PS1 Solutions

Jingle Fu

1 Consumption Allocation

Problem Setup

The Home agent's consumption basket is given by

$$C_t = \left(\frac{C_{T,t}}{\gamma}\right)^{\gamma} \left(\frac{C_{N,t}}{1-\gamma}\right)^{1-\gamma},$$

where:

- $C_{T,t}$ is the quantity of the traded good (its price is normalized to 1),
- $C_{N,t}$ is the quantity of the domestic non-traded good (price $P_{N,t}$),
- γ is the expenditure share on the traded good.

The consumer minimizes total expenditure subject to attaining a given consumption level C_t. The problem is

$$\begin{aligned}
&\min_{C_{T,t},\, C_{N,t}} \; P_t C_t = C_{T,t} + P_{N,t}\, C_{N,t} \\
&\text{s.t.} \quad C_t = \left(\frac{C_{T,t}}{\gamma}\right)^{\gamma} \left(\frac{C_{N,t}}{1-\gamma}\right)^{1-\gamma}.
\end{aligned}$$

Define the Lagrangian function:

$$\mathcal{L} = C_{T,t} + P_{N,t}\, C_{N,t} + \lambda\Big(C_t - \left(\frac{C_{T,t}}{\gamma}\right)^{\gamma} \left(\frac{C_{N,t}}{1-\gamma}\right)^{1-\gamma}\Big).$$

The FOCs with respect to $C_{T,t}$ and $C_{N,t}$ are:

$$\begin{aligned}
\mathcal{L}_{C_{T,t}} &= 1 - \lambda\, \gamma \left(\frac{C_{T,t}}{\gamma}\right)^{\gamma-1} \frac{1}{\gamma} \left(\frac{C_{N,t}}{1-\gamma}\right)^{1-\gamma} = 0, \\
\mathcal{L}_{C_{N,t}} &= P_{N,t} - \lambda\,(1-\gamma) \left(\frac{C_{T,t}}{\gamma}\right)^{\gamma} \frac{1}{1-\gamma} \left(\frac{C_{N,t}}{1-\gamma}\right)^{-\gamma} = 0 \\
\Rightarrow \quad \frac{1}{P_{N,t}} &= \frac{\gamma}{1-\gamma}\, \frac{C_{N,t}}{C_{T,t}}.
\end{aligned}$$

The dual (expenditure) minimization problem yields the unit cost function (composite price index) for the consumption bundle:

$$P_t\, C_t = \min\left\{ C_{T,t} + P_{N,t}\, C_{N,t} :\ C_t = \left(\frac{C_{T,t}}{\gamma}\right)^{\gamma} \left(\frac{C_{N,t}}{1-\gamma}\right)^{1-\gamma} \right\}.$$

So, we have:

$$\begin{aligned} P_t C_T &= C_{T,t} + P_{N,t}\, C_{N,t} \\ &= C_{T,t} + \frac{1-\gamma}{\gamma} C_{T,t} \\ \Rightarrow C_{T,t} &= \gamma\, P_t\, C_t \\ \Rightarrow C_{N,t} &= \frac{1-\gamma}{\gamma}\frac{C_{T,t}}{P_{N,t}} \\ &= (1-\gamma)\, P_t\, C_t. \end{aligned}$$

As P_t is the minimum expenditure required to attain the given consumption level $C_t = 1$, we have:

$$\begin{aligned} & \left(\frac{C_{T,t}}{\gamma}\right)^{\gamma} \left(\frac{C_{N,t}}{1-\gamma}\right)^{1-\gamma} = 1 \\ \Rightarrow \quad & (P_t C_t)^{\gamma} \left(\frac{P_t C_t}{P_N, t}\right)^{1-\gamma} = 1 \\ \Rightarrow \quad & P_t = (P_{N,t})^{1-\gamma}. \end{aligned}$$

Analogously, for the Foreign agent, we have

$$\begin{aligned} C^*_{T,t} &= \gamma\, P^*_t\, C^*_t \\ C^*_{N,t} &= (1-\gamma)\, P^*_t\, C^*_t \\ P^*_t &= (P^*_{N,t})^{1-\gamma}. \end{aligned}$$

Economic Intuition

- The parameter γ reflects the expenditure share on the traded good.
- Since the traded good is the numéraire (price normalized to 1), its cost enters directly, while the cost of the non-traded good is weighted by its price $P_{N,t}$.
- The composite price index P_t is a weighted geometric mean of the individual prices. With the traded goods price equal to 1, we have $P_t = (P_{N,t})^{1-\gamma}$.
- The optimal consumption choices allocate expenditure in a way that equates the marginal rate of substitution to the ratio of prices.

2 Market Clearing

Under market clearing, the quantities of traded and non-traded goods produced must equal the quantities consumed. We have:

Non-Traded Goods Market

For Home, market clearing in non-traded goods is:

$$n\,C_{N,t} = A_{N,t}\,(L_{N,t})^{1-\alpha}.$$

Substituting $C_{N,t} = (1-\gamma)\frac{P_t C_t}{P_{N,t}}$ with $P_t = (P_{N,t})^{1-\gamma}$, we obtain:

$$\boxed{n(1-\gamma)\,(P_{N,t})^{-\gamma}\,C_t = A_{N,t}\,(L_{N,t})^{1-\alpha}.}$$

For Foreign, the market clearing condition is: $(1-n)C^*_{N,t} = A^*_{N,t}(L^*_{N,t})^{(1-\alpha)}$

$$\boxed{(1-n)(1-\gamma)\,(P^*_{N,t})^{-\gamma}\,C^*_t = A^*_{N,t}\,(L^*_{N,t})^{1-\alpha}.}$$

Traded Goods Market

Global market clearing for traded goods is:

$$n\,C_{T,t} + (1-n)\,C^*_{T,t} = A_{T,t}\,(n - L_{N,t})^{1-\alpha} + A^*_{T,t}\left((1-n) - L^*_{N,t}\right)^{1-\alpha}.$$

Substituting $C_{T,t} = \gamma\,P_t\,C_t$ with $P_t = (P_{N,t})^{1-\gamma}$ (and similarly for Foreign), we have:

$$\boxed{n\gamma\,(P_{N,t})^{1-\gamma}\,C_t + (1-n)\gamma\,(P^*_{N,t})^{1-\gamma}\,C^*_t = A_{T,t}\,(n - L_{N,t})^{1-\alpha} + A^*_{T,t}\left((1-n) - L^*_{N,t}\right)^{1-\alpha}.}$$

Intuition: Non-traded goods are produced and consumed domestically, while traded goods are balanced across countries.

3 Intertemporal Allocation

Home Optimization

The Home agent maximizes:

$$U_t = \sum_{s=0}^{\infty}(\beta_{H,t+s})^s \ln C_{t+s},$$

subject to the period-t budget constraint:

$$n\,P_t\,C_t + n\,B_{t+1} = A_{T,t}\,(n - L_{N,t})^{1-\alpha} + P_{N,t}\,A_{N,t}\,(L_{N,t})^{1-\alpha} + n(1+r_t)B_t.$$

Focusing on a single period (the infinite-horizon structure is standard), we write:

$$\mathcal{L} = \ln C_t + \beta_{H,t+1}\ln C_{t+1} - \lambda_t\Big[A_{T,t}(n-L_{N,t})^{1-\alpha} + P_{N,t}A_{N,t}(L_{N,t})^{1-\alpha} + n(1+r_t)B_t - n\,P_t\,C_t - n\,B_{t+1}\Big].$$

Take the FOCs:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial C_t} &= \frac{1}{C_t} - \lambda_t n P_t = 0 \quad \Rightarrow \quad \lambda_t = \frac{1}{nP_t\,C_t}\\ \frac{\partial\mathcal{L}}{\partial B_{t+1}} &= -\lambda_t + \beta_{H,t+1}(1+r_{t+1})\,\lambda_{t+1} = 0.\end{aligned}$$

Substitute the expressions for λ_t and λ_{t+1}:

$$\frac{1}{nP_t\,C_t} = \beta_{H,t+1}(1+r_{t+1})\frac{1}{nP_{t+1}\,C_{t+1}}.$$

Cancel n and rearrange:

$$\frac{1}{C_t} = \beta_{H,t+1}(1+r_{t+1})\frac{P_t}{P_{t+1}}\frac{1}{C_{t+1}}.$$

Set:

$$1 + r^C_{t+1} \equiv (1+r_{t+1})\frac{P_t}{P_{t+1}},$$

so the Euler equation becomes:

$$\boxed{C_{t+1} = \beta_{H,t+1}(1+r^C_{t+1})\,C_t.}$$

From Question (1), we know that $P_t = (P_{N,t})^{(1-\gamma)}$, so, we have:

$$1 + r^C_{t+1} = (1+r_{t+1})\frac{P_t}{P_{t+1}} = (1+r_{t+1})\Big(\frac{P_{N,t}}{P_{N,t+1}}\Big)^{1-\gamma}.$$

Since

$$C_{T,t} = \gamma\,P_t\,C_t,$$

it follows that

$$C_{T,t+1} = \gamma\,P_{t+1}\,C_{t+1} = \gamma\,P_{t+1}\,\beta_{H,t+1}(1+r^C_{t+1})\,C_t.$$

As we note that

$$\beta_{H,t+1}(1+r^C_{t+1}) = \beta_{H,t+1}(1+r_{t+1})\frac{P_t}{P_{t+1}},$$

so simplifying, we obtain:

$$\boxed{C_{T,t+1} = \beta_{H,t+1}(1 + r_{t+1})\, C_{T,t}.}$$

Remark. The Foreign agent's optimization yields analogous Euler equations:

$$C^*_{t+1} = \beta_{F,t+1}(1 + r^*_{C,t+1})\, C^*_t, \quad C^*_{T,t+1} = \beta_{F,t+1}(1 + r_{t+1})\, C^*_{T,t},$$

with

$$1 + r^{*C}_{t+1} = (1 + r_{t+1}) \frac{P^*_t}{P^*_{t+1}}.$$

Foreign Optimization

Similarly, for the Foreign country:

$$\boxed{C^*_{t+1} = \beta_{F,t+1}\,(1 + r^*_{C,t+1})\, C^*_t, \qquad C^*_{T,t+1} = \beta_{F,t+1}\,(1 + r_{t+1})\, C^*_{T,t},}$$

with

$$1 + r^{*C}_{t+1} = (1 + r_{t+1}) \left(\frac{P^*_{N,t}}{P^*_{N,t+1}} \right)^{1-\gamma}.$$

Intuition: Households equate the marginal utility cost of consuming today versus tomorrow, with intertemporal decisions affected by relative price changes.

4 Labor Allocation

The production functions are given by:

$$Y_{T,t} = A_{T,t}(n - L_{N,t})^{1-\alpha}, \quad Y_{N,t} = A_{N,t}(L_{N,t})^{1-\alpha}.$$

Compute the marginal product of labor in each sector:

$$\frac{\partial Y_{T,t}}{\partial (n - L_{N,t})} = (1 - \alpha) A_{T,t}(n - L_{N,t})^{-\alpha},$$

$$\frac{\partial Y_{N,t}}{\partial L_{N,t}} = (1 - \alpha) A_{N,t}(L_{N,t})^{-\alpha}.$$

The Home agent allocates labor so that the marginal value product in the traded sector equals the marginal value product (adjusted by the non-traded price) in the non-traded sector:

$$(1 - \alpha) A_{T,t}(n - L_{N,t})^{-\alpha} = P_{N,t}\,(1 - \alpha) A_{N,t}(L_{N,t})^{-\alpha}.$$

Cancel the common factor $1-\alpha$ and rearrange:

$$\boxed{A_{T,t}(n - L_{N,t})^{-\alpha} = P_{N,t}\,A_{N,t}(L_{N,t})^{-\alpha}.}$$

The analogous condition for the Foreign country is:

$$\boxed{A^*_{T,t}\big((1-n) - L^*_{N,t}\big)^{-\alpha} = P^*_{N,t}\,A^*_{N,t}(L^*_{N,t})^{-\alpha}.}$$

5 Resource Constraints and the Real Exchange Rate

Resource Constraints

Recall the Home budget constraint:

$$n\,P_t\,C_t + n\,B_{t+1} = A_{T,t}(n - L_{N,t})^{1-\alpha} + P_{N,t}A_{N,t}(L_{N,t})^{1-\alpha} + n(1+r_t)B_t.$$

From Question 1, we have:

$$C_{N,t} = (1-\gamma)\,\frac{P_t}{P_{N,t}}\,C_t.$$

Since $P_t = (P_{N,t})^{1-\gamma}$, then:

$$C_{N,t} = (1-\gamma)(P_{N,t})^{-\gamma}C_t.$$

Given that non-traded goods are produced solely for domestic consumption, we also have the production identity (from Question 2):

$$n\,(1-\gamma)(P_{N,t})^{-\gamma}C_t = A_{N,t}(L_{N,t})^{1-\alpha}.$$

Thus, the expenditure on traded goods (which uses the consumption price index) plus net asset accumulation must equal traded output plus bond returns:

$$\boxed{n\gamma\,(P_{N,t})^{1-\gamma}C_t + n\,B_{t+1} = A_{T,t}(n - L_{N,t})^{1-\alpha} + n(1+r_t)B_t.}$$

Similarly, for Foreign we obtain:

$$\boxed{(1-n)\gamma\,(P^*_{N,t})^{1-\gamma}C^*_t - n\,B_{t+1} = A^*_{T,t}\big((1-n) - L^*_{N,t}\big)^{1-\alpha} - n(1+r_t)B_t.}$$

Then, we define the real exchange rate as the ratio of Foreign to Home consumption price indices:

$$Q_t \equiv \frac{P^*_t}{P_t}.$$

Since

$$P_t = (P_{N,t})^{1-\gamma} \quad \text{and} \quad P^*_t = (P^*_{N,t})^{1-\gamma},$$

we have:

$$\boxed{Q_t = \left(\frac{P^*_{N,t}}{P_{N,t}}\right)^{1-\gamma}.}$$

6 Steady State

In steady state, consumption is constant so that $C_{t+1} = C_t$. The Euler equation for the Home agent is

$$C_{t+1} = \beta_0(1 + r^C_{t+1})\, C_t.$$

But by definition the real return in consumption units is

$$1 + r^C_{t+1} = (1 + r_{t+1}) \left(\frac{P_t}{P_{t+1}}\right)^{1-\gamma}.$$

In steady state prices do not change ($P_t = P_{t+1}$) so that

$$1 + r^C_{t+1} = 1 + r_{t+1} \quad \Rightarrow \quad C_t = \beta_0(1 + r_0)C_t.$$

Dividing by $C_t > 0$ yields:

$$1 = \beta_0(1 + r_0).$$

Step 2. Price Normalization

By convention we normalize the steady state price of non-tradables to one:

$$P_{N,0} = 1, \quad \text{and similarly} \quad P^*_{N,0} = 1.$$

Then the consumption price indices become

$$P_0 = (P_{N,0})^{1-\gamma} = 1, \quad P^*_0 = 1.$$

Step 3. Labor Allocation in the Home Country

The Home intratemporal labor allocation condition is:

$$A_{T,0}\,(n - L_{N,0})^{-\alpha} = P_{N,0}\, A_{N,0}\,(L_{N,0})^{-\alpha}.$$

Since $P_{N,0} = 1$, this simplifies to:

$$A_{T,0}\,(n - L_{N,0})^{-\alpha} = A_{N,0}\,(L_{N,0})^{-\alpha}.$$

Rearrange by dividing both sides by $A_{T,0}$ and by $(L_{N,0})^{-\alpha}$:

$$\left(\frac{n - L_{N,0}}{L_{N,0}}\right)^{-\alpha} = \frac{A_{N,0}}{A_{T,0}}.$$

Taking the reciprocal and then the $1/\alpha$-th root,

$$\frac{n - L_{N,0}}{L_{N,0}} = \left(\frac{A_{T,0}}{A_{N,0}}\right)^{1/\alpha}.$$

Now, using the calibration

$$A_{N,0} = A_{T,0}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha},$$

we have

$$\frac{A_{T,0}}{A_{N,0}} = \left(\frac{\gamma}{1-\gamma}\right)^{\alpha}.$$

Then,

$$\frac{n - L_{N,0}}{L_{N,0}} = \left[\left(\frac{\gamma}{1-\gamma}\right)^{\alpha}\right]^{1/\alpha} = \frac{\gamma}{1-\gamma}.$$

Thus,

$$\boxed{L_{N,0} = n(1-\gamma).}$$

By symmetry, the corresponding condition for Foreign is:

$$A^*_{T,0}\Big((1-n) - L^*_{N,0}\Big)^{-\alpha} = P^*_{N,0}\, A^*_{N,0}\, (L^*_{N,0})^{-\alpha}.$$

Since $P^*_{N,0} = 1$, the same steps lead to:

$$\frac{(1-n) - L^*_{N,0}}{L^*_{N,0}} = \frac{\gamma}{1-\gamma},$$

so that

$$\boxed{L^*_{N,0} = (1-n)(1-\gamma).}$$

We derive the steady-state consumption from the market clearing conditions for non-traded and traded goods. The non-traded goods clearing condition is

$$n\, C_{N,0} = A_{N,0}\, (L_{N,0})^{1-\alpha}.$$

From Question 1 we have

$$C_{N,0} = (1-\gamma)\,\frac{P_0}{P_{N,0}}\, C_0.$$

Since $P_0 = 1$ and $P_{N,0} = 1$, it follows that

$$C_{N,0} = (1-\gamma)\, C_0.$$

Substitute into the clearing condition:

$$n\,(1-\gamma)\,C_0 = A_{N,0}\,(L_{N,0})^{1-\alpha}.$$

Recall that $L_{N,0} = n(1-\gamma)$, so

$$n\,(1-\gamma)\,C_0 = A_{N,0}\,\big[n(1-\gamma)\big]^{1-\alpha}.$$

Solve for C_0:

$$C_0 = A_{N,0}\,\big[n(1-\gamma)\big]^{-\alpha}.$$

We then derive from the traded goods clearing condition:

$$\begin{aligned}
& n\gamma\,(P_{N,t})^{1-\gamma}\,C_t + (1-n)\gamma\,(P^*_{N,t})^{1-\gamma}\,C^*_t = A_{T,t}\,(n-L_{N,t})^{1-\alpha} + A^*_{T,t}\left((1-n)-L^*_{N,t}\right)^{1-\alpha} \\
\Rightarrow\quad & n\gamma\,C_0 + \frac{\gamma}{1-\gamma}A^*_{N,0}(L^*_{N,0})^{1-\alpha} = A_{T,0}\,(n-L_{N,0})^{1-\alpha} + A^*_{T,0}\,(1-n-L^*_{N,0})^{1-\alpha} \\
\Rightarrow\quad & n\gamma\,C_0 + \frac{\gamma}{1-\gamma}A^*_{N,0}(1-n)^{1-\alpha}(1-\gamma)^{1-\alpha} = A_{T,0}\,(n-n(1-\gamma))^{1-\alpha} + \\
& A^*_{T,0}\,(1-n-(1-n)(1-\gamma))^{1-\alpha} \\
\Rightarrow\quad & n\gamma\,C_0 + \frac{\gamma}{1-\gamma}A^*_{N,0}(1-n)^{1-\alpha}(1-\gamma)^{1-\alpha} = A_{T,0}\,(n\gamma)^{1-\alpha} + A^*_{T,0}\,\Big[(1-n)\gamma\Big]^{1-\alpha} \\
\Rightarrow\quad & n\gamma\,C_0 + \frac{\gamma}{1-\gamma}A_{T,0}\Big(\frac{1-n}{n}\Big)^{\alpha}\Big(\frac{1-\gamma}{\gamma}\Big)^{\alpha}(1-n)^{1-\alpha}(1-\gamma)^{1-\alpha} = A_{T,0}\,(n\gamma)^{1-\alpha} + \\
& A_{T,0}\Big(\frac{1-n}{n}\Big)^{\alpha}\Big[(1-n)\gamma\Big]^{1-\alpha} \\
\Rightarrow\quad & n\gamma\,C_0 + A_{T,0}(1-n)n^{-\alpha}\gamma^{1-\alpha} = A_{T,0}\,(n\gamma)^{1-\alpha} + A_{T,0}(1-n)n^{-\alpha}\gamma^{1-\alpha} \\
\Rightarrow\quad & C_0 = A_{T,0}\,(n\gamma)^{-\alpha}.
\end{aligned}$$

We take a weighted geometric mean with weights γ and $1-\gamma$. That is,

$$C_0 = (C_0)^{1-\gamma}\cdot(C_0)^{\gamma},$$

so that

$$C_0 = \big[A_{N,0}\,n^{-\alpha}(1-\gamma)^{-\alpha}\big]^{1-\gamma}\,\big[A_{T,0}\,n^{-\alpha}\,\gamma^{-\alpha}\big]^{\gamma}.$$

Combine exponents:

$$C_0 = (A_{T,0})^{\gamma}(A_{N,0})^{1-\gamma}\,n^{-\alpha(\gamma+1-\gamma)}\,\gamma^{-\alpha\gamma}\,(1-\gamma)^{-\alpha(1-\gamma)}.$$

We obtain:

$$\boxed{C_0 = (A_{T,0})^{\gamma}(A_{N,0})^{1-\gamma}\,\big[n\gamma^{\gamma}(1-\gamma)^{1-\gamma}\big]^{-\alpha}.}$$

A similar derivation for Foreign (noting that the population is $1-n$) gives:

$$\boxed{C^*_0 = (A^*_{T,0})^{\gamma}(A^*_{N,0})^{1-\gamma}\,\big[(1-n)\gamma^{\gamma}(1-\gamma)^{1-\gamma}\big]^{-\alpha}.}$$

Because of the calibration (and the fact that the relative productivities satisfy

$$\frac{A_{N,0}}{A_{T,0}} = \left(\frac{1-\gamma}{\gamma}\right)^{\alpha} \quad \text{and} \quad \frac{A^*_{N,0}}{A^*_{T,0}} = \left(\frac{1-n}{n}\right)^{\alpha}\left(\frac{1-\gamma}{\gamma}\right)^{\alpha}),$$

one can check that indeed

$$\frac{C_0}{C_0^*} = 1.$$

7 Log-Linear Approximation

We linearize the equilibrium conditions around the steady state. Denote for any variable x_t its deviation from steady state by

$$\hat{x}_t = \frac{x_t - x_0}{x_0}.$$

We also define cross-country differences later but for now we linearize the Home equations.

A. Non-Traded Goods Market

The Home non-traded goods market clearing condition is:

$$n(1-\gamma)(P_{N,t})^{-\gamma}C_t = A_{N,t}(L_{N,t})^{1-\alpha}.$$

Taking logarithms, we have

$$\ln n + \ln(1-\gamma) - \gamma \ln P_{N,t} + \ln C_t = \ln A_{N,t} + (1-\alpha)\ln L_{N,t}.$$

Linearizing around the steady state (and noting that constants vanish in the difference), we obtain:

$$-\gamma\, \hat{P}_{N,t} + \hat{C}_t = \hat{A}_{N,t} + (1-\alpha)\, \hat{L}_{N,t}.$$

B. Resource Constraint

The Home resource constraint is:

$$n\gamma\, (P_{N,t})^{1-\gamma}C_t + nB_{t+1} = A_{T,t}(n - L_{N,t})^{1-\alpha} + n(1+r_t)B_t.$$

Taking logs and linearizing (and assuming that in steady state $B_t = 0$, so only percentage deviations matter), we have:

$$(1-\gamma)\, \hat{P}_{N,t} + \hat{C}_t + \hat{B}_{t+1} = \hat{A}_{T,t} - \frac{(1-\alpha)(n - L_{N,0})}{n - L_{N,0}}\, \hat{L}_{N,t} + \frac{1}{\beta_0}\, \hat{B}_t.$$

Since in steady state $n - L_{N,0} = n\gamma$, the coefficient on $\hat{L}_{N,t}$ becomes $(1-\alpha)$. Thus,

$$\boxed{(1-\gamma)\,\hat{P}_{N,t} + \hat{C}_t + \hat{B}_{t+1} = \hat{A}_{T,t} - (1-\alpha)\,\hat{L}_{N,t} + \frac{1}{\beta_0}\,\hat{B}_t.} \tag{7b}$$

C. Euler Equation

The Home Euler equation is:

$$C_{t+1} = \beta_0(1 + r^C_{t+1})\,C_t.$$

Taking logs,

$$\ln C_{t+1} - \ln C_t = \ln \beta_0 + \ln(1 + r^C_{t+1}).$$

Linearizing (and noting $\ln \beta_0$ is constant), we have

$$\hat{C}_{t+1} - \hat{C}_t = \hat{\beta}_{H,t+1} + \beta_0\,\hat{r}_{C,t+1}.$$

Recall that the real rate in consumption terms is related to the non-traded price by

$$1 + r^C_{t+1} = (1 + r_{t+1})\left(\frac{P_{N,t}}{P_{N,t+1}}\right)^{1-\gamma}.$$

Taking logs and linearizing,

$$\ln(1 + r^C_{t+1}) \approx \ln(1 + r_{t+1}) + (1-\gamma)(\ln P_{N,t} - \ln P_{N,t+1}),$$

so that

$$\hat{r}_{C,t+1} \approx \hat{r}_{t+1} + (1-\gamma)(\hat{P}_{N,t} - \hat{P}_{N,t+1}).$$

Thus, we can write the Euler equation as:

$$\boxed{\hat{C}_{t+1} - \hat{C}_t = (1-\gamma)(\hat{P}_{N,t} - \hat{P}_{N,t+1}) + \hat{\beta}_{H,t+1} + \beta_0\,\hat{r}_{t+1}.} \tag{7c}$$

D. Labor Allocation

The intratemporal condition is:

$$A_{T,t}(n - L_{N,t})^{-\alpha} = P_{N,t}\,A_{N,t}(L_{N,t})^{-\alpha}.$$

Taking logarithms:

$$\ln A_{T,t} - \alpha \ln(n - L_{N,t}) = \ln P_{N,t} + \ln A_{N,t} - \alpha \ln L_{N,t}.$$

Linearize around steady state:

$$\hat{A}_{T,t} - \alpha \frac{n - L_{N,t}}{n - L_{N,0}} (n - \hat{} L_{N,t}) = \hat{P}_{N,t} + \hat{A}_{N,t} - \alpha \hat{L}_{N,t}.$$

Using the fact that in steady state $n - L_{N,0} = n\gamma$, a careful linearization (which involves the derivative of $\ln(n - L_{N,t})$) yields:

$$\boxed{\hat{A}_{T,t} + \frac{\alpha}{\gamma} \hat{L}_{N,t} = \hat{P}_{N,t} + \hat{A}_{N,t}.} \tag{7d}$$

E. Real Exchange Rate

Since

$$P_t = (P_{N,t})^{1-\gamma}, \quad P_t^* = (P_{N,t}^*)^{1-\gamma},$$

taking logs gives

$$\hat{P}_t = (1-\gamma) \hat{P}_{N,t}, \quad \hat{P}_t^* = (1-\gamma) \hat{P}_{N,t}^*.$$

Thus, the log deviation of the real exchange rate defined by

$$Q_t = \frac{P_t^*}{P_t}$$

is

$$\hat{Q}_t = \hat{P}_t^* - \hat{P}_t = (1-\gamma)(\hat{P}_{N,t}^* - \hat{P}_{N,t}).$$

8 Worldwide Equilibrium

Define world aggregates as population-weighted averages (e.g., $\hat{C}_t^W = n \hat{C}_t + (1-n) \hat{C}_t^*$). Then, from the above log-linearized equations one can show:

- $\hat{L}_{N,t}^W = 0$, i.e., aggregate non-traded labor remains fixed.
- The world non-traded price satisfies

$$\hat{P}_{N,t}^W = \hat{A}_{T,t}^W - \hat{A}_{N,t}^W.$$

- World consumption is given by

$$\hat{C}_t^W = \gamma \hat{A}_{T,t}^W + (1-\gamma) \hat{A}_{N,t}^W.$$

- The world Euler equation becomes

$$\beta_0 \hat{r}_{t+1} = -\hat{\beta}_{t+1}^W + \left(\hat{A}_{T,t+1}^W - \hat{A}_{T,t}^W \right).$$

Intuition: World aggregates respond only to symmetric shocks, with the real interest rate driven by global productivity changes and aggregate patience.

9 Cross-Country Differences

Define differences as (Home minus Foreign) for a variable x by $\tilde{x}_t = \hat{x}_t - \hat{x}^*_t$. Then:
Non-Traded Goods Market (Difference):

$$\frac{\gamma}{1-\gamma}\,\hat{Q}_t + \tilde{C}_t = \tilde{A}_{N,t} + (1-\alpha)\,\tilde{L}_{N,t}. \tag{9a}$$

Resource Constraints (Difference):

$$-\hat{Q}_t + \tilde{C}_t + \frac{\hat{B}_{t+1}}{1-n} = \tilde{A}_{T,t} - \frac{(1-\alpha)(1-\gamma)}{\gamma}\,\tilde{L}_{N,t}. \tag{9b}$$

Euler Equation (Difference):

$$\tilde{C}_{t+1}-\tilde{C}_t = (1-\gamma)\Big[(\hat{P}_{N,t}-\hat{P}^*_{N,t})-(\hat{P}_{N,t+1}-\hat{P}^*_{N,t+1})\Big]+(\hat{\beta}_{H,t+1}-\hat{\beta}_{F,t+1})+\beta_0\Big(\hat{r}_{C,t+1}-\hat{r}^*_{C,t+1}\Big). \tag{9c}$$

Labor Allocation (Difference):

$$\frac{\alpha}{\gamma}\,\tilde{L}_{N,t} = -\frac{1}{1-\gamma}\,\hat{Q}_t - \tilde{A}_{N,t}. \tag{9d}$$

Intuition: These equations link cross-country differences in consumption, labor, and the real exchange rate to differences in productivity and intertemporal preferences.

10 Long-Run Allocation (Period $t+1$)

Assume that from $t+1$ onward the economy reaches a new steady state with no further discount factor shocks ($\hat{\beta}_{H,t+2} = \hat{\beta}_{F,t+2} = 0$). Taking the cross-country asset position $\hat{B}_{t+1}$ as given, one can show:

$$\hat{Q}_{t+1} = -(1-\gamma)\Big[(\hat{A}_{T,t+1}-\hat{A}^*_{T,t+1})-(\hat{A}_{N,t+1}-\hat{A}^*_{N,t+1})\Big] - \frac{\alpha(1-\gamma)}{1-\beta_0}\frac{1}{\beta_0}\frac{\hat{B}_{t+1}}{1-n},$$

$$\tilde{L}_{N,t+1} = \frac{\gamma}{1-\beta_0}\frac{1}{\beta_0}\frac{\hat{B}_{t+1}}{1-n},$$

$$\tilde{C}_{t+1} = \gamma\big(\hat{A}_{T,t+1}-\hat{A}^*_{T,t+1}\big) + (1-\gamma)\big(\hat{A}_{N,t+1}-\hat{A}^*_{N,t+1}\big) + \frac{\gamma}{1-\beta_0}\frac{1}{\beta_0}\frac{\hat{B}_{t+1}}{1-n}.$$

Interpretation:

- A positive $\hat{B}_{t+1}$ (Home wealthier) implies higher relative consumption and a lower Q_{t+1} (Homes goods become relatively more expensive).
- Permanent productivity differences affect steady state consumption and prices directly.

11 Short-Run Allocation (Period t)

Assume initially $\hat{B}_t = 0$. Solving the system (starting with the labor and non-traded market conditions, then using the resource constraint and Euler equations) yields:

$$\frac{\hat{B}_{t+1}}{1-n} = \frac{\beta_0}{\gamma+\alpha(1-\gamma)}\Big[(\hat{\beta}_{H,t+1}-\hat{\beta}_{F,t+1})-(\hat{A}_{T,t+1}-\hat{A}^*_{T,t+1})+(\hat{A}_{T,t}-\hat{A}^*_{T,t})\Big], \tag{11a}$$

$$\tilde{C}_t = \gamma(\hat{A}_{T,t}-\hat{A}^*_{T,t})+(1-\gamma)(\hat{A}_{N,t}-\hat{A}^*_{N,t})-\frac{\gamma\,\beta_0}{\gamma+\alpha(1-\gamma)}\Big[(\hat{\beta}_{H,t+1}-\hat{\beta}_{F,t+1})-(\hat{A}_{T,t+1}-\hat{A}^*_{T,t+1})+(\hat{A}_{T,t}-\hat{A}^*_{T,t})\Big], \tag{11b}$$

$$\hat{Q}_t = -(1-\gamma)(\hat{A}_{T,t}-\hat{A}^*_{T,t})+(1-\gamma)(\hat{A}_{N,t}-\hat{A}^*_{N,t})+\frac{(1-\gamma)\,\alpha\,\beta_0}{\gamma+\alpha(1-\gamma)}\Big[(\hat{\beta}_{H,t+1}-\hat{\beta}_{F,t+1})-(\hat{A}_{T,t+1}-\hat{A}^*_{T,t+1})+(\hat{A}_{T,t}- \tag{11c}$$

$$\tilde{L}_{N,t} = -\frac{\gamma\,\beta_0}{\gamma+\alpha(1-\gamma)}\Big[(\hat{\beta}_{H,t+1}-\hat{\beta}_{F,t+1})-(\hat{A}_{T,t+1}-\hat{A}^*_{T,t+1})+(\hat{A}_{T,t}-\hat{A}^*_{T,t})\Big]. \tag{11d}$$

Interpretation:

- A temporary increase in Home patience (i.e. $\hat{\beta}_{H,t+1}-\hat{\beta}_{F,t+1}>0$) leads to $\hat{B}_{t+1}>0$ (Home runs a current account surplus), lower relative consumption, and a real exchange rate movement consistent with a trade surplus.
- Temporary traded or non-traded productivity shocks affect the current account and labor allocation differently.
- For permanent shocks ($\hat{A}_{T,t}=\hat{A}_{T,t+1}$), intertemporal balance is restored with $\hat{B}_{t+1}=0$ and immediate adjustment to the new steady state.

12 Summary of Key Economic Insights

- **Consumption and Prices:** The structure of the consumption basket implies that a rise in the nontraded good price $P_{N,t}$ increases the overall consumption price P_t and shifts the consumption mix.
- **Market Clearing:** Non-traded goods are produced and consumed domestically, whereas traded goods are allocated internationally, linking domestic production choices to the real exchange rate.

- **Intertemporal Choices:** The Euler equations show that higher patience or higher real returns lead to deferred consumption. Cross-country differences in patience drive current account imbalances.

- **Labor Allocation:** Labor is reallocated across sectors until the marginal value products are equalized; productivity shifts affect both output composition and relative prices.

- **Steady State and Log-Linearization:** In a symmetric steady state, relative prices and allocations are balanced. Log-linearization permits analysis of small shocks and their propagation.

- **Short-Run vs. Long-Run Dynamics:** Temporary shocks generate current account imbalances and short-run reallocation, while permanent shocks adjust consumption and prices directly with no asset accumulation.

- **Wealth Effects:** A positive net asset position (Home wealthier) implies higher steady-state consumption and a relatively stronger (appreciated) currency.