Macroeconomics A; EI060

Short problems

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1 Euler under uncertainty

Question: he consumer maximizes the following utility:

$$u(C_1) + \beta E u(C_2)$$

where k denotes the state of nature of probability $\pi(k)$. He can invest in a bond giving an interest rate $r = (1/\beta - 1)$. Output is an endowment, which can vary in period 2. The budget constraints are:

$$\begin{aligned} C_1 + B_2 &= Y_1 \\ C_2(k) &= \frac{1}{\beta} B_2 + Y_2(k) \end{aligned}$$

Show that:

$$u'(C_1) = E[u'(C_2)]$$

2 Approximation of budget constraint

Question: We approximate the system around an allocation where $C_2(k) = C_1 = Y_2(k) = Y_1 = \overline{C}$

Define the approximation, scaled by the steady state value. For instance:

$$\widehat{C}_1 = \frac{C_1 - \overline{C}}{\overline{C}}$$

Show that the budget constraint in period 2 in state k can be written as the linear approximation:

$$\widehat{C}_2(k) = \frac{1}{\beta}\left(\widehat{Y}_1 - \widehat{C}_1\right) + \widehat{Y}_2(k)$$

3 Quadratic approximation of Euler condition

Question: Recal that a general function of consumption can be expanded as such up to a quadratic term:

$$f(C) = f\left(\overline{C}\right) + f'\left(\overline{C}\right)\left(C - \overline{C}\right) + \frac{1}{2}f''\left(\overline{C}\right)\left(C - \overline{C}\right)^2$$

Show that:

$$\begin{aligned} u'(C_1) &= u'\left(\overline{C}\right) + u''\left(\overline{C}\right)\overline{C}\widehat{C}_1 + \frac{1}{2}u'''\left(\overline{C}\right)\left(\overline{C}\right)^2\left(\widehat{C}_1\right)^2 \\ u'(C_2(k)) &= u'\left(\overline{C}\right) + u''\left(\overline{C}\right)\overline{C}\widehat{C}_2(k) + \frac{1}{2}u'''\left(\overline{C}\right)\left(\overline{C}\right)^2\left(\widehat{C}_2(k)\right)^2 \end{aligned}$$

Show that the quadratic approximation of the Euler is:

$$\widehat{C}_1 = E\left[\widehat{C}_2\right] + \frac{1}{2}\frac{u'''\left(\overline{C}\right)}{u''\left(\overline{C}\right)}\overline{C}\left[-\left(\widehat{C}_1\right)^2 + E\left[\left(\widehat{C}_2\right)^2\right]\right]$$

4 First-order solution

Question: We can represent the deviation of variable from the reference point as composed to several orders. Focus on the first two:

$$\widehat{C} = \widehat{C}_{O(1)} + \widehat{C}_{O(2)}$$

$\widehat{C}_{O(1)}$ is proportional to the innovations of shocks, while $\widehat{C}_{O(2)}$ is proportional to the squares innovations of shocks (which in expected terms are their variance).

When taking the quadratic approximation of the function $f(C)$, we can split it between its first and second order components:

$$\begin{aligned} f(C) - f\left(\overline{C}\right) &= f'\left(\overline{C}\right)\overline{C}\widehat{C} + \frac{1}{2}f''\left(\overline{C}\right)\left(\overline{C}\right)^2\left(\widehat{C}\right)^2 \\ \left[f(C) - f\left(\overline{C}\right)\right]_{O(1)} &= f'\left(\overline{C}\right)\overline{C}\widehat{C}_{O(1)} \\ \left[f(C) - f\left(\overline{C}\right)\right]_{O(2)} &= f'\left(\overline{C}\right)\overline{C}\widehat{C}_{O(2)} + \frac{1}{2}f''\left(\overline{C}\right)\left(\overline{C}\right)^2\left(\widehat{C}_{O(1)}\right)^2 \end{aligned}$$

Note that $\left(\widehat{C}_{O(1)}\right)^2$ is of order 2.

We consider that output in the first period is constant. Output in period 2 can fluctuate, but the expected value of these shocks is zero.

Show that:

$$\widehat{C}_{1,O(1)} = E\left[\widehat{C}_{2,O(1)}\right] = 0$$

Does the uncertainty of output in period 2 has an effect?

5 Precautionary savings

Question: We now turn to the second-order dimension of the Euler condition. Show that:

$$\widehat{C}_{1,O(2)} = E\left[\widehat{C}_{2,O(2)}\right] - \left(\frac{1}{2}\frac{u'''\left(\overline{C}\right)}{-u''\left(\overline{C}\right)}\overline{C}\right)E\left[\left(\widehat{Y}_{2,O(1)}(k)\right)^2\right]$$

How does the volatility of output matters for initial consumption?

What are the characteristics of the utility function that are needed to have an effect? Is it the case for the following two utilities (assume a is small):

$$\begin{aligned} u(C) &= C - \frac{a}{2}C^2 \\ u(C) &= \frac{(C)^{1-\sigma}}{1-\sigma} \end{aligned}$$