Exercise 3:

(a) $$Y_2 \begin{cases} Y_1 + \varepsilon & \text{w/ prob. } 1/2 \\ Y_1 - \varepsilon & \text{w/ prob. } 1/2 \end{cases}$$

$$E[Y_2] = 1/2(Y_1 + \varepsilon) + 1/2(Y_1 - \varepsilon) = Y_1$$

Budget constraint :

$$C_1 = Y_1 - B_2$$
$$C_2 = Y_2 + B_2(1+r)$$

Utility ~~function~~ function

$$E[U] = \ln(C_1) + \beta E[C_2]$$

$$E[C_2] = E[Y_2 + B_2(1+r)]$$
$$= Y_1 + B_2(1+r)$$

It is simplest to plug the constraints into the utility function to solve for B_2

$$E[U] = \ln(Y_1 - B_2) + \beta(Y_1 + B_2(1+r))$$

$$\frac{\partial U}{\partial B_2} = 0 \implies \frac{1}{Y_1 - B_2} = \beta(1+r)$$

$$\underline{B_2 = Y_1 - 1}$$

(b) Now we have that

$$E[u] = \ln(C_1) + \beta E[\ln(C_2)]$$

$$= \ln(C_1) + \tfrac{1}{2}\beta\ln(Y_1 + \varepsilon + (1+r)B_2) + \tfrac{1}{2}\beta\ln(Y_1 - \varepsilon + (1+r)B_2)$$

Taking the FOC w.r.t. B_2
(recalling that $C_1 = Y_1 - B_2$)

$$\frac{\partial u}{\partial B_2} = 0 \implies \frac{1}{Y_1 - B_2} = \frac{\beta}{2}\left(\frac{1}{Y_1 + \varepsilon + (1+r)B_2} + \frac{1}{Y_1 - \varepsilon + (1+r)B_2}\right)$$

Now let's assume that $\beta = 1$ and $r = 0$
(this follows from the setup)

$$\frac{1}{Y_1 - B_2} = \frac{1}{2}\left(\frac{1}{Y_1 + \varepsilon + B_2} + \frac{1}{Y_1 - \varepsilon + B_2}\right)$$

$$\frac{Y_1 + \varepsilon + B_2}{Y_1 - B_2} = \frac{1}{2}\left(1 + \frac{Y_1 + \varepsilon + B_2}{Y_1 - \varepsilon + B_2}\right)$$

$$= \frac{1}{\cancel{2}}\left(\cancel{2} + \frac{\cancel{2}\varepsilon}{Y_1 - \varepsilon + B_2}\right)$$

$$= \frac{Y_1 + B_2}{Y_1 - \varepsilon + B_2}$$

$$(Y_1 + B_2)^2 - \varepsilon^2 = (Y_1 + B_2)(Y_1 - B_2)$$

$$\cancel{Y_1^2} + 2Y_1 B_2 + B_2^2 - \varepsilon^2 = \cancel{Y_1^2} - B_2^2$$

$$B_2(B_2 + Y_1) = \frac{\varepsilon^2}{2}$$

If $\varepsilon = 0 \implies B_2 = 0$

If $\varepsilon \neq 0 \implies B_2 > 0 \quad (?)$