Macroeconomics A; EI056

Short problems

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1 Solow model in discrete time

1.1 Capital dynamics

Question: Consider the model in discrete time. The production function is:

$$Y_t = (K_t)^{\alpha} (A_t L_t)^{1-\alpha} \Rightarrow y_t = (k_t)^{\alpha}$$

where $y_t$and k_t are scaled by effective labor $A_t L_t$. A fraction s_K of output is saved, so the output dynamics are:

$$K_{t+1} - K_t = s_K Y_t - \delta K_t$$

Labor grows at a rate n and productivity at a rate g (that is $L_{t+1} = (1+n) L_t$ and $A_{t+1} = (1+g) A_t$).

Show that the dynamics of scaled capital are:

$$k_{t+1} - k_t = \frac{1}{(1+n)(1+g)} s_K y_t - \frac{(1+n)(1+g) - (1-\delta)}{(1+n)(1+g)} k_t$$

Answer: The dynamics of capital are:

$$\begin{aligned}
K_{t+1} - K_t &= s_K Y_t - \delta K_t \\
A_{t+1} L_{t+1} k_{t+1} - A_t L_t k_t &= s_K A_t L_t y_t - \delta A_t L_t k_t \\
(1+n)(1+g) A_t L_t k_{t+1} - A_t L_t k_t &= s_K A_t L_t y_t - \delta A_t L_t k_t \\
(1+n)(1+g) k_{t+1} - k_t &= s_K y_t - \delta k_t \\
k_{t+1} - \frac{1}{(1+n)(1+g)} k_t &= \frac{1}{(1+n)(1+g)} s_K y_t - \frac{\delta}{(1+n)(1+g)} k_t \\
k_{t+1} - k_t - \frac{1-(1+n)(1+g)}{(1+n)(1+g)} k_t &= \frac{1}{(1+n)(1+g)} s_K y_t - \frac{\delta}{(1+n)(1+g)} k_t \\
k_{t+1} - k_t &= \frac{1}{(1+n)(1+g)} s_K y_t + \frac{(1-\delta) - (1+n)(1+g)}{(1+n)(1+g)} k_t \\
k_{t+1} - k_t &= \frac{1}{(1+n)(1+g)} s_K y_t - \frac{(1+n)(1+g) - (1-\delta)}{(1+n)(1+g)} k_t
\end{aligned}$$

1.2 Steady state

Question: Show that in the steady state:

$$\begin{aligned} k^* &= \left[\frac{s_K}{(1+n)(1+g)-(1-\delta)}\right]^{\frac{1}{1-\alpha}} \\ y^* &= \left[\frac{s_K}{(1+n)(1+g)-(1-\delta)}\right]^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

Answer: In the steady-state, $k_{t+1} = k_t = k^*$, so the dynamics imply:

$$\begin{aligned} 0 &= k_{t+1} - k_t \\ 0 &= \frac{1}{(1+n)(1+g)} s_K y^* - \frac{(1+n)(1+g)-(1-\delta)}{(1+n)(1+g)} k^* \\ 0 &= s_K y^* - [(1+n)(1+g)-(1-\delta)]\, k^* \\ 0 &= s_K (k^*)^{\alpha} - [(1+n)(1+g)-(1-\delta)]\, k^* \\ 0 &= s_K (k^*)^{\alpha-1} - [(1+n)(1+g)-(1-\delta)] \\ 0 &= s_K - [(1+n)(1+g)-(1-\delta)]\,(k^*)^{1-\alpha} \\ [(1+n)(1+g)-(1-\delta)]\,(k^*)^{1-\alpha} &= s_K \\ k^* &= \left[\frac{s_K}{(1+n)(1+g)-(1-\delta)}\right]^{\frac{1}{1-\alpha}} \end{aligned}$$

Using the production function, we get the output:

$$\begin{aligned} y^* &= (k^*)^{\alpha} \\ y^* &= \left[\frac{s_K}{(1+n)(1+g)-(1-\delta)}\right]^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

1.3 Approximation

Question: Show that a linear expansion of the capital dynamics around the steady state implies:

$$\hat{k}_{t+1} - \hat{k}_t = -\frac{(1+n)(1+g)-(1-\delta)}{(1+n)(1+g)}(1-\alpha)\,\hat{k}_t$$

where $\hat{k}_t = ln(k_t) - ln(k^*)$. You may find useful to use the fact that $k_t = exp(ln(k_t))$.

What can you say about the dynamics of capital? How is the speed of movement affected by α, n, g, δ and s_K?

Answer: We re-express the capital dynamics in logs:

$$\begin{aligned} k_{t+1} - k_t &= \frac{1}{(1+n)(1+g)} s_K y_t - \frac{(1+n)(1+g)-(1-\delta)}{(1+n)(1+g)} k_t \\ exp(ln(k_{t+1})) - exp(ln(k_t)) &= \frac{1}{(1+n)(1+g)} s_K (exp(ln(k_t)))^{\alpha} - \frac{(1+n)(1+g)-(1-\delta)}{(1+n)(1+g)} exp(ln(k_t)) \end{aligned}$$

The first order expansion of $exp\left(ln\left(k_t\right)\right)$ is:

$$
\begin{aligned}
exp\left(ln\left(k_t\right)\right) &= exp\left(ln\left(k^*\right)\right) + exp\left(ln\left(k^*\right)\right)\left[ln\left(k_t\right) - ln\left(k^*\right)\right] \\
exp\left(ln\left(k_t\right)\right) &= exp\left(ln\left(k^*\right)\right)\left(1+\hat{k}_t\right) \\
exp\left(ln\left(k_t\right)\right) &= k^*\left(1+\hat{k}_t\right)
\end{aligned}
$$

Similarly:

$$
\begin{aligned}
\left(exp\left(ln\left(k_t\right)\right)\right)^\alpha &= \left(exp\left(ln\left(k^*\right)\right)\right)^\alpha + \alpha\left(exp\left(ln\left(k^*\right)\right)\right)^{\alpha-1}\left[exp\left(ln\left(k^*\right)\right)\left[ln\left(k_t\right) - ln\left(k^*\right)\right]\right] \\
\left(exp\left(ln\left(k_t\right)\right)\right)^\alpha &= \left(exp\left(ln\left(k^*\right)\right)\right)^\alpha + \alpha\left(exp\left(ln\left(k^*\right)\right)\right)^{\alpha}\left[ln\left(k_t\right) - ln\left(k^*\right)\right] \\
\left(exp\left(ln\left(k_t\right)\right)\right)^\alpha &= \left(exp\left(ln\left(k^*\right)\right)\right)^\alpha\left(1+\alpha\hat{k}_t\right) \\
\left(exp\left(ln\left(k_t\right)\right)\right)^\alpha &= \left(k^*\right)^\alpha\left(1+\alpha\hat{k}_t\right)
\end{aligned}
$$

Putting it all together:

$$
\begin{aligned}
exp\left(ln\left(k_{t+1}\right)\right) - exp\left(ln\left(k_t\right)\right) &= \frac{1}{\left(1+n\right)\left(1+g\right)}s_K\left(exp\left(ln\left(k_t\right)\right)\right)^\alpha - \frac{\left(1+n\right)\left(1+g\right)-\left(1-\delta\right)}{\left(1+n\right)\left(1+g\right)}exp\left(ln\left(k_t\right)\right) \\
k^*\left(1+\hat{k}_{t+1}\right) - k^*\left(1+\hat{k}_t\right) &= \frac{1}{\left(1+n\right)\left(1+g\right)}s_K\left(k^*\right)^\alpha\left(1+\alpha\hat{k}_t\right) \\
&\quad - \frac{\left(1+n\right)\left(1+g\right)-\left(1-\delta\right)}{\left(1+n\right)\left(1+g\right)}k^*\left(1+\hat{k}_t\right) \\
\left(1+\hat{k}_{t+1}\right) - \left(1+\hat{k}_t\right) &= \frac{1}{\left(1+n\right)\left(1+g\right)}s_K\left(k^*\right)^{\alpha-1}\left(1+\alpha\hat{k}_t\right) \\
&\quad - \frac{\left(1+n\right)\left(1+g\right)-\left(1-\delta\right)}{\left(1+n\right)\left(1+g\right)}\left(1+\hat{k}_t\right)
\end{aligned}
$$

Using our result for k^*, we have $\left(k^*\right)^{\alpha-1} = \frac{(1+n)(1+g)-(1-\delta)}{s_K}$, thus:

$$
\begin{aligned}
\left(1+\hat{k}_{t+1}\right) - \left(1+\hat{k}_t\right) &= \frac{1}{\left(1+n\right)\left(1+g\right)}s_K\frac{\left(1+n\right)\left(1+g\right)-\left(1-\delta\right)}{s_K}\left(1+\alpha\hat{k}_t\right) \\
&\quad - \frac{\left(1+n\right)\left(1+g\right)-\left(1-\delta\right)}{\left(1+n\right)\left(1+g\right)}\left(1+\hat{k}_t\right) \\
\left(1+\hat{k}_{t+1}\right) - \left(1+\hat{k}_t\right) &= \frac{\left(1+n\right)\left(1+g\right)-\left(1-\delta\right)}{\left(1+n\right)\left(1+g\right)}\left(1+\alpha\hat{k}_t\right) \\
&\quad - \frac{\left(1+n\right)\left(1+g\right)-\left(1-\delta\right)}{\left(1+n\right)\left(1+g\right)}\left(1+\hat{k}_t\right) \\
\hat{k}_{t+1} - \hat{k}_t &= \frac{\left(1+n\right)\left(1+g\right)-\left(1-\delta\right)}{\left(1+n\right)\left(1+g\right)}\left(1+\alpha\hat{k}_t - 1 - \hat{k}_t\right) \\
\hat{k}_{t+1} - \hat{k}_t &= \frac{\left(1+n\right)\left(1+g\right)-\left(1-\delta\right)}{\left(1+n\right)\left(1+g\right)}\left(\alpha-1\right)\hat{k}_t \\
\hat{k}_{t+1} - \hat{k}_t &= -\frac{\left(1+n\right)\left(1+g\right)-\left(1-\delta\right)}{\left(1+n\right)\left(1+g\right)}\left(1-\alpha\right)\hat{k}_t \\
\hat{k}_{t+1} - \hat{k}_t &= -\Theta\hat{k}_t
\end{aligned}
$$

This shows that capital reverts to the steady state: if $\hat{k}_t > 0$ then $\hat{k}_{t+1} < \hat{k}_t$.

This reversion if faster (Θ is larger) if α small (capital is not a big part of production), g or n is large (the capital ratio is rapidly diluted on its own), and δ is high (capital rapidly wears out on its own.

$$\begin{aligned}
\frac{\partial\Theta}{\partial\alpha} &= -\frac{(1+n)(1+g)-(1-\delta)}{(1+n)(1+g)}<0\\
\frac{\partial\Theta}{\partial g} &= \frac{(1-\delta)}{(1+n)(1+g)^2}(1-\alpha)>0\\
\frac{\partial\Theta}{\partial n} &= \frac{(1-\delta)}{(1+n)^2(1+g)}(1-\alpha)>0\\
\frac{\partial\Theta}{\partial\delta} &= \frac{1}{(1+n)(1+g)}(1-\alpha)>0
\end{aligned}$$

The savings rate s_K has no impact on the speed of convergence however.

2 Endogenous growth

2.1 Dynamics of capital and output

Question: Consider the Solow model where labor is constant ($L_t = 1$) and technology is affected by capital:

$$A_t = A_t^{exog} K_t^{\eta}$$

In the standard model we assume $\eta = 0$. Consider that the exogenous component of productivity is constant $A_t^{exog} = 1$. Capital accumulation is as in the previous section:

$$K_{t+1} - K_t = s_K Y_t - \delta K_t$$

Show that the growth rates of capital and output are (α is the weight of capital in the production function, as in the previous question):

$$\begin{aligned}
\frac{K_{t+1}}{K_t} &= s_K (K_t)^{(1-\alpha)(\eta-1)} + (1-\delta)\\
\frac{Y_{t+1}}{Y_t} &= \left(\frac{K_{t+1}}{K_t}\right)^{(1-\alpha)(\eta-1)}
\end{aligned}$$

Answer: Take the capital accumulation equation:

$$\begin{aligned}
K_{t+1} - K_t &= s_K Y_t - \delta K_t\\
K_{t+1} - K_t &= s_K (K_t)^{\alpha} (A_t L_t)^{1-\alpha} - \delta K_t\\
K_{t+1} - K_t &= s_K (K_t)^{\alpha} (K_t^{\eta})^{1-\alpha} - \delta K_t\\
\frac{K_{t+1} - K_t}{K_t} &= s_K (K_t)^{\alpha-1} (K_t^{\eta})^{1-\alpha} - \delta\\
\frac{K_{t+1} - K_t}{K_t} &= s_K (K_t)^{(1-\alpha)(\eta-1)} - \delta\\
\frac{K_{t+1}}{K_t} &= s_K (K_t)^{(1-\alpha)(\eta-1)} + (1-\delta)
\end{aligned}$$

Using the technology:

$$\begin{aligned}
Y_t &= (K_t)^{\alpha} (A_t L_t)^{1-\alpha}\\
Y_t &= (K_t)^{\alpha} (K_t^{\eta})^{1-\alpha}\\
Y_t &= (K_t)^{(1-\alpha)(\eta-1)}
\end{aligned}$$

This implies:

$$
\begin{aligned}
Y_{t+1} - Y_t &= (K_{t+1})^{(1-\alpha)(\eta-1)} - (K_t)^{(1-\alpha)(\eta-1)} \\
\frac{Y_{t+1} - Y_t}{Y_t} &= \frac{(K_{t+1})^{(1-\alpha)(\eta-1)} - (K_t)^{(1-\alpha)(\eta-1)}}{(K_t)^{(1-\alpha)(\eta-1)}} \\
\frac{Y_{t+1} - Y_t}{Y_t} &= \left(\frac{K_{t+1}}{K_t}\right)^{(1-\alpha)(\eta-1)} - 1 \\
\frac{Y_{t+1}}{Y_t} &= \left(\frac{K_{t+1}}{K_t}\right)^{(1-\alpha)(\eta-1)}
\end{aligned}
$$

2.2 Impact of savings

Question: How does the saving rate s_k affects the growth rate in the long run (along a steady growth path)?

Contrast the general results with the standard Solow model.

Answer: The Standard Solow model is the case where $\eta = 0$. In that case, the dynamics are inversely proportional to the level of capital, i.e. K_{t+1}/K_t is small when K_t is high:

$$
\frac{K_{t+1}}{K_t} = s_K (K_t)^{-(1-\alpha)} + (1-\delta)
$$

The steady state in that case is $K_{t+1} = K_t = K^*$:

$$
\begin{aligned}
1 &= s_K (K^*)^{-(1-\alpha)} + (1-\delta) \\
\delta &= s_K (K^*)^{-(1-\alpha)} \\
(K^*)^{1-\alpha} &= \frac{s_K}{\delta} \\
K^* &= \left[\frac{s_K}{\delta}\right]^{\frac{1}{1-\alpha}}
\end{aligned}
$$

The savings rate has not impact in the long run once capital has reached K^*. In the short run, a higher savings rates raises K_{t+1}/K_t for a given value of K_t. But as seen in the previous question, it has no impact on the speed of convergence towards the steady state.

The results of the standard Solow model also hold when $\eta \in (0,1)$. The steady state is simply modified to:

$$
\begin{aligned}
1 &= s_K (K^*)^{(1-\alpha)(\eta-1)} + (1-\delta) \\
\delta &= s_K (K^*)^{-(1-\alpha)(1-\eta)} \\
(K^*)^{(1-\alpha)(1-\eta)} &= \frac{s_K}{\delta} \\
K^* &= \left[\frac{s_K}{\delta}\right]^{\frac{1}{(1-\alpha)(1-\eta)}}
\end{aligned}
$$

Things change if the impact of capital on growth is so strong so that $\eta = 1$. In that case, the dynamics are:

$$
\begin{aligned}
\frac{K_{t+1}}{K_t} &= s_K (K_t)^{(1-\alpha)(\eta-1)} + (1-\delta) \\
\frac{K_{t+1}}{K_t} &= s_K + (1-\delta)
\end{aligned}
$$

There is not steady state value of K^* because the production function is now linear in capital:

$$\begin{aligned} Y_t &= (K_t)^{\alpha}\left(K_t^{\eta}\right)^{1-\alpha} \\ Y_t &= (K_t)^{\alpha}\left(K_t\right)^{1-\alpha} \\ Y_t &= K_t \end{aligned}$$

The decreasing returns to scale at the heart of the Solow "brake" mechanism are now absent. The savings rate now affects the growth rate of output.