Lecture Notes:
Econometrics I

Based on lectures by **Marko Mlikota** in Autumn semester, 2024

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These lecture notes were taken in the course *Econometrics I* taught by **Marko Mlikota** at Graduate of International and Developoment Studies, Geneva as part of the International Economics program (Semester I, 2024).

Currently, these are just drafts of the lecture notes. There can be typos and mistakes anywhere. So, if you find anything that needs to be corrected or improved, please inform at jingle.fu@graduateinstitute.ch.

Contents

[1] This part is borrowed from Xiang Ao, March 24, 2009, *An Introduction to censored, truncated or sample-selected data*, Harvard Business School.

[3] This lecture is not required in class, I personally borrowed contents from different books and papers to form the part, based on Amemiya(1985)[1] and Hayashi(2000)[2]. Extremum estimators are a wide class of estimators for parametric models that are calculated through maximization (or minimization) of a certain objective function, which depends on the data. The general theory of extremum estimators was developed by Amemiya (1985)[1].

Lecture 1.

Statistical Inference

$$x : X \sim p(x|\theta)$$

θ is the parameter setting the shape of the distribution.

Definition 1.0.1. Point Estimator $\delta(X)$

A mapping δ from sample space of X to the parameter space Θ: $\delta : X \to \Theta$

Given X, what's the best Θ. $\delta(x)$ is an estimate.

1.1 Methods to get $\delta(X)$

1.1.1 LS Estimation

$$X_1 \sim p(x|\theta), \quad \mathbb{E}[X_1|\theta] = \theta, \quad X_1|\theta \sim N(0,1)$$

Point estimator $\hat{\theta}$ is the argument that minimizes the objective function

$$\hat{\theta}_{\text{LS}} = \arg\min_{\theta} \sum_{i=1}^{n} (x_i - \theta)^2$$

where we assume that $\theta = \mathbb{E}[x_i|\theta]$. Using the First Order Condition (FOC) to solve, we have

$$\frac{\partial(\cdot)}{\partial\theta} = \sum_{i=1}^{n} -2(x_i - \theta) = 0$$

We get $\hat{\theta}_{\text{LS}} = \frac{1}{n}\sum_{i=1}^{n} x_i$

1.1.2 Method of Moments

Find $\hat{\theta}$ such that

$$\mathbb{E}[X|\hat{\theta}_{\text{MM}}] = \frac{1}{n}\sum_{i=1}^{n} x_i$$

Or, such that

$$\mathbb{E}[X^2|\theta]_{\theta=\hat{\theta}_{\text{MM}}} = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

$\mathbb{V}[X|\theta] = 1$, thus $\mathbb{E}[X^2|\theta] = \mathbb{V}[X|\theta] + (\mathbb{E}[X|\theta]^2) = 1 + \theta^2$.

$$1 + \hat{\theta}^2_{\text{MM}} = \frac{1}{n}\sum_{i=1}^{n} x_i^2$$

We get

$$\hat{\theta}_{\text{MM}} = \sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2 - 1}$$

Note. Choose $\hat{\theta}_{\text{MM}}$ s.t. $\mathbb{E}[h(X)|\theta]$ under $\theta = \hat{\theta}_{\text{MM}}$ is the mean of samples $\frac{1}{n}\sum_{i=1}^{n} x_i^2$.

1.1.3 Maximum Likelihood

$$\hat{\theta}_{\text{ML}} = \arg\max_{\theta} \mathcal{L}(\theta|x)$$

where

$$\mathcal{L}(\theta|x) = p(x|\theta)$$

is the PDF of the RV $X|\theta$.

Note. Specify the whole distribution.

With X as i.i.d. distribution, we have

$$\begin{aligned}
\mathcal{L}(\theta|x) &= p(x|\theta) \\
&= \prod_{i=1}^{n} p(x_i|\theta) \\
&= \prod_{i=1}^{n} (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right\} \\
&= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2\right\} \\
&\sim N(0,\sigma^2)
\end{aligned}$$

Then, let's define $\ell(\theta|x) = \log \mathcal{L}(\theta|x) = -\frac{n}{2}\log(2\pi) - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2$

$$\hat{\theta}_{\text{ML}} = \frac{1}{n}\sum_{i=1}^{n} x_i = \hat{\theta}_{\text{MM}} = \hat{\theta}_{\text{LS}}$$

$$\begin{aligned}
\hat{\theta}_{\text{ML}} &= \arg\max_{\theta} -\frac{n}{2}\log(2\pi) - \frac{1}{2\sigma^2}\sum(x_i-\mu)^2 \\
&= \arg\min_{\theta} \sum(x_i-\mu)^2
\end{aligned}$$

Definition 1.1.1. $\delta(X)$ is unbiased if $\mathbb{E}[\delta(X)|\theta] = \theta$.

e.g. $\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i$ is unbiased because

$$\begin{aligned}
\mathbb{E}[\hat{\theta}|\theta] &= \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} X_i|\theta\right] \\
&= \frac{1}{n}\mathbb{E}\left[\sum X_i\right] \\
&= \frac{1}{n}\sum \mathbb{E}[X_i] \\
&= \theta
\end{aligned}$$

However, $\hat{\theta}_* = \frac{1}{n-1}\sum_{i=1}^{n} X$ is biased, because

$$\begin{aligned}\mathbb{E}[\hat{\theta}_*|\theta] &= \mathbb{E}\left[\frac{1}{n-1}\sum_{i=1}^{n} X_i|\theta\right] \\ &= \frac{1}{n-1}\mathbb{E}\left[\sum X_i\right] \\ &= \frac{n}{n-1}\theta \\ &\neq \theta\end{aligned}$$

For the variance,

$$\mathbb{V}[\hat{\theta}|\theta] = \mathbb{V}\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{V}[X_i|\theta] = \frac{\sigma^2}{n}$$

and,

$$\mathbb{V}[\hat{\theta}_*|\theta] = \mathbb{V}\left[\frac{n}{n-1}\hat{\theta}\right] = \frac{n^2}{(n-1)^2}\mathbb{V}[\hat{\theta}] = \frac{n\sigma^2}{(n-1)^2}$$

To get the estimation of $\mathbb{E}$ and $\mathbb{V}$, we have to assume the mean and variance of the distribution.

Under $X_i|\theta \sim N(0,1)$, we have $\hat{\theta} = \frac{1}{n}\sum X_i \sim N\left(0, \frac{1}{n}\right)$.

1.2 Asymptotic Properties

Definition 1.2.1. Point estimator $\delta(X)$ is consistent if $\delta(X) \xrightarrow{p} \theta$.

By Weak Law of Large Numbers (WLLN),

$$\hat{\theta} = \frac{1}{n}\sum X_i \xrightarrow{p} \mathbb{E}[X] = \theta$$

Now let's look at $\hat{\theta}_*$ again,

$$\hat{\theta}_* = \frac{1}{n-1}\sum X_i = \frac{n}{n-1}\frac{1}{n}\sum X_i = \frac{n}{n-1}\mathbb{E}[X] = \frac{n}{n-1}\hat{\theta} \xrightarrow{p} \theta$$

as $\lim_{n\to\infty} \frac{n}{n-1} = 1$ and $\hat{\theta} \xrightarrow{p} \theta$. Slutsky's theorem tells us that we can form the limit of their product as the product of the limits.

Central Limit Theorem (CLT):

$$\begin{aligned}&\sqrt{n}\frac{\frac{1}{n}\sum X_i - \mathbb{E}[X_i]}{\sqrt{\mathbb{V}[X_i]}} \xrightarrow{p} N(0,1) \\ \Rightarrow &\sqrt{n}\frac{\bar{X}_n - \mu}{\sigma} \xrightarrow{d} N(0,1) \text{ Set the variance to 1} \\ \Rightarrow &\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N(0,1) \\ \Rightarrow &\sqrt{n}(\hat{\theta} - \theta) \overset{\text{approx}}{\sim} N(0,1) \text{ in our finite sample of } n \\ \Rightarrow &(\hat{\theta} - \theta) \sim N\left(0, \frac{1}{n}\right) \\ \Rightarrow &\hat{\theta} \overset{\text{approx}}{\sim} N\left(\theta, \frac{1}{n}\right)\end{aligned}$$

RECALL: Statistical inference given x, what can we say?

1. Point inference, best guess for θ

2. Hypothesis testing, is θ larger than 1 or not?

3. Interval inference, give an interval where you are sure that θ lies in.

Lecture 2.

Problem Set 1

Solution.

For the Least Squares estimation, we need to solve the following problem:

$$\min \sum_{i=1}^{n} (x_i - \theta)^2$$

Denote $F = \sum_{i=1}^{n} (x_i - \theta)^2$ and take the first order derivative with respect to θ, we have the FOC:

$$\begin{aligned} \frac{\partial F}{\partial \theta} &= \sum -2(x_i - \theta) = 0 \\ \Rightarrow \hat{\theta} &= \frac{1}{n} \sum_{i=1}^{n} x_i \end{aligned}$$

Solution.

The mean of $\hat{\theta}$ is its expectation, so we calculate:

$$\mathbb{E}[\hat{\theta}] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{1}{n}\mathbb{E}\left[\sum_{i=1}^{n} x_i\right] \overset{1}{=} \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[x_i] \overset{2}{=} \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[u_i] = \frac{1}{n}\cdot n\theta = \theta$$

1. We are using the property of $\mathbb{E}$: $\mathbb{E}\left[\sum_{i=1}^{n} x_i\right] = \sum_{i=1}^{n}\mathbb{E}[x_i]$, no matter x_i are independent or not.
2. We are using the property of $\mathbb{E}$: $\mathbb{E}[\theta + u_i] = \mathbb{E}[u_i] + \theta$ if θ is a constant number.

Thus, $\hat{\theta}$ is unbiased and we make no other assumptions on pdf of $x_i|\theta$ and the sample $\{x_i\}_{i=1}^n$.

Solution.

$$\mathbb{V}[\hat{\theta}] = \mathbb{V}\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{1}{n^2}\mathbb{V}\left[\sum_{i=1}^{n} x_i\right] = \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{V}[x_i] = \frac{\sigma^2}{n}$$

By having this result, we make the following 2 assumptions:

1. For sample set $\{x_i\}_{i=1}^n$, we assume that for all samples x_i, they are mutually independent, which gives that $\mathbb{V}\left[\sum_{i=1}^{n} u_i\right] = \sum_{i=1}^{n}\mathbb{V}[u_i]$.
2. For the pdf of $x_i|\theta$, we assume that x_i are independently distributed, which means that $\mathbb{V}[x_i] = \mathbb{V}[u_i] = \sigma^2$

Lecture 3.

Hypothesis Testing

3.1 Some Basic Concepts

Definition 3.1.1. Null Hypothesis

The null hypothesis $\mathcal{H}_0$ is the set $\theta = \theta_0$ or $\beta \in \mathcal{B}_0$.

Or, we denote it as:

$$\mathcal{H}_0 : \theta \in \Theta_0$$

For econometrics, we usually set $\mathcal{H}_0 : \beta = 0$.

Definition 3.1.2. Alternative Hypothesis

The alternative hypothesis $\mathcal{H}_1$ is the set $\{\theta \in \Theta : \theta \neq \theta_0\}$ or $\{\beta \in \mathcal{B} : \beta \notin \mathcal{B}_0\}$.

Or, we denote it as:

$$\mathcal{H}_1 : \theta \in \Theta_1$$

For econometrics, we usually set $\mathcal{H}_1 : \beta \neq 0$. Often Θ_1 is the complement of Θ_0.

Note. Point estimator of $\mathbf{1}\{\theta \in \Theta_0\}$ (if $\theta \in \Theta_0$, the function equals 1).

	$\mathcal{H}_0$ is true	$\mathcal{H}_0$ is false
Accept $\mathcal{H}_0$	✓: $1-\alpha$	×: $1-\beta$
Reject $\mathcal{H}_0$	×: α	✓: β

α is the Type I error, $1-\beta$ is the Type II error.

Definition 3.1.3. A hypothesis test $\varphi \in \{0,1\}$ is a rule that specifies when we reject and when we accept (do not reject) $\mathcal{H}_0$, with $\varphi = 0$ indicating rejection.

Definition 3.1.4. Power Function

$$\beta(\theta) = \mathbb{P}[\text{rejecting}|\theta \text{ is true}] = \mathbb{P}[\varphi = 0|\theta]$$

Definition 3.1.5. Size of a test

The size of a test is α if $\sup_{\theta \in \Theta_0} \beta(\theta_0) = \alpha$, $\alpha \in (0,1)$.

Generic form:

$$\varphi(x;\alpha) = \mathbf{1}\{T(x) < c_\alpha\}$$

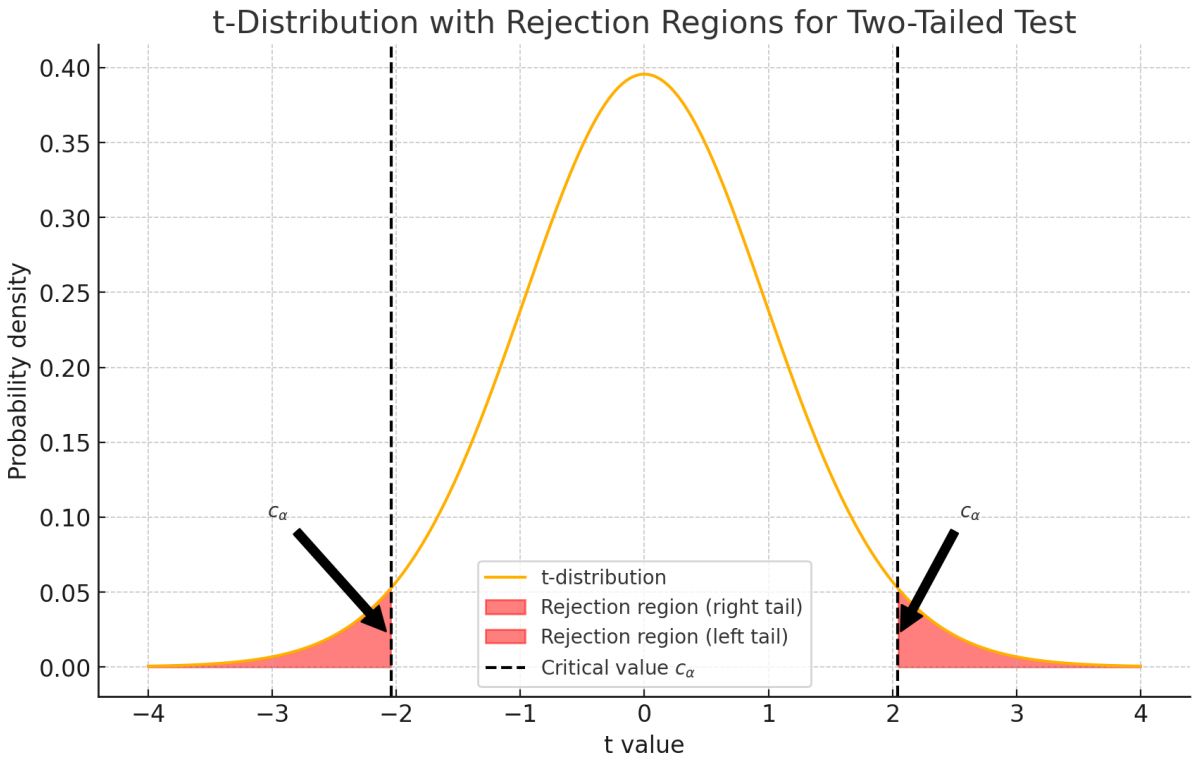
3.2 T-test

Definition 3.2.1. Suppose $\hat{\theta}|\theta \sim N(\theta, v^2)$, and we are testing a point hypothesis $\mathcal{H}_0 : \theta = \theta_0$. Under the alternative $\mathcal{H}_1 : \theta \neq \theta_0$, the two-sided t-test is

$$\varphi_t(x) = \mathbf{1}\left\{\left|\frac{\hat{\theta} - \theta_0}{v}\right| < c\right\}$$

Note. The test-statistic $T(X) = \left|\frac{\hat{\theta}-\theta_0}{v}\right|$ is a function of data X because our estimator θ, $\hat{\theta}$, is a function of X.

Example 1. Let $X|\theta \sim N(\theta, v^2)$, under $\mathcal{H}_0 : \hat{\theta} \sim N(\theta_0, v^2) \to \frac{\hat{\theta}-\theta_0}{v} \sim N(0,1)$



$$\begin{aligned}
\beta(\theta) &= \mathbb{P}[\text{rejecting}|\theta \text{ is true}] \\
&= \mathbb{P}[\varphi = 0|\theta] \\
&= \mathbb{P}[T(X) > c_\alpha|\theta] \\
&= \mathbb{P}\left[\left|\frac{\hat{\theta} - \theta_0}{v}\right| > c_\alpha|\theta\right] \\
&= 1 - \mathbb{P}\left[\left|\frac{\hat{\theta} - \theta_0}{v}\right| \leq c_\alpha|\theta\right] \\
&= 1 - \mathbb{P}\left[-c_\alpha \leq \frac{\hat{\theta} - \theta_0}{v} \sim N(0,1) \leq c_\alpha|\theta\right] \\
&= 1 - \left(\mathbb{P}\left[\frac{\hat{\theta} - \theta_0}{v} \leq c_\alpha|\theta\right] - \mathbb{P}\left[\frac{\hat{\theta} - \theta_0}{v} \leq -c_\alpha|\theta\right]\right) \\
&= 1 - [\Phi(c_\alpha) - \Phi(-c_\alpha)]
\end{aligned}$$

$$
\begin{aligned}
&= 1 - [\Phi(c_\alpha) - (1 - \Phi(c_\alpha))] \\
&= 2 - 2\Phi(c_\alpha) \\
&= \alpha
\end{aligned}
$$

Under $\alpha = 0.05$, we get $c_\alpha = 1.64$, $\alpha = 0.1$, we get $c_\alpha = 1.96$.

To compute the power of this test, we need to think about what happens if $\mathcal{H}_0$ is false. Assuming that $\tilde{\theta}$ is the true value of θ.

$$
\begin{aligned}
\beta(\tilde{\theta}) &= \mathbb{P}[\text{rejecting}|\tilde{\theta} \text{ is true}] \\
&= \mathbb{P}[\varphi = 0|\tilde{\theta}] \\
&= \mathbb{P}[T(X) > c_\alpha|\tilde{\theta}] \\
&= \mathbb{P}\left[\left|\frac{\hat{\theta} - \theta_0}{v}\right| > c_\alpha|\tilde{\theta}\right]
\end{aligned}
$$

To find this, under $\tilde{\theta}$ is true, $\tilde{\theta} \sim N(\tilde{\theta}, v^2)$, $\hat{\theta} - \tilde{\theta} \sim N(0, v^2)$, $\hat{\theta} - \theta_0 \sim N(\tilde{\theta} - \theta_0, v^2)$, $\frac{\hat{\theta}-\theta_0}{v} \sim N(\tilde{\theta} - \theta_0, 1)$, $\frac{\hat{\theta}-\theta_0}{v} - (\tilde{\theta} - \theta_0) \sim N(0, 1)$

So,

$$
\begin{aligned}
\beta(\tilde{\theta}) &= \mathbb{P}\left[\left|\frac{\hat{\theta} - \theta_0}{v}\right| > c_\alpha|\tilde{\theta}\right] \\
&= 1 - \mathbb{P}\left[\left|\frac{\hat{\theta} - \theta_0}{v}\right| \leq c_\alpha|\theta\right] \\
&= 1 - \mathbb{P}\left[-c_\alpha \leq \frac{\hat{\theta} - \theta_0}{v} \leq c_\alpha|\theta\right] \\
&= 1 - \mathbb{P}[-c_\alpha - (\tilde{\theta} - \theta_0) \leq z \sim N(0,1) \leq c_\alpha - (\tilde{\theta} - \theta_0)] \\
&= 1 - (\Phi[c_\alpha - (\tilde{\theta} - \theta_0)] - \Phi[-c_\alpha - (\tilde{\theta} - \theta_0)])
\end{aligned}
$$

The higher the probability of wrongly accepting (or failing to reject) $\mathcal{H}_0$. It is common to be rather conservative (i.e. erring on the side of not rejecting $\mathcal{H}_0$) and report test results for sizes of 10%, 5% and 1%.

3.3 Likelihood Ratio Test

Definition 3.3.1.

$$
\varphi_{\text{LR}}(x) = \mathbf{1}\left\{\frac{\sup\limits_{\theta\in\Theta_1} p(x|\theta)}{\sup\limits_{\theta\in\Theta_0} p(x|\theta)} < c_\alpha\right\}, T_{\text{LR}}(X) = \frac{\sup\limits_{\theta\in\Theta_1} p(x|\theta)}{\sup\limits_{\theta\in\Theta_0} p(x|\theta)}.
$$

So, if there are points in Θ_0 for which observed x is more likely than points in Θ_1, the ratio is small — the test is likely to accept $\mathcal{H}_0$.

Under $\mathcal{H}_0 : \theta = \theta_0$ and the alternative $\mathcal{H}_1 : \theta \neq \theta_0$,

$$
\varphi_{\text{LR}}(x) = \mathbf{1}\left\{\frac{p(x|\hat{\theta}_{\text{ML}})}{p(x|\theta_0)} < c\right\}
$$

Example 2. $\{x_i\}_{i=1}^n$, $x_i|\theta \sim N(\theta, 1)$,

$$p(x|\theta) = \prod_i p(x_i|\theta) = (2\pi)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x-\theta)^2\right\}$$

if $n = 1$. As $x = \hat{\theta}$, $p(x|\hat{\theta}) = (2\pi)^{-\frac{1}{2}}$

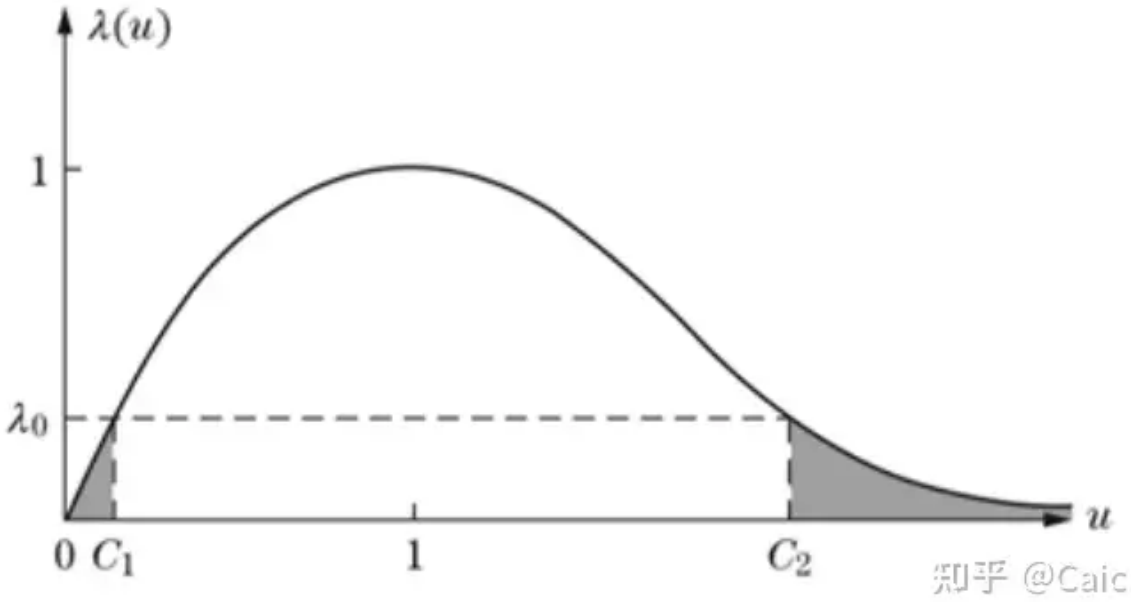
$$T(x) = \frac{p(x|\hat{\theta}_{\text{ML}})}{p(x|\theta_0)} = \frac{(2\pi)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x-\hat{\theta})^2\right\}}{(2\pi)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x-\theta_0)^2\right\}} = \exp\left\{\frac{1}{2}(x-\theta_0)^2\right\}$$

as $x = \hat{\theta}$.

$$\varphi_{\text{LR}} = \mathbf{1}\left\{\exp\left\{\frac{1}{2}(x-\theta_0)^2\right\} < c_\alpha\right\}$$

$$\begin{aligned}
\alpha &= \mathbb{P}\{\text{Reject}|\theta_0 \text{ is actually true}\} \\
&= \mathbb{P}\left\{\exp\left\{\frac{1}{2}(x-\theta_0)^2\right\} \geq c_\alpha|\theta_0\right\} \\
&= \mathbb{P}\{(x-\theta_0)^2 \geq 2\log c_\alpha|\theta_0\} \\
&= 1 - \mathbb{P}\left\{(x-\theta_0)^2 < 2\log c_\alpha|\theta_0\right\}
\end{aligned}$$

If $x \sim N(\theta_0, 1)$, $(x-\theta_0)^2 \sim \chi_1^2$



Note. **Uniformly most powerful test** Highest probability of rejection, of wrong acceptance.

$$\varphi(x) = \mathbf{1}\{T(x) < c_\alpha\}$$

$$\alpha = \mathbb{P}\{T(x) > c_\alpha|\theta_0\}(\text{reject rule})$$

3.3.1 Numerical Hypothesis Testing

Algorithm 1: Numerical Hypothesis Testing

Input : Distribution $N(\theta_0, 1)$, sample size M, significance level α
Output: Decision to accept or reject H_0

for $m = 1$ **to** M **do**
  Draw $x^m \sim N(\theta_0, 1)$;
  Compute $T(x^m)$;
end
Sort $\{T(x^m)\}_{m=1}^M$ in ascending order;
Set c_α to the $100(1-\alpha)$ quantile of $\{T(x^m)\}_{m=1}^M$;
Compute $T(x)$ for your observed realization x;
if $T(x) \leq c_\alpha$ **then**
  Accept H_0;
else
  Reject H_0;
end

Small p-value means $\mathcal{H}_0$ is likely to be rejected, larger p-value means it's likely to be true.

3.4 Coverage Sets

3.4.1 Frequentist Confidence Sets

A confidence set $C(X) \subseteq \Theta$ is a (random) set that should cover the true θ with a prespecified probability:

$$\inf_{\theta \in \Theta} \mathbb{P}[\theta \in C(X)|\theta] = 1 - \alpha$$

$$C(x) = \{\theta_0 \in \Theta : \varphi(x; \theta_0) = 1\}$$

contains all the values of θ_0 that we would accept.

Consider $\mathcal{H}_0 : \theta = \theta_0$ vs. $\mathcal{H}_1 : \theta \neq \theta_0$, $\varphi_\alpha(x) = \mathbf{1}\{T(x) < c_{\alpha;\theta_0}\}$.

Example 3. T-test: $T(x) = \left|\frac{\hat{\theta}-\theta_0}{v}\right| \Rightarrow \varphi\{x; \theta_0, \alpha\} = \mathbf{1}\{T(x) < c\}$

$$\begin{aligned} C(x) &= \{\theta_0 \in \Theta, \varphi\{x; \theta_0, \alpha\} = 1\} \\ &= \left\{\theta_0 \in \Theta, \left|\frac{\hat{\theta} - \theta_0}{v}\right| < c\right\} \\ &= \left\{\theta_0 \in \Theta, -c < \frac{\hat{\theta} - \theta_0}{v} < c\right\} \\ &= \{\theta_0 \in \Theta, -cv + \hat{\theta} < \theta_0 < cv + \hat{\theta}\} \end{aligned}$$

Example 4. LR-test: $\varphi(x; \theta_0, \alpha) = \mathbf{1}\{(x - \theta_0)^2 < \tilde{c_\alpha}\}$

$$\begin{aligned} C(x) &= \{\theta_0 \in \Theta, \varphi\{x; \theta_0, \alpha\} = 1\} \\ &= \{\theta_0 \in \Theta, (x - \theta_0)^2 < \tilde{c_\alpha}\} \\ &= \left\{\theta_0 \in \Theta, -\sqrt{\tilde{c_\alpha}} < x - \theta_0 < \sqrt{\tilde{c_\alpha}}\right\} \end{aligned}$$

$$= \left\{\theta_0 \in \Theta, x - \sqrt{\tilde{c_\alpha}} < \theta_0 < x + \sqrt{\tilde{c_\alpha}}\right\}$$

3.4.2 Numerical Confidence Set Construction

Algorithm 2: Numerical Confidence Set Construction

Data: Choose a grid $\mathcal{T}$ of values for θ_0

for *each* $\theta_0 \in \mathcal{T}$ **do**
  for $m = 1$ **to** M **do**
    Draw $x^m \sim N(\theta_0, 1)$; `// Distribution of` X `under` $H_0 : \theta = \theta_0$
    Compute $T(x^m, \theta_0)$;
  end
  Get the critical value $c_\alpha(\theta_0)$ as the $(1-\alpha)$th quantile of $\{T(x^m, \theta_0)\}_{m=1}^M$;
  Compute $T(x; \theta_0)$ for observed x;
  if $T(x; \theta_0) \leq c_\alpha(\theta_0)$ **then**
    $\theta_0 \in C(x)$;
  else
    $\theta_0 \notin C(x)$;
  end
end

Lecture 4.

Problem Set 2

Solution.

Step 1: Write the pdf of observations $x_i|\theta$

Since $x_i = \theta + u_i$, and we assume $u_i \sim \mathcal{N}(0, \sigma^2)$, then $x_i \sim \mathcal{N}(\theta, \sigma^2)$, and we have:

$$p(x_i|\theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2}\frac{(x_i-\theta)^2}{\sigma^2}\right\}.$$

Step 2: Define Likelihood Function

We have assumed that observations in the sample are independent. Thus,

$$L_n(\theta) = \prod_{i=1}^{n} p(x_i|\theta) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2\right\}$$

Log-linearize the function, and we define the log-likelihood function:

$$\ell_n(\theta) = n\log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2$$

Step 3: Define the Likelihood Estimation problem and find the $\hat{\theta}$

For maximum likelihood estimation, we need to solve the following problem:

$$\hat{\theta}_{ML} = \arg\max_{\theta\in\Theta} L_n(\theta) = \arg\max_{\theta\in\Theta} \ell_n(\theta)$$

So, we need to maximize:

$$\ell_n(\theta) = n\log\left(\frac{1}{\sqrt{2\pi\sigma^2}}\right) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\theta)^2$$

Take the derivative of $\ell_n(\theta)$ with respect to θ, and set it to zero for maximization,

$$\begin{aligned}\frac{\partial \ell_n(\theta)}{\partial\theta} &= \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\theta)\\ &= \frac{1}{\sigma^2}(\sum_{i=1}^{n}x_i - n\theta)\\ &= 0\end{aligned}$$

Thus, we have

$$\hat{\theta}_{ML} = \frac{1}{n}\sum_{i=1}^{n}x_i$$

Solution.

Step 1: Find the likelihood function

For $\mathcal{H}_0 : \theta = \theta_0$, the likelihood function is:

$$L_n(\theta_0) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \theta_0)^2\right\}$$

For $\mathcal{H}_1 : \theta \neq \theta_0$, the maximum likelihood estimator is $\hat{\theta}_{ML} = \frac{1}{n}\sum_{i=1}^{n} x_i = \hat{\theta}$. The likelihood function is:

$$L_n(\theta_1) = \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^n \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \hat{\theta})^2\right\}$$

Step 2: Define the likelihood ratio test and c_α

The likelihood ratio and the test is firstly defined as follows(we'll simplify to another version later):

$$\varphi_{LR}(x) = \mathbf{1}\{LR_n < c\}$$

$$\begin{aligned} LR_n &= \frac{L_n(\theta_1)}{L_n(\theta_0)} \\ &= \exp\left\{\frac{1}{2\sigma^2}\left[\sum_{i=1}^{n}(x_i - \theta_0)^2 - \sum_{i=1}^{n}(x_i - \hat{\theta})^2\right]\right\} \end{aligned}$$

Denote

$$D = \sum_{i=1}^{n}(x_i - \theta_0)^2 - \sum_{i=1}^{n}(x_i - \hat{\theta})^2$$

Using the identity:

$$\sum_{i=1}^{n}(x_i - \theta_0)^2 = \sum_{i=1}^{n}(x_i - \hat{\theta} + \hat{\theta} - \theta_0)^2 = \sum_{i=1}^{n}(x_i - \hat{\theta})^2 + n(\hat{\theta} - \theta_0)^2$$

Thus,

$$\begin{aligned} D &= \sum_{i=1}^{n}(x_i - \theta_0)^2 - \sum_{i=1}^{n}(x_i - \hat{\theta})^2 \\ &= \sum_{i=1}^{n}(x_i - \hat{\theta})^2 + n(\hat{\theta} - \theta_0)^2 - \sum_{i=1}^{n}(x_i - \hat{\theta})^2 \\ &= n(\hat{\theta} - \theta_0)^2 \end{aligned}$$

So, the likelihood ratio LR_n is:

$$LR_n = \exp\left\{\frac{n}{2\sigma^2}(\hat{\theta} - \theta_0)^2\right\}$$

Then, we simplify the expression and define the test statistic $T(x)$ as below:

$$T(x) = 2\log(LR_n) = \frac{n}{\sigma^2}(\hat{\theta} - \theta_0)^2$$

And our LR test would be:

$$\varphi_{LR}(x) = \mathbf{1}\left\{T(x) = \frac{n}{\sigma^2}(\hat{\theta} - \theta_0)^2 < c'\right\}$$

where $c' = 2\log(c)$. To get a size α test, we find c' so as to set the Type I error to α, which is:

$$\mathbb{P}[T(x) \geq c' | \mathcal{H}_0] = \alpha$$

we can denote that $c' = c_\alpha$.

Step 3: Determine the distribution of $T(x)$ under $\mathcal{H}_0$ and find the value of c_α

Under $\mathcal{H}_0$, $\hat{\theta} \sim \mathcal{N}(\theta_0, \frac{\sigma^2}{n})$, because:

$$\mathbb{E}[\hat{\theta}] = \mathbb{E}\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}[x_i] = \frac{1}{n}\cdot n\theta_0 = \theta_0$$
$$\mathbb{V}[\hat{\theta}] = \mathbb{V}\left[\frac{1}{n}\sum_{i=1}^{n} x_i\right] = \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{V}[x_i] = \frac{1}{n}\cdot n\sigma^2 = \frac{\sigma^2}{n}$$

Then, standardizing $\hat{\theta}$, we'll have:

$$Z = \frac{\hat{\theta} - \theta_0}{\sqrt{\sigma^2/n}} \sim \mathcal{N}(0,1)$$

Using the hint, we know that

$$Z^2 = \left(\frac{\hat{\theta} - \theta_0}{\sqrt{\sigma^2/n}}\right)^2 = \frac{n}{\sigma^2}(\hat{\theta} - \theta_0)^2 \sim \chi_1^2$$

Therefore, under $\mathcal{H}_0$,

$$T(x) = \frac{n}{\sigma^2}(\hat{\theta} - \theta_0)^2 = Z^2 \sim \chi_1^2$$

Given $\alpha = 0.05$,

$$c_\alpha = \chi^2_{1,0.95} \approx 3.8415$$

Step 4: Set the decision rule

- Reject $\mathcal{H}_0$: $T(x) > c_\alpha = 3.8415$
- Do not reject $\mathcal{H}_0$: $T(x) \leq c_\alpha = 3.8415$

Solution.

We have $\sigma^2 = 6$, $n = 4$, $x_1 = 178$, $x_2 = 161$,$x_3 = 168$, $x_4 = 172$, $\theta_0 = 175$, so $\hat{\theta} = 169.75$.

Put this data back into our $T(x)$ and LR test, we have:

$$T(x) = \frac{n}{\sigma^2}(\hat{\theta} - \theta_0)^2 = \frac{4}{6}(169.75 - 175)^2 = 18.735 > 3.8415$$

We reject $\mathcal{H}_0$.

Solution.

Numerical approximation of c_α: 3.6266

Analyticalc_α from chi-squared distribution: 3.8415

Difference between numerical and analytical c_α: 0.2148, which is about 5.6% of the analytical c_α, so our approximation is not very close to the true value c_α.

I expect the estimated approximation to get closer to the real analytical value of c_α as M is larger.

Since the $T(x)$ we get is 18.735 which is greatly larger than 3.84 and 3.62, which is our numerical result, the conclusion from previous exercise doesn't change, we still reject $\mathcal{H}_0$.

Solution.

Based on our previous LR test, we have:

$$\varphi_{LR}(x) = \mathbf{1}\left\{T(x) = \frac{n}{\sigma^2}(\hat{\theta} - \theta_0)^2 < c_\alpha\right\}$$
$$= \mathbf{1}\left\{(\hat{\theta} - \theta_0)^2 < \frac{c_\alpha \sigma^2}{n}\right\}$$

$$= \mathbf{1}\left\{-\sqrt{\frac{c_\alpha \sigma^2}{n}} < (\hat{\theta} - \theta_0) < \sqrt{\frac{c_\alpha \sigma^2}{n}}\right\}$$
$$= \mathbf{1}\left\{\hat{\theta} - \sqrt{\frac{c_\alpha \sigma^2}{n}} < \theta_0 < \hat{\theta} + \sqrt{\frac{c_\alpha \sigma^2}{n}}\right\}$$

Thus, we can define $C(X)$ as:

$$C(X) = \left[\hat{\theta} - \sqrt{\frac{c_\alpha \sigma^2}{n}}, \hat{\theta} + \sqrt{\frac{c_\alpha \sigma^2}{n}}\right]$$

Apply our previous data: $\sigma^2 = 6$, $n = 4$, $x_1 = 178$, $x_2 = 161$,$x_3 = 168$, $x_4 = 172$, $\theta_0 = 175$, $\hat{\theta} = 169.75$, and $c_\alpha = 3.8415$, we have:

$$C(X) = [169.75 - 2.4, 169.75 + 2.4] = [167.35, 172.15]$$

$\theta_0 = 175$ is not in this interval.

Because we rejected $\mathcal{H}_0 : \theta = 175$, it's consistent that 175 is not within the 95% confidence interval.

Lecture 5.

Least Squares Estimation of the Linear Regression Model

5.1 Finite Sample Properties

$$y_i = x_i'\beta + u$$
$$Y = X\beta + U$$

where

$$Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1}, \quad X = \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix}_{n\times k}, \quad U = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}_{n\times 1}$$

Assumption 5.1.1. (Independent Sampling). Observations $z_i = \{y_t, x_i\}_{i=1}^n$ are independent across i.

Assumption 5.1.2. (Full rank). The matrix $X'X = \sum x_i x_i'$ is of full rank.

Assumption 5.1.3. (Conditional Independence). $\mathbb{E}[u_i|x_i] = 0$.
$\mathbb{E}[y_i] = \mathbb{E}[x_i'\beta + u_i|x_i] = \mathbb{E}[x_i'\beta|x_i] + \mathbb{E}[u_i|x_i] = x_i'\beta$

Assumption 5.1.4. (Homoskedasticity). $\mathbb{V}[u_i|x_i] = \sigma^2$ for all i.
$\mathbb{V}[y_i] = \mathbb{V}[x_i'\beta + u_i|x_i] = \sigma^2$

The OLS estimator:

$$\hat{\beta}_{\text{OLS}} = \arg\min_{\beta\in\mathbb{R}^k} \sum u_i^2 = \arg\min_{\beta\in\mathbb{R}^k} \sum_{i=1}^n (y_i - x_i'\beta)^2 = \arg\min_{\beta\in\mathbb{R}^k} (Y - X\beta)'(Y - X\beta)$$

$$\begin{aligned} \frac{\partial (Y - X\beta)'(Y - X\beta)}{\partial \beta} &= X'(Y - X\beta) = 0 \\ &\Rightarrow X'Y - X'X\beta = 0 \\ &\Rightarrow (X'X)^{-1}X'Y = \hat{\beta} \end{aligned}$$

$$\hat{Y} = X\hat{\beta} = X(X'X)^{-1}X'Y = P_X Y$$

where $P_X = X(X'X)^{-1}X'$ is the projection matrix.

$$y = x\hat{\beta} + \hat{u} \rightarrow y = \hat{y} + \hat{u}$$

thus,

$$\hat{U} = Y - \hat{Y} = Y - P_X Y = (I - P_X)Y = M_X Y$$

where M_X is another projection matrix.

$$Y = P_X Y + M_X Y = (P_X + M_X)Y.$$

In another sense:

$$\hat{U} = Y - \hat{Y} = X\beta + U - X(X'X)^{-1}X'(X\beta + U) = (I_n - X(X'X)^{-1}X')U = (I_n - P_X)U = M_X U.$$

P_X and M_X are idempotent: $P_X = P_X'$ and $P_X P_X = P_X$, and are orthogonal to each other: $P_X M_X = M_X P_X = 0$.

The total sum of squares (SST) is given by:

$$\sum_{i=1}^{n} y_i^2 = Y'Y$$

It measures the variability in y_i across observations i.

We can decompose it into the explained sum of squares (SSE) and the residual sum of squares (SSR)

$$\begin{aligned}
\text{SST} = Y'Y &= (P_X Y + M_X Y)'(P_X Y + M_X Y) \\
&= Y'P_X P_X Y + Y'P_X M_X Y + Y'M_X P_X Y + Y'M_X M_X Y \\
&= Y'P_X P_X Y + Y'M_X M_X Y \\
&= \hat{Y}'\hat{Y} + \hat{U}'\hat{U} \\
&= \text{SSE} + \text{SSR}
\end{aligned}$$

Based on that, we get the R^2-statistic as a measure of how well X accounts for the variation in Y in the linear regression model:

$$R^2 = \frac{\hat{Y}'\hat{Y}}{Y'Y} = 1 - \frac{\hat{U}'\hat{U}}{Y'Y} \in [0,1] \left(\frac{\text{SSE}}{\text{SST}} = 1 - \frac{\text{SSR}}{\text{SST}} \right)$$

Look at **Assumption 5.1.3** again, it always gives product have intercept $x_{i1} = 1$.

$$\begin{aligned}
\mathbb{E}[\hat{\beta}|X] &= \mathbb{E}[(X'X)^{-1}X'Y|X] \\
&= \mathbb{E}[(X'X)^{-1}X'(X\beta + U)|X] \\
&= \mathbb{E}[\beta|X] + \mathbb{E}[(X'X)^{-1}X'U|X] \\
&= \beta + (X'X)^{-1}X'\mathbb{E}[U|X] \\
&= \beta \\
\Rightarrow \mathbb{E}[\hat{\beta}] &= \mathbb{E}[\mathbb{E}[\hat{\beta}|X]] = \beta
\end{aligned}$$

For **Assumption 5.1.4**, the conditional variance of $\hat{\beta}_{OLS}$ is given by:

$$\begin{aligned}
\mathbb{V}[\hat{\beta}|X] &= \mathbb{E}[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'|X] \\
&= \mathbb{E}[((X'X)^{-1}X'U)((X'X)^{-1}X'U)'|X] \\
&= \mathbb{E}[(X'X)^{-1}X'UU'X(X'X)^{-1}|X] \\
&= (X'X)^{-1}X'\mathbb{E}[UU'|X]X(X'X)^{-1} \\
&= (X'X)^{-1}X'\sigma^2 X(X'X)^{-1} \\
&= \sigma^2(X'X)^{-1}
\end{aligned}$$

Note.

$$UU' = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} u_1 & \cdots & u_n \end{bmatrix} = \begin{bmatrix} u_1^2 & u_1u_2 & \cdots & u_1u_n \\ u_2u_1 & u_2^2 & \cdots & u_2u_n \\ \vdots & \vdots & \ddots & \vdots \\ u_nu_1 & u_nu_2 & \cdots & u_n^2 \end{bmatrix}$$

$$\mathbb{E}[UU'|X] = \begin{bmatrix} \mathbb{E}[u_1^2|X] & \mathbb{E}[u_1u_2|X] & \cdots & \mathbb{E}[u_1u_n|X] \\ \mathbb{E}[u_2u_1|X] & \mathbb{E}[u_2^2|X] & \cdots & \mathbb{E}[u_2u_n|X] \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{E}[u_nu_1|X] & \mathbb{E}[u_nu_2|X] & \cdots & \mathbb{E}[u_n^2|X] \end{bmatrix}$$

By **Assumption 5.1.3** and **Assumption 5.1.4**, we have: $\mathbb{E}[u_i|X] = 0$ and $\mathbb{E}[u_i^2|X] = \sigma^2$. Furthermore, $\mathbb{E}[u_iu_j|X] = 0$ for $i \neq j$.

$$\mathbb{E}[UU'|X] = \sigma^2 I_n$$

By LIE again, the unconditional variance of $\hat{\beta}_{OLS}$ is given by:

$$\begin{aligned} \mathbb{V}[\hat{\beta}] &= \mathbb{E}[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'] \\ &= \mathbb{E}[\mathbb{E}[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'|X]] \\ &= \mathbb{E}[\mathbb{V}[\hat{\beta}|X]] \\ &= \mathbb{E}[\sigma^2(X'X)^{-1}] \\ &= \sigma^2\mathbb{E}[(X'X)^{-1}] \end{aligned}$$

Theorem 5.1.1 (Gauss-Markov Theorem).

If **Assumption 5.1.1** to **Assumption 5.1.4** hold, then the OLS estimator $\hat{\beta}_{OLS}$ is the best linear unbiased estimator (**BLUE**) of β.

Note.

- **Best** means that the OLS estimator has the smallest variance among all linear unbiased estimators. $\mathbb{V}[\hat{\beta}_{OLS}] \leq \mathbb{V}[\hat{\beta}]$ for all linear unbiased estimators $\hat{\beta}$.
- **Linear** means that the estimator is a linear function of the dependent variable. $\hat{\beta} = c + dY$.
- **Unbiased** means that the expected value of the estimator is equal to the true value of the parameter. $\mathbb{E}[\hat{\beta}] = \beta$.

If, we know $\hat{\beta}$, $\mathbb{E}[\hat{\beta}]$ and $\mathbb{V}[\hat{\beta}]$. To find the unconditional expectation of $\hat{\beta}_{OLS}$, we could only use asymptotic properties of the OLS estimator. We can show that the OLS estimator is consistent and asymptotically normal.

$$\begin{aligned} \hat{\beta}_{OLS} &= (X'X)^{-1}X'Y \\ &= (X'X)^{-1}X'(X\beta + U) \\ &= \beta + (X'X)^{-1}X'U \\ &= \beta + \left(\sum x_ix_i'\right)^{-1}\left(\sum x_iu_i\right) \end{aligned}$$

$$
\begin{aligned}
&= \beta + \left(\frac{1}{n}\sum x_i x_i'\right)^{-1} \frac{1}{n}\left(\sum x_i u_i\right) \\
&\xrightarrow{p} \beta + \mathbb{E}[x_i x_i']^{-1}\mathbb{E}[x_i u_i]
\end{aligned}
$$

Note. By WLLN, we know that $\frac{1}{n}\sum z_i \xrightarrow{p} \mathbb{E}[z_i]$. So, $\frac{1}{n}\sum x_i u_i \xrightarrow{p} \mathbb{E}[x_i u_i] = Q$, and $\left[\frac{1}{n}\sum x_i x_i'\right]^{-1} \xrightarrow{p} \{\mathbb{E}[x_i x_i']\}^{-1} \xrightarrow{p} \mathbb{E}[x_i x_i']^{-1} = Q^{-1}$.

So, we have:

$$
\begin{aligned}
\hat{\beta}_{OLS} - \beta &\xrightarrow{p} \mathbb{E}[x_i x_i']^{-1}\mathbb{E}[x_i u_i] \\
&= \mathbb{E}[x_i x_i']^{-1}\mathbb{E}[\mathbb{E}[x_i u_i | x_i]] \\
&= \mathbb{E}[x_i x_i']^{-1}\mathbb{E}[x_i \mathbb{E}[u_i | x_i]] \\
&= \mathbb{E}[x_i x_i']^{-1}\mathbb{E}[x_i \cdot 0] \\
&= 0
\end{aligned}
$$

Note. By the Central Limit Theorem, we know that:

$$
\begin{aligned}
\sqrt{n}(\hat{\beta}_{OLS} - \beta) &= (X'X)^{-1}X'U \\
&= \underbrace{\left(\frac{1}{n}\sum x_i x_i'\right)^{-1}}_{\mathbb{E}[x_i x_i']^{-1}} \sqrt{n}\ \underbrace{\frac{1}{n}\sum x_i u_i}_{\xrightarrow{d}\mathcal{N}(0,\mathbb{V}[x_i u_i])} \\
&\xrightarrow{p} \mathbb{E}[x_i x_i']^{-1}\mathcal{N}(0, \mathbb{V}[x_i u_i])
\end{aligned}
$$

Thus, we have:

$$
\begin{aligned}
\sqrt{n}(\hat{\beta}_{OLS} - \beta) &\xrightarrow{d} \mathcal{N}(0, \mathbb{E}[x_i x_i']^{-1}\mathbb{V}[x_i u_i]\mathbb{E}[x_i x_i']^{-1'}) \\
\mathbb{V}[x_i u_i] &= \mathcal{N}(0, \mathbb{E}\left[(x_i u_i - \mathbb{E}[x_i u_i])(x_i u_i - \mathbb{E}[x_i u_i])'\right]) \\
&= \mathcal{N}(0, \mathbb{E}[x_i u_i u_i' x_i] - \mathbb{E}[x_i u_i]\mathbb{E}[x_i u_i]') \\
&= \mathcal{N}(0, \mathbb{E}[x_i u_i u_i' x_i']) \\
&= \mathcal{N}(0, \mathbb{E}[\mathbb{E}[u_i^2 | x_i] x_i x_i']) \\
&= \mathcal{N}(0, \mathbb{E}[\sigma^2 x_i x_i']) \\
&= \sigma^2 \mathbb{E}[x_i x_i'] \\
&= \sigma^2 Q \\
\Rightarrow \sqrt{n}(\hat{\beta}_{OLS} - \beta) &\xrightarrow{d} \mathcal{N}(0, \mathbb{E}[x_i x_i']^{-1}\sigma^2\mathbb{E}[x_i x_i']\mathbb{E}[x_i x_i']^{-1}) = \mathcal{N}(0, \sigma^2 Q^{-1})
\end{aligned}
$$

Then, we could say that:

- For finite n, $\sqrt{n}(\hat{\beta}_{OLS} - \beta) \overset{approx}{\sim} \mathcal{N}(0, \sigma^2 Q^{-1})$;
- $\hat{\beta} \overset{approx}{\sim} \mathcal{N}(\beta, \frac{\sigma^2}{n} Q^{-1})$;
- Replace unknown σ^2 and Q^{-1} by $\hat{\sigma}^2$ and $\hat{Q}^{-1}$ to get the t-distribution. We would have:

$$
\hat{\beta} \overset{approx}{\sim} \mathcal{N}\left(\beta, \frac{\hat{\sigma}^2}{n}\hat{Q}^{-1}\right).
$$

5.2 Hypothesis Testing

As $\hat{\beta} \overset{approx}{\sim} \mathcal{N}\left(\beta, \frac{\hat{\sigma}^2}{n}\hat{Q}^{-1}\right)$, we know that:

$$\hat{\beta}_j \overset{approx}{\sim} \mathcal{N}(\beta_j, \frac{\hat{\sigma}^2}{n}[\hat{Q}^{-1}]_{jj})$$

for a single parameter $\beta_j \in \beta$ where $[\hat{Q}^{-1}]_{jj}$ is the j-th diagonal element of $\hat{Q}^{-1}$.

This enables us to test a point hypothesis $\mathcal{H}_0 : \beta_j = \beta_{j,0}$ against the alternative $\mathcal{H}_1 : \beta_j \neq \beta_{j,0}$ using the t-test:

$$\varphi_t(x) = \mathbf{1}\left\{T_x < c\right\}, \text{with } T_t = \left|\frac{\hat{\beta}_{j,0} - \beta_j}{\hat{\sigma}_{\beta_{j,0}}}\right|,$$

where $\hat{\sigma}_{\beta_{j,0}} = \sqrt{\frac{\hat{\sigma}^2}{n}\hat{Q}^{-1}_{jj}}$.

Because the distribution of $\hat{\beta}_j$ is not exact, but only asymptotically valid, so too does the resulting test-statistic only asymptotically converge to a standard Normal distribution:

$$t = \frac{\hat{\beta}_j - \beta_{j,0}}{\sqrt{\frac{\hat{\sigma}^2}{n}[\hat{Q}^{-1}]_{jj}}} \xrightarrow{d} \mathcal{N}(0,1)$$

$$\hat{\beta}_j \overset{approx}{\sim} \mathcal{N}\left(\beta_j, \underbrace{\frac{\sigma^2}{n}\left(\frac{1}{n}\sum x_i x_i'\right)^{-1}_{jj}}_{V_j}\right)$$

$$t = \frac{\hat{\beta}_j - \beta_j}{\sqrt{V_j}} \overset{approx}{\sim} \mathcal{N}(0,1)$$

Definition 5.2.1 (Wald Test).

The asymptotic distribution of the Wald-test-statistic, T_W, follows from asymptotic Normality of $\hat{\beta}$, the Delta Method and the fact that $(X-\mu)'\Sigma^{-1}(X-\mu) \sim \chi^2_{dim(X)}$ for $X \sim N(\mu, \Sigma)$.

Using $\sqrt{n}(\hat{\beta} - \beta_0) \xrightarrow{d} \mathcal{N}(0, V)$ and the Delta method, we get

$$\sqrt{n}\left(g(\hat{\beta}) - g(\beta_0)\right) \xrightarrow{d} G \cdot \mathcal{N}(0,V) = N(0, GVG'), \quad \text{with} \quad G = \left.\frac{\partial g(\beta)}{\partial \beta}\right|_{\beta=\beta_0}.$$

Therefore,

$$\sqrt{n}\left(g(\hat{\beta}) - g(\beta_0)\right)' [GVG']^{-1} \sqrt{n}\left(g(\hat{\beta}) - g(\beta_0)\right) \xrightarrow{d} \chi^2_m.$$

Under $\mathcal{H}_0 : g(\beta_0) = 0$. Also, because we do not know β_0, we replace G with $G(\hat{\beta})$, as $\hat{\beta}$ is our estimator of β_0.

More general hypotheses $\mathcal{H}_0 : g(\beta) = 0$ vs. $\mathcal{H}_1 : g(\beta) \neq 0$ for some function $g : \mathbb{R}^k \to \mathbb{R}^m$ (i.e. $m \leq k$ restrictions) can be tested using the Wald test. It uses the following statistic:

$$T_W = ng\left(\hat{\beta}_{OLS}\right)' \left[G\left(\hat{\beta}_{OLS}\right)\hat{V}G\left(\hat{\beta}_{OLS}\right)'\right]^{-1} g\left(\hat{\beta}_{OLS}\right) \xrightarrow{d} \chi^2_m,$$

where $\hat{V} = \hat{\sigma}^2 \hat{Q}^{-1}$ and where $G\left(\hat{\beta}_{OLS}\right) = \partial g(\beta)/\,\partial\beta|_{\beta=\hat{\beta}_{OLS}}$ is the $m \times k$ matrix of derivatives of g with respect to β evaluated at $\hat{\beta}_{OLS}$. The short derivation in the Appendix illustrates that the Wald test-statistic is based on the idea that if $\mathcal{H}_0$ is true, then the difference between $g\left(\hat{\beta}_{OLS}\right)$ and $g(\beta) = 0$ should be small. Suppose we are interested in testing $\mathcal{H}_0 : \{\beta_2 + \beta_3 = 5, \beta_4 = 0\}$ under a five-dimensional vector β. Then we would take

$$g(\beta) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \beta - \begin{bmatrix} 5 \\ 0 \end{bmatrix}, \quad \text{with} \quad G(\beta) = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

If $g(\beta) = 0$ is s.t. it tests only $\beta_j = \beta_{j,0}$ for a single β_j , then the Wald test is equivalent to the t-test: $\varphi_W = \varphi_t$.

Theorem 5.2.1 (Delta Method).

$X \xrightarrow{d} \mathcal{N}(\mu, \sigma^2)$, and $g : \mathbb{R}^k \to \mathbb{R}^q$ is a differentiable function. Then, $g(X) \xrightarrow{d} \mathcal{N}(g(\mu), \sigma^2(g'(\mu))^2)$.

Let $\beta \in \mathbb{R}^k$ and $g : \mathbb{R}^k \to \mathbb{R}^q$ be a differentiable function. If $\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} \xi$, then

$$\sqrt{n}(g(\hat{\beta}) - g(\beta)) \xrightarrow{d} G'\xi,$$

where $G = G(\beta) = \frac{\partial}{\partial\beta} g(\beta)'$.

In particular, if $\xi \sim \mathcal{N}(0, V)$, then

$$\sqrt{n}(g(\hat{\beta}) - g(\beta)) \xrightarrow{d} \mathcal{N}(0, GVG').$$

By previous results, if we have $V = \sigma^2 Q^{-1}$, then

$$\sqrt{n}(g(\hat{\beta}) - g(\beta)) \xrightarrow{d} \mathcal{N}(0, G\sigma^2 Q^{-1} G')$$

where $G(u) = \frac{\partial}{\partial u} g(u)'$ and $G = G(\beta)$.

5.3 Violations of Ideal Conditions

First of all, note that while unbiasedness requires the conditional independence assumption 3 to hold, both consistency and asymptotic Normality go through even under the weaker exogeneity assumption $\mathbb{E}[u_i x_i] = 0$.[1]

5.3.1 Singular $X'X$

If $X'X$ is not of full rank, then the OLS estimator is not even defined. There are two reasons that lead to this case.

5.3.2 Heteroskedasticity

Suppose we replace the Assumption 5.1.4 with the weaker assumption that $\mathbb{V}[u_i|x_i] = \sigma_i^2$ for all i. Then, the OLS estimator is still unbiased, but the variance of the OLS estimator is now given by:

$$\begin{aligned} \mathbb{V}[x_i u_i] &= \mathbb{E}[(x_i u_i - \mathbb{E}[x_i u_i])\,(x_i u_i - \mathbb{E}[x_i u_i])'] \\ &= \mathbb{E}[x_i u_i u_i' x_i'] - \mathbb{E}[x_i u_i]\mathbb{E}[x_i u_i]' \end{aligned}$$

[1] This is because it's implied by the conditional independence assumption.

$$
\begin{aligned}
&= \mathbb{E}[\mathbb{E}[u_i^2|x_i]x_i x_i'] \\
&= \mathbb{E}[x_i x_i' u_i^2] \\
&= \mathbb{E}[x_i x_i' \sigma_i^2] \\
\Rightarrow \sqrt{n}(\hat{\beta}_{OLS} - \beta) &\xrightarrow{d} \mathcal{N}(0, Q^{-1}\mathbb{E}[x_i x_i' u_i^2]Q^{-1})
\end{aligned}
$$

The asymptotic variance can again be estimated by replacing $\mathbb{E}[x_i x_i' u_i^2]$ with its sample analogue: as a consistent estimator: $\frac{1}{n}\sum_{i=1}^n x_i x_i' \hat{u}_i^2$.

Note that if the variances $\{\sigma_i^2\}_{i=1}^n$ were known, we could transform the heteroskedastic model into a homoskedastic one by writing the regression as:

$$\frac{y_i}{\sigma_i} = \frac{x_i'}{\sigma_i}\beta + \frac{u_i}{\sigma_i}.$$

In this model, observations are weighted by the inverses of their standard deviations and, as a result, less noisy observations are given more weight as they are more informative about the relation between X and Y. Let $\mathbb{V}[U|X] = \Sigma = diag(\sigma_1^2, \cdots, \sigma_n^2)$. We can then write the regression in matrix form as: $\Sigma^{-\frac{1}{2}}Y = \Sigma^{-\frac{1}{2}}X\beta + \Sigma^{-\frac{1}{2}}U$, with $\mathbb{V}[\Sigma^{-\frac{1}{2}}U|X] = I$.

The OLS estimator in this weighted regression model is referred to as the Generalized Least Squares (GLS) estimator. It is given by:

$$\hat{\beta}_{GLS} = \left(\left(\Sigma^{-\frac{1}{2}}X\right)'\left(\Sigma^{-\frac{1}{2}}X\right)\right)\left(\Sigma^{-\frac{1}{2}}X\right)'\left(\Sigma^{-\frac{1}{2}}Y\right) = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}Y.$$

Under otherwise the same conditions as for OLS, this estimator is unbiased[2] and consistent[3]. and has variance:

$$\mathbb{V}[\hat{\beta}_{GLS}] = \mathbb{E}\left[(X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}UU'\Sigma^{-1}X(X'\Sigma^{-1}X)^{-1}\right] = \mathbb{E}\left[(X'\Sigma^{-1}X)^{-1}\right].$$

5.3.3 Endogeneity

Omitted Variables

Suppose the true model is given by:

$$y_i = x_i'\beta + z_i'\delta + \varepsilon_i \text{ , with } \mathbb{E}[x_i\varepsilon_i] = 0,$$

i.e. exogeneity holds in this true model, whereas the researcher estimates

$$y_i = x_i'\gamma + u_i.$$

Notice that we have written the coefficient as γ rather than β and the error as u rather than ε. Goldberger (1991) introduced the catchy labels long regression and short regression to emphasize the distinction. Typically, $\beta \neq \gamma$,except in special cases. To see this, we calculate

$$
\begin{aligned}
\gamma &= \left(\mathbb{E}\left[XX'\right]\right)^{-1}\mathbb{E}\left[XY\right] \\
&= \left(\mathbb{E}\left[XX'\right]\right)^{-1}\mathbb{E}\left[X\left(X'\beta + Z'\delta + \varepsilon\right)\right] \\
&= \beta + \left(\mathbb{E}\left[XX'\right]\right)^{-1}\mathbb{E}\left[XZ'\right]\delta
\end{aligned}
$$

[2] $\mathbb{E}\left[\hat{\beta}_{GLS}|X\right] = \mathbb{E}\left[\left(X'\Sigma^{-1}X\right)^{-1}\left(X\beta + U\right)|X\right] = \beta + \mathbb{E}\left[\left(X'\Sigma^{-1}X\right)^{-1}U|X\right] = \beta.$

[3] $\hat{\beta}_{GLS} - \beta = (X'\Sigma^{-1}X)^{-1}X'\Sigma^{-1}U \xrightarrow{p} \frac{1}{n}\mathbb{E}\left[\frac{x_i x_i'}{\sigma_i^2}\right]\frac{1}{n}\mathbb{E}\left[\frac{x_i u_i}{\sigma_i^2}\right] \xrightarrow{p} 0.$

$$= \beta + \Gamma\delta$$

where $\Gamma = Q^{-1}Q_{XZ}$ is the coefficient matrix from a projection of Z on X.

Observe that $\gamma = \beta + \Gamma\delta \neq \beta$ unless $\Gamma = 0$ or $\delta = 0$. Thus the short and long regressions have different coefficients. They are the same only under one of two conditions. First, if the projection of Z on X yields a set of zero coefficients (they are uncorrelated), or second, if the coefficient on Z in is zero. The difference $\Gamma\delta$ between γ and β is known as omitted variable bias. It is the consequence of omission of a relevant correlated variable.

Lecture 6.

Likelihood-Based Inference

In previous lectures we have discussed the least squares estimation of the linear regression model. In this lecture we will discuss the likelihood-based inference.

	ch2	ch3(LRM)
LS	$\mathbb{E}[y_i\|\theta]$	$\mathbb{E}[y_i\|x_i,\beta]$
ML	$p(y_i\|\theta)$	$p(y_i\|x_i,\beta)$

6.1 ML for LRM

$$y_i = x_i'\beta + u_i$$

For LS, we assume $\mathbb{E}[u_i|x_i] = 0$, which gives $\mathbb{E}[y_i|x_i] = x_i'\beta$. For ML, we assume $u_i \sim N(0,\sigma^2)$, which gives $y_i|x_i \sim N(x_i'\beta,\sigma^2)$.

$$p(y_i|x_i) = (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2}(y_i - x_i'\beta)^2\right\}$$

We use θ to represent parameters (β,σ^2).

$$\begin{aligned}
\Rightarrow \mathcal{L}(\theta|y_i,x) &= p(y|x_i,\theta) \\
&= \prod_{i=1}^{n} p(y_i|x_i,\theta) \\
&= \prod_{i=1}^{n} (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2}(y_i - x_i'\beta)^2\right\} \\
&= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - x_i'\beta)^2\right\}
\end{aligned}$$

Then, we can get the log-likelihood function:

$$\begin{aligned}
\ell(\theta|y_i,x) &= \log \mathcal{L}(\theta|y_i,x) \\
&= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}(Y - X\beta)'(Y - X\beta) \\
\Rightarrow \hat{\theta}_{ML} &= (\hat{\beta}_{ML},\hat{\sigma}^2_{ML}) = \arg\max_{\theta=(\beta,\sigma^2)} \ell(\theta|y_i,x)
\end{aligned}$$

Take the first order condition (FOC) of β and σ^2:

$$\begin{aligned}
\beta &: \frac{1}{\sigma^2}X'(Y - X\beta) = 0 \\
\Rightarrow \hat{\beta}_{ML} &= (X'X)^{-1}X'Y \\
\sigma^2 &: -\frac{n}{2\sigma^2} + \frac{2}{(2\sigma^2)^2}(Y - X\beta)'(Y - X\beta) = 0 \\
\Rightarrow \hat{\sigma}^2_{ML} &= \frac{1}{n}(Y - X\beta)'(Y - X\beta) = \frac{1}{n}\sum_{i=1}^{n}\hat{u}_i^2 = \hat{\sigma}^2_{LS}
\end{aligned}$$

As we know that:

$$\hat{\beta} = (X'X)^{-1}X'Y = \beta + (X'X)^{-1}X'U = \beta + \left(\frac{1}{n}\sum x_i x_i'\right)^{-1}\sum x_i u_i$$

$$\Rightarrow \hat{\beta}|X \sim \mathcal{N}(\beta, V)$$

where

$$\begin{aligned} V &= \mathbb{V}\left[(X'X)^{-1}X'U|X\right] \\ &= (X'X)^{-1}\mathbb{V}[X'U|X](X'X)^{-1} \\ &= (X'X)^{-1}X'\mathbb{V}[U|X]X(X'X)^{-1} \\ &= (X'X)^{-1}X'\sigma^2 IX(X'X)^{-1} \\ &= \sigma^2(X'X)^{-1} \end{aligned}$$

We define the **Score Function** as:

Definition 6.1.1 (Score Function).

$$\begin{aligned} S(\theta) &= \frac{\partial \ell(\theta|y_i, x)}{\partial \theta} \\ &= \begin{bmatrix} \frac{\partial \ell(\theta|y_i,x)}{\partial \beta} \\ \frac{\partial \ell(\theta|y_i,x)}{\partial \sigma^2} \end{bmatrix} \end{aligned}$$

As we know that:

$$\begin{aligned} \frac{\partial \ell(\theta|y_i, x)}{\partial \beta} &= \frac{1}{\sigma^2}X'(Y - X\beta) \\ \frac{\partial \ell(\theta|y_i, x)}{\partial \sigma^2} &= -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2}(Y - X\beta)'(Y - X\beta) \end{aligned}$$

We take the Hessians of β and σ^2:

$$\begin{aligned} \mathcal{H}(\theta) &= \frac{\partial^2 \ell(\theta|y_i, x)}{\partial\theta\partial\theta'} = \frac{\partial S(\theta)}{\partial \theta'} \\ &= \begin{bmatrix} \frac{\partial^2 \ell(\theta|y_i,x)}{\partial\beta\partial\beta} & \frac{\partial^2 \ell(\theta|y_i,x)}{\partial\beta\partial\sigma^2} \\ \frac{\partial^2 \ell(\theta|y_i,x)}{\partial\sigma^2\partial\beta} & \frac{\partial^2 \ell(\theta|y_i,x)}{\partial\sigma^2\partial\sigma^2} \end{bmatrix} \\ \frac{\partial^2 \ell(\theta|y_i, x)}{\partial\beta\partial\beta'} &= -\frac{1}{\sigma^2}X'X \\ \frac{\partial^2 \ell(\theta|y_i, x)}{\partial\sigma^2\partial\sigma^2} &= \frac{n}{2(\sigma^2)^2} - \frac{1}{(\sigma^2)^3}(Y - X\beta)'(Y - X\beta) \\ \frac{\partial^2 \ell(\theta|y_i, x)}{\partial\beta\partial\sigma^2} &= \frac{1}{(\sigma^2)^2}X'(Y - X\beta) \end{aligned}$$

Then, we can get the **Information Matrix** as:

Definition 6.1.2 (Information Matrix).

$$I(\theta) = \mathbb{E}[s(\theta)s(\theta)'] = -\mathbb{E}\left[\frac{\partial^2 \ell(\theta|y_i, x)}{\partial\theta\partial\theta'}\right]$$

$$= -\mathbb{E}\left[\begin{bmatrix} \frac{\partial^2 \ell(\theta|y_i,x)}{\partial\beta\partial\beta} & \frac{\partial^2 \ell(\theta|y_i,x)}{\partial\beta\partial\sigma^2} \\ \frac{\partial^2 \ell(\theta|y_i,x)}{\partial\sigma^2\partial\beta} & \frac{\partial^2 \ell(\theta|y_i,x)}{\partial\sigma^2\partial\sigma^2} \end{bmatrix}\right]$$

$$\begin{aligned}
\mathbb{E}[s(\beta)s(\beta)'] &= -\mathbb{E}\left[\frac{1}{\sigma^2}X'(Y-X\beta)\left[\frac{1}{\sigma^2}X'(Y-X\beta)\right]'\right] \\
&= \mathbb{E}\left[\frac{1}{\sigma^4}X'UU'X\right] \\
&= \frac{1}{\sigma^4}X'\mathbb{E}[UU']X \\
&= \frac{1}{\sigma^4}X'\sigma^2 IX \\
&= \frac{1}{\sigma^2}X'X \\
&= \mathbb{E}[-\mathcal{H}(\beta)]
\end{aligned}$$

Then, we could have the **Cramer-Rao Lower Bound**:

Definition 6.1.3 (Cramer-Rao Lower Bound)**.**

Let $\tilde{\theta}$ be an unbiased estimator of θ, then:

$$\begin{aligned}
\mathbb{V}[\tilde{\theta}|X] &\geq I^{-1}(\theta) \\
&= \sigma^2(X'X)^{-1}
\end{aligned}$$

Take a model $\ell(\theta|y)$, and

$$\begin{aligned}
\hat{\theta} &= \underset{\theta}{\operatorname{argmax}}\, \ell(\theta|y) \\
\bar{\theta} &= \underset{\theta, g(\theta)=0}{\operatorname{argmax}}\, \ell(\theta|y)
\end{aligned}$$

and $\sqrt{n}(\hat{\theta_0} - \theta) \xrightarrow{d} \mathcal{N}(0, V)$. We want to test: $\mathcal{H}_0 : g(\theta) = 0$. Previously, we have three tests:

- t-test: $t = \frac{\hat{\theta}-\theta_0}{\sqrt{\frac{1}{n}\hat{V}}} \xrightarrow{d} \mathcal{N}(0,1)$
- Wald Test: $W = ng(\hat{\theta})'[G(\hat{\theta})\hat{V}G(\hat{\theta})']^{-1}g(\hat{\theta}) \sim \chi^2_k$
- LR Test: $LR = -2(\ell(\hat{\theta}_0) - \ell(\bar{\theta})) \sim \chi^2_k$
- LM Test: $LM = S(\bar{\theta})'I(\bar{\theta})^{-1}S(\bar{\theta}) \sim \chi^2_k$

Lecture 7.

Likelihood-Based Inference(2)

7.1 Binary Choice: Logit Model & Probit Model

Suppose $y_i \in \{0, 1\}$, the common approach is still:

$$\begin{aligned} y_i &= x_i'\beta + u_i \\ \mathbb{E}[u_i|x_i] &= 0 \end{aligned}$$

But, the linear regression is not attractive, because $\mathbb{E}[y_i|x_i] = \mathbb{P}[y_i = 1|x_i]$ is bounded between 0 and 1, while the linear regression is unbounded.

We define a new regression model:

$$\begin{aligned} y_i^* &= x_i'\beta + u_i \\ u_i|x_i &\sim \mathcal{N}(0, 1) \end{aligned}$$

and assume we have: $y_i = \mathbf{1}\{y_i^* \geq 0\}$. If utility is positive $y_i = 1$, if it is negative, $y_i = 0$.

Then, we have the probability of $y_i = 1$ as:

$$\mathbb{P}[y_i = 1] = \mathbb{P}[x_i'\beta + u_i \geq 0] = \mathbb{P}[u_i \geq -x_i'\beta] = 1 - \Phi(-x_i'\beta) = \Phi(x_i'\beta)$$

and $\mathbb{P}[y_i = 0] = 1 - \Phi(x_i'\beta) = \Phi(-x_i'\beta)$, where Φ is the CDF of a standard normal RV.

Hence, y_i had the PDF:

$$p(y_i|x_i, \beta) = \begin{cases} \Phi(x_i'\beta) & y_i = 1 \\ \Phi(-x_i'\beta) & y_i = 0 \end{cases} = \Phi(x_i'\beta)^{y_i}\Phi(-x_i'\beta)^{1-y_i}$$

which is the Bernoulli distribution with probability of success $\Phi(x_i'\beta)$.

This leads to our likelihood function:

$$\begin{aligned} \mathcal{L}(\beta|Y, X) &= \prod_{i=1}^{n} p(y_i|x_i, \beta) \\ &= \prod_{i=1}^{n} \Phi(x_i'\beta)^{y_i}\Phi(-x_i'\beta)^{1-y_i} \end{aligned}$$

Then, we can get the log-likelihood function:

$$\begin{aligned} \ell(\beta|Y, X) &= \log \mathcal{L}(\beta|Y, X) \\ &= \sum_{i=1}^{n} \{y_i \log \Phi(x_i'\beta) + (1 - y_i) \log \Phi(-x_i'\beta)\} \end{aligned}$$

where we have the estimator defines as: $\hat{\beta} = \arg\max_\beta \ell(\beta|Y, X)$.

Then, we can have:

$$\hat{y}_i = \mathbb{E}[y_i|x_i', \beta] = \Phi(x_i'\hat{\beta}) \Rightarrow R^2 = \frac{\hat{Y}'\hat{Y}}{Y'Y}.$$

The partial effect of X on Y is:

$$\begin{aligned}\delta &= \mathbb{E}[y_i|x_i = x_2, \beta] - \mathbb{E}[y_i|x_i = x_1, \beta] \\ &= \Phi(x_2'\beta) - \Phi(x_1'\beta)\end{aligned}$$

approximately,

$$\frac{\partial \mathbb{E}[y_i|x_i\beta]}{\partial x_i} = \frac{\partial \Phi(x_i'\beta)}{\partial x_i} = \phi(x_i'\beta)\beta.$$

which is:

$$\Delta\mathbb{E}[y_i|x_i\beta] = \phi(x_i'\beta)\beta\Delta x_i.$$

Since $\Phi(\cdot)$ is a strictly increasing function, the sign of β_j reveals the sign of the partial effect of x_j, but the size of β_j is not interpretable. Only the relative sizes of two coefficients β_k and β_l have (qualitative) meaning. Nevertheless, we can test for the partial effect of x_j being zero by testing $\mathcal{H}_0 : \beta_j = 0$, because the former is zero iff $\beta_j = 0$.

7.2 Censored Outcomes: Tobit Model

Suppose y_i is censored at 0, i.e., $y_i \geq 0$. We can deal with this by assuming:

$$\begin{aligned}y_i^* &= x_i'\beta + u_i \\ u_i|x_i &\sim \mathcal{N}(0, \sigma^2) \\ y_i &= y_i^* \mathbf{1}\{y_i^* \geq 0\}\end{aligned}$$

In this case, the probability of observing $y_i = 0$ is:

$$\mathbb{P}[y_i = 0] = \mathbb{P}[y_i^* < 0] = \mathbb{P}[x_i'\beta + u_i < 0] = \Phi\left(-\frac{x_i'\beta}{\sigma}\right).$$

To get the PDF $p(y_i)$ for $y_i > 0$, we derive the CDF:

$$\mathbb{P}[y_i < y] = \mathbb{P}[y_i^* < y] = \mathbb{P}\left[\frac{u_i}{\sigma} < \frac{y_i - x_i'\beta}{\sigma}\right] = \Phi\left(\frac{y_i - x_i'\beta}{\sigma}\right),$$

which gives that

$$p(y) = \frac{\partial \mathbb{P}[y_i < y]}{\partial y} = \frac{1}{\sigma}\phi\left(\frac{y - x_i'\beta}{\sigma}\right)$$

for $y > 0$.

Hence, the PDF observations y_i is:

$$p(y_i|x_i, \beta, \sigma) = \begin{cases} \Phi\left(-\frac{x_i'\beta}{\sigma}\right) & y_i = 0 \\ \frac{1}{\sigma}\phi\left(\frac{y_i - x_i'\beta}{\sigma}\right) & y_i > 0 \end{cases}$$

Then, we can get the likelihood function:

$$\mathcal{L}(\beta, \sigma|Y, X) = \prod_{i=1}^{n} p(y_i|x_i, \beta, \sigma)$$

$$= \prod_{i=1}^{n} \left\{ \Phi\left(-\frac{x_i'\beta}{\sigma}\right) \right\}^{1\{y_i=0\}} \left\{ \frac{1}{\sigma}\phi\left(\frac{y_i - x_i'\beta}{\sigma}\right) \right\}^{1\{y_i>0\}}$$

Then, we can get the log-likelihood function:

$$\begin{aligned}\ell(\beta, \sigma|Y, X) &= \log \mathcal{L}(\beta, \sigma|Y, X) \\ &= \sum_{i=1}^{n} \left\{ \mathbf{1}\{y_i = 0\} \log \Phi\left(-\frac{x_i'\beta}{\sigma}\right) + \mathbf{1}\{y_i > 0\} \log\left(\frac{1}{\sigma}\phi\left(\frac{y_i - x_i'\beta}{\sigma}\right)\right) \right\}\end{aligned}$$

Let $\mathcal{G} = \{i, y_i = 0\}$, we can wirte the log-likelihood function as:

$$\ell(\beta, \sigma|Y, X) = \sum_{i\in\mathcal{G}} \left\{ \log \Phi\left(-\frac{x_i'\beta}{\sigma}\right) \right\} + \sum_{i\notin\mathcal{G}} \left\{ \log\left(\frac{1}{\sigma}\phi\left(\frac{y_i - x_i'\beta}{\sigma}\right)\right) \right\}$$

Our estimator is defined as:

$$\hat{\theta}_{ML} = (\hat{\beta}_{ML}, \hat{\sigma}^2_{ML}) = \arg \max_{\theta=(\beta,\sigma)} \ell(\theta|Y, X).$$

To compute partial effects, we need to derive the (conditional) expectation $\mathbb{E}[y_i|x_i]$. Using the result that for $Z \sim N(0, 1)$, $\mathbb{E}[Z|Z > -c] = \phi(c)/\Phi(c)$ (inverse Mills ratio), we get

$$\mathbb{E}[y_i|y_i > 0] = \mathbb{E}[x_i'\beta + u_i|u_i > -x_i'\beta] = x_i'\beta + \sigma\phi\left(\frac{x_i'\beta}{\sigma}\right)/\Phi\left(\frac{x_i'\beta}{\sigma}\right).$$

> **Note.** The inverse Mills ratio is the ratio of the probability density function to the complementary cumulative distribution function of a distribution. Its use is often motivated by the following property of the truncated normal distribution. If X is a random variable having a normal distribution with mean μ and variance σ^2, then
>
> $$\mathbb{E}[\, X \mid X > \alpha \,] = \mu + \sigma \frac{\phi\left(\frac{\alpha-\mu}{\sigma}\right)}{1 - \Phi\left(\frac{\alpha-\mu}{\sigma}\right)},$$
> $$\mathbb{E}[\, X \mid X < \alpha \,] = \mu - \sigma \frac{\phi\left(\frac{\alpha-\mu}{\sigma}\right)}{\Phi\left(\frac{\alpha-\mu}{\sigma}\right)}$$
>
> where α is a constant, ϕ denotes the standard normal density function, and Φ is the standard normal cumulative distribution function.
>
> The two fractions are the inverse Mills ratios.

Then, we obtain

$$\begin{aligned}\mathbb{E}[y_i] &= \mathbb{E}[y_i|y_i \geq 0]\mathbb{P}[y_i \geq 0] \\ &= \mathbb{E}[y_i|y_i = 0]\mathbb{P}[y_i = 0] + \mathbb{E}[y_i|y_i > 0]\mathbb{P}[y_i > 0] \\ &= \mathbb{E}[y_i|y_i > 0](1 - \mathbb{P}[y_i = 0]) \\ &= \mathbb{E}[y_i^*|y_i^* > 0]\left(1 - \Phi\left(\frac{-x_i'\beta}{\sigma}\right)\right) \\ &= \mathbb{E}[x_i'\beta + u_i|u_i + x_i'\beta > 0]\left(1 - \Phi\left(\frac{-x_i'\beta}{\sigma}\right)\right) \\ &= \mathbb{E}[x_i'\beta + u_i|u_i > -x_i'\beta]\left(1 - \Phi\left(\frac{-x_i'\beta}{\sigma}\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= x_i'\beta + \sigma \frac{\phi(x_i'\beta/\sigma)}{\Phi(x_i'\beta/\sigma)} \left(1 - \Phi\left(\frac{-x_i'\beta}{\sigma}\right)\right) \\
&= \Phi\left(\frac{x_i'\beta}{\sigma}\right) \left(x_i'\beta + \sigma \frac{\phi\left(\frac{x_i'\beta}{\sigma}\right)}{\Phi\left(\frac{x_i'\beta}{\sigma}\right)}\right) \\
&= \Phi\left(\frac{x_i'\beta}{\sigma}\right) x_i'\beta + \sigma\phi\left(\frac{x_i'\beta}{\sigma}\right).
\end{aligned}
$$

7.3 Marginal Effects of Nonlinear Models(Probit, Logit, Tobit. etc.)

> **Note.**
>
> β_{MLE} is not the marginal effect of y_i, but it can be regarded as the marginal effect of potential variable y_i^*.

Expectation and marginal effect of y_i^*

$$
\begin{aligned}
\mathbb{E}[y_i^*|x_i] &= x_i'\beta \\
\frac{\partial \mathbb{E}[y_i^*|x_i]}{\partial x_{ij}} &= \beta_j
\end{aligned}
$$

Truncated Models

$$
\begin{aligned}
\mathbb{E}[y_i \mid y_i > 0, x_i] &= x_i'\beta + \mathbb{E}[u_i \mid u_i > -x_i'\beta] \\
&= x_i'\beta + \sigma\lambda\left(\frac{x_i'\beta}{\sigma}\right)
\end{aligned}
$$

where $\lambda = \frac{\phi(x_i'\beta/\sigma)}{\Phi(x_i'\beta/\sigma)}$ is the inverse Mills ratio.

$$
\begin{aligned}
\frac{\partial \mathbb{E}[y_i \mid y_i > 0, x_i]}{\partial x_j} &= \beta_j + \sigma \frac{\partial \lambda}{\partial x_j} \\
&= \beta_j + \sigma \frac{\partial \lambda}{\partial c} \frac{\partial c}{\partial x_j} \\
&= \beta_j + \frac{\partial \lambda}{\partial c} \beta_j \\
&= \beta_j \left\{1 - \lambda\left(\frac{x_i'\beta}{\sigma}\right)\left[\frac{x_i'\beta}{\sigma} + \lambda\left(\frac{x_i'\beta}{\sigma}\right)\right]\right\}
\end{aligned}
$$

So, we could tell that the effect of x_j on y_i is not only affected by β_j, but also affected by $\frac{x_i'\beta}{\sigma}$.

Censored Models

$$
\begin{aligned}
\mathbb{E}[y_i \mid x_i] &= \mathbb{P}[y_i > 0 \mid x_i] \cdot \mathbb{E}[y_i \mid y_i > 0, x_i] \\
&= \Phi\left(\frac{x_i'\beta}{\sigma}\right) \mathbb{E}[y_i \mid y_i > 0, x_i] \\
\frac{\partial \mathbb{E}[y_i \mid x_i]}{\partial x_j} &= \frac{\partial \mathbb{P}[y_i > 0 \mid x_i]}{x_j} \mathbb{E}[y_i \mid y_i > 0, x_i] + \mathbb{P}[y_i > 0 \mid x_i] \frac{\partial \mathbb{E}[y_i \mid y_i > 0, x_i]}{x_j} \\
&= \beta_j \Phi\left(\frac{x_i'\beta}{\sigma}\right)
\end{aligned}
$$

Derivative Function	**Gradient**
Optimal Response y^*	$\frac{\partial E(y^*\|x)}{\partial x_j} = \beta_j$
Positive Response y (censored at 0)	$\frac{\partial E(y\|y>0,X)}{\partial x_j} = \beta_j\{1 - \lambda(c)[c + \lambda(c)]\}$
Non-zero Response y (censored at 0)	$\frac{\partial E(y\|x)}{\partial x_j} = \beta_j \Phi(c)$

Table 7.1: Note: $c = \frac{x\beta}{\sigma}$

7.4 Censored, truncated or sample-selected data [1]

Table 7.2: Comparison of censored, truncated, and sample-selected cases. (borrowed from Richard Breen)

Sample	Y **Variable**	X **Variable**	**Example**
Censored	y is known exactly only if some criterion defined in terms of y is met.	x variables are observed for the entire sample, regardless of whether y is observed exactly	Determinants of income; income is measured exactly only if it above the poverty line. All other incomes are reported at the poverty line
Selected	y is observed only if a criteria defined in terms of some other random variable (Z) is met.	x and w (the determinants of whether $Z = 1$) are observed for the entire sample, regardless of whether y is observed	Survey data with item or unit non-response
Truncated	y is known only if some criterion defined in terms of y is met.	x variables are observed only if y is observed.	Donations to political campaigns

Definition 7.4.1 (Truncated). A sample is truncated if some observations are systematically excluded from the sample.

Example 5.

Suppose we are interested in relationship between people's income(y) and education(x). If we have observations of both y and x only for people whose income is above \$20,000 per year, then we have a truncated sample.

Definition 7.4.2 (Censored).

A sample is censored if no observations have been systematically excluded but some of information contained in them has been suppressed.

Example 6.

If we don't observe people's income (y) and education(x) with income lower than \$20,000 per year, then we have a truncated sample. If we have observations of x for people whose income above AND below that limit, but no information on exactly how much their income is for those whose income below the limit (only thing we know is that it's lower than \$20,000 per year), then we have a censored sample.

[1] This part is borrowed from Xiang Ao, March 24, 2009, *An Introduction to censored, truncated or sample-selected data*, Harvard Business School.

2 An illustration for truncated or censored data

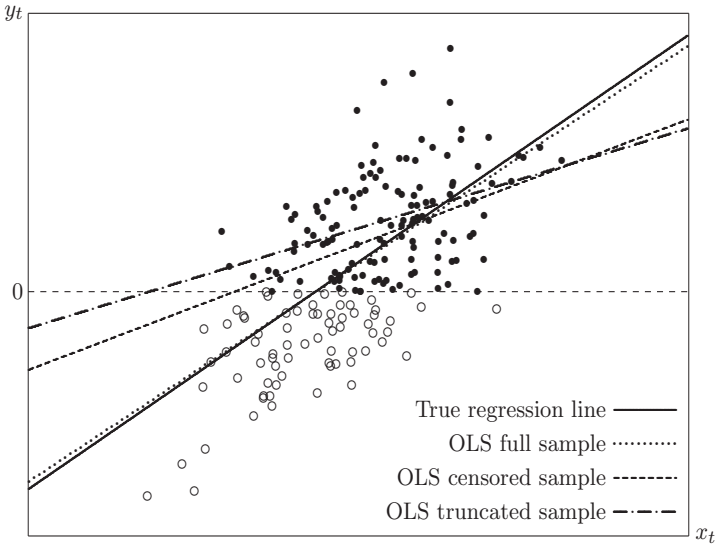
Consider the model

$$y_t^0 = \beta_1 + \beta_2 x_t + u_t,\ u_t \sim NID(0, \sigma^2), \tag{1}$$

where y_t^0 is a latent variable. We actually observe y_t, which differs from y_t^0 , because it's either truncated or censored.

Suppose that censorship or truncation occurs whenever y_t^0 is less than 0. Clearly, the larger the error term u_t, the larger is y_t^0 and thus the greater must be the probability that $y_t^0 \geq 0$. This probability also depends on x_t. So for the sample we observe, u_t does not have conditional mean 0 and is not uncorrelated with x_t. OLS using truncated or censored samples (OLS of y_t on x_t) yields biased and inconsistent estimators. Normally we would like to draw inference on the population which is represented by the full sample. Shown in figure 1, ideally we would have the OLS regression line (if we had the full sample), which is very close to the "true" regression line (which is the mechnism generates the data). We would have the "OLS censored sample" line if we run OLS on the censored sample; we would have "OLS truncated sample" if we run OLS on the truncated sample. It shows that both of them are severely biased from the "true" regression line.

Figure 1: Effects of truncation and censoring from Davidson and MacKinnon



3 Truncated Models

For truncated data, a consistent estimator comes from maximum likelihood estimator (MLE). If we assume the error terms in the latent variable model has a known distribution, then MLE estimator can be applied. The most popular choice is Gaussian.

$$\begin{aligned} \Pr(y_t^0 \geq 0) &= \Pr(\mathbf{X}_t\beta + u_t \geq 0) \\ &= 1 - \Pr(u_t/\sigma < -\mathbf{X}_t\beta/\sigma) \\ &= 1 - \Phi(-\mathbf{X}_t\beta/\sigma) = \Phi(\mathbf{X}_t\beta/\sigma) \end{aligned} \tag{2}$$

The density of y_t is proportional to the density of y_t^0 when $y_t^0 \geq 0$ and y_t is observed. It is 0 elsewhere. The factor of proportionality, which is needed to ensure that the density integrates to unity, is the inverse of the probability that $y_t^0 \geq 0$. The density of y_t can be written as

$$\frac{\sigma^{-1}\phi((y_t - \mathbf{X}_t\beta)/\sigma)}{\Phi(\mathbf{X}_t\beta/\sigma)} \tag{3}$$

This implies the log-likelihood function,

$$\ell(\mathbf{y}, \beta, \sigma) = -\frac{n}{2}\log(2\pi) - n\log(\sigma) - \frac{1}{2\sigma^2}\sum_{t=1}^{n}(y_t - \mathbf{X}_t\beta)^2 - \sum_{t=1}^{n}\log\Phi(\mathbf{X}_t\beta/\sigma). \tag{4}$$

This can be estimated by MLE. In Stata, truncreg is the command to do truncated regression model.

4 Censored Models

The most popular censored model is Tobit model.

$$\begin{aligned} y_t^0 &= \mathbf{X}_t + u_t, \quad u_t \sim NID(0, \sigma^2) \\ y_t &= y_t^0 \text{ if } y_t^0 > 0;\ y_t = 0 \text{ otherwise.} \end{aligned} \tag{5}$$

We see that

$$\begin{aligned} \Pr(y_t = 0) &= \Pr(y_t^0 \leq 0) = \Pr(\mathbf{X}_t\beta + u_t \leq 0) \\ &= \Pr(u_t/\sigma < -\mathbf{X}_t\beta/\sigma) \\ &= \Phi(-\mathbf{X}_t\beta/\sigma) \end{aligned} \tag{6}$$

The contribution to the log-likelihood function made by observations with $y_t = 0$ is

$$\ell_t(y_t, \beta, \sigma) = \log\Phi(-\mathbf{X}_t\beta/\sigma). \tag{7}$$

If y_t is positive, the contribution to the log-likelihood is the logarithm of the density,

$$\log(\frac{1}{\sigma}\phi((y_t - \mathbf{X}_t\beta)/\sigma)). \tag{8}$$

The log-likelihood function of the tobit model is

$$\sum_{y_t=0} \log \Phi(-\mathbf{X}_t\beta/\sigma) + \sum_{y_t>0} \log(\frac{1}{\sigma}\phi((y_t - \mathbf{X}_t\beta)/\sigma))) \tag{9}$$

This can be estimated by MLE. In Stata, tobit or intreg can be used for censoring models.

5 Sample Selection

The sample selection models differ from censored model in that it involves a different variable (from y itself) to determine the censorship (selection).

Suppose that y_t^0 and z_t^0 are two latent variables, generated by the bivariate process

$$\begin{bmatrix} y_t^0 \\ z_t^0 \end{bmatrix} = \begin{bmatrix} \mathbf{X}_t\beta \\ \mathbf{W}_t\gamma \end{bmatrix} + \begin{bmatrix} u_t \\ v_t \end{bmatrix}, \ \begin{bmatrix} u_t \\ v_t \end{bmatrix} \sim NID(\mathbf{0}, \begin{bmatrix} \sigma^2 & \rho \\ \rho & 1 \end{bmatrix}) \tag{10}$$

We observe y_t and z_t:

$$\begin{aligned} y_t &= y_t^0 \text{ if } z_t^0 > 0; \ y_t \text{ unobservable otherwise;} \\ z_t &= 1 \text{ if } z_t^0 > 0; \ z_t = 0 \text{ otherwise.} \end{aligned} \tag{11}$$

There are two types of observations, ones we observe $y_t = y_t^0$ and $z_t = 1$, along with both $\mathbf{X_t}$ and $\mathbf{W_t}$, and ones we observe only $z_t = 0$ and $\mathbf{W_t}$.

Each observation contributes to the likelihood function by

$$I(z_t = 0)\Pr(z_t = 0) + I(z_t = 1)\Pr(z_t = 1)f(y_t^0|z_t = 1), \tag{12}$$

Under the normality assumption,

$$u_t = \rho v_t + e_t \tag{13}$$

where e_t is independent of $v_t \sim N(0, 1)$. A useful fact about the standard normal distribution is that

$$\mathrm{E}(v_t|v_t > -x) = \lambda(x) = \frac{\phi(x)}{\Phi(x)} \tag{14}$$

and the function $\lambda(x)$ is called the inverse Mills ratio.

The log-likelihood function can be shown to be

$$\sum_{z_t=0} \log \Phi(-\mathbf{W}_t\gamma) + \sum_{z_t=1} \log(\frac{1}{\sigma}\phi((y_t - \mathbf{X}_t\beta)/\sigma)) + \sum_{z_t=1} \log \Phi(\frac{\mathbf{W}_t\gamma + \rho(y_t - \mathbf{X}_t\beta)/\sigma}{(1-\rho^2)^{1/2}}) \tag{15}$$

So this model can be estimated by MLE.

However, it is popular to use Heckman's two-step method.

Heckman's method is based on the fact that the original latent model can be rewritten as

$$y_t = \mathbf{X}_t\beta + \rho v_t + e_t \tag{16}$$

Here the error term u_t is divided into two parts, one perfectly correlated with v_t, and one independent of v_t.

In the first step, an ordinary probit model is used to obtain consistent estimates $\hat{\gamma}$ of the parameters of the selection equation.

In the second step, the unobserved v_t is replaced by the selectivity regressor $\frac{\phi(\mathbf{W}_t\hat{\gamma})}{\Phi(\mathbf{W}_t\hat{\gamma})}$.

Therefore, the regression becomes

$$y_t = \mathbf{X}_t\beta + \rho\frac{\phi(\mathbf{W}_t\hat{\gamma})}{\Phi(\mathbf{W}_t\hat{\gamma})} + e_t \tag{17}$$

Or,

$$y_t = \mathbf{X}_t\beta + \rho\hat{\lambda}_t + e_t \tag{18}$$

One thing to note in Heckman model is that in many situations, $\hat{\lambda}$ is believed to be highly collinear with $\mathbf{X_t}$ (See Olsen), if $\mathbf{W_t}$ is the same as $\mathbf{X_t}$. However, the model can still be estimated due to the nonlinearity of the model. Nevertheless, it becomes a regular practice to require $\mathbf{W_t}$ contains at least one extra variable than $\mathbf{X_t}$, which is sometimes called exclusion restriction. In many situations, $\mathbf{W_t}$ contains all $\mathbf{X_t}$ variables, and at least one more variable. The reason to contain all $\mathbf{X_t}$ variables is because the selection is "endogenous" in the sense that y_t is a factor in determining selection; this is modeled by including all the $\mathbf{X_t}$'s.

In Stata, heckman is the command to do sample selection models. It has options to do a Heckman two-step estimation or MLE estimation. Usually MLE is preferred.

6 Switching Regression (Treatment-Effects Model)

There is another situation that is similar to sample selection model: we observe y for $z = 1$ and $z = 0$. In the example of effect of education on income, we observe both union member's income and non-member's income. In this case, we have switching regression model or treatment-effect model. The treatment effect model estimates the effect of an endogenous binary treatmentz_t (treatment, program participation, etc.) on a continuous, fully-observed variable,y_t, conditional on the independent variables x_t and w_t.

Suppose z_t^0 is a latent variable.

$$\begin{bmatrix} y_t \\ z_t^0 \end{bmatrix} = \begin{bmatrix} \mathbf{X}_t\beta \\ \mathbf{W}_t\gamma \end{bmatrix} + \delta \begin{bmatrix} z_t \\ 0 \end{bmatrix} + \begin{bmatrix} u_t \\ v_t \end{bmatrix}, \begin{bmatrix} u_t \\ v_t \end{bmatrix} \sim NID(\mathbf{0}, \begin{bmatrix} \sigma^2 & \rho \\ \rho & 1 \end{bmatrix}) \tag{19}$$

We observe y_t and z_t:

$$z_t = 1 \text{ if } z_t^0 > 0;\ z_t = 0 \text{ otherwise.} \tag{20}$$

Notice that the only difference between the switching regression and selection model is that we observe y_t when $z_t = 0$ and $z_t = 1$. It's not a selection, but a regime switching.

Both Stata (treatreg) and SAS (qlim) can estimate the switching regression. The default is by MLE.

Lecture 8.

Topics in Econometrics(1)

8.1 Numerical Estimation

For previous estomation methods, we have:

$$\hat{\theta} = \arg\min_{\theta} \mathcal{Q}_n(\theta; Y^n).$$

Take the first-order condition to higher orders, we have:

$$\begin{aligned}
\mathcal{Q}^{(1)}(\theta) = \frac{\partial \mathcal{Q}(\theta)}{\partial \theta} &= \begin{bmatrix} \frac{\partial \mathcal{Q}(\theta)}{\partial \theta_1} \\ \vdots \\ \frac{\partial \mathcal{Q}(\theta)}{\partial \theta_k} \end{bmatrix} \\
\Rightarrow \mathcal{Q}^{(1)}(\theta^{m+1}) &= \mathcal{Q}^{(1)}(\theta^m) + \mathcal{Q}^{(2)}(\theta^m)(\theta^{m+1} - \theta^m) + \frac{1}{2}\mathcal{Q}^{(3)}(\theta^m)(\theta^{m+1} - \theta^m)^2 + \cdots \\
&\approx \mathcal{Q}^{(1)}(\theta^m) + \mathcal{Q}^{(2)}(\theta^m)(\theta^{m+1} - \theta^m) \\
&= 0. \\
\Rightarrow \theta^{m+1} &= \theta^m - \left[\mathcal{Q}^{(2)}(\theta^m)\right]^{-1} \mathcal{Q}^{(1)}(\theta^m).
\end{aligned}$$

Note (Newton-Raphson Method).

The Newton-Raphson method is a root-finding algorithm that uses the first few terms of the Taylor series of a function $f(x)$ in the vicinity of a starting point x_0 to find the root of the function. The method is based on the idea that a continuous and differentiable function can be approximated by a straight line tangent to it. The method is iterative and converges quadratically to the root.

Algorithm 3: Newton-Raphson Method

Input: Initialize θ^0, tolerence level $\varepsilon > 0$

for $m = 0$ **to** M **do**
  Given θ^m, compute $\mathcal{Q}^{(2)}(\theta^m, Y^n)^{-1}$ and $\mathcal{Q}^{(1)}(\theta^m, Y^n)$;
  Set $\theta^{m+1} = \theta^m - \mathcal{Q}^{(2)}(\theta^m, Y^n)^{-1}\mathcal{Q}^{(1)}(\theta^m, Y^n)$;
  if $\left\|\theta^{m+1} - \theta^m\right\| < \varepsilon$ **then**
    $\hat{\theta} = \theta^{m+1}$;
  else
    Proceed to the next iteration;
  end
end

In some cases, we are not able to find the joint expectation of β and Σ, but may know identically.

Example 7.

$$\hat{\theta} = \arg\min_{\theta} \mathcal{Q}(\theta; Y^n)$$

cannot be obtained analytically.

As $\theta = [\theta_1, \theta_2]'$, we know $\hat{\theta}_1 \mid \theta_2$ and $\hat{\theta}_2 \mid \theta_1$

Let's consider the GLS model:

$$y_i = x_i'\beta + u_i, \quad u_i|x_i \sim \mathcal{N}(0, \sigma_i^2).$$

Then, the likelihood function is:

$$\begin{aligned}
\hat{\theta}\left(\hat{\beta}, \hat{\Sigma}\right) &= \arg\max_{\theta} p\left(y \mid \beta, \Sigma\right) \\
\Rightarrow p\left(y \mid \beta, \Sigma\right) &= \prod_{i=1}^{n} p\left(y_i \mid \beta, \Sigma\right) \\
&= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp\left\{-\frac{1}{2\sigma_i^2}\left(y_i - x_i'\beta\right)^2\right\} \\
&= (2\pi)^{-\frac{n}{2}} \prod_{i=1}^{n} \sigma_i^{-1} \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\frac{1}{\sigma_i^2}\left(y_i - x_i'\beta\right)^2\right\} \\
&= (2\pi)^{-\frac{n}{2}} \left(\prod_{i=1}^{n} \sigma_i^{-1}\right) \exp\left\{-\frac{1}{2}\left(Y - X\beta\right)'\Sigma^{-1}\left(Y - X\beta\right)\right\} \\
&= (2\pi)^{-\frac{n}{2}} |\Sigma|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(Y - X\beta\right)'\Sigma^{-1}\left(Y - X\beta\right)\right\}. \\
\Rightarrow \hat{\beta} \mid \Sigma &= \left(X'\Sigma^{-1}X\right)^{-1} X'\Sigma^{-1}Y, \\
\hat{\Sigma} \mid \beta &= \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\sigma_i^2}\left(y_i - x_i'\hat{\beta}\right)^2 = diag(Y - X\beta)(Y - X\beta)'.
\end{aligned}$$

We can obtain the estimator $\hat{\theta}$ by iterating between $\hat{\beta} \mid \hat{\Sigma}$ and $\hat{\Sigma} \mid \hat{\beta}$.

Note (Meng and Rubin (1993) Algorithm).

Algorithm 4: Meng and Rubin (1993) Algorithm

Input: Initialize Σ^0(i.e. $= I$), tolerence level $\varepsilon > 0$

for $m = 1$ **to** M *given* Σ^m **do**
  Compute $\beta^{m+1} = \hat{\beta} \mid \hat{\Sigma}^m$;
  Compute $\Sigma^{m+1} = \hat{\Sigma} \mid \hat{\beta}^{m+1}$;
  if $\left\|\theta^{m+1} - \theta^m\right\| < \varepsilon$ **then**
    $\hat{\theta} = \theta^{m+1}$;
  else
    Proceed to the next iteration;
  end
end

8.2 Bootstrapping

For some point estimator $\hat{\theta}$, we only have the aympototic distribution of $\hat{\theta}$, but not the finite-sample distribution.

Example 8. Suppose we have the following model:

$$y_i = x_i'\beta + u_i, \quad u_i|x_i \sim \mathcal{N}(0, \sigma^2).$$

Then, the estimator $\hat{\beta}$ is:

$$\hat{\beta} = (X'X)^{-1} X'Y.$$

We know that:

$$\sqrt{n}\left(\hat{\beta} - \beta\right) \xrightarrow{d} \mathcal{N}\left(0, \sigma^2 (X'X)^{-1}\right).$$

However, we do not know the finite-sample distribution of $\hat{\beta}$.

But, we could use the bootstrapping method to estimate the finite-sample distribution of $\hat{\beta}$.

Note (Bootstrapping Method).

Algorithm 5: Bootstrapping Method

Input: Sample $Y^n = \{y_1, \cdots, y_n\}$, number of bootstrap samples B

1 **for** $m = 1$ **to** M **do**

2 Generate a bootstrap sample of n_B observations by sampling with replacement from $\{z_i\}_{i=1}^n$;

3 Compute the bootstrap estimator $\hat{\theta}^m$ using $\{z_i^m\}_{i=1}^{n_B}$;

4 **end**

5 The set $\{\hat{\theta}_m\}_{m=1}^M$ approximates the finite-sample distribution of $\hat{\theta} \mid \theta$ for sample size n_B;

6 Compute the bootstrap standard error $\hat{\sigma}_{\hat{\theta}} = \sqrt{\frac{1}{B}\sum_{m=1}^B \left(\hat{\theta}^m - \bar{\hat{\theta}}\right)^2}$;

8.3 Extremum Estimation

8.3.1 Standard Asymptotics

We have a general form of question:

$$\hat{\theta} = \arg\min_{\theta \in \Theta} \mathcal{Q}_n(\theta; Y^n).$$

Problem.

- Consistency: $\hat{\theta} \xrightarrow{p} \theta_0$?
- Distribution: $\sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} \mathcal{N}(0, V) \to \hat{\theta}_0 \overset{approx}{\to} \mathcal{N}\left(0, \frac{1}{n}\hat{V}\right)$?

Example 9 (Probit Model).

$$\hat{\beta} = \arg\min_{\beta} -\frac{1}{n}\sum_{i=1}^n \{y_i \log \Phi(x_i'\beta) + (1 - y_i) \log(-\Phi(x_i'\beta))\} = \ell(\beta \mid y).$$

Proposition 8.3.1 (Consistency).

Assumption 8.3.1.

- Θ is compact.
- $\mathcal{Q}_n(\theta, Y^n)$ converges uniformly in probability to $\mathcal{Q}(\theta)$ uniformly in θ; i.e.
$$\forall \varepsilon > 0, \mathbb{P}\left[\sup_{\theta\in\Theta} |\mathcal{Q}_n(\theta, Y^n) - \mathcal{Q}(\theta)| < \varepsilon\right] \to 1;$$
- $\mathcal{Q}(\theta)$ is continuous in Θ;
- $\mathcal{Q}(\theta)$ is uniquely minimized by θ_0, i.e. $\mathcal{Q}(\theta) > \mathcal{Q}(\theta_0) \quad \forall \theta \in \Theta, \theta \neq \theta_0$.

Then, $\hat{\theta} \xrightarrow{p} \theta_0$.

For our original model:

$$\{y_i\}_{i=1}^n \quad \text{with} \quad \mathbb{E}[y_i] = 0.$$

we have:

$$\hat{\theta} = \arg\min_{\theta} \sum_{i=1}^n (y_i - \theta)^2$$
$$\Rightarrow \hat{\theta} = \frac{1}{n}\sum_{i=1}^n y_i \xrightarrow{p} \mathbb{E}[y_i] = 0.$$

In this case, we have:

- Θ is compact, take $\Theta = [-c, c]$ for some large c.
- $\mathcal{Q}_n(\theta, Y^n) = \sum_{i=1}^n (y_i - \theta)^2 \xrightarrow{p} \underbrace{\mathbb{E}\left[(y_{i-\theta})^2\right]}_{Q(\theta)}$ by LLN;
- $Q(\theta) = \mathbb{E}[y_i^2] - 2\theta\mathbb{E}[y_i] + \theta^2 = \mathbb{V}[y_i] + (\mathbb{E}[y_i] - \theta)^2$ is continuous.
- $Q(\theta)$ is uniquely minimized by $\theta_0 = \mathbb{E}[y_i]$.
- $\hat{\theta} = \frac{1}{n}\sum_{i=1}^n y_i \xrightarrow{p} \mathbb{E}[y_i] = 0$.

Proposition 8.3.2 (Uniform Law of Large Numbers)**.**

Assumption 8.3.2.

- x_i are i.i.d;
- Parameter space Θ is compact;
- Object function $m(x;\theta)$ is continuous in θ;
- $\mathbb{E}\left[\sup_{\theta\in\Theta} \|m(x;\theta)\|\right] < \infty$.

Then,

$$\sup_{\theta\in\Theta}\left|\frac{1}{n}\sum_{i=1}^n m(x_i;\theta) - \mathbb{E}[m(x_i;\theta)]\right| \xrightarrow{p} 0.$$

Example 10 (Nonlinear Least Squares(NLS)).

Consider the nonlinear least squares (NLS) estimation of the regression model:

$$y_i = (x_i'\beta)^3 + u_i, \quad \mathbb{E}[u_i \mid x_i] = 0.$$

Define $\mathscr{B} = \{\beta \in \mathbb{R}^k : \|\beta\| \leq c\}$ for some $c > 0$ large and let

$$\hat{\beta} = \arg\min_{\beta \in \mathscr{B}} Q_n(\beta, Y^n) = \arg\min_{\beta \in \mathcal{B}} \frac{1}{2n} \sum_{i=1}^{n} \left(y_i - (x_i'\beta)^3\right)^2.$$

To show $\hat{\beta} \xrightarrow{p} \beta_0$, we show uniform convergence in probability of Q_n:

- $\mathscr{B}$ is compact in $\mathbb{R}^k$.
- $\mathcal{Q}_n(\beta) = \frac{1}{n}\sum \frac{1}{2}\left(y_i - (x_i'\beta)^3\right)^2 \xrightarrow{p} \mathbb{E}\left[\frac{1}{2}\left(y_i - (x_i'\beta)^3\right)^2\right] = \mathcal{Q}(\beta)$; [a]
- We know $\mathbb{E}\left[(y_i - h(x_i))^2\right]$ is uniquely minimized at $h(x_i) = \mathbb{E}[y_i \mid x_i] = (x_i'\beta_0)^3$. Thus, $Q_n(\beta) = \frac{1}{2}\mathbb{E}\left[\left(y_i - (x_i'\beta)^3\right)^2\right]$ is uniqely minimized at $\beta = \beta_0$.
- $Q(\theta)$ is continuous

[a] $Q_n(\beta) = \frac{1}{2}\frac{1}{n}\sum_{i=1}^{n} m((x_i, y_i), \beta)$ converges uniformly in probability to $Q(\beta) = \frac{1}{2}\mathbb{E}[m(x;\theta)] = \mathbb{E}\left[\left(y_i - (x_i'\beta)^3\right)^2\right]$ because $m((x_i, y_i), \beta) = \left(y_i - (x_i'\beta)^3\right)^2$ satisfies the conditions for the ULLN; the first three are obvious, and for the fourth it is sufficient to assume $\mathbb{E}\left[\|u_i\|^2\right] < \infty$ and $\mathbb{E}\left[\|x_i\|^6\right] < \infty$, along with $\|\beta\| \leq c$:

$$\mathbb{E}\left[\sup_{\beta \in \mathscr{B}} \|m((x_i, y_i), \beta)\|\right] \leqslant \mathbb{E}\left[|y_i|^2\right] + \sup_{\beta \in \mathscr{B}} 2\mathbb{E}\left[|y_i|\,|x_i|^3\,\|\beta\|^3\right] + \sup_{\beta \in \mathscr{B}}\left[|x_i|^6\,\|\beta\|^6\right] < \infty.$$

In general, there are three ways to show that θ_0 is the unique minimizer of $Q(\theta)$. First, one can write out $Q(\theta)$ to see it explicitly by looking at FOCs (and SOCs) as in the first example above. Second, one can use the conditional-expectation-argument as in the second example above. Third, one can show that $Q(\tilde{\theta}) - Q(\theta_0) > 0 \quad \forall \tilde{\theta} \neq \theta_0$.

Proposition 8.3.3 (Asymptotic Normality).

In addition to the conditions in 8.3.1, we assume:

Assumption 8.3.3.

- $\theta_0 \in int(\Theta)$;
- $\sqrt{n}\mathcal{Q}_n^{(1)}(\theta_0, Y^n) \xrightarrow{d} \mathcal{N}(0, M)$.
- $\mathcal{Q}_n(\theta, Y^n) \in \mathcal{C}^2$ w.r.t. $\theta \quad \forall Y^n$. Also, $\exists H$ s.t. $\mathcal{Q}_n^{(2)}(\theta_0, Y^n) \xrightarrow{p} H \quad \forall \theta_n \xrightarrow{p} \theta_0$.

Then,

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} \mathcal{N}(0, H^{-1}MH^{-1}).$$

Example 11 (Nonlinear Least Squares(NLS)).

Let's look at the NLS example again. We have:

- $\beta_0 \in int(\mathscr{B})$ for large c.

- By CLT, we have:

$$\begin{aligned}\mathcal{Q}_n(\beta) &= \frac{1}{2n}\sum_{i=1}^{n}\left(y_i - (x_i'\beta)^3\right)^2\\ \sqrt{n}\mathcal{Q}_n^{(1)}(\beta_0, Y^n) &= -\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(y_i - (x_i'\beta_0)^3\right)3\left(x_i'\beta_0\right)^2 x_i\\ &\xrightarrow{d} \mathbb{E}\left[9u_i^2(x_i'\beta_0)^4 x_i x_i'\right]\\ &\equiv \mathcal{N}(0, M).\end{aligned}$$

- $\mathcal{Q}_n(\beta) \in \mathcal{C}^2$ w.r.t. β.

$$\begin{aligned}\mathcal{Q}_n^{(2)}(\beta_0, Y^n) &= \frac{1}{n}\sum_{i=1}^{n} -\left(y_i - (x_i'\beta_0)^3\right)\left[6(x_i'\beta_0)x_i'x_i\right] + 9\left(x_i'\beta_0\right)^4 x_i x_i'\\ &\xrightarrow{d} \mathbb{E}\left[9(x_i'\beta_0)^4 x_i x_i'\right]\\ &\equiv H.\end{aligned}$$

As we know $M = \mathbb{E}\left[9u_i^2(x_i'\beta_0)^4 x_i x_i'\right]$ and $H = \mathbb{E}\left[9(x_i'\beta_0)^4 x_i x_i'\right]$, we have:

$$\sqrt{n}(\hat{\beta} - \beta_0) \xrightarrow{d} \mathcal{N}(0, \underbrace{H^{-1}MH^{-1}}_{V}).$$

where

$$\begin{aligned}V &= \left(\mathbb{E}\left[9(x_i'\beta_0)^4 x_i x_i'\right]\right)^{-1}\mathbb{E}\left[9u_i^2(x_i'\beta_0)^4 x_i x_i'\right]\left(\mathbb{E}\left[9(x_i'\beta_0)^4 x_i x_i'\right]\right)^{-1}\\ \Rightarrow \hat{V} &= \frac{1}{9}\left(\frac{1}{n}\sum_{i=1}^{n}(x_i'\hat{\beta})^4 x_i x_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}(x_i'\hat{\beta})^4 x_i x_i'\hat{u}_i^2\right)\left(\frac{1}{n}\sum_{i=1}^{n}(x_i'\hat{\beta})^4 x_i x_i'\right)^{-1}\\ \overset{\mathbb{E}[u_i^2|x_i]=0}{\Longrightarrow} \hat{V} &= \frac{1}{9}\left(\frac{1}{n}\sum_{i=1}^{n}(x_i'\hat{\beta})^4 x_i x_i'\right)^{-1}\sigma^2\mathbb{E}\left[(x_i'\beta)^4 x_i x_i'\right]\left(\frac{1}{n}\sum_{i=1}^{n}(x_i'\hat{\beta})^4 x_i x_i'\right)^{-1}\\ &= \frac{\hat{\sigma}^2}{9}\left(\mathbb{E}\left[(x_i'\beta)^4 x_i x_i'\right]\right)^{-1}\\ &= \frac{\hat{\sigma}^2}{9}\left[\frac{1}{n}\sum(x_i'\beta)^4 x_i x_i'\right]^{-1}\end{aligned}$$

Example 12 (Maximum-likelihood Estimation).

Consider the normal maximum likelihood estimation:

$$\hat{\theta} = \arg\min_{\theta} -\frac{1}{n}\sum_{i=1}^{n}\log f(y_i;\theta).$$

We have:

$$Q_n(\theta) = -\frac{1}{n}\sum_{i=1}^{n}\log f(y_i;\theta).$$

Then, we have:

$$\begin{aligned}\sqrt{n}Q_n^{(1)}(\theta_0) &= -\frac{1}{n}\sum_{i=1}^{n}\frac{\partial \log f(y_i;\theta_0)}{\partial\theta} \\ &= -\frac{1}{\sqrt{n}}\sum_{i=1}^{n}s_i(\theta_0) \\ &\xrightarrow{d} \mathcal{N}(0,\mathbb{V}[s_i(\theta_0)]).\end{aligned}$$

where

$$\mathbb{V}[s_i(\theta_0)] = \mathbb{E}[s_i(\theta_0)s_i(\theta_0)'] - \mathbb{E}[s_i(\theta_0)]^2 \overset{\mathbb{E}[s_i(\theta_0)]=0}{=} \mathbb{E}[s_i(\theta_0)s_i(\theta_0)'].$$

Also, we have:

$$\begin{aligned}Q_n^{(2)}(\theta_0) &= -\frac{1}{n}\sum_{i=1}^{n}\frac{\partial^2 \log f(y_i;\theta_0)}{\partial\theta\partial\theta'} \\ &= -\frac{1}{n}\sum_{i=1}^{n}H_i(\theta_0) \\ &\xrightarrow{p} \mathbb{E}[-H_i(\theta_0)] = I(\theta).\end{aligned}$$

We know that:

$$\begin{aligned}s(\theta) &= \frac{1}{\sigma^2}(y_i - x_i\theta)x_i \\ H(\theta) &= -\frac{1}{\sigma^2}x_ix_i' \\ \Rightarrow \mathbb{E}[s_i(\theta_0)s_i(\theta_0)'] &= \frac{1}{\sigma^4}\mathbb{E}[x_i'(y_i - x_i\theta_0)(y_i - x_i\theta_0)'x_i] \\ &= \frac{1}{\sigma^2}\mathbb{E}[x_ix_i']\end{aligned}$$

Thus, we have: $M = H$. Hence we get:

$$\sqrt{n}(\hat{\theta}_{MLE} - \theta) \xrightarrow{d} \mathcal{N}(0, H^{-1})$$

Lecture 9.

General Theory of Extremum Estimation* [1]

9.1 Asymptotic Normality of M-Estimators

The objective function for M-estimators is

$$Q_n(\theta) = -\frac{1}{n}\sum_{t=1}^{n} m(w_t;\theta). \tag{9.1}$$

It will be convenient to give symbols to the gradient (vector of first derivatives) and the Hessian (matrix of second derivatives) of the m function as

$$s(w_t;\theta) = \frac{\partial m(w_t;\theta)}{\partial \theta} \tag{9.2}$$

$$H(w_t;\theta) = \frac{\partial s(w_t;\theta)}{\partial \theta'} = \frac{\partial^2 m(w_t;\theta)}{\partial \theta \partial \theta'}. \tag{9.3}$$

In analogy to ML, $s(w_t;\theta)$ will be referred to as the **score vector for observation** t. This $s(w_t;\theta)$ should not be confused with the score in the usual sense, which is the gradient of the objective function $Q_n(\theta)$. The score in the latter sense will be denoted $s_n(\theta)$ later in this chapter. The same applies to the Hessian: $H(w_t;\theta)$ will be referred to as the **Hessian for observation** t, and the Hessian of $Q_n(\theta)$ will be denoted $H_n(\theta)$.

The goal of this subsection is the asymptotic normality of $\hat{\theta}$ described below. In the process of deriving it, we will make a number of assumptions, which will be collected in Proposition 7.8 below. Assume that $m(w_t;\theta)$ is differentiable in θ and that $\hat{\theta}$ is in the interior of Θ. So $\hat{\theta}$, being the interior solution to the problem of maximizing $Q_n(\theta)$, satisfies the first-order conditions

$$0 = \frac{\partial Q_n(\hat{\theta})}{\partial \theta} = \frac{1}{n}\sum_{t=1}^{n} s(w_t;\hat{\theta}). \tag{9.4}$$

We now use the following result from calculus:

Theorem 9.1.1 (Mean Value Theorem)**.**

Let $h : \mathbb{R}^p \to \mathbb{R}^q$ be continuously differentiable. Then $h(x)$ admits the mean value expansion

$$h(x) = h(x_0) + \frac{\partial h(\bar{x})}{\partial x'}(x - x_0), \tag{9.5}$$

where $\tilde{x}$ is a mean value lying between x and x_0. [a]

[a] The Mean Value Theorem only applies to individual elements of h, so that $\bar{x}$ actually differs from element to element of the vector equation. This complication does not affect the discussion in the text.

Setting $q = p$, $x = \hat{\theta}$, $x_0 = \theta_0$, and $h(\cdot) = \frac{\partial Q_n(\cdot)}{\partial \theta}$ in the Mean Value Theorem, we obtain the following

[1] This lecture is not required in class, I personally borrowed contents from different books and papers to form the part, based on Amemiya(1985)[1] and Hayashi(2000)[2]. Extremum estimators are a wide class of estimators for parametric models that are calculated through maximization (or minimization) of a certain objective function, which depends on the data. The general theory of extremum estimators was developed by Amemiya (1985)[1].

mean value expansion:

$$\frac{\partial Q_n(\hat{\theta})}{\partial \theta} = \frac{\partial Q_n(\theta_0)}{\partial \theta} + \frac{\partial^2 Q_n(\bar{\theta})}{\partial \theta \partial \theta'}(\hat{\theta} - \theta_0) \tag{9.6}$$

$$= \frac{1}{n}\sum_{t=1}^{n} s(w_t;\theta_0) + \left[\frac{1}{n}\sum_{t=1}^{n} H(w_t;\bar{\theta})\right](\hat{\theta} - \theta_0), \tag{9.7}$$

where $\tilde{\theta}$ is a mean value that lies between $\hat{\theta}$ and θ_0. The continuous differentiability requirement of the Mean Value Theorem is satisfied if $m(w_t;\theta)$ is twice continuously differentiable with respect to θ. Combining this equation with the first-order condition above, we obtain

$$0 = \frac{1}{n}\sum_{t=1}^{n} s(w_t;\theta_0) + \left[\frac{1}{n}\sum_{t=1}^{n} H(w_t;\bar{\theta})\right](\hat{\theta} - \theta_0). \tag{9.8}$$

Assuming that $\frac{1}{n}\sum_{t=1}^{n} H(w_t;\bar{\theta})$ is nonsingular, this equation can be solved for $\hat{\theta} - \theta_0$ to yield

$$\sqrt{n}(\hat{\theta} - \theta_0) = -\left[\frac{1}{n}\sum_{t=1}^{n} H(w_t;\bar{\theta})\right]^{-1} \frac{1}{\sqrt{n}}\sum_{t=1}^{n} s(w_t;\theta_0). \tag{9.9}$$

This expression for ($\sqrt{n}$ times) the sampling error will be referred to as the mean value expansion for the sampling error. Note that the score vector $s(w_t;\theta_0)$ is evaluated at the true parameter value θ_0.

Now, since $\bar{\theta}$ lies between θ_0 and $\hat{\theta}$, $\bar{\theta}$ is consistent for θ_0 if $\hat{\theta}$ is. If $\{w_t\}$ is ergodic stationary, it is natural to conjecture that

$$\frac{1}{n}\sum_{t=1}^{n} H(w_t;\bar{\theta}) \to \mathbb{E}[H(w_t;\theta_0)]. \tag{9.10}$$

The ergodic stationarity of w_t and consistency of $\hat{\theta}$ alone, however, are not enough to ensure this; some technical condition needs to be assumed. One such technical condition is the uniform convergence of $\frac{1}{n}\sum_{t=1}^{n} H(w_t;\cdot)$ to $\mathbb{E}[H(w_t;\cdot)]$ in a neighborhood of θ_0. By the Uniform Convergence Theorem, the uniform convergence of $\frac{1}{n}\sum_{t=1}^{n} H(w_t;\cdot)$ is satisfied if the following dominance condition is satisfied for the Hessian: for some neighborhood $\mathcal{N}$ of θ_0, $\mathbb{E}\left[\sup_{\mu\in\mathcal{N}} \|H(w_t;\theta)\|\right] < \infty$. This is a sufficient condition for (9.10); if you can directly verify (9.10), then there is no need to verify the dominance condition for asymptotic normality.

Finally, if (9.10) holds and if

$$\frac{1}{\sqrt{n}}\sum_{t=1}^{n} s(w_t;\theta_0) \xrightarrow{d} N(0,\Sigma), \tag{9.11}$$

then by the Slutsky theorem we have

$$\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N\left(0, (\mathbb{E}[H(w_t;\theta_0)])^{-1}\,\Sigma\,(\mathbb{E}[H(w_t;\theta_0)])^{-1}\right). \tag{9.12}$$

Proposition 9.1.1 (Asymptotic normality of M-estimators).

Suppose that the conditions of either Proposition 7.3 or Proposition 7.4 are satisfied, so that $\{w_t\}$ is ergodic stationary and the M-estimator $\hat{\theta}$ defined by (7.1.1) and (7.1.2) is consistent. Suppose, further, that

(1) θ_0 is in the interior of Θ,

(2) $m(w_t;\theta)$ is twice continuously differentiable in θ for any w_t,

(3) $\frac{1}{\sqrt{n}} \sum_{t=1}^{n} s(w_t; \theta_0) \xrightarrow{d} N(0, \Sigma)$, Σ positive definite, where $s(w_t; \theta)$ is defined previously,

(4) **Local dominance condition on the Hessian** for some neighborhood $\mathcal{N}$ of θ_0,

$$\mathbb{E}\left[\sup_{\theta \in \mathcal{N}} \|H(w_t; \theta)\|\right] < \infty,$$

so that for any consistent estimator $\tilde{\theta}$,

$$\frac{1}{n} \sum_{t=1}^{n} H(w_t; \tilde{\theta}) \xrightarrow{p} \mathbb{E}[H(w_t; \theta_0)],$$

where $H(w_t; \theta)$ is defined previously.

(5) $\mathbb{E}[H(w_t; \theta_0)]$ is nonsingular.

Then $\hat{\theta}$ is asymptotically normal with

$$\text{Avar}(\hat{\theta}) = \left(\mathbb{E}[H(w_t; \theta_0)]\right)^{-1} \Sigma \left(\mathbb{E}[H(w_t; \theta_0)]\right)^{-1}.$$

(This is Theorem 4.1.3 of Amemiya (1985)[1] adapted to M-estimators). Two remarks are in order.

Remark.

- Of the assumptions we have made in the derivation of asymptotic normality, the following are not listed in the proposition:

  (i) $\hat{\theta}$ is an interior point,

  (ii) $\frac{1}{n} \sum_{t=1}^{n} H(w_t; \hat{\theta})$ is nonsingular.

  It is intuitively clear that these conditions hold because $\hat{\theta}$ converges in probability to an interior point θ_0, and $\frac{1}{n} \sum_{t=1}^{n} H(w_t; \hat{\theta})$ converges in probability to a nonsingular matrix $\mathbb{E}[H(w_t; \theta_0)]$.

  See Newey and McFadden (1994, p. 2152) for a rigorous proof. This sort of technicality will be ignored in the rest of this chapter.

- If w_t is ergodic stationary, then so is $s(w_t; \theta_0)$ and the matrix Σ is the long run variance matrix of $\{s(w_t; \theta_0)\}$. A sufficient condition for (3) is Gordin's condition introduced in Section 6.5. So condition (3) in the proposition can be replaced by Gordin's condition on $\{s(w_t; \theta_0)\}$.

  It is satisfied, for example, if w_t is i.i.d. and $\mathbb{E}[s(w_t; \theta_0)] = 0$. The assumption that Σ is positive definite is not really needed for the conclusion of the proposition, but we might as well assume it here because in virtually all applications it is satisfied (or assumed) and also because it will be required in the discussion of hypothesis testing later in this chapter.

9.1.1 Consistent Asymptotic Variance Estimation

To use this asymptotic result for hypothesis testing, we need a consistent estimate of

$$\text{Avar}(\hat{\theta}) = \left(\mathbb{E}[H(w_t; \theta_0)]^{-1} \Sigma \mathbb{E}[H(w_t; \theta_0)]^{-1}\right)^{-1}.$$

The long-run variance was introduced in Section 6.5.

Since $\hat{\theta} \xrightarrow{p} \theta_0$, condition (4) of Proposition 9.1.1 implies that

$$\frac{1}{n}\sum_{t=1}^{n} H(w_t;\hat{\theta}) \xrightarrow{p} \mathbb{E}[H(w_t;\theta_0)].$$

Therefore, provided that there is available a consistent estimator $\hat{\Sigma}$ of Σ,

$$\text{Avar}(\hat{\theta}) = \left[\frac{1}{n}\sum_{t=1}^{n} H(w_t;\hat{\theta})\right]^{-1} \hat{\Sigma} \left[\frac{1}{n}\sum_{t=1}^{n} H(w_t;\hat{\theta})\right]^{-1}$$

is a consistent estimator of the asymptotic variance matrix.

To obtain $\hat{\Sigma}$, the methods introduced in Section 6.6, such as the VARHAC, can be applied to the estimated series $\{s(w_t;\hat{\theta})\}$ under some suitable technical conditions.

9.1.2 Asymptotic Normality of Conditional ML

Lecture 10.

Topics in Econometrics(2) - Cross-Sectional Data

10.1 Recall

For sample $X : \{x_i\}_{i=1}^n$, we draw $p(x;\theta)$ giving us :

- point estimator $\hat{\theta}$:
  - $p_n(\hat{\theta})$ in finite samples
  - $p(\hat{\theta})$ in Asymptotic samples, where $\operatorname{plim}_{n\to\infty}(\hat{\theta})$ and $\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{d} \mathcal{N}(0,V)$.
- Hypothesis testing: $\mathcal{H}_0 : \theta = \theta_0 \to g(\theta) : \theta - \theta_0 = 0$.
- CI construction:

10.2 Parameter Transformation

Example 13.

LRM: $y_i = x_i'\beta + u_i$, interest in: $\beta : \tilde{x}'\beta = \mathbb{E}[y_i \mid x_i = \tilde{x}]$.

Probit:

$$\begin{aligned} y_i^* &= x_i'\beta + u_i \\ y_i &= \mathbf{1}\{y_i^* > 0\}. \end{aligned}$$

interest (mostly) in $\mathbb{E}[y_i \mid x_i = \tilde{x}] = \mathbb{E}[y_i = 1 \mid x_i = \tilde{x}] = \Phi(\tilde{x}_i'\beta)$.

1. Point estimator
2. Hypothesis testing & CI construction:
   - We know how to test: $\mathcal{H}_0 : g(\theta) = 0$, apply to $\mathcal{H}_0 : f(\theta) = f_0$, rewrite $\mathcal{H}_0 : g(\theta) = f(\theta) - f_0 = 0$.
   - We know that $CI : \{\mathcal{H}_0 : f(\theta) = f_0 \text{ is accepted}\}$. But finding this set can be very hard.

     Example 14.

     In the linear regression model, we have $\hat{\beta} \mid \beta \sim \mathcal{N}(\beta, V)$ with $V = \sigma^2\mathbb{E}[(X'X)^{-1}]$, therefore $x'\hat{\beta} \mid x'\beta \sim \mathcal{N}(x'\beta, x'Vx)$.

     $$\begin{aligned} T_w &= n g(\hat{\theta})'[g(\hat{\theta})g(\hat{\theta})']^{-1} g(\hat{\theta}) < c \\ &= n\left[f(\hat{\theta}) - f_0\right]'[GVG']^{-1}\left[f(\hat{\theta}) - f_0\right] < c \\ &\xrightarrow{d} \chi^2(r), \end{aligned}$$

     where $r = \operatorname{rank}[g(\theta)]$
   - It's easier if $f(\theta)$ is a scalar. Try to find the distribution of $f(\hat{\theta})$.

– Analytically, in finite samples:

If $\hat{\beta} \mid X \sim \mathcal{N}\left(\beta_0, \underbrace{\sigma^2(X'X)^{-1}}_{V}\right)$, then,

$$\underbrace{\tilde{x}'\hat{\beta}}_{\hat{\delta}} \mid X \sim \mathcal{N}\left(\tilde{x}'\beta_0, \tilde{x}'V\tilde{x}\right)$$

$$\hat{\delta} \sim \mathcal{N}(\delta_0, V_\delta)$$

$$\Rightarrow T_t(x) = \left|\frac{\delta - \delta_0}{\sqrt{V_\delta}}\right|$$

– Analytically, in asymptotic properties:

Example 15. $\Phi(\tilde{x}'\hat{\beta})$

* No finite sample distribution
* Asymptotic distribution based on Delta Method:

$$\begin{aligned}
\sqrt{n}(\hat{\theta} - \theta) &\xrightarrow{d} \mathcal{N}(0, V) \\
\Rightarrow \sqrt{n}\left(g(\hat{\theta}) - g(\theta)\right) &\xrightarrow{d} \mathcal{N}(0, GVG') \\
\Rightarrow \sqrt{n}(\hat{\beta} - \beta) &\xrightarrow{d} \mathcal{N}(0, V_\beta) \\
\Rightarrow \sqrt{n}\left(\Phi(\tilde{x}'\hat{\beta}) - \Phi(\tilde{x}'\beta_0)\right) &\xrightarrow{d} \mathcal{N}\left(0, \phi(\tilde{x}'\beta_0)\tilde{x}'V\tilde{x}\phi(\tilde{x}'\beta_0)'\right) \\
\Rightarrow \Phi(\tilde{x}'\hat{\beta}) &\xrightarrow{d} \mathcal{N}\left(\Phi(\tilde{x}'\beta_0), \frac{\phi^2(\tilde{x}'\beta_0)\tilde{x}'V\tilde{x}}{n}\right)
\end{aligned}$$

Example 16 (Probit Model)**.**
In Probit Model, the marginal effect of one variable x_c is measured by $g(\beta_c) = \phi(x'\beta)\beta_c$, then, we implement the parameter transformation to get the distribution of $g(\hat{\beta}_c)$.

$$\begin{aligned}
G(\hat{\beta}) = \frac{\partial g(\hat{\beta})}{\partial \beta'} &= \phi(x'\beta) + \phi'(x'\beta)\beta \\
&= \phi(x'\beta) - (x'\beta)\phi(x'\beta)x'\beta \\
&= \phi(x'\beta)\left(I - (x'\beta)^2\right)
\end{aligned}$$

Then, using the Theorem[5.2.1](Delta Method), we get the distribution of $g(\hat{\beta}_c)$:

$$\sqrt{n}(g(\hat{\beta}) - g(\beta)) \xrightarrow{d} \mathcal{N}\left(0, G(\theta)VG(\theta)'\right)$$

– Bootstrapping: to get distribution of $f(\hat{\theta})$, take $\left\{\hat{\theta}^m\right\}_{i=1}^{M}$ as the estimator applied to data $\{x_i^m\}_{i=1}^{n_B}$. Then, we get the distribution of $f(\hat{\theta})$: take $\left\{f(\hat{\theta^m})\right\}_{i=1}^{M}$.

10.3 Instrumental Variables

Background: $y_i = x_i'\beta + u_i$, with $\mathbb{E}[x_i u_i] \neq 0$, meaning x_i is endogenous. β is consistent if $\mathbb{E}[x_i u_i] = 0$.

Example 17.

So, we want to find a way to estimate β consistently when x_i is endogenous.

$$\begin{aligned} y_i &= x_i'\beta + u_i \\ x_i &= z_i'\gamma + e_i \end{aligned}$$

1. z_i is exogenous to error term u_i: $\mathbb{E}[z_i u_i] = 0$.
2. z_i is relevant to regressor x_i: $\mathbb{E}[z_i x_i'] \neq 0$

Then, we have the 2SLS method:

1. Estimate $\hat{\gamma}$ from $x_i = z_i'\gamma + e_i$. $\hat{\gamma} = (Z'Z)^{-1}Z'X$ and

$$\begin{aligned} \hat{x}_i &= z_i'\hat{\gamma} \\ \hat{X} &= Z\hat{\gamma} = Z(Z'Z)^{-1}Z'X \\ &= Z\gamma + Z(Z'Z)^{-1}Z'e \\ &= X + Z(Z'Z)^{-1}Z'e \end{aligned}$$

2. Estimate $\hat{\beta}$ from $y_i = \hat{x}_i'\beta + u_i^*$. This gives us

$$\begin{aligned} \hat{\beta}_{2SLS} &= \left(\hat{X}'\hat{X}\right)^{-1}\hat{X}'Y \\ &= \left((P_Z X)'P_Z X\right)^{-1}(P_Z X)'Y \\ &= \left(X'P_Z'P_Z X\right)^{-1}X'P_Z Y \\ &= \left(X'Z(Z'Z)^{-1}Z'X\right)^{-1}X'Z(Z'Z)^{-1}Z'Y \\ &= \beta + (\cdots)\underbrace{Z'U}_{0} \\ &\xrightarrow{p} \beta \end{aligned}$$

$$\begin{aligned} \hat{\beta}_{IV} &= \left(\sum_{i=1}^n z_i x_i'\right)^{-1}\sum_{i=1}^n z_i y_i \\ \sqrt{n}(\hat{\beta}_{IV} - \beta) &\xrightarrow{d} \mathcal{N}(0, V_{IV}) \end{aligned}$$

- V_{IV} is not easy to find.
- CI : $\{\mathcal{H}_0 : \beta = \beta_0 \text{ is accepted}\}$.

Appendix

Recommended Resources

Books

[1] Takeshi Amemiya. *Advanced Econometrics*. Cambridge, MA: Harvard University Press, 1985 (pp. 45, 47)

[2] Fumio Hayashi. *Econometrics*. Princeton, New Jersey: Princeton University Press, 2000 (p. 45)

[3] James H. Stock and Mark W. Watson. *Introduction to Econometrics*. 4th ed. New York: Pearson, 2003

[4] Jeffrey M. Wooldridge. *Introductory Econometrics: A Modern Approach*. 7th ed. Cengage Learning, 2020

[5] Bruce E. Hansen. *Econometrics*. Princeton, New Jersey: Princeton University Press, 2022

[6] Jeffrey M. Wooldridge. *Econometric Analysis of Cross Section and Panel Data*. 2nd ed. Cambridge, Massachusetts: The MIT Press, 2010

[7] Joshua Chan et al. *Bayesian Econometric Methods*. 2nd ed. Cambridge, United Kingdom: Cambridge University Press, 2019

[8] Badi H. Baltagi. *Econometric Analysis of Panel Data*. 6th ed. Cham, Switzerland: Springer, 2021

[9] James D. Hamilton. *Time Series Analysis*. Princeton, New Jersey: Princeton University Press, 1994. ISBN: 9780691042893

Others

[10] Roger Bowden. "The Theory of Parametric Identification". In: *Econometrica* 41.6 (1973), pp. 1069–1074. DOI: 10.2307/1914036

[11] Robert I. Jennrich. "Asymptotic Properties of Non-linear Least Squares Estimators". In: *The Annals of Mathematical Statistics* 40.2 (1969), pp. 633–643. DOI: 10.1214/aoms/1177697731

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[13] Thomas J. Rothenberg. "Identification in Parametric Models". In: *Econometrica* 39.3 (1971), pp. 577–591. DOI: 10.2307/1913267

[14] George Tauchen. "Diagnostic Testing and Evaluation of Maximum Likelihood Models". In: *Journal of Econometrics* 30 (1985), pp. 415–443. DOI: 10.1016/0304-4076(85)90149-6

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[16] Halbert White. "Maximum Likelihood Estimation of Misspecified Models". In: *Econometrica* 50.1 (1982), pp. 1–25. DOI: 10.2307/1912526