Lecture 5: Existence and uniqueness of trade models

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1 Introduction

Over the past two weeks, we have seen that there exist a number of microeconomic justifications for the gravity trade model. To recap, we have derived the following four gravity equations for bilateral trade flows X_{ij}. The Armington model of Anderson (1979) (with perfect competition and identical preferences) yields:

$$X_{ij} = \tau_{ij}^{1-\sigma} w_i^{1-\sigma} A_i^{\sigma-1} E_j P_j^{\sigma-1},$$

where $P_j \equiv \left(\sum_{i\in S} \tau_{ij}^{1-\sigma} w_i^{1-\sigma} A_i^{\sigma-1}\right)^{\frac{1}{1-\sigma}}$, so that:

$$X_{ij} = \frac{\tau_{ij}^{1-\sigma} A_i^{\sigma-1} w_i^{1-\sigma}}{\sum_{k\in S} \tau_{kj}^{1-\sigma} A_k^{\sigma-1} w_k^{1-\sigma}} E_j \tag{1}$$

The Krugman (1980) model yields:

$$X_{ij} = \frac{1}{\sigma} \tau_{ij}^{1-\sigma} w_i^{1-\sigma} A_i^{\sigma-1} \frac{L_i}{f_i^e} E_j P_j^{\sigma-1},$$

where $P_j \equiv \left(\sum_{i\in S} \tau_{ij}^{1-\sigma} \frac{L_i}{f_i^e} w_i^{1-\sigma} A_i^{\sigma-1}\right)^{\frac{1}{1-\sigma}}$, so that:

$$X_{ij} = \frac{\frac{1}{\sigma} \tau_{ij}^{1-\sigma} w_i^{1-\sigma} A_i^{\sigma-1} \frac{L_i}{f_i^e}}{\sum_{k\in S} \tau_{kj}^{1-\sigma} \frac{L_k}{f_k^e} A_k^{\sigma-1} w_k^{1-\sigma}} E_j \tag{2}$$

The Melitz (2003) model (with a Pareto distribution and exogenous entry) yields:

$$X_{ij} = \tau_{ij}^{-\theta_i} w_i^{-\theta_i} f_{ij}^{\frac{\sigma-\theta_i-1}{\sigma-1}} M_i E_j^{\frac{\theta_i}{\sigma-1}} P_j^{\theta},$$

where $P_j \equiv \left(E_j^{-\frac{\sigma-\theta_i-1}{\sigma-1}} \sum_{i \in S} M_i w_i^{-\theta} \tau_{ij}^{-\theta} f_{ij}^{\frac{\sigma-\theta_i-1}{\sigma-1}} \right)^{-\frac{1}{\theta_i}}$, so that:

$$X_{ij} = \frac{\tau_{ij}^{-\theta_i} f_{ij}^{\frac{\sigma-\theta_i-1}{\sigma-1}} M_i w_i^{-\theta_i}}{\sum_{k \in S} \tau_{ij}^{-\theta} f_{kj}^{\frac{\sigma-\theta_i-1}{\sigma-1}} M_k w_k^{-\theta}} E_j \tag{3}$$

The Eaton and Kortum (2002) model yields:

$$X_{ij} = \tau_{ij}^{-\theta} w_i^{-\theta} T_i E_j P_j^{\theta},$$

where $P_j \equiv \left(\sum_{i \in S} T_i w_i^{-\theta} \tau_{ij}^{-\theta} \right)^{-\frac{1}{\theta}}$, so that:

$$X_{ij} = \frac{\tau_{ij}^{-\theta} T_i w_i^{-\theta}}{\sum_{k \in S} \tau_{kj}^{-\theta} T_k w_k^{-\theta}} E_j \tag{4}$$

Then note that all of the gravity models above can be written as:

$$X_{ij} = \frac{K_{ij} w_i^{\alpha}}{\sum_{k \in S} K_{kj} w_k^{\alpha}} E_j, \tag{5}$$

for some (exogenous) model parameter $K_{ij} > 0$, (exogenous) trade elasticity $\alpha < 0$, (endogenous) wage w_i, and (endogenous) income Y_j. This week, we will be examining the properties of equation (5).

2 General equilibrium conditions

In this section, we define the three equilibrium conditions that determine equilibrium.

The first equilibrium condition is that the total income in a country is equal to the income earned by labor, i.e.:

$$w_i L_i = Y_i. \tag{6}$$

We refer to this as the l**abor market clearing condition.**

The second equilibrium condition is that the total income in a country is equal to the income earned from trade, i.e.:

$$Y_i = \sum_{j \in S} X_{ij}. \tag{7}$$

We refer to this as the **goods market clearing condition**.

The third equilibrium condition is that the total expenditure is equal to to the income earned by from trade, i.e.:

$$Y_i = E_i = \sum_{j \in S} X_{ji}. \tag{8}$$

We refer to this as the **balanced trade condition**.

3 The excess demand function

Combining the general version of the gravity equation (equation (5)) with the labor market clearing condition (equation (6)) and the balanced trade condition (equation (8)) yields:

$$X_{ij} = \frac{K_{ij} w_i^{\alpha}}{\sum_{k \in S} K_{kj} w_k^{\alpha}} w_j L_j. \tag{9}$$

Now we sum across all destinations and apply the goods market clearing condition (equation (7)) and the labor market clearing condition (equation (6)) once again:

$$\begin{aligned}
X_{ij} &= \frac{K_{ij} w_i^{\alpha}}{\sum_{k \in S} K_{kj} w_k^{\alpha}} w_j L_j \implies \\
\sum_{j \in S} X_{ij} &= \sum_{j \in S} \frac{K_{ij} w_i^{\alpha}}{\sum_{k \in S} K_{kj} w_k^{\alpha}} w_j L_j \iff \\
Y_i &= \sum_{j \in S} \frac{K_{ij} w_i^{\alpha}}{\sum_{k \in S} K_{kj} w_k^{\alpha}} w_j L_j \iff \\
w_i L_i &= \sum_{j \in S} \frac{K_{ij} w_i^{\alpha}}{\sum_{k \in S} K_{kj} w_k^{\alpha}} w_j L_j
\end{aligned} \tag{10}$$

Note that equation (10) depends only on exogenous parameters and the equilibrium wages. Furthermore, because it holds for all $i \in S$, there is one equation for each unknown wage.

For all $i \in S$ define the function $Z_i(\mathbf{w})$ as:

$$Z_i(\mathbf{w}) \equiv \frac{1}{w_i} \left(\sum_{j \in S} \frac{K_{ij} w_i^{\alpha}}{\sum_{k \in S} K_{kj} w_k^{\alpha}} w_j L_j - w_i L_i \right) \tag{11}$$

Note that if $Z_i(\mathbf{w}^*) = 0$ for all $i \in S$, then $\mathbf{w}^*$ is an equilibrium vector of prices. Note too that if $Z_i(\mathbf{w}) > 0$, then that means that country i is selling more than it earns, so we can consider $Z_i(\cdot)$ to be the *excess demand function* of goods from country $i \in S$. (We divide the right hand size of Z_i so that how much is sold and earned is in terms of the quantity of labor in country i).

4 Existence and uniqueness

Using the excess demand equation (11), it is possible to show that the equilibrium wages exist and are unique. The following derivations rely heavily on Alvarez and Lucas (2007), which themselves rely heavily on Mas-Colell, Whinston, and Green (1995).

4.1 Existence

Let us note some properties of $Z_i(\cdot)$:

1. For all $\mathbf{w} \gg 0$ (i.e. for all $\mathbf{w}$ such that $w_i > 0$ for all $i \in S$) and for all $i \in S$, $Z_i(\cdot)$ is continuous. This is immediately evident from equation (11).

2. For all $i \in S$, $Z_i(\cdot)$ is homogeneous of degree zero. To see this, note that for any $\beta > 0$:

$$\begin{aligned}
Z_i(\beta \mathbf{w}) &= \frac{1}{\beta w_i}\left(\sum_{j \in S} \frac{K_{ij}(\beta w_i)^\alpha}{\sum_{k \in S} K_{kj}(\beta w_k)^\alpha} \beta w_j L_j - \beta w_i L_i\right) \iff \\
&= \frac{1}{w_i}\left(\sum_{j \in S} \frac{K_{ij} w_i^\alpha}{\sum_{k \in S} K_{kj} w_k^\alpha} w_j L_j - w_i L_i\right) \iff \\
&= Z_i(\mathbf{w})
\end{aligned}$$

Intuitively, the units that the wages are measured in do not matter (this is reassuring, as it would be shocking if the equilibrium depended on the correct choice of units!).

3. For all $\mathbf{w} \gg 0$, we have:

$$\sum_{i \in S} w_i Z_i(\mathbf{w}) = 0$$

To see this, note that:

$$\begin{aligned}
\sum_{i \in S} w_i Z_i(\mathbf{w}) &= \sum_{i \in S} w_i \frac{1}{w_i}\left(\sum_{j \in S} \frac{K_{ij} w_i^\alpha}{\sum_{k \in S} K_{kj} w_k^\alpha} w_j L_j - w_i L_i\right) \iff \\
&= \sum_{i \in S}\left(\sum_{j \in S} \frac{K_{ij} w_i^\alpha}{\sum_{k \in S} K_{kj} w_k^\alpha} w_j L_j - w_i L_i\right) \iff \\
&= \sum_{i \in S}\sum_{j \in S} \frac{K_{ij} w_i^\alpha}{\sum_{k \in S} K_{kj} w_k^\alpha} w_j L_j - \sum_{i \in S} w_i L_i \iff \\
&= \sum_{i \in S}\sum_{j \in S} X_{ij} - \sum_{i \in S} Y_i \iff \\
&= \sum_{i \in S} Y_i - \sum_{i \in S} Y_i \iff \\
&= 0.
\end{aligned}$$

4. For all $\mathbf{w} \gg 0$, there exists an $s > 0$ such that $Z_i(\mathbf{w}) > -s$ for all $i \in S$. To see this,

$$\begin{aligned}
Z_i(\mathbf{w}) &= \frac{1}{w_i}\left(\sum_{j \in S} \frac{K_{ij} w_i^\alpha}{\sum_{k \in S} K_{kj} w_k^\alpha} w_j L_j - w_i L_i\right) \iff \\
&= \frac{1}{w_i}\sum_{j \in S} \frac{K_{ij} w_i^\alpha}{\sum_{k \in S} K_{kj} w_k^\alpha} w_j L_j - L_i \implies \\
&> -L_i,
\end{aligned}$$

since $\frac{1}{w_i}\sum_{j \in S} \frac{K_{ij} w_i^\alpha}{\sum_{k \in S} K_{kj} w_k^\alpha} w_j L_j > 0$ for all $\mathbf{w} \gg 0$. Hence if we define $s \equiv \max_{i \in S} L_i$, $Z_i(\mathbf{w}) > -s$ for all $i \in S$.

5. Consider any $\mathbf{w} \in \mathbb{R}^{||S||}$ such that there exists an $l \in S$ where $w_l = 0$ and an $l' \in S$ where $w_{l'} > 0$. Consider any sequence of wages such that $\mathbf{w}^n \to \mathbf{w}$ as $n \to \infty$. Then:

$$\max_{i \in S} Z_i\left(\mathbf{w}^n\right) \to \infty.$$

To see this, note that:

$$\begin{aligned}
Z_i\left(\mathbf{w}\right) &= \max_{i \in S} \frac{1}{w_i}\left(\sum_{j \in S} \frac{K_{ij} w_i^\alpha}{\sum_{k \in S} K_{kj} w_k^\alpha} w_j L_j - w_i L_i\right) \iff \\
\max_{i \in S} Z_i\left(\mathbf{w}^n\right) &= \max_{i \in S}\left(\frac{1}{w_i}\sum_{j \in S} \frac{K_{ij} w_i^\alpha}{\sum_{k \in S} K_{kj} w_k^\alpha} w_j L_j - L_i\right) \implies \\
\max_{i \in S} Z_i\left(\mathbf{w}^n\right) &> \max_{i \in S} \frac{1}{w_i} \max_{j \in S} \frac{K_{ij} w_i^\alpha}{\sum_{k \in S} K_{kj} w_k^\alpha} w_j L_j - \max_{i \in S} L_i \implies \\
\max_{i \in S} Z_i\left(\mathbf{w}^n\right) &> \max_{i,j \in S} \frac{w_j}{w_i} \frac{K_{ij} w_i^\alpha}{\sum_{k \in S} K_{kj} w_k^\alpha} L_j - \max_{i \in S} L_i
\end{aligned}$$

Hence if it is the case that $\max_{i,j \in S} \frac{w_j}{w_i} \frac{K_{ij} w_i^\alpha}{\sum_{k \in S} K_{kj} w_k^\alpha} L_j \to \infty$, then because $\max_{i \in S} Z_i\left(\mathbf{w}^n\right)$ is bounded below by it, it must be that $\max_{i \in S} Z_i\left(\mathbf{w}^n\right) \to \infty$. Note that:

$$\begin{aligned}
\max_{i,j \in S} \frac{w_j}{w_i} \frac{K_{ij} w_i^\alpha}{\sum_{k \in S} K_{kj} w_k^\alpha} L_j &> \left(\min_{l \in S} L_s\right) \max_{i,j \in S} w_j \frac{K_{ij} w_i^{\alpha-1}}{\sum_{k \in S} K_{kj} w_k^\alpha} \iff \\
\max_{i,j \in S} \frac{w_j}{w_i} \frac{K_{ij} w_i^\alpha}{\sum_{k \in S} K_{kj} w_k^\alpha} L_j &> C \max_{i \in S} \frac{K_{ij} w_i^{\alpha-1}}{\sum_{k \in S} K_{kj} w_k^\alpha} \implies \\
\max_{i,j \in S} \frac{w_j}{w_i} \frac{K_{ij} w_i^\alpha}{\sum_{k \in S} K_{kj} w_k^\alpha} L_j &> C \max_{i \in S} \frac{K_{ij} w_i^{\alpha-1}}{\sum_{k \in S} K_{kj} \left(\min_{l \in S} w_l\right)^\alpha} \iff \\
\max_{i,j \in S} \frac{w_j}{w_i} \frac{K_{ij} w_i^\alpha}{\sum_{k \in S} K_{kj} w_k^\alpha} L_j &> C \frac{K_{ij} \left(\min_{l \in S} w_l\right)^{\alpha-1}}{\sum_{k \in S} K_{kj} \left(\min_{l \in S} w_l\right)^\alpha} \iff \\
\max_{i,j \in S} \frac{w_j}{w_i} \frac{K_{ij} w_i^\alpha}{\sum_{k \in S} K_{kj} w_k^\alpha} L_j &> C \times \frac{K_{ij}}{\sum_{k \in S} K_{kj}} \times \left(\min_{l \in S} w_l\right)^{-1} \qquad (12)
\end{aligned}$$

where $C \equiv \left(\min_{l \in S} L_s\right)\left(\max_{j \in S} w_j\right) > 0$. [A brief run down of the derivation: the first line we ignore the population since the population is positive everywhere; the second line used the maximization across j to choose the highest wage (which is only necessary to ensure a positive wage); the third line made the denominator as large as possible, relying on the fact that $\alpha < 0$, the fourth line used the maximization across i to make the numerator as large as possible, and the fifth line canceled like terms.] Since there exists an $l \in S$ such that $w_l^n \to 0$ as $n \to \infty$, equation (12) implies that $\max_{i,j \in S} \frac{w_j}{w_i} \frac{K_{ij} w_i^\alpha}{\sum_{k \in S} K_{kj} w_k^\alpha} L_j \to \infty$ as $n \to \infty$, which in turn implies that $\max_{i \in S} Z_i\left(\mathbf{w}^n\right) \to \infty$ as $n \to \infty$.

Theorem. *If a function $\left\{Z_i\left(\cdot\right)\right\}_{i \in S}$ satisfies the above five conditions, then there exists a $\mathbf{w}^* \gg 0$ such that $Z_i\left(\mathbf{w}^*\right) = 0$ for all $i \in S$.*

Proof. This theorem comes directly from Mas-Colell, Whinston, and Green (1995) Proposition 17.C.1 on p.585, so I will refer you there for the complete proof. The basic idea of the proof is simple, however. Because $\{Z_i(\cdot)\}_{i\in S}$ is homogeneous of degree zero, it is sufficient for us to search for an equilibrium vector of wages within the simplex $\mathbf{w} \in \Delta^{||S||}$, which is a closed and bounded set. One can then construct a correspondence between $\Delta^{||S||}$ and $\Delta^{||S||}$ that (loosely speaking) assigns higher wages to those countries where the excess demand is higher. The fifth property we established above ensures that as the wage in any country approaches zero, the excess demand in some region approaches infinity, which (loosely speaking) keeps the equilibrium wage away from the boundaries of the simplex, and (more formally) can be used to prove the upper hemicontinuity of the correspondence. All this together allows one to apply Kakutani's fixed point theorem, which guarantees existence. $\square$

4.2 Uniqueness

A final property of the excess demand function $\{Z_i(\cdot)\}_{i\in S}$ is necessary to establish the uniqueness of the equilibrium. A (differentiable) excess demand function $\{Z_i(\cdot)\}_{i\in S}$ is said to satisfy the **gross substitute property** if for all $i \in S$:

$$\frac{\partial Z_i(\mathbf{w})}{\partial w_j} > 0 \tag{13}$$

for all $j \neq i$. Intuitively, as the wage in other countries rises, the demand for goods from country i increases.

Theorem. *If a function $\{Z_i(\cdot)\}_{i\in S}$ satisfies the gross substitute property (and is homogeneous of degree zero), then the equilibrium $\mathbf{w}^* \gg 0$ such that $Z_i(\mathbf{w}^*) = 0$ for all $i \in S$ is unique (to scale).*

Proof. This theorem comes directly from Mas-Colell, Whinston, and Green (1995) Proposition 17.F.3 on p.613, but is simple enough to replicate here. Suppose there exist $\mathbf{w}^*$ and $\tilde{\mathbf{w}}^*$ that are not collinear such that $Z_i(\mathbf{w}^*) = 0 = Z_i(\tilde{\mathbf{w}}^*)$ for all $i \in S$, i.e. both wages are distinct equilibria. Because $Z_i(\cdot)$ is homogeneous of degree zero, we can re-scale the equilibrium wages such that $w_l^* = \tilde{w}_l^*$ for some $l \in S$ and $w_i^* \geq \tilde{w}_i^*$ for all $i \neq l$. Now let us try to go from $\tilde{w}_i^*$ to w_i^*. From the gross substitute property, increasing the wage in any country will increase the excess demand country l, so that it must be that $Z_l(\mathbf{w}^*) > Z_l(\tilde{\mathbf{w}}^*)$, which is a contradiction. $\square$

We now show that the excess demand function $\{Z_i(\cdot)\}_{i\in S}$ does indeed satisfy the gross substitute property of equation (13). Differentiating equation (11) yields:

$$\begin{aligned}\frac{\partial Z_i(\mathbf{w})}{\partial w_j} &= \frac{\partial}{\partial w_j}\frac{1}{w_i}\left(\sum_{l\in S}\frac{K_{il}w_i^\alpha}{\sum_{k\in S}K_{kl}w_k^\alpha}w_lL_l - w_iL_i\right) \iff \\ &= \frac{1}{w_i}\left(\frac{K_{ij}w_i^\alpha}{\sum_{k\in S}K_{kj}w_k^\alpha}L_j - \alpha w_j^{\alpha-1}\sum_{k\in S}\frac{K_{jk}K_{ik}w_i^\alpha w_kL_k}{\left(\sum_{j\in S}K_{jk}w_j^\alpha\right)^2}\right) > 0\end{aligned}$$

because the first term in the parentheses is positive and the second term is also positive since $\alpha < 0$.

5 A "universal" approach

Thus far we have shown that for any microeconomic foundation of a trade model that yields a gravity expression as in equation (5), there will exist a unique set of equilibrium wages such that the labor market clears (i.e. equation (6) holds) and the goods market clears (i.e. equation (7) holds). This is extremely helpful, but what remains unclear from such an approach is how necessary the common assumptions of the four models we have thus far studied (e.g. CES preferences) are for the equilibrium.

An alternative "top down" approach is to start by considering all models which yield the following "general" gravity equation:

$$X_{ij} = K_{ij}\gamma_i\delta_j, \tag{14}$$

where $K_{ij} > 0$ is the (exogenous) bilateral trade friction, γ_i is an (endogenous) origin fixed effect, and δ_j is an (endogenous) destination fixed effect.

I now define a number of equilibrium conditions. I say that the **goods market clearing condition** holds if for all countries, the income is equal to total sales, i.e. for all $i \in S$:

$$Y_i = \sum_{j\in S} X_{ij}. \tag{15}$$

I say that **trade is balanced** if for all countries, income is equal to expenditures, i.e. for all $i \in S$:

$$Y_i = \sum_{j\in S} X_{ji}. \tag{16}$$

I say that the **generalized labor market clearing** condition holds if for all countries, the equilibrium income can be written as a log linear function of the origin and destination fixed effects:

$$Y_i = \bar{B}_i\gamma_i^{\alpha}\delta_i^{\beta}, \tag{17}$$

where $\bar{B}_i > 0$ and α and β are exogenous parameters.

5.1 Existence and uniqueness when $\beta = 0$ and $\alpha < 0$

Suppose that $\beta = 0$ so that:

$$Y_i = \bar{B}_i\gamma_i^{\alpha}.$$

We say that **trade is balanced** if for all $j \in S$ the income in j is equal to its expenditure, i.e.:

$$Y_j = \sum_{i\in S} X_{ij} \tag{18}$$

Substituting the general gravity equation (14) into the balanced trade condition (18) yields:

$$
\begin{aligned}
Y_j &= \sum_{i \in S} X_{ij} \iff \\
Y_j &= \sum_{i \in S} K_{ij} \gamma_i \delta_j \iff \\
Y_j &= \delta_j \sum_{i \in S} K_{ij} \gamma_i \iff \\
\delta_j &= \frac{Y_j}{\sum_{k \in S} K_{kj} \gamma_k}
\end{aligned} \tag{19}
$$

Substituting (19) back into the generalized gravity equation (14) then yields:

$$
\begin{aligned}
X_{ij} &= K_{ij} \gamma_i \delta_j \iff \\
X_{ij} &= \frac{K_{ij} \gamma_i}{\sum_{k \in S} K_{kj} \gamma_k} Y_j,
\end{aligned} \tag{20}
$$

which is equivalent to the gravity expression (5) with $\gamma_i = w_i^\alpha$.

Substituting the generalized labor market clearing condition into the gravity equation (20) yields:

$$
\begin{aligned}
X_{ij} &= \frac{K_{ij} \gamma_i}{\sum_{k \in S} K_{kj} \gamma_k} Y_j \iff \\
X_{ij} &= \frac{K_{ij} \gamma_i}{\sum_{k \in S} K_{kj} \gamma_k} \bar{B}_j \gamma_j^\alpha
\end{aligned} \tag{21}
$$

Summing equation (21) over all destinations and applying the good market clearing condition (equation (7)) then yields:

$$
\begin{aligned}
\sum_{j \in S} X_{ij} &= \sum_{j \in S} \frac{K_{ij} \gamma_i}{\sum_{k \in S} K_{kj} \gamma_k} \bar{B}_j \gamma_j^\alpha \iff \\
Y_i &= \sum_{j \in S} \frac{K_{ij} \gamma_i}{\sum_{k \in S} K_{kj} \gamma_k} \bar{B}_j \gamma_j^\alpha \iff \\
\bar{B}_i \gamma_i^\alpha &= \sum_{j \in S} \frac{K_{ij} \gamma_i}{\sum_{k \in S} K_{kj} \gamma_k} \bar{B}_j \gamma_j^\alpha \iff \\
\bar{B}_i \tilde{\gamma}_i &= \sum_{j \in S} \frac{K_{ij} \tilde{\gamma}_i^{\frac{1}{\alpha}}}{\sum_{k \in S} K_{kj} \tilde{\gamma}_k^{\frac{1}{\alpha}}} \bar{B}_j \tilde{\gamma}_j,
\end{aligned} \tag{22}
$$

where $\tilde{\gamma}_i \equiv \gamma_i^\alpha$. Comparing expression (22) to equation (10) which formed the basis for the excess demand function, it is immediately clear that the two expressions are mathematically identical; the only difference is that equation (22) provides a solution for the origin fixed effect in the generalized gravity equation (14) rather than the wage, and the labor supply is replaced by $\bar{B}_i$. As a result, we can use the same proof as above to prove the following theorem:

Theorem. *Consider any trade model that yields the generalized gravity equation (14) where $\beta = 0$ and $\alpha < 0$. Then there exists a unique (to scale) set of origin and destination fixed effects such that (1) the goods market clears; (2) trade is balanced; and (3) the generalized labor market clearing condition holds.*

Proof. The proof follows the same method as above; the only thing to note is that once the equilibrium (to-scale) set of origin fixed effects has been determined from equation (22), the equilibrium destination fixed effects can be determined from equation (19) and the generalized labor market clearing condition:

$$\delta_j = \frac{\bar{B}_j \gamma_j^{\alpha}}{\sum_{k \in S} K_{kj} \gamma_k}.$$

□

5.2 Existence and uniqueness when trade costs are quasi-symmetric

In this section, we generalize the set of gravity models considered using a different mathematical tools in the special case where trade costs are "quasi-symmetric." I say that trade costs are **quasi-symmetric** when for all $i, j \in S$ we have:

$$K_{ij} = \tilde{K}_{ij} K_i^A K_j^B,$$

where $\tilde{K}_{ij} = \tilde{K}_{ji}$. Note that symmetric trade costs (where $K_{ij} = K_{ji}$) are quasi-symmetric, but there also exist non-symmetric trade costs that are quasi-symmetric.

5.2.1 A helpful mathematical proposition

The following is a very helpful mathematical proposition that we will use below.

Proposition 1. *Consider the system of equations:*

$$x_i = \lambda \sum_{j \in S} K_{ij} x_j^{\alpha} \; \forall i \in S, \tag{23}$$

where $K_{ij} \in (a_{low}, a_{hi})$ for all $i, j \in S$, where $0 < a_{low} < a_{hi} < \infty$ Then: (i) there exists a $\lambda > 0$ and a set of x_i that solves equation (23) such that $x_i > 0$ for all $i \in S$ and $\sum_{i \in S} x_i = B$ for some B (this is just an arbitrary normalization); and (ii) the solution is unique if $\alpha \in [-1, 1]$.

Proof. This theorem is a collection of mathematical results, some of which are pretty complicated. Rather than presenting the proof, I reference each of the results in the following table: □

Value of α	Existence (continuous)	Uniqueness (continuous)
$\alpha > 1$	Guo and Lakshmikantham (1988)	N/A
$\alpha = 1$	Jentzsch's theorem (Birkhoff, 1957)	
$\alpha \in (0, 1)$	Zabreyko, Koshelev, Krasnosel'skii, Mikhlin, Rakovshchik, and Stetsenko (1975)	
$\alpha \in [-1, 0)$	Karlin and Nirenberg (1967)	
$\alpha < 1$	Karlin and Nirenberg (1967)	N/A
Value of α	*Existence (discrete)*	*Uniqueness (discrete)*
$\alpha > 1$	Guo and Lakshmikantham (1988)	N/A
$\alpha = 1$	Perron-Frobenius Theorem	
$\alpha \in (0, 1)$	Fujimoto and Krause (1985)	
$\alpha \in [-1, 0)$	Karlin and Nirenberg (1967)	
$\alpha < 1$	Karlin and Nirenberg (1967)	N/A

5.2.2 Proposition 2

We now present the main result that arises from assuming that trade costs are quasi-symmetric.

Proposition 2. *For any gravity trade model with quasi-symmetric trade costs, if there exist strictly positive and bounded origin and destination fixed effects such that the goods market clears and trade is balanced, then it must be the case that:*

$$K_i^A \gamma_i = \kappa K_i^B \delta_i \tag{24}$$

for some $\kappa > 0$.

Proof. Equating equations (15) and (16) yields:

$$\sum_{j \in S} X_{ij} = \sum_{j \in S} X_{ji}. \tag{25}$$

Substituting in the gravity equation (equation (14)) into equation 25 and imposing that trade frictions are quasi-symmetric yields:

$$\begin{gathered}
\sum_{j \in S} X_{ij} = \sum_{j \in S} X_{ji} \iff \\
\sum_{j \in S} \tilde{K}_{ij} K_i^A K_j^B \gamma_i \delta_j = \sum_{j \in S} \tilde{K}_{ij} K_j^A K_i^B \gamma_j \delta_i \iff \\
K_i^A \gamma_i \sum_{j \in S} \tilde{K}_{ij} K_j^B \delta_j = K_i^B \delta_i \sum_{j \in S} \tilde{K}_{ij} K_j^A \gamma_j \iff \\
\frac{K_i^A \gamma_i}{K_i^B \delta_i} = \frac{\sum_{j \in S} \tilde{K}_{ij} K_j^A \gamma_j}{\sum_{j \in S} \tilde{K}_{ij} K_j^B \delta_j}.
\end{gathered}$$

Define $\phi_i \equiv \frac{K_i^A \gamma_i}{K_i^B \delta_i}$, $\lambda = 1$, and $F_{ij} \equiv \frac{\tilde{K}_{ij} K_j^B \delta_j}{\sum_{k \in S} \tilde{K}_{ik} K_k^B \delta_k}$. Note that F_{ij} is strictly positive and bounded for all $i, j \in S$. Then note that we can write:

$$\begin{aligned}
\frac{K_i^A \gamma_i}{K_i^B \delta_i} &= \frac{\sum_{j \in S} \tilde{K}_{ij} K_j^A \gamma_j}{\sum_{j \in S} \tilde{K}_{ij} K_j^B \delta_j} \iff \\
\phi_i &= \frac{\sum_{j \in S} \tilde{K}_{ij} K_j^B \delta_j \phi_j}{\sum_{j \in S} \tilde{K}_{ij} K_j^B \delta_j} \iff \\
\phi_i &= \sum_{j \in S} \frac{\tilde{K}_{ij} K_j^B \delta_j}{\sum_{j \in S} \tilde{K}_{ij} K_j^B \delta_j} \phi_j \iff \\
\phi_i &= \lambda \sum_{j \in S} F_{ij} \phi_j. \qquad (26)
\end{aligned}$$

From Proposition 1, there exists a unique (to-scale) strictly positive$\{\phi_i\}$ and $\lambda > 0$ that solves equation 26. Hence, if we find a solution to 26, it will be the unique solution. We guess that $\lambda = 1$ and $\phi_i = \kappa$ for some $\kappa > 0$. Then note that equation (26) implies:

$$\begin{aligned}
\kappa &= \sum_{j \in S} \frac{\tilde{K}_{ij} K_j^B \delta_j}{\sum_{k \in S} \tilde{K}_{ik} K_k^B \delta_k} \kappa \iff \\
\sum_{k \in S} \tilde{K}_{ik} K_k^B \delta_k &= \sum_{j \in S} \tilde{K}_{ij} K_j^B \delta_j,
\end{aligned}$$

so that the guess holds. Given the definition of ϕ_i, we then have:

$$\begin{aligned}
\phi_i &= \kappa \iff \\
K_i^A \gamma_i &= \kappa K_i^B \delta_i,
\end{aligned}$$

as required. □

What does Proposition (2) imply for bilateral trade flows? To answer this note that:

$$\begin{aligned}
\frac{X_{ij}}{X_{ji}} &= \frac{K_{ij} \gamma_i \delta_j}{K_{ji} \gamma_j \delta_i} \iff \\
\frac{X_{ij}}{X_{ji}} &= \frac{\tilde{K}_{ij} K_i^A K_j^B \gamma_i \delta_j}{\tilde{K}_{ji} K_j^A K_i^B \gamma_j \delta_i} \iff \\
\frac{X_{ij}}{X_{ji}} &= \frac{K_i^A K_j^B \gamma_i \left(\frac{1}{\kappa} \frac{K_j^A}{K_j^B} \gamma_j \right)}{K_j^A K_i^B \gamma_j \left(\frac{1}{\kappa} \frac{K_i^A}{K_i^B} \gamma_i \right)} \iff \\
\frac{X_{ij}}{X_{ji}} &= 1.
\end{aligned}$$

Hence, in equilibrium when trade is balanced, the goods market clears, and bilateral trade costs are quasi-symmetric, then the value of trade flows from any origin to any destination are exactly equal to the value of trade flows in the other direction. Since trade is balanced between any two partners, trade will be trivially balanced overall. The surprising result here is that it turns out with symmetric trade costs, having trade be bilaterally balanced is the unique way to ensure overall trade is balanced.

5.2.3 Existence and Uniqueness

Proposition (2) provides a very elegant way of proving the existence and uniqueness of gravity trade models with quasi symmetric trade costs.

Theorem. *For any gravity trade model with trade costs are quasi-symmetric, there exists a set of origin and destination fixed effects such that (1) the goods market clears; (2) trade is balanced; and (3) the generalized labor market clearing condition holds. Furthermore, this equilibrium is unique if $\alpha + \beta \geq 2$ or $\alpha + \beta \leq 0$.*

Proof. Combining the goods market clearing condition (15) and the gravity equation (14) yields:

$$\begin{aligned} Y_i &= \sum_{j \in S} X_{ij} \iff \\ Y_i &= \sum_{j \in S} K_{ij} \gamma_i \delta_j. \end{aligned} \tag{27}$$

From the generalized labor market clearing condition, we can write equation (27) as:

$$\begin{aligned} \bar{B}_i \gamma_i^{\alpha} \delta_i^{\beta} &= \sum_{j \in S} K_{ij} \gamma_i \delta_j \iff \\ \bar{B}_i \gamma_i^{\alpha-1} \delta_i^{\beta} &= \sum_{j \in S} K_{ij} \delta_j. \end{aligned} \tag{28}$$

Because trade costs are quasi-symmetric, the goods market clears, and trade is balanced, we can apply the results of Theorem 2. Substituting equation 24 into equation (28) yields:

$$\begin{aligned} \bar{B}_i \gamma_i^{\alpha-1} \delta_i^{\beta} &= \sum_{j \in S} K_{ij} \delta_j \iff \\ \bar{B}_i \gamma_i^{\alpha-1} \left(\frac{1}{\kappa} \frac{K_i^A}{K_i^B} \gamma_i \right)^{\beta} &= \sum_{j \in S} K_{ij} \left(\frac{1}{\kappa} \frac{K_j^A}{K_j^B} \gamma_j \right) \iff \\ \gamma_i^{\alpha+\beta-1} &= \kappa^{\beta-1} \sum_{j \in S} K_{ij} \left(\frac{K_j^A}{K_j^B} \right) \left(\frac{K_i^A}{K_i^B} \right)^{-\beta} \frac{1}{\bar{B}_i} \gamma_j \iff \\ \tilde{\gamma}_i &= \lambda \sum_{j \in S} F_{ij} \tilde{\gamma}_j^{\frac{1}{\alpha+\beta-1}}, \end{aligned} \tag{29}$$

where $\tilde{\gamma}_i \equiv \gamma_i^{\alpha+\beta-1}$, $\lambda \equiv \kappa^{\beta-1} > 0$, and $F_{ij} \equiv K_{ij} \left(\frac{K_j^A}{K_j^B} \right) \left(\frac{K_i^A}{K_i^B} \right)^{-\beta} \frac{1}{\bar{B}_i} > 0$. From Theorem 1, there exists a strictly positive set of $\{\tilde{\gamma}_i\}$ that solve equation (29) and the solution is unique if $\frac{1}{\alpha+\beta-1} \in [-1, 1]$, i.e. $\alpha + \beta \geq 2$ or $\alpha + \beta \leq 0$. $\square$

Theorem 5.2.3 generalizes the gross substitutes result in several important ways: (1) it proves existence for any value of α (not just $\alpha < 0$); (2) it provides information on where equilibria may not be unique; and (3) it allow for $\beta \neq 0$, which as see below allows for us to incorporate the possibility of intermediate inputs in production.[1]

5.3 Existence and uniqueness: the general case

Finally, we consider the general case. For simplicity, I state the theorem and refer the interested students to Allen, Arkolakis, and Takahashi (2014) for the proof.

Theorem. *Consider any general equilibrium gravity model. If $\alpha + \beta \neq 1$, then:*
i) The model has a positive solution and all possible solutions are positive;
ii) If $\alpha, \beta \leq 0$ or $\alpha, \beta \geq 1$, then the solution is unique.

Proof. See Theorem 1 of Allen, Arkolakis, and Takahashi (2014).

Hence, we now have a complete characterization of the existence and uniqueness of gravity models! □

6 Next steps

In the next class we will delve into the welfare properties of gravity trade models.

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[1] In Alvarez and Lucas (2007) (upon which the gross substitutes proof is based), the model allows for intermediate goods to be used in production; however, for the result to apply with intermediate goods, it must be the case that there is a non-tradable sector that is (loosely speaking) "more important" than the intermediate goods. In the models considered here, there are no non-tradables, in which case gross substitutes will not hold for any amount of intermediate goods.

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