Problem 1: We will solve the problem in two stages. First we will solve for any output level $q \geq 0$ the cost minimization problem $\min_{z\geq 0} wz$ s.t. $f(z_1, z_2) \geq q$. In a second step we solve the profit maximization problem $\max_{q\geq 0} pq - c(q, w)$.

i) In this case f is strictly increasing and the inputs are perfect substitutes. Thus, the firm will use only the cheaper of the two inputs and so if $w_1 \leq w_2$ the conditional factor demand is

$$z(w_1, w_2, q) = \begin{cases} (q^2, 0) & \text{if } w_1 < w_2 \\ \{z \geq 0 : z_1 + z_2 = q^2\} & \text{if } w_1 = w_2 \\ (0, q^2) & \text{if } w1 > w2. \end{cases}$$

while the cost function is given by $c(q, w) = \min\{w_1, w_2\}$. The profit maximization problem is given by

$$\max_{q\geq 0} pq - \min\{w_1, w_2\}q^2.$$

This problem is strictly concave and thus the point $q = p/(2\min\{w_1, w_2\})$ which satisfies the FOC solves the profit maximization problem. The supply function and the (unconditional) factor demand function are given by $q(w_1, w_2, p) = \frac{p}{2\min\{w_1,w_2\}}$ and

$$z(p, w_1, w_2) = \begin{cases} ((p/2w_1)^2, 0) & \text{if } w_1 < w_2 \\ \{(z_1, z_2) : z_1 + z_2 = (p/2w_1)^2\} & \text{if } w_1 = w_2 \\ (0, (p/2w_1)^2) & \text{if } w_1 > w_2. \end{cases}$$

ii) The production function is quasi-concave and the production factors are perfect complements. In particular, the two factors have to be employed in the proportion 1 : 1. Thus the conditional factor demand and the cost function are given by $z(w, q) = (q^{1/\alpha}, q^{1/\alpha})$ and $c(w, q) = (w_1 + w_2)q^{1/\alpha}$. Since $\alpha \in (0, 1)$ it follows that the profit maximization problem $\max_{q\geq 0} pq - (w_1 + w_2)q^{1/\alpha}$ is strictly concave. Therefore, the optimum is determined by the FOC. The FOC delivers $q(w, p) = (\alpha p/(w_1 + w_2))^{\alpha/(1-\alpha)}$ and thus the unconditional demand for factor $l = 1, 2$ is $z_l(w; p) = (\alpha p/(w_1 + w_2))^{\alpha/(1-\alpha)}$.

iii) The production function f is of the CES type. It is easily verified that f exhibits constant returns to scale. Therefore the cost function must be linear in q. For $\rho = 1$, f is linear and so we are again in the case of perfect substitutes.

If $\rho < 1$ then f is strictly quasi-concave (you should verify this). Since the objective function is linear, this means that the cost minimization problem has a unique solution. In particular, strict quasi-concavity implies that if there is a point on the isoquant for the output level q where the marginal rate of technical substitution is equal to the ratio of the input prices, then that point constitutes the input combination which minimizes the costs for the output level.

It is easily verified that the system of equations

$$\begin{aligned} f(z) &= q \\ \frac{\partial f(z)/\partial z_1}{\partial f(z)/\partial z_1} &= \frac{w_1}{w_2} \end{aligned}$$

admits the solution

$$z_l(q,w) = q w_l^{1/(\rho-1)}[w_1 + w_2^{\rho/(\rho-1)}]^{(-1/\rho)}.$$

As for the cost function we therefore obtain

$$c(q,w) = w_1 z_1(q,w) + w_2 z_2(q,w) = q\,\underbrace{[w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)}]^{(\rho-1)/\rho}}_{\phi(w)}.$$

The profit maximization problem $\max_{q\geq 0} pq - c(q,w)$ has a solution if and only if $\phi(w) \geq p$. In particular, if $\phi(w) = p$ then $q(p,w) = \mathbb{R}_+$ (and thus also the unconditional factor demand in this case is the set of all vectors (z_1, z_2) such that $z_1/z_2 = w_2/w_1$). If instead $\phi(w) > p$ then $q(p,w) = 0$ (obviously also the unconditional factor demand vanishes).

Problem 2:

i) We know that cost functions are concave and linear homogeneous in prices. Thus

$$\begin{aligned} & c((1/2)w' + (1/2)w'', q) &\geq& \ (1/2)c(w',q) + (1/2)c(w'',q) \\ \Leftrightarrow \quad & (1/2)c(w' + w'', q) &\geq& \ (1/2)c(w',q) + (1/2)c(w'',q). \end{aligned}$$

Multiplying both sides of this inequality with 2 yields the desired result.

ii) The answer to this question is a qualified 'Yes'. First observe that if f is continuous then for any q the constraint $f(z) \geq q$ must hold with equality at the cost-minimal input combination (if for some q we have $f(z) > q$, then by continuity there must be a $t \in (0,1)$ such that $f(tz) > q$; but then since $w(tz) < wz$ it follows that z cannot be the cost-minimal input combination for q). Now assume that in a neighborhood of the point $(\bar{w}, \bar{q})$ the cost-minimization problem has a unique interior solution, $z(q,w)$, which is differentiable in q. Then, since $c(q,w) = wz(q,w)$, we have

$$\frac{\partial c(\bar{q}, \bar{w}}{\partial q} = \sum_l \bar{w} \frac{\partial z_l(\bar{q}, \bar{w})}{\partial q}.$$

Given that $z(\bar{w}, \bar{q}) > 0$ we know that for all l the FOC

$$w_l = \lambda(\bar{w}, \bar{q}) \frac{f(z(\bar{q}, \bar{w}))}{z_l}.$$

must be satisfied. Moreover, since $q = f(z(q,w))$ for all (w,q) we know that

$$\sum_l \bar{w} \frac{\partial f(z(\bar{q}, \bar{w}))}{\partial z_l} \frac{\partial z_l(\bar{q}, \bar{w})}{\partial q} = 1.$$

Combining these conditions we get

$$\sum_l \lambda(\bar{w}, \bar{q}) \frac{\partial f(z(\bar{q}, \bar{w}))}{\partial z_l} \frac{\partial z_l(\bar{q}, \bar{w})}{\partial q} = \lambda(\bar{w}, \bar{q}).$$

Problem 3: For the statements which are compatible with cost minimization we give examples of production functions for which the statements are true at least for some values of factor prices and output levels.

i) The conditions $\partial z_2(q,w)/\partial w_1 > 0$ and $\partial z_3(q,w)/\partial w_1 > 0$ tell us that the input factors two and three are both substitutes of input factor 1. A production function which delivers factor demand functions of this type is $f(z_1,z_2,z_3) = z_1 z_2 z_3$.

ii) The conditions $\partial z_2(q,w)/\partial w_1 > 0$ and $\partial z_3(q,w)/\partial w_1 < 0$ tell us that input factor 1 is a complement of input factor 3 and a substitute of input factor 2. It is easily verified that the following production function produces conditional input demand functions with this property: $f(z_1,z_2,z_3) = \min\{z_1,z_3\}z_2$.

iii) This property can never be satisfied. In order to see this, consider an increase of the output from q to $q' > q$; moreover, let z and z' be the optimal input combinations for the two output levels. Since $f(z') \geq q' > q$ it follows that z' is an input combination which belongs to the feasible set of the cost minimization problem for the output level q. This implies $c(q',w) - c(q,w) = wz' - wz = w(z'-z) \geq 0$. But this of course means that we cannot have $z'_l < z_l$ for all l.

iv) Take $f(z_1,z_2,z_3) = \sum_l z_l$. For this production function we have that $z_1(q,w) = 0$ for all q whenever $w_1 > w_2, w_3$.

v) Take $f(z_1,z_2,z_3) = \min\{z_1,z_2,z_3\}$. The cost and conditional factor demand functions are in this case $c(q,w) = (w_1+w_2+w_3)q$ and $z_l(q,w) = q$. Thus $z_1(q,w)/z_2(q,w) = 1$ for all (q,w).

Problem 4:

i) By the law of supply the (net) supply of any good reacts positively to an increase of its own price. Thus, the demand for an input must decrease as its own price increases.

ii) Denote the quantity and price of unskilled labor by z_u and w_u respectively. Moreover, denote the output quantity and the output price by q and p, respectively. Then, if the factor demand is a differentiable function we have that

$$\frac{\partial q}{\partial w_u} = \frac{\partial}{\partial w_u}\left[\frac{\partial \pi}{\partial p}\right] = \frac{\partial}{\partial p}\left[\frac{\partial \pi}{\partial w_u}\right] = -\frac{\partial z_u}{\partial p} > 0,$$

where the first and the third equality signs follow from Hotelling's Lemma while the second one is implied by the symmetry of the (supply) substitution matrix.

Problem 5: Remember that a cost function is the mathematical analogue of an expenditure function. Thus, if we want to calculate production functions which produce the given cost functions we simply have to follow the same procedure which we have applied in Problem 5 of Problem set 4, where we have seen how to obtain a utility function from an expenditure function.

We will not spell out here all the details once more, but simply state the results. You should try to get to those results by yourself!!!

i) $f(z_1, z_2) = Bz_1^{\alpha} z_2^(1 - \alpha)$, where $B = 1/(A\alpha^{\alpha}(1 - \alpha)^(1 - \alpha)$;

ii) $f(z_1, z_2) = z_1 + z_2$;

iii) $f(z_1, z_2) = \min\{z_1, z_2\}$.

Problem 6: The optimization problem of the firm is given by

$$\max_{z \geq 0} f(z) \qquad \text{s.t.} \qquad pf(z) - wz \geq 0.$$

The solution set and the value function for this problem are denoted by $z(p, w)$ and $q(p, w)$, respectively. Notice that if the production function exhibits non-decreasing returns to this problem will either be trivial (i.e. the only feasible point and thus the solution is 0) or it will have no solution at all (if for given (p, w) the profit is non-negative for some z then it will be non-negative along the whole ray λz, $\lambda > 1$). Thus, in what follows we will assume that the production function is such that the feasible set of our optimization problem, $Z(p, w) = \{z : pf(z) - wz \geq 0\}$,is bounded for any (p, w). If we also assume that f is continuous and that $f(0) = 0$ then $Z(p, w)$ is non-empty ($0 \in Z(p, w)$ for all (p, w)) and closed (by continuity of $pf(z) - wz$). Together these assumptions guarantee the existence of a solution for our optimization problem. Finally, it is also convenient to assume that f is strongly monotonic in z (that is, if $z' \geq z$ and $z' \neq z$ then $f(z') > f(z)$).

Given all these assumptions we can show that $z(p, w)$ and $q(p, w)$ satisfy the following properties:

i) $q(p, w)$ is continuous and $z(p, w)$ is upper hemi-continuous;

ii) $q(p, w)$ and $z(p, w)$ are homogeneous of degree zero in (p, w);

iii) if $q(p, w) > 0$ and $p' > p$ then $q(p', w) > q(p, w)$. Moreover, q is non-increasing in w_l, $l = 1, \ldots, L$;

iv) $z_l(p, w)$ is non-increasing in w_l;

v) if f is concave then $z(p, w)$ is convex for all (p, w); moreover, if f is strictly concave, then $z(p, w)$ is a continuous function;

vi) $q(p, w)$ is quasi-convex in (p, w);

vii) under appropriate uniqueness and differentiability assumptions we have

$$z_l(p, w) = \frac{q(p, w)[\partial q(p, w)/\partial w_l]}{\partial q(p, w)/\partial p}.$$

Proof:

i) In order to show continuity of q we can rely on Berge's Theorem of the Maximum. This theorem requires that the objective function is continuous and that the correspondence which describes the feasible set for any parameter, i.e. in our case the correspondence $(p; w) \to Z(p, w)$, is continuous. Both these conditions are satisfied in the case of our optimization problem. Since we have never introduced the notions of upper hemi-continuity or continuity for correspondences we will not verify this in detail.

ii) As for homogeneity of degree zero of q and z, simply observe that the feasible set of the optimization problem does not change if prices are scaled by the same strictly positive factor, i.e. $Z(p, w) = Z(\lambda p, \lambda w)$ for all $\lambda > 0$. Since the objective function does not depend on the parameters (p, w) this implies that the solution set and the value of the maximization problem cannot change with λ.

iii) We first argue that monotonicity of f implies that $pf(z) - wz = 0$ for all $z \in z(p, w)$. Suppose that $pf(z) - wz > 0$ for some (p, w) and $z \in z(p, w)$. Then by continuity of the profit there exists a neighborhood around z such that any point belonging to this neighborhood satisfies this inequality too (that is any point of the neighborhood would be feasible). Such a neighborhood contains also points $z' \neq z$ such that $z' \geq z$. But then $f(z') > f(z)$, which contradicts the assumption that z is an optimal point.

Now suppose that $q(p, w) > 0$ and $p' > p$. Then $p'f(z) - wz > pf(z) - wz = 0$ for all $z \in z(p, w)$. Thus, for any $z \in z(p, w)$ there exists a $t > 1$ such that $p'f(tz) - w(tz) \geq 0$ (i.e. $tz \in Z(p', w)$). But then $\max_{z' \in Z(p', w)} f(z') \geq f(tz) > f(z) = q(p, w)$.

That q is non-increasing in w_l can be shown by an analogous argument.

iv) Let w and w' be two input price vectors which differ only in their l-th component; in particular, let $w'_l > w_l$. Let z and z' be two optimal input vectors for these two prices. Then $pf(z) - wz = pf(z') - w'z = 0$, and thus $p(f(z) - f(z')) - (wz - w'z') = p(f(z) - f(z')) - (w_l z_l - w'_l z'_l) = 0$. We have argued earlier that the output is non-increasing in factor prices. Thus $p(f(z) - f(z')) \geq 0$. But then we cannot have $z'_l > z_l$ since that would imply $p(f(z) - f(z')) - (w_l z_l - w'_l z'_l) > 0$.

v) We first show that concavity of f implies that $Z(p, w)$ is convex for all (p, w). So assume that f is concave. Moreover, let $z, z' \in Z(p, w)$ and write $z(\lambda) = \lambda z + (1 - \lambda)z'$. Then, we have

$$\begin{aligned} pf(z(\lambda)) - wz(\lambda) &\geq p(\lambda f(z) + (1 - \lambda)f(z')) - \lambda wz - (1 - \lambda)wz' \\ &= \lambda(pf(z) - wz) + (1 - \lambda)(pf(z') - wz') \geq 0 \end{aligned}$$

and so $z(\lambda) \in Z(p, w)$. Since f is concave it is also quasi-concave. But if we maximize a function with convex upper contour sets over a convex set, then the solution set must be convex. Formally: $z, z' \in z(p, w)$ implies that $z, z' \in Z(p, w)$ and thus $z(\lambda) \in Z(p, w)$. But by quasi-concavity (or concavity) $f(z(\lambda)) \geq f(z) = f(z')$ and thus $z(\lambda) \in z(p, w)$.

Suppose f is strictly concave and that $z(p, w)$ is not a singleton. Then, if $z, z' \in z(p, w)$, $z \neq z'$, by strict concavity we have that $pf(z(\lambda)) - wz(\lambda) > pf(z) - wz = pf(z') - wz' = 0$ for all $\lambda \in (0, 1)$. Thus, $z(\lambda)$ is an interior point of $Z(p, w)$. But then (by continuity) there is a $t > 1$ such that $tz(\lambda) \in Z(p, w)$. Since $f(tz(\lambda)) > f(z(\lambda)) > f(z) = f(z')$ we get a contradiction with our assumption that z and z' are optimal at (p, w).

Continuity of $z(\cdot, \cdot)$ follows again from Berge's Theorem of the maximum.

vi) We have to show that for any two pairs (p, w) and (p', w') we have

$$q(p(\lambda), w(\lambda)) \leq \max\{q(p, w), q(p', w')\}.$$

In order to show this we just have to prove that any $z \in Z(p(\lambda), w(\lambda))$ must belong either to $Z(p, w)$ or $Z(p', w')$. If that is the case then, the maximum which can be reached when we have to choose from $Z(p(\lambda), w(\lambda))$ cannot be larger than the larger one among the maxima which can be reached when choosing from $Z(p, w)$ and $Z(p', w')$. So assume that $z \in Z(p(\lambda), w(\lambda))$. Then

$$\begin{aligned} 0 \leq p(\lambda) f(z) - w(\lambda) z &= (\lambda p + (1-\lambda) p') f(z) - (\lambda w + (1-\lambda) w') z \\ &= \lambda (p f(z) - w z) + (1-\lambda)(p' f(z) - w' z). \end{aligned}$$

The sum after the second equality sign can be non-negative only if at least one of its components is non-negative. But that means that either $z \in Z(p, w)$ or $z \in Z(p', w')$.