Macroeconomics A
Problem Set 3

Johannes Boehm

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1 A simplified real-business cycle model with additive technology shocks

Consider an economy consisting of a constant population of inifinitely lived individuals. The representative individual maximizes the expected value of $\sum_{t=0}^{\infty} u\left(C_t\right)/\left(1+\rho\right)^t, \quad \rho > 0$. The instantaneous utility function, $u\left(C_t\right)$, is $u\left(C_t\right) = C_t - \theta C_t^2$, $\theta > 0$. Assume that C is always in the range where $u'\left(C\right)$ is positive.

Output is linear in capital, plus an additive disturbance: $Y_t = AK_t + e_t$. There is no depreciation: thus $K_{t+1} = K_t + Y_t - C_t$ and the interest rate is A. Assume $A = \rho$. Finally, the disturbance follows a first-order autoregressive process: $e_t = \phi e_{t-1} + \varepsilon_t$ where $-1 < \phi < 1$ and where the ε_t's are mean-zero, i.i.d shocks.

1. Find the first-order condition (Euler equation) relating C_t and expectations of C_{t+1}.
2. Guess that consumption takes the form $C_t = \alpha + \beta K_t + \gamma e_t$. Given this guess, what is K_{t+1} as a function of K_t and e_t?
3. What values must the parameters α, β, and γ have for the first-order condition in part 1 to be satisfied for all values of K_t and e_t?
4. What are the effects of a one-time shock to ε on the paths of Y, K, and C?

2 Solving the RBC model by finding the social optimum

Consider the RBC model with full depreciation that we discussed in the lecture ("a very special case"):

$$\begin{aligned} U\left(C_t, L\right) &= \log C_t + b\log\left(1-L_t\right) \\ Z_t F(K_t, 1) &= Z_t K_t^{\alpha} \\ \delta &= 1 \end{aligned}$$

(Assume furthermore that there is no technology or population growth.) Denote by

$$V\left(K_t, A_t\right) = \max_{\{C_t, L_t\}_{t=0}^{\infty}} E_t \sum_{t=0}^{\infty} \beta^{\tau} U\left(C_{t+\tau}, L_{t+\tau}\right)$$

the *value function* of the household, which captures the expected present discounted value from the present period forward, into the infinite future, under the optimal consumption and labor allocations (and taking the budget constraint etc into account). The arguments of the value function are the current *state*: assets K_t and the state of productivity A_t. Hence the value function tells you, for every state, what the maximum of the household's objective function is under the optimal choice.

1. Explain intuitively why V must satisfy

$$V\left(K_t, A_t\right) = \max_{C_t, L_t} \left\{ \log C_t + b \log\left(1 - L_t\right) + \beta E_t\left[V\left(K_{t+1}, A_{t+1}\right)\right]\right\}.$$

   This equation is known as the *Bellman equation*.

2. Given the log-linear structure of the model, let us guess that V takes the form

$$V\left(K_t, A_t\right) = \gamma_0 + \gamma_K \log K_t + \gamma_A \log A_t$$

   where the values of the β's are to be determined. Substituting this conjectured form and the facts that $K_{t+1} = Y_t - C_t$ and $E_t\left[\log A_{t+1}\right] = \rho \log A_t$ into the Bellman equation yields

$$V\left(K_t, A_t\right) = \max_{C_t, L_t} \left\{ \log C_t + b \log\left(1 - L_t\right) + \beta\left[\gamma_0 + \gamma_K \log\left(Y_t - C_t\right) + \gamma_A \log A_t\right]\right\}.$$

   Find the first-order condition for C_t. Show that it implies that C_t/Y_t does not depend on K_t or A_t.

3. Find the first-order condition for L_t. Use this condition and the result in part 2 to show that L_t does not depend on K_t or A_t.

4. Substitute the production function and the results in questions 2 and 3 for the optimal C_t and L_t into the equation above for V, and show that the resulting expression has the form $V\left(K_t, A_t\right) = \gamma_0' + \gamma_K' \log K_t + \gamma_A' \log A_t$.

5. What must γ_K and γ_A be so that $\gamma_K' = \gamma_K$ and $\gamma_A' = \gamma_A$?

6. What are the implied values of C/Y and L? Are they the same as those found in the lecture?