Lecture 9: Calibration and Counterfactuals in Gravity Trade Models

Treb Allen and Costas Arkolakis

Spring 2015

Northwestern ECON 460: Graduate International Trade

1 Introduction

In this lecture, we (finally!) turn to the data. In particular, suppose that we observe bilateral trade flows between a set of countries. Given this data, we would like to ask what the model predicts would happen if something changed in the world (e.g. there is a drop in trade costs between two countries), i.e. we would like to be able to use the model to perform *counterfactuals*. In order to answer this question, however, we first need to know several things. Is it possible for one to find a set of model parameters so that the model is consistent with the data we observe? If so, is there only one set of data-consistent model parameters or are there many? These two questions are the basis of the study of the *identification* of gravity trade models. Given answers to these questions, we can then ask how one can find model parameters that are consistent with the data we observe, i.e. how can we *calibrate* the model to the data. This is the focus of this lecture, which is based on the papers Eaton and Kortum (2002), Dekle, Eaton, and Kortum (2008), and Allen, Arkolakis, and Takahashi (2014).

2 The model and the data

Let me briefly remind you the class of models we are considering and discuss the data we are assumed to observe.

2.1 Universal gravity model (reminded)

Suppose we are interested bringing a general equilibrium gravity model to the data. The general equilibrium gravity model satisfies the following equilibrium conditions:

1. The value of trade flows between any two locations satisfy the **gravity equation**:

$$X_{ij} = K_{ij}\gamma_i\delta_j, \tag{1}$$

where $\mathbf{K} \equiv \{K_{ij}\}$ is assumed to be exogenous (i.e. it is a model parameter), while $\{\gamma_i\}$ and $\{\delta_j\}$ are endogenous.

2. In all locations, the **goods market clears**:

$$Y_i = \sum_{j\in S} X_{ij}; \tag{2}$$

3. In all locations, **trade is balanced**:

$$Y_j = \sum_{i\in S} X_{ij}; \tag{3}$$

4. In all locations, the **generalized labor market clearing condition** holds:

$$Y_i = B_i\gamma_i^{\alpha}\delta_i^{\beta}. \tag{4}$$

As a reminder, the parameters governing the model are the bilateral matrix of frictions $\mathbf{K}$, the vector of income shifters $\vec{B}$, and the scalars α and β.

2.2 The data

Suppose we observe the trade flows between all locations, i.e. we observe $\{X_{ij}\}$ for all $i \in S$ and $j \in S$. We would like to ask to what extent we can recover the exogenous model parameters $\mathbf{K}$, $\vec{B}$, and scalars α and β. (Note that with N locations, there are N^2 observed trade flows and $N^2 + N + 2$ model parameters. Not surprisingly, it will turn out that the model is underidentified).

Before proceeding to these results, however, we must address an issue familiar to trade empiricists: in contrast to equilibrium condition 3 above, trade data is usually not balanced. It is not obvious how one ought to address unbalanced trade (which we view as a dynamic phenomenon) in the context of a static model. As a result, we consider the following two options as short-cuts that ought to be addressed more seriously in the future.

The first option is to treat the trade deficit as exogenous (as in Dekle, Eaton, and Kortum (2008)). Define $E_i \equiv \sum_{j\in S} X_{ji}$ to be the expenditure in location $i \in S$, $Y_i \equiv \sum_{j\in S} X_{ij}$ to be the output in location $i \in S$ and $\bar{D}_i \equiv E_i - Y_i$ to be the (exogenous) trade deficit. In this case, we can rewrite the trade balance condition (3) as:

$$Y_j + \bar{D}_j = \sum_{i\in S} X_{ij}; \tag{5}$$

One of the disadvantages of allowing for exogenous deficits is that the theoretical results discussed earlier in the course (in particular, the uniqueness of the equilibrium) do not necessarily hold.

The second option is to make the trade data balanced. One simple way of doing so is to ignore the observed level of trade flows and instead treat the observed import *shares* $\lambda_{ij} \equiv \frac{X_{ij}}{\sum_i X_{ij}}$ as the true data. We can then find the unique set of incomes that are consistent with those import shares and balanced trade by solving the following linear system of equations:

$$Y_i = \sum_j \lambda_{ij} Y_j.$$

By the Perron-Frobenius theorem, there exists a unique (to-scale) set of Y_i;[1] we pin down the scale with the normalization that $\sum_{i \in S} Y_i = 1$. Given these equilibrium Y_i, we then define the balanced trade flows $X_{ij}^b = \lambda_{ij} Y_j$. It is straightforward to see that these trade flows are balanced:

$$\sum_j X_{ji}^b = \sum_j \lambda_{ji} Y_i = Y_i \sum_j \frac{X_{ji}}{\sum_j X_{ji}} = Y_i = \sum_j \lambda_{ij} Y_j = \sum_j X_{ij}^b.$$

One of the disadvantages of treating the import shares as true data instead of trade levels is that we could have just as easily treated export shares as the true data; a similar procedure in that case would actually imply different balanced trade flows (unless observed trade flows were already balanced) for the same reason that the eigenvectors of a transposed matrix are not necessarily equal to the eigenvectors of the matrix itself.

In what follows, we discuss the identification with exogenous deficits, as that nests the case when we observe (or have constructed) balanced trade data.

3 Identification and Calibration

Given any set of observed trade flows $\{X_{ij}\}$ and gravity constants α and β, we show to what extent model fundamentals such as bilateral trade frictions can be recovered? We summarize the identification result in the following proposition:

Proposition 1. *Take as given any (possibly unbalanced) set of observed trade flows $\{X_{ij}\}$. Choose any gravity constants α and β, set of income shifters $\{B_i\}$ and set of own trade flow frictions $\{K_{ii}\}$. Then there exists a unique set of $\{K_{ij}\}_{i \neq j}$ (and set of exogenous trade deficits $\left\{\bar{D}_i\right\}$) that, given the chosen parameters, yield equilibrium trade flows that equal to the observed trade flows.*

Proof. We need to show that there exist a unique set of $\{K_{ij}\}_{i \neq j}$ such that observed trade flows satisfy the model equilibrium conditions. Substituting the labor market clearing condition (4) into the goods market clearing condition (2) and the (modified) balanced trade

[1] Recall the Perron-Frobenius theorem guarantees that there exists a unique (to-scale) strictly positive vector that solves $Y_i = \kappa \sum_j \lambda_{ij} Y_j$ for the largest value of $\kappa > 0$. Since import shares sum to one, it is straightforward to show that $\kappa = 1$ in this case: $\kappa = \frac{\sum_i Y_i}{\sum_i \sum_j \lambda_{ij} Y_j} = \frac{\sum_i Y_i}{\sum_j Y_j \sum_i \lambda_{ij}} = 1$.

condition (5) simplifies the equilibrium into the following three conditions.

$$K_{ij}\gamma_i\delta_j = X_{ij} \tag{6}$$
$$B_i\gamma_i^{\alpha}\delta_i^{\beta} = \sum_j X_{ij} \tag{7}$$
$$B_i\gamma_i^{\alpha}\delta_i^{\beta} + \bar{D}_i = \sum_j X_{ji}, \tag{8}$$

where recall that $\{X_{ij}\}$ are observed and $\{B_i\}$,$\{K_{ii}\}$, α, and β have already been chosen. The goal is to find a unique set of $\{K_{ij}\}_{i\neq j}$ and deficits $\left\{\bar{D}_i\right\}$ that satisfy these conditions.

We proceed by construction. Choose:

$$\begin{aligned}
\bar{D}_i &\equiv \sum_j X_{ji} - \sum_j X_{ij} \\
Y_i &\equiv \sum_j X_{ij} \\
\gamma_i &\equiv \left(\frac{X_{ii}}{K_{ii}}\right)^{-\frac{\beta}{\alpha-\beta}} \left(\frac{B_i}{Y_i}\right)^{-\frac{1}{\alpha-\beta}} \\
\delta_i &\equiv \left(\frac{X_{ii}}{K_{ii}}\right)^{\frac{\alpha}{\alpha-\beta}} \left(\frac{B_i}{Y_i}\right)^{\frac{1}{\alpha-\beta}} \\
K_{ij} &\equiv \frac{X_{ij}}{\gamma_i\delta_j}
\end{aligned}$$

We verify that these definitions satisfy equilibrium conditions given observed trade flows. First, note that (6) is satisfied by the construction of K_{ij}.

Second, note that:

$$\begin{aligned}
B_i\gamma_i^{\alpha}\delta_i^{\beta} &= B_i\left(\left(\frac{X_{ii}}{K_{ii}}\right)^{-\frac{\beta}{\alpha-\beta}} \left(\frac{B_i}{Y_i}\right)^{-\frac{1}{\alpha-\beta}}\right)^{\alpha} \left(\left(\frac{X_{ii}}{K_{ii}}\right)^{\frac{\alpha}{\alpha-\beta}} \left(\frac{B_i}{Y_i}\right)^{\frac{1}{\alpha-\beta}}\right)^{\beta} \iff \\
B_i\gamma_i^{\alpha}\delta_i^{\beta} &= B_i\left(\frac{B_i}{Y_i}\right)^{\frac{\beta-\alpha}{\alpha-\beta}} \iff \\
B_i\gamma_i^{\alpha}\delta_i^{\beta} &= \sum_j X_{ij},
\end{aligned}$$

so that equilibrium condition (7) is satisfied.

Finally, note that:

$$\begin{aligned}
B_i\gamma_i^{\alpha}\delta_i^{\beta} + \bar{D}_i &= \sum_j X_{ij} + \sum_j X_{ji} - \sum_j X_{ij} \iff \\
B_i\gamma_i^{\alpha}\delta_i^{\beta} + \bar{D}_i &= \sum_j X_{ji},
\end{aligned}$$

so that equilibrium condition (8) is satisfied.. $\square$

Proposition 1 shows that general equilibrium gravity models are fundamentally under-identified in two ways. First, there exists an inability to determine which model parameter is responsible for the level of trade flows. In particular, the scale of the bilateral trade frictions and the income shifters cannot be separately identified: intuitively, a larger value of the income shifter can be counteracted with lower bilateral trade frictions without affecting the equilibrium. Second, the observed trade flows can be rationalized by the model for any chosen value of α and β (as long as $\alpha \neq \beta$). That is, the gravity constants cannot be identified using trade flow data alone. This result underpins why previous attempts to estimate (transformations of) these gravity constants have relied on additional sources of data such as prices (see e.g. Eaton and Kortum (2002), Simonovska and Waugh (2009), and Waugh (2010)).

4 Counterfactuals

You might be concerned that if the parameters of gravity models are unidentified from observed trade data, it is impossible (without additional information) to use them to make counterfactual predictions. It turns out that this concern is only half valid: the fact that the trade data are consistent with any set of income shifters $\{B_i\}_i$ and own trade frictions $\{K_{ii}\}_i$ does *not* affect counterfactuals. However, the parameters α and β have very important implications for counterfactual predictions. (As a result, we will devote the next lecture to how to estimate these gravity constants using additional data).

4.1 Global counterfactuals

Let us first consider the effect of an arbitrary change in trade costs. Throughout this section, we will use the "exact hat algebra" (pioneered by Dekle, Eaton, and Kortum (2008)). Let $\hat{x}_i \equiv \frac{x_i'}{x_i}$, where x_i is the (observed) current value of variable x_i and x_i' is the (unobserved) counterfactual value of variable x_i. Hence, $\hat{x}_i$ reflects the change in x_i due to the counterfactual.

Suppose that the trade friction matrix (exogenously) changes from $\mathbf{K}$ to $\mathbf{K}'$ and the income shifter exogenous changes from $\vec{B}$ to $\vec{B}'$. (Note that this is sufficiently general to capture any possible change in model fundamentals). What will be the affect on trade flows, incomes, and welfare in all countries?

To answer this question, we first simplify the equilibrium system of equations. By substituting the gravity condition (1) and labor market clearing condition (4) into both the goods market clearing condition (2) and the (modified) balanced trade condition (5), we are left with the two systems of equations and two unknowns:

$$B_i \gamma_i^{\alpha} \delta_i^{\beta} = \sum_{j \in S} K_{ij} \gamma_i \delta_j \tag{9}$$

$$B_i \gamma_i^{\alpha} \delta_i^{\beta} + \bar{D}_i = \sum_{j \in S} K_{ji} \gamma_j \delta_i. \tag{10}$$

Note that these equations will also hold in the counterfactual equilibrium:

$$B_i' \left(\gamma_i'\right)^{\alpha} \left(\delta_i'\right)^{\beta} = \sum_{j \in S} K_{ij}' \gamma_i' \delta_j' \tag{11}$$

$$B_i' \left(\gamma_i'\right)^{\alpha} \left(\delta_i'\right)^{\beta} + \bar{D}_i = \sum_{j \in S} K_{ji}' \gamma_j' \delta_i', \tag{12}$$

where note that we are holding $\bar{D}_i$ fixed. We can then divide equation (11) by (9), which yields:

$$\begin{aligned}
\frac{B_i' \left(\gamma_i'\right)^{\alpha} \left(\delta_i'\right)^{\beta}}{B_i \gamma_i^{\alpha} \delta_i^{\beta}} &= \frac{\sum_{j \in S} K_{ij}' \gamma_i' \delta_j'}{\sum_{j \in S} K_{ij} \gamma_i \delta_j} \iff \\
\hat{B}_i \hat{\gamma}_i^{\alpha} \hat{\delta}_i^{\beta} &= \sum_{j \in S} \frac{1}{\sum_{k \in S} K_{ik} \gamma_i \delta_k} K_{ij}' \gamma_i' \delta_j' \iff \\
\hat{B}_i \hat{\gamma}_i^{\alpha} \hat{\delta}_i^{\beta} &= \sum_{j \in S} \frac{K_{ij} \gamma_i \delta_j}{\sum_{k \in S} K_{ik} \gamma_i \delta_k} \hat{K}_{ij} \hat{\gamma}_i \hat{\delta}_j \iff \\
\hat{B}_i \hat{\gamma}_i^{\alpha} \hat{\delta}_i^{\beta} &= \sum_{j \in S} \left(\frac{X_{ij}}{Y_i}\right) \hat{K}_{ij} \hat{\gamma}_i \hat{\delta}_j. \qquad (13)
\end{aligned}$$

Similarly, we can divide equation (12) by (10), which yields:

$$\begin{aligned}
\frac{B_i' \left(\gamma_i'\right)^{\alpha} \left(\delta_i'\right)^{\beta} + \bar{D}_i}{B_i \gamma_i^{\alpha} \delta_i^{\beta} + \bar{D}_i} &= \frac{\sum_{j \in S} K_{ji}' \gamma_j' \delta_i'}{\sum_{j \in S} K_{ji} \gamma_j \delta_i} \iff \\
B_i' \left(\gamma_i'\right)^{\alpha} \left(\delta_i'\right)^{\beta} \frac{1}{E_i} + \frac{\bar{D}_i}{E_i} &= \sum_{j \in S} \frac{1}{\sum_{k \in S} K_{ki} \gamma_k \delta_i} K_{ji}' \gamma_j' \delta_i' \iff \\
\hat{B}_i \hat{\gamma}_i^{\alpha} \hat{\delta}_i^{\beta} \frac{B_i \gamma_i^{\alpha} \delta_i^{\beta}}{E_i} + \frac{\bar{D}_i}{E_i} &= \sum_{j \in S} \frac{K_{ji} \gamma_j \delta_i}{\sum_{k \in S} K_{ki} \gamma_k \delta_i} \hat{K}_{ji} \hat{\gamma}_j \hat{\delta}_i \iff \\
\hat{B}_i \hat{\gamma}_i^{\alpha} \hat{\delta}_i^{\beta} \frac{Y_i}{E_i} + \frac{\bar{D}_i}{E_i} &= \sum_{j \in S} \left(\frac{X_{ji}}{E_i}\right) \hat{K}_{ji} \hat{\gamma}_j \hat{\delta}_i. \qquad (14)
\end{aligned}$$

Recall that trade flows $\{X_{ij}\}$, expenditure $\{E_i\}$, income $\{Y_i\}$, and deficits $\left\{\bar{D}_i\right\}$ are observed and the particular counterfactual chosen defines the change in income shifters $\left\{\hat{B}_i\right\}$ and the change in bilateral trade frictions $\left\{\hat{K}_{ij}\right\}$. As a result, equations (13) and (14) can be jointly solved to determine the equilibrium changes in the origin sizes $\{\hat{\gamma}_i\}$ and destination sizes $\left\{\hat{\delta}_i\right\}$. Indeed, if trade is balanced (so $\bar{D}_i = 0$ and $Y_i = E_i$ for all $i \in S$), equations (13) and (14) simplify to:

$$\hat{B}_i \hat{\gamma}_i^{\alpha} \hat{\delta}_i^{\beta} = \sum_{j \in S} \left(\frac{X_{ij}}{Y_i}\right) \hat{K}_{ij} \hat{\gamma}_i \hat{\delta}_j \tag{15}$$

$$\hat{B}_i \hat{\gamma}_i^{\alpha} \hat{\delta}_i^{\beta} = \sum_{j \in S} \left(\frac{X_{ji}}{Y_i}\right) \hat{K}_{ji} \hat{\gamma}_j \hat{\delta}_i. \tag{16}$$

Equations (15) and (16) are mathematically identical to the equilibrium system in levels (where the income shifter is replaced with the change in income shifters and the bilateral trade friction is replaced with the product of the change in bilateral trade frictions and either the export share or the import share). As a result, the sufficient conditions for the equilibrium to exist and be unique imply that the counterfactual will also exist and be unique.

Similarly, calculating the change in the equilibrium given a certain counterfactual defined by $\left\{\hat{B}_i\right\}$ and $\left\{\hat{K}_{ij}\right\}$ and observed trade flows $\{X_{ij}\}$ proceeds identically to how one calculates the equilibrium itself given $\{B_i\}$ and $\{K_{ij}\}$. Note that once the change in origin size $\{\hat{\gamma}_i\}$ and destination size $\left\{\hat{\delta}_i\right\}$ are determined, the change in trade flows and incomes can be immediately derived from the following expressions:

$$\hat{Y}_i = \hat{B}_i \hat{\gamma}_i^{\alpha} \hat{\delta}_i^{\beta}$$
$$\hat{X}_{ij} = \hat{K}_{ij} \hat{\gamma}_i \hat{\delta}_j.$$

If we can write welfare as $W_i = C_i \left(\frac{X_{ii}}{Y_i}\right)^{-\rho}$, we can also immediately derive the change in welfare:

$$\hat{W}_i = \hat{C}_i \left(\frac{\hat{B}_i}{\hat{K}_{ii}} \hat{\gamma}_i^{\alpha-1} \hat{\delta}_i^{\beta-1}\right)^{\rho}.$$

4.2 Local counterfactuals

The advantage of the global counterfactual equations (13) and (14) is that they can be applied to an arbitrary change in trade costs (or income shifters). The disadvantage is that the entire system of equations has to be calculated separately for each change in trade costs one is interested in. If we consider small changes in trade costs, however, it turns out that we can simultaneously calculate how *any* change in trade costs affects *all* bilateral trade flows, incomes, and welfare, i.e. we can simultaneously derive the full N^4 set of elasticities:

$$\frac{\partial \ln X_{ij}}{\partial \ln K_{kl}} \; \forall i, j, k, l \in S.$$

To do so, we re-write equations (9) and (10) in vector form and apply the implicit function theorem. Define the function $f(\vec{x}, \mathbf{K}) : \mathbb{R}^{2N+N^2} \to \mathbb{R}^{2N}$, where $\vec{x} = \begin{pmatrix} \tilde{\gamma}_i \\ \tilde{\delta}_i \end{pmatrix}$, $\tilde{\gamma}_i \equiv \log \gamma_i$ and $\tilde{\delta}_i \equiv \log \delta_i$, $\tilde{K}_{ij} \equiv \log K_{ij}$, and f is defined as follows:

$$f\left(\vec{x}, \tilde{\mathbf{K}}\right) = \begin{pmatrix} B_i \exp\left(\alpha \tilde{\gamma}_i + \beta \log \tilde{\delta}_i\right) - \sum_{j \in S} \exp\left(\tilde{K}_{ij} + \tilde{\gamma}_i + \tilde{\delta}_j\right) \\ B_i \exp\left(\alpha \tilde{\gamma}_i + \beta \log \tilde{\delta}_i\right) + \bar{D}_i - \sum_{j \in S} \exp\left(\tilde{K}_{ji} + \tilde{\gamma}_j + \tilde{\delta}_i\right) \end{pmatrix}. \tag{17}$$

Note that if $f\left(\vec{x}, \tilde{K}\right) = \vec{0}$, then $\vec{x}$ is an equilibrium for a trade model with (log) bilateral trade friction matrix $\tilde{\mathbf{K}}$. Writing $\vec{x}$ as an implicit function of $\tilde{\mathbf{K}}$, i.e. $\vec{x}\left(\tilde{\mathbf{K}}\right)$ and fully differentiating f with respect to $\tilde{\mathbf{K}}$ at the equilibrium yields:

$$\mathbf{D}_x f \mathbf{D}_{\tilde{K}} \vec{x} + \mathbf{D}_{\tilde{K}} f = \vec{0}, \tag{18}$$

where:

$$\mathbf{D}_x f \equiv \left[\frac{\partial f_i}{\partial x_j}\right]_{ij} \implies$$
$$\mathbf{D}_x f = \begin{pmatrix} (\alpha-1)\,\mathbf{Y} & \beta\mathbf{Y}-\mathbf{X} \\ \alpha\mathbf{Y}-\mathbf{X}^T & (\beta-1)\,\mathbf{Y} \end{pmatrix},$$

where $\mathbf{Y}$ is a $N \times N$ diagonal matrix whose i^{th} diagonal is equal to Y_i and $\mathbf{X}$ is the $N \times N$ trade matrix.

Similarly, $\mathbf{D}_{\tilde{K}} f$ is a $2N \times N^2$ matrix that depends only on trade flows:

$$\mathbf{D}_{\tilde{K}} f \equiv \left[\frac{\partial f_i}{\partial \tilde{\mathbf{K}}_j}\right]_{ij} \implies$$

$$\mathbf{D}_{\tilde{K}} f = -\begin{pmatrix} X_{11} & \cdots & X_{1N} & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & X_{21} & \cdots & X_{2N} & \cdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & X_{N1} & \cdots & X_{NN} \\ X_{11} & \cdots & 0 & X_{21} & \cdots & 0 & \cdots & X_{N1} & \cdots & 0 \\ 0 & \ddots & \vdots & 0 & \ddots & \vdots & \cdots & 0 & \ddots & \vdots \\ 0 & \cdots & X_{1N} & 0 & \cdots & X_{2N} & \cdots & 0 & \cdots & X_{NN} \end{pmatrix}$$

If $\mathbf{D}_x f$ was of full rank, we could immediately invert equation (18) (i.e. apply the implicit function theorem) to yield:

$$\mathbf{D}_{\tilde{K}} \vec{x} = -\left(\mathbf{D}_x f\right)^{-1} \mathbf{D}_{\tilde{K}} f. \tag{19}$$

However, because Walras Law holds and we can without loss of generality apply a normalization to $\{\gamma_i\}$ and $\{\delta_i\}$, we effectively have $N-1$ equations and $N-1$ unknowns. Hence, there exists an infinite number of solutions to equation (18), each corresponding to a different normalization. Let us define the generalized (pseudo) inverse of $\mathbf{D}_x f$ as $\mathbf{A}^+$, i.e.:

$$\mathbf{A}^+ = \begin{pmatrix} (\alpha-1)\,\mathbf{Y} & \beta\mathbf{Y}-\mathbf{X} \\ \alpha\mathbf{Y}-\mathbf{X}^T & (\beta-1)\,\mathbf{Y} \end{pmatrix}^+, \tag{20}$$

and, with slight abuse of notation, denote the $\langle i,j \rangle$ element of $\mathbf{A}^+$ as A^+_{ij}. The generalized inverse is one of the infinitely many solutions solving equation (19) (see James (1978)). Given the functional form of $\mathbf{D}_K f$, we can then calculate the elasticities of the origin and destination sizes to changes in the bilateral trade costs up to some scalar $c \in \mathbb{R}$:

$$\frac{\partial \ln \gamma_l}{\partial \ln K_{ij}} = X_{ij} \times \left(A^+_{l,i} + A^+_{N+l,j} - c\right) \tag{21}$$
$$\frac{\partial \ln \delta_l}{\partial \ln K_{ij}} = X_{ij} \times \left(A^+_{N+l,i} + A^+_{l,j} - c\right). \tag{22}$$

To pin down c, we have to specify a normalization. In the case where the world income is the numeraire, we can write:

$$\sum_l B_l \gamma_l^{\alpha} \delta_l^{\beta} = 1 \implies$$

$$\sum_l Y_l \left(\alpha \frac{\partial \ln \gamma_l}{\partial \ln K_{ij}} + \beta \frac{\partial \ln \delta_l}{\partial \ln K_{ij}} \right) = 0. \tag{23}$$

In this case, it turns out that $c \equiv \frac{1}{(\alpha+\beta)} \sum_l Y_l \left(\alpha \left(A^+_{l,i} + A^+_{N+l,j} \right) + \beta \left(A^+_{N+l,i} + A^+_{l,j} \right) \right)$. To see this, note that:

$$\begin{aligned}
\sum_l Y_l \left(\alpha \frac{\partial \ln \gamma_l}{\partial \ln K_{ij}} + \beta \frac{\partial \ln \delta_l}{\partial \ln K_{ij}} \right) &= \begin{array}{c} \sum_l Y_l (\alpha \left(X_{ij} \times \left(A^+_{l,i} + A^+_{N+l,j} - c \right) \right) + \\ +\beta \left(X_{ij} \times \left(A^+_{N+l,i} + A^+_{l,j} - c \right) \right)) \end{array} \\
&= \begin{array}{r} X_{ij} \sum_l Y_l \left(\alpha \left(X_{ij} \times \left(A^+_{l,i} + A^+_{N+l,j} \right) \right) + \beta \left(X_{ij} \times \left(A^+_{N+l,i} + A^+_{l,j} \right) \right) \right) \\ -X_{ij} c \left(\alpha + \beta \right) \sum_l Y_l \end{array} \\
&= \begin{array}{c} X_{ij} \sum_l Y_l \left(\alpha \left(X_{ij} \times \left(A^+_{l,i} + A^+_{N+l,j} \right) \right) + \beta \left(X_{ij} \times \left(A^+_{N+l,i} + A^+_{l,j} \right) \right) \right) - \\ \left(\frac{1}{(\alpha+\beta)} X_{ij} \sum_l Y_l \left(\alpha \left(A^+_{l,i} + A^+_{N+l,j} \right) + \beta \left(A^+_{N+l,i} + A^+_{l,j} \right) \right) \right) \left(\alpha + \beta \right) Y^W \end{array} \\
&= 0,
\end{aligned}$$

i.e. equation (23) also holds. More generally, different choices of c correspond to different normalizations. A particularly simple example is if we choose the normalization $\gamma_1 = 1$. Since this implies that $\frac{\partial \ln \gamma_1}{\partial \ln K_{ij}} = 0$, $c = X_{ij} \times \left(A^+_{1,i} + A^+_{N+1,j} \right)$. In this case, however, an alternative procedure is even simpler: the elasticities for all $i > 1$ can be calculated directly by inverting the $(2N-1) \times (2N-1)$ matrix generated by removing the first row and first column of $\mathbf{D}_x f$.

To summarize, determining the complete set of elasticities of the origin and destination sizes to all bilateral trade costs (at least up to scale) can be accomplished by a single inversion of the $2N \times 2N$ matrix given in equation (20). As with the global comparative statics, once one has the elasticities $\left\{ \frac{\partial \ln \gamma_l}{\partial \ln K_{ij}} \right\}_{l,i,j \in S}$ and $\left\{ \frac{\partial \ln \delta_l}{\partial \ln K_{ij}} \right\}_{l,i,j \in S}$, the elasticity of trade flows, income, and welfare to any change in bilateral trade costs can be immediately determined:

$$\begin{aligned}
\frac{\partial \ln Y_l}{\partial \ln K_{ij}} &= \alpha \frac{\partial \ln \gamma_i}{\partial \ln K_{ij}} + \beta \frac{\partial \ln \delta_i}{\partial \ln K_{ij}} \\
\frac{\partial \ln X_{kl}}{\partial \ln K_{ij}} &= \frac{\partial \ln K_{kl}}{\partial \ln K_{ij}} + \frac{\partial \ln \gamma_k}{\partial \ln K_{ij}} + \frac{\partial \ln \delta_l}{\partial \ln K_{ij}} \\
\frac{\partial \ln W_l}{\partial \ln K_{ij}} &= \rho \left((\alpha - 1) \frac{\partial \ln \gamma_l}{\partial \ln K_{ij}} + (\beta - 1) \frac{\partial \ln \delta_l}{\partial \ln K_{ij}} - \frac{\partial \ln K_{ll}}{\partial \ln K_{ij}} \right),
\end{aligned}$$

where note that $\frac{\partial \ln K_{kl}}{\partial \ln K_{ij}} = 1$ if $k = i$ and $l = j$ and 0 otherwise.

5 Conclusion and next steps

At this point, you have almost all the tools necessary to bring a structural gravity model to the data. There is just one nagging thing that we have thus far ignored: while it is reasonable to assume you will observe the bilateral trade flow matrix $\mathbf{X}$, the gravity constants α and β are not observed (and, given the Proposition above, cannot be identified from trade data alone). Furthermore, we have seen that the gravity constants play a crucial role in determining the results of counterfactual analysis. Given their importance, in the next lecture, we will discuss how one can identify these parameters.

References

Allen, T., C. Arkolakis, and Y. Takahashi (2014): "Universal gravity," *NBER Working Paper*, (w20787).

Dekle, R., J. Eaton, and S. Kortum (2008): "Global Rebalancing with Gravity: Measuring the Burden of Adjustment," *IMF Staff Papers*, 55(3), 511–540.

Eaton, J., and S. Kortum (2002): "Technology, Geography and Trade," *Econometrica*, 70(5), 1741–1779.

James, M. (1978): "The generalised inverse," *The Mathematical Gazette*, pp. 109–114.

Simonovska, I., and M. Waugh (2009): "The Elasticity of Trade: Estimates and Evidence," *Manuscript, University of California, Davis and New York University.*

Waugh, M. (2010): "International Trade and Income Differences," *forthcoming, American Economic Review.*