Macroeconomics A; EI056

Short problems

Cédric Tille

Class of November 28, 2023

1 Debt dynamics

1.1 Flow constraints

Question: In nominal terms, the government spends G_t to purchase goods and services, and pays an interest rate i_t on its debt B_t. This is finance by taxes T_t or new debt, $B_{t+1} - B_t$:

$$G_t + i_t B_t = T_t + B_{t+1} - B_t$$

Nominal GDP Y_t grows at a rate $g_t = Y_t / Y_{t-1} - 1$. The primary deficit is $D_t^{\text{prim}} = G_t - T_t$.

Show that the flow constraint in terms of variables scaled by GDP (for example $d_t^{\text{prim}} = D_t^{\text{prim}} / Y_t$) is:

$$b_{t+1} - b_t = \frac{d_t^{\text{prim}}}{1 + g_{t+1}} + \frac{i_t - g_{t+1}}{1 + g_{t+1}} b_t$$

Answer: The flow budget constraint is written as follows:

$$\begin{aligned}
G_t + i_t B_t &= T_t + B_{t+1} - B_t \\
\frac{G_t}{Y_t} + i_t \frac{B_t}{Y_t} &= \frac{T_t}{Y_t} + \frac{B_{t+1}}{Y_{t+1}} \frac{Y_{t+1}}{Y_t} - \frac{B_t}{Y_t} \\
\frac{G_t - T_t}{Y_t} + i_t \frac{B_t}{Y_t} &= \frac{B_{t+1}}{Y_{t+1}} (1 + g_{t+1}) - \frac{B_t}{Y_t} \\
d_t^{\text{prim}} + i_t b_t &= b_{t+1} (1 + g_{t+1}) - b_t \\
d_t^{\text{prim}} + i_t b_t - g_{t+1} b_t + g_{t+1} b_t &= b_{t+1} (1 + g_{t+1}) - b_t \\
d_t^{\text{prim}} + (i_t - g_{t+1}) b_t &= (1 + g_{t+1}) (b_{t+1} - b_t) \\
b_{t+1} - b_t &= \frac{d_t^{\text{prim}}}{1 + g_{t+1}} + \frac{i_t - g_{t+1}}{1 + g_{t+1}} b_t
\end{aligned}$$

1.2 Steady state

Question: Consider that all variables are constant. Is a primary surplus, $d^{\text{prim}} < 0$, needed?

Answer: If all variables are constant, we get:

$$\begin{aligned}
b-b &= \frac{d^{\text{prim}}}{1+g}+\frac{i-g}{1+g}b \\
0 &= d^{\text{prim}}+(i-g)\,b \\
d^{\text{prim}} &= -(i-g)\,b
\end{aligned}$$

If $i>g$ debt requires a primary surplus ($d^{\text{prim}}<0$) to be kept constant. If $i<g$ one can run a deficit.

1.3 Intertemporal constraint

Question: Consider that i_t and g_t are constant. Show that:

$$b_t=\frac{1}{1+i}\sum_{s=0}^{\infty}\left(\frac{1+g}{1+i}\right)^{s}\left(-d_{t+s}^{\text{prim}}\right)+\lim_{k\to\infty}\left(\frac{1+g}{1+i}\right)^{k}b_{t+k+1}$$

Do we have the usual transversality condition, as in last week?

Answer: We first derive the expression in general terms. The flow constraint is:

$$\begin{aligned}
b_{t+1}-b_t &= \frac{d_t^{\text{prim}}}{1+g_{t+1}}+\frac{i_t-g_{t+1}}{1+g_{t+1}}b_t \\
b_{t+1} &= \frac{d_t^{\text{prim}}}{1+g_{t+1}}+\frac{1+i_t}{1+g_{t+1}}b_t \\
\frac{1+i_t}{1+g_{t+1}}b_t &= b_{t+1}-\frac{d_t^{\text{prim}}}{1+g_{t+1}} \\
b_t &= \frac{1+g_{t+1}}{1+i_t}b_{t+1}-\frac{1}{1+i_t}d_t^{\text{prim}}
\end{aligned}$$

Iterate forward:

$$\begin{aligned}
b_t &= \frac{1+g_{t+1}}{1+i_t}b_{t+1}-\frac{1}{1+i_t}d_t^{\text{prim}} \\
b_t &= -\frac{1}{1+i_t}d_t^{\text{prim}}+\frac{1+g_{t+1}}{1+i_t}\left[-\frac{1}{1+i_{t+1}}d_{t+1}^{\text{prim}}+\frac{1+g_{t+2}}{1+i_{t+1}}b_{t+2}\right] \\
b_t &= -\frac{1}{1+i_t}d_t^{\text{prim}}-\frac{1}{1+i_{t+1}}\frac{1+g_{t+1}}{1+i_t}d_{t+1}^{\text{prim}}+\frac{1+g_{t+1}}{1+i_t}\frac{1+g_{t+2}}{1+i_{t+1}}b_{t+2} \\
b_t &= -\frac{1}{1+i_t}d_t^{\text{prim}}-\frac{1}{1+i_{t+1}}\frac{1+g_{t+1}}{1+i_t}d_{t+1}^{\text{prim}}-\frac{1+g_{t+1}}{1+i_t}\frac{1+g_{t+2}}{1+i_{t+1}}\frac{1}{1+i_{t+2}}d_{t+2}^{\text{prim}} \\
&\quad+\frac{1+g_{t+1}}{1+i_t}\frac{1+g_{t+2}}{1+i_{t+1}}\frac{1+g_{t+3}}{1+i_{t+2}}b_{t+3} \\
b_t &= -\frac{1}{1+i_t}d_t^{\text{prim}}-\frac{1}{1+i_{t+1}}R_{t,t}^{g-i}d_{t+1}^{\text{prim}}-\frac{1}{1+i_{t+2}}R_{t,t+1}^{g-i}d_{t+2}^{\text{prim}}+R_{t,t+2}^{g-i}b_{t+3}
\end{aligned}$$

where $R_{t,t+s}^{g-i}=\frac{1+g_{t+1}}{1+i_t}\frac{1+g_{t+2}}{1+i_{t+1}}...\frac{1+g_{t+s+1}}{1+i_{t+s}}=\prod_{i=0}^{s}\frac{1+g_{t+s+1}}{1+i_{t+s}}$. If we take the limit going to infinity, we get:

$$b_t = -\sum_{s=0}^{\infty}\frac{1}{1+i_{t+s+1}}R_{t,t+s}^{g-i}d_{t+s}^{\text{prim}}+\lim_{k\to\infty}R_{t,t+k}^{g-i}b_{t+k+1}$$

With constant i and g we have $R^{g-i}_{t,t+s} = \frac{1+g_{t+1}}{1+i_t}\frac{1+g_{t+2}}{1+i_{t+1}}...\frac{1+g_{t+s+1}}{1+i_{t+s}} = \left(\frac{1+g}{1+i}\right)^s$ and:

$$\begin{aligned}
b_t &= -\sum_{s=0}^{\infty}\frac{1}{1+i}\left(\frac{1+g}{1+i}\right)^s d^{\text{prim}}_{t+s} + \lim_{k\to\infty}\left(\frac{1+g}{1+i}\right)^k b_{t+k+1} \\
b_t &= \frac{1}{1+i}\sum_{s=0}^{\infty}\left(\frac{1+g}{1+i}\right)^s \left(-d^{\text{prim}}_{t+s}\right) + \lim_{k\to\infty}\left(\frac{1+g}{1+i}\right)^k b_{t+k+1}
\end{aligned}$$

If $g < i$, $\frac{1+g}{1+i} < 1$ and therefore the weight on the final debt, and deficits far in the future, becomes negligible. In that case, we get the usual transversality condition. If $g > i$ however $\left(\frac{1+g}{1+i}\right)^k$ gets bigger the longer the horizon k, and we don't have the transversality condition.

1.4 Alternative discount

Question: Show that we can rewrite the flow budget constraint as:

$$b_{t+1} - b_t = \frac{d^{\text{prim}}_t}{1+g_{t+1}} + \frac{i_t - m_t}{1+g_{t+1}}b_t + \frac{m_t - g_{t+1}}{1+g_{t+1}}b_t$$

where m_t is the private sector budget discount factor, that we can think of as the rate of return on private capital. We assume that this rate of return is higher than the GDP growth rate.

Assuming that i_t, g_t and m_t are constant, show that we get

$$b_t = \frac{1}{1+m}\sum_{s=0}^{\infty}\left(\frac{1+g}{1+m}\right)^s \left[-d^{\text{prim}}_{t+s} + (m-i)\,b_{t+s}\right] + \lim_{k\to\infty}\left(\frac{1+g}{1+m}\right)^k b_{t+k+1}$$

Do we have the usual transversality condition, as in last week? Is there a new source of income for the government.

Answer: Start from the flow budget constraint:

$$\begin{aligned}
d^{\text{prim}}_t + i_t b_t &= b_{t+1}\left(1+g_{t+1}\right) - b_t \\
d^{\text{prim}}_t + \left(i_t - m_t + m_t\right) b_t &= b_{t+1}\left(1+g_{t+1}\right) - b_t \\
d^{\text{prim}}_t + \left(i_t - m_t\right) b_t + m_t b_t &= b_{t+1}\left(1+g_{t+1}\right) - b_t \\
d^{\text{prim}}_t + \left(i_t - m_t\right) b_t + \left(m_t - g_{t+1}\right) b_t + g_{t+1} b_t &= b_{t+1}\left(1+g_{t+1}\right) - b_t \\
d^{\text{prim}}_t + \left(i_t - m_t\right) b_t + \left(m_t - g_{t+1}\right) b_t &= \left(1+g_{t+1}\right)\left(b_{t+1} - b_t\right) \\
b_{t+1} - b_t &= \frac{d^{\text{prim}}_t}{1+g_{t+1}} + \frac{i_t - m_t}{1+g_{t+1}}b_t + \frac{m_t - g_{t+1}}{1+g_{t+1}}b_t
\end{aligned}$$

We now follow similar steps as in the previous question.

$$\begin{aligned}
b_{t+1} - b_t &= \frac{d^{\text{prim}}_t}{1+g_{t+1}} + \frac{i_t - m_t}{1+g_{t+1}}b_t + \frac{m_t - g_{t+1}}{1+g_{t+1}}b_t \\
b_{t+1} &= \frac{d^{\text{prim}}_t}{1+g_{t+1}} + \frac{i_t - m_t}{1+g_{t+1}}b_t + \frac{1 + m_t}{1+g_{t+1}}b_t \\
\frac{1+m_t}{1+g_{t+1}}b_t &= b_{t+1} - \frac{d^{\text{prim}}_t}{1+g_{t+1}} - \frac{i_t - m_t}{1+g_{t+1}}b_t \\
b_t &= \frac{1+g_{t+1}}{1+m_t}b_{t+1} - \frac{1}{1+m_t}\left[d^{\text{prim}}_t + \left(i_t - m_t\right) b_t\right]
\end{aligned}$$

Iterate forward:

$$
\begin{aligned}
b_t &= \frac{1+g_{t+1}}{1+m_t} b_{t+1} - \frac{1}{1+m_t}\left[d_t^{\text{prim}} + (i_t - m_t)\, b_t\right] \\
b_t &= -\frac{1}{1+m_t}\left[d_t^{\text{prim}} + (i_t - m_t)\, b_t\right] + \frac{1+g_{t+1}}{1+m_t}\left[-\frac{1}{1+m_{t+1}}\left[d_{t+1}^{\text{prim}} + (i_{t+1} - m_{t+1})\, b_{t+1}\right] + \frac{1+g_{t+2}}{1+m_{t+1}} b_{t+2}\right] \\
b_t &= -\frac{1}{1+m_t}\left[d_t^{\text{prim}} + (i_t - m_t)\, b_t\right] - \frac{1}{1+m_{t+1}}\frac{1+g_{t+1}}{1+m_t}\left[d_{t+1}^{\text{prim}} + (i_{t+1} - m_{t+1})\, b_{t+1}\right] \\
&\quad + \frac{1+g_{t+1}}{1+m_t}\frac{1+g_{t+2}}{1+m_{t+1}} b_{t+2} \\
b_t &= -\frac{1}{1+m_t}\left[d_t^{\text{prim}} + (i_t - m_t)\, b_t\right] - \frac{1}{1+m_{t+1}}\frac{1+g_{t+1}}{1+m_t}\left[d_{t+1}^{\text{prim}} + (i_{t+1} - m_{t+1})\, b_{t+1}\right] \\
&\quad - \frac{1+g_{t+1}}{1+m_t}\frac{1+g_{t+2}}{1+m_{t+1}}\frac{1}{1+m_{t+2}}\left[d_{t+2}^{\text{prim}} + (i_{t+2} - m_{t+2})\, b_{t+2}\right] \\
&\quad + \frac{1+g_{t+1}}{1+m_t}\frac{1+g_{t+2}}{1+m_{t+1}}\frac{1+g_{t+3}}{1+m_{t+2}} b_{t+3} \\
b_t &= -\frac{1}{1+m_t}\left[d_t^{\text{prim}} + (i_t - m_t)\, b_t\right] - \frac{1}{1+m_{t+1}} R_{t,t}^{g-m}\left[d_{t+1}^{\text{prim}} + (i_{t+1} - m_{t+1})\, b_{t+1}\right] \\
&\quad - \frac{1}{1+m_{t+2}} R_{t,t+1}^{g-m}\left[d_{t+2}^{\text{prim}} + (i_{t+2} - m_{t+2})\, b_{t+2}\right] + R_{t,t+2}^{g-m} b_{t+3}
\end{aligned}
$$

where $R_{t,t+s}^{g-m} = \prod_{i=0}^{s} \frac{1+g_{t+s+1}}{1+m_{t+s}}$. If we take the limit going to infinity, we get:

$$
b_t = -\sum_{s=0}^{\infty} \frac{1}{1+m_{t+s+1}} R_{t,t+s}^{g-m}\left[d_{t+s}^{\text{prim}} + (i_{t+s} - m_{t+s})\, b_{t+s}\right] + \lim_{k\to\infty} R_{t,t+k}^{g-m} b_{t+k+1}
$$

With constant i, g, and m we have $R_{t,t+s}^{g-m} = \left(\frac{1+g}{1+m}\right)^s$ and:

$$
\begin{aligned}
b_t &= -\sum_{s=0}^{\infty} \frac{1}{1+m}\left(\frac{1+g}{1+m}\right)^s \left[d_{t+s}^{\text{prim}} + (i-m)\, b_{t+s}\right] + \lim_{k\to\infty}\left(\frac{1+g}{1+m}\right)^k b_{t+k+1} \\
b_t &= \frac{1}{1+m}\sum_{s=0}^{\infty}\left(\frac{1+g}{1+m}\right)^s \left[-d_{t+s}^{\text{prim}} + (m-i)\, b_{t+s}\right] + \lim_{k\to\infty}\left(\frac{1+g}{1+m}\right)^k b_{t+k+1}
\end{aligned}
$$

As $m > g$ the ration $\left(\frac{1+g}{1+m}\right)^k$ converges to zero, and we get the usual transversality condition:

$$
b_t = \frac{1}{1+m}\sum_{s=0}^{\infty}\left(\frac{1+g}{1+m}\right)^s \left[-d_{t+s}^{\text{prim}} + (m-i)\, b_{t+s}\right]
$$

The initial debt has to be offset by primary surpluses ($d_{t+s}^{\text{prim}} < 0$) or by the spread difference between private capital and government bonds, $m > i$, times the size of public debt. This is the "debt revenue" of Reis.

2 Political economy of deficit

2.1 Preferences, and final choice

Question: Consider the model of heterogeneous preferences seen in class. We have two periods. In each there is an endowment W. Government spending is used to purchases two types of goods,

M and N. The government's budget constraints in the two periods are (B denotes debt):

$$\begin{aligned} M_1 + N_1 &= W + B \\ M_2 + N_2 &= W - B \end{aligned}$$

Individuals are different depending on their preferences for the two types of goods the government can purchase. Specifically, the utility of individual i is:

$$V_i = E \sum_{t=1}^{2} \left[\alpha_i ln \left(M_t \right) + \left(1 - \alpha_i \right) ln \left(N_t \right) \right]$$

Individuals differ according to their preference for good M which is captured by the weight $\alpha_i \in [0, 1]$.

Show that in the second period:

$$M_2 = \alpha_{med,2} \left(W - B \right) \qquad ; \qquad N_2 = \left(1 - \alpha_{med,2} \right) \left(W - B \right)$$

Answer: In the second period, the median voter maximizes

$$\alpha_{med,2} ln \left(M_2 \right) + \left(1 - \alpha_{med,2} \right) ln \left(N_2 \right) = \alpha_{med,2} ln \left(M_2 \right) + \left(1 - \alpha_{med,2} \right) ln \left(W - B - M_2 \right)$$

The first-order condition with respect to M_2 is:

$$\begin{aligned} 0 &= \alpha_{med,2} \frac{1}{M_2} - \left(1 - \alpha_{med,2} \right) \frac{1}{W - B - M_2} \\ \alpha_{med,2} \left(W - B - M_2 \right) &= \left(1 - \alpha_{med,2} \right) M_2 \\ M_2 &= \alpha_{med,2} \left(W - B \right) \end{aligned}$$

which implies:

$$\begin{aligned} N_2 &= W - B - M_2 \\ N_2 &= \left(1 - \alpha_{med,2} \right) \left(W - B \right) \end{aligned}$$

2.2 Initial choice

Question: In the first period, the median voter maximizes:

$$\begin{aligned} & E \sum_{t=1}^{2} \left[\alpha_{med,1} ln \left(M_t \right) + \left(1 - \alpha_{med,1} \right) ln \left(N_t \right) \right] \\ &= \alpha_{med,1} ln \left(M_1 \right) + \left(1 - \alpha_{med,1} \right) ln \left(N_1 \right) + E \left[\alpha_{med,1} ln \left(M_2 \right) + \left(1 - \alpha_{med,1} \right) ln \left(N_2 \right) \right] \end{aligned}$$

Show that this utility is:

$$\begin{aligned} & \alpha_{med,1} ln \left(M_1 \right) + \left(1 - \alpha_{med,1} \right) ln \left(W + B - M_1 \right) + ln \left(W - B \right) \\ & + \alpha_{med,1} E ln \left(\alpha_{med,2} \right) + \left(1 - \alpha_{med,1} \right) E ln \left(1 - \alpha_{med,2} \right) \end{aligned}$$

Show that the optimal choice for M_1 is:

$$M_1 = \alpha_{med,1} \left(W + B \right) \qquad ; \qquad N_1 = \left(1 - \alpha_{med,1} \right) \left(W + B \right)$$

Answer: In the first period, the median voter maximizes:

$$
\begin{aligned}
&E\sum_{t=1}^{2}\left[\alpha_{med,1}ln\left(M_t\right)+\left(1-\alpha_{med,1}\right)ln\left(N_t\right)\right]\\
=&\alpha_{med,1}ln\left(M_1\right)+\left(1-\alpha_{med,1}\right)ln\left(N_1\right)+E\left[\alpha_{med,1}ln\left(M_2\right)+\left(1-\alpha_{med,1}\right)ln\left(N_2\right)\right]
\end{aligned}
$$

Using the results for period 2, this becomes:

$$
\begin{aligned}
&\alpha_{med,1}ln\left(M_1\right)+\left(1-\alpha_{med,1}\right)ln\left(N_1\right)+\alpha_{med,1}Eln\left(M_2\right)+\left(1-\alpha_{med,1}\right)Eln\left(N_2\right)\\
=&\alpha_{med,1}ln\left(M_1\right)+\left(1-\alpha_{med,1}\right)ln\left(N_1\right)+\alpha_{med,1}Eln\left(\alpha_{med,2}\left(W-B\right)\right)+\left(1-\alpha_{med,1}\right)Eln\left(\left(1-\alpha_{med,2}\right)\left(W-B\right)\right)\\
=&\alpha_{med,1}ln\left(M_1\right)+\left(1-\alpha_{med,1}\right)ln\left(W+B-M_1\right)+ln\left(W-B\right)\\
&+\alpha_{med,1}Eln\left(\alpha_{med,2}\right)+\left(1-\alpha_{med,1}\right)Eln\left(1-\alpha_{med,2}\right)
\end{aligned}
$$

The first-order condition with respect to M_1 is:

$$
\begin{aligned}
0 &= \alpha_{med,1}\frac{1}{M_1}-\left(1-\alpha_{med,1}\right)\frac{1}{W+B-M_1}\\
\alpha_{med,1}\left(W+B-M_1\right) &= \left(1-\alpha_{med,1}\right)M_1\\
M_1 &= \alpha_{med,1}\left(W+B\right)
\end{aligned}
$$

which implies:

$$N_1=W+B-M_1=\left(1-\alpha_{med,1}\right)\left(W+B\right)$$

2.3 Intertemporal choice

Question: Using the results, show that the utility is:

$$
\begin{aligned}
&ln\left(W+B\right)+ln\left(W-B\right)+\alpha_{med,1}ln\left(\alpha_{med,1}\right)+\left(1-\alpha_{med,1}\right)ln\left(1-\alpha_{med,1}\right)\\
&+\alpha_{med,1}Eln\left(\alpha_{med,2}\right)+\left(1-\alpha_{med,1}\right)Eln\left(1-\alpha_{med,2}\right)
\end{aligned}
$$

What is the optimal debt level B?

Answer: The intertemporal utility is:

$$
\begin{aligned}
&\alpha_{med,1}ln\left(M_1\right)+\left(1-\alpha_{med,1}\right)ln\left(W+B-M_1\right)+ln\left(W-B\right)\\
&+\alpha_{med,1}Eln\left(\alpha_{med,2}\right)+\left(1-\alpha_{med,1}\right)Eln\left(1-\alpha_{med,2}\right)\\
=&\alpha_{med,1}ln\left(\alpha_{med,1}\left(W+B\right)\right)+\left(1-\alpha_{med,1}\right)ln\left(\left(1-\alpha_{med,1}\right)\left(W+B\right)\right)+ln\left(W-B\right)\\
&+\alpha_{med,1}Eln\left(\alpha_{med,2}\right)+\left(1-\alpha_{med,1}\right)Eln\left(1-\alpha_{med,2}\right)\\
=&\alpha_{med,1}ln\left(W+B\right)+\left(1-\alpha_{med,1}\right)ln\left(W+B\right)+ln\left(W-B\right)\\
&+\alpha_{med,1}ln\left(\alpha_{med,1}\left(W+\right)\right)+\left(1-\alpha_{med,1}\right)ln\left(1-\alpha_{med,1}\right)\\
&+\alpha_{med,1}Eln\left(\alpha_{med,2}\right)+\left(1-\alpha_{med,1}\right)Eln\left(1-\alpha_{med,2}\right)\\
=&ln\left(W+B\right)+ln\left(W-B\right)+\alpha_{med,1}ln\left(\alpha_{med,1}\right)+\left(1-\alpha_{med,1}\right)ln\left(1-\alpha_{med,1}\right)\\
&+\alpha_{med,1}Eln\left(\alpha_{med,2}\right)+\left(1-\alpha_{med,1}\right)Eln\left(1-\alpha_{med,2}\right)
\end{aligned}
$$

The first-order condition with respect to B is:

$$\begin{aligned} 0 &= \frac{1}{W+B} - \frac{1}{W-B} \\ W-B &= W+B \\ 0 &= 2B \\ B &= 0 \end{aligned}$$

There is thus no debt, hence no deficit problem. This is because the motive to starve the future government (which may have different preferences from the current one) of resources is exactly offset by the need to ensure that it spends enough on the goods that the current government cares about.