Macroeconomics A; EI060

Short problems

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1 Exchange rate and money

Question: The model consists of the uncovered interest parity, money demand, and purchasing power parity. In linearized terms, omitting variables that are not relevant, we have:

$$\begin{aligned} i_{t+1}^{H} &= \mathbb{E}_t\left(e_{t+1}\right) - e_t \\ m_t - p_t &= -\lambda i_{t+1}^{H} \\ p_t &= e_t \end{aligned}$$

Show that:

$$e_t = \frac{\lambda}{1+\lambda}\mathbb{E}_t\left(e_{t+1}\right) + \frac{1}{1+\lambda}m_t$$

Then show that:

$$e_t = \frac{1}{1+\lambda}\sum_{s=t}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s-t}\mathbb{E}_t\left(m_s\right)\right]$$

Answer: We write the uncovered parity as follows, using the PPP to substitute for the price level in the money demand:

$$\begin{aligned} i_{t+1}^{H} &= \mathbb{E}_t\left(e_{t+1}\right) - e_t \\ \frac{1}{\lambda}\left(-m_t + p_t\right) &= \mathbb{E}_t\left(e_{t+1}\right) - e_t \\ \frac{1}{\lambda}\left(-m_t + e_t\right) &= \mathbb{E}_t\left(e_{t+1}\right) - e_t \\ -\frac{1}{\lambda}m_t + \left(\frac{1}{\lambda}+1\right)e_t &= \mathbb{E}_t\left(e_{t+1}\right) \\ -m_t + \left(1+\lambda\right)e_t &= \lambda\mathbb{E}_t\left(e_{t+1}\right) \\ e_t &= \frac{\lambda}{1+\lambda}\mathbb{E}_t\left(e_{t+1}\right) + \frac{1}{1+\lambda}m_t \end{aligned}$$

We iterate forward:

$$\begin{aligned}
e_t &= \frac{\lambda}{1+\lambda}\mathbb{E}_t\left(e_{t+1}\right) + \frac{1}{1+\lambda}m_t \\
e_t &= \frac{1}{1+\lambda}m_t + \frac{\lambda}{1+\lambda}\mathbb{E}_t\left(\frac{1}{1+\lambda}m_{t+1} + \frac{\lambda}{1+\lambda}e_{t+2}\right) \\
e_t &= \frac{1}{1+\lambda}m_t + \frac{\lambda}{1+\lambda}\frac{1}{1+\lambda}\mathbb{E}_t\left(m_{t+1}\right) \\
&\quad + \left(\frac{\lambda}{1+\lambda}\right)^2\mathbb{E}_t\left(\frac{1}{1+\lambda}m_{t+2} + \frac{\lambda}{1+\lambda}e_{t+3}\right) \\
e_t &= \frac{1}{1+\lambda}\left(m_t + \frac{\lambda}{1+\lambda}\mathbb{E}_t\left(m_{t+1}\right) + \left(\frac{\lambda}{1+\lambda}\right)^2\mathbb{E}_t\left(m_{t+2}\right)\right) \\
&\quad + \left(\frac{\lambda}{1+\lambda}\right)^3\mathbb{E}_t\left(e_{t+3}\right) \\
e_t &= \frac{1}{1+\lambda}\sum_{s=t}^{\infty}\left(\left(\frac{\lambda}{1+\lambda}\right)^s\mathbb{E}_t\left(m_s\right)\right) + lim_{s\to\infty}\left(\frac{\lambda}{1+\lambda}\right)^s\mathbb{E}_t\left(e_s\right)
\end{aligned}$$

The transversality condition implies that the last term is zero, hence:

$$e_t = \frac{1}{1+\lambda}\sum_{s=t}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s-t}\mathbb{E}_t\left(m_s\right)\right]$$

2 Constant money growth rate: useful expressions

Question: The money supply grows at a constant rate μ:

$$m_s = m_t + \mu\left(s-t\right)$$

We first derive some useful properties. Recall that:

$$\sum_{s=0}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^s\right] = \frac{1}{1-\frac{\lambda}{1+\lambda}} = 1+\lambda$$

The expression $\left(\frac{\lambda}{1+\lambda}\right)^{s-1}s$ is the derivative of $\left(\frac{\lambda}{1+\lambda}\right)^s$ with respect to $\frac{\lambda}{1+\lambda}$. As the sum is a linear function, the sum of a derivative is the derivative of the sum. Show that:

$$\sum_{s=0}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s-1}s\right] = \frac{\partial\left(1+\lambda\right)}{\partial\left(\frac{\lambda}{1+\lambda}\right)}$$

The recall the chain rule $\frac{\partial f(\lambda)}{\partial g(\lambda)} = \frac{\partial f(\lambda)}{\partial\lambda}/\left[\frac{\partial g(\lambda)}{\partial\lambda}\right]$ and show:

$$\sum_{s=0}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s-1}s\right] = \left(1+\lambda\right)^2$$

Answer: The sum of $\left(\frac{\lambda}{1+\lambda}\right)^{s-1}s$ is:

$$\sum_{s=0}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s-1}s\right] = \sum_{s=0}^{\infty}\left[\frac{\partial\left(\frac{\lambda}{1+\lambda}\right)^s}{\partial\left(\frac{\lambda}{1+\lambda}\right)}\right]$$

$$
\begin{aligned}
\sum_{s=0}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s-1} s\right] &= \frac{\partial}{\partial\left(\frac{\lambda}{1+\lambda}\right)}\left[\sum_{s=0}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s}\right]\right] \\
\sum_{s=0}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s-1} s\right] &= \frac{\partial}{\partial\left(\frac{\lambda}{1+\lambda}\right)}\left[1+\lambda\right]
\end{aligned}
$$

Using the chain rule:

$$
\begin{aligned}
\sum_{s=0}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s-1} s\right] &= \frac{\partial\left(1+\lambda\right)}{\partial\lambda}\frac{1}{\partial\left(\frac{\lambda}{1+\lambda}\right)/\partial\lambda} \\
\sum_{s=0}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s-1} s\right] &= \frac{1}{\partial\left(\frac{\lambda}{1+\lambda}\right)/\partial\lambda} \\
\sum_{s=0}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s-1} s\right] &= \frac{1}{\frac{1+\lambda-\lambda}{(1+\lambda)^2}} \\
\sum_{s=0}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s-1} s\right] &= \left(1+\lambda\right)^2
\end{aligned}
$$

3 Constant money growth rate: useful expressions

Question: Using the results above, show that:

$$
e_t = m_t + \lambda\mu
$$

Answer: With money growing at a constant rate, the exchange rate solution is:

$$
\begin{aligned}
e_t &= \frac{1}{1+\lambda}\sum_{s=t}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s-t}\left(m_t+\mu\left(s-t\right)\right)\right] \\
e_t &= \frac{1}{1+\lambda}m_t\sum_{s=t}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s-t}\right] + \frac{1}{1+\lambda}\mu\sum_{s=t}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s-t}\left(s-t\right)\right] \\
e_t &= \frac{1}{1+\lambda}m_t\sum_{s=0}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s}\right] + \frac{1}{1+\lambda}\frac{\lambda}{1+\lambda}\mu\sum_{s=0}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s-1} s\right]
\end{aligned}
$$

Using our results derived previously, this is:

$$
\begin{aligned}
e_t &= \frac{1}{1+\lambda}m_t\sum_{s=0}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s}\right] + \frac{1}{1+\lambda}\frac{\lambda}{1+\lambda}\mu\sum_{s=0}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s-1} s\right] \\
e_t &= \frac{1}{1+\lambda}m_t\left(1+\lambda\right) + \frac{1}{1+\lambda}\frac{\lambda}{1+\lambda}\mu\left(1+\lambda\right)^2 \\
e_t &= m_t + \lambda\mu
\end{aligned}
$$

4 Autoregressive money process

Question: In terms of movements around a steady state, money follows an autoregressive process:

$$m_{t+1} = \rho m_t + \epsilon_{t+1}$$

Show that

$$\mathbb{E}_t(m_s) = \rho^{s-t} m_t$$

And show that the exchange rate is:

$$e_t = m_t \frac{1}{1+\lambda(1-\rho)}$$

Answer: Re-arrang

Consider that the money supply follows an autoregressive process around $m = 0$:

$$m_{t+1} = \rho m_t + \epsilon_{t+1}$$

Iterating forward:

$$\begin{aligned}
m_{t+2} &= \rho m_{t+1} + \epsilon_{t+2} \\
&= \rho^2 m_t + \rho\epsilon_{t+1} + \epsilon_{t+2} \\
m_{t+3} &= \rho m_{t+2} + \epsilon_{t+3} \\
&= \rho^3 m_t + \rho^2\epsilon_{t+1} + \rho\epsilon_{t+2} + \epsilon_{t+3}
\end{aligned}$$

Taking expectations:

$$\begin{aligned}
\mathbb{E}_t(m_{t+2}) &= \rho^2 m_t \\
\mathbb{E}_t(m_{t+3}) &= \rho^3 m_t
\end{aligned}$$

In general, this means:

$$\mathbb{E}_t(m_s) = \rho^{s-t} m_t$$

The exchange rate general solution is then:

$$\begin{aligned}
e_t &= \frac{1}{1+\lambda}\sum_{s=t}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s-t}\mathbb{E}_t(m_s)\right] \\
e_t &= \frac{1}{1+\lambda}\sum_{s=t}^{\infty}\left[\left(\frac{\lambda}{1+\lambda}\right)^{s-t}\rho^{s-t}m_t\right] \\
e_t &= m_t\frac{1}{1+\lambda}\sum_{s=t}^{\infty}\left[\left(\frac{\lambda\rho}{1+\lambda}\right)^{s-t}\right] \\
e_t &= m_t\frac{1}{1+\lambda}\frac{1}{1-\frac{\lambda\rho}{1+\lambda}} \\
e_t &= m_t\frac{1}{1+\lambda-\lambda\rho} \\
e_t &= m_t\frac{1}{1+\lambda(1-\rho)}
\end{aligned}$$