_{X|y_i^0,p(X)}\left\{E_{D|y^0,X}[D_i \mid y_i^0, X_i] \mid y_i^0, p(X_i)\right\} \end{aligned} \tag{9}$$

where the first equality comes from direct application of the Law of Iterated Expectations and the second equality trivially holds because $p(X_i)$ is a function of X_i, so that conditioning on the latter makes the conditioning on the first redundant.

Next, we can apply the hypothesis of the theorem, i.e. conditional independence of D_i and y_i^0 given X_i, to get rid of the conditioning on y_i^0 in equation 9:

$$\begin{aligned} E_{X|y_i^0,p(X)}\left\{E_{D|y^0,X}[D_i \mid y_i^0, X_i] \mid y_i^0, p(X_i)\right\} &= \\ E_{X|y_i^0,p(X)}\left\{E_{D|X}[D_i \mid X_i] \mid y_i^0, p(X_i)\right\} &= \\ E_{X|y_i^0,p(X)}\left[p(x_i) \mid y_i^0, p(X_i)\right] &= p(X_i) \end{aligned} \tag{10}$$

where the first equality comes from the hypothesis of the theorem, the second from the definition of propensity score and the third trivially from the fact that the expected value of any random variable conditional on itself is simply equal to itself.

The combination of equations 9 and 10 shows that $E[D_i \mid y_i^0, p(X_i)] = p(X_i)$, which together with equation 8, proves the theorem.

□