Macroeconomics A; EI056

Technical appendix: Solution of the Ramsey model of intertemporal optimization

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1 Introduction

This appendix goes through the intertemporal optimization based on Romer Ch. 2. We start with the continuous time approach as is presented in the textbook.

We then present the discrete time approach. This is the one done in clas, because discrete time is more common in macroeconomics. In the discrete time approach, we solve for the steady state of the model, as well as the linear approximation of dynamics. The solution is presented using the method of undetermined coefficients, as well as the Blanchard-Kahn matrix method.

The appendix ends with the presentation of the Tobin Q theory of investment (Romer Ch. 9).

2 The Ramsey model

2.1 The household's problem

The representative household supplies an exogenous amount of labor L_t that grows at a rate n (in the notation of Romer we set $H = 1$ for simplicity), so $L_t = e^{nt} L_0$ (L_0 is the initial population which we set to 1). The utility of the representative household is:

$$U_0 = \int_0^\infty e^{-\rho t} \frac{(c_t)^{1-\theta}}{1-\theta} L_t dt = \int_0^\infty e^{-(\rho - n)t} \frac{(c_t)^{1-\theta}}{1-\theta} dt$$

where c_t is consumption per capita. ρ is the time discount. We assume $\rho > n$ so the population growth is small enough not to offset the time discount.

The household supplies labor paid at a wage w_t and capital to firms, with capital earning a rental rate r_t, from which we deduct depreciation δ. Income from wage and capital is used to fund consumption, replace the depreciated capital δK_t, and accumulate capital dK_t:

$$\begin{aligned}
C_t + \delta K_t + \frac{dK_t}{dt} &= w_t L_t + r_t K_t \\
\frac{C_t}{L_t} + \frac{dK_t}{dt}\frac{1}{L_t} &= w_t + (r_t - \delta)\frac{K_t}{L_t} \\
c_t + \frac{dK_t}{dt}\frac{1}{L_t} &= w_t + (r_t - \delta)\, k_t
\end{aligned}$$

The growth of the capital labor-ratio is:

$$\begin{aligned}
\dot{k}_t &= \frac{dk_t}{dt} \\
\dot{k}_t &= \frac{d(K_t/L_t)}{dt} \\
\dot{k}_t &= \frac{dK_t}{dt}\frac{1}{L_t} - \frac{K_t}{(L_t)^2}\frac{dL_t}{dt} \\
\dot{k}_t &= \frac{dK_t}{dt}\frac{1}{L_t} - k_t n
\end{aligned}$$

The resource constraint is thus:

$$\begin{aligned}
c_t + \frac{dK_t}{dt}\frac{1}{L_t} &= w_t + (r_t - \delta)\, k_t \\
c_t + \dot{k}_t + n k_t &= w_t + (r_t - \delta)\, k_t \\
\dot{k}_t &= w_t + (r_t - (\delta + n))\, k_t - c_t \qquad (1)
\end{aligned}$$

2.1.1 Solving the Hamiltonian

The problem is expressed as the following Hamiltonian:

$$H = e^{-(\rho - n)t}\left[\frac{(c_t)^{1-\theta}}{1-\theta} + \lambda_t\left[w_t + (r_t - (\delta + n))\, k_t - c_t\right]\right]$$

$e^{-\rho t} L_t \lambda_t$ is the value of marginal output in utility terms. The first optimality condition is:

$$\begin{aligned}
\frac{\partial H}{\partial c_t} &= 0 \\
e^{-(\rho - n)t}(c_t)^{-\theta} &= e^{-(\rho - n)t}\lambda_t \\
(c_t)^{-\theta} &= \lambda_t \\
\frac{d(c_t)^{-\theta}}{dt} &= \dot{\lambda}_t \\
-\theta (c_t)^{-\theta - 1}\dot{c}_t &= \dot{\lambda}_t \\
-\theta \lambda_t (c_t)^{-1}\dot{c}_t &= \dot{\lambda}_t
\end{aligned}$$

$(c_t)^{-\theta} = \lambda_t$ means that the marginal utility of consumption is the marginal value of capital, both expressed from the point of view of the same period.

The derivative with respect to capital is:

$$\frac{\partial H}{\partial k_t} = e^{-(\rho-n)t}\lambda_t \left(r_t - (\delta+n)\right)$$

The second optimality condition is:

$$\begin{aligned}
\frac{\partial H}{\partial k_t} &= -\frac{d}{dt}\left(e^{-(\rho-n)t}\lambda_t\right) \\
\frac{\partial H}{\partial k_t} &= -\left(-(\rho-n)\,e^{-(\rho-n)t}\lambda_t + e^{-(\rho-n)t}L_0\dot{\lambda}_t\right) \\
\frac{\partial H}{\partial k_t} &= -e^{-(\rho-n)t}\left(-(\rho-n)\,\lambda_t + \dot{\lambda}_t\right) \\
e^{-(\rho-n)t}\lambda_t \left(r_t - (\delta+n)\right) &= e^{-(\rho-n)t}\left((\rho-n)\,\lambda_t - \dot{\lambda}_t\right) \\
\lambda_t \left(r_t - (\delta+n)\right) &= (\rho-n)\,\lambda_t - \dot{\lambda}_t \\
\lambda_t \left(r_t - \delta\right) &= \rho\lambda_t - \dot{\lambda}_t \\
\lambda_t \left(\rho + \delta - r_t\right) &= \dot{\lambda}_t
\end{aligned}$$

Combining the two conditions we get:

$$\begin{aligned}
\dot{\lambda}_t &= \lambda_t \left(\rho+\delta-r_t\right) \\
-\theta\lambda_t \left(c_t\right)^{-1}\dot{c}_t &= \lambda_t \left(\rho+\delta-r_t\right) \\
-\theta\frac{\dot{c}_t}{c_t} &= \left(\rho+\delta-r_t\right) \\
\frac{\dot{c}_t}{c_t} &= \frac{1}{\theta}\left(r_t - (\rho+\delta)\right) \qquad (2)
\end{aligned}$$

2.2 The firm's problem

The representative firm produces output using capital and labor (for simplicity we abstract from productivity growth and set $A_t = 1$ in Romer's notation):

$$Y_t = F\left(K_t, L_t\right)$$

The technology is characterized by constant returns to scale:

$$\begin{aligned}
\frac{Y_t}{L_t} &= F\left(\frac{K_t}{L_t}, 1\right) \\
y_t &= f\left(k_t\right)
\end{aligned}$$

The firm pays a wage w_t and a rental rate of capital r_t. It chooses capital and labor to maximize its profits:

$$\begin{aligned}
\Pi_t &= F\left(K_t, L_t\right) - w_t L_t - r_t K_t \\
\Pi_t &= L_t\left[f\left(k_t\right) - w_t - r_t k_t\right]
\end{aligned}$$

The first order condition with respect to capital k_t is:

$$\begin{aligned}
0 &= \frac{\partial \Pi_t}{\partial k_t} \\
0 &= L_t \left[f'(k_t) - r_t \right] \\
0 &= f'(k_t) - r_t \\
f'(k_t) &= r_t
\end{aligned}$$

The first order condition with respect to labor L_t is:

$$\begin{aligned}
0 &= \frac{\partial \Pi_t}{\partial L_t} \\
0 &= \left[f(k_t) - w_t - r_t k_t \right] + L_t \frac{\partial \left[f(k_t) - w_t - r_t k_t \right]}{\partial k_t} \frac{\partial k_t}{\partial L_t} \\
0 &= \left[f(k_t) - w_t - r_t k_t \right] + L_t \left[f'(k_t) - r_t \right] \frac{\partial k_t}{\partial L_t} \\
0 &= f(k_t) - w_t - r_t k_t \\
w_t &= f(k_t) - r_t k_t \\
w_t &= f(k_t) - k_t f'(k_t)
\end{aligned}$$

where we used the fact that $f'(k_t) = r_t$.

Therefore the capital dynamics (1) become:

$$\begin{aligned}
\dot{k}_t &= w_t + (r_t - (\delta + n)) k_t - c_t \\
\dot{k}_t &= f(k_t) - k_t f'(k_t) + (f'(k_t) - (\delta + n)) k_t - c_t \\
\dot{k}_t &= f(k_t) - (\delta + n) k_t - c_t \qquad (3)
\end{aligned}$$

Note that as the production function has constant returns to scale, the sum of all factors payments is equal to output:

$$\begin{aligned}
Y_t &= w_t L_t + r_t K_t \\
y_t &= w_t + r_t k_t
\end{aligned}$$

2.3 A two-equation model

The model boils down to two equations. The first is the dynamics of consumption (2), using the fact that the marginal product of labor is equal to the real interest rate:

$$\begin{aligned}
\frac{\dot{c}_t}{c_t} &= \frac{1}{\theta} (r_t - (\rho + \delta)) \\
\frac{\dot{c}_t}{c_t} &= \frac{1}{\theta} (f'(k_t) - (\rho + \delta)) \qquad (4)
\end{aligned}$$

The second is the dynamics of capital (3).

Constant consumption in (4) defines a unique capital stock:

$$\dot{c}_t = 0 \Rightarrow f'(k_t) = \rho + \delta \Rightarrow k_t = (f')^{-1}(\rho + \delta)$$

Consumption increases if the capital is below this value

$$k_t < (f')^{-1}(\rho + \delta) \Rightarrow f'(k_t) > \rho + \delta \Rightarrow \dot{c}_t > 0$$

Constant capital in (3) defines a bell curve in a capital-consumption space:

$$\begin{aligned} \dot{k}_t &= 0 \Rightarrow c_t = f(k_t) - k_t(\delta + n) \\ &\Rightarrow \frac{\partial c_t}{\partial k_t} = f'(k_t) - (\delta + n) < 0 \qquad \text{if} \qquad f'(k_t) \text{ is low, i.e. } k_t \text{ is high} \end{aligned}$$

Capital increases if consumption is small enough:

$$c_t < f(k_t) - k_t(\delta + n) \Rightarrow \dot{k}_t > 0$$

In the steady state both consumption and capital are constant:

$$c^* = f(k^*) - k^*(\delta + n) \qquad\qquad f'(k^*) = \rho + \delta$$

Note that as $\rho > n$, the capital ratio is below the level that would maximize consumption (the golden rule):

$$\max_{k_t} c_t = f(k_t) - k_t(\delta + n)$$

implies:

$$\begin{aligned} 0 &= f'(k_t) - (\delta + n) \\ f'(k_t) &= \delta + n < \rho + \delta \end{aligned}$$

2.4 Introducing government

The government consumes G_t, so the per-capita resource constraint becomes:

$$\dot{k}_t = f(k_t) - c_t - g_t - k_t(\delta + n)$$

Government expenditures, including interest on the debt, are financed by per capita lump-sum taxes τ_t and debt issuance:

$$\begin{aligned} G_t + r_t B_t &= L_t \tau_t + \frac{dB_t}{dt} \\ \frac{dB_t}{L_t dt} &= g_t - \tau_t + r_t b_t \\ \dot{b}_t &= g_t - \tau_t + (r_t - n) b_t \end{aligned}$$

where $B > 0$ represents government debt. The consumer pays taxes and holds the government debt:

$$\begin{aligned} C_t + \frac{d\left(B_t + K_t\right)}{dt} &= w_t L_t - L_t \tau_t + r_t \left(B_t + K_t\right) - \delta K_t \\ c_t + \frac{d\left(B_t + K_t\right)}{L_t dt} &= w_t - \tau_t + r_t \left(b_t + k_t\right) - \delta k_t \\ c_t + \dot{b}_t + \dot{k}_t &= w_t - \tau_t + \left(r_t - n\right)\left(b_t + k_t\right) - \delta k_t \end{aligned}$$

Combining these show that the presence of government debt is irrelevant:

$$\begin{aligned} c_t + \dot{b}_t + \dot{k}_t &= w_t - \tau_t + \left(r_t - n\right)\left(b_t + k_t\right) - \delta k_t \\ c_t + g_t - \tau_t + \left(r_t - n\right) b_t + \dot{k}_t &= w_t - \tau_t + \left(r_t - n\right)\left(b_t + k_t\right) - \delta k_t \\ c_t + g_t + \dot{k}_t &= w_t + \left(r_t - \left(n + \delta\right)\right) k_t \end{aligned}$$

We now add a distortionary tax on capital, with the consumer receiving only a fraction $1 - \tau_k$ of the return on capital. For simplicity the government issues no debt and purchases no good, instead it just repays the proceeds of the tax on capital income in a lump sum way:

$$\tau_k r_t k_t + \tau_t = 0$$

The consumer's resource constraint is not affected by taxes:

$$\begin{aligned} c_t + \dot{k}_t &= w_t - \tau_t + \left(\left(1 - \tau_k\right) r_t - \left(n + \delta\right)\right) k_t \\ &= w_t + \left(r_t - \left(n + \delta\right)\right) k_t - \left[\tau_t + \tau_k r_t k_t\right] \\ &= w_t + \left(r_t - \left(n + \delta\right)\right) k_t \end{aligned}$$

$$\Rightarrow \dot{k}_t = f\left(k_t\right) - c_t - \left(n + \delta\right) k_t$$

The dynamics of consumption are however affected:

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}\left[\left(1 - \tau_k\right) f'\left(k_t\right) - \left(\rho + \delta\right)\right]$$

3 A discrete-time version

3.1 The household's problem

We usually deal with discrete time models in macroeconomics. It is thus useful to consider the optimization in that case. The utility is:

$$U_0 = \sum_{t=0}^{\infty} \frac{1}{\left(1 + \rho\right)^t} \frac{\left(c_t\right)^{1-\theta}}{1 - \theta} L_t$$

I expressed the discount factor as $\left(1 + \rho\right)^{-t}$, as this will simplify the notation below. It is not exacty equal to writing $e^{-\rho t}$ as in the textbook, but is equal to a first-order approximation when ρ is close to zero. It also common to write the discount factor as β^t where $\beta = (1+\rho)^{-1}$ is between 0 and 1. Note that whichever notation we take does not matter for the interpretation of the model.

The resource constraint is:

$$w_t L_t + r_t K_t = C_t + K_{t+1} - K_t + \delta K_t$$

We express this in per capita terms. First, notice that:

$$\begin{aligned}
k_{t+1} - k_t &= \frac{K_{t+1}}{L_{t+1}} - \frac{K_t}{L_t} \\
&= \frac{K_{t+1}}{L_{t+1}} - \frac{K_t}{L_{t+1}} + \frac{K_t}{L_{t+1}} - \frac{K_t}{L_t} \\
&= \frac{K_{t+1} - K_t}{L_t\,(1+n)} + \frac{K_t}{L_t\,(1+n)} - \frac{K_t}{L_t} \\
&= \frac{K_{t+1} - K_t}{L_t\,(1+n)} - \frac{n}{1+n}\frac{K_t}{L_t} \\
&= \frac{1}{1+n}\left(\frac{K_{t+1} - K_t}{L_t} - k_t n\right)
\end{aligned}$$

We therefore write:

$$\frac{K_{t+1} - K_t}{L_t} = (1+n)\,(k_{t+1} - k_t) + k_t n$$

We then write the constraint in per capita terms as:

$$\begin{aligned}
w_t + r_t \frac{K_t}{L_t} &= \frac{C_t}{L_t} + \frac{K_{t+1} - K_t}{L_t} + \delta \frac{K_t}{L_t} \\
w_t + r_t k_t &= c_t + (1+n)\,(k_{t+1} - k_t) + k_t\,(n+\delta) \\
(1+n)\,(k_{t+1} - k_t) &= w_t + (r_t - (n+\delta))\,k_t - c_t \qquad (5)
\end{aligned}$$

which is the discrete time equivalent to (1).

The standard approach to solving the problem is to write the following Lagrangian:

$$\begin{aligned}
\mathcal{L} = &\sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} L_t \frac{(c_t)^{1-\theta}}{1-\theta} \\
&+ \sum_{t=0}^{\infty} \frac{1}{(1+\rho)^t} L_t \lambda_t \left[w_t + r_t k_t - c_t - (1+n)\,(k_{t+1} - k_t) - k_t\,(n+\delta)\right]
\end{aligned}$$

There is one resource constraint for each period t, with associated multiplier λ_t. The first order conditions with respect to consumption is:

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial c_t} \\
0 &= \frac{1}{(1+\rho)^t} L_t\,(c_t)^{-\theta} - \frac{1}{(1+\rho)^t} L_t \lambda_t \\
(c_t)^{-\theta} &= \lambda_t
\end{aligned}$$

The first order conditions with respect to capital is:

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial k_{t+1}} \\
0 &= \frac{\partial}{\partial k_{t+1}} \left[\begin{array}{c} \frac{1}{(1+\rho)^t} L_t \frac{(c_t)^{1-\theta}}{1-\theta} \\ +\frac{1}{(1+\rho)^t} L_t \lambda_t \left[\begin{array}{c} w_t + r_t k_t - c_t \\ -(1+n)(k_{t+1}-k_t) - k_t(n+\delta) \end{array} \right] \\ +\frac{1}{(1+\rho)^{t+1}} L_{t+1} \frac{(c_{t+1})^{1-\theta}}{1-\theta} \\ +\frac{1}{(1+\rho)^{t+1}} L_{t+1} \lambda_{t+1} \left[\begin{array}{c} w_{t+1} + r_{t+1} k_{t+1} - c_{t+1} \\ -(1+n)(k_{t+2}-k_{t+1}) - k_{t+1}(n+\delta) \end{array} \right] \\ +... \end{array} \right] \\
0 &= -\frac{1}{(1+\rho)^t} L_t \lambda_t (1+n) + \frac{1}{(1+\rho)^{t+1}} L_{t+1} \lambda_{t+1} \left[r_{t+1} + (1+n) - (n+\delta) \right] \\
\lambda_t (1+n) &= \frac{L_{t+1}}{L_t} \frac{1}{1+\rho} \lambda_{t+1} (1 + r_{t+1} - \delta) \\
\lambda_t &= \lambda_{t+1} \frac{1 + r_{t+1} - \delta}{1+\rho}
\end{aligned}
$$

Combining the two conditions, we get:

$$
\begin{aligned}
(c_t)^{-\theta} &= (c_{t+1})^{-\theta} \frac{1 + r_{t+1} - \delta}{1+\rho} \\
\frac{c_{t+1}}{c_t} &= \left(\frac{1 + r_{t+1} - \delta}{1+\rho} \right)^{\frac{1}{\theta}} \qquad (6)
\end{aligned}
$$

which is the discrete time equivalent to (2).

3.2 A two-equation model

The firm's problem is exactly the same one as in continuous time, because it is basically a static problem. We thus get the same optimality conditions:

$$
\begin{aligned}
r_t &= f'(k_t) \\
w_t &= f(k_t) - k_t f'(k_t)
\end{aligned}
$$

Using these results, we rewrite the Euler condition (6) and the dynamics of capital (5) as:

$$
\begin{aligned}
\frac{c_{t+1}}{c_t} &= \left(\frac{1 + f'(k_{t+1}) - \delta}{1+\rho} \right)^{\frac{1}{\theta}} \\
(1+n)(k_{t+1} - k_t) &= f(k_t) - (n+\delta) k_t - c_t
\end{aligned}
$$

This is the discrete time equivalent to the system we analyzed in the phase diagram.

3.3 Explicit solution: the steady state

To be able to pursue the solution further, let's consider a specific form for the production function:

$$Y_t = F(K_t, L_t) = (K_t)^{\alpha} (L_t)^{1-\alpha}$$

which implies:

$$y_t = (k_t)^{\alpha}$$

Our two equations system is then:

$$\begin{aligned} \frac{c_{t+1}}{c_t} &= \left(\frac{1 + \alpha (k_{t+1})^{\alpha-1} - \delta}{1+\rho} \right)^{\frac{1}{\theta}} && (7) \\ (1+n)(k_{t+1} - k_t) &= (k_t)^{\alpha} - (n+\delta) k_t - c_t && (8) \end{aligned}$$

We first solve for the steady state where the variables are constant. We denote these variables by an asterisk. With constant variables, the system is:

$$\begin{aligned} 1 &= \left(\frac{1 + \alpha (k^*)^{\alpha-1} - \delta}{1+\rho} \right)^{\frac{1}{\theta}} \\ 0 &= (k^*)^{\alpha} - (n+\delta) k^* - c^* \end{aligned}$$

The first equation immediately gives the capital k^*:

$$\begin{aligned} 1 &= \left(\frac{1 + \alpha (k^*)^{\alpha-1} - \delta}{1+\rho} \right)^{\frac{1}{\theta}} \\ 1 &= \frac{1 + \alpha (k^*)^{\alpha-1} - \delta}{1+\rho} \\ 1+\rho &= 1 + \alpha (k^*)^{\alpha-1} - \delta \\ \frac{\rho+\delta}{\alpha} &= (k^*)^{\alpha-1} \\ k^* &= \left(\frac{\alpha}{\rho+\delta} \right)^{\frac{1}{1-\alpha}} \end{aligned}$$

Consumption c^* follows from the second equation:

$$\begin{aligned} c^* &= (k^*)^{\alpha} - (n+\delta) k^* \\ c^* &= \left(\frac{\alpha}{\rho+\delta} \right)^{\frac{\alpha}{1-\alpha}} - (n+\delta) \left(\frac{\alpha}{\rho+\delta} \right)^{\frac{1}{1-\alpha}} \\ c^* &= \left(\frac{\rho+\delta}{\alpha} - (n+\delta) \right) \left(\frac{\alpha}{\rho+\delta} \right)^{\frac{1}{1-\alpha}} \end{aligned}$$

Note that the real interest rate is:

$$\begin{aligned}
r^* &= f'(k^*) \\
r^* &= \alpha (k^*)^{\alpha-1} \\
r^* &= \rho + \delta
\end{aligned}$$

The real interest rate thus simply offsets the time preference and depreciation. We thus have solved for the steady state.

3.4 Explicit solution: log-linear approximation

Our next step is to write a log-linear approximation of (7)-(8). This is done by differentiating the system. We denote the log deviation of a variable from the steady state by:

$$\hat{x}_t = \frac{x_t - x^*}{x^*}$$

We start by writing the approximation of (7):

$$\frac{1}{c_t}(c_{t+1} - c^*) - \frac{c_{t+1}}{(c_t)^2}(c_t - c^*) = \frac{1}{\theta}\left(\frac{1 + \alpha (k_{t+1})^{\alpha-1} - \delta}{1+\rho}\right)^{\frac{1}{\theta}-1} \frac{\alpha (k_{t+1})^{\alpha-2}(\alpha - 1)}{1+\rho}(k_{t+1} - k^*)$$

Now recall that we evaluate all derivatives at the steady state values:

$$\begin{aligned}
\frac{1}{c^*}(c_{t+1} - c^*) - \frac{c^*}{(c^*)^2}(c_t - c^*) &= \frac{1}{\theta}\left(\frac{1 + \alpha (k^*)^{\alpha-1} - \delta}{1+\rho}\right)^{\frac{1}{\theta}-1} \frac{\alpha (k^*)^{\alpha-2}(\alpha - 1)}{1+\rho}(k_{t+1} - k^*) \\
\frac{c_{t+1} - c^*}{c^*} - \frac{c_t - c^*}{c^*} &= \frac{1}{\theta}\left(\frac{1 + \alpha (k^*)^{\alpha-1} - \delta}{1+\rho}\right)^{\frac{1}{\theta}-1} \frac{\alpha (k^*)^{\alpha-1}(\alpha - 1)}{1+\rho}\frac{k_{t+1} - k^*}{k^*} \\
\hat{c}_{t+1} - \hat{c}_t &= \frac{1}{\theta}\left(\frac{1 + r^* - \delta}{1+\rho}\right)^{\frac{1}{\theta}-1} \frac{r^*(\alpha - 1)}{1+\rho}\hat{k}_{t+1} \\
\hat{c}_{t+1} - \hat{c}_t &= \frac{1}{\theta}\left(\frac{1 + r^* - \delta}{1 + r^* - \delta}\right)^{\frac{1}{\theta}-1} \frac{r^*}{1 + r^* - \delta}(\alpha - 1)\hat{k}_{t+1} \\
\hat{c}_{t+1} - \hat{c}_t &= -\frac{1}{\theta}\frac{r^*}{1 + r^* - \delta}(1 - \alpha)\hat{k}_{t+1} \qquad (9)
\end{aligned}$$

(8) is approximated as follows:

$$
\begin{aligned}
(1+n)\left((k_{t+1}-k^*)-(k_t-k^*)\right) &= \alpha\,(k_t)^{\alpha-1}(k_t-k^*)-(n+\delta)(k_t-k^*)-(c_t-c^*)\\
(1+n)\,k^*\left(\frac{k_{t+1}-k^*}{k^*}-\frac{k_t-k^*}{k^*}\right) &= \alpha\,(k^*)^{\alpha}\,\frac{k_t-k^*}{k^*}-(n+\delta)\,k^*\frac{k_t-k^*}{k^*}-c^*\frac{c_t-c^*}{c^*}\\
(1+n)\,k^*\left(\hat{k}_{t+1}-\hat{k}_t\right) &= \alpha\,(k^*)^{\alpha}\,\hat{k}_t-(n+\delta)\,k^*\hat{k}_t-\left((k^*)^{\alpha}-(n+\delta)\,k^*\right)\hat{c}_t\\
\hat{k}_{t+1} &= \frac{\alpha\,(k^*)^{\alpha-1}+(1-\delta)}{1+n}\hat{k}_t-\frac{(k^*)^{\alpha-1}-(n+\delta)}{1+n}\hat{c}_t\\
\hat{k}_{t+1} &= \frac{1+r^*-\delta}{1+n}\hat{k}_t-\frac{1}{\alpha}\frac{r^*-\alpha\,(n+\delta)}{1+n}\hat{c}_t \qquad (10)
\end{aligned}
$$

3.5 Explicit solution: undetermined coefficients

We conjecture that the solution for (7)-(10) is of the form:

$$
\hat{c}_t=\eta_{ck}\hat{k}_t \qquad ; \qquad \hat{k}_{t+1}=\eta_{kk}\hat{k}_t
$$

where η_{ck} and η_{kk} are coefficients that we need to solve for. (10) is then written as:

$$
\begin{aligned}
\hat{k}_{t+1} &= \frac{1+r^*-\delta}{1+n}\hat{k}_t-\frac{1}{\alpha}\frac{r^*-\alpha\,(n+\delta)}{1+n}\hat{c}_t\\
\eta_{kk}\hat{k}_t &= \frac{1+r^*-\delta}{1+n}\hat{k}_t-\frac{1}{\alpha}\frac{r^*-\alpha\,(n+\delta)}{1+n}\eta_{ck}\hat{k}_t\\
\eta_{kk} &= \frac{1+r^*-\delta}{1+n}-\frac{1}{\alpha}\frac{r^*-\alpha\,(n+\delta)}{1+n}\eta_{ck}\\
(1+n)\,\eta_{kk} &= 1+r^*-\delta-\frac{r^*-\alpha\,(n+\delta)}{\alpha}\eta_{ck}\\
\eta_{ck} &= \alpha\frac{1+r^*-\delta-(1+n)\,\eta_{kk}}{r^*-\alpha\,(n+\delta)}
\end{aligned}
$$

This gives η_{ck} as a function of η_{kk}.

(7) is written as:

$$
\begin{aligned}
\hat{c}_{t+1}-\hat{c}_t &= -\frac{1}{\theta}\frac{r^*}{1+r^*-\delta}(1-\alpha)\,\hat{k}_{t+1}\\
\eta_{ck}\left(\hat{k}_{t+1}-\hat{k}_t\right) &= -\frac{1}{\theta}\frac{r^*}{1+r^*-\delta}(1-\alpha)\,\hat{k}_{t+1}\\
\eta_{ck}\left(\eta_{kk}\hat{k}_t-\hat{k}_t\right) &= -\frac{1}{\theta}\frac{r^*}{1+r^*-\delta}(1-\alpha)\,\eta_{kk}\hat{k}_t\\
\eta_{ck}\,(\eta_{kk}-1) &= -\frac{1-\alpha}{\theta}\frac{r^*}{1+r^*-\delta}\eta_{kk}\\
\alpha\frac{1+r^*-\delta-(1+n)\,\eta_{kk}}{r^*-\alpha\,(n+\delta)}(\eta_{kk}-1) &= -\frac{1-\alpha}{\theta}\frac{r^*}{1+r^*-\delta}\eta_{kk}
\end{aligned}
$$

This is a quadratic polynomial in η_{kk}:

$$\begin{aligned}
0 &= \alpha\frac{1+r^*-\delta-(1+n)\,\eta_{kk}}{r^*-\alpha\,(n+\delta)}\,(\eta_{kk}-1)+\frac{1-\alpha}{\theta}\frac{r^*}{1+r^*-\delta}\eta_{kk} \\
0 &= \theta\,(1+r^*-\delta)\,\alpha\,(1+r^*-\delta-(1+n)\,\eta_{kk})\,(\eta_{kk}-1)+(r^*-\alpha\,(n+\delta))\,(1-\alpha)\,r^*\eta_{kk} \\
0 &= -\theta\alpha\,(1+r^*-\delta)\,(1+n)\,(\eta_{kk})^2 \\
&\quad+\left[\begin{array}{c}(r^*-\alpha\,(n+\delta))\,(1-\alpha)\,r^*+\theta\alpha\,(1+r^*-\delta)^2 \\ +\theta\alpha\,(1+r^*-\delta)\,(1+n)\end{array}\right]\eta_{kk} \\
&\quad-\theta\alpha\,(1+r^*-\delta)^2
\end{aligned}$$

Solving the value of the roots gives a complex expression. We can however get insights on the properties. Denote the polynomial as follows:

$$\begin{aligned}
Pol\,(\eta_{kk}) &= -\theta\alpha\,(1+r^*-\delta)\,(1+n)\,(\eta_{kk})^2 \\
&\quad+\left[\begin{array}{c}(r^*-\alpha\,(n+\delta))\,(1-\alpha)\,r^*+\theta\alpha\,(1+r^*-\delta)^2 \\ +\theta\alpha\,(1+r^*-\delta)\,(1+n)\end{array}\right]\eta_{kk} \\
&\quad-\theta\alpha\,(1+r^*-\delta)^2
\end{aligned}$$

If η_{kk} goes to plus or minus infinity, the $(\eta_{kk})^2$ terms dominates, and it is negative. Thus:

$$Pol\,(-\infty)<0 \qquad ; \qquad Pol\,(+\infty)<0$$

If we evaluate the polynomial at $\eta_{kk}=0$ we get a negative value:

$$Pol\,(0)=-\theta\alpha\,(1+r^*-\delta)^2<0$$

If we evaluate the polynomial at $\eta_{kk}=1$ we get a positive value (assuming that $n+\delta$ is small enough):

$$\begin{aligned}
Pol\,(1) &= -\theta\alpha\,(1+r^*-\delta)\,(1+n)+(r^*-\alpha\,(n+\delta))\,(1-\alpha)\,r^* \\
&\quad+\theta\alpha\,(1+r^*-\delta)^2+\theta\alpha\,(1+r^*-\delta)\,(1+n)-\theta\alpha\,(1+r^*-\delta)^2 \\
&= (r^*-\alpha\,(n+\delta))\,(1-\alpha)\,r^*>0
\end{aligned}$$

So, if we graph the polynomial as a function of η_{kk} we start with negative values when η_{kk} is a large negative number, remain at a negative value when $\eta_{kk}=0$, cross to a positive value somewhere in the range $\eta_{kk}\in(0,1)$ to reach a positive value when $\eta_{kk}=1$, and then cross to a negative value somewhere in the range $\eta_{kk}>1$ to get to negative values when η_{kk} is a large positive number. There are thus two values of η_{kk} for which $Pol\,(\eta_{kk})=0$. One is $\eta_{kk}>1$, but this implies an explosive path for capital. The other is $0<\eta_{kk}<1$ which leads to a stable system.

As $\eta_{kk}<1$ our results imply that $\eta_{ck}>0$.

We can consider a simple example where the rate of tame preference is $\rho=3.5\%$, depreciation is $\delta=1.5\%$, so the steady state real interest rate is $r^*=5\%$. Population grows at a rate $n=2\%$, and we have a log utility of consumption ($\theta = 1$). If we set $\alpha = 1/3$ we get $\eta_{kk} = 0.948$ and $\eta_{ck} = 0.5897$. In the steady state, consumption amounts to three quarters of GDP: $c^*/y^* = 76.7\%$.

3.6 Explicit solution: Blanchard and Kahn

As we do not have stochastic shocks, the system is a simple version of Blanchard-Kahn. $\hat{k}_t$ is the predetermined state variable, and $\hat{c}_t$ the control variable. The two equations are:

$$\begin{aligned}
\hat{k}_{t+1} &= \frac{1+r^*-\delta}{1+n}\hat{k}_t - \frac{1}{\alpha}\frac{r^*-\alpha\,(n+\delta)}{1+n}\hat{c}_t \\
\hat{c}_{t+1} - \hat{c}_t &= -\frac{1}{\theta}\frac{r^*}{1+r^*-\delta}\,(1-\alpha)\,\hat{k}_{t+1}
\end{aligned}$$

We write this in a matrix form as follows:

$$\begin{aligned}
\begin{vmatrix} 1 & 0 \\ \frac{1}{\theta}\frac{r^*}{1+r^*-\delta}\,(1-\alpha) & 1 \end{vmatrix}\begin{vmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{vmatrix} &= \begin{vmatrix} \frac{1+r^*-\delta}{1+n} & -\frac{1}{\alpha}\frac{r^*-\alpha(n+\delta)}{1+n} \\ 0 & 1 \end{vmatrix}\begin{vmatrix} \hat{k}_t \\ \hat{c}_t \end{vmatrix} \\
X\begin{vmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{vmatrix} &= Y\begin{vmatrix} \hat{k}_t \\ \hat{c}_t \end{vmatrix} \\
\begin{vmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{vmatrix} &= X^{-1}Y\begin{vmatrix} \hat{k}_t \\ \hat{c}_t \end{vmatrix} \\
\begin{vmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{vmatrix} &= A\begin{vmatrix} \hat{k}_t \\ \hat{c}_t \end{vmatrix}
\end{aligned}$$

The eigenvalue-eigenvector decomposition of A is $A = C^{-1}\Lambda C$:

$$\begin{aligned}
A &= C^{-1}\Lambda C \\
CA &= \begin{vmatrix} J_1 & 0 \\ 0 & J_2 \end{vmatrix}\begin{vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{vmatrix}
\end{aligned}$$

where the matrix is ordered such that $J_1 > 1 > J_2$. I choose this order because Matlab orders them that way. Be aware though that in Seton Leonard's *An Introductory Guide to Macroeconomic Theory* they are ordered the other way round. The way you order the eigenvalues does not matter, but it is important that the steps described below be done with the correct blocks of the matrix system. I.e. the step that implies J_1 should be done with the block of eigenvalues that are indeed larger than unity.

We then write the system as:

$$
\begin{aligned}
\begin{vmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{vmatrix} &= A \begin{vmatrix} \hat{k}_t \\ \hat{c}_t \end{vmatrix} \\
\begin{vmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{vmatrix} &= C^{-1}\Lambda C \begin{vmatrix} \hat{k}_t \\ \hat{c}_t \end{vmatrix} \\
C \begin{vmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{vmatrix} &= \Lambda C \begin{vmatrix} \hat{k}_t \\ \hat{c}_t \end{vmatrix} \\
\begin{vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{vmatrix} \begin{vmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{vmatrix} &= \begin{vmatrix} J_1 & 0 \\ 0 & J_2 \end{vmatrix} \begin{vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{vmatrix} \begin{vmatrix} \hat{k}_t \\ \hat{c}_t \end{vmatrix} \\
\begin{vmatrix} C_{11}\hat{k}_{t+1} + C_{12}\hat{c}_{t+1} \\ C_{21}\hat{k}_{t+1} + C_{22}\hat{c}_{t+1} \end{vmatrix} &= \begin{vmatrix} J_1 & 0 \\ 0 & J_2 \end{vmatrix} \begin{vmatrix} C_{11}\hat{k}_t + C_{12}\hat{c}_t \\ C_{21}\hat{k}_t + C_{22}\hat{c}_t \end{vmatrix} \\
\begin{vmatrix} z_{t+1} \\ q_{t+1} \end{vmatrix} &= \begin{vmatrix} J_1 & 0 \\ 0 & J_2 \end{vmatrix} \begin{vmatrix} z_t \\ q_t \end{vmatrix}
\end{aligned}
$$

The first row $z_{t+s} = (J_1)^s z_t$ which goes to infinity as $J_1 > 1$. It thus must be that $z_t = 0$, which implies:

$$
\begin{aligned}
0 &= C_{11}\hat{k}_t + C_{12}\hat{c}_t \\
\hat{c}_t &= -(C_{12})^{-1} C_{11}\hat{k}_t
\end{aligned}
$$

This corresponds η_{ck} in the undetermined coefficients.

The second row of the matrix system is:

$$
\begin{aligned}
q_{t+1} &= J_2 q_t \\
C_{21}\hat{k}_{t+1} + C_{22}\hat{c}_{t+1} &= J_2 \left(C_{21}\hat{k}_t + C_{22}\hat{c}_t \right) \\
C_{21}\hat{k}_{t+1} - C_{22}(C_{12})^{-1} C_{11}\hat{k}_{t+1} &= J_2 \left(C_{21}\hat{k}_t - C_{22}(C_{12})^{-1} C_{11}\hat{k}_t \right) \\
\hat{k}_{t+1} &= \left(C_{21} - C_{22}(C_{12})^{-1} C_{11} \right)^{-1} J_2 \left(C_{21} - C_{22}(C_{12})^{-1} C_{11} \right) \hat{k}_t
\end{aligned}
$$

This corresponds η_{kk} in the undetermined coefficients.

Note that in the simple example there is only one control variable and one state variable. The method is the same when there are many. $\hat{k}$ and $\hat{c}$ are then vectors of variables, and the C's matrices.

4 Adjustment cost in investment

4.1 Firm's optimization

For simplicity, we consider that labor is constant ($n = 0$) and set it to be 1 without loss of generality. The firm faces a constant wage w and a constant real interest rate r at which profits are discounted.

The value of output produced is $f(k_t)$. Capital increases with investment, i_t (for simplicity we abstract from depreciation)

$$k_{t+1} = i_t + k_t$$

The firm faces an adjustment cost $C(i_t)$ when its investment differs from zero:

$$C(0) = 0 \qquad\qquad C'(0) = 0 \qquad\qquad C'' > 0$$

The firm's profit at time t are sales, minus the wage bill, minus the cost of investment including adjustment:

$$\Pi_t = f(k_t) - w_t - (i_t + C(i_t))$$

The Lagrangian for the firm's optimization is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \frac{1}{(1+r)^t} [f(k_t) - w_t - i_t - C(i_t)] + \sum_{t=0}^{\infty} \frac{q_t}{(1+r)^t} [i_t - k_{t+1} + k_t]$$

The first-order condition with respect to i_t is:

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial i_t} \\
0 &= \frac{\partial}{\partial i_t} \left[\frac{1}{(1+r)^t} [f(k_t) - w_t - i_t - C(i)] + \frac{q_t}{(1+r)^t} [i_t - k_{t+1} + k_t] \right] \\
0 &= -\frac{1 + C'(i_t)}{(1+r)^t} + \frac{q_t}{(1+r)^t} \\
1 + C'(i_t) &= q_t
\end{aligned}$$

The first-order condition with respect to k_{t+1} is:

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial k_{t+1}} \\
0 &= \frac{\partial}{\partial k_{t+1}} \left[\begin{array}{c} \frac{1}{(1+r)^{t+1}} [f(k_{t+1}) - w_{t+1} - i_{t+1} - C(i_{t+1})] \\ + \frac{q_{t+1}}{(1+r)^{t+1}} [i_{t+1} - k_{t+2} + k_{t+1}] + \frac{q_t}{(1+r)^t} [i_t - k_{t+1} + k_t] \end{array} \right] \\
0 &= \frac{1}{(1+r)^{t+1}} f'(k_{t+1}) + \frac{q_{t+1}}{(1+r)^{t+1}} - \frac{q_t}{(1+r)^t} \\
0 &= f'(k_{t+1}) + q_{t+1} - (1+r) q_t \\
r q_t &= f'(k_{t+1}) + q_{t+1} - q_t
\end{aligned}$$

We thus have a dynamic system in two equations:

$$\begin{aligned}
q_t - 1 &= C'(k_{t+1} - k_t) \\
r q_t &= f'(k_{t+1}) + q_{t+1} - q_t
\end{aligned}$$

The first shows that capital is constant when $q_t = 1$, and increasing when $q_t > 1$. The second shows that a constant q implies a negative relation between q and k: $rq_t = f'(k_{t+1})$. A higher value of capital given q_t implies $q_{t+1} > q_t$.

In the steady state $q^* = 1$ and $r = f'(k^*)$.