# Problem Set 1

Mathematics and Statistics for Economists

Due date: 11:59pm Sunday 1rd September 2024

## Essential Math

**Question 1** Sandra pays income tax according to the schedule

$$T(X) = \begin{cases} 0 & \text{if } X < E \\ t(X - E) & \text{if } X \geq E \end{cases}$$

where $X$ is her pre-tax income; $E$ and $t$ are positive constants with $t < 1$. Sandra is also eligible for an income-related transfer

$$B(X) = \begin{cases} s(P - X) & \text{if } X < P \\ 0 & \text{if } X \geq P \end{cases}$$

where $P$ and $s$ are constants such that $P > 0$ and $t < s < 1$. Sandra's disposable income $Y$ is therefore equal to $F(X)$, where

$$F(X) = X - T(X) + B(X)$$

Sketch the graph $Y = F(X)$ in each of the following cases:
(i) $E > P$;
(ii) $E < P, s + t < 1$;
(iii) $E < P, s + t > 1$.

**Question 2** Given the function

$$y = \frac{x^2}{x^2 - x + 1}$$

(i) Use the quotient rule to calculate $\frac{dy}{dx}$
(ii) Obtain the same result by writing $y = u^{-1}$, where $u = 1 - x^{-1} + x^{-2}$, and using the composite function rule.

**Question 3** A data set consists of $n$ observations on $k$ variables $x_1, x_2, \ldots x_k$; the $i$ th observation is denoted $(x_{1i}, x_{2i}, \ldots, x_{ki})$. Let $\mathbf{X}$ be the $n \times k$ matrix whose $i$ th row is the $i$ th observation. Calculate the matrix $\mathbf{X}'\mathbf{X}$, expressing its entries in $\sum$ -notation, and verify that it is symmetric.

**Question 4** Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be invertible matrices of the same order. Simplify the expressions

$$(\mathbf{I} + \mathbf{A})\mathbf{A}^{-1}(\mathbf{I} - \mathbf{A}), \quad \mathbf{A}\left(3\mathbf{A}^{-1} + 4\mathbf{B}^{-1}\right)\mathbf{B}, \quad \left(\mathbf{A}\mathbf{B}^{-1}\mathbf{C}\right)^{-1}$$

**Question 5** Given a quadratic function:

$$f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \mathbf{A}\mathbf{x} + \mathbf{b}^\top \mathbf{x} + c$$

where: $\mathbf{x} \in \mathbb{R}^{3\times 1}$ $\mathbf{b} \in \mathbb{R}^{3\times 1}$ are column vectors, $\mathbf{A} \in \mathbb{R}^{3\times 3}$ is a symmetric matrix, $c \in \mathbb{R}$ is a scalar.
(i) Derive the gradient of $f(\mathbf{x})$ with respect to $\mathbf{x}$.
(ii) Derive the Hessian of $f(\mathbf{x})$ with respect to $\mathbf{x}$.

## Mathematical Analysis

**Question 6** Suppose the function from our In-class Exercise

$$y = -x^4 + 4x^3 - 6x^2 + 8x + 3$$

is defined only for $x \geq 0$. Derive the coordinates of the local minimum point and the global maximum point. What happens if the function is defined only for (i) $x \geq 1$, (ii) $x \geq 2$, (iii) $x \geq 3$ ?

**Question 7** Which of the following functions of $x$ are convex? Which are concave?
(i) $(2x-1)^6$
(ii) $\sqrt{1+x^2}$
(iii) $x^5 - x$

**Question 8** Compute the first and second derivatives of each of the following functions:
(i) $e^{x^2 \cdot 3x - 2}$
(ii) $\ln\left(x^4+2\right)^2$
(iii) $\frac{x}{\ln x}$

**Question 9** let

$$f(x) = x^2 \ln x$$

(i) Find the linear and quadratic approximations to $f(x)$ for values of $x$ close to 2. Construct a numerical table similar to our In-class Exercise for $x = 1.80, 1.95, 2.02, 2.10$ and $2.25$ .
(ii) Find third-order approximations to $f(x)$ for values of $x$ close to 1 and for values of $x$ close to 2 . Use these approximations to extend the numerical tables of Example 1 and part (a) of this exercise by an additional row.

**Question 10** Consider the standard consumption Euler equation that emerges from household optimization problems with CRRA utility:

$$\left(\frac{c_{t+1}}{c_t}\right)^\sigma = \beta\left(1 + r_t\right)$$

Log-linearize the above equation.