Macroeconomics A; EI056

Technical appendix: Solow growth model

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1 Main assumptions

The model is set in continuous time. This is the limit of discrete time when the length of each period goes to zero. Consider the changes in a variable L over an interval Δ: $L_{t+\Delta} - L_t$. In discrete time $\Delta = 1$, while the continuous time is the limit of $\Delta \to 0$. The instantaneous growth rate n is the speed of change per unit of elapsed time, scaled by the initial level of the variable:

$$L_{t+\Delta} - L_t = nL_t\Delta$$

In continuous time:

$$\dot{L}_t = \lim_{\Delta \to 0} \frac{L_{t+\Delta} - L_t}{\Delta} = nL_t \Rightarrow \frac{\dot{L}_t}{L_t} = n$$

The technology has constant returns to scale. It uses labor L, physical capital K, human capital H, with an exogenous productivity A:

$$\begin{aligned} Y_t &= F(K_t, H_t, A_t L_t) \\ cY_t &= F(cK_t, cH_t, cA_t L_t) \end{aligned}$$

We can scale everything by effective labor (that is: $c = 1/(A_t L_t)$):

$$\begin{aligned} y_t &= f(k_t, h_t) \\ y_t &= \frac{Y_t}{A_t L_t} \text{ and } k_t = \frac{K_t}{A_t L_t} \text{ and } h_t = \frac{H_t}{A_t L_t} \text{ and } f(k_t, h_t) = F(k_t, h_t, 1) \end{aligned}$$

The technology is such that with respect to each capital $f' > 0$, $f'' < 0$, $f'(0)$ is very large, $f(0,0) = 0$. The standard specification is a Cobb-Douglas case:

$$Y_t = (K_t)^\alpha (H_t)^\beta (A_t L_t)^{1-\alpha-\beta} \Rightarrow y_t = (k_t)^\alpha (h_t)^\beta$$

The growth rates of the labor and technology are exogenous:

$$\dot{L}_t = nL_t \qquad\qquad \dot{A}_t = gA_t$$

2 Dynamics of capital stocks

Savings are exogenous with a fraction s_K of output going to adding physical capital and a fraction s_H going to adding human capital. Both capital depreciates are a rate δ. The laws of motion of capital are:

$$\begin{aligned}
\dot{K}_t &= s_K Y_t - \delta K_t = A_t L_t \left(s_K y_t - \delta k_t\right) \\
\dot{H}_t &= s_H Y_t - \delta H_t = A_t L_t \left(s_H y_t - \delta h_t\right)
\end{aligned}$$

This can be written in per-capita terms. Think of the variables as functions of time, and take a differential of k with respect to time, using the chain rule:

$$\begin{aligned}
\dot{k}_t &= \frac{\partial k_t}{\partial t} = \frac{\partial}{\partial t}\frac{K_t}{A_t L_t} = \frac{1}{A_t L_t}\frac{\partial K_t}{\partial t} - \frac{K_t}{A_t \left(L_t\right)^2}\frac{\partial L_t}{\partial t} - \frac{K_t}{L_t \left(A_t\right)^2}\frac{\partial A_t}{\partial t} \\
&= \frac{1}{A_t L_t}\dot{K}_t - k_t \frac{1}{L_t}\dot{L}_t - k_t \frac{1}{A_t}\dot{A}_t = \frac{1}{A_t L_t}\dot{K}_t - (n+g)\, k_t
\end{aligned}$$

$$\Rightarrow \dot{K}_t = A_t L_t \left(\dot{k}_t + (n+g)\, k_t\right)$$

We then write the laws of motion of per capita capital:

$$\begin{aligned}
\dot{K}_t &= A_t L_t \left(s_K y_t - \delta k_t\right) \\
\dot{k}_t + (n+g)\, k_t &= s_K y_t - \delta k_t \\
\dot{k}_t &= s_K y_t - (\delta + n + g)\, k_t \\
\dot{k}_t &= s_K \left(k_t\right)^\alpha \left(h_t\right)^\beta - (\delta + n + g)\, k_t
\end{aligned}$$

and similarly:

$$\dot{h}_t = s_H \left(k_t\right)^\alpha \left(h_t\right)^\beta - (\delta + n + g)\, h_t$$

3 Steady state

Take the law of motions:

$$\begin{aligned}
\dot{k}_t &= s_K \left(k_t\right)^\alpha \left(h_t\right)^\beta - (\delta + n + g)\, k_t \\
\dot{h}_t &= s_H \left(k_t\right)^\alpha \left(h_t\right)^\beta - (\delta + n + g)\, h_t
\end{aligned}$$

The right-hand side of the first equation has a concave first term in k_t and a linear second term in k_t. If k_t is small the first dominates and $\dot{k}_t > 0$, so capital increases. If k_t is large the second dominates and $\dot{k}_t < 0$, so capital decreases.

In the steady state $\dot{k}_t = \dot{h}_t = 0$ and $k_t = k^*$ and $h_t = h^*$. This implies:

$$
\begin{aligned}
0 &= s_K (k^*)^\alpha (h^*)^\beta - (\delta + n + g) k^* \\
0 &= s_H (k^*)^\alpha (h^*)^\beta - (\delta + n + g) h^*
\end{aligned}
$$

Notice the complementarity between the two capital measures:

$$
\begin{aligned}
k^* &= \left(\frac{s_K}{\delta + n + g}\right)^{\frac{1}{1-\alpha}} (h^*)^{\frac{\beta}{1-\alpha}} \Rightarrow \frac{\partial k^*}{\partial h^*} > 0 \qquad \frac{\partial^2 k^*}{\partial h^* \partial h^*} < 0 \\
k^* &= \left(\frac{\delta + n + g}{s_H}\right)^{\frac{1}{\alpha}} (h^*)^{\frac{1-\beta}{\alpha}} \Rightarrow \frac{\partial k^*}{\partial h^*} > 0 \qquad \frac{\partial^2 k^*}{\partial h^* \partial h^*} > 0
\end{aligned}
$$

The solution is:

$$
\begin{aligned}
\left(\frac{s_K}{\delta + n + g}\right)^{\frac{1}{1-\alpha}} (h^*)^{\frac{\beta}{1-\alpha}} &= \left(\frac{\delta + n + g}{s_H}\right)^{\frac{1}{\alpha}} (h^*)^{\frac{1-\beta}{\alpha}} \\
(h^*)^{\frac{\beta}{1-\alpha} - \frac{1-\beta}{\alpha}} &= \left(\frac{\delta + n + g}{s_H}\right)^{\frac{1}{\alpha}} \left(\frac{\delta + n + g}{s_K}\right)^{\frac{1}{1-\alpha}} \\
(h^*)^{\frac{\beta\alpha - (1-\alpha)(1-\beta)}{\alpha(1-\alpha)}} &= (\delta + n + g)^{\frac{1-\alpha+\alpha}{\alpha(1-\alpha)}} \left(\frac{1}{s_H}\right)^{\frac{1}{\alpha}} \left(\frac{1}{s_K}\right)^{\frac{1}{1-\alpha}} \\
(h^*)^{\beta\alpha - (1-\alpha)(1-\beta)} &= (\delta + n + g)^{1-\alpha+\alpha} \left(\frac{1}{s_H}\right)^{(1-\alpha)} \left(\frac{1}{s_K}\right)^{\alpha} \\
(h^*)^{-(1-\alpha-\beta)} &= (\delta + n + g) \left(\frac{1}{s_H}\right)^{(1-\alpha)} \left(\frac{1}{s_K}\right)^{\alpha} \\
(h^*)^{1-\alpha-\beta} &= \frac{1}{\delta + n + g} (s_K)^\alpha (s_H)^{1-\alpha} \\
h^* &= \left[\frac{(s_K)^\alpha (s_H)^{1-\alpha}}{\delta + n + g}\right]^{\frac{1}{1-\beta-\alpha}}
\end{aligned}
$$

We then get k^* as:

$$\begin{aligned}
k^* &= \left(\frac{s_K}{\delta+n+g}\right)^{\frac{1}{1-\alpha}} (h^*)^{\frac{\beta}{1-\alpha}} \\
k^* &= \left(\frac{s_K}{\delta+n+g}\right)^{\frac{1}{1-\alpha}} \left[\frac{(s_K)^{\alpha}(s_H)^{1-\alpha}}{\delta+n+g}\right]^{\frac{1}{1-\beta-\alpha}\frac{\beta}{1-\alpha}} \\
k^* &= (s_K)^{\frac{1}{1-\alpha}} (s_K)^{\frac{\alpha}{1-\beta-\alpha}\frac{\beta}{1-\alpha}} (s_H)^{(1-\alpha)\frac{1}{1-\beta-\alpha}\frac{\beta}{1-\alpha}} \left(\frac{1}{\delta+n+g}\right)^{\frac{1}{1-\beta-\alpha}\frac{\beta}{1-\alpha}+\frac{1}{1-\alpha}} \\
k^* &= (s_K)^{\frac{1}{1-\alpha}\left[1+\frac{\alpha\beta}{1-\beta-\alpha}\right]} (s_H)^{\frac{\beta}{1-\beta-\alpha}} \left(\frac{1}{\delta+n+g}\right)^{\frac{1}{1-\alpha}\left[1+\frac{\beta}{1-\beta-\alpha}\right]} \\
k^* &= (s_K)^{\frac{1}{1-\alpha}\frac{1-\beta-\alpha+\alpha\beta}{1-\beta-\alpha}} (s_H)^{\frac{\beta}{1-\beta-\alpha}} \left(\frac{1}{\delta+n+g}\right)^{\frac{1}{1-\alpha}\frac{1-\beta-\alpha+\beta}{1-\beta-\alpha}} \\
k^* &= (s_K)^{\frac{1}{1-\alpha}\frac{(1-\beta)-\alpha(1-\beta)}{1-\beta-\alpha}} (s_H)^{\frac{\beta}{1-\beta-\alpha}} \left(\frac{1}{\delta+n+g}\right)^{\frac{1}{1-\alpha}\frac{1-\alpha}{1-\beta-\alpha}} \\
k^* &= (s_K)^{\frac{1}{1-\alpha}\frac{(1-\beta)(1-\alpha)}{1-\beta-\alpha}} (s_H)^{\frac{\beta}{1-\beta-\alpha}} \left(\frac{1}{\delta+n+g}\right)^{\frac{1}{1-\beta-\alpha}} \\
k^* &= (s_K)^{\frac{1-\beta}{1-\beta-\alpha}} (s_H)^{\frac{\beta}{1-\beta-\alpha}} \left(\frac{1}{\delta+n+g}\right)^{\frac{1}{1-\beta-\alpha}} \\
k^* &= \left[\frac{(s_K)^{1-\beta}(s_H)^{\beta}}{\delta+n+g}\right]^{\frac{1}{1-\beta-\alpha}}
\end{aligned}$$

The solution is thus:

$$k^* = \left[\frac{(s_K)^{1-\beta}(s_H)^{\beta}}{\delta+n+g}\right]^{\frac{1}{1-\beta-\alpha}} \qquad h^* = \left[\frac{(s_K)^{\alpha}(s_H)^{1-\alpha}}{\delta+n+g}\right]^{\frac{1}{1-\beta-\alpha}}$$

Output is:

$$y^* = (k^*)^{\alpha}(h^*)^{\beta} = (s_K)^{\frac{\alpha}{1-\beta-\alpha}} (s_H)^{\frac{\beta}{1-\beta-\alpha}} \left[\frac{1}{\delta+n+g}\right]^{\frac{\alpha+\beta}{1-\beta-\alpha}}$$

Note that output per capita increases in the steady state:

$$z_t = \frac{Y_t}{L_t} = A_t y_t \qquad \frac{\dot{z}_t}{z_t} = g + \frac{\dot{y}_t}{y_t} = g$$

4 The Golden rule

Determine the savings rate that maximizes consumption, i.e. output minus savings:

$$c^* = (1 - s_K - s_H)\, y^* = (k^*)^{\alpha}(h^*)^{\beta} - (\delta+n+g)(k^*+h^*)$$

Take the first order conditions:

$$\begin{aligned}
\frac{\partial c^*}{\partial k^*} &= 0 \Rightarrow \alpha \left(k^*\right)^{\alpha-1} \left(h^*\right)^{\beta} = (\delta + n + g) \\
\frac{\partial c^*}{\partial h^*} &= 0 \Rightarrow \beta \left(k^*\right)^{\alpha} \left(h^*\right)^{\beta-1} = (\delta + n + g)
\end{aligned}$$

Using the solutions for k^* and h^* this implies:

$$s_K = \alpha \qquad\qquad s_H = \beta$$

Abstract from human capital for simplicity, and consider a permanent increase in s_K. k gradually increases, so $\dot{k}_t$ jumps and gradually comes down to zero. As consumption is output minus savings, it falls initially. In the long run:

$$\begin{aligned}
c^* &= \left(k^*\right)^{\alpha} - (\delta + n + g)\, k^* \\
\frac{\partial c^*}{\partial s_K} &= \left[\alpha \left(k^*\right)^{\alpha-1} - (\delta + n + g)\right] \frac{\partial k^*}{\partial s_K}
\end{aligned}$$

If $\alpha \left(k^*\right)^{\alpha-1} > (\delta + g + n)$ i.e. capital is low, higher savings increase long run consumption (at a cost of short run consumption dip of course). If $\alpha \left(k^*\right)^{\alpha-1} < (\delta + g + n)$ higher savings reduce consumption (or conversely lower savings increase long run consumption, and also short run consumption, that is the economy is dynamically inefficient). If $\alpha \left(k^*\right)^{\alpha-1} = (\delta + g + n)$ there is no effect (golden rule).

5 The impact of savings rates

The steady state capitals and output in logs are:

$$\begin{aligned}
\ln k^* &= \frac{1-\beta}{1-\beta-\alpha} \ln (s_K) + \frac{\beta}{1-\beta-\alpha} \ln (s_H) - \frac{1}{1-\beta-\alpha} \ln (\delta + n + g) \\
\ln h^* &= \frac{\alpha}{1-\beta-\alpha} \ln (s_K) + \frac{1-\alpha}{1-\beta-\alpha} \ln (s_H) - \frac{1}{1-\beta-\alpha} \ln (\delta + n + g) \\
\ln y^* &= \frac{\alpha}{1-\beta-\alpha} \ln (s_K) + \frac{\beta}{1-\beta-\alpha} \ln (s_H) - \frac{\beta+\alpha}{1-\beta-\alpha} \ln (\delta + n + g)
\end{aligned}$$

The elasticity with respect to savings into physical capital are:

$$\frac{\partial \ln k^*}{\partial \ln (s_K)} = \frac{1-\beta}{1-\beta-\alpha} \qquad\qquad \frac{\partial \ln y^*}{\partial \ln (s_K)} = \frac{\alpha}{1-\beta-\alpha}$$

The labor share $1-\alpha$ is $2/3$. Without human capital this implies $\alpha = 1/3$ and $\beta = 0$, hence:

$$\frac{\partial \ln y^*}{\partial \ln (s_K)} = 0.5$$

For instance if s goes from 0.2 to 0.22 (a 10% increase) output increases only by 5%.

If labor consists substantially of human capital ($\alpha = \beta = 1/3$) we get:

$$\frac{\partial \ln y^*}{\partial \ln (s_K)} = 1$$

So and increase of s from 0.2 to 0.22 raises output by 10%.

6 A quick note on Taylor expansions

A function $F(x)$ can be written as a Taylor expansion around $x = x^*$:

$$F(x) = F(x^*) + F'(x^*)[x - x^*] + \frac{1}{2!}F''(x^*)[x - x^*]^2 + ...$$

A linear expansion focuses on the first order terms, namely:

$$F(x) \simeq F(x^*) + F'(x^*)[x - x^*] \tag{1}$$

In a bivariate case we have:

$$F(x, y) = F(x^*, y^*) + \left.\frac{\partial F(x, y)}{\partial x}\right|_{x=x^*,\ y=y^*} [x - x^*] + \left.\frac{\partial F(x, y)}{\partial y}\right|_{x=x^*,\ y=y^*} [y - y^*]$$

We apply this to the capital accumulation (for simplicity we abstract from human capital for this illustration of the method):

$$\dot{k}_t = s_K (k_t)^\alpha - (\delta + n + g) k_t \tag{2}$$

In the steady state we have:

$$\begin{aligned} \dot{k}^* &= 0 \\ s_K (k^*)^\alpha &= (\delta + g + n) k^* \end{aligned} \tag{3}$$

Apply (1) to $F(x) = F\left(\dot{k}_t\right) = \dot{k}_t$:

$$\begin{aligned} F\left(\dot{k}_t\right) &= F\left(\dot{k}^*\right) + F'\left(\dot{k}^*\right)\left[\dot{k}_t - \dot{k}^*\right] \\ \dot{k}_t &= \dot{k}^* + 1 \times \left[\dot{k}_t - \dot{k}^*\right] \\ \dot{k}_t &= \dot{k}_t \end{aligned} \tag{4}$$

This looks trivial, but it's no surprise as we took a linear approximation of a linear function, and, obviously, get back the function itself.

Apply (1) to $F(x) = F(k_t) = (\delta + g + n) k_t$:

$$\begin{aligned} F(k_t) &= F(k^*) + F'(k^*)[k_t - k^*] \\ (\delta + g + n) k_t &= (\delta + g + n) k^* + (\delta + g + n)[k_t - k^*] \end{aligned} \tag{5}$$

Finally, apply (1) to $F(x) = F(k_t, h_t) = s_K (k_t)^\alpha$:

$$\begin{aligned} F(k_t) &= F(k^*) + F'(k^*)[k_t - k^*] \\ s_K (k_t)^\alpha &= s_K (k^*)^\alpha + s_K \alpha (k^*)^{\alpha-1}[k_t - k^*] \end{aligned} \tag{6}$$

Now combine (4)-(5)-(6) to rewrite the elements of (2), using (3) in the fourth line:

$$\begin{aligned}
\dot{k}_t &= s_K (k_t)^\alpha - (\delta + g + n) k_t \\
\dot{k}_t &= s_K (k^*)^\alpha + s_K \alpha (k^*)^{\alpha-1}[k_t - k^*] - (\delta + g + n) k^* - (\delta + g + n)[k_t - k^*] \\
\dot{k}_t &= [s_K (k^*)^\alpha - (\delta + g + n) k^*] + \left[s_K \alpha (k^*)^{\alpha-1} - (\delta + g + n)\right][k_t - k^*] \\
\dot{k}_t &= \left[s_K \alpha (k^*)^{\alpha-1} - (\delta + g + n)\right][k_t - k^*] \\
\dot{k}_t &= \left[\frac{(\delta + g + n) k^*}{(k^*)^\alpha} \alpha (k^*)^{\alpha-1} - (\delta + g + n)\right][k_t - k^*] \\
\dot{k}_t &= (\delta + g + n)[\alpha - 1][k_t - k^*] \\
\dot{k}_t &= -(\delta + g + n)[1 - \alpha][k_t - k^*]
\end{aligned}$$

Notice that:

$$\frac{\partial [k_t - k^*]}{\partial t} = \frac{\partial k_t}{\partial t} - \underbrace{\frac{\partial k^*}{\partial t}}_{=0:\ k^* \text{ is constant}} = \frac{\partial k_t}{\partial t} = \dot{k}_t$$

7 Speed of convergence

The dynamics of output are written as:

$$\begin{aligned} y_t &= (k_t)^\alpha (h_t)^\beta \\ \dot{y}_t &= \alpha (k_t)^{\alpha-1} (h_t)^\beta \dot{k}_t + \beta (k_t)^\alpha (h_t)^{\beta-1} \dot{h}_t \end{aligned}$$

Approximate this around the steady state where $\dot{k}_t = \dot{h}_t = 0$ and $k_t = k^*$ and $h_t = h^*$:

$$\dot{y}_t = \alpha (k^*)^{\alpha-1} (h^*)^\beta \dot{k}_t + \beta (k^*)^\alpha (h^*)^{\beta-1} \dot{h}_t$$

The idea is to compute $\dot{k}_t$ and $\dot{h}_t$ as functions of the distance from the steady state, $(k_t - k^*)$ and $(h_t - h^*)$.

We start by expanding the law of motions around the steady state:

$$\begin{aligned}
\dot{k}_t &= \alpha s_K (k^*)^{\alpha-1} (h^*)^\beta (k_t - k^*) + \beta s_K (k^*)^\alpha (h^*)^{\beta-1} (h_t - h^*) \\
&\quad - (\delta + n + g)(k_t - k^*) \\
\dot{h}_t &= \alpha s_H (k^*)^{\alpha-1} (h^*)^\beta (k_t - k^*) + \beta s_H (k^*)^\alpha (h^*)^{\beta-1} (h_t - h^*) \\
&\quad - (\delta + n + g)(h_t - h^*)
\end{aligned}$$

From the solutions for k^* and h^* we can write the savings rates as functions of steady state capital stocks:

$$s_K = \frac{k^* [\delta + n + g]}{(k^*)^\alpha (h^*)^\beta} \qquad s_H = \frac{h^* [\delta + n + g]}{(k^*)^\alpha (h^*)^\beta}$$

The approximated laws of motion are then:

$$\begin{aligned}
\dot{k}_t &= \alpha [\delta + n + g] (k_t - k^*) + \beta [\delta + n + g] \frac{k^*}{h^*} (h_t - h^*) \\
&\quad - (\delta + n + g) (k_t - k^*) \\
\dot{k}_t &= (\delta + n + g) \left[(\alpha - 1) (k_t - k^*) + \beta \frac{k^*}{h^*} (h_t - h^*) \right]
\end{aligned}$$

and:

$$\begin{aligned}
\dot{h}_t &= \alpha [\delta + n + g] \frac{h^*}{k^*} (k_t - k^*) + \beta [\delta + n + g] (h_t - h^*) \\
&\quad - (\delta + n + g) (h_t - h^*) \\
\dot{h}_t &= (\delta + n + g) \left[\alpha \frac{h^*}{k^*} (k_t - k^*) + (\beta - 1) (h_t - h^*) \right]
\end{aligned}$$

Next use this in the approximated output dynamics:

$$\begin{aligned}
\dot{y}_t &= \alpha (k^*)^{\alpha-1} (h^*)^\beta \dot{k}_t + \beta (k^*)^\alpha (h^*)^{\beta-1} \dot{h}_t \\
\dot{y}_t &= \alpha (k^*)^{\alpha-1} (h^*)^\beta (\delta + n + g) \left[(\alpha - 1) (k_t - k^*) + \beta \frac{k^*}{h^*} (h_t - h^*) \right] \\
&\quad + \beta (k^*)^\alpha (h^*)^{\beta-1} (\delta + n + g) \left[\alpha \frac{h^*}{k^*} (k_t - k^*) + (\beta - 1) (h_t - h^*) \right] \\
\dot{y}_t &= (\delta + n + g) (\alpha + \beta - 1) \left[\begin{array}{c} \alpha (k^*)^{\alpha-1} (h^*)^\beta (k_t - k^*) \\ + \beta (k^*)^\alpha (h^*)^{\beta-1} (h_t - h^*) \end{array} \right]
\end{aligned}$$

The technology is expanded around the steady state as:

$$y_t - y^* = \alpha (k^*)^{\alpha-1} (h^*)^\beta (k_t - k^*) + \beta (k^*)^\alpha (h^*)^{\beta-1} (h_t - h^*)$$

Using this, the approximated output dynamics become:

$$\dot{y}_t = -(1 - \alpha - \beta) (\delta + n + g) (y_t - y^*)$$

Converting this growth rate in level we write:

$$y_t - y^* = (y_0 - y^*) \exp \{ -(1 - \beta - \alpha) (\delta + n + g) t \}$$

hence the speed of convergence is given by $(1-\alpha-\beta)(\delta+n+g)$. The half-life of a deviation is:

$$\begin{aligned}\frac{1}{2} &= \exp\{-(1-\beta-\alpha)(\delta+n+g)t\} \\ -\ln[2] &= -(1-\beta-\alpha)(\delta+n+g)t \\ t &= \frac{\ln[2]}{(1-\beta-\alpha)(\delta+n+g)}\end{aligned}$$

Set $\delta+n+g=6\%$ and $\alpha=1/3$. Without human capital ($\beta=0$) convergence is fast ($t=17$), but if $\beta=1/3$ convergence is slower ($t=35$).

8 Income differences

Compare two countries in the steady state. Relative outputs are:

$$\ln\left(\frac{y_2^*}{y_1^*}\right)=\alpha\ln\left(\frac{k_2^*}{k_1^*}\right)+\beta\ln\left(\frac{h_2^*}{h_1^*}\right)$$

The marginal return of capital is:

$$MPK=\frac{\partial Y}{\partial K}=\alpha(K)^{\alpha-1}(H)^{\beta}(AL)^{1-\alpha-\beta}=\alpha(k^*)^{\alpha-1}(h^*)^{\beta}=\alpha\frac{y^*}{k^*}$$

Therefore:

$$\frac{MPK_2}{MPK_1}=\frac{y_2^*}{y_1^*}\left(\frac{k_2^*}{k_1^*}\right)^{-1}$$

Consider that country 2 has an output per worker 10 times as large as country 1 ($y_2^*/y_1^*=10$), and $\alpha=1/3$. Without human capital, this requires $k_2^*/k_1^*=1'000$, and the marginal return to capital that is 100 times larger in country 1. With $\beta=1/3$ capital (human and physical) only has to be 30 times as large in country 2 as in country 1, while the return on capital in country 1 is 3 times as large as in country 2.

9 Empirical evidence

In terms of regression. We have ($g=0.05$):

$$\ln y^*=\underbrace{\frac{\alpha}{1-\beta-\alpha}}_{0.73}[\ln(s_K)-\ln(n+0.05)]+\underbrace{\frac{\beta}{1-\beta-\alpha}}_{0.67}[\ln(s_H)-\ln(n+0.05)]$$

this implies $\alpha=0.31$, $\beta=0.28$, $\alpha+\beta=0.6$. Without human capital, we would estimate:

$$\ln y^*=\underbrace{\frac{\alpha}{1-\alpha}}_{1.48}[\ln(s)-\ln(n+0.05)]$$

which implies $\alpha=0.6$.