Lecture Notes:
Real Analysis

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Draft updated on June 4, 2025

This is the lecture note taken in the course *Real Analysis* taught by **So-Chin Chen** at National Tsing Hua University, Taiwan in the semester I of 2021. The course is designed to provide a comprehensive understanding of the fundamental concepts and theories in Real Analysis.

Currently, these are just drafts of the lecture notes. There can be typos and mistakes anywhere. So, if you find anything that needs to be corrected or improved, please inform at `jingle.fu@graduateinstitute.ch`.

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Lecture 1.

Preliminaries

Real analysis originates from calculus. When we were studying calculus, we deal with two sides: differentiation and integration, which are inverse operations. Real analysis is the study of these two sides in a more rigorous way: starting from generalizing the Riemann integral, but why?

Recall that when we were studying Riemann integral, we define $f : [a, b] \to \mathbb{R}$ as a bounded function on a closed interval $[a, b]$.

$$\int_a^b f(x)\,dx \text{ exists?} \tag{1.1}$$

The Riemann integral is then defined as the limit of Riemann sums as the partition of the interval becomes finer. We take the partition of the interval $[a, b]$ into n subintervals, $P = \{a < x_0 < x_1 < \cdots < x_n = b\}$, take $t_k \in [x_{k-1}, x_k]$ as a sample point in each subinterval, and then we form the Riemann sum:

$$R(f, P) = \sum_{k=1}^{n} f(t_k)\Delta x_k \tag{1.2}$$

where $\Delta x_k = x_k - x_{k-1}$ is the width of the k-th subinterval. The Riemann integral is then defined as:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} R(f, P) \tag{1.3}$$

if the limit exists.

In Darboux integral, we define the upper and lower sums:

$$L(f, P) = \sum_{k=1}^{n} m_k \Delta x_k, \quad U(f, P) = \sum_{k=1}^{n} M_k \Delta x_k \tag{1.4}$$

where $m_k = \inf_{x\in[x_{k-1},x_k]} f(x)$ and $M_k = \sup_{x\in[x_{k-1},x_k]} f(x)$. The Darboux integral is then defined as:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} L(f, P) = \lim_{n\to\infty} U(f, P) \tag{1.5}$$

if the limit exists. The Riemann integral and the Darboux integral are equivalent, but the Darboux integral is more general.

Example 1.

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

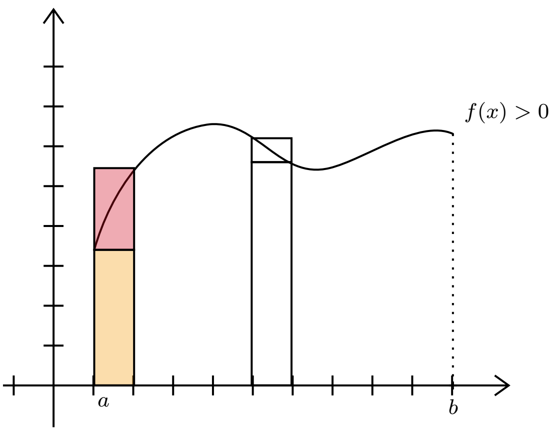
for $x \in [0, 1]$. The Riemann integral does not exist, but the Darboux integral exists and is equal to 0.

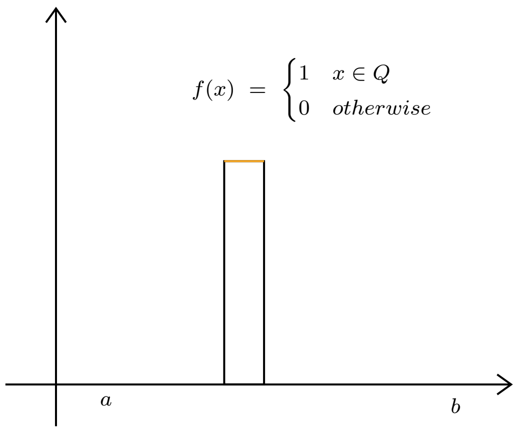
$$L(f, P) = 0, \quad U(f, P) = 1$$

Lebesgue extends the idea of Riemann integral to more general functions.

> **Theorem 1.0.1** (Lebesgue's Criterion for Integrability).
>
> Set $\mathcal{D} = \{x \in [a, b] | f \text{ is not continuous at } x\}$. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Then f is Lebesgue is integrable if and only if $\mathcal{D}$ is of measure zero.

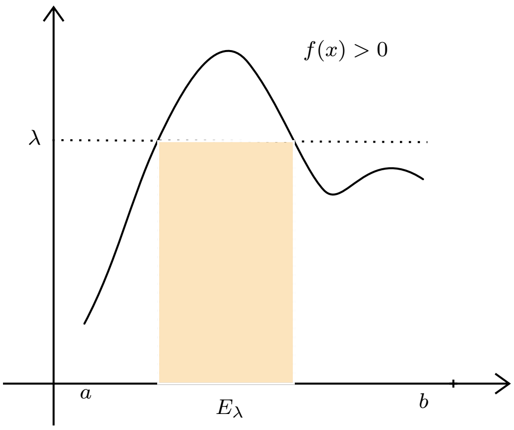
We first show the difference of Riemann integral and Darboux integral in the graph(left). Then let's look back at example 1 (right), we can see that the function is not continuous at any point in $[0, 1]$. By the definition of Riemann integral, we can see that the Riemann integral does not exist.





What do we really want by taking integrals? Basically speaking, we want to find the area under the curve. In the case of Riemann and Darboux integrals, we partitioned the x-axis into subintervals, but we encountered problems.

It's quite common that we start to try partitioning the y-axis instead. We define the Lebesgue measure as $E_\lambda = \{x \in [a, b] | f(x) > \lambda\}$.



Assuming that E_λ is measurable, we know that the shaded area is $\lambda|E_\lambda|$.

Lecture 2.

Measure Theory

2.1 Outer Measure

Outer measure can be defined on every set.

Definition 2.1.1 (Outer Measure).

$E \subseteq \mathbb{R}^n$ is a set, I is a closed interval: $I = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n | a_i \leq x_i \leq b_i, i = 1, \cdots, n\}$, and $v(I)$ is the volume of the interval I,

$$v(I) = \begin{cases} \prod_{i=1}^{n}(b_i - a_i), & \text{if } a_i \leq b_i; \\ 0, & \text{otherwise.} \end{cases}$$

For set E, consider a *countable* collection of open, nounded intervals that cover E, $S = \{I_i\}_{i=1}^{\infty}$, in the sense that $E \subseteq \bigcup_{i=1}^{\infty} I_i$. For each such collection, consider the sum of the volumes of the intervals in the collection. We define

$$\sigma(S) = \sum_{i=1}^{\infty} v(I_i) \tag{2.1}$$

The outer measure of E, denoted by $m^*(E)$, is

$$m^*(E) = \inf \sigma(S) \tag{2.2}$$

the infimum is taken over all countable collections of closed intervals S.

Lemma 2.1.1.

If I is a closed interval, then $m^*(I) = v(I)$.

Proof.

By definition, I covers itself, so $m^*(I) \leq v(I)$. Given any $\varepsilon > 0$, $\exists S = \{I_i\}_{i=1}^{\infty}$, a closed interval cover, such that $\sigma(S) \leq m^*(I) + \varepsilon$. We need to show that $v(I) \leq \sum_{i=1}^{\infty} v(I_i) = \sigma(S)$. For each i, choose a bigger I_i^*, such that $I \subseteq int(I_i^*)$ and $v(I_i^*) \leq v(I_i)(1+\varepsilon)$. Then we have $I \subseteq \bigcup_{i=1}^{\infty} int(I_i^*)$. By compactness of I, (The Heine-Borel theorem), we can find an integer N such that $I \subseteq \bigcup_{i=1}^{N} int(I_i^*)$, hence

$$v(I) \leq \sum_{i=1}^{N} v(I_i^*) \leq (1+\varepsilon) \sum_{i=1}^{N} v(I_i) \leq (1+\varepsilon)\sigma(S).$$

So $v(I) \leq \sigma(S)$, if we take infimum over all S, we have $v(I) \leq m^*(I)$. $\square$

Lecture 3.

Appendix

Recommended Resources

Books

[1] Richard L. Wheeden Antoni Zygmund. *Measure and Integral An Introduction to Real Analysis*. 2nd ed. New York: CRC Press, 2015

[2] Walter Rudin. *Real and Complex Analysis*. 3rd ed. Singapore: McGraw-Hill, 1987

[3] Gerald B. Folland. *Real Analysis Modern Techniques and Their Applications*. 2nd ed. Toronto, Canada: Wiley, 1999

[4] P. M. Fitzpatrick Halsey L. Royden. *Real Analysis*. 5th ed. New Jersey: Pearson Education, Inc., 2023

[5] Elias M. Stein and Rami Shakarchi. *Real Analysis: Measure Theory, Integration, and Hilbert Spaces*. Princeton, New Jersey: Princeton University Press, 2005

[6] Sheldon Axler. *Measure, Integration & Real Analysis*. Springer Open, 2022