Macroeconomics A; EI056

Technical appendix: The real business cycles (RBC) model with productivity and population trends

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1 Building blocks of the model

Consider a closed economy where output is produced using labor and capital:

$$Y_t = (K_t)^{\alpha} (A_t L_t)^{1-\alpha} \qquad ; \qquad 0 < \alpha < 1 \tag{1}$$

where A is an exogenous stochastic productivity parameter. Output can be consumed, used by the government, or invested:

$$Y_t = C_t + G_t + I_t$$

Capital depreciates at a rate δ, so the capital dynamics are:

$$K_{t+1} = (1-\delta) K_t + I_t \tag{2}$$

Output is produced by a firm. At period t the firms buys K_t units of capital from the consumers at a price 1 (the capital good and the consumption goods are the same). It borrows at a real interest rate r_t to finance this purchase. At the end of the period the firms sell the remaining units of capital, $(1-\delta) K_t$, to the consumers. The firm's profits are then:

$$\Pi_t = (K_t)^{\alpha} (A_t L_t)^{1-\alpha} - w_t L_t - [(1+r_t) - (1-\delta)] K_t$$

where w is the real wage. The maximization of profits with respect to labor and capital leads to:

$$\frac{\partial \Pi_t}{\partial L_t} = 0 \Rightarrow w_t = (1-\alpha) \left(\frac{K_t}{A_t L_t} \right)^{\alpha} A_t \tag{3}$$

$$\frac{\partial \Pi_t}{\partial K_t} = 0 \Rightarrow r_t = \alpha \left(\frac{K_t}{A_t L_t} \right)^{\alpha-1} - \delta \tag{4}$$

The economy is subjected to shocks in government spending and productivity. Both variables have a trend growth rate and fluctuate around it (they are trend-reverting). The growth rate for productivity is g and for government spending it is $g+n$ (n being the population growth rate):

$$\ln A_t = \bar{A} + gt + \hat{A}_t \quad ; \quad \hat{A}_t = \rho_A \hat{A}_{t-1} + \varepsilon_{A,t} \tag{5}$$

$$\ln G_t = \bar{G} + (g+n)\,t + \hat{G}_t \quad ; \quad \hat{G}_t = \rho_G \hat{G}_{t-1} + \varepsilon_{G,t} \tag{6}$$

where ρ_A and ρ_G are the persistence of shocks (both are between -1 and 1).

Consumers maximize an intertemporal utility in consumption and leisure:

$$U_t = E_t \sum_{s=0}^{\infty} e^{-\rho s} \left[\ln c_{t+s} + b \ln (1 - l_{t+s})\right] N_{t+s}$$

where c and l are per capita consumption and labor ($C = Nc$, $L = Nl$) and N is the population that grows at a rate n (we set the H in Romer equal to one):

$$\ln N_t = \bar{N} + nt$$

This is maximized subject to the budget constraint:

$$C_t + I_t = w_t L_t + \left[(1 + r_t) - (1 - \delta)\right] K_t - T_t \tag{7}$$

where T denotes lump-sum taxes. We assume that the government runs a balances budget, so $T_t = G_t$. Consumers earn money from wages and renting out capital to firms, pay taxes, and use their disposable income to consume and invest in capital. Using (2), we write (7) as:

$$C_t + K_{t+1} = w_t L_t + (1 + r_t) K_t - G_t \tag{8}$$

As there is uncertainty, the future variables need to be indexes not only by time, but also by the state of nature in which the economy can find itself in the future. We index the possible states at time $t+s$ by x_{t+s}. The probability of being in state x_{t+s} is $\pi(x_{t+s})$, and the value of a variable z in that state is $z(x_{t+s})$. From the point of view of the current period, the consumer maximizes the expected utility, which we write as the sum across state weighted by their probability:

$$\begin{aligned}
& E_t \sum_{s=0}^{\infty} e^{-\rho s} \left[\ln c_{t+s} + b \ln (1 - l_{t+s})\right] N_{t+s} \\
= & \sum_{x_{t+s}} \pi(x_{t+s}) \left(\sum_{s=0}^{\infty} e^{-\rho s} \left[\ln c(x_{t+s}) + b \ln (1 - l(x_{t+s}))\right] N(x_{t+s}) \right) \\
= & \sum_{s=0}^{\infty} e^{-\rho s} \sum_{x_{t+s}} \pi(x_{t+s}) \left[\left[\ln c(x_{t+s}) + b \ln (1 - l(x_{t+s}))\right] N(x_{t+s})\right] \\
= & \sum_{s=0}^{\infty} e^{-\rho s} N_{t+s} \sum_{x_{t+s}} \pi(x_{t+s}) \left[\ln c(x_{t+s}) + b \ln (1 - l(x_{t+s}))\right]
\end{aligned}$$

where we use the fact that as population growth is not random, then for any state of nature in period $t+s$ we have: $N(x_{t+s}) = N_{t+s}$. The budget constraint in a specific state of nature at $t+s$ is:

$$N_{t+s}c(x_{t+s}) + K(x_{t+s+1}) = w(x_{t+s})N_{t+s}l(x_{t+s}) + (1+r(x_{t+s}))K(x_{t+s}) - G(x_{t+s})$$

The optimization is then written as the following Lagrangian:

$$\mathcal{L} = \sum_{s=0}^{\infty} e^{-\rho s} \sum_{x_{t+s}} \pi(x_{t+s}) \left\{ \begin{array}{c} N_{t+s}\ln(c(x_{t+s})) + b\ln(1-l(x_{t+s}))N_{t+s} \\ -\varphi(x_{t+s}) \left[\begin{array}{c} K(x_{t+s+1}) + N_{t+s}c(x_{t+s}) + G(x_{t+s}) \\ -w(x_{t+s})N_{t+s}l(x_{t+s}) - (1+r(x_{t+s}))K(x_{t+s}) \end{array} \right] \end{array} \right\}$$

We now take the first-order conditions, focusing on variables at time t and $t+1$ for simplicity. The first-order condition with respect to current consumption is:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial c(x_t)} \\ 0 &= N_t \left\{ \frac{1}{c(x_t)} - \varphi(x_t) \right\} \\ \frac{1}{c(x_t)} &= \varphi(x_t) \end{aligned}$$

where we used $\pi(x_t) = 1$ as at period t we know which state we are in. The first-order condition with respect to future consumption in a specific state is:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial c(x_{t+1})} \\ 0 &= e^{-\rho}\pi(x_{t+1})N_{t+1} \left\{ \frac{1}{c(x_{t+1})} - \varphi(x_{t+1}) \right\} \\ \frac{1}{c(x_{t+1})} &= \varphi(x_{t+1}) \end{aligned}$$

The first-order condition with respect to current labor is:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial l(x_t)} \\ 0 &= N_t \left\{ -\frac{b}{1-l(x_t)} + \varphi(x_t)w(x_t) \right\} \\ \frac{b}{1-l(x_t)} &= \varphi(x_t)w(x_t) \end{aligned}$$

The first-order condition with respect to future labor in a specific state is:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial l(x_{t+1})} \\ 0 &= e^{-\rho}\pi(x_{t+1})N_{t+1} \left\{ -\frac{b}{1-l(x_{t+1})} + \varphi(x_{t+1})w(x_{t+1}) \right\} \end{aligned}$$

$$\frac{b}{1-l\left(x_{t+1}\right)} \quad = \quad \varphi\left(x_{t+1}\right) w\left(x_{t+1}\right)$$

Finally, we write the first-order condition with respect to capital $K\left(x_{t+1}\right)$. Note that this future capital stock is determined in period t and will apply to all possible states of nature in period $t+1$:

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial K\left(x_{t+1}\right)} \\
0 &= -\varphi\left(x_t\right)+e^{-\rho} \sum_{x_{t+1}} \pi\left(x_{t+1}\right) \varphi\left(x_{t+1}\right)\left(1+r\left(x_{t+1}\right)\right) \\
\varphi\left(x_t\right) &= e^{-\rho} \sum_{x_{t+1}} \pi\left(x_{t+1}\right) \varphi\left(x_{t+1}\right)\left(1+r\left(x_{t+1}\right)\right)
\end{aligned}$$

As the state of nature is known at time t, we can denote variables simply by the time subscript (for instance $w\left(x_t\right)=w_t$). Combining the first order conditions with respect to current labor and consumption gives the labor supply:

$$\begin{aligned}
\frac{b}{1-l_t} &= w_t \varphi_t \\
\frac{b}{1-l_t} &= w_t \frac{1}{c_t} \\
w_t &= \frac{b}{1-l_t} c_t \qquad (9)
\end{aligned}$$

The expectation of a variable is the sum of its values across states of natures time the probabilities of these states:

$$E_t z_{z+1}=\sum_{x_{t+1}} \pi\left(x_{t+1}\right) z\left(x_{t+1}\right)$$

Combining the optimality conditions with respect to capital and consumptions, we get the Euler condition:

$$\begin{aligned}
\varphi_t &= e^{-\rho} E_t \varphi_{t+1}\left(1+r_{t+1}\right) \\
\frac{1}{c_t} &= e^{-\rho} E_t\left[\frac{1}{c_{t+1}}\left(1+r_{t+1}\right)\right] \qquad (10)
\end{aligned}$$

Combining (3)-(4) with (9)-(10), we compute two of the key relations, namely the Euler and the labor market equilibrium:

$$\begin{aligned}
\frac{b}{1-l_t} c_t &= w_t=(1-\alpha)\left(\frac{K_t}{A_t L_t}\right)^\alpha A_t \qquad (11) \\
\frac{1}{c_t} &= e^{-\rho} E_t\left[\left(1+r_{t+1}\right) \frac{1}{c_{t+1}}\right] \\
&= e^{-\rho} E_t\left[\frac{1}{c_{t+1}}\left(1+\alpha\left(\frac{K_{t+1}}{A_{t+1} L_{t+1}}\right)^{\alpha-1}-\delta\right)\right] \qquad (12)
\end{aligned}$$

(1) and (8) imply:

$$\begin{aligned}
K_{t+1} &= w_t L_t + (1+r_t) K_t - G_t - C_t \\
&= w_t L_t + (r_t + \delta) K_t + (1-\delta) K_t - G_t - C_t \\
&= (1-\alpha)\left(\frac{K_t}{A_t L_t}\right)^{\alpha} A_t L_t + \alpha \left(\frac{K_t}{A_t L_t}\right)^{\alpha-1} K_t + (1-\delta) K_t - G_t - C_t \\
&= (K_t)^{\alpha} (A_t L_t)^{1-\alpha} + (1-\delta) K_t - G_t - C_t \qquad (13)
\end{aligned}$$

The model is summarized by (11)-(13) and the shocks (5)-(6).

Investment is computed from the capital accumulation (2)

$$\begin{aligned}
I_t &= K_{t+1} - (1-\delta) K_t \\
\frac{I_t}{K_t} &= \frac{K_{t+1}}{K_t} - (1-\delta)
\end{aligned}$$

2 The steady growth path

2.1 Analytical results

Because productivity and government spending per capita grow at a trend rate g and population grows at a rate n, the system has a drift built into it. We start by solving for the steady growth path (i.e. the trend-steady state). Along that path we denote variables with an upper bar. All variables grow at steady rates. Specifically, per-capita consumption c grows at the rate of productivity g ($ln(\bar{c}_{t+1}) - ln(\bar{c}_t) = g$).

We start by writing (12) as:

$$\begin{aligned}
\frac{1}{\bar{c}_t} &= e^{-\rho} \frac{1}{\bar{c}_{t+1}} \left(1 + \alpha \left(\frac{\bar{K}_{t+1}}{\bar{A}_{t+1}\bar{L}_{t+1}}\right)^{\alpha-1} - \delta\right) \\
1 &= e^{-\rho} \frac{\bar{c}_t}{\bar{c}_{t+1}} \left(1 + \alpha \left(\frac{\bar{K}_{t+1}}{\bar{A}_{t+1}\bar{L}_{t+1}}\right)^{\alpha-1} - \delta\right) \\
1 &= e^{-\rho} \frac{\exp[\ln \bar{c}_t]}{\exp[\ln \bar{c}_{t+1}]} \left(1 + \alpha \left(\frac{\bar{K}_{t+1}}{\bar{A}_{t+1}\bar{L}_{t+1}}\right)^{\alpha-1} - \delta\right) \\
1 &= e^{-\rho} \frac{\exp[\ln \bar{c}_t]}{\exp[\ln \bar{c}_t + g]} \left(1 + \alpha \left(\frac{\bar{K}_{t+1}}{\bar{A}_{t+1}\bar{L}_{t+1}}\right)^{\alpha-1} - \delta\right) \\
1 &= e^{-\rho} \frac{1}{\exp[g]} \left(1 + \alpha \left(\frac{\bar{K}_{t+1}}{\bar{A}_{t+1}\bar{L}_{t+1}}\right)^{\alpha-1} - \delta\right) \\
1 &= e^{-\rho-g} \left(1 + \alpha \left(\frac{\bar{K}_{t+1}}{\bar{A}_{t+1}\bar{L}_{t+1}}\right)^{\alpha-1} - \delta\right)
\end{aligned}$$

This implies that the ratio $\bar{K}_{t+1}/\left(\bar{A}_{t+1}\bar{L}_{t+1}\right)$ is constant, and we denote it by $\bar{k}$:

$$\bar{k} = \left(\frac{\alpha}{e^{\rho+g} - (1-\delta)}\right)^{\frac{1}{1-\alpha}} \tag{14}$$

Capital K thus grows at a rate $g+n$.(11) is then written as:

$$\begin{aligned}
\frac{b}{1-\bar{l}_t}\bar{c}_t &= (1-\alpha)\left(\bar{k}\right)^{\alpha}\bar{A}_t \\
\frac{b}{1-\bar{l}_t}\exp\left[c_0 + gt\right] &= (1-\alpha)\left(\bar{k}\right)^{\alpha}\exp\left[a_0 + gt\right] \\
\frac{b}{1-\bar{l}_t} &= (1-\alpha)\left(\bar{k}\right)^{\alpha} e^{a_0 - c_0}
\end{aligned}$$

The labor input per capita is thus constant:

$$\bar{l} = 1 - \frac{b}{(1-\alpha)\left(\bar{k}\right)^{\alpha}}\frac{C_0}{A_0} \tag{15}$$

As l is constant capital grows at a rate $g+n$, and:

$$\frac{\bar{K}_{t+1}}{\bar{A}_{t+1}\bar{L}_{t+1}} = \frac{\bar{K}_{t+1}}{\bar{A}_{t+1}\bar{N}_{t+1}\bar{l}} = \frac{\exp\left[k_0 + (g+n)(t+1)\right]}{\exp\left[a_0 + g(t+1) + n_0 + n(t+1)\right]\bar{l}} = \frac{K_0}{A_0 N_0}\frac{1}{\bar{l}}$$

which gives $e^{k_0 - a_0 - n_0}$:

$$\frac{K_0}{A_0 N_0} = \bar{l}\bar{k} \tag{16}$$

(13) is written as:

$$\begin{aligned}
\bar{K}_{t+1} &= (1-\delta)\bar{K}_t + \left(\bar{K}_t\right)^{\alpha}\left(\bar{A}_t\bar{L}_t\right)^{1-\alpha} - \bar{C}_t - \bar{G}_t \\
\frac{\bar{K}_{t+1}}{\bar{K}_t} &= (1-\delta) + \left(\frac{\bar{A}_t\bar{L}_t}{\bar{K}_t}\right)^{1-\alpha} - \frac{N_c\bar{c}_t}{\bar{K}_t} - \frac{\bar{G}_t}{\bar{K}_t} \\
\frac{\exp\left[k_0 + (g+n)(t+1)\right]}{\exp\left[k_0 + (g+n)t\right]} &= (1-\delta) + \left(\frac{1}{\bar{k}}\right)^{1-\alpha} \\
&\quad - \frac{\exp\left[n_0 + c_0 + (g+n)t\right]}{\exp\left[k_0 + (g+n)t\right]} - \frac{\exp\left[g_0 + (g+n)t\right]}{\exp\left[k_0 + (g+n)t\right]} \\
e^{g+n} &= (1-\delta) + \left(\bar{k}\right)^{\alpha-1} - e^{n_0 + c_0 - k_0} - e^{g_0 - k_0} \\
e^{g+n} &= (1-\delta) + \left(\bar{k}\right)^{\alpha-1} - \frac{N_0 C_0}{K_0} - \frac{G_0}{K_0}
\end{aligned} \tag{17}$$

Investment is written as:

$$\begin{aligned}
\frac{I_t}{K_t} &= \frac{K_{t+1}}{K_t} - (1-\delta) \\
\frac{I_t}{K_t} &= \frac{K_{t+1}}{A_{t+1}L_{t+1}}\frac{A_t L_t}{K_t}\frac{A_{t+1}L_{t+1}}{A_t L_t} - (1-\delta) \\
\frac{I_t}{K_t} &= \frac{\bar{k}}{\bar{k}}\frac{\exp\left[a_0 + n_0 + (g+n)(t+1)\right]}{\exp\left[a_0 + n_0 + (g+n)(t)\right]} - (1-\delta)
\end{aligned}$$

$$\frac{I_t}{K_t} = e^{g+n} - (1-\delta)$$

(3) and (4) imply:

$$\begin{aligned} w_t &= (1-\alpha)\left(\bar{k}\right)^{\alpha}\exp\left[a_0 + gt\right] \\ r_t &= \bar{r} = \alpha\left(\bar{k}\right)^{\alpha-1} - \delta \end{aligned} \tag{18}$$

The real wage grows at a rate g and the real interest rate is constant.

2.2 Calibration

The various expressions can be calibrated using some key variables in the data. We set values of α and δ. Taking the average real interest rate in the data $\bar{r}$ then gives $\bar{k}$ from (18):

$$\bar{k} = \left(\frac{\alpha}{\bar{r}+\delta}\right)^{\frac{1}{1-\alpha}}$$

Combining this with (14) we solve for $\rho + g$ as: $e^{\rho+g} = 1 + \bar{r}$. We can then set a value of g and infer ρ.

Recall that the production function is:

$$\begin{aligned} Y_t &= (K_t)^{\alpha}(A_t L_t)^{1-\alpha} \\ \frac{Y_t}{K_t} &= \left(\frac{A_t L_t}{K_t}\right)^{1-\alpha} \\ \frac{Y_t}{K_t} &= \frac{\bar{Y}}{\bar{K}} = \frac{Y_0}{K_0} = \left(\bar{k}\right)^{\alpha-1} = \frac{\bar{r}+\delta}{\alpha} \end{aligned}$$

Next, we set the steady state ratio of government consumption to GDP, $\bar{G}/\bar{Y} = G_0/Y_0$ and compute:

$$\frac{G_0}{K_0} = \frac{G_0}{Y_0}\frac{Y_0}{K_0} = \frac{G_0}{Y_0}\frac{\bar{r}+\delta}{\alpha}$$

Setting the value of the population growth rate n. (17) then gives $N_0 C_0/K_0$:

$$\frac{N_0 C_0}{K_0} = (1-\delta) + \left(\bar{k}\right)^{\alpha-1} - e^{g+n} - \frac{G_0}{K_0}$$

3 Log linearization

We expand the variables around the steady growth path. Define tilde variables as log deviations:

$$\hat{X}_t = \frac{X_t - \bar{X}_t}{\bar{X}_t}$$

(5)-(6) are already in the right form:

$$\hat{A}_t = \rho_A \hat{A}_{t-1} + \varepsilon_{A,t} \qquad ; \qquad \hat{G}_t = \rho_G \hat{G}_{t-1} + \varepsilon_{G,t}$$

The left-hand side of (11) is linearized as follows:

$$\begin{aligned}
\frac{b}{1-l_t}c_t &= \frac{b}{1-\bar{l}}\bar{c}_t+\frac{b}{1-\bar{l}}\left(c_t-\bar{c}_t\right)-b\bar{c}_t\left(\frac{1}{1-\bar{l}}\right)^2(-1)\left(l_t-\bar{l}\right)\\
&= \frac{b}{1-\bar{l}}\bar{c}_t+\frac{b}{1-\bar{l}}\bar{c}_t\frac{c_t-\bar{c}_t}{\bar{c}_t}-\frac{b\bar{c}_t}{1-\bar{l}}\frac{1}{1-\bar{l}}(-1)\,\bar{l}\frac{l_t-\bar{l}}{\bar{l}}\\
&= \frac{b}{1-\bar{l}}\bar{c}_t\left(1+\frac{c_t-\bar{c}_t}{\bar{c}_t}+\frac{\bar{l}}{1-\bar{l}}\frac{l_t-\bar{l}}{\bar{l}}\right)\\
&= \frac{b}{1-\bar{l}}\bar{c}_t\left(1+\hat{c}_t+\frac{\bar{l}}{1-\bar{l}}\hat{l}_t\right)
\end{aligned}$$

The right-hand side of (11) is linearized as follows (N never deviates from the steady growth path):

$$\begin{aligned}
&(1-\alpha)\left(\frac{K_t}{A_tN_tl_t}\right)^\alpha A_t\\
=\;&(1-\alpha)\left(\bar{k}\right)^\alpha\bar{A}_t+(1-\alpha)\left(\bar{k}\right)^\alpha\left(A_t-\bar{A}_t\right)\\
&+(1-\alpha)\,\bar{A}_t\alpha\left[\left(\bar{k}\right)^{\alpha-1}\frac{K_t-\bar{K}_t}{\bar{A}_t\bar{N}_t\bar{l}}-\left(\bar{k}\right)^{\alpha-1}\bar{k}\frac{1}{\bar{A}_t}\left(A_t-\bar{A}_t\right)\right]\\
&-(1-\alpha)\,\bar{A}_t\alpha\left(\bar{k}\right)^{\alpha-1}\bar{k}\frac{1}{\bar{l}}\left(l_t-\bar{l}\right)\\
=\;&(1-\alpha)\left(\bar{k}\right)^\alpha\bar{A}_t\left(1+\frac{A_t-\bar{A}_t}{\bar{A}_t}\right)\\
&+(1-\alpha)\,\bar{A}_t\alpha\left[\left(\bar{k}\right)^\alpha\frac{K_t-\bar{K}_t}{\bar{K}_t}-\left(\bar{k}\right)^\alpha\frac{A_t-\bar{A}_t}{\bar{A}_t}-\left(\bar{k}\right)^\alpha\frac{l_t-\bar{l}}{\bar{l}}\right]\\
=\;&(1-\alpha)\left(\bar{k}\right)^\alpha\bar{A}_t\left(1+\hat{A}_t+\alpha\left[\hat{K}_t-\hat{A}_t-\hat{l}_t\right]\right)\\
=\;&(1-\alpha)\left(\bar{k}\right)^\alpha\bar{A}_t\left(1+(1-\alpha)\,\hat{A}_t+\alpha\hat{K}_t-\alpha\hat{l}_t\right)
\end{aligned}$$

(11) is then:

$$\begin{aligned}
\frac{b}{1-l_t}c_t &= (1-\alpha)\left(\frac{K_t}{A_tN_tl_t}\right)^\alpha A_t\\
\frac{b}{1-\bar{l}}\bar{c}_t\left(1+\hat{c}_t+\frac{\bar{l}}{1-\bar{l}}\hat{l}_t\right) &= (1-\alpha)\left(\bar{k}\right)^\alpha\bar{A}_t\left(1+(1-\alpha)\,\hat{A}_t+\alpha\hat{K}_t-\alpha\hat{l}_t\right)\\
\hat{c}_t+\frac{\bar{l}}{1-\bar{l}}\hat{l}_t &= (1-\alpha)\,\hat{A}_t+\alpha\hat{K}_t-\alpha\hat{l}_t\\
\hat{c}_t+\left(\frac{\bar{l}}{1-\bar{l}}+\alpha\right)\hat{l}_t &= (1-\alpha)\,\hat{A}_t+\alpha\hat{K}_t \qquad (19)
\end{aligned}$$

Next turn to (13). The left hand side is expanded as:

$$\begin{aligned}
K_{t+1} &= \bar{K}_{t+1}+K_{t+1}-\bar{K}_{t+1}\\
&= \bar{K}_{t+1}+\bar{K}_{t+1}\frac{K_{t+1}-\bar{K}_{t+1}}{\bar{K}_{t+1}}\\
&= \bar{K}_{t+1}\left(1+\hat{K}_{t+1}\right)
\end{aligned}$$

The right-hand side of (13) is expanded as:

$$
\begin{aligned}
& (1-\delta) K_t + (K_t)^\alpha (A_t N_t l_t)^{1-\alpha} - N_t c_t - G_t \\
= \;& (1-\delta) \bar{K}_t + \left(\bar{K}_t\right)^\alpha \left(\bar{A}_t \bar{N}_t \bar{l}\right)^{1-\alpha} - \bar{C}_t - \bar{G}_t + (1-\delta)\left(K_t - \bar{K}_t\right) \\
& +\alpha \left(\bar{K}_t\right)^{\alpha-1} \left(\bar{A}_t \bar{N}_t \bar{l}\right)^{1-\alpha} \left(K_t - \bar{K}_t\right) + (1-\alpha)\left(\bar{K}_t\right)^\alpha \left(\bar{A}_t \bar{N}_t \bar{l}\right)^{-\alpha} \bar{N}_t \bar{l} \left(A_t - \bar{A}_t\right) \\
& + (1-\alpha)\left(\bar{K}_t\right)^\alpha \left(\bar{A}_t \bar{N}_t \bar{l}\right)^{-\alpha} \bar{A}_t \bar{N}_t \left(l_t - \bar{l}\right) - \bar{N}_t \left(c_t - \bar{c}_t\right) - \left(G_t - \bar{G}_t\right) \\
= \;& (1-\delta) \bar{K}_t + \left(\bar{K}_t\right)^\alpha \left(\bar{A}_t \bar{N}_t \bar{l}\right)^{1-\alpha} - \bar{C}_t - \bar{G}_t + (1-\delta) \bar{K}_t \frac{K_t - \bar{K}_t}{\bar{K}} \\
& +\alpha \left(\bar{K}_t\right)^\alpha \left(\bar{A}_t \bar{N}_t \bar{l}\right)^{1-\alpha} \frac{K_t - \bar{K}_t}{\bar{K}} + (1-\alpha)\left(\bar{K}_t\right)^\alpha \left(\bar{A}_t \bar{N}_t \bar{l}\right)^{1-\alpha} \frac{A_t - \bar{A}_t}{\bar{A}_t} \\
& + (1-\alpha)\left(\bar{K}_t\right)^\alpha \left(\bar{A}_t \bar{N}_t \bar{l}\right)^{1-\alpha} \frac{l_t - \bar{l}}{\bar{l}} - \bar{C}_t \frac{c_t - \bar{c}_t}{\bar{c}_t} - \bar{G}_t \frac{G_t - \bar{G}_t}{\bar{G}_t} \\
= \;& (1-\delta) \bar{K}_t + \left(\bar{K}_t\right)^\alpha \left(\bar{A}_t \bar{N}_t \bar{l}\right)^{1-\alpha} - \bar{C}_t - \bar{G}_t + (1-\delta) \bar{K}_t \hat{K}_t \\
& + \left(\bar{K}_t\right)^\alpha \left(\bar{A}_t \bar{N}_t \bar{l}\right)^{1-\alpha} \left[\alpha \hat{K}_t + (1-\alpha)\hat{A}_t + (1-\alpha)\hat{l}_t\right] - \bar{C}_t \hat{c}_t - \bar{G}_t \hat{G}_t
\end{aligned}
$$

Putting both sides together, (13) becomes:

$$
\begin{aligned}
K_{t+1} &= (1-\delta) K_t + (K_t)^\alpha (A_t N_t l_t)^{1-\alpha} - N_t c_t - G_t \\
\bar{K}_{t+1}\left(1 + \hat{K}_{t+1}\right) &= (1-\delta) \bar{K}_t + \left(\bar{K}_t\right)^\alpha \left(\bar{A}_t \bar{N}_t \bar{l}\right)^{1-\alpha} - \bar{C}_t - \bar{G}_t + (1-\delta) \bar{K}_t \hat{K}_t \\
& \quad + \left(\bar{K}_t\right)^\alpha \left(\bar{A}_t \bar{N}_t \bar{l}\right)^{1-\alpha} \left[\alpha \hat{K}_t + (1-\alpha)\hat{A}_t + (1-\alpha)\hat{l}_t\right] - \bar{C}_t \hat{c}_t - \bar{G}_t \hat{G}_t \\
\bar{K}_{t+1}\hat{K}_{t+1} &= (1-\delta) \bar{K}_t \hat{K}_t + \left(\bar{K}_t\right)^\alpha \left(\bar{A}_t \bar{N}_t \bar{l}\right)^{1-\alpha} \left[\alpha \hat{K}_t + (1-\alpha)\hat{A}_t + (1-\alpha)\hat{l}_t\right] \\
& \quad - \bar{C}_t \hat{c}_t - \bar{G}_t \hat{G}_t \\
\bar{K}_{t+1}\hat{K}_{t+1} &= \bar{K}_t \left[(1-\delta)\hat{K}_t + \left(\bar{k}\right)^{\alpha-1} \left[\alpha \hat{K}_t + (1-\alpha)\hat{A}_t + (1-\alpha)\hat{l}_t\right]\right] \\
& \quad - \bar{K}_t \left[\frac{\bar{C}_t}{\bar{K}_t}\hat{c}_t + \frac{\bar{G}_t}{\bar{K}_t}\hat{G}_t\right] \\
\hat{K}_{t+1} &= \frac{\bar{K}_t}{\bar{K}_{t+1}} \left[(1-\delta)\hat{K}_t + \left(\bar{k}\right)^{\alpha-1} \left[\alpha \hat{K}_t + (1-\alpha)\hat{A}_t + (1-\alpha)\hat{l}_t\right]\right] \\
& \quad - \frac{\bar{K}_t}{\bar{K}_{t+1}} \left[\frac{\bar{C}_t}{\bar{K}_t}\hat{c}_t + \frac{\bar{G}_t}{\bar{K}_t}\hat{G}_t\right] \\
\hat{K}_{t+1} &= \frac{1}{e^{g+n}} \left[(1-\delta)\hat{K}_t + \left(\bar{k}\right)^{\alpha-1} \left[\alpha \hat{K}_t + (1-\alpha)\hat{A}_t + (1-\alpha)\hat{l}_t\right]\right] \\
& \quad - \frac{1}{e^{g+n}} \left[\frac{\bar{C}_t}{\bar{K}_t}\hat{c}_t + \frac{\bar{G}_t}{\bar{K}_t}\hat{G}_t\right] \\
\hat{K}_{t+1} &= \eta_{KK}\hat{K}_t + \eta_{KA}\hat{A}_t + \eta_{KC}\hat{c}_t + \eta_{KL}\hat{l}_t + \eta_{KG}\hat{G}_t \qquad (20)
\end{aligned}
$$

where:

$$
\begin{aligned}
\eta_{KK} &= \frac{1}{e^{g+n}} \left[(1-\delta) + \left(\bar{k}\right)^{\alpha-1}\alpha\right] = \frac{1+\bar{r}}{e^{g+n}} \\
\eta_{KA} &= \frac{1}{e^{g+n}} \left(\bar{k}\right)^{\alpha-1}(1-\alpha) = \frac{\bar{r}+\delta}{e^{g+n}} \frac{1-\alpha}{\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\eta_{KC} &= -\frac{1}{e^{g+n}}\frac{N_0 C_0}{K_0} \\
\eta_{KL} &= \eta_{KA} \\
\eta_{KG} &= -\frac{1}{e^{g+n}}\frac{G_0}{K_0}
\end{aligned}
$$

The left-hand side of (12) is expanded as:

$$
\begin{aligned}
\frac{1}{c_t} &= \frac{1}{\bar{c}_t} - \left(\frac{1}{\bar{c}_t}\right)^2 (c_t - \bar{c}_t) \\
&= \frac{1}{\bar{c}_t} - \frac{1}{\bar{c}_t}\frac{c_t - \bar{c}_t}{\bar{c}_t} = \frac{1}{\bar{c}_t}(1 - \hat{c}_t)
\end{aligned}
$$

The right-hand side of 12) is expanded as:

$$
\begin{aligned}
& e^{-\rho} E_t \left[\frac{1}{c_{t+1}}\left(1 + \alpha\left(\frac{K_{t+1}}{A_{t+1}N_{t+1}l_{t+1}}\right)^{\alpha-1} - \delta\right)\right] \\
= \; & e^{-\rho} E_t \left[\frac{1}{\bar{c}_{t+1}}\left(1 + \alpha\left(\bar{k}\right)^{\alpha-1} - \delta\right)\right] \\
& + e^{-\rho}\left(1 + \alpha\left(\bar{k}\right)^{\alpha-1} - \delta\right) E_t \left[-\left(\frac{1}{\bar{c}_{t+1}}\right)^2 (c_{t+1} - \bar{c}_{t+1})\right] \\
& + e^{-\rho} E_t \left[\frac{1}{\bar{c}_{t+1}}\alpha(\alpha - 1)\left[\begin{array}{c} \left(\bar{k}\right)^{\alpha-2}\frac{K_{t+1}-\bar{K}_{t+1}}{\bar{A}_{t+1}\bar{N}_{t+1}\bar{l}} - \left(\bar{k}\right)^{\alpha-2}\bar{k}\frac{A_{t+1}-\bar{A}_{t+1}}{\bar{A}_{t+1}} \\ - \left(\bar{k}\right)^{\alpha-2}\bar{k}\frac{l_{t+1}-\bar{l}}{\bar{l}} \end{array}\right]\right]
\end{aligned}
$$

$$
\begin{aligned}
= \; & e^{-\rho} E_t \left[\frac{1}{\bar{c}_{t+1}}\left(1 + \alpha\left(\bar{k}\right)^{\alpha-1} - \delta\right)\right] \\
& - e^{-\rho}\frac{1}{\bar{c}_{t+1}}\left(1 + \alpha\left(\bar{k}\right)^{\alpha-1} - \delta\right) E_t \frac{c_{t+1} - \bar{c}_{t+1}}{\bar{c}_{t+1}} \\
& + e^{-\rho}\frac{1}{\bar{c}_{t+1}}\alpha(\alpha - 1)\left(\bar{k}\right)^{\alpha-1} E_t \left[\frac{K_{t+1} - \bar{K}_{t+1}}{\bar{K}_{t+1}} - \frac{A_{t+1} - \bar{A}_{t+1}}{\bar{A}_{t+1}} - \frac{l_{t+1} - \bar{l}}{\bar{l}}\right] \\
= \; & e^{-\rho} E_t \left[\frac{1}{\bar{c}_{t+1}}\left(1 + \alpha\left(\bar{k}\right)^{\alpha-1} - \delta\right)\right] \\
& - e^{-\rho}\frac{1}{\bar{c}_{t+1}}\left(1 + \alpha\left(\bar{k}\right)^{\alpha-1} - \delta\right) E_t \hat{c}_{t+1} \\
& + e^{-\rho}\frac{1}{\bar{c}_{t+1}}\alpha(\alpha - 1)\left(\bar{k}\right)^{\alpha-1} E_t \left[\hat{K}_{t+1} - \hat{A}_{t+1} - \hat{l}_{t+1}\right]
\end{aligned}
$$

Combining both sides (12) becomes:

$$
\begin{aligned}
\frac{1}{c_t} &= e^{-\rho} E_t \left[\frac{1}{c_{t+1}}\left(1 + \alpha\left(\bar{k}\right)^{\alpha-1} - \delta\right)\right] \\
\frac{1}{\bar{c}_t}(1 - \hat{c}_t) &= e^{-\rho} E_t \left[\frac{1}{\bar{c}_{t+1}}\left(1 + \alpha\left(\bar{k}\right)^{\alpha-1} - \delta\right)\right] \\
& \quad - e^{-\rho}\frac{1}{\bar{c}_{t+1}}\left(1 + \alpha\left(\bar{k}\right)^{\alpha-1} - \delta\right) E_t \hat{c}_{t+1}
\end{aligned}
$$

$$
\begin{aligned}
& +e^{-\rho}\frac{1}{\bar{c}_{t+1}}\alpha\left(\alpha-1\right)\left(\bar{k}\right)^{\alpha-1}E_t\left[\hat{K}_{t+1}-\hat{A}_{t+1}-\hat{l}_{t+1}\right] \\
-\hat{c}_t &= -E_t\hat{c}_{t+1}+e^{-\rho}\frac{\bar{c}_t}{\bar{c}_{t+1}}\alpha\left(\alpha-1\right)\left(\bar{k}\right)^{\alpha-1}E_t\left[\hat{K}_{t+1}-\hat{A}_{t+1}-\hat{l}_{t+1}\right] \\
\hat{c}_t &= E_t\hat{c}_{t+1}-e^{-\rho}\frac{\bar{c}_t}{\bar{c}_{t+1}}\alpha\left(\alpha-1\right)\left(\bar{k}\right)^{\alpha-1}E_t\left[\hat{K}_{t+1}-\hat{A}_{t+1}-\hat{l}_{t+1}\right] \\
\hat{c}_t &= E_t\hat{c}_{t+1}-\frac{\alpha\left(\alpha-1\right)\left(\bar{k}\right)^{\alpha-1}}{1+\alpha\left(\bar{k}\right)^{\alpha-1}-\delta}E_t\left[\hat{K}_{t+1}-\hat{A}_{t+1}-\hat{l}_{t+1}\right]
\end{aligned}
$$

Recall that along the steady growth path the real interest is:

$$
\bar{r}=\alpha\left(\bar{k}\right)^{\alpha-1}-\delta
$$

We then write the Euler relation as:

$$
\begin{aligned}
\hat{c}_t &= E_t\hat{c}_{t+1}-\frac{1}{1+\alpha\left(\bar{k}\right)^{\alpha-1}-\delta}\alpha\left(\alpha-1\right)\left(\bar{k}\right)^{\alpha-1}E_t\left[\hat{K}_{t+1}-\hat{A}_{t+1}-\hat{l}_{t+1}\right] \\
\hat{c}_t &= E_t\hat{c}_{t+1}-\frac{1}{1+\bar{r}}\alpha\left(\alpha-1\right)\left(\bar{k}\right)^{\alpha-1}E_t\left[\hat{K}_{t+1}-\hat{A}_{t+1}-\hat{l}_{t+1}\right] \\
\hat{c}_t &= E_t\hat{c}_{t+1}-\frac{\bar{r}+\delta}{1+\bar{r}}\left(\alpha-1\right)E_t\left[\hat{K}_{t+1}-\hat{A}_{t+1}-\hat{l}_{t+1}\right] \\
\hat{c}_t &= E_t\hat{c}_{t+1}+\frac{\bar{r}+\delta}{1+\bar{r}}\left(1-\alpha\right)E_t\left[\hat{K}_{t+1}-\hat{A}_{t+1}-\hat{l}_{t+1}\right] \qquad (21)
\end{aligned}
$$

Investment is expanded as follows:

$$
\begin{aligned}
\frac{I_t}{K_t} &= \frac{K_{t+1}}{K_t}-\left(1-\delta\right) \\
\frac{\bar{I}_t}{\bar{K}_t}+\frac{I_t-\bar{I}_t}{\bar{K}_t}-\frac{\bar{I}_t}{\bar{K}_t}\frac{K_t-\bar{K}_t}{\bar{K}_t} &= \frac{\bar{K}_{t+1}}{\bar{K}_t}-\left(1-\delta\right)+\frac{\bar{K}_{t+1}}{\bar{K}_t}\frac{K_{t+1}-\bar{K}_{t+1}}{\bar{K}_{t+1}}-\frac{\bar{K}_{t+1}}{\bar{K}_t}\frac{K_t-\bar{K}_t}{\bar{K}_t} \\
\frac{\bar{I}_t}{\bar{K}_t}\frac{I_t-\bar{I}_t}{\bar{I}_t}-\frac{\bar{I}_t}{\bar{K}_t}\frac{K_t-\bar{K}_t}{\bar{K}_t} &= \frac{\bar{K}_{t+1}}{\bar{K}_t}\frac{K_{t+1}-\bar{K}_{t+1}}{\bar{K}_{t+1}}-\frac{\bar{K}_{t+1}}{\bar{K}_t}\frac{K_t-\bar{K}_t}{\bar{K}_t} \\
\frac{\bar{I}_t}{\bar{K}_t}\left(\hat{I}_t-\hat{K}_t\right) &= \frac{\bar{K}_{t+1}}{\bar{K}_t}\left(\hat{K}_{t+1}-\hat{K}_t\right) \\
\hat{I}_t-\hat{K}_t &= \frac{e^{g+n}}{e^{g+n}-\left(1-\delta\right)}\left(\hat{K}_{t+1}-\hat{K}_t\right) \\
\hat{I}_t-\hat{K}_t &= \frac{e^{g+n}}{e^{g+n}-\left(1-\delta\right)}\left(\begin{array}{c}\left(\eta_{KK}-1\right)\hat{K}_t+\eta_{KA}\hat{A}_t \\ +\eta_{KC}\hat{c}_t+\eta_{KL}\hat{l}_t+\eta_{KG}\hat{G}_t\end{array}\right) \\
\hat{I}_t &= \left(1+\frac{e^{g+n}}{e^{g+n}-\left(1-\delta\right)}\left(\eta_{KK}-1\right)\right)\hat{K}_t \\
& +\frac{e^{g+n}}{e^{g+n}-\left(1-\delta\right)}\left(\eta_{KA}\hat{A}_t+\eta_{KC}\hat{c}_t+\eta_{KL}\hat{l}_t+\eta_{KG}\hat{G}_t\right)
\end{aligned}
$$

4 Solution with undetermined coefficients

4.1 Analytical steps

The model is summarized by (19)-(21) and (5)-(6):

$$\begin{aligned}
\hat{c}_t + \left(\frac{\bar{l}}{1-\bar{l}} + \alpha\right)\hat{l}_t &= (1-\alpha)\hat{A}_t + \alpha\hat{K}_t \\
\hat{K}_{t+1} &= \eta_{KK}\hat{K}_t + \eta_{KA}\hat{A}_t + \eta_{KC}\hat{c}_t + \eta_{KL}\hat{l}_t + \eta_{KG}\hat{G}_t \\
\hat{c}_t &= E_t\hat{c}_{t+1} + \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)E_t\left[\hat{K}_{t+1} - \hat{A}_{t+1} - \hat{l}_{t+1}\right] \\
\hat{A}_t &= \rho_A\hat{A}_{t-1} + \varepsilon_{A,t} \\
\hat{G}_t &= \rho_G\hat{G}_{t-1} + \varepsilon_{G,t}
\end{aligned}$$

There are three state variables: $\hat{A}$, $\hat{G}$ and $\hat{K}$ and two control variables: $\hat{C}$ and $\hat{l}$. The endogenous variables should then be linear functions of the state variables, with coefficients to be determined:

$$\begin{aligned}
\hat{C}_t &= a_{CK}\hat{K}_t + a_{CA}\hat{A}_t + a_{CG}\hat{G}_t \\
\hat{l}_t &= a_{LK}\hat{K}_t + a_{LA}\hat{A}_t + a_{LG}\hat{G}_t \\
\hat{K}_{t+1} &= b_{KK}\hat{K}_t + b_{KA}\hat{A}_t + b_{KG}\hat{G}_t
\end{aligned}$$

From (19) we write:

$$\begin{aligned}
\hat{C}_t &= -\left(\frac{\bar{l}}{1-\bar{l}} + \alpha\right)\hat{l}_t + (1-\alpha)\hat{A}_t + \alpha\hat{K}_t \\
a_{CK}\hat{K}_t + a_{CA}\hat{A}_t + a_{CG}\hat{G}_t &= -\left(\frac{\bar{l}}{1-\bar{l}} + \alpha\right)\left[a_{LK}\hat{K}_t + a_{LA}\hat{A}_t + a_{LG}\hat{G}_t\right] + (1-\alpha)\hat{A}_t + \alpha\hat{K}_t
\end{aligned}$$

This implies three restrictions on the coefficients:

$$\begin{aligned}
a_{CK} &= -\left(\frac{\bar{l}}{1-\bar{l}} + \alpha\right)a_{LK} + \alpha \\
a_{CA} &= -\left(\frac{\bar{l}}{1-\bar{l}} + \alpha\right)a_{LA} + (1-\alpha) \\
a_{CG} &= -\left(\frac{\bar{l}}{1-\bar{l}} + \alpha\right)a_{LG}
\end{aligned} \tag{22}$$

From (20) we write:

$$\begin{aligned}
\hat{K}_{t+1} &= \eta_{KK}\tilde{K}_t + \eta_{KA}\hat{A}_t + \eta_{KC}\hat{C}_t + \eta_{KL}\hat{l}_t + \eta_{KG}\hat{G}_t \\
b_{KK}\hat{K}_t + b_{KA}\hat{A}_t + b_{KG}\hat{G}_t &= \eta_{KK}\tilde{K}_t + \eta_{KA}\hat{A}_t + \eta_{KC}\left[a_{CK}\hat{K}_t + a_{CA}\hat{A}_t + a_{CG}\hat{G}_t\right] \\
&\quad + \eta_{KL}\left[a_{LK}\hat{K}_t + a_{LA}\hat{A}_t + a_{LG}\hat{G}_t\right] + \eta_{KG}\hat{G}_t
\end{aligned}$$

Using (22) to substitute for the coefficients on consumption, this implies three restrictions:

$$\begin{aligned}
b_{KK} &= \eta_{KK} + \eta_{KC} a_{CK} + \eta_{KL} a_{LK} \\
&= \eta_{KK} + \eta_{KC} \alpha + \left[\eta_{KL} - \eta_{KC}\left(\frac{\bar{l}}{1-\bar{l}} + \alpha\right)\right] a_{LK} \\
b_{KA} &= \eta_{KA} + \eta_{KC} a_{CA} + \eta_{KL} a_{LA} \\
&= \eta_{KA} + \eta_{KC}(1-\alpha) + \left[\eta_{KL} - \eta_{KC}\left(\frac{\bar{l}}{1-\bar{l}} + \alpha\right)\right] a_{LA} \qquad (23) \\
b_{KG} &= \eta_{KC} a_{CG} + \eta_{KL} a_{LG} + \eta_{KG} \\
&= \eta_{KG} + \left[\eta_{KL} - \eta_{KC}\left(\frac{\bar{l}}{1-\bar{l}} + \alpha\right)\right] a_{LG}
\end{aligned}$$

Given a_{LK}, a_{LA} and a_{LG} (22)-(23) allow us to compute a_{CK}, a_{CA} and a_{CG} and b_{KK}, b_{KA} and b_{KG}.

We now turn to (21):

$$\begin{aligned}
\hat{c}_t &= E_t\hat{c}_{t+1} + \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha) E_t\left[\hat{K}_{t+1} - \hat{A}_{t+1} - \hat{l}_{t+1}\right] \\
a_{CK}\hat{K}_t + a_{CA}\hat{A}_t + a_{CG}\hat{G}_t &= E_t\left[a_{CK}\hat{K}_{t+1} + a_{CA}\hat{A}_{t+1} + a_{CG}\hat{G}_{t+1}\right] \\
&\quad + \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha) E_t\left[\hat{K}_{t+1} - \hat{A}_{t+1} - a_{LK}\hat{K}_{t+1} - a_{LA}\hat{A}_{t+1} - a_{LG}\hat{G}_{t+1}\right] \\
a_{CK}\hat{K}_t + a_{CA}\hat{A}_t + a_{CG}\hat{G}_t &= \left[a_{CA} - \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)(1+a_{LA})\right] E_t\hat{A}_{t+1} \\
&\quad + \left[a_{CG} - \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha) a_{LG}\right] E_t\hat{G}_{t+1} \\
&\quad + \left[a_{CK} + \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)\left[1-a_{LK}\right]\right] E_t\hat{K}_{t+1}
\end{aligned}$$

Using (5)-(6) this becomes:

$$\begin{aligned}
& a_{CK}\hat{K}_t + a_{CA}\hat{A}_t + a_{CG}\hat{G}_t \\
= & \left[a_{CA} - \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)(1+a_{LA})\right]\rho_A\hat{A}_t \\
& + \left[a_{CG} - \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha) a_{LG}\right]\rho_G\hat{G}_t \\
& + \left[a_{CK} + \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)\left[1-a_{LK}\right]\right] E_t\hat{K}_{t+1} \\
= & \left[a_{CA} - \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)(1+a_{LA})\right]\rho_A\hat{A}_t + \left[a_{CG} - \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha) a_{LG}\right]\rho_G\hat{G}_t \\
& + \left[a_{CK} + \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)\left[1-a_{LK}\right]\right]\left[b_{KK}\hat{K}_t + b_{KA}\hat{A}_t + b_{KG}\hat{G}_t\right]
\end{aligned}$$

This implies three restrictions:

$$a_{CK} = \left[a_{CK} + \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)\left[1-a_{LK}\right]\right] b_{KK}$$

$$
\begin{aligned}
a_{CA} &= \left[a_{CA} - \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)(1+a_{LA})\right]\rho_A + \left[a_{CK} + \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)[1-a_{LK}]\right] b_{KA} \\
a_{CG} &= \left[a_{CG} - \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)\, a_{LG}\right]\rho_G + \left[a_{CK} + \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)[1-a_{LK}]\right] b_{KG}
\end{aligned}
$$

Using (22)-(23) the last two of these restrictions become:

$$
\begin{aligned}
0 &= \left(\frac{\bar{l}}{1-\bar{l}}+\alpha\right) a_{LA} - (1-\alpha) \\
&+ \left[-\left(\frac{\bar{l}}{1-\bar{l}}+\alpha+\frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)\right) a_{LA} + \frac{1-\delta}{1+\bar{r}}(1-\alpha)\right]\rho_A \\
&+ \left[-\left(\frac{\bar{l}}{1-\bar{l}}+\alpha+\frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)\right) a_{LK} + \alpha + \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)\right] \\
&\times \left[\eta_{KA} + \eta_{KC}(1-\alpha) + \left[\eta_{KL} - \eta_{KC}\left(\frac{\bar{l}}{1-\bar{l}}+\alpha\right)\right] a_{LA}\right]
\end{aligned}
$$

and:

$$
\begin{aligned}
0 &= \left(\frac{\bar{l}}{1-\bar{l}}+\alpha\right) a_{LG} \\
&- \left(\frac{\bar{l}}{1-\bar{l}}+\alpha+\frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)\right) a_{LG}\rho_G \\
&+ \left[-\left(\frac{\bar{l}}{1-\bar{l}}+\alpha+\frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)\right) a_{LK} + \alpha + \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)\right] \\
&\times \left[\eta_{KG} + \left[\eta_{KL} - \eta_{KC}\left(\frac{\bar{l}}{1-\bar{l}}+\alpha\right)\right] a_{LG}\right]
\end{aligned}
$$

These two relations give linear solutions for a_{LA} and a_{LG} conditional on a_{LK}.

The first of the two restrictions is a quadratic polynomial in a_{LK}:

$$
\begin{aligned}
0 &= \left(\frac{\bar{l}}{1-\bar{l}}+\alpha\right) a_{LK} - \alpha \\
&+ \left[-\left(\frac{\bar{l}}{1-\bar{l}}+\alpha+\frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)\right) a_{LK} + \alpha + \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)\right] \\
&\times \left[\eta_{KK} + \eta_{KC}\alpha + \left[\eta_{KL} - \eta_{KC}\left(\frac{\bar{l}}{1-\bar{l}}+\alpha\right)\right] a_{LK}\right]
\end{aligned}
$$

We pick the value of a_{LK} such that b_{KK} is smaller than one, so the system converges back to the steady state.

4.2 A numerical illustration

We take the following parameters:

$$
\alpha = \frac{1}{3},\ g = 0.005,\ n = 0.0025,\ \delta = 0.025,\ \rho_A = \rho_G = 0.95,\ \bar{r} = 0.015,\ \frac{\bar{G}}{\bar{Y}} = 0.2,\ \bar{l} = \frac{1}{3}
$$

These values imply the following:

$$
\begin{aligned}
\bar{k} &= 24.056 \\
\frac{Y_0}{K_0} &= 0.12 \quad ; \quad \frac{G_0}{K_0} = 0.024 \\
\frac{C_0}{K_0} &= 0.06347 \quad ; \quad \frac{C_0}{Y_0} = 0.5289 \\
\frac{I_0}{K_0} &= 0.0325 \quad ; \quad \frac{I_0}{Y_0} = 0.2711
\end{aligned}
$$

The numerical values of the various coefficients are then:

$$
\begin{aligned}
\eta_{KK} &= 1.0074 \quad ; \quad \eta_{KA} = 0.0794 \quad ; \quad \eta_{KC} = -0.063 \\
\eta_{KL} &= 0.0794 \quad ; \quad \eta_{KG} = -0.024
\end{aligned}
$$

And:

$$
\begin{aligned}
a_{CK} &= 0.59 \quad ; \quad a_{CA} = 0.38 \quad ; \quad a_{CG} = -0.13 \\
a_{LK} &= -0.31 \quad ; \quad a_{LA} = 0.35 \quad ; \quad a_{LG} = 0.15 \\
b_{KK} &= 0.95 \quad ; \quad b_{KA} = 0.08 \quad ; \quad b_{KG} = -0.004
\end{aligned}
$$

5 Solution with Blanchard and Kahn

5.1 Overall system

The model can be solved using the matrix representation that we used with the Ramsey model. We recall the linear system (19)-(21) and (5)-(6):

$$
\begin{aligned}
\hat{c}_t + \left(\frac{\bar{l}}{1-\bar{l}} + \alpha\right)\hat{l}_t &= (1-\alpha)\,\hat{A}_t + \alpha\hat{K}_t \\
\hat{K}_{t+1} &= \eta_{KK}\hat{K}_t + \eta_{KA}\hat{A}_t + \eta_{KC}\hat{c}_t + \eta_{KL}\hat{l}_t + \eta_{KG}\hat{G}_t \\
\hat{c}_t &= E_t\hat{c}_{t+1} + \frac{\bar{r}+\delta}{1+\bar{r}}\,(1-\alpha)\,E_t\left[\hat{K}_{t+1} - \hat{A}_{t+1} - \hat{l}_{t+1}\right] \\
\hat{A}_{t+1} &= \rho_A\hat{A}_t + \varepsilon_{A,t+1} \\
\hat{G}_{t+1} &= \rho_G\hat{G}_t + \varepsilon_{G,t+1}
\end{aligned}
$$

where:

$$
\begin{aligned}
\eta_{KK} &= \frac{1+\bar{r}}{e^{g+n}} \\
\eta_{KA} &= \eta_{KL} = \frac{\bar{r}+\delta}{e^{g+n}}\frac{1-\alpha}{\alpha} \\
\eta_{KC} &= -\frac{1}{e^{g+n}}\frac{N_0C_0}{K_0} \\
\eta_{KG} &= -\frac{1}{e^{g+n}}\frac{G_0}{K_0}
\end{aligned}
$$

Before writing a matrix system, we need one more step- Notice that the first equation of the system (the labor market clearing) only contains variables at time t and none at time $t+1$. This will lead to a problem of non-invertible matrix. To avoid this, we write labor for this condition as:

$$
\begin{aligned}
\hat{l}_t &= \frac{(1-\alpha)}{\frac{\bar{l}}{1-\bar{l}}+\alpha}\hat{A}_t + \frac{\alpha}{\frac{\bar{l}}{1-\bar{l}}+\alpha}\hat{K}_t - \frac{1}{\frac{\bar{l}}{1-\bar{l}}+\alpha}\hat{c}_t \\
\hat{l}_t &= \eta_{LA}\hat{A}_t + \eta_{LK}\hat{K}_t + \eta_{LC}\hat{c}_t
\end{aligned}
$$

We can then write the system in a tighter form:

$$
\begin{aligned}
\hat{K}_{t+1} &= \left(\eta_{KK}+\eta_{KL}\eta_{LK}\right)\hat{K}_t + \left(\eta_{KA}+\eta_{KL}\eta_{LA}\right)\hat{A}_t + \left(\eta_{KC}+\eta_{KL}\eta_{LC}\right)\hat{c}_t + \eta_{KG}\hat{G}_t \\
\hat{c}_t &= \left(1-\eta_{LC}\frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)\right)E_t\hat{c}_{t+1} + \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)E_t\left[\left(1-\eta_{LK}\right)\hat{K}_{t+1} - \left(1+\eta_{LA}\right)\hat{A}_{t+1}\right] \\
\hat{A}_{t+1} &= \rho_A\hat{A}_t + \varepsilon_{A,t+1} \\
\hat{G}_{t+1} &= \rho_G\hat{G}_t + \varepsilon_{G,t+1}
\end{aligned}
$$

This shows that when using the method you first need to "tighten" the system to avoid issues of non-invertible matrices.

The vectors of state variables, control variables, and shocks are:

$$
S_t = \begin{vmatrix} \hat{K}_t \\ \hat{A}_t \\ \hat{G}_t \end{vmatrix} \quad ; \quad P_t = |\hat{c}_t| \quad ; \quad V_{t+1} = \begin{vmatrix} \varepsilon_{A,t+1} \\ \varepsilon_{G,t+1} \end{vmatrix}
$$

They system is (X and Y are 4x4 matrices and Z is a 4x2 matrix):

$$
X\begin{vmatrix} S_{t+1} \\ E_tP_{t+1} \end{vmatrix} = Y\begin{vmatrix} S_t \\ P_t \end{vmatrix} + ZV_{t+1}
$$

where:

$$
\begin{aligned}
X &= \begin{vmatrix} 1 & 0 & 0 & 0 \\ \frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)(1-\eta_{LK}) & -\frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha)(1+\eta_{LA}) & 0 & 1-\eta_{LC}\frac{\bar{r}+\delta}{1+\bar{r}}(1-\alpha) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{vmatrix} \\
Y &= \begin{vmatrix} \eta_{KK}+\eta_{KL}\eta_{LK} & \eta_{KA}+\eta_{KL}\eta_{LA} & \eta_{KG} & \eta_{KC}+\eta_{KL}\eta_{LC} \\ 0 & 0 & 0 & 1 \\ 0 & \rho_A & 0 & 0 \\ 0 & 0 & \rho_G & 0 \end{vmatrix} \\
Z &= \begin{vmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{vmatrix}
\end{aligned}
$$

We rewrite the system as:

$$
\begin{aligned}
X\begin{vmatrix} S_{t+1} \\ E_tP_{t+1} \end{vmatrix} &= Y\begin{vmatrix} S_t \\ P_t \end{vmatrix} + ZV_{t+1} \\
\begin{vmatrix} S_{t+1} \\ E_tP_{t+1} \end{vmatrix} &= X^{-1}Y\begin{vmatrix} S_t \\ P_t \end{vmatrix} + X^{-1}ZV_{t+1} \\
\begin{vmatrix} S_{t+1} \\ E_tP_{t+1} \end{vmatrix} &= A\begin{vmatrix} S_t \\ P_t \end{vmatrix} + BV_{t+1} \\
\begin{vmatrix} S_{t+1} \\ E_tP_{t+1} \end{vmatrix} &= C^{-1}\Lambda C\begin{vmatrix} S_t \\ P_t \end{vmatrix} + BV_{t+1}
\end{aligned}
$$

where Λ is the diagonal matrix of eigenvalues of A and C the matrix of eigenvectors (in the rows). We split$CA = \Lambda C$ along the lines of state and control variables::

$$
\Lambda C = \begin{vmatrix} J_1 & 0 \\ 0 & J_2 \end{vmatrix}\begin{vmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{vmatrix}
$$

where J_1 is a 3x3 matrix, J_2 is a 1x1 matrix, C_{11} is a 3x3 matrix, C_{12} is a 3x1 matrix, C_{21} is a 1x3 matrix, C_{22} is a 1x1 matrix. We also define:

$$
Q_t = C\begin{vmatrix} S_t \\ P_t \end{vmatrix} = \begin{vmatrix} C_{11}S_t + C_{12}P_t \\ C_{21}S_t + C_{22}P_t \end{vmatrix} = \begin{vmatrix} Q_{1,t} \\ Q_{2,t} \end{vmatrix}
$$

where $Q_{1,t}$ is a 3x1 vector and $Q_{1,t}$ is a 1x1 vector.

5.2 Solving for control variables

Take the expectation of the system from the point of view of t:

$$
\begin{vmatrix} E_tS_{t+1} \\ E_tP_{t+1} \end{vmatrix} = C^{-1}\Lambda C\begin{vmatrix} S_t \\ P_t \end{vmatrix}
$$

The eigenvalues in J_1 are smaller than 1 and the ones in J_2 are larger than one (we should of course ensure that this is indeed the case). We can write:

$$
\begin{aligned}
C\begin{vmatrix} E_tS_{t+1} \\ E_tP_{t+1} \end{vmatrix} &= \Lambda C\begin{vmatrix} S_t \\ P_t \end{vmatrix} \\
E_tQ_{t+1} &= \Lambda Q_t \\
\begin{vmatrix} E_tQ_{1,t+1} \\ E_tQ_{2,t+1} \end{vmatrix} &= \begin{vmatrix} J_1 & 0 \\ 0 & J_2 \end{vmatrix}\begin{vmatrix} Q_{1,t} \\ Q_{2,t} \end{vmatrix}
\end{aligned}
$$

Take the bottow row of this system:

$$E_t Q_{2,t+1} = J_2 Q_{2,t}$$

As $J_2 > 1$ this is explosive, unless $Q_{2,t} = 0$:

$$\begin{aligned} 0 &= Q_{2,t} \\ 0 &= C_{21} S_t + C_{22} P_t \\ P_t &= -\left(C_{22}\right)^{-1} C_{21} S_t \end{aligned}$$

This gives the control variables as a function of the state variables.

5.3 Solving for state variables

Take the top row of the systemt in expected terms:

$$\begin{aligned} E_t Q_{1,t+1} &= J_1 Q_{1,t} \\ C_{11} E_t S_{t+1} + C_{12} E_t P_{t+1} &= J_1 \left[C_{11} S_t + C_{12} P_t\right] \\ \left[C_{11} - C_{12}\left(C_{22}\right)^{-1} C_{21}\right] E_t S_{t+1} &= J_1 \left[C_{11} - C_{12}\left(C_{22}\right)^{-1} C_{21}\right] S_t \\ E_t S_{t+1} &= (\Omega)^{-1} J_1 \Omega S_t = D S_t \end{aligned}$$

where: $\Omega = C_{11} - C_{12}\left(C_{22}\right)^{-1} C_{21}$.

The state variables are also affected by shocks. This component is drawn directly from the initial matrix system:

$$\begin{aligned} \begin{vmatrix} S_{t+1} \\ E_t P_{t+1} \end{vmatrix} &= C^{-1} \Lambda C \begin{vmatrix} S_t \\ P_t \end{vmatrix} + B V_{t+1} \\ \begin{vmatrix} S_{t+1} \\ E_t P_{t+1} \end{vmatrix} &= \begin{vmatrix} E_t S_{t+1} \\ E_t P_{t+1} \end{vmatrix} + B V_{t+1} \end{aligned}$$

This implies:

$$\begin{aligned} S_{t+1} &= E_t S_{t+1} + B_T V_{t+1} \\ S_{t+1} &= D S_t + F V_{t+1} \end{aligned}$$

where B_T are the top rows of B, taking as many rows as they are state variables