L9. Strategic Interactions II

Yuan Zi

EI037: Microeconomics
Geneva Graduate Institute (IHEID)

Outline Recap

- Basic elements
- Simultaneous-move games
- Dynamic games

Dynamic games

Introduction

- In the previous lecture we studied simultaneous-move games.
- Most economic situations however involve players **choosing actions over time**.
  - Labor unions and firms may make repeated offers and counter-offers to each other when negotiating over a new contract.
  - You may bargain with sellers at a second-hand market.
  - (Some markets are "set" like that): auctions, WTO negotiations.
- This lecture: dynamic games.
  - We only consider games with finite moves (i.e. finite games).

Big Picture

Central issue compared to simultaneous-move games: **the credibility of a player's strategy**.

Example: Your parents declare today that if you don't get A in your EI037 Microeconomics course, your allowance will be cut to half.

- Your parents are playing a strategy (i.e., threatening with a contingent payoff), hoping to affect your (optimal) **strategy**.
- Is your parents' threat credible? Why?
- How will this affect your strategy/action?

Big Picture

We want to exclude "empty threats" from dynamic games.

$\rightarrow$ thus people propose a "reasonable" equilibrium (or solution) concept to do so.

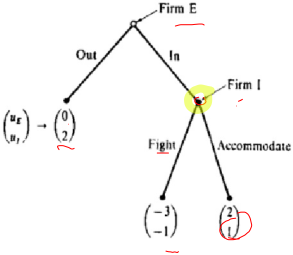
- Subgame Perfect Nash Equilibrium (SPNE)
- Weak Perfect Bayesian Equilibrium (WPBE)
- Perfect Bayesian Equilibrium (PBE)

Subgame Perfect Nash Equilibrium

We first consider dynamic games **with perfect information**.

Turns out that the Nash equilibrium concept we studied before does not suffice to rule out non-credible strategies.

Example (The Predation Game):

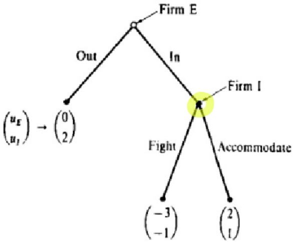


		Firm I	
		Fight if Firm E Plays "In"	Accommodate if Firm E Plays "In"
Firm E	Out	0, 2	0, 2
	In	−3, −1	2, 1

Figure 9.B.1
Extensive and normal forms for Example 9.B.1. The Nash equilibrium $(\sigma_E, \sigma_I) =$ (out, fight if firm E plays "in") involves a noncredible threat.

- What are Firm E (entrant) and I (incumbent)'s sets of (pure) strategies, respectively?
- What are the NE?
- Do you think any of the equilibria is likely to happen in reality? Anything "not reasonable"?

Example (The Predation Game):



		Firm I	
		Fight if Firm E Plays "In"	Accommodate if Firm E Plays "In"
Firm E	Out	0, 2	0, 2
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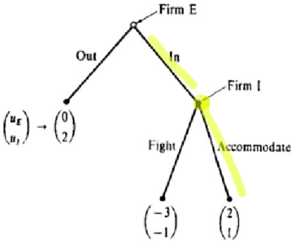
Figure 9.B.1
Extensive and normal forms for Example 9.B.1. The Nash equilibrium $(\sigma_E, \sigma_I) =$ (out, fight if firm E plays "in") involves a noncredible threat.

- As we can see from this example, in dynamic games the Nash equilibrium concept may not give sensible predictions. The concept is permitting the incumbent to make an empty threat that the entrant nevertheless takes seriously when choosing its strategy.
- To rule out predictions such as (out, fight if firm E plays "in"), we want to insist that players' equilibrium strategies satisfy what might be called the **Principle of Sequential Rationality**: a player's strategy should be optimal actions at every point in the game tree.

Example (The Predation Game):



		Firm I: Fight if Firm E Plays "In"	Firm I: Accommodate if Firm E Plays "In"
Firm E	Out	0, 2	0, 2
	In	−3, −1	2, 1

Figure 9.B.1
Extensive and normal forms for Example 9.B.1. The Nash equilibrium $(\sigma_E, \sigma_I) =$ (out, fight if firm E plays "in") involves a noncredible threat.

- This principle is intimately related to the procedure of **Backward Induction**. To find a strategy that satisfies sequential rationality, this naturally involves solving first for the optimal behavior at the "end" of the game, then the optimal behavior at the "second-to-last" stage of the game given the anticipation of this later behavior,..., and so on.
- In the predation game, which strategy(ies) do not satisfy the principle of rationality?

Backward Induction in Finite Games of Perfect Information

In fact, for any **finite games with perfect information**, backward induction can be directly applied to capture the idea of sequential rationality with great generality and power.

Proposition 9.1 (Zermelo's Theorem) Every finite game of perfect information Γ_E has a pure strategy Nash equilibrium that can be derived through backward induction. Moreover, if no players has the same payoffs at any two terminal nodes, then there is a unique Nash equilibrium that can be derived in this manner.

Note that incidentally Zermelo's theorem establishes the existence of a pure strategy Nash equilibrium in all finite games with perfect information.

Backward Induction and Reduced Game:

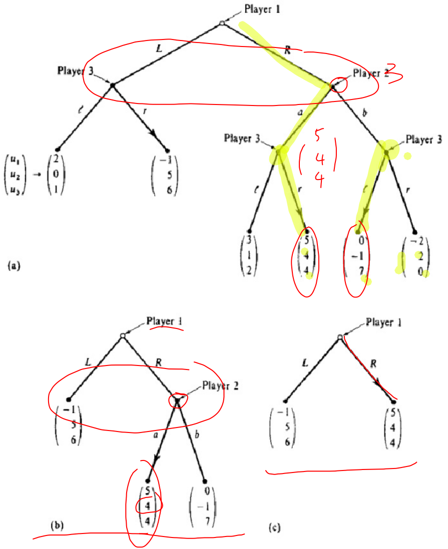


Figure 9.B.3
Reduced games in a backward induction procedure for a finite game of perfect information.
(a) Original game.
(b) First reduced game. (c) Second reduced game.

R a r

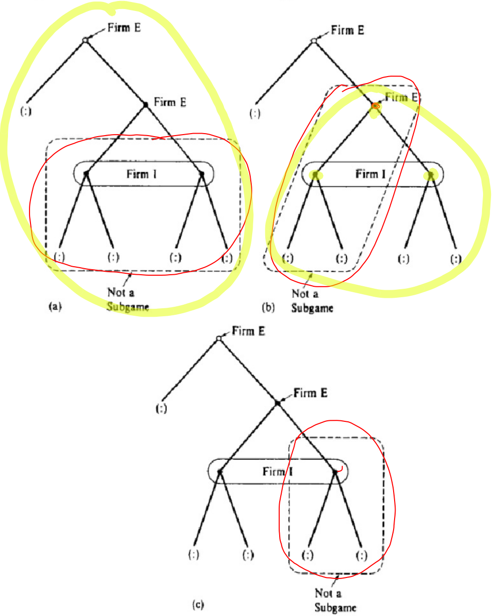
Subgame Perfect Nash Equilibrium

It is clear enough how to apply the principle of sequential rationality in finite games with perfect information. However, to apply the principle in more general games involving imperfect information, we need to first specify what is a "subsequent decision unit", i.e., **subgame**.

Definition 9.1 (Subgame) A subgame of an extensive form game Γ_E is a subset of the game having the following properties:

1. It begins with an information set containing a single decision node, contains all the decision nodes that are successors (both immediate and later) of this node, and contains *only* these nodes.
2. If the decision node x is in the subgame, then every $x' \in H(x)$ is also in the subgame, where $H(x)$ is the information set that contains decision node x. (That is, there are no "broken" information sets).

Example: Three parts of a game that are not subgames.



The key feature of a subgame is that, contemplated in isolation, is it a game in its own right.

We can therefore apply to it the idea of Nash equilibrium predictions:

When we required that the equilibrium concept satisfy sequential rationality in all subgames, we basically introduced the definition of **Subgame Perfect Nash Equilibirum**

Definition 9.2 (Subgame Perfect Nash Equilibrium) A player's strategy $\sigma = (\sigma_1, ..., \sigma_I)$ in an I-player extensive form game Γ_E is a subgame perfect Nash equilibrium (SPNE) if it induces a Nash equilibrium in every subgame of Γ_E.

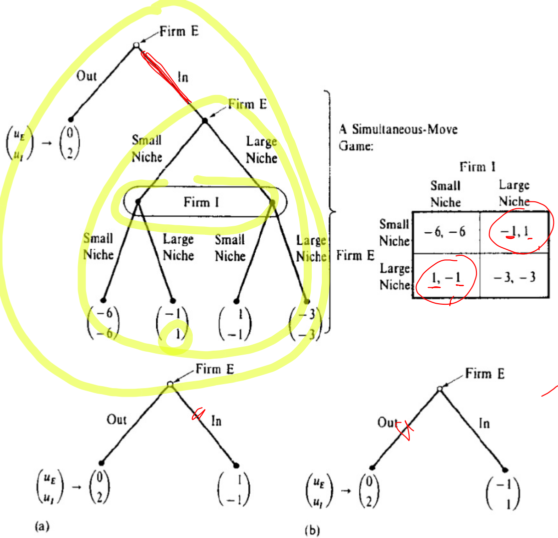
Proposition 9.2 Every finite game of perfect information Γ_E has a pure strategy subgame perfect Nash equilibrium. Moreover, if no players have the same payoffs at any two terminal nodes, then there is a unique subgame perfect Nash equilibrium.

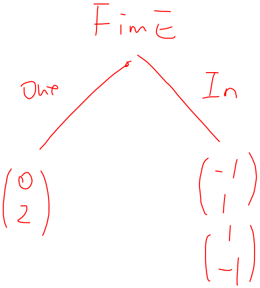
To identify the set of SPNE in a general (finite) dynamic game Γ_E, we can use a **generalization of the backward induction procedure**:

1. Start at the end of the game tree, identify the Nash equilibria for each of the final subgames.
2. Select one Nash equilibrium in each of these subgames, and derive the reduced extensive form game in which these final subgames are replaced by the payoffs that result in these subgames when players use these equilibrium strategies.
3. Repeat step 1 and 2 backward, until every move in Γ_E is terminated.
4. If multiple equilibria are never encountered in any step of this process, this profile of strategies is the unique SPNE. If multiple equilibria are encountered, the full set of SPNEs is identified by repeating the procedure for each possible equilibrium that could occur for the subgames in question.

Generalized Backward Induction for Games with Imperfect Information

Example: The generalized backward induction to find SPNE in dynamic games with imperfect information

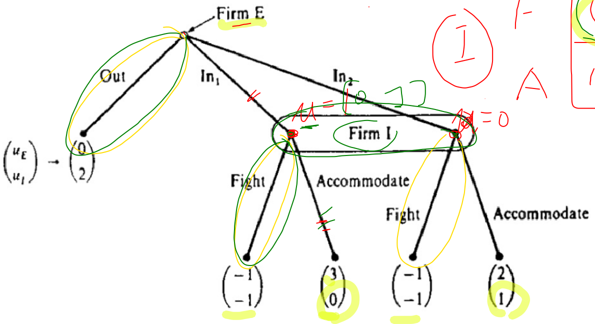




Beliefs and Sequential Rationality

Although subgame perfection is often very useful in capturing the principle of sequential rationality, sometimes it is not enough. Consider the following variation of the entry game:

Example:



E

I	O	I_1	I_2
F	(0, 2)	(−1, −1)	(−1, −1)
A	(0, 2)	(3, 0)	(2, 1)

WPBE

PBE

WPBE

Figure 9.C.1
Extensive form for Example 9.C.1. The SPNE concept may fail to insure sequential rationality.

- the criterion of subgame perfection is of no use here, because the only subgame is the game as a whole, all pure strategy NE are SPNE.
- Find out all SPNE. Are strategies containing "fight" reasonable for Firm I to play?

Beliefs and Sequential Rationality

To exclude empty threats in this case, we introduce a new solution concept: **weak perfect Bayesian equilibrium** (or can be called weak sequential equilibrium).

This extends the principle of sequential rationality by formally introducing the notion of **beliefs**.

Definition 9.3 (Belief) A system of beliefs μ in extensive form game Γ_E is a specification of a probability $\mu(x) \in [0,1]$ for each decision node x in Γ_E such that

$$\sum_{x \in H} \mu(x) = 1$$

for all information set H.

- Can you design a set of beliefs for the entry game in the previous slide?

Definition 9.4 (sequential rational i.t.o information set) A strategy profile $\sigma = (\sigma_1, ..., \sigma_I)$ in extensive form game Γ_E is sequentially rational at information set H given a system of beliefs μ if, denoting by $\iota(H)$ the player who moves at information set H, we have

$$E(U_{\iota(H)} \mid H, \mu, \sigma_{\iota(H)}, \sigma_{-\iota(H)}) \geq E(U_{\iota(H)} \mid H, \mu, \tilde{\sigma}_{\iota(H)}, \sigma_{-\iota(H)})$$

for all information set H.

$\rightarrow$ In words, a strategy profile is sequentially rational if no player finds it worthwhile, once one of her information sets has been reached, to revise her strategy given her beliefs about what has already occurred (as embodied in μ) and her rivals' strategies.

Weak Perfect Bayesian Equilibrium

With these two notations (Definition 9.3 and 9.4), we can now define a weak perfect Bayesian equilibrium. The definition involves two conditions:

1. Strategies must be sequentially rational given beliefs.
2. Whenever possible, beliefs should be consistent with strategies.
   - *In equilibrium*, players should have correct beliefs about their opponents' strategy choices.
   - We give an example in the next slides.

Correct Beliefs

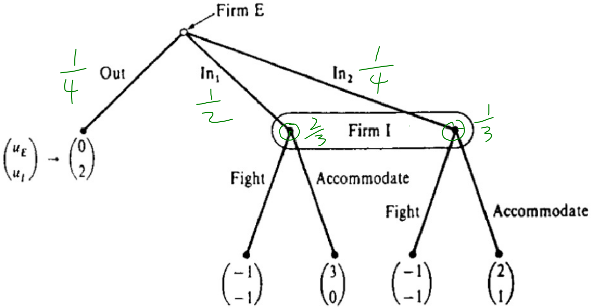


Figure 9.C.1
Extensive form for Example 9.C.1. The SPNE concept may fail to insure sequential rationality.

- Suppose firm E uses the completely mixed strategy: $(out = \frac{1}{4}, in_1 = \frac{1}{2}, in_2 = \frac{1}{4})$.
- Then firm I must have correct beliefs that his information set will be reached with probability $\frac{3}{4}$.
- Conditional on the information set having been reached, firm I's belief on which node he is in should follow Bayes' rule: the conditional probability of him being at the left node is $\frac{2}{3}$, at the right node is $\frac{1}{3}$.

Correct Beliefs

- The more difficult issue arrives when firm E is not using completely mixed strategies.

- In this case, some information sets may no longer be reached with positive probability, so we cannot use Bayes' rule to compute conditional probabilities for nodes in these information sets.

- Intuitively, even if players were to play the game repetitively, the equilibrium play would generate no experience on which they could "validate" their beliefs at information sets that are off the equilibrium path.

1. The **weak perfect Bayesian equilibrium** concept takes an agnostic view towards what players should believe in these information sets: their beliefs can be any.

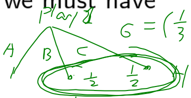
2. The **perfect Bayesian equilibrium**, on the other hand, requires players' beliefs to be robust when they accidentally arrive at off-equilibrium sets (we study in the end).

Definition 9.5 (Weak Perfect Bayesian Equilibrium) A profile of strategies and system of beliefs (σ, μ) is a weak perfect Bayesian equilibrium (weak PBE) in extensive form game Γ_E if it has the following properties:

i The strategy profile σ is sequentially rational given belief system μ.

ii The system of beliefs μ is derived from strategy profile σ through Bayes' rule whenever possible. That is, for any information set H such that $prob(H \mid \sigma) > 0$, we must have

$$\mu(x) = \frac{Prob(x \mid \sigma)}{Prob(H \mid \sigma)}, \text{for all } x \in H$$



$\sigma = (\frac{1}{3} \quad \frac{1}{3} \quad \frac{1}{3})$

$Prob(x \mid \sigma) = \frac{1}{3}$

A . B . C

$Prob(H \mid \sigma) = \frac{2}{3}$

$\mu(B) = \frac{1/3}{2/3} = \frac{1}{2}$

- Note that the definition formally incorporates beliefs as part of an equilibrium.

Find out weak PBEs for the following games. What are the system of beliefs supporting each weak PBEs?

Example 1:

$Pr(In1 \mid \sigma_E) = 1$

$Pr(In2 \mid \sigma_E) = 0$

$Pr(In1 \mid H) = 1$

Out F

A In1 $\mu = 1$

SPNE $\rightarrow$ μ

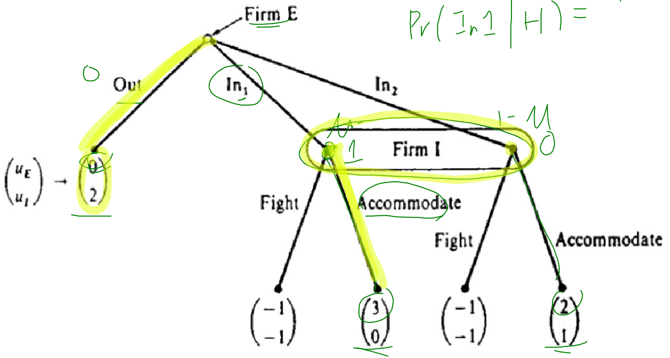


Figure 9.C.1

Extensive form for Example 9.C.1. The SPNE concept may fail to insure sequential rationality.

Example 2:

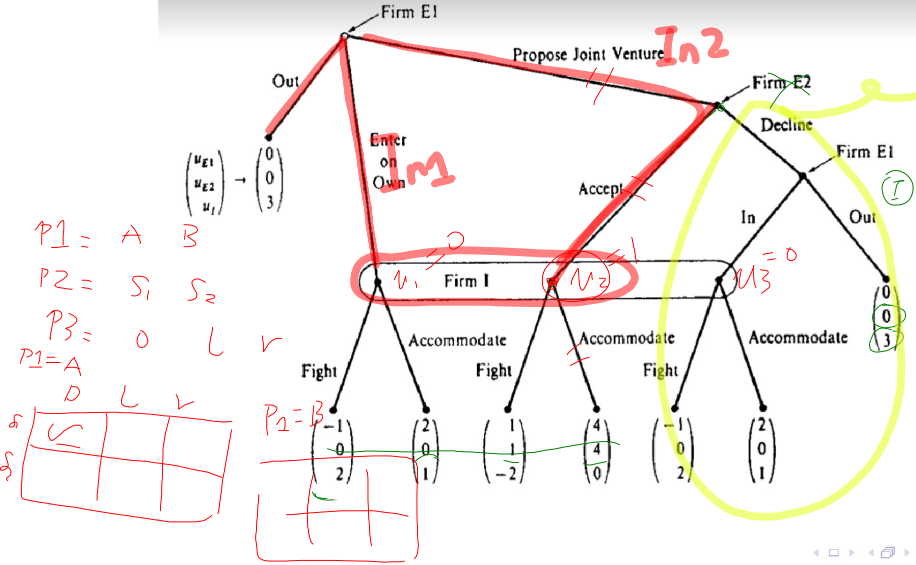


Figure 9.C.2
Extensive form for Example 9.C.2.

E1

I

	Out	E	J
F	(0, 3)	(-1, 2)	(1, -2)
A	(0, 3)	(2, 1)	(4, 0)

WPBE

$\mu_2 = 1$

Example 3:

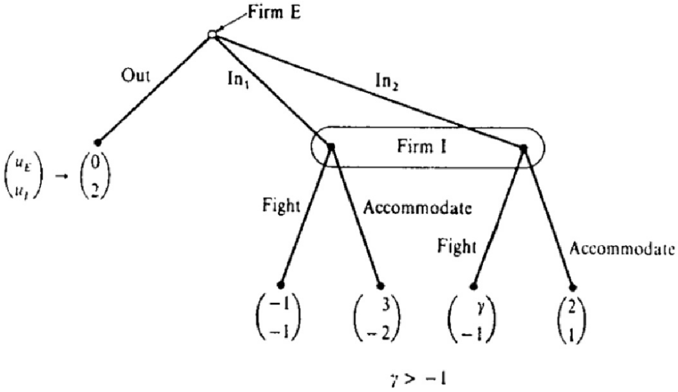


Figure 9.C.3

Extensive form for Example 9.C.3.

Strengthening of the Weak Perfect Bayesian Equilibrium Concept

Sometimes the concept of weak PBE can be too weak.

Example:

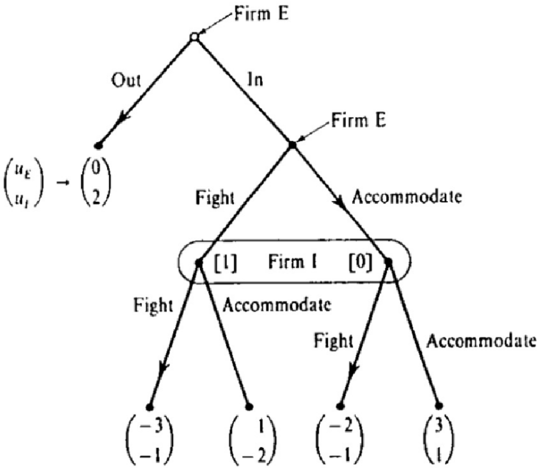


Figure 9.C.5

Extensive form for Example 9.C.5. A weak PBE may not be subgame perfect.

One weak PBE of this game is $(\sigma_E, \sigma_I) = $ ([out, "accommodate" if in], ["fight" if firm E players "in"]) combined with beliefs for firm I that assign probability 1 to firm E having played flight.

Note that these strategies are not subgame perfect!

The problem is that firm I's post-entry belief about E's post-entry play is unrestricted by weak PBE, because firm I's information set is off the equilibrium path.

– Note that weak PBE only requires sequential rationality for any information set H such that $prob(H \mid \sigma) > 0$, but here, the information set of I is such that $prob(H \mid \sigma) = 0$!

Thus in literature extra consistency restrictions on beliefs are often added to the weak PBE concept to avoid these problems, with the resulting solution concept referred to as Perfect Bayesian Equilibrium (or to be called sequential equilibrium).

Perfect Bayesian Equilibrium

Definition 9.6 (Perfect Bayesian Equilibrium) A profile of strategies and system of beliefs (σ, μ) is a Perfect Bayesian equilibrium of extensive form game Γ_E if it has the following properties:

i The strategy profile is sequentially rational given the belief system μ.

ii There exists a sequence of completely mixed strategies that is close to the equilibrium strategy σ (i.e., with small perturbation of the equilibrium strategies), such that the players can (approximately) justify their beliefs even in some small probability scenario that players make mistakes in choosing their strategies.

Example:

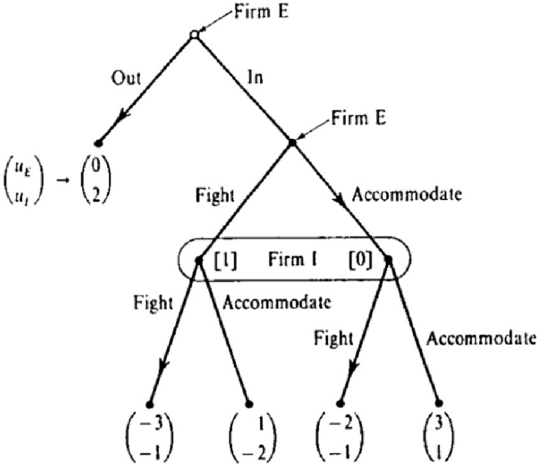


Figure 9.C.5
Extensive form for Example 9.C.5. A weak PBE may not be subgame perfect.

- What is(are) the PBE(s) of this game?

To see why ([out, “accommodate” if in], [“fight” if firm E players “in”]) is not PBE:

1. If somehow (even with a small probability) firm I reached his information set (action node), he will know that firm E played “In”.
2. Being rational, once E played “In” there is no reason for E to play “fight” because it gives lower payoffs regardless of I’s action. So firm I can guess that E must have played “accomodate”.
3. Given this reasonable guess, [“fight” if firm E players “in”] is not the optimal strategy for firm I.
4. This inconsistency implies that ([out, “accommodate” if in], [“fight” if firm E players “in”]) is not a PBE.

⇒ The unique PBE in the previous game are those of the unique SPNE: ([in, accommodate if in],[accommodate if firm I plays in])