Demystifying DSGE Models

John Cuddy
The Graduate Institute
Geneva

Some Introductory Points

About the course

- ***6-credit*** course in 10 weeks
- → learning curve quite ***steep***
- ***6*** take-home problem sets → ***80%*** of final mark
- May be done in groups ***but***
- Each individual ***must*** submit her/his own problem set with some ***personal*** commentary/features to distinguish it from others
- ***In-class 3-hour*** final exam in week following last lecture → ***20%*** of final mark

Problem sets

- Due ***weekly*** before class in following week (unless otherwise indicated – eg, mid-term exams in other macro courses!)
- First set requires ***brain + Dynare***

Dynare

- requires ***Matlab***
- ➔ ***Tutorial on Dynare*** later this week

Class Moodle

- ***Maria*** will post on Moodle
  - Lecture Powerpoints: the night before each lecture
  - Problem Sets: each week after lecture
  - Problem Set Solutions: each subsequent week after lecture
  - Additional Notes: as and when required

Tutorials

- ***Maria*** will organise these as and when required

Final Exam

- ***Maria*** will organise in week following last lecture

Outline

I. Under the Hood – Nuts and Bolts of a DSGE Model
II. Adding Bells and Whistles – The NK DSGE in all its Glory
III. Case Study – The Canonical Smets-Wouters DSGE
IV. Bringing DSGE Models to the Data – Beyond Calibration and Simulation
V. Extensions to the NK DSGE - The Financial and Housing Sectors
VI. Extensions (2) – Unemployment and Environment
VII. Internationality – The Open Economy

Demystifying DSGE Models

1. Under the Hood – Nuts and Bolts of a DSGE Model

- ***What?***
- ***DSGE model*** is:
  - ***Dynamic*** – traces out ***transition*** from one period to the next
  - ***Stochastic*** – subject to ***random fluctuations*** coming from external ***shocks***
  - ***General Equilibrium*** – all markets clear and everything depends on everything else in a coherent fashion (***Walras***)
- "Real Business Cycle" (RBC) model - Kydland and Prescott 1982 - is ***original DSGE*** model [won ***Nobel Prize***]; already middle-aged (> 40 years!)
- ***Unrealistic assumptions*** but ***methodology*** has endured and forms basis of NK DSGEs

- ***Why?***
- ***Lucas critique*** [also won ***Nobel Prize***]:
- Standard ***simultaneous equations model*** assumes parameters are ***constant***
- ***But people are not stupid***
- They ***adjust*** their economic behaviour to ***anticipated*** policy of authorities
- ➔ need to develop model in which parameters are ***inherently*** stable ("***deep parameters***")

How?

Micro ➔ ***stable behavioural parameters***

- Recall your first-year ***undergraduate micro*** class
- ***Consumer assumed to maximise her utility*** (derived from consumption of goods and of leisure time) ***subject to her budget constraint***
- In Micro 101, this was done in a ***single*** time period (***static***)
- Now introduce a ***dynamic*** element
- Define ***intertemporal utility function***

$$U(C_1, C_2, C_3, \ldots)$$

- ***Assume*** that preferences are ***additively separable***:
- $U(C_1, C_2, C_3, \ldots) = U(C_1) + \beta\, U(C_2) + \beta^2\, U(C_3) + \ldots$
- where
  - $\beta = 1/(1 + \theta)$ is ***intertemporal discount factor***
  - $\theta > 0$ is ***intertemporal preference rate [2% ➔ 0.02]***
- **β < 1** since consumption ***today*** is more valuable than consumption ***tomorrow*** [1/1.02 = 0.98]
- ➔ utility function type used in RBC model

- ***Basic RBC*** model assumes
  - ***decentralised decisions*** by firms and households
  - ***perfectly functioning competitive markets***
- ➔ ***subject to constraints***, households maximise overall ***utility function***:

$$E_t\left[\sum_{i=0}^{\boxed{\infty}} \beta^i \left(U\left(C_{t+i}\right) - V\left(L_{t+i}\right)\right)\right]$$

Eg, annual discount rate of 7%
➔quarterly β = 0.983
➔ β^i = 0.067 after 40 years [= working life]

- where
  - C_t is consumption
  - L_t is hours worked (thus enters ***negatively*** in utility)
  - β is representative household's intertemporal discount factor [β marginally < 1, say 0.98]
  - Household lasts ***forever*** (sum to ∞)

But, after several periods, discount is very substantial, so little ***practical*** difference with a ***finite*** sum

- ***What constraints***?
- ***First***, household's ***intertemporal budget constraint***
- Constraint for j-th consumer

$$P_t\left(C_{j,t}+I_{j,t}\right)=W_tL_{j,t}+R_tK_{j,t}$$

- where
  - P = general ***price level*** (***average*** over period t)
  - I = level of ***investment*** (made ***during*** period t)
  - W = return on human capital (***wage rate***)
  - K = ***capital stock*** (at ***beginning*** of period t)
  - R = (gross) ***return on physical capital*** (over period t)
- LHS ➔ ***expenditure;*** RHS ➔ ***revenue***

- ***Second*** constraint***:*** Technology and investment ➔

$$Y_t = C_t + I_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$

$$K_{t+1} = I_t + (1-\delta) K_t$$

Assuming Cobb-Douglas; there are others!

- Exogenous AR(1) process for technology term A_t:

$$\log A_t = \rho \log A_{t-1} + \epsilon_t$$

Note that depending on the value of ρ this makes technology shock more or less ***persistent*** – which ➔ model a ***fluctuating*** appearance

- That's it!
- No government, no trade, no money, no population [i.e., use ***per capita*** variables]
- Clearly, this is **D**ynamic (t + i), **S**tochastic (ϵ) and **G**eneral **E**quilibrium (only technology is exogenous)

Solution Secret 1:
The Lagrangian

- Solve DSGE model using ***Lagrangian*** you are used to from Micro 101 or Bootcamp
- Requires finding and solving "First Order Conditions" ***(FOCs)***
  - ***Assume*** objective function is ***convex*** ➔ ***Second*** Order Conditions ***satisfied*** ➔ forget about them
- More ***complicated*** than usual Lagrangian maximisation
  - because it involves ***infinite*** series of time points
- Wriggle out of this difficulty with some ingenuity

- Standard utility function: ***Constant Relative Risk Aversion*** (CRRA) function defined over consumption

$$U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma}$$

$\sigma \equiv$ risk aversion parameter = inverse of intertemporal rate of substitution

NB: if $\sigma = 1$ ➔ log(C)

U'(C) = $C^{-\sigma}$ **!!**

- In DSGEs, ***also*** standard to assume:
- ∃ ***disutility*** derived from performing ***labour***
- ***separable*** from utility derived from consumption
- ➔ utility function of form

φ is inverse of "Frisch elasticity of labour supply", which is percentage change in ***hours*** arising from a given percentage change in ***wages***

$$U(C_t) - V(L_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\varphi}}{1+\varphi}$$

V'(L) = L^{φ} **!!**

- ***Household j's*** Lagrangian then becomes:

$$\mathscr{L} = E_t \sum_{t=0}^{\infty} \beta^t \left\{ \left[\frac{C_{j,t}^{1-\sigma}}{1-\sigma} - \frac{L_{j,t}^{1+\varphi}}{1+\varphi} \right] - \lambda_{j,t} \left[P_t C_{j,t} + \underbrace{P_t K_{j,\boxed{t+1}} - P_t(1-\delta) K_{j,t}}_{= P_t I_t} - W_t L_{j,t} - R_t K_{j,t} \right] \right\}$$

- Formidable!
- Use a ***trick: endogenous*** variables ***at time t*** appear only ***once*** in the infinite sum ➔ easy to differentiate to obtain FOCs if we index by ***t***
- But here we also have a one-period ***lead*** on capital
- ➔ easier to consider FOC wrt capital (K) at time ***t+1*** instead of ***t***

- ***Repeating***:

$$\mathscr{L} = E_t \sum_{t=0}^{\infty} \beta^t \left\{ \left[\frac{C_{j,t}^{1-\sigma}}{1-\sigma} - \frac{L_{j,t}^{1+\varphi}}{1+\varphi} \right] - \lambda_{j,t} \left[P_t C_{j,t} + P_t K_{j,t+1} - P_t(1-\delta) K_{j,t} - W_t L_{j,t} - R_t K_{j,t} \right] \right\}$$

- FOCs:

$$\frac{\partial \mathscr{L}}{\partial C_{j,t}} = C_{j,t}^{-\sigma} - \lambda_{j,t} P_t = 0$$

$$\frac{\partial \mathscr{L}}{\partial L_{j,t}} = -L_{j,t}^{\varphi} + \lambda_{j,t} W_t = 0$$

These simple FOC forms are reason why we divide by (1-σ) and (1+φ)

$$\frac{\partial \mathscr{L}}{\partial K_{j,t+1}} = -\lambda_{j,t} P_t + \underbrace{\beta E_t \lambda_{j,t+1} \left[(1-\delta) E_t P_{t+1} + E_t R_{t+1} \right]}_{\text{Since } K_t \text{ is also in Lagrangian, need to lead by 1 to get } K_{t+1}} = 0$$

Lagrange Multiplier

- Very simple model = FOCs derived above
- But elements and procedures are identical for ***all*** DSGE models, however complex:
- Just as in Micro 101:
  - ***define utility***
  - then ***derive FOCs*** to maximise it
- We are almost finished
- But we need to define equations for supply and demand for ***factors of production***
- And (in)***famous Euler Equation [➔ dynamic (ie, intertermporal) equilibrium]***

- →Household j's ***labour supply function*** derived taking ratio of first two FOCs (***intratemporal*** optimisation):
- $$L_{j,t}^{\varphi} = \lambda_{j,t} W_t$$
- and

$$C_{j,t}^{-\sigma} = \lambda_{j,t} P_t$$

- → $$C_{j,t}^{\sigma} L_{j,t}^{\varphi} = \frac{W_t}{P_t}$$ = ***Real Wage***

- Household j's ***Euler Equation*** derived from first and third FOCs:

$$C_{j,t}^{-\sigma} = \lambda_{j,t} P_t$$

$$-\lambda_{j,t} P_t + \beta E_t \lambda_{j,t+1} \left[(1-\delta) E_t P_{t+1} + E_t R_{t+1} \right] = 0$$

- ➔

$$-C_{j,t}^{-\sigma} + \beta E_t \left\{ \left(\frac{C_{j,t+1}^{-\sigma}}{P_{t+1}} \right) \left[(1-\delta) P_{t+1} + E_t R_{t+1} \right] \right\} = 0$$

- Hence

LHS: Relative value of (expected) consumption tomorrow vs (actual) today
RHS: Discounted value of investment made from foregone consumption

Euler: $$\left(\frac{E_t C_{j,t+1}}{C_{j,t}} \right)^{\sigma} = \beta \left[(1-\delta) + E_t \left(\frac{R_{t+1}}{P_{t+1}} \right) \right]$$

Intertemporal optimisation

- ***Production*** sector
- Maximise ***Profit Function***, choosing amounts of each factor input (L_t,K_t):

$$\max_{L_{j,t},K_{j,t}} \Pi_{j,t} = A_t K_{j,t}^{\alpha} L_{j,t}^{1-\alpha} P_{j,t} - W_t L_{j,t} - R_t K_{j,t}$$

- FOCs

From production function, this is just $Y_{j,t}/K_{j,t}$

$$\frac{\partial \Pi_{j,t}}{\partial K_{j,t}} = \alpha \boxed{A_t K_{j,t}^{\alpha-1} L_{j,t}^{1-\alpha}} P_{j,t} - R_t = 0$$

And this is $Y_{j,t}/L_{j,t}$

$$\frac{\partial \Pi_{j,t}}{\partial L_{j,t}} = (1-\alpha) \boxed{A_t K_{j,t}^{\alpha} L_{j,t}^{-\alpha}} P_{j,t} - W_t = 0$$

- ➔ equations for ***demand*** for capital and for labour (***intratemporal*** optimisation):

$$\frac{\partial \Pi_{j,t}}{\partial K_{j,t}} \implies \alpha \boxed{A_t K_{j,t}^{\alpha-1} L_{j,t}^{1-\alpha}} P_{j,t} = R_t \qquad \frac{\partial \Pi_{j,t}}{\partial L_{j,t}} \implies (1-\alpha) \boxed{A_t K_{j,t}^{\alpha} L_{j,t}^{-\alpha}} P_{j,t} = W_t$$

$$\underbrace{\frac{R_t}{P_{j,t}}}_{\text{Real MCK}} = \underbrace{\alpha \boxed{\frac{Y_{j,t}}{K_{j,t}}}}_{\text{MPK}} \qquad \underbrace{\frac{W_t}{P_{j,t}}}_{\text{Real MCL}} = \underbrace{(1-\alpha) \boxed{\frac{Y_{j,t}}{L_{j,t}}}}_{\text{MPL}}$$

- ***Total Cost*** for producing firm is

$$TC_{j,t} = W_t L_{j,t} + R_t K_{j,t}$$

- which with a little algebra can be shown to equal

$$TC_{j,t} = \frac{Y_{j,t}}{A_t}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_t}{\alpha}\right)^{\alpha}$$

- ➔ ***Marginal Cost*** ($\partial TC/\partial Y$)

$$MC_{j,t} = \frac{1}{A_t}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_t}{\alpha}\right)^{\alpha}$$

- ***Markups*** over this Marginal Cost will turn out to be ***very important*** in NK DSGE models

- In RBC, marginal cost assumed ***same for all firms***
  - ➔ $MC_{j,t} \equiv MC_t$
- ***RBC*** economy also assumes ***perfect competition***
- ➔ $P_t = MC_t$ [ie, ***no*** markup in RBC !!]
- ➔ ***general price level***

$$P_t = \frac{1}{A_t}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_t}{\alpha}\right)^{\alpha}$$

- Together with (1) law of motion of capital, (2) equilibrium condition and (3) definition of productivity shock, this completes ***basic RBC model***

- Model becomes:

$C_t^\sigma L_t^\varphi = \frac{W_t}{P_t}$ (Labour supply)

$\left(\frac{E_t C_{t+1}}{C_t}\right)^\sigma = \beta\left[(1-\delta) + E_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right]$ (Euler Equation)

$K_{t+1} = (1-\delta)K_t + I_t$ (Law of motion of capital)

$Y_t = A_t K_t^\alpha L_t^{1-\alpha}$ (Production function)

$K_t = \alpha MC_t \frac{Y_t}{R_t}$ (Demand for capital)

$L_t = (1-\alpha)MC_t \frac{Y_t}{W_t}$ (Demand for labour)

$P_t = \frac{1}{A_t}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_t}{\alpha}\right)^\alpha$ (Price level)

$I_t = Y_t - C_t$ (Equilibrium condition)

$\log A_t = \rho_A \log A_{t-1} + \epsilon_t$ (Productivity shock)

- We may now write down this ***non-linear*** model directly in **Dynare**
- **Dynare's timing convention:**
  - The value of the capital stock used in production at time t is determined by investment at time $t-1$ and therefore predetermined

$$K_t = (1-\delta)K_{t-1} + I_t \tag{1}$$

$$Y_t = K_{t-1}^{\alpha} L_t^{1-\alpha} \tag{2}$$

  - To use the **stock at the beginning of period notation**

$$\boxed{\begin{aligned} K_{t+1} &= (1-\delta)K_t + I_t && (3) \\ Y_t &= K_t^{\alpha} L_t^{1-\alpha} && (4) \end{aligned}}$$

  you can declare a particular variable as predetermined using the `predetermined_variables` declaration

- **➔ we use command "predetermined_variables K;" after "var ...;"**

- **var** C I Y K L R W P A;
- **predetermined_variables** K;

$$\left(\frac{E_t C_{j,t+1}}{C_{j,t}}\right)^{\sigma} = \beta\left[(1-\delta) + E_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right]$$

- % Model equations block

$$C^{\sigma}_{j,t} L^{\varphi}_{j,t} = \frac{W_t}{P_t}$$

- **model;**
- C^sigma*L^phi = W/P;
- (C(+1)/C)^sigma = beta*((1-delta)+R(+1)/P(+1));
- K(+1) = (1-delta)*K+I;

$$A_t K^{\alpha}_{j,t} L^{1-\alpha}_{j,t}$$

- Y = A*(K^alpha)*(L^(1-alpha));
- K = alpha*Y/(R/P);

$$P_t = \frac{1}{A_t}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_t}{\alpha}\right)^{\alpha}$$

- L = (1-alpha)*Y/(W/P);
- P = (1/A)*(W/(1-alpha))^(1-alpha)*(R/alpha)^alpha;
- I = Y - C;
- log(A) = rhoA*log(A(-1)) + e;
- **end;**

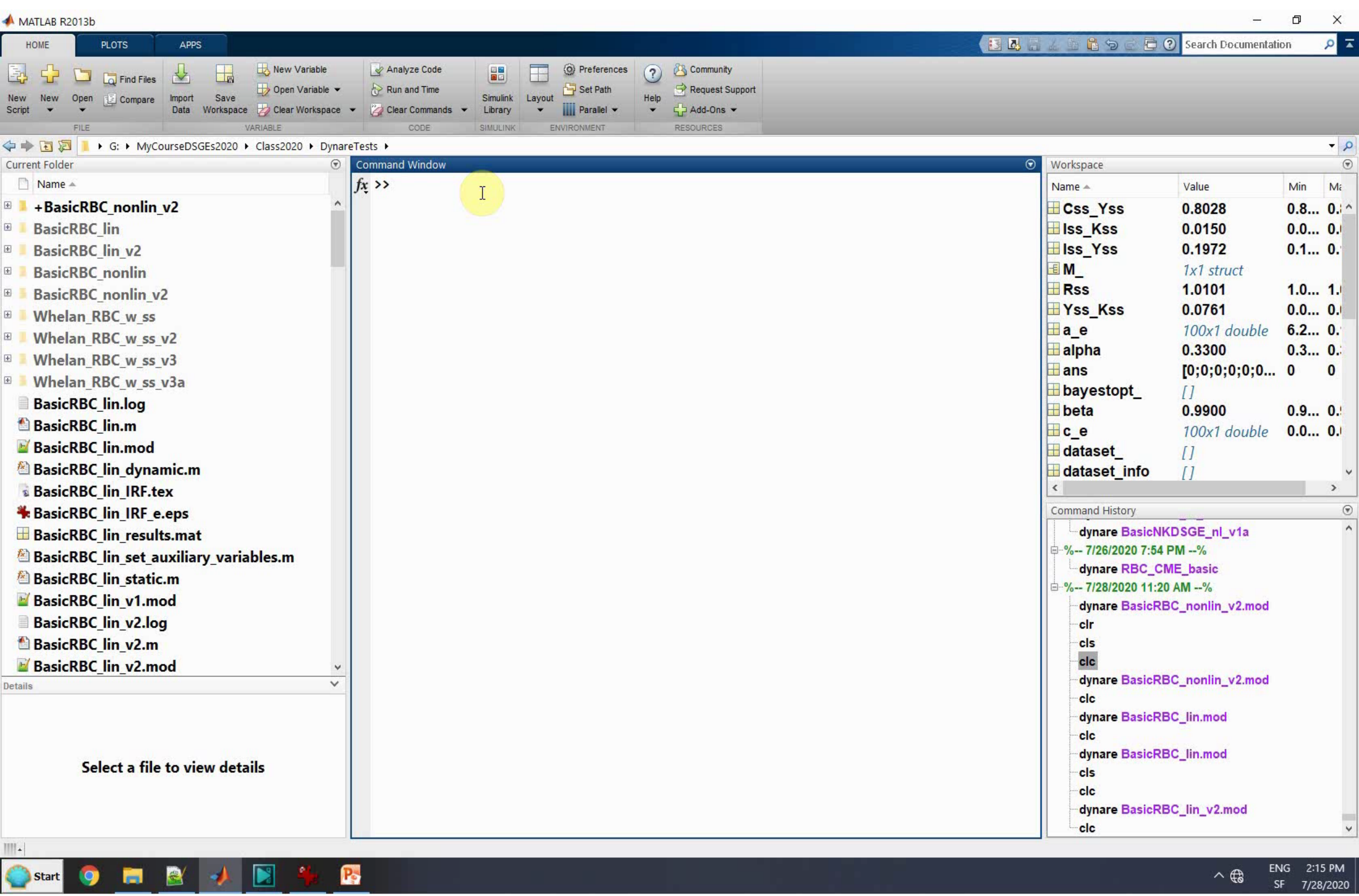
- Simulating this ***nonlinear*** model requires that we introduce ***steady-state*** values [ie, those to which model variables "tend" to return after a perturbation – ie, ***shock***]
- This is required so that **Dynare** can check for fulfilment of ***Blanchard-Kahn*** conditions

Blanchard and Kahn established local conditions for existence and uniqueness of a solution to a DSGE model, in terms of eigenvalues *computed at steady state* of model

- If (as is ***usual*** in large-scale models) no ***analytical*** steady-state solution exists, we may ask **Dynare** to attempt to solve model itself for steady-state values
- But this requires that we provide **Dynare** with ***initial guesses*** at those steady-state values

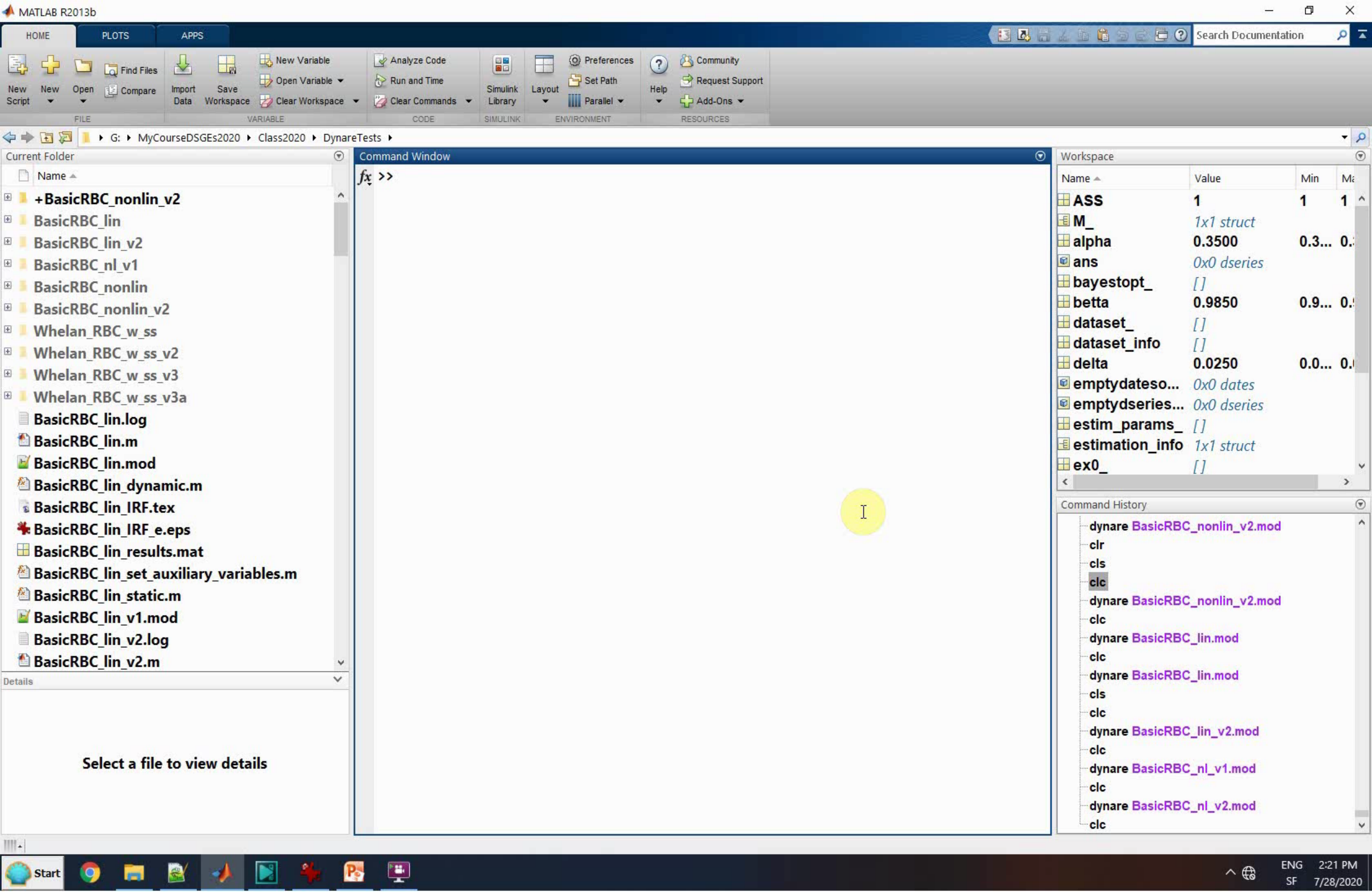
- %*************************************
- % Block 4: Initial values
- %*************************************

- % Initial values - these are used to start Dynare's steady-state solver
- **initval;**
- % could (should?) write a full-scale steady-state function....
- Y = 1; % arbitrary guess
- C = .8; % follows from guess at Y, with mpc = 0.8
- L = 0.7; % arbitrary guess
- K = 3; % follows from Y and guess at capital-output ratio
- I = Y-C; % definitional relationship
- W = (1-alpha)*Y/L; % from FOC
- R = alpha*Y/K + (1 – delta); % from FOC
- A = 1; % ==> log(A)=0 ==> no shock until subsequently imposed
- P = 1; % Normalisation from Walras' Law (k goods ➔ k-1 independent equations in equilibrium ➔ may use *relative* prices ➔ may set aggregate price level to 1)
- e = 0;
- **end;**


MATLAB R2013b
HOME
PLOTS
APPS
Search Documentation
New Script
New
Open
Find Files
Compare
Import Data
Save Workspace
New Variable
Open Variable
Clear Workspace
Analyze Code
Run and Time
Clear Commands
Simulink Library
Layout
Preferences
Set Path
Parallel
Help
Community
Request Support
Add-Ons
FILE
VARIABLE
CODE
SIMULINK
ENVIRONMENT
RESOURCES
G: ▸ MyCourseDSGEs2020 ▸ Class2020 ▸ DynareTests ▸
Current Folder
Name
+BasicRBC_nonlin_v2
BasicRBC_lin
BasicRBC_lin_v2
BasicRBC_nonlin
BasicRBC_nonlin_v2
Whelan_RBC_w_ss
Whelan_RBC_w_ss_v2
Whelan_RBC_w_ss_v3
Whelan_RBC_w_ss_v3a
BasicRBC_lin.log
BasicRBC_lin.m
BasicRBC_lin.mod
BasicRBC_lin_dynamic.m
BasicRBC_lin_IRF.tex
BasicRBC_lin_IRF_e.eps
BasicRBC_lin_results.mat
BasicRBC_lin_set_auxiliary_variables.m
BasicRBC_lin_static.m
BasicRBC_lin_v1.mod
BasicRBC_lin_v2.log
BasicRBC_lin_v2.m
BasicRBC_lin_v2.mod
Details
Select a file to view details
Command Window
fx >>
Workspace
Name
Value
Min
Css_Yss 0.8028
Iss_Kss 0.0150
Iss_Yss 0.1972
M_ 1x1 struct
Rss 1.0101
Yss_Kss 0.0761
a_e 100x1 double
alpha 0.3300
ans [0;0;0;0;0;0...
bayestopt_ []
beta 0.9900
c_e 100x1 double
dataset_ []
dataset_info []
Command History
dynare BasicNKDSGE_nl_v1a
%-- 7/26/2020 7:54 PM --%
dynare RBC_CME_basic
%-- 7/28/2020 11:20 AM --%
dynare BasicRBC_nonlin_v2.mod
clr
cls
clc
dynare BasicRBC_nonlin_v2.mod
clc
dynare BasicRBC_lin.mod
clc
dynare BasicRBC_lin.mod
cls
clc
dynare BasicRBC_lin_v2.mod
clc
Start
ENG
SF
2:15 PM
7/28/2020

- Ugh!
- But if we change the initial values to Y=2.3, C = 1.8 and K = 20 …

[J:\MyCourseDSGEs2025\Tests\BasicRBC2025_nl_v3.mod]


MATLAB R2013b
HOME
PLOTS
APPS
Search Documentation
New Script
New
Open
Find Files
Compare
Import Data
Save Workspace
New Variable
Open Variable
Clear Workspace
Analyze Code
Run and Time
Clear Commands
Simulink Library
Layout
Preferences
Set Path
Parallel
Help
Community
Request Support
Add-Ons
FILE
VARIABLE
CODE
SIMULINK
ENVIRONMENT
RESOURCES
G: ▸ MyCourseDSGEs2020 ▸ Class2020 ▸ DynareTests ▸
Current Folder
Name
+BasicRBC_nonlin_v2
BasicRBC_lin
BasicRBC_lin_v2
BasicRBC_nl_v1
BasicRBC_nonlin
BasicRBC_nonlin_v2
Whelan_RBC_w_ss
Whelan_RBC_w_ss_v2
Whelan_RBC_w_ss_v3
Whelan_RBC_w_ss_v3a
BasicRBC_lin.log
BasicRBC_lin.m
BasicRBC_lin.mod
BasicRBC_lin_dynamic.m
BasicRBC_lin_IRF.tex
BasicRBC_lin_IRF_e.eps
BasicRBC_lin_results.mat
BasicRBC_lin_set_auxiliary_variables.m
BasicRBC_lin_static.m
BasicRBC_lin_v1.mod
BasicRBC_lin_v2.log
BasicRBC_lin_v2.m
Details
Select a file to view details
Command Window
fx >>
Workspace
Name Value Min Max
ASS 1 1 1
M_ 1x1 struct
alpha 0.3500 0.3... 0.3
ans 0x0 dseries
bayestopt_ []
betta 0.9850 0.9... 0.9
dataset_ []
dataset_info []
delta 0.0250 0.0... 0.0
emptydateso... 0x0 dates
emptydseries... 0x0 dseries
estim_params_ []
estimation_info 1x1 struct
ex0_ []
Command History
dynare BasicRBC_nonlin_v2.mod
clr
cls
clc
dynare BasicRBC_nonlin_v2.mod
clc
dynare BasicRBC_lin.mod
clc
dynare BasicRBC_lin.mod
cls
clc
dynare BasicRBC_lin_v2.mod
clc
dynare BasicRBC_nl_v1.mod
clc
dynare BasicRBC_nl_v2.mod
clc
Start
ENG
SF
2:21 PM
7/28/2020

- ***Non-linear*** **model simulation** ➔

STEADY-STATE RESULTS:

Y	2.33763
C	1.82917
I	0.508452
K	20.3381
L	0.729242

➔
C/Y = 0.78
I/Y = 0.22
K/Y = 8.7

So in this case, Dynare was able to find Steady-State on its own with no input from us

.
.
.

EIGENVALUES:

Modulus	Real	Imaginary
0.95	0.95	0
0.9614	0.9614	0
1.056	1.056	0
1.737e+09	-1.737e+09	0
5.935e+18	5.935e+18	0

There are 3 eigenvalue(s) larger than 1 in modulus for 3 forward-looking variable(s)
The rank condition is verified.

- ***Non-linear*** model simulation ➔ following IRFs

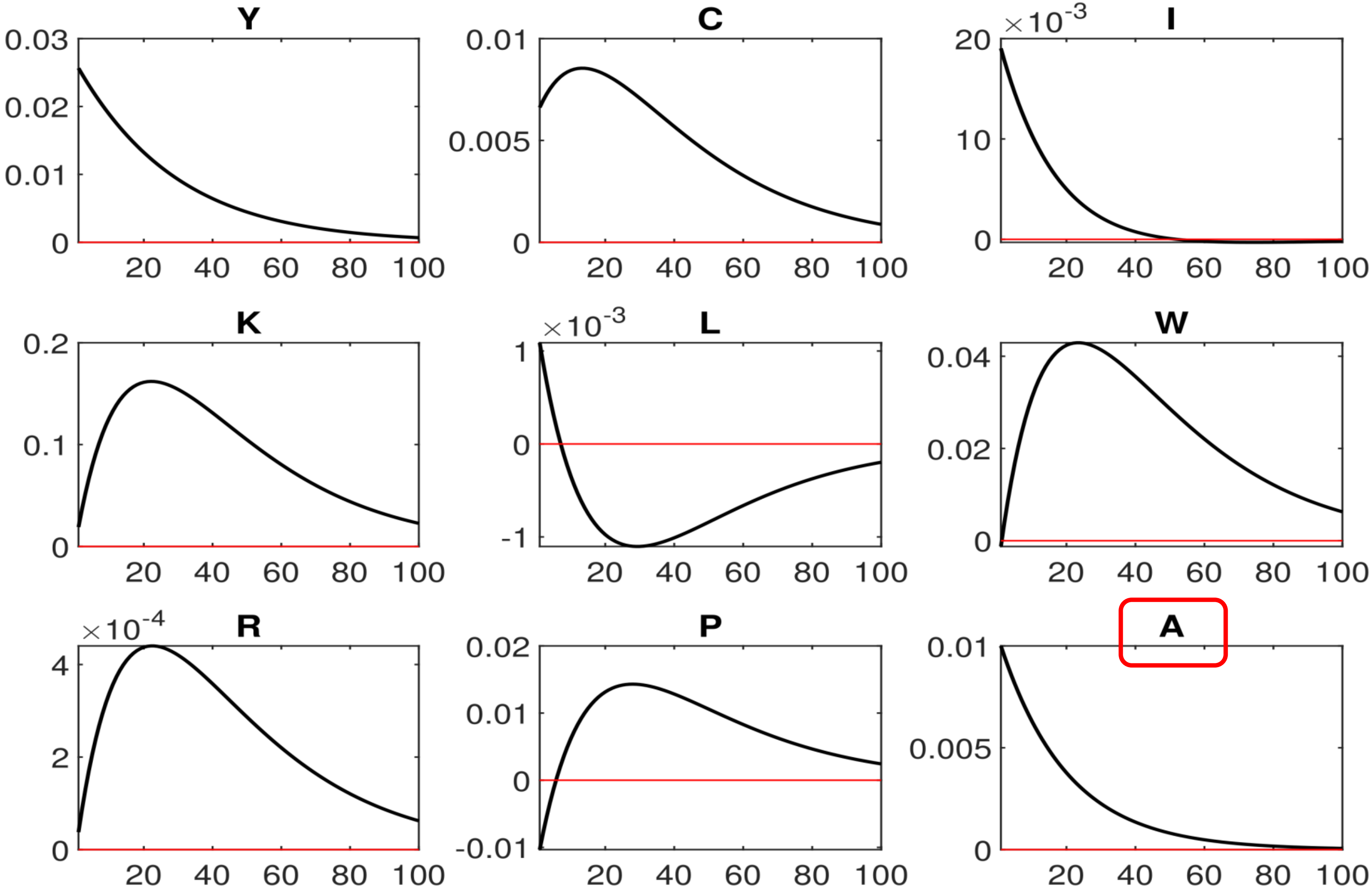


Figure 1: Impulse response functions (orthogonalized shock to e).

Business Cycle Facts

Some business cycle facts (for US 1947Q1 – 2018Q3)

	Volatility		Comovement	Persistence
	Standard deviation	Relative standard deviation	Contemporaneous correlation with output	First-order autocorrelation
Output	1.61	1	1	0.85
Consumption	1.28	0.80	0.89	0.86
Investment	7.42	4.61	0.89	0.80
Output per hour	1.08	0.67	0.44	0.72
Hours per worker	0.53	0.33	0.73	0.82
Real wages	0.90	0.56	0.17	0.74
Employment	1.38	0.86	0.80	0.92
Unemployment	13.24	8.22	-0.82	0.89

Note: All variables are in logarithms and have been detrended using a Hodrick-Prescott filter. Output, Consumption and Investment are in per capita terms.

Some business cycle facts (for US 1947Q1 – 2018Q3)

Table 2: Behavior of components of output in recessions

Components	ave. share in GDP	ave. share in fall in GDP
Consumption		
– durables	8%	16%
– nondurables	26%	11%
– services	30%	9%
Investment		
– residential	5%	21%
– fixed business	11%	12%
– inventories	0.7%	41%
Net Exports	–0.4%	–12%
Government Purchases	21%	3%

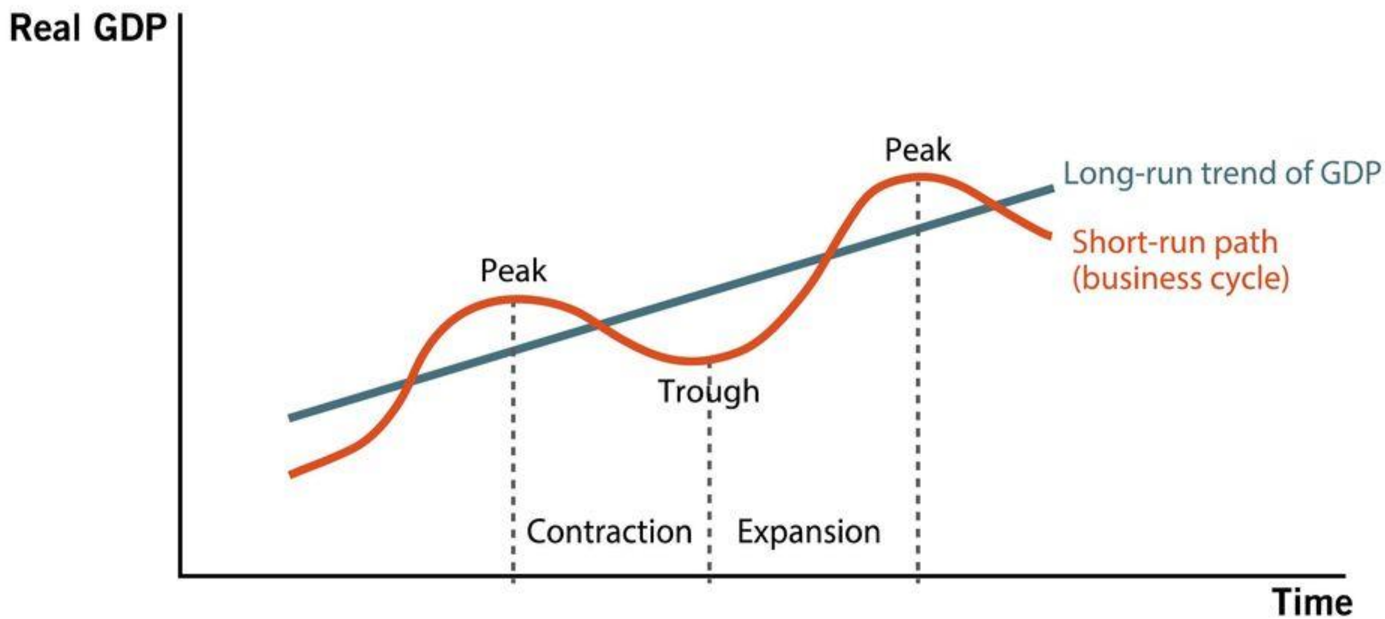
Trade actually *improved* GDP during **recessions** as imports fell more sharply than exports

Great ratios:

- $Y = C + I + G + X - M$
- $0.6 < C/Y < 0.8$
- $0.05 < I/Y < 0.25$
- $0.1 < G/Y < 0.5$
- $-0.1 < (X - M)/Y < 0.1$
- $3 < K/Y < 6$
- Macro-economy will seldom go outside these observed ***empirical bounds***

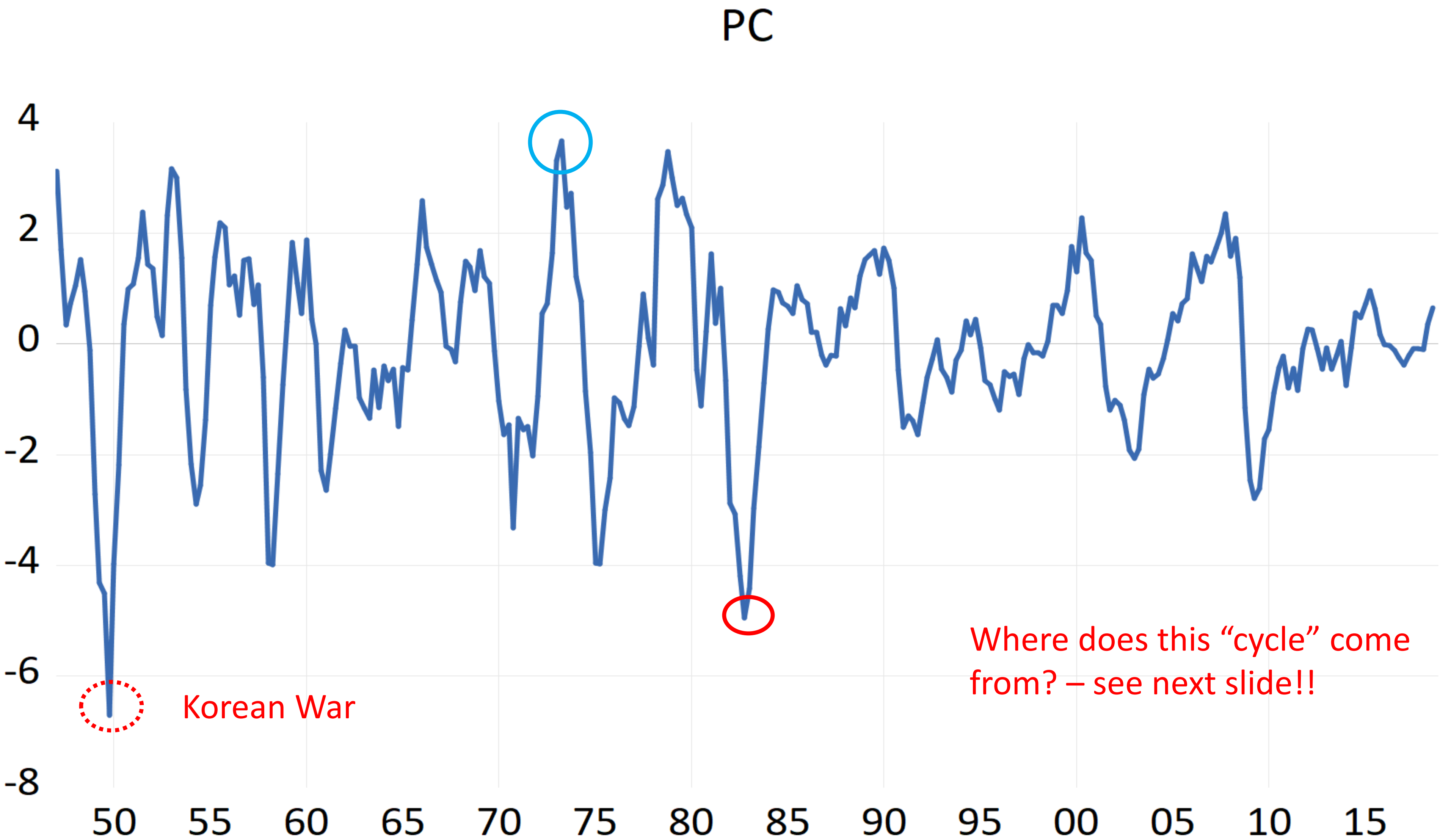
- “Long-run trend of GDP” = Balanced Growth Path

The Business Cycle

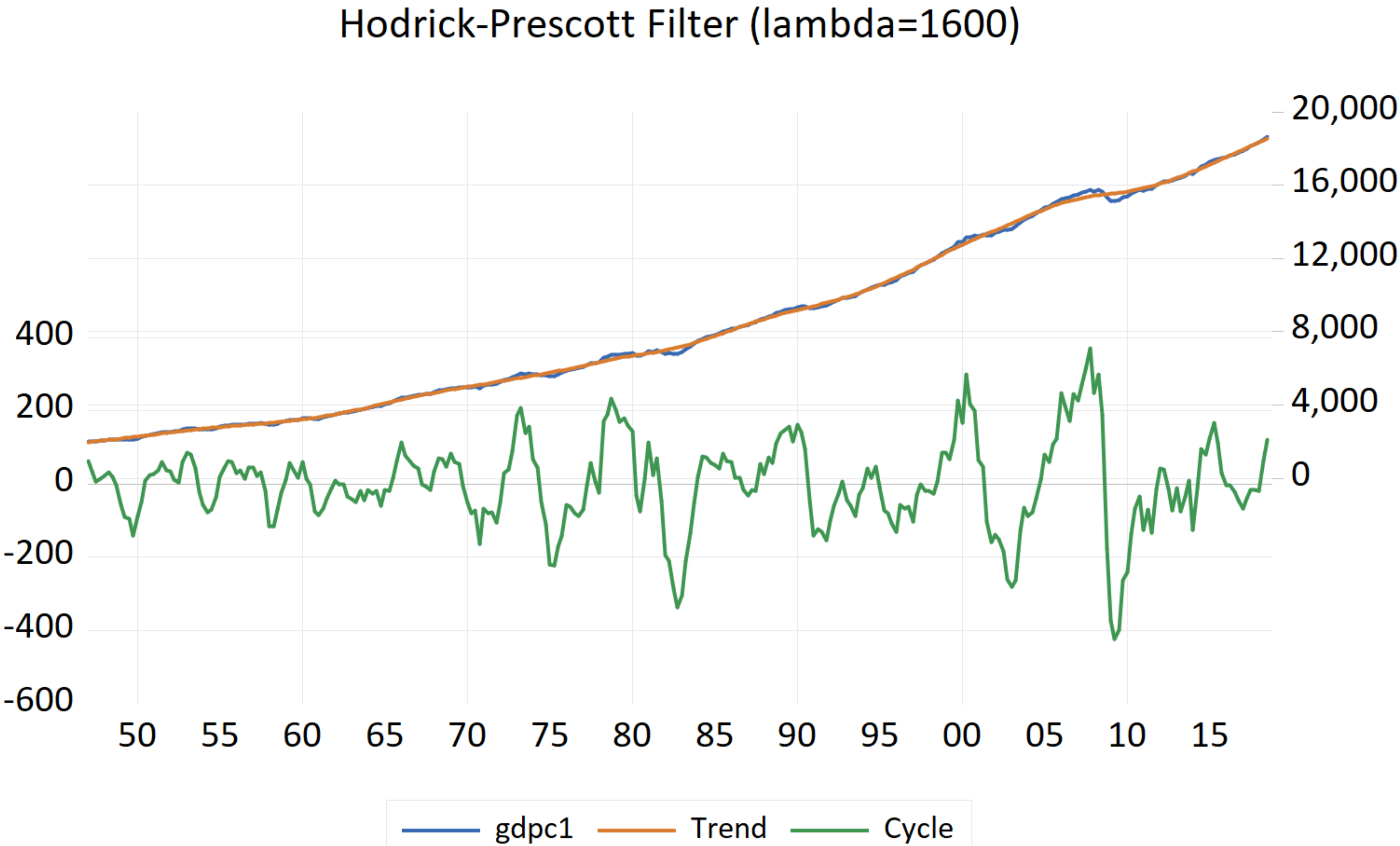


- Typically, fluctuations around long-run growth path will be ***relatively small***
  - recessions and booms do not usually amount to more than 1 - 5% changes vis-à-vis that path [Covid was an exception!!]
  - Next slides demonstrate this:
  - output is up to ***4% above trend*** in cyclical ***peaks***
  - And up to ***5% below trend*** in cyclical ***troughs*** [exception: Korean War]

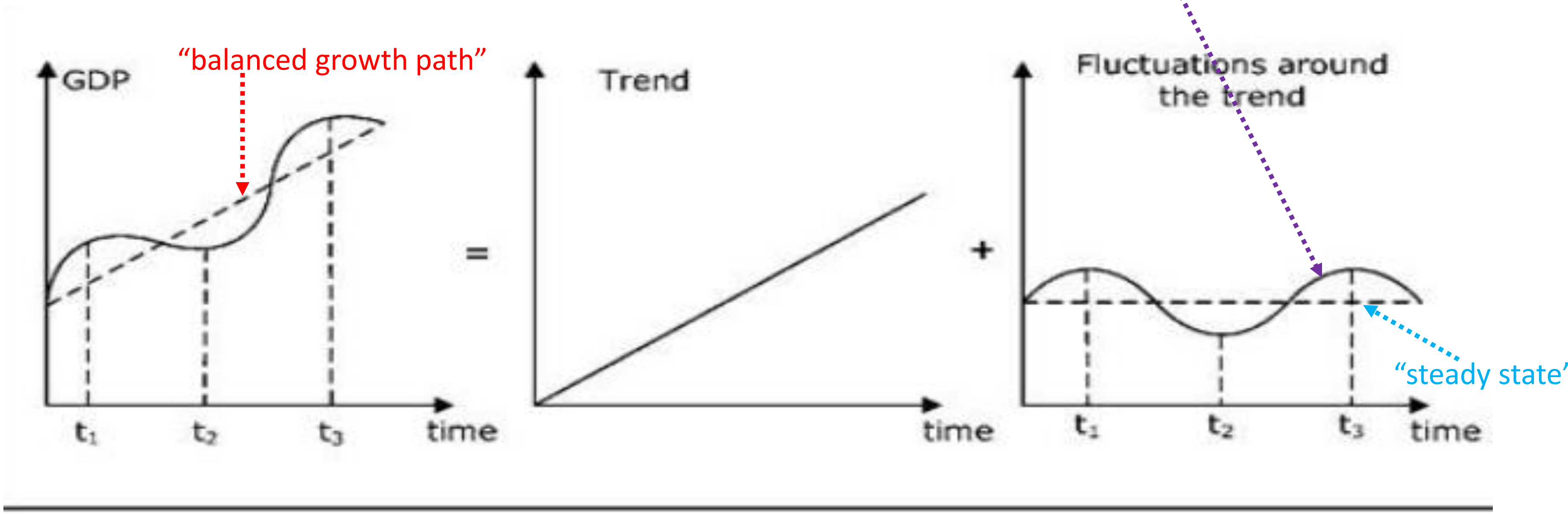
This shows ***percentage*** by which ***cyclical*** part of US real GDP per capita varies from ***actual level*** over period 1947Q1 – 2018Q3: ***max***=+3.7; ***min***=-6.7



This shows ***real GDP per capita*** for the US over the period 1947Q1 – 2018Q3 (blue), the HP filter trend (brown) and the difference between the two (the "cycle") in green



- Suppose we model these ***deviations from trend*** ➔ model now represents ***business-cycle component*** of ***real*** economy (hence "RBC")



- Figure says just "GDP", but to ensure ***stationarity*** of variables, typically we use (logs of) ***real per capita*** variables
- Note how balanced growth path becomes steady state

Solution Secret 2: Log-Linearisation

- In earlier simple model, Dynare was able to solve (once we gave it "good" initial values)
- But in many cases, ***nonlinear*** model is ***too complicated*** for Dynare to solve on its own
- ➔ simplify by ***approximating*** it
- Doing so requires use of ***Taylor series approximation*** you remember from secondary-school algebra (or Bootcamp)
- This Taylor series approximation may be taken to ***first order***, ***second order***, ***third order*** and so on, depending on how close one needs approximation to be; ***simplest*** is ***linear***

- ***Recall (Taylor Series)***: ***Any*** nonlinear function $F(x_t, y_t)$ can be ***approximated around any point*** (x_t^*, y_t^*) – *eg, steady state* - using formula:

$$
\begin{aligned}
F(x_t, y_t) = & F(x_t^*, y_t^*) + F_x(x_t^*, y_t^*)(x_t - x_t^*) + F_y(x_t^*, y_t^*)(y_t - y_t^*) \\
& + F_{xx}(x_t^*, y_t^*)(x_t - x_t^*)^2 + F_{xy}(x_t^*, y_t^*)(x_t - x_t^*)(y_t - y_t^*) \\
& + F_{yy}(x_t^*, y_t^*)(y_t - y_t^*)^2 + \ldots
\end{aligned}
$$

- If ***gap*** between (x_t, y_t) and (x_t^*, y_t^*) is ***small***, second and higher powers and cross-terms can be ***ignored*** ➔ ***linear*** approximation (all * known)

$$
F(x_t, y_t) \approx F(x_t^*, y_t^*) + F_x(x_t^*, y_t^*)(x_t - x_t^*) + F_y(x_t^*, y_t^*)(y_t - y_t^*)
$$

$$F\left(x_t, y_t\right) \approx F\left(x_t^*, y_t^*\right) + F_x\left(x_t^*, y_t^*\right)\left(x_t - x_t^*\right) + F_y\left(x_t^*, y_t^*\right)\left(y_t - y_t^*\right)$$

- or $F(x_t, y_t) \approx \alpha + \beta_1 (x_t - x_t^*) + \beta_2 (y_t - y_t^*)$
- But x_t^* and y_t^* are often ***known*** *[eg, steady state]* so they can also be subsumed into parameters
- ➔ $F(x_t, y_t) \approx \delta + \gamma_1 x_t + \gamma_2 y_t$
- Using this, ***transform*** nonlinear model into simple set of ***linear*** equations

- Need a few ***tricks*** to do log-linearisation
- ***Lower-case letters*** define log-deviations of variables X_t from their steady-state values X*

$$x_t \equiv log X_t - log X^*$$

- ***First trick:*** any variable can be written as

$$X_t = X^* \frac{X_t}{X^*} = X^* e^{x_t}$$

Since $x_t = \log(X_t/X^*)$

- ***Second trick:*** apply ***first-order Taylor*** approximation of e^{x_t}

$$e^{x_t} \approx (1 + x_t)$$

- ➔ $$X_t \approx X^* (1 + x_t)$$

NB: Lower Case x

GDP GDP$_{SS}$ GDP growth rate

- ***Third trick:*** ignore ***cross-product terms***

$$X_t Y_t \approx X^* Y^* (1 + x_t)(1 + y_t) \approx X^* Y^* (1 + x_t + y_t)$$

- Substitute these approximations ➔ lots of terms ***cancel out***
- ➔linear system in deviations of logged variables from steady-state values
- ***That's it!***

- Slides above demonstrated that (log-) ***linear approximation*** around steady-state is in most cases reasonably ***valid***
- ➔ set of linear equations in ***deviations of logs*** of model variables from steady-state values
- ***And*** log-differences ≈ percentage deviations:

steady-state value

$$x_t = \log(X_t) - \log(\boxed{X}) = \log(X_t/X) = \log(1 + \%\text{ change }) \simeq \%\text{ change}$$

- ➔system expresses variables in terms of ***percentage deviations from steady-state paths*** – i.e. ***growth rates***

NB: % deviations in DSGE are measured in **decimals** !!

If steady-state values are themselves growing, then obviously need to de-trend these also! We will see how to do this when looking at estimation data

- ***Example***: Production function (***Cobb-Douglas***):

$$Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$$

recall, $X_t = X^* e^{x_t}$

- ➔ in terms of steady-states and log-deviations

$$Y^* e^{y_t} = (A^* e^{a_t})\,(K^*)^{\alpha}\, e^{\alpha k_t}\,(L^*)^{1-\alpha}\, e^{(1-\alpha) l_t}$$

- Steady-state values also obey identities ➔

$$Y^* = (A^*)\,(K^*)^{\alpha}\,(L^*)^{1-\alpha}$$

- Cancelling terms ➔

$$e^{y_t} = e^{a_t} e^{\alpha k_t} e^{(1-\alpha) l_t}$$

- ➔(through Taylor series approximation) $e^{x_t} \approx (1 + x_t)$

$$(1 + y_t) = (1 + a_t)\,(1 + \alpha k_t)\,(1 + (1-\alpha) l_t)$$

- Ignoring cross-products of log-deviations (since ***linear*** approximation) ➔

$$y_t = a_t + \alpha k_t + (1 - \alpha) l_t$$

- ***Tedious***, but very simple ***five-step*** matter:
  1. rewrite equation in terms of steady-states and log-deviations
  2. use fact that steady-state values obey identities
  3. cancel terms
  4. apply first-order Taylor series approximations
  5. ignore cross-products of log-deviations
- Let’s complete log-linearising of this model:

- Household's ***labour supply*** function

$$C_{j,t}^{\sigma} L_{j,t}^{\varphi} = \frac{W_t}{P_t}$$

- First divide by steady-states (C*, L*, W*, P*)
- But steady-states also have to obey equations! ➔ $C^{*\sigma} L^{*\varphi} = W^*/P^*$
- Then take ***logs*** and drop subscripts:
- σ ***log***(C/C*) + φ ***log***(L/L*) = ***log***(W/W*) – ***log***(P/P*)
- ➔ $\sigma\, c + \varphi\, \ell = w - p$

  (where ***lower-case = log deviations from steady-state***, recall)

- ***Labour demand*** function

$$\underbrace{\frac{W_t}{P_{j,t}}}_{\text{Real MCL}} = \underbrace{(1-\alpha)\frac{Y_{j,t}}{L_{j,t}}}_{\text{MPL}}$$

- First divide by steady-states (Y*, L*, W*, P*)
- But steady-states also have to obey equations! ➔ W*/P* = (1 – α) Y*/L*
- Then take ***logs*** [note that (1 – α) cancels]:
- ***log***(W/W*) - ***log***(P/P*) = ***log***(Y/Y*) – ***log***(L/L*)
- ➔ w - p = y - ℓ or ℓ = y – (w – p)

- ***Capital demand*** function

$$\underbrace{\frac{R_t}{P_{j,t}}}_{\text{Real MCK}} = \underbrace{\alpha\frac{Y_{j,t}}{K_{j,t}}}_{\text{MPK}}$$

- First divide by steady-states (Y*, K*, R*, P*)
- R*/P* = α Y*/K*
- Then take ***logs*** [note that α cancels]:
- ***log***(R/R*) - ***log***(P/P*) = ***log***(Y/Y*) – ***log***(K/K*)
- ➔ r - p = y - k or k = y – (r – p)

- General ***price level***

$$P_t = \frac{1}{A_t}\left(\frac{W_t}{1-\alpha}\right)^{1-\alpha}\left(\frac{R_t}{\alpha}\right)^{\alpha}$$

- First divide by steady-states (P*, W*, R*, A*)
- $P^* = (1/A^*)(W^*/(1-\alpha))^{1-\alpha}(R^*/\alpha)^{\alpha}$
- Then take ***logs***:
- ***log***(P/P*) = (1 - α) ***log***(W/W*) + α ***log***(R/R*) – ***log***(A/A*)
- ➔ p = (1 - α) w + α r - a

- Finally, Consumption ***Euler Equation***

$$\left(\frac{E_t C_{j,t+1}}{C_{j,t}}\right)^{\sigma} = \beta\left[(1-\delta) + E_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right]$$

- $\rightarrow$ via $\boxed{X_t = X^* \frac{X_t}{X^*} = X^* e^{x_t}}$

$$\frac{1}{\beta}\left(\frac{C_{ss}^{\sigma}}{C_{ss}^{\sigma}}\right) e^{(\sigma E_t c_{t+1} - \sigma c_t)} = (1-\delta) + \frac{R_{ss}}{P_{ss}} e^{[E_t(r_{t+1} - p_{t+1})]}$$

$\boxed{e^{x_t} \approx (1 + x_t)}$

- $\rightarrow$

$$\boxed{\frac{1}{\beta}[1}+\sigma\left(E_t c_{t+1} - c_t\right)] = \boxed{(1-\delta) + \frac{R_{ss}}{P_{ss}}[1} + E_t\left(r_{t+1} - p_{t+1}\right)]$$

- $\rightarrow$

$$\frac{\sigma}{\beta}\left(E_t c_{t+1} - c_t\right) = \frac{R_{ss}}{P_{ss}} E_t\left(r_{t+1} - p_{t+1}\right)$$

In SS, $C_{j,t} = C_{j,t+1}$ so Euler $\rightarrow$ this $=1/\beta$ $\rightarrow$ cancel !

- Also, ***Equilibrium condition*** $Y_t = C_t + I_t$ ➔

$$Y_{ss}e^{y_t} = C_{ss}e^{c_t} + I_{ss}e^{i_t}$$

- ➔

$$Y_{ss}(1 + y_t) = C_{ss}(1 + c_t) + I_{ss}(1 + i_t)$$

- But $Y_{ss} = C_{ss} + I_{ss}$ ➔

$$Y_{ss}y_t = C_{ss}c_t + I_{ss}i_t$$

- or

$$y_t = \frac{C^*}{Y^*}c_t + \frac{I^*}{Y^*}i_t$$

- ➔ **Dynare** simulation model equations
- (J:\MyCourseDSGEs2025\Powerpoints\Wk1\BasicRBC2025_lin_4PPWk1_v1.mod)

- ***Summarising our log-linearised model:***

$$\sigma c_t + \varphi l_t = w_t - p_t \qquad \text{(Labour supply)}$$

$$\frac{\sigma}{\beta}\left(c_{t+1} - c_t\right) = \frac{R_{ss}}{P_{ss}}\left(r_{t+1} - p_{t+1}\right) \qquad \text{(Euler equation)}$$

$$k_{t+1} = (1-\delta)\, k_t + \delta i_t \qquad \text{(Law of motion of capital)}$$

$$y_t = a_t + \alpha k_t + (1-\alpha)\, l_t \qquad \text{(Production function)}$$

$$k_t = y_t - (r_t - p_t) \qquad \text{(Demand for capital)}$$

$$l_t = y_t - (w_t - p_t) \qquad \text{(Demand for labour)}$$

$$p_t = (1-\alpha)\, w_t + \alpha r_t - a_t \qquad \text{(Price level)}$$

$$y_t = \frac{C_{ss}}{Y_{ss}} c_t + \frac{I_{ss}}{Y_{ss}} i_t \qquad \text{(Equilibrium condition)}$$

$$a_t = \rho_A a_{t-1} + e_t \qquad \text{(Productivity shock)}$$

- ***Dynare*** **simulation model equations :**

(J:\MyCourseDSGEs2025\Powerpoints\Wk1\BasicRBC2025_lin.mod)

```
sigma*c + phi*l = w - p; % 1 Labour Supply
(sigma/beta)*(c(+1)-c)=(Rss/Pss)*(r(+1)-p(+1)); % 2 Euler equation
k(+1) = (1-delta)*k + delta*i; % 3 Law of motion of capital
y = a + alpha*k + (1-alpha)*l; % 4 Production function
k = y - (r - p); % 5 Demand for capital
l = y - (w - p); % 6 Labour demand
p = (1-alpha)*w + alpha*r - a; % 7 Price Level
Yss*y = Css*c + Iss*i; % 8 Equilibrium condition
a = rhoa*a(-1) + e; % 9 Productivity shock
```

$$\sigma c_t + \varphi l_t = w_t - p_t$$

$$\frac{\sigma}{\beta}\left(c_{t+1} - c_t\right) = \frac{R_{ss}}{P_{ss}}\left(r_{t+1} - p_{t+1}\right)$$

$$k_{t+1} = (1-\delta)\, k_t + \delta i_t$$

$$y_t = a_t + \alpha k_t + (1-\alpha)\, l_t$$

$$k_t = y_t - (r_t - p_t)$$

$$l_t = y_t - (w_t - p_t)$$

$$p_t = (1-\alpha)\, w_t + \alpha r_t - a_t$$

$$y_t = \frac{C_{ss}}{Y_{ss}} c_t + \frac{I_{ss}}{Y_{ss}} i_t$$

$$a_t = \rho_A a_{t-1} + e_t$$

Simulating Model's Behaviour

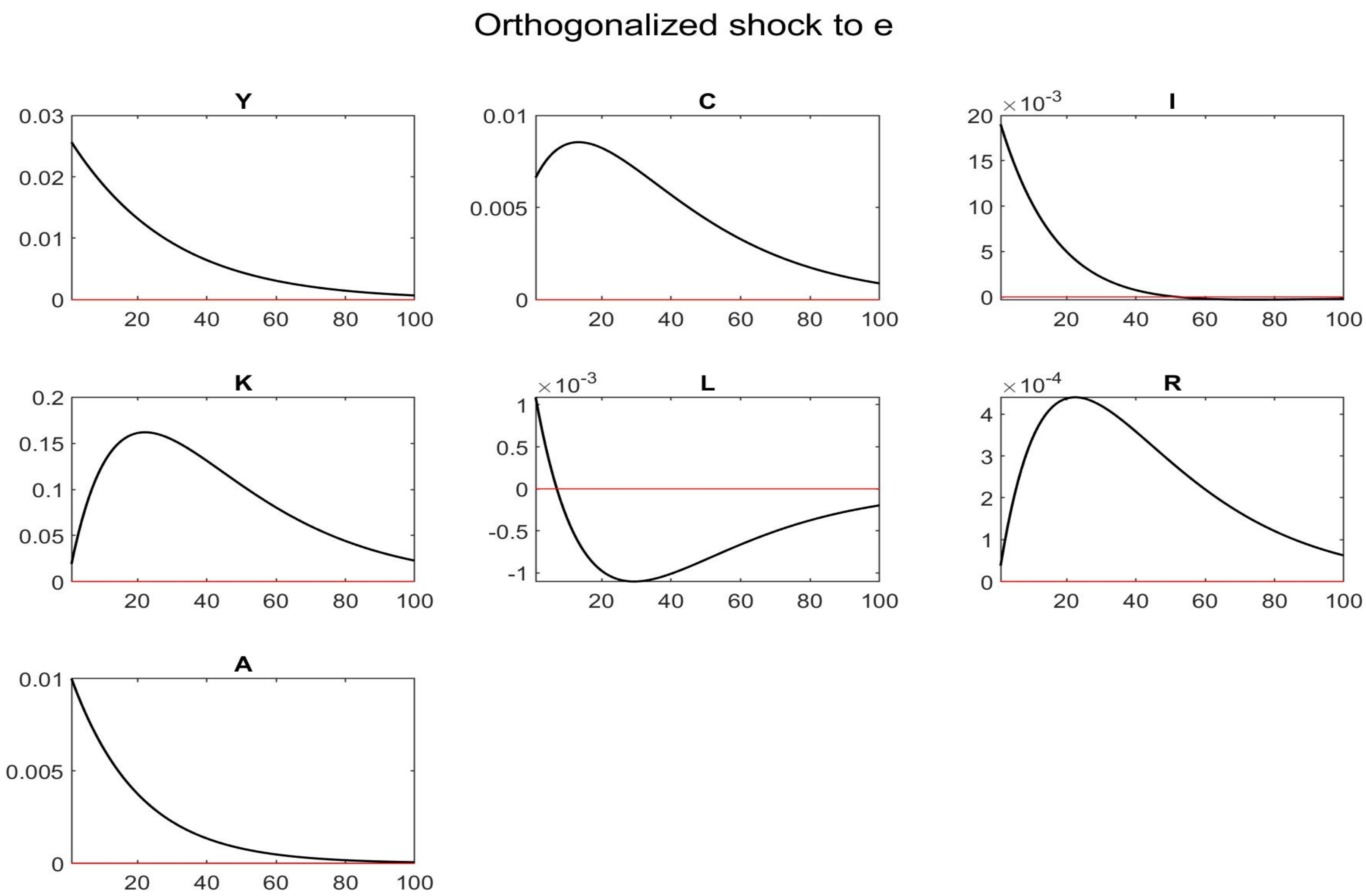
- Using same "calibrated" parameters as for non-linear version of model (assume ***quarterly***):

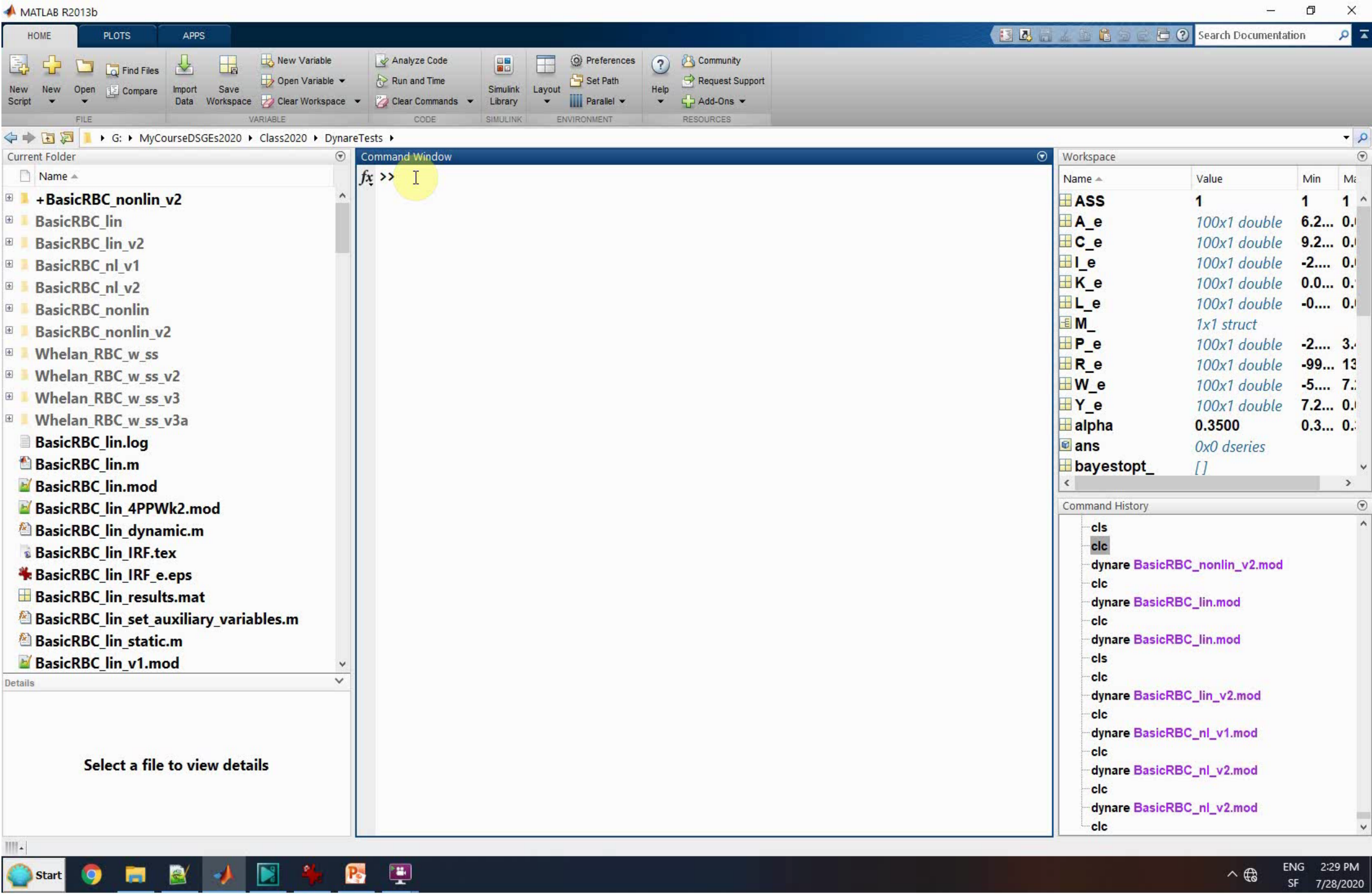
Table 5: Parameter Values

Parameter	Value	Description
α	0.350	capital share
β	0.985	discount factor
δ	0.025	depreciation rate
σ	2.000	relative risk aversion
ϕ	1.500	Elasticity of substitution
ρ	0.950	persistence of TFP shock
$\frac{Yss}{Kss}$	0.115	SS output-capital ratio
$\frac{Iss}{Kss}$	0.025	SS investment-capital ratio
$\frac{Iss}{Yss}$	0.218	SS investment-output ratio
$\frac{Css}{Yss}$	0.782	SS consumption-output ratio
Rss	1.015	SS return on capital

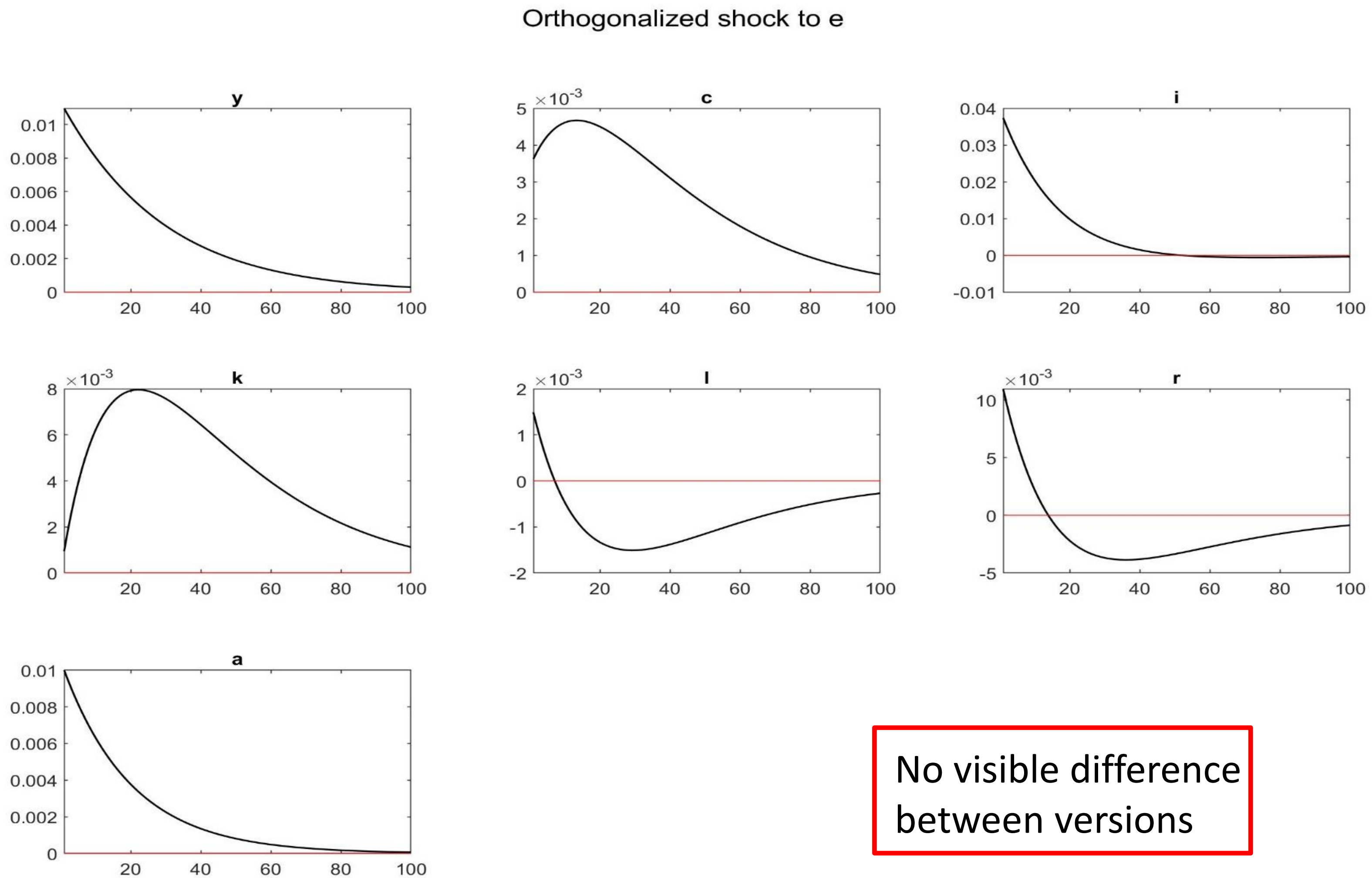
➔ annual rate 6%

- ***Non-linear*** model simulation ➔ following IRFs



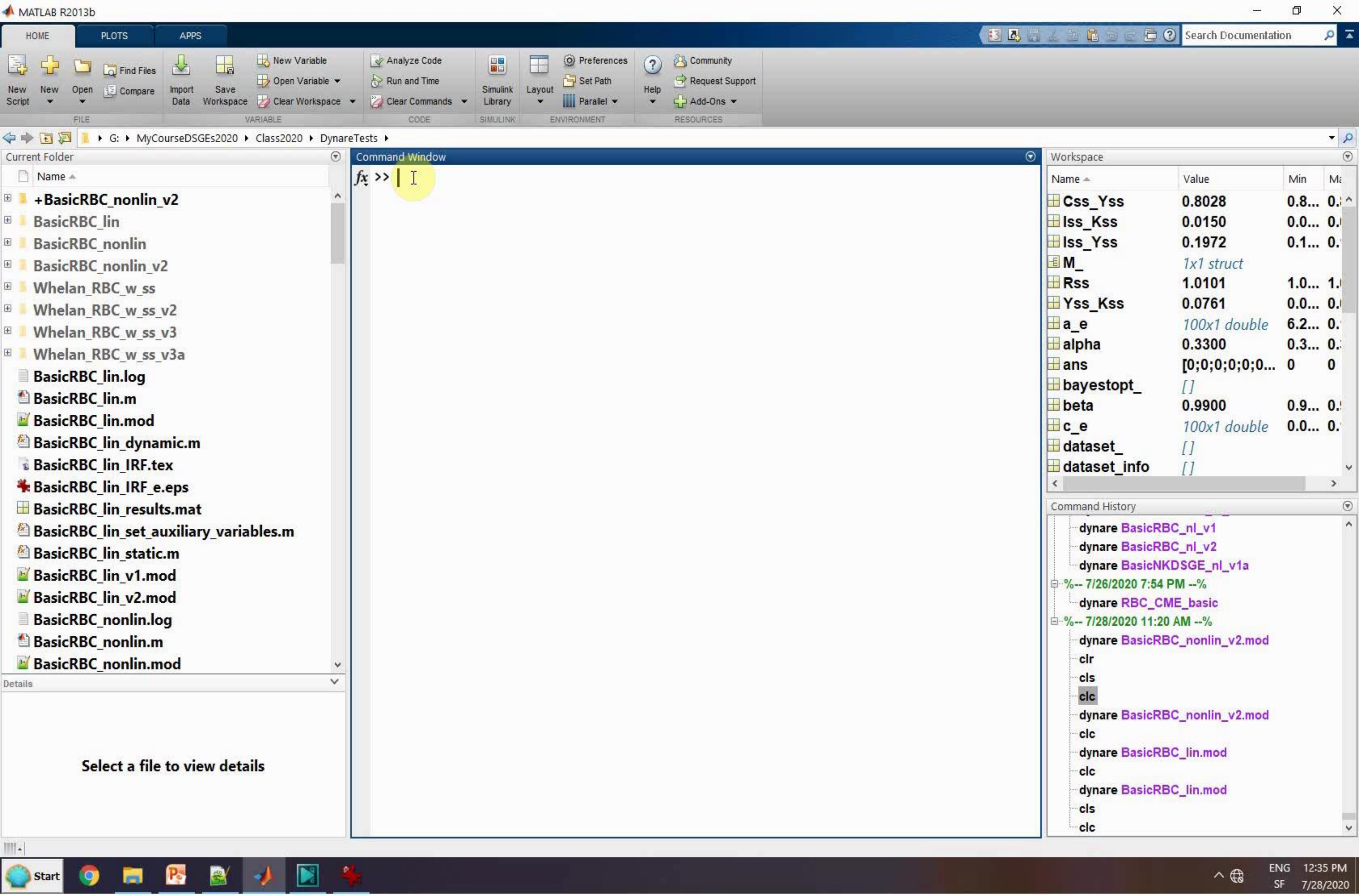

MATLAB R2013b
HOME
PLOTS
APPS
Search Documentation
New Script
New
Open
Find Files
Compare
FILE
Import Data
Save Workspace
New Variable
Open Variable
Clear Workspace
VARIABLE
Analyze Code
Run and Time
Clear Commands
CODE
Simulink Library
SIMULINK
Layout
Preferences
Set Path
Parallel
ENVIRONMENT
Help
Community
Request Support
Add-Ons
RESOURCES
G: ▸ MyCourseDSGEs2020 ▸ Class2020 ▸ DynareTests ▸
Current Folder
Name
+BasicRBC_nonlin_v2
BasicRBC_lin
BasicRBC_lin_v2
BasicRBC_nl_v1
BasicRBC_nl_v2
BasicRBC_nonlin
BasicRBC_nonlin_v2
Whelan_RBC_w_ss
Whelan_RBC_w_ss_v2
Whelan_RBC_w_ss_v3
Whelan_RBC_w_ss_v3a
BasicRBC_lin.log
BasicRBC_lin.m
BasicRBC_lin.mod
BasicRBC_lin_4PPWk2.mod
BasicRBC_lin_dynamic.m
BasicRBC_lin_IRF.tex
BasicRBC_lin_IRF_e.eps
BasicRBC_lin_results.mat
BasicRBC_lin_set_auxiliary_variables.m
BasicRBC_lin_static.m
BasicRBC_lin_v1.mod
Details
Select a file to view details
Command Window
fx >>
Workspace
Name Value Min
ASS 1 1 1
A_e 100x1 double 6.2...
C_e 100x1 double 9.2...
I_e 100x1 double -2....
K_e 100x1 double 0.0...
L_e 100x1 double -0....
M_ 1x1 struct
P_e 100x1 double -2....
R_e 100x1 double -99...
W_e 100x1 double -5....
Y_e 100x1 double 7.2...
alpha 0.3500 0.3...
ans 0x0 dseries
bayestopt_ []
Command History
cls
clc
dynare BasicRBC_nonlin_v2.mod
clc
dynare BasicRBC_lin.mod
clc
dynare BasicRBC_lin.mod
cls
clc
dynare BasicRBC_lin_v2.mod
clc
dynare BasicRBC_nl_v1.mod
clc
dynare BasicRBC_nl_v2.mod
clc
dynare BasicRBC_nl_v2.mod
clc
Start
ENG SF
2:29 PM 7/28/2020

- ***Linear*** model simulation ➔ following IRFs

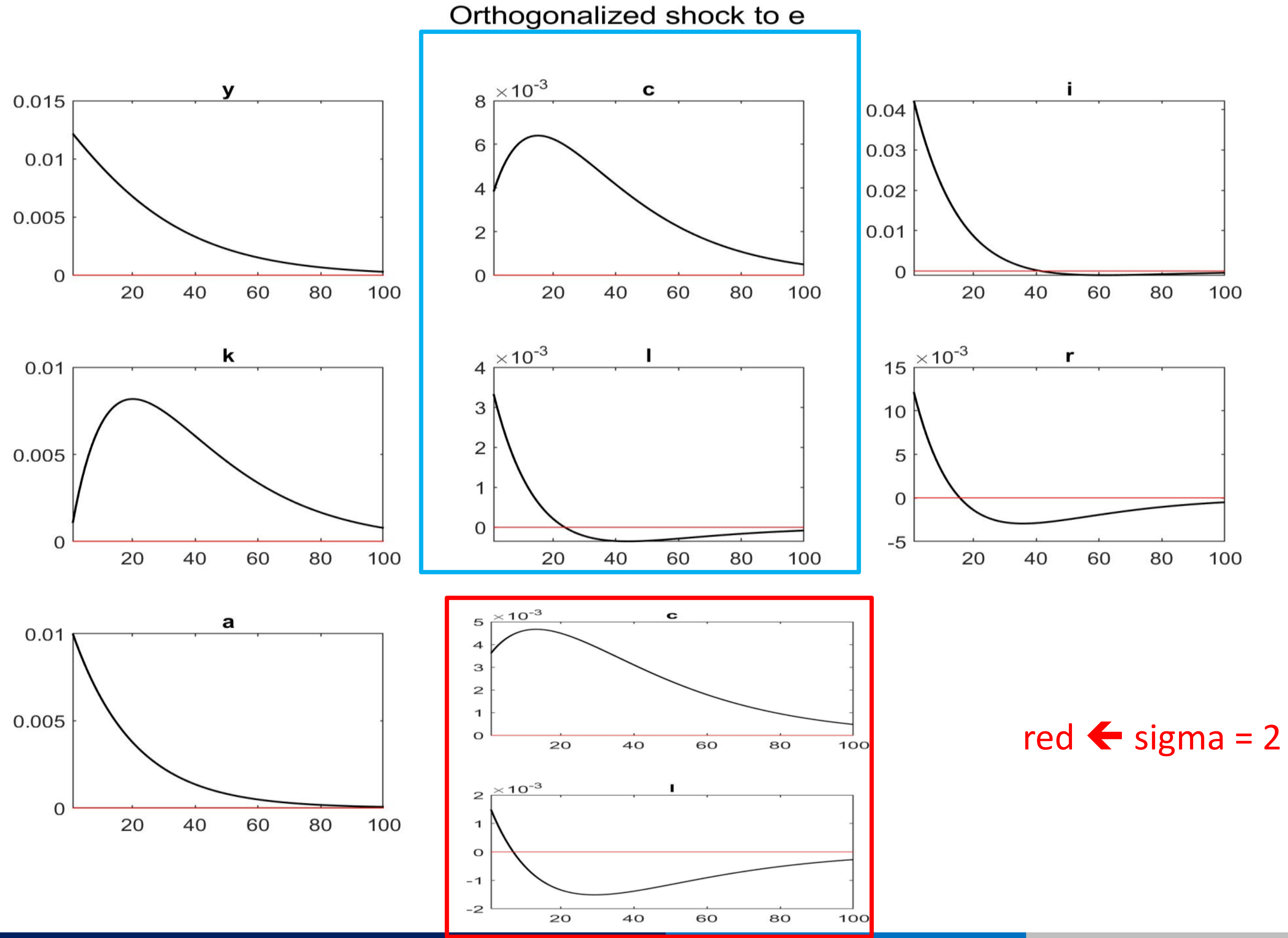


No visible difference between versions

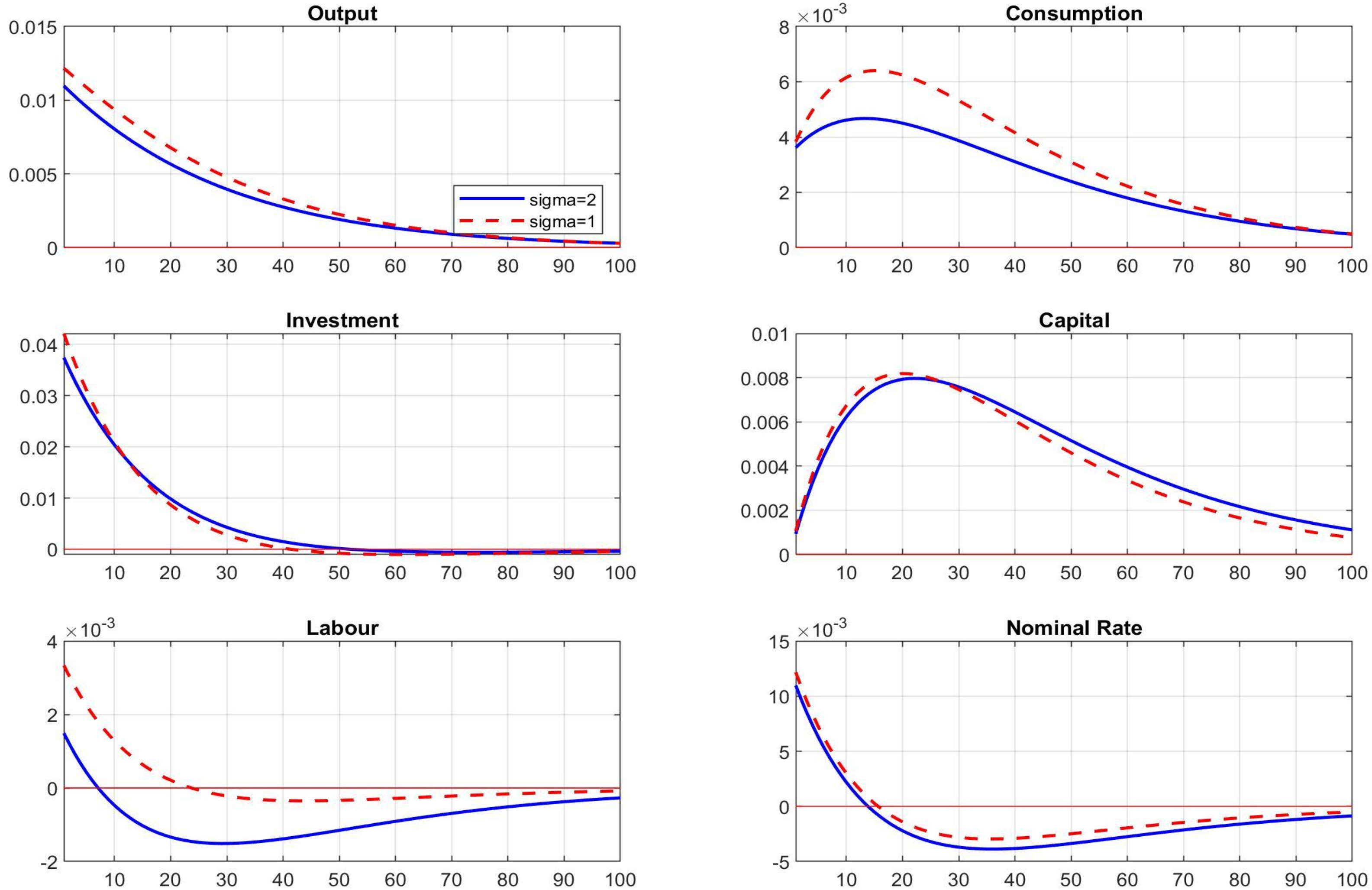
- What happens if we use $\sigma = 1$? [J:\MyCourseDSGEs2025\Tests\BasicRBC2025_lin_v2.mod]
- recall this ➔ U = log(C) - ... :


MATLAB R2013b
HOME
PLOTS
APPS
Search Documentation
New Script
New
Open
Find Files
Compare
FILE
Import Data
Save Workspace
New Variable
Open Variable
Clear Workspace
VARIABLE
Analyze Code
Run and Time
Clear Commands
CODE
Simulink Library
SIMULINK
Layout
Preferences
Set Path
Parallel
ENVIRONMENT
Help
Community
Request Support
Add-Ons
RESOURCES
G: ▸ MyCourseDSGEs2020 ▸ Class2020 ▸ DynareTests ▸
Current Folder
Name
+BasicRBC_nonlin_v2
BasicRBC_lin
BasicRBC_nonlin
BasicRBC_nonlin_v2
Whelan_RBC_w_ss
Whelan_RBC_w_ss_v2
Whelan_RBC_w_ss_v3
Whelan_RBC_w_ss_v3a
BasicRBC_lin.log
BasicRBC_lin.m
BasicRBC_lin.mod
BasicRBC_lin_dynamic.m
BasicRBC_lin_IRF.tex
BasicRBC_lin_IRF_e.eps
BasicRBC_lin_results.mat
BasicRBC_lin_set_auxiliary_variables.m
BasicRBC_lin_static.m
BasicRBC_lin_v1.mod
BasicRBC_lin_v2.mod
BasicRBC_nonlin.log
BasicRBC_nonlin.m
BasicRBC_nonlin.mod
Details
Select a file to view details
Command Window
fx >>
Workspace
Name | Value | Min | Ma
Css_Yss 0.8028 0.8... 0.
Iss_Kss 0.0150 0.0... 0.
Iss_Yss 0.1972 0.1... 0.
M_ 1x1 struct
Rss 1.0101 1.0... 1.
Yss_Kss 0.0761 0.0... 0.
a_e 100x1 double 6.2... 0.
alpha 0.3300 0.3... 0.
ans [0;0;0;0;0;0... 0 0
bayestopt_ []
beta 0.9900 0.9... 0.
c_e 100x1 double 0.0... 0.
dataset_ []
dataset_info []
Command History
dynare BasicRBC_nl_v1
dynare BasicRBC_nl_v2
dynare BasicNKDSGE_nl_v1a
%-- 7/26/2020 7:54 PM --%
dynare RBC_CME_basic
%-- 7/28/2020 11:20 AM --%
dynare BasicRBC_nonlin_v2.mod
clr
cls
clc
dynare BasicRBC_nonlin_v2.mod
clc
dynare BasicRBC_lin.mod
clc
dynare BasicRBC_lin.mod
cls
clc
Start
ENG
SF
12:35 PM
7/28/2020

- Response of c and ℓ *(l)* are different, as expected



- Comparing responses on one graph:



J:\MyCourseDSGEs2025\Powerpoints\Wk1\BasicRBC_Walsh.m

Appendix: Calculating Steady State

- The equilibrium condition has coefficients that are values relating to ***steady-state path:***

$$y_t = \frac{C^*}{Y^*} c_t + \frac{I^*}{Y^*} i_t$$

- Here C*/Y* and I*/Y* “calibrated”
- Where did these values come from?
- Here, they were determined from so-called “great ratios”
- To obtain those, need to do some calculations:

- Take original non-linearised system and figure out what things look like along a ***zero growth path*** (ie, "steady" state)
- Start with steady-state interest rate
- ➔ in steady-state $C_t = C_{t+1} = C^*$ and $R_t = R_{t+1} = R^*$ and we arbitrarily define MC*[= P* in RBC] = 1
- ➔ from Euler,

$$\left(\frac{E_t C_{j,t+1}}{C_{j,t}}\right)^{\sigma} = \beta\left[(1-\delta) + E_t\left(\frac{R_{t+1}}{P_{t+1}}\right)\right]$$

- ➔ $1 = \beta[R^* + 1 - \delta]$ or $R^* = 1/\beta - 1 + \delta$
- ➔ in steady state economy, ***gross*** rate of return on capital R is determined uniquely by rate of time preference β and depreciation rate δ

- But we know from capital demand that

$$K_t = \alpha MC_t \frac{Y_t}{R_t}$$

- Steady state ➔
- $R^* = 1/\beta - 1 + \delta = \alpha Y^*/K^*$ since assume $MC^* = P^* = 1$

- ➔ $$\frac{Y^*}{K^*} = \frac{\beta^{-1} + \delta - 1}{\alpha}$$

- So we have one steady-state ratio (= inverse of familiar ***capital-output ratio***)

- Next, equation for law of motion of capital

$$K_{t+1} = I_t + (1-\delta)K_t$$

- In steady-state $K_t = K_{t+1} = K^*$ ➔

$$\frac{I^*}{K^*} = \delta$$

- Combine with previous steady-state ratio ➔

$$\frac{I^*}{Y^*} = \frac{\frac{I^*}{K^*}}{\frac{Y^*}{K^*}} = \frac{\alpha\delta}{\beta^{-1}+\delta-1}$$

- Y = C + I ➔

$$\frac{C^*}{Y^*} = 1 - \frac{\alpha\delta}{\beta^{-1}+\delta-1}$$

- ➔ define values of steady-state ratios in terms of ***"deep" parameters*** of model:

```
sigma = 2;
phi = 1.5;
alpha = 0.35;
beta = 0.985;
delta = 0.025;
rhoa = 0.95;
Rss = 1/beta;
Yss_Kss = ((1/beta)+delta-1)/alpha;
Iss_Kss = delta;
Iss_Yss = (Iss_Kss)/(Yss_Kss);
Css_Yss = 1 - (Iss_Yss);
```

This is what we calibrated these parameters to be. Look familiar?

$\frac{Yss}{Kss}$	0.115	SS output-capital ratio
$\frac{Iss}{Kss}$	0.025	SS investment-capital ratio
$\frac{Iss}{Yss}$	0.218	SS investment-output ratio
$\frac{Css}{Yss}$	0.782	SS consumption-output ratio
Rss	1.015	SS return on capital

Calculated as:

1.015

0.115

0.025

0.218

0.782

Appendix: Some History

- Workhorse models of macroeconomics until the 1980s: ***simultaneous equation models*** – pioneered by Tinbergen in the 1930s
- Set of behavioural equations describing ***aggregate variables***
  - interdependent
  - reciprocal causality (*simultaneous* or *lagged*)
- see examples below from Tinbergen (1939), Klein-Goldberger (1955), Brookings (1965) and Cuddy (1968)

- Tinbergen, J. (1939). Statistical Testing of Business Cycle Theories: Part II: Business Cycles in the United States of America, 1919-1932. Books (Jan Tinbergen). Agaton Press, New York.

The following variables would then, by *a priori* reasoning, seem to be of importance for the explanation of the rest of consumption fluctuations:

Wages and salaries $(L_w + L_s)$;
Urban non-workers' income E;
Capital gains G;
The rate of increase in farm prices $p^f - p^f_{-1}$, or Δp^f, as an indication of speculative profits, which are not included in E but may nevertheless have influenced consumption (agricultural prices have been selected as they are especially subject to speculative influences);
Some measure of the degree of inequality of income distribution, for which PARETO's α has been taken;[4]
Cost of living p;
A trend, standing for slow changes in habits, population growth and changes in population structure.

- The consumption function found by Tinbergen was

$$U' - E'_F = 0.95\,(L_w + L_s) + (0.77 \pm 0.32)\,E + (0.28 \pm 0.13)\,G \\ + (0.05 \pm 0.02)\Delta p^f + (0.03 \pm 0.09)p + 0.37t, \qquad (2.1)$$

where the left-hand member represents urban consumption outlay.

Klein, Lawrence R.; Goldberger, Arthur S. (1955). An Econometric Model for the United States, 1929–1952. Amsterdam: North-Holland

Using this revised glossary and rearranging several of the identities, we may rewrite the Klein-Goldberger model as follows:

$$(2.0.1)\; C = -22.26 + 0.55(W_1 + W_2 - T_W) + 0.41(P - T_C - T_N - S_C) + 0.34\,(R_1 + R_2 - T_R) + 0.26\,C_{-1} + 0.072(L_1)_{-1} + 0.26\,N_P$$

$$(2.0.2)\quad I = -16.71 + 0.78(P - T_C - T_N + R_1 + R_2 - T_R + D)_{-1} - 0.073\,K_{-1} + 0.14(L_2)_{-1}$$

W_1 = Private wages; W_2 = Government wages; P = Nonwage nonfarm income
R_1 = Farm income; R_2 = Farm subsidies; L_1 = Household liquid assets; N_p = Population
L_2 = Business liquid assets; D = Depreciation; T_i = Taxes less transfers, sector i

Duesenberry, J., Fromm, G., Klein, L., Kuh, E. (eds.), The Brookings Quarterly Econometric Model of the United States, North-Holland, 1965

Consumer demand

$$C_{\mathrm{DEA}}^{54} = -\underset{(11.97)}{22.93} + \underset{(0.0372)}{0.2415} Y_{\mathrm{D}}^{54} - \underset{(8.16)}{10.62}\left[\frac{P_{\mathrm{CDEA}}}{P_{\mathrm{C}}}\right] - \underset{(0.0170)}{0.07405}\left[K_{\mathrm{CDEA}}^{54}\right]_{-1}.$$

$$K_{\mathrm{CDEA}}^{54} = \left[K_{\mathrm{CDEA}}^{54}\right]_{-1} + C_{\mathrm{DEA}}^{54} - \tfrac{1}{48}\left\{\sum_{i=-48}^{-1}\left[C_{\mathrm{DEA}}^{54}\right]_{i}\right\}.$$

THE OPTIMISATION OF MACROECONOMIC POLICY:
A STUDY OF CANADA

A Dissertation Presented
by
John David Arnold Cuddy
of
King's College
in Partial Fulfilment of the Requirements
for the Degree of
Doctor of Philosophy
Faculty of Economics and Politics
University of Cambridge
1968

Avoiding non-stationarity

"This problem is avoided in the analysis presented here by the use of the dlog (relative first difference) transformation, rather than absolute levels, for the variables of interest. This transformation also eliminates almost all problems of multicollinearity, and thus sharpens the parameter estimates quite substantially."

Note how this model anticipates use of dlogs in DSGE models ...

$$(IV.2.6) \quad \dot{c}_t = .728\ \dot{y}_{dw_t} + .141\ \dot{y}_{dn_t} + .200\ \dot{y}_{dn_{t-1}}$$

$$- .0738\ \dot{od}_{t-1} - .361\ \Delta\tilde{u}_t$$

od = consumer outstanding debt
Δu = change in unemployment rate

$\dot{}$ = dlog

$$(IV.3.16) \quad \dot{i}_{nr_t} = 4.39\ \Delta\dot{k}_t + 378.\ \Delta cu_t + .625(\dot{cf-t_c})_{t-1}$$

$$+ .557\ (\dot{cf-t_c})_{t-2} + .305\ \Delta\dot{p}_{i_t}$$

Δcu = change in capacity utilisation
cf = corporate cash flow
k = fixed capital stock plus inventories

$$(IV.4.18) \quad \Delta h_t = .083 - .263\ s_t + .144\ s_{t-1} - .508\ h_{t-1}$$

$$- .048\ r_{1_t}$$

h = non-farm inventories
s = final sales
r_l = 10yr interest rate

- "Real Business Cycle" (RBC) model - Kydland and Prescott 1982 - is ***original DSGE*** model
- In ***RBC model***
  - SEM equations replaced by ***first-order conditions for solving*** intertemporal problems faced by households and firms
  - SEM ***ad hoc assumptions*** on formation of expectations (usually "adaptive") replaced by "**rational expectations**"
- ***RE***: agents' expectations
  - may be wrong, but are correct ***on average*** over time
  - not ***systematically biased***
  - collectively use ***all relevant information***

- ***Economic agents***
- ***SEMs***: individuals – whether consumers or firms – do not exist
- Only ***aggregates*** are used (eg, consumer non-durable expenditures, total non-residential construction)
- ***RBC:*** **∃** individual agents – but these are ***heterogeneous*** by nature
- Modelling of each economic agent's individual choices is clearly ***impossible***

- ***Solution***: ***group*** economic agents into larger ***categories*** with similar consumption characteristics: called ***"representative agent"***
  - households [possibly of several different types]
  - firms [also possibly of several different types]
  - government [monetary and fiscal authorities separately]
  - financial sector + housing sector
  - foreign sector
- Significant simplification of reality
- But permits ***interlinking*** each type of agent to see how they interact
- Also allows for detailed analysis of macroeconomic policy choices

- ***Lifespan:*** temporal reference that agents use to make their decisions
- In RBC/DSGE models, ***assume*** agents have ***infinite*** time horizons
- Although ***individual*** consumer has ***finite*** lifespan, ”family” or “household” RA contains members who are periodically born and die
- Hence lifespan of ***“representative household”*** can be considered infinite
- Ditto ***firms*** and ***governmen***t: when deciding budgets ***do not consider*** that they will go bankrupt/lose election in near future

Appendix: DSGE Matrix Algebra

- This section is based on Fanelli's "Estimation of Quasi-Rational DSGE Monetary Models"
- Let $Z_t \equiv (Z_{1,t}, Z_{2,t}, \cdots, Z_{n,t})'$ be a $(n \times 1)$ vector of ***endogenous*** variables
- Linearised NK macroeconomic system of equations can then be expressed as

$$\Gamma_0 Z_t = \Gamma_f \mathcal{E}_t Z_{t+1} + \Gamma_b Z_{t-1} + c + v_t \quad (1)$$

- where Γ_i, $i = 0,f,b$ are $(n \times n)$ matrices of ***structural parameters*** γ^s [an $(m_s \times 1)$ vector]
- $c(\gamma^s)$ is a $(n \times 1)$ constant
- v_t is a $n \times 1$ vector of structural shocks

- Repeating $\Gamma_0 Z_t = \Gamma_f \mathcal{E}_t Z_{t+1} + \Gamma_b Z_{t-1} + c + v_t$ (1)
- If $\Gamma_f = \varnothing$, this is just traditional simultaneous system of equations (***SEM***)
- whereas with $\Gamma_b = \varnothing$, one gets a 'purely forward-looking' specification
- Let $X_t := (X_{1,t}, X_{2,t}, \cdots, X_{p,t})'$ be a $(p \times 1)$ vector of ***observable*** variables, $p \leq n$
- ***Measurement*** equations

$$X_t \equiv GZ_t \quad (2)$$

- link ***observable*** X_t to ***endogenous*** Z_t variables, where G is $(p \times n)$

- A typical ***completion*** of system (1) is obtained through autoregressive ***AR(1)*** specification

$$v_t = \Theta v_{t-1} + u_t \quad (3)$$

- where Θ is a $(p \times p)$ - possibly diagonal - stable matrix (i.e. with eigenvalues inside the unit disk)
- u_t is a ***white noise*** with covariance matrix Σ_u
- ***Assumption*** that structural shocks are ***autocorrelated*** is ***common*** in literature but is ***not*** generally derived from first-principles
- Further assumption of an AR(1) specification for shocks is clearly ***restrictive***

- ***Unique solution*** of system (1)-(3), ***if it exists***, can be cast in form

$$X_t = \Phi_1 X_{t-1} + \mu + \Psi v_t \qquad (4)$$

- where (p × p) matrices $\Phi_1(\gamma^s)$ and $\Psi(\gamma^s)$ and (p × 1) constant μ (γ^s) are highly ***nonlinear functions*** of γ^s
- and fulfill a number of ***cross-equation restrictions*** (CER)
- Once these CER are deduced, structural parameters γ^s can be estimated by ***maximising*** one of various approximations of ***likelihood function*** of system (1) – (3)

- Time series structure of v_t is related to dynamic structure of DSGE model: when (3) – AR(1) - is substituted into (4) ➔ ***VAR(2)***:

$$X_t = (\Phi_1 + \Psi\,\Theta\,\Psi^{-1})X_{t-1} - \Psi\,\Theta\,\Psi^{-1}X_{t-2} + \mu + \Psi v_t$$

- Similarly, if v_t is arbitrarily specified as AR(2), implied reduced form equilibrium of DSGE reads as a constrained ***VAR(3)***, and so forth
- ➔ if VAR which best fits data is a ***VAR(k)***

$$X_t = \Omega_1\,X_{t-1} + \cdots + \Omega_k\,X_{t-k} + \mu + \varepsilon_t$$

- then clearly (4) ➔ ***restricting*** $\Omega_i = 0\ \forall\ i > 2$

- ➔ best-fitting model for data can not be regarded as reduced form solution of DSGE model (1)-(3), ***unless*** time series structure of v_t is properly adapted ("***missing dynamics***")
- ***DSGE-VAR framework*** is a method to deal with this ***empirical*** problem

Appendix: Covid-19 and DSGEs

- ***SEIR Model*** is basic model used to investigate propagation of viruses – eg ***Covid-19***
- The model consists of ***five groups ("states")***:
  - S = number of susceptible
  - E = number of exposed
  - I = number of infectious
  - R = number of recovered (or immune)
  - D = number of deceased
- SEIR is a ***DSGE***:
- ***Dynamic***: S(t), E(t), I(t), R(t) and D(t)
- ***Stochastic***: time spent in particular group = f(exponential distribution)
- ***GE***: circular feedback for entire population

- The spread of the epidemic is described by the following ***dynamic*** system of equations:

β_0 = transmission rate

$\mathcal{X}_t$ = avg contacts p.p./t

λ_j j in $\{\mathcal{E},\mathcal{I}\}$ = transition rate out of state $\mathcal{E}$ or $\mathcal{I}$

γ = death rate

Susceptible
$$\mathcal{S}_{t+1} = \mathcal{S}_t \boxed{- \beta_0 \mathcal{S}_t \frac{\mathcal{I}_t}{\mathcal{N}_t} \mathcal{X}_t}$$

Exposed
$$\mathcal{E}_{t+1} = \mathcal{E}_t \boxed{- \lambda_{\mathcal{E}} \mathcal{E}_t} + \beta_0 \mathcal{S}_t \frac{\mathcal{I}_t}{\mathcal{N}_t} \mathcal{X}_t$$

Infected
$$\mathcal{I}_{t+1} = \mathcal{I}_t \boxed{- \lambda_{\mathcal{I}} \mathcal{I}_t} + \lambda_{\mathcal{E}} \mathcal{E}_t$$

Recovered
$$\mathcal{R}_{t+1} = \mathcal{R}_t + (1 - \gamma) \lambda_{\mathcal{I}} \mathcal{I}_t$$

Dead
$$\mathcal{D}_{t+1} = \mathcal{D}_t + \gamma \lambda_{\mathcal{I}} \mathcal{I}_t$$

Total Population
$$\mathcal{N}_{t+1} = \mathcal{S}_{t+1} + \mathcal{E}_{t+1} + \mathcal{I}_{t+1} + \mathcal{R}_{t+1}$$

- Parameter β_0 is ‘basic’ transmission rate, i.e. probability of disease transmission in a contact between a susceptible and an infectious person
- Define aggregate exposure, $\mathcal{X}_t$ = average number of contacts per person in period t
- ➔ $\beta_t = \beta_0 \mathcal{X}_t$ is ‘effective’ transmission rate: depends on contacts $\mathcal{X}_t$ in period t and ‘basic’ transmission rate
- Once agent enters Exposed state, she moves through the $\mathcal{E},\mathcal{I}$ states, as defined by transmission rates λ_j
- Until she exits as either Recovered or Dead
- It is assumed that the ***time*** each single subject remains in any of the ***epidemic states*** is a random variable with ***exponential distribution***

- The dynamics of the infectious class ($\mathcal{I}$) depends on the so-called ***basic reproduction number*** $\mathcal{R}_0$
- $\mathcal{R}_0$ = ***expected number of new infections*** from a ***single*** infection in a population where all subjects are susceptible
- In this model, $\mathcal{R}_0$ can be derived as

$$\mathcal{R}_0 = \beta_0\mu/[\gamma(\gamma + \lambda_I)]$$

where μ is the population ***birth rate***

- It turns out that $\mathcal{R}_0$ has ***threshold properties***:
- If $\mathcal{R}_0 \leq 1$, there is a ***disease-free equilibrium***
- If $\mathcal{R}_0 > 1$, there is an ***Endemic Equilibrium*** (the disease is not totally eradicated and ***remains*** in the population)