

## PS1 Solutions

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### 1 Consumption Allocation

#### Problem Setup

The Home agent's consumption basket is given by

$$C_t = \left( \frac{C_{T,t}}{\gamma} \right)^\gamma \left( \frac{C_{N,t}}{1-\gamma} \right)^{1-\gamma},$$

where:

- $C_{T,t}$  is the quantity of the traded good (its price is normalized to 1),
- $C_{N,t}$  is the quantity of the domestic non-traded good (price  $P_{N,t}$ ),
- $\gamma$  is the expenditure share on the traded good.

The consumer minimizes total expenditure subject to attaining a given consumption level  $C_t$ . The problem is

$$\begin{aligned} \min_{C_{T,t}, C_{N,t}} \quad & P_t C_t = C_{T,t} + P_{N,t} C_{N,t} \\ \text{s.t.} \quad & C_t = \left( \frac{C_{T,t}}{\gamma} \right)^\gamma \left( \frac{C_{N,t}}{1-\gamma} \right)^{1-\gamma}. \end{aligned}$$

Define the Lagrangian function:

$$\mathcal{L} = C_{T,t} + P_{N,t} C_{N,t} + \lambda \left( C_t - \left( \frac{C_{T,t}}{\gamma} \right)^\gamma \left( \frac{C_{N,t}}{1-\gamma} \right)^{1-\gamma} \right).$$

The FOCs with respect to  $C_{T,t}$  and  $C_{N,t}$  are:

$$\begin{aligned} \mathcal{L}_{C_{T,t}} &= 1 - \lambda \gamma \left( \frac{C_{T,t}}{\gamma} \right)^{\gamma-1} \frac{1}{\gamma} \left( \frac{C_{N,t}}{1-\gamma} \right)^{1-\gamma} = 0, \\ \mathcal{L}_{C_{N,t}} &= P_{N,t} - \lambda (1-\gamma) \left( \frac{C_{T,t}}{\gamma} \right)^\gamma \frac{1}{1-\gamma} \left( \frac{C_{N,t}}{1-\gamma} \right)^{-\gamma} = 0 \\ \Rightarrow \quad & \frac{1}{P_{N,t}} = \frac{\gamma}{1-\gamma} \frac{C_{N,t}}{C_{T,t}}. \end{aligned}$$

The dual (expenditure) minimization problem yields the unit cost function (composite price index) for the consumption bundle:

$$P_t C_t = \min \left\{ C_{T,t} + P_{N,t} C_{N,t} : C_t = \left( \frac{C_{T,t}}{\gamma} \right)^\gamma \left( \frac{C_{N,t}}{1-\gamma} \right)^{1-\gamma} \right\}.$$

So, we have:

$$\begin{aligned} P_t C_t &= C_{T,t} + P_{N,t} C_{N,t} \\ &= C_{T,t} + \frac{1-\gamma}{\gamma} C_{T,t} \\ \Rightarrow C_{T,t} &= \gamma P_t C_t \\ \Rightarrow C_{N,t} &= \frac{1-\gamma}{\gamma} \frac{C_{T,t}}{P_{N,t}} \\ &= (1-\gamma) P_t C_t. \end{aligned}$$

As  $P_t$  is the minimum expenditure required to attain the given consumption level  $C_t = 1$ , we have:

$$\begin{aligned} \left( \frac{C_{T,t}}{\gamma} \right)^\gamma \left( \frac{C_{N,t}}{1-\gamma} \right)^{1-\gamma} &= 1 \\ \Rightarrow (P_t C_t)^\gamma \left( \frac{P_t C_t}{P_{N,t}} \right)^{1-\gamma} &= 1 \\ \Rightarrow P_t &= (P_{N,t})^{1-\gamma}. \end{aligned}$$

Analogously, for the Foreign agent, we have

$$\begin{aligned} C_{T,t}^* &= \gamma P_t^* C_t^* \\ C_{N,t}^* &= (1-\gamma) P_t^* C_t^* \\ P_t^* &= (P_{N,t}^*)^{1-\gamma}. \end{aligned}$$

## Economic Intuition

- The parameter  $\gamma$  reflects the expenditure share on the traded good.
- Since the traded good is the numéraire (price normalized to 1), its cost enters directly, while the cost of the non-traded good is weighted by its price  $P_{N,t}$ .
- The composite price index  $P_t$  is a weighted geometric mean of the individual prices. With the traded goods price equal to 1, we have  $P_t = (P_{N,t})^{1-\gamma}$ .
- The optimal consumption choices allocate expenditure in a way that equates the marginal rate of substitution to the ratio of prices.

## 2 Market Clearing

Under market clearing, the quantities of traded and non-traded goods produced must equal the quantities consumed. We have:

### Non-Traded Goods Market

For Home, market clearing in non-traded goods is:

$$nC_{N,t} = A_{N,t}(L_{N,t})^{1-\alpha}.$$

Substituting  $C_{N,t} = (1 - \gamma)\frac{P_t C_t}{P_{N,t}}$  with  $P_t = (P_{N,t})^{1-\gamma}$ , we obtain:

$$n(1 - \gamma)(P_{N,t})^{-\gamma}C_t = A_{N,t}(L_{N,t})^{1-\alpha}.$$

For Foreign, the market clearing condition is:  $(1 - n)C_{N,t}^* = A_{N,t}^*(L_{N,t}^*)^{1-\alpha}$

$$(1 - n)(1 - \gamma)(P_{N,t}^*)^{-\gamma}C_t^* = A_{N,t}^*(L_{N,t}^*)^{1-\alpha}.$$

### Traded Goods Market

Global market clearing for traded goods is:

$$nC_{T,t} + (1 - n)C_{T,t}^* = A_{T,t}(n - L_{N,t})^{1-\alpha} + A_{T,t}^*((1 - n) - L_{N,t}^*)^{1-\alpha}.$$

Substituting  $C_{T,t} = \gamma P_t C_t$  with  $P_t = (P_{N,t})^{1-\gamma}$  (and similarly for Foreign), we have:

$$n\gamma(P_{N,t})^{1-\gamma}C_t + (1 - n)\gamma(P_{N,t}^*)^{1-\gamma}C_t^* = A_{T,t}(n - L_{N,t})^{1-\alpha} + A_{T,t}^*((1 - n) - L_{N,t}^*)^{1-\alpha}.$$

**Intuition:** Non-traded goods are produced and consumed domestically, while traded goods are balanced across countries.

## 3 Intertemporal Allocation

### Home Optimization

The Home agent maximizes:

$$U_t = \sum_{s=0}^{\infty} (\beta_{H,t+s})^s \ln C_{t+s},$$

subject to the period- $t$  budget constraint:

$$nP_t C_t + nB_{t+1} = A_{T,t}(n - L_{N,t})^{1-\alpha} + P_{N,t}A_{N,t}(L_{N,t})^{1-\alpha} + n(1 + r_t)B_t.$$

Focusing on a single period (the infinite-horizon structure is standard), we write:

$$\mathcal{L} = \ln C_t + \beta_{H,t+1} \ln C_{t+1} - \lambda_t \left[ A_{T,t}(n - L_{N,t})^{1-\alpha} + P_{N,t}A_{N,t}(L_{N,t})^{1-\alpha} + n(1 + r_t)B_t - nP_t C_t - nB_{t+1} \right].$$

Take the FOCs:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C_t} &= \frac{1}{C_t} - \lambda_t n P_t = 0 \quad \Rightarrow \quad \lambda_t = \frac{1}{nP_t C_t} \\ \frac{\partial \mathcal{L}}{\partial B_{t+1}} &= -\lambda_t + \beta_{H,t+1}(1 + r_{t+1})\lambda_{t+1} = 0. \end{aligned}$$

Substitute the expressions for  $\lambda_t$  and  $\lambda_{t+1}$ :

$$\frac{1}{nP_t C_t} = \beta_{H,t+1}(1 + r_{t+1}) \frac{1}{nP_{t+1} C_{t+1}}.$$

Cancel  $n$  and rearrange:

$$\frac{1}{C_t} = \beta_{H,t+1}(1 + r_{t+1}) \frac{P_t}{P_{t+1}} \frac{1}{C_{t+1}}.$$

Set:

$$1 + r_{t+1}^C \equiv (1 + r_{t+1}) \frac{P_t}{P_{t+1}},$$

so the Euler equation becomes:

$$\boxed{C_{t+1} = \beta_{H,t+1}(1 + r_{t+1}^C)C_t.}$$

From Question (1), we know that  $P_t = (P_{N,t})^{(1-\gamma)}$ , so, we have:

$$1 + r_{t+1}^C = (1 + r_{t+1}) \frac{P_t}{P_{t+1}} = (1 + r_{t+1}) \left( \frac{P_{N,t}}{P_{N,t+1}} \right)^{1-\gamma}.$$

Since

$$C_{T,t} = \gamma P_t C_t,$$

it follows that

$$C_{T,t+1} = \gamma P_{t+1} C_{t+1} = \gamma P_{t+1} \beta_{H,t+1} (1 + r_{t+1}^C) C_t.$$

As we note that

$$\beta_{H,t+1}(1 + r_{t+1}^C) = \beta_{H,t+1}(1 + r_{t+1}) \frac{P_t}{P_{t+1}},$$

so simplifying, we obtain:

$$C_{T,t+1} = \beta_{H,t+1}(1 + r_{t+1})C_{T,t}.$$

**Remark.** The Foreign agent's optimization yields analogous Euler equations:

$$C_{t+1}^* = \beta_{F,t+1}(1 + r_{C,t+1}^*)C_t^*, \quad C_{T,t+1}^* = \beta_{F,t+1}(1 + r_{t+1})C_{T,t}^*,$$

with

$$1 + r_{t+1}^{*C} = (1 + r_{t+1}) \frac{P_t^*}{P_{t+1}^*}.$$

## Foreign Optimization

Similarly, for the Foreign country:

$$C_{t+1}^* = \beta_{F,t+1}(1 + r_{C,t+1}^*)C_t^*, \quad C_{T,t+1}^* = \beta_{F,t+1}(1 + r_{t+1})C_{T,t}^*,$$

with

$$1 + r_{t+1}^{*C} = (1 + r_{t+1}) \left( \frac{P_{N,t}^*}{P_{N,t+1}^*} \right)^{1-\gamma}.$$

**Intuition:** Households equate the marginal utility cost of consuming today versus tomorrow, with intertemporal decisions affected by relative price changes.

## 4 Labor Allocation

The production functions are given by:

$$Y_{T,t} = A_{T,t}(n - L_{N,t})^{1-\alpha}, \quad Y_{N,t} = A_{N,t}(L_{N,t})^{1-\alpha}.$$

Compute the marginal product of labor in each sector:

$$\frac{\partial Y_{T,t}}{\partial (n - L_{N,t})} = (1 - \alpha)A_{T,t}(n - L_{N,t})^{-\alpha},$$

$$\frac{\partial Y_{N,t}}{\partial L_{N,t}} = (1 - \alpha)A_{N,t}(L_{N,t})^{-\alpha}.$$

The Home agent allocates labor so that the marginal value product in the traded sector equals the marginal value product (adjusted by the non-traded price) in the non-traded sector:

$$(1 - \alpha)A_{T,t}(n - L_{N,t})^{-\alpha} = P_{N,t}(1 - \alpha)A_{N,t}(L_{N,t})^{-\alpha}.$$

Cancel the common factor  $1 - \alpha$  and rearrange:

$$A_{T,t}(n - L_{N,t})^{-\alpha} = P_{N,t}A_{N,t}(L_{N,t})^{-\alpha}.$$

The analogous condition for the Foreign country is:

$$A_{T,t}^*((1 - n) - L_{N,t}^*)^{-\alpha} = P_{N,t}^*A_{N,t}^*(L_{N,t}^*)^{-\alpha}.$$

## 5 Resource Constraints and the Real Exchange Rate

### Resource Constraints

Recall the Home budget constraint:

$$nP_tC_t + nB_{t+1} = A_{T,t}(n - L_{N,t})^{1-\alpha} + P_{N,t}A_{N,t}(L_{N,t})^{1-\alpha} + n(1 + r_t)B_t.$$

From Question 1, we have:

$$C_{N,t} = (1 - \gamma) \frac{P_t}{P_{N,t}} C_t.$$

Since  $P_t = (P_{N,t})^{1-\gamma}$ , then:

$$C_{N,t} = (1 - \gamma)(P_{N,t})^{-\gamma} C_t.$$

Given that non-traded goods are produced solely for domestic consumption, we also have the production identity (from Question 2):

$$n(1 - \gamma)(P_{N,t})^{-\gamma} C_t = A_{N,t}(L_{N,t})^{1-\alpha}.$$

Thus, the expenditure on traded goods (which uses the consumption price index) plus net asset accumulation must equal traded output plus bond returns:

$$n\gamma(P_{N,t})^{1-\gamma}C_t + nB_{t+1} = A_{T,t}(n - L_{N,t})^{1-\alpha} + n(1 + r_t)B_t.$$

Similarly, for Foreign we obtain:

$$(1 - n)\gamma(P_{N,t}^*)^{1-\gamma}C_t^* - nB_{t+1} = A_{T,t}^*((1 - n) - L_{N,t}^*)^{1-\alpha} - n(1 + r_t)B_t.$$

Then, we define the real exchange rate as the ratio of Foreign to Home consumption price indices:

$$Q_t \equiv \frac{P_t^*}{P_t}.$$

Since

$$P_t = (P_{N,t})^{1-\gamma} \quad \text{and} \quad P_t^* = (P_{N,t}^*)^{1-\gamma},$$

we have:

$$Q_t = \left( \frac{P_{N,t}^*}{P_{N,t}} \right)^{1-\gamma}.$$

## 6 Steady State

In steady state, consumption is constant so that  $C_{t+1} = C_t$ . The Euler equation for the Home agent is

$$C_{t+1} = \beta_0(1 + r_{t+1}^C)C_t.$$

But by definition the real return in consumption units is

$$1 + r_{t+1}^C = (1 + r_{t+1}) \left( \frac{P_t}{P_{t+1}} \right)^{1-\gamma}.$$

In steady state prices do not change ( $P_t = P_{t+1}$ ) so that

$$1 + r_{t+1}^C = 1 + r_{t+1} \Rightarrow C_t = \beta_0(1 + r_0)C_t.$$

Dividing by  $C_t > 0$  yields:

$$1 = \beta_0(1 + r_0).$$

### Step 2. Price Normalization

By convention we normalize the steady state price of non-tradables to one:

$$P_{N,0} = 1, \quad \text{and similarly} \quad P_{N,0}^* = 1.$$

Then the consumption price indices become

$$P_0 = (P_{N,0})^{1-\gamma} = 1, \quad P_0^* = 1.$$

### Step 3. Labor Allocation in the Home Country

The Home intratemporal labor allocation condition is:

$$A_{T,0}(n - L_{N,0})^{-\alpha} = P_{N,0}A_{N,0}(L_{N,0})^{-\alpha}.$$

Since  $P_{N,0} = 1$ , this simplifies to:

$$A_{T,0}(n - L_{N,0})^{-\alpha} = A_{N,0}(L_{N,0})^{-\alpha}.$$

Rearrange by dividing both sides by  $A_{T,0}$  and by  $(L_{N,0})^{-\alpha}$ :

$$\left(\frac{n - L_{N,0}}{L_{N,0}}\right)^{-\alpha} = \frac{A_{N,0}}{A_{T,0}}.$$

Taking the reciprocal and then the  $1/\alpha$ -th root,

$$\frac{n - L_{N,0}}{L_{N,0}} = \left(\frac{A_{T,0}}{A_{N,0}}\right)^{1/\alpha}.$$

Now, using the calibration

$$A_{N,0} = A_{T,0} \left(\frac{1 - \gamma}{\gamma}\right)^\alpha,$$

we have

$$\frac{A_{T,0}}{A_{N,0}} = \left(\frac{\gamma}{1 - \gamma}\right)^\alpha.$$

Then,

$$\frac{n - L_{N,0}}{L_{N,0}} = \left[\left(\frac{\gamma}{1 - \gamma}\right)^\alpha\right]^{1/\alpha} = \frac{\gamma}{1 - \gamma}.$$

Thus,

$$\boxed{L_{N,0} = n(1 - \gamma).}$$

By symmetry, the corresponding condition for Foreign is:

$$A_{T,0}^* \left((1 - n) - L_{N,0}^*\right)^{-\alpha} = P_{N,0}^* A_{N,0}^* (L_{N,0}^*)^{-\alpha}.$$

Since  $P_{N,0}^* = 1$ , the same steps lead to:

$$\frac{(1 - n) - L_{N,0}^*}{L_{N,0}^*} = \frac{\gamma}{1 - \gamma},$$

so that

$$\boxed{L_{N,0}^* = (1 - n)(1 - \gamma).}$$

We derive the steady-state consumption from the market clearing conditions for non-traded and traded goods. The non-traded goods clearing condition is

$$nC_{N,0} = A_{N,0}(L_{N,0})^{1-\alpha}.$$

From Question 1 we have

$$C_{N,0} = (1 - \gamma) \frac{P_0}{P_{N,0}} C_0.$$

Since  $P_0 = 1$  and  $P_{N,0} = 1$ , it follows that

$$C_{N,0} = (1 - \gamma) C_0.$$



Substitute into the clearing condition:

$$n(1 - \gamma)C_0 = A_{N,0}(L_{N,0})^{1-\alpha}.$$

Recall that  $L_{N,0} = n(1 - \gamma)$ , so

$$n(1 - \gamma)C_0 = A_{N,0}[n(1 - \gamma)]^{1-\alpha}.$$

Solve for  $C_0$ :

$$C_0^N = A_{N,0}[n(1 - \gamma)]^{-\alpha}.$$

We then derive from the traded goods clearing condition:

$$\begin{aligned} n\gamma(P_{N,t})^{1-\gamma}C_t + (1 - n)\gamma(P_{N,t}^*)^{1-\gamma}C_t^* &= A_{T,t}(n - L_{N,t})^{1-\alpha} + A_{T,t}^*((1 - n) - L_{N,t}^*)^{1-\alpha} \\ \Rightarrow n\gamma C_0 + \frac{\gamma}{1 - \gamma}A_{N,0}^*(L_{N,0}^*)^{1-\alpha} &= A_{T,0}(n - L_{N,0})^{1-\alpha} + A_{T,0}^*(1 - n - L_{N,0}^*)^{1-\alpha} \\ \Rightarrow n\gamma C_0 + \frac{\gamma}{1 - \gamma}A_{N,0}^*(1 - n)^{1-\alpha}(1 - \gamma)^{1-\alpha} &= A_{T,0}(n - n(1 - \gamma))^{1-\alpha} + \\ &A_{T,0}^*(1 - n - (1 - n)(1 - \gamma))^{1-\alpha} \\ \Rightarrow n\gamma C_0 + \frac{\gamma}{1 - \gamma}A_{N,0}^*(1 - n)^{1-\alpha}(1 - \gamma)^{1-\alpha} &= A_{T,0}(n\gamma)^{1-\alpha} + A_{T,0}^*[(1 - n)\gamma]^{1-\alpha} \\ \Rightarrow n\gamma C_0 + \frac{\gamma}{1 - \gamma}A_{T,0}\left(\frac{1 - n}{n}\right)^\alpha \left(\frac{1 - \gamma}{\gamma}\right)^\alpha (1 - n)^{1-\alpha}(1 - \gamma)^{1-\alpha} &= A_{T,0}(n\gamma)^{1-\alpha} + \\ &A_{T,0}\left(\frac{1 - n}{n}\right)^\alpha [(1 - n)\gamma]^{1-\alpha} \\ \Rightarrow n\gamma C_0 + A_{T,0}(1 - n)n^{-\alpha}\gamma^{1-\alpha} &= A_{T,0}(n\gamma)^{1-\alpha} + A_{T,0}(1 - n)n^{-\alpha}\gamma^{1-\alpha} \\ \Rightarrow C_0^T &= A_{T,0}(n\gamma)^{-\alpha}. \end{aligned}$$

We take a weighted geometric mean with weights  $\gamma$  and  $1 - \gamma$ . That is,

$$C_0 = (C_0^N)^{1-\gamma} \cdot (C_0^T)^\gamma,$$

so that

$$C_0 = [A_{N,0}n^{-\alpha}(1 - \gamma)^{-\alpha}]^{1-\gamma} [A_{T,0}n^{-\alpha}\gamma^{-\alpha}]^\gamma.$$

We obtain:

$$C_0 = (A_{T,0})^\gamma (A_{N,0})^{1-\gamma} [n\gamma^\gamma(1 - \gamma)^{1-\gamma}]^{-\alpha}.$$

A similar derivation for Foreign (noting that the population is  $1 - n$ ) gives:

$$C_0^* = (A_{T,0}^*)^\gamma (A_{N,0}^*)^{1-\gamma} [(1 - n)\gamma^\gamma(1 - \gamma)^{1-\gamma}]^{-\alpha}.$$

Because of the calibration (and the fact that the relative productivities satisfy

$$\frac{A_{N,0}}{A_{T,0}} = \left( \frac{1-\gamma}{\gamma} \right)^\alpha \quad \text{and} \quad \frac{A_{N,0}^*}{A_{T,0}^*} = \left( \frac{1-n}{n} \right)^\alpha \left( \frac{1-\gamma}{\gamma} \right)^\alpha,$$

we can check that indeed

$$\frac{C_0}{C_0^*} = 1.$$

## 7 Log-Linear Approximation

We linearize the equilibrium conditions around the steady state. Denote for any variable  $x_t$  its deviation from steady state by

$$\hat{x}_t = \frac{x_t - x_0}{x_0}.$$

We also define cross-country differences later but for now we linearize the Home equations.

### A. Non-Traded Goods Market

The Home non-traded goods market clearing condition is:

$$n(1-\gamma)(P_{N,t})^{-\gamma}C_t = A_{N,t}(L_{N,t})^{1-\alpha}.$$

Taking logarithms, we have

$$\ln n + \ln(1-\gamma) - \gamma \ln P_{N,t} + \ln C_t = \ln A_{N,t} + (1-\alpha) \ln L_{N,t}.$$

Linearizing around the steady state (and noting that constants vanish in the difference), we obtain:

$$\boxed{-\gamma \widehat{P_{N,t}} + \widehat{C}_t = \widehat{A_{N,t}} + (1-\alpha) \widehat{L_{N,t}}.} \quad (7a)$$

### B. Resource Constraint

The Home resource constraint is:

$$n\gamma(P_{N,t})^{1-\gamma}C_t + nB_{t+1} = A_{T,t}(n - L_{N,t})^{1-\alpha} + n(1+r_t)B_t.$$

Taking logs and linearizing (and assuming that in steady state  $B_t = 0$ , so only percentage deviations matter), we have:

$$(1-\gamma)\widehat{P_{N,t}} + \widehat{C}_t + \widehat{B_{t+1}} = \widehat{A_{T,t}} - \frac{(1-\alpha)(n - L_{N,0})}{n - L_{N,0}} \widehat{L_{N,t}} + \frac{1}{\beta_0} \widehat{B}_t.$$

Since in steady state  $n - L_{N,0} = n\gamma$ , the coefficient on  $\widehat{L_{N,t}}$  becomes  $(1 - \alpha)$ . Thus,

$$(1 - \gamma)\widehat{P_{N,t}} + \widehat{C_t} + \widehat{B_{t+1}} = \widehat{A_{T,t}} - (1 - \alpha)\widehat{L_{N,t}} + \frac{1}{\beta_0}\widehat{B_t}. \quad (7c)$$

## C. Euler Equation

Recall that:

$$C_{t+1} = C_t \beta_{H,t+1} (1 + r_{t+1}) \left( \frac{P_{N,t}}{P_{N,t+1}} \right)^{1-\gamma}$$

Taking logs and linearizing, we have:

$$\widehat{C_{t+1}} = \widehat{C_t} + \widehat{\beta_{H,t+1}} + \frac{r_{t+1} - r_0}{1 + r_0} - (1 - \gamma)(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}).$$

As  $\beta_0(1 + r_0) = 1$ ,

$$\begin{aligned} \widehat{r_t} &= r_t - \frac{1 - \beta_0}{\beta_0} \\ &= r_t - \frac{1 - \frac{1}{1+r_0}}{\frac{1}{1+r_0}} \\ &= r_t - r_0 \end{aligned}$$

So, the Euler equation becomes:

$$\widehat{C_{t+1}} = \widehat{C_t} + \widehat{\beta_{H,t+1}} + \beta_0 \widehat{r_{t+1}} - (1 - \gamma)(\widehat{P_{N,t}} - \widehat{P_{N,t+1}}). \quad (7e)$$

## D. Labor Allocation

The intratemporal condition is:

$$A_{T,t}(n - L_{N,t})^{-\alpha} = P_{N,t}A_{N,t}(L_{N,t})^{-\alpha}.$$

Taking logarithms:

$$\ln A_{T,t} - \alpha \ln(n - L_{N,t}) = \ln P_{N,t} + \ln A_{N,t} - \alpha \ln L_{N,t}.$$

Linearize around steady state:

$$\widehat{A_{T,t}} - \alpha \frac{L_{N,t} - L_{N,0}}{n - L_{N,0}} = \widehat{P_{N,t}} + \widehat{A_{N,t}} - \alpha \widehat{L_{N,t}}.$$

Using the fact that in steady state  $n - L_{N,0} = n\gamma$ :

$$\widehat{A_{T,t}} - \alpha \frac{\widehat{L_{N,t}} n (1 - \gamma)}{n\gamma} = \widehat{P_{N,t}} + \widehat{A_{N,t}} - \alpha \widehat{L_{N,t}}.$$

$$\boxed{\widehat{A}_{T,t} + \frac{\alpha}{\gamma} \widehat{L}_{N,t} = \widehat{P}_{N,t} + \widehat{A}_{N,t}.} \quad (7g)$$

## E. Real Exchange Rate

As we know:

$$1 + r_{t+1}^C = (1 + r_{t+1}) \left( \frac{P_{N,t}}{P_{N,t+1}} \right)^{1-\gamma},$$

take logs and linearize both sides, we have:

$$\begin{aligned} \frac{r_{t+1}^C - r_0^C}{1 + r_0^C} &= \frac{r_{t+1} - r_0}{1 + r_0} + (1 - \gamma)(\widehat{P}_{N,t} - \widehat{P}_{N,t+1}). \\ \Rightarrow \beta_0 \widehat{r_{t+1}^C} &= \beta_0 \widehat{r_{t+1}} + (1 - \gamma)(\widehat{P}_{N,t} - \widehat{P}_{N,t+1}). \\ \boxed{\widehat{r_{t+1}^C} &= \widehat{r_{t+1}} + (1 - \gamma) \frac{1}{\beta_0} (\widehat{P}_{N,t} - \widehat{P}_{N,t+1}).} \end{aligned} \quad (7i)$$

## 8 Worldwide Equilibrium

Define world aggregates as population-weighted averages (e.g.,  $\widehat{C}_t^W = n\widehat{C}_t + (1 - n)\widehat{C}_t^*$ ). Then, from the above log-linearized equations one can show:

- **Non-Traded Goods Market:**

$$n \times (7a) + (1 - n) \times (7b) \Rightarrow -\gamma \widehat{P}_{N,t}^W + \widehat{C}_t^W = \widehat{A}_{N,t}^W + (1 - \alpha) \widehat{L}_{N,t}^W. \quad (8a)$$

- **Resource Constraint:**

$$n \times (7c) + (1 - n) \times (7d) \Rightarrow (1 - \gamma) \widehat{P}_{N,t}^W + \widehat{C}_t^W = \widehat{A}_{T,t}^W - (1 - \alpha) \frac{1 - \gamma}{\gamma} \widehat{L}_{N,t}^W. \quad (8b)$$

- **Euler Equation:**

$$n \times (7e) + (1 - n) \times (7f) \Rightarrow \widehat{C}_{t+1}^W = \widehat{C}_t^W + (1 - \gamma)(\widehat{P}_{N,t}^W - \widehat{P}_{N,t+1}^W) + \widehat{\beta_{H,t+1}^W} + \beta_0 \widehat{r_{t+1}}. \quad (8c)$$

- **Labor Allocation:**

$$n \times (7g) + (1 - n) \times (7h) \Rightarrow \widehat{A}_{T,t}^W + \frac{\alpha}{\gamma} \widehat{L}_{N,t}^W = \widehat{P}_{N,t}^W + \widehat{A}_{N,t}^W. \quad (8d)$$

- **Real Exchange Rate:**

$$n \times (7i) + (1 - n) \times (7j) \Rightarrow \beta_0 \widehat{r_{t+1}^{CW}} = (1 - \gamma)(\widehat{P}_{N,t}^W - \widehat{P}_{N,t+1}^W) + \beta_0 \widehat{r_{t+1}}. \quad (8e)$$

Let (8a)-(8d), we get:

$$(1 - \gamma)\widehat{P_{N,t}^W} + \widehat{C_t^W} = \widehat{A_{T,t}^W} - (1 - \alpha + \frac{\alpha}{\gamma})\widehat{L_{N,t}^W}.$$

Combine with (8b), we have:

$$\begin{aligned} (1 - \alpha + \frac{\alpha}{\gamma})\widehat{L_{N,t}^W} &= -(1 - \alpha)\frac{1 - \gamma}{\gamma}\widehat{L_{N,t}^W} \\ &= (1 - \alpha)\widehat{L_{N,t}^W} - \frac{1 - \alpha}{\gamma}\widehat{L_{N,t}^W} \\ &\Rightarrow \frac{1}{\gamma}\widehat{L_{N,t}^W} = 0. \end{aligned}$$

Let (8b)-(8a), we have:

$$\begin{aligned} \widehat{P_{N,t}^W} &= \widehat{A_{T,t}^W} - \widehat{A_{N,t}^W} - (1 - \alpha)\left(\frac{1 - \gamma}{\gamma} + 1\right)\widehat{L_{N,t}^W} \\ &= \widehat{A_{T,t}^W} - \widehat{A_{N,t}^W}. \end{aligned}$$

Take  $(1 - \gamma) \times (8a) + \gamma(8b)$ , we have:

$$(1 - \gamma)\widehat{C_t^W} + \gamma\widehat{C_t^W} = \widehat{C_t^W} = (1 - \gamma)\widehat{A_{N,t}^W} + \gamma\widehat{A_{T,t}^W}.$$

Finally, from (8c), we know that:

$$\begin{aligned} \beta_0\widehat{r_{t+1}} + \widehat{\beta_{H,t+1}^W} &= \widehat{C_{t+1}^W} - \widehat{C_t^W} - (1 - \gamma)(\widehat{P_{N,t}^W} - \widehat{P_{N,t+1}^W}) \\ &= \gamma\widehat{A_{t+1}^W} + (1 - \gamma)\widehat{A_{N,t+1}^W} - \gamma\widehat{A_{T,t+1}^W} - (1 - \gamma)\widehat{A_{N,t+1}^W} \\ &\quad - (1 - \gamma)(\widehat{A_{T,t}^W} - \widehat{A_{N,t}^W}) + (1 - \gamma)(\widehat{A_{T,t+1}^W} - \widehat{A_{N,t+1}^W}) \\ &= \widehat{A_{T,t+1}^W} - \widehat{A_{T,t}^W}. \end{aligned}$$

**Intuition:** World aggregates respond only to symmetric shocks, with the real interest rate driven by global productivity changes and aggregate patience.

## 9 Cross-Country Differences

As we know that  $Q_t = \left(\frac{P_{N,t}^*}{P_{N,t}}\right)^{1-\gamma}$ , log-linearize the equation, we have:

$$\widehat{Q_t} = (1 - \gamma)(\widehat{P_{N,t}^*}) - \widehat{P_{N,t}}.$$

**Intuition:** These equations link cross-country differences in consumption, labor, and the real exchange rate to differences in productivity and intertemporal preferences.

## 10 Long-Run Allocation (Period $t + 1$ )

Assume that from  $t + 1$  onward the economy reaches a new steady state with no further discount factor shocks ( $\hat{\beta}_{H,t+2} = \hat{\beta}_{F,t+2} = 0$ ). Taking the cross-country asset position  $\widehat{B}_{t+1}$  as given, one can show:

$$\hat{Q}_{t+1} = -(1 - \gamma) \left[ (\hat{A}_{T,t+1} - \hat{A}_{T,t+1}^*) - (\widehat{A}_{N,t+1} - \hat{A}_{N,t+1}^*) \right] - \frac{\alpha(1 - \gamma)}{1 - \beta_0} \frac{1}{\beta_0} \frac{\widehat{B}_{t+1}}{1 - n},$$

$$\tilde{L}_{N,t+1} = \frac{\gamma}{1 - \beta_0} \frac{1}{\beta_0} \frac{\widehat{B}_{t+1}}{1 - n},$$

$$\tilde{C}_{t+1} = \gamma(\hat{A}_{T,t+1} - \hat{A}_{T,t+1}^*) + (1 - \gamma)(\widehat{A}_{N,t+1} - \hat{A}_{N,t+1}^*) + \frac{\gamma}{1 - \beta_0} \frac{1}{\beta_0} \frac{\widehat{B}_{t+1}}{1 - n}.$$

**Interpretation:**

- A positive  $\widehat{B}_{t+1}$  (Home wealthier) implies higher relative consumption and a lower  $Q_{t+1}$  (Homes goods become relatively more expensive).
- Permanent productivity differences affect steady state consumption and prices directly.

## 11 Short-Run Allocation (Period $t$ )

Assume initially  $\widehat{B}_t = 0$ . Solving the system (starting with the labor and non-traded market conditions, then using the resource constraint and Euler equations) yields:

$$\frac{\widehat{B}_{t+1}}{1 - n} = \frac{\beta_0}{\gamma + \alpha(1 - \gamma)} \left[ (\widehat{\beta}_{H,t+1} - \hat{\beta}_{F,t+1}) - (\hat{A}_{T,t+1} - \hat{A}_{T,t+1}^*) + (\widehat{A}_{T,t} - \hat{A}_{T,t}^*) \right], \quad (11a)$$

$$\tilde{C}_t = \gamma(\widehat{A}_{T,t} - \hat{A}_{T,t}^*) + (1 - \gamma)(\widehat{A}_{N,t} - \hat{A}_{N,t}^*) - \frac{\gamma\beta_0}{\gamma + \alpha(1 - \gamma)} \left[ (\widehat{\beta}_{H,t+1} - \hat{\beta}_{F,t+1}) - (\hat{A}_{T,t+1} - \hat{A}_{T,t+1}^*) + (\widehat{A}_{T,t} - \hat{A}_{T,t}^*) \right], \quad (11b)$$

$$\hat{Q}_t = -(1 - \gamma)(\widehat{A}_{T,t} - \hat{A}_{T,t}^*) + (1 - \gamma)(\widehat{A}_{N,t} - \hat{A}_{N,t}^*) + \frac{(1 - \gamma)\alpha\beta_0}{\gamma + \alpha(1 - \gamma)} \left[ (\widehat{\beta}_{H,t+1} - \hat{\beta}_{F,t+1}) - (\hat{A}_{T,t+1} - \hat{A}_{T,t+1}^*) + (\widehat{A}_{T,t} - \hat{A}_{T,t}^*) \right], \quad (11c)$$

$$\tilde{L}_{N,t} = -\frac{\gamma\beta_0}{\gamma + \alpha(1 - \gamma)} \left[ (\widehat{\beta}_{H,t+1} - \hat{\beta}_{F,t+1}) - (\hat{A}_{T,t+1} - \hat{A}_{T,t+1}^*) + (\widehat{A}_{T,t} - \hat{A}_{T,t}^*) \right]. \quad (11d)$$

**Interpretation:**

- A temporary increase in Home patience (i.e.  $\widehat{\beta}_{H,t+1} - \hat{\beta}_{F,t+1} > 0$ ) leads to  $\widehat{B}_{t+1} > 0$  (Home runs a current account surplus), lower relative consumption, and a real exchange rate movement consistent with a trade surplus.
- Temporary traded or non-traded productivity shocks affect the current account and labor allocation differently.

- For permanent shocks ( $\widehat{A}_{T,t} = \widehat{A}_{T,t+1}$ ), intertemporal balance is restored with  $\widehat{B}_{t+1} = 0$  and immediate adjustment to the new steady state.

## 12 Summary of Key Economic Insights

- **Consumption and Prices:** The structure of the consumption basket implies that a rise in the nontraded good price  $P_{N,t}$  increases the overall consumption price  $P_t$  and shifts the consumption mix.
- **Market Clearing:** Non-traded goods are produced and consumed domestically, whereas traded goods are allocated internationally, linking domestic production choices to the real exchange rate.
- **Intertemporal Choices:** The Euler equations show that higher patience or higher real returns lead to deferred consumption. Cross-country differences in patience drive current account imbalances.
- **Labor Allocation:** Labor is reallocated across sectors until the marginal value products are equalized; productivity shifts affect both output composition and relative prices.
- **Steady State and Log-Linearization:** In a symmetric steady state, relative prices and allocations are balanced. Log-linearization permits analysis of small shocks and their propagation.
- **Short-Run vs. Long-Run Dynamics:** Temporary shocks generate current account imbalances and short-run reallocation, while permanent shocks adjust consumption and prices directly with no asset accumulation.
- **Wealth Effects:** A positive net asset position (Home wealthier) implies higher steady-state consumption and a relatively stronger (appreciated) currency.