Lecture 8: Trade and Migration

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1 Introduction

In quantitative general equilibrium trade models (e.g. Anderson (1979), Krugman (1980), Eaton and Kortum (2002), and Melitz (2003)) labor is assumed to be fixed in each location. Conversely, in economic geography models (e.g. Krugman (1991), Helpman (1998), and Allen and Arkolakis (2014)), labor is assumed to be perfectly mobile across locations. In this note, we develop a quantitative dynamic economic geography model where the movement of both goods and workers is costly across space. The framework nests both general equilibrium trade models (when migration costs are assumed to infinitely high) and economic geography models (where migration costs are assumed to be zero). In intermediate cases when migration costs are positive but finite, the framework delivers bilateral "gravity" type equations for the flow of workers across space, which is consistent with empirical evidence on migration flows. To make the model tractable, we borrow tools from Artuç, Chaudhuri, and McLaren (2010) (who empirically analyze the movement of workers across sectors) to write down a model with infinite lived agents who move across space subject to bilateral migration frictions. We show that the system of steady-state equilibrium conditions are a special case of the mathematical systems of equations analyzed in Allen, Arkolakis, and Takahashi (2014), which allows us to characterize the properties of the steady state equilibrium.

2 The model

In this section, we present the model.

2.1 Setup

Geography

There is a world comprised of a compact set S of locations and countably infinite time periods indexed by the integers $t \in \mathbb{N}_0$. The world is inhabited by an exogenous measure $\bar{L}$

of infinitely lived workers. Let L_{it} denote the number of workers in location i at time t. The distribution of agents across locations in period 0, i.e. $\{L_{i0}\}$, is assumed to be exogenous.

Production

Workers in location i produce a differentiated variety with (composite) productivity A_{it} and earn an (endogenous) wage w_{it} for doing so. Product markets are assumed to be perfectly competitive. Workers have CES preferences over varieties produced in all locations with elasticity of substitution σ. We assume that the productivity of a particular place depends on an exogenous composite (the "fundamental" productivity) and the local population:

$$A_{it} = \bar{A}_i L_{it}^{\alpha}. \tag{1}$$

Allen and Arkolakis (2014) show how particular values of the "productivity spillover" α create isomorphisms between this Armington setup and other familiar frameworks (e.g. monopolistic competition with free entry as in Krugman (1980)).

Trade

Because workers have CES preferences over varieties and each location produces a differentiated variety, workers will consume varieties produced in other locations. We assume that trade between locations is subject to "iceberg" trade costs such that $\tau_{ij} \geq 1$ units of a good produced in location $i \in S$ must be shipped in order for one unit to arrive in location $j \in S$.

Given the setup of the model, the value of trade from i to j, X_{ij}, can be written as:

$$X_{ijt} = \tau_{ij}^{1-\sigma} \left(\frac{w_{it}}{A_{it}}\right)^{1-\sigma} P_{jt}^{\sigma-1} E_{jt}, \tag{2}$$

where E_j is the total expenditure in location j.

Preferences

A worker ν living in location $i \in S$ in time $t \in \mathbb{N}_0$ receives period utility:

$$U_{it}(\nu) = u_{it} \frac{w_{it}}{P_{it}},$$

where u_{it} is the (composite) amenity value of location i in time t, $\frac{w_{it}}{P_{it}}$ is the workers real wage (where P_{it} is the Dixit-Stiglitz ideal price index). We assume that amenity of a particular place depends on an exogenous term and the local population:

$$u_{it} = \bar{u}_i L_{it}^{\beta}. \tag{3}$$

As above, Allen and Arkolakis (2014) show how particular values of the "amenity spillover" β create isomorphisms between this setup and other familiar frameworks (e.g. the inclusion of "housing" in the utility function as in Helpman (1998) and Redding (2012)).

Worker ν receives lifetime utility:

$$U(\nu) = \sum_{t \geq 0} \delta^t \log\left(U_{i(\nu,t)t}\right),$$

where δ is the worker's discount factor and $i(\nu, t)$ is the location that worker ν lives in time t (which is chosen optimally). Note that the worker takes $i(\nu, 0)$ (i.e. her initial location) as exogenous.

Migration

The timing of the model is as follows. In any period, given a distribution of labor across space, markets clear and workers consume in their current location. Workers then receive their idiosyncratic location-specific preference shifter for the subsequent period. Workers then choose where to migrate to for the subsequent period. Moving from location i to location j causes a worker to incur an additive (utility) migration cost $\tilde{\mu}_{ij} \geq 0$.

2.2 Assumptions

In order to make the model tractable, we make the following three assumptions:

Assumption 1. *Trade costs are quasi-symmetric, i.e. for all* $i \in S$ *and* $j \in S$, $\tau_{ij} = \tau_{ij}^S \tau_i^A \tau_j^B$, *where* $\tau_{ij}^S = \tau_{ji}^S$, $\tau_i^A > 0$, $\tau_j^B > 0$.

Assumption 2. *Migration costs are quasi-symmetric, i.e. for all* $i \in S$ *and* $j \in S$, $\tilde{\mu}_{ij} = \tilde{\mu}_{ij}^S + \tilde{\mu}_i^A + \tilde{\mu}_j^B$, *where* $\tilde{\mu}_{ij}^S = \tilde{\mu}_{ji}^S$, $\tilde{\mu}_i^A \in \mathbb{R}$, $\tilde{\mu}_i^B \in \mathbb{R}$.

Assumptions 1 and 2 allow us to simplify the system of equations governing the equilibrium using the results of Allen, Arkolakis, and Takahashi (2014). While restrictive, it should be noted that most empirical applications of the gravity equation assume quasi-symmetry, except perhaps in an error term, e.g. Waugh (2010).

Assumption 3. *The idiosyncratic location-specific preference shifter* $\varepsilon_{it}(\nu)$ *is distributed independently and identically across time, locations, and individuals, with a Type I (Gumbel) extreme value distribution, i.e.* $\Pr\{\varepsilon_{it}(\nu) \leq \varepsilon\} = \exp\{-\exp(-\varepsilon\theta)\}$, *where a higher value of* θ *indicates less dispersion in preferences.*

As we will see below, Assumption 3 allows us to derive closed form solutions for the worker's value function, similar to how Artuç, Chaudhuri, and McLaren (2010) derive the value function for workers switching sectors.

2.3 Distribution of economic activity in each period

We now derive the equilibrium. To facilitate the analysis, we first focus on the spatial equilibrium in a particular time period. We then turn to the dynamic migration problem of workers.

Spatial equilibrium in a particular time period

Given an (endogenous) distribution of workers across locations in time t, $\{L_{it}\}$, the equilibrium wage and price index in each location is determined by the following three equilibrium conditions:

1. For all time periods $t \in \mathbb{N}_0$ and all locations $i \in S$, income is equal to the total sales, i.e. $Y_{it} = \sum_{j \in S} X_{ijt}$.

2. For all time periods $t \in \mathbb{N}_0$ and all locations $i \in S$, income is equal to expenditure, i.e. $Y_{it} = E_{it} = \sum_{j \in S} X_{jit}$.

3. For all time periods $t \in \mathbb{N}_0$ and all locations $i \in S$, income is equal to the wage bill, i.e. $Y_{it} = w_{it} L_{it}$.

Note that equilibrium conditions #1 and #2 together imply that trade is balanced. This spatial equilibrium is well studied (see, e.g. Anderson and Van Wincoop (2003), Alvarez and Lucas (2007), and Allen, Arkolakis, and Takahashi (2014)). Combining equilibrium conditions #1 and #3 with the productivity equation (1) and the gravity trade flow equation (2) yields the first per period spatial equilibrium condition:

$$
\begin{aligned}
w_i L_i &= \sum_j \tau_{ij}^{1-\sigma} A_i^{\sigma-1} w_i^{1-\sigma} P_j^{\sigma-1} w_j L_j \iff \\
w_i L_i &= \sum_j \tau_{ij}^{1-\sigma} \bar{A}_i^{\sigma-1} w_i^{1-\sigma} L_i^{\alpha(\sigma-1)} P_j^{\sigma-1} w_j L_j \qquad (4)
\end{aligned}
$$

Similarly, combining equilibrium conditions #2 and #3 with the productivity equation (1) and the gravity trade flow equation (2) yields the second per period spatial equilibrium condition:

$$
\begin{aligned}
P_i^{1-\sigma} &= \sum_j \tau_{ji}^{1-\sigma} A_j^{\sigma-1} w_j^{1-\sigma} \iff \\
P_i^{1-\sigma} &= \sum_j \tau_{ji}^{1-\sigma} \bar{A}_j^{\sigma-1} w_j^{1-\sigma} L_j^{\alpha(\sigma-1)}. \qquad (5)
\end{aligned}
$$

From Thereom 2 of Allen, Arkolakis, and Takahashi (2014), we know that if trade costs are quasi-symmetric (i.e. if Assumption 1 holds), equilibrium conditions #1 and #2 (along with the gravity equation) will ensure that the origin and destination fixed effects will be equal up to scale, i.e.:

$$
A_i^{\sigma-1} w_i^{1-\sigma} \tau_i^A = \kappa \tau_i^B P_i^{\sigma-1} w_i L_i \qquad (6)
$$

for some $\kappa > 0$ (which is determined by the normalization of wages).

Finally, it will prove to be helpful in what follows to write the equilibrium conditions as functions of the welfare in each location. Given the indirect utility function, we have that welfare can be written as:

$$
W_i = \frac{w_i}{P_i} u_i \qquad (7)
$$

so that equation (6) can be written as:

$$
\begin{aligned}
A_i^{\sigma-1} w_i^{1-\sigma} &= \kappa \frac{\tau_i^B}{\tau_i^A} \left(\frac{w_i}{W_i} u_i \right)^{\sigma-1} w_i L_i \iff \\
w_i &= \left(\kappa \frac{\tau_i^B}{\tau_i^A} A_i^{1-\sigma} W_i^{1-\sigma} u_i^{\sigma-1} L_i \right)^{\frac{1}{1-2\sigma}} \qquad (8)
\end{aligned}
$$

Substituting equations (7) and (8) into the equilibrium market clearing condition (4) (after some tedious algebra) yields a single equation governing the equilibrium relationship between the population and welfare in each location:

$$
\begin{aligned}
w_i L_i &= \sum_j \tau_{ij}^{1-\sigma} \bar{A}_i^{\sigma-1} w_i^{1-\sigma} L_i^{\alpha(\sigma-1)} P_j^{\sigma-1} w_j L_j \iff \\
W_{it}^{\sigma\tilde{\sigma}} L_{it}^{\tilde{\sigma}(1-\alpha(\sigma-1)-\sigma\beta)} &= \sum_j \tau_{ij}^{1-\sigma} \left(\frac{\tau_i^B}{\tau_i^A} \frac{\tau_j^A}{\tau_j^B} \right)^{\frac{\sigma}{\sigma-1}\tilde{\sigma}} \left(\bar{A}_i \bar{u}_j \right)^{(\sigma-1)\tilde{\sigma}} \left(\bar{u}_i \bar{A}_j \right)^{\tilde{\sigma}\sigma} W_{jt}^{(1-\sigma)\tilde{\sigma}} L_{jt}^{\tilde{\sigma}(1+\beta(\sigma-1)+\sigma\alpha)},
\end{aligned}
\tag{9}
$$

where $\tilde{\sigma} \equiv \frac{1-\sigma}{1-2\sigma}$. Note that equation (9) allows us to determine the welfare in all locations from the distribution of population across all locations.

2.4 Migration choice

We now turn to the decision of agents of how to migrate between different locations. Let $U_{it}(\vec{\varepsilon}_t)$ denote the value of a worker living in location i after having received her vector of extreme value (Gumbel) moving cost draws for her decision of where to move in the subsequent period. Her value function can be written recursively as the sum of the (period) welfare she receives living in location i in time t plus a discounted expected value she gets in the following period:

$$
\begin{aligned}
U_{it}(\varepsilon_t) &= \log W_{it} + \delta \max_j \left\{ E_t \left[U_{jt+1}(\vec{\varepsilon}_t) \right] - \tilde{\mu}_{ij} + \varepsilon_{ijt} \right\} \iff \\
U_{it}(\varepsilon_t) &= \log W_{it} + \delta \max_j \left\{ \tilde{V}_{jt+1} - \tilde{\mu}_{ij} + \varepsilon_{ijt} \right\},
\end{aligned}
\tag{10}
$$

where $\tilde{V}_{it} \equiv E_{t-1}\left[U_{jt}(\varepsilon)\right]$ is the expected value of a worker living in location i at time t prior to learning her idiosyncratic draws. We can then solve for this expected value of residence by taking expectations over the extreme value draws using equation (10):

$$
\begin{aligned}
\tilde{V}_{it} &= E\left[U_{it}(\varepsilon_t)\right] \\
\tilde{V}_{it} &= \log W_{it} + \delta E\left[\max_j \left\{ \tilde{V}_{jt+1} - \tilde{\mu}_{ij} + \varepsilon_{ijt} \right\} \right] \\
\tilde{V}_{it} &= \log W_{it} + \frac{\delta}{\theta} \log \left(\sum_k \exp\left(\tilde{V}_{kt+1} - \tilde{\mu}_{ik} \right)^{\theta} \right) \iff \\
\exp \tilde{V}_{it} &= W_{it} \left(\sum_k \left(\frac{\exp\left(\tilde{V}_{kt+1}\right)}{\exp\left(\tilde{\mu}_{ij}\right)} \right)^{\theta} \right)^{\frac{\delta}{\theta}} \iff \\
V_{it} &= W_{it} \left(\sum_k V_{kt+1}^{\theta} \mu_{ij}^{-\theta} \right)^{\frac{\delta}{\theta}},
\end{aligned}
\tag{11}
$$

where the third line used the property of a Gumbel distribution for the expected utility (see e.g. Train (2009)) which is the analog of the property of the Frechet distribution we saw when examining the Eaton and Kortum (2002) model $V_{it} \equiv \exp \tilde{V}_{it}$, and $\mu_{ij} \equiv \exp \tilde{\mu}_{ij}$. Equation (11) provides an equilibrium relationship between the dynamic value function V_{it} and the welfare W_{it} that the agent receives in each period.

Like the Frechet distribution, the Gumbel distribution also provides a simple formulation for the fraction of people moving from i to j between t and $t+1$, π_{ijt}:

$$\pi_{ijt} = \frac{\exp\left(\tilde{V}_{jt+1} - \tilde{\mu}_{ij}\right)^{\theta}}{\sum_k \exp\left(\tilde{V}_{kt+1} - \tilde{\mu}_{ik}\right)^{\theta}} \iff$$

$$\pi_{ijt} = \frac{\left(V_{jt+1}/\mu_{ij}\right)^{\theta}}{\left(\sum_k V_{kt+1}/\mu_{ik}\right)^{\theta}}. \tag{12}$$

Hence, the fraction of people living in location i in time t who choose to move to location j (and will arrive there in time $t+1$) can be written as:

$$L_{ijt} = \pi_{ijt} L_{it} = \mu_{ij}^{-\theta} L_{it} \Pi_{it+1}^{-\theta} V_{jt+1}, \tag{13}$$

where:

$$\Pi_{it} \equiv \left(\sum_j \mu_{ij}^{-\theta} V_{jt}^{\theta}\right)^{\frac{1}{\theta}} \tag{14}$$

is the option value of living in location i at time t. Equation (13) is a gravity equation for migration: all else equal, there will be greater flows from location i to location j in time t the lower the bilateral migration costs, μ_{ij}, the higher the value of living in location j, V_{jt+1}, and the lower the option value of remaining in location i, Π_{it+1}.

Given the transition probabilities, we can also determine the population in each location as a function of the distribution of the population in the previous period, since the number of people living in location j in period $t+1$ is simply the number of people who have moved there from all locations so that:

$$L_{jt+1} V_{jt+1}^{-\theta} = \sum_i \mu_{ij}^{-\theta} \Pi_{it+1}^{-\theta} L_{it}, \tag{15}$$

Equation (15) provides a law of motion for population across space.

2.5 Equilibrium

We can now define an equilibrium in this model. Given productivities $\left\{\bar{A}_i\right\}_{i\in S}$, amenities $\left\{\bar{u}_i\right\}_{i\in S}$, (symmetric) trade costs $\left\{\tau_{ij}\right\}_{i,j\in S}$, migration costs $\left\{\mu_{ij}\right\}_{i,j\in S}$, model parameters $\{\alpha, \beta, \sigma, \theta\}$, and an initial distribution of labor $\left\{L_{i0}\right\}_{i\in S}$equilibrium is defined as the population in every location in every time period $\left\{L_{it}\right\}_{i\in S, t\in\mathbb{N}}$, the (period) welfare in every location in every time period $\left\{W_{it}\right\}_{i\in S, t\in\mathbb{N}}$, the present discounted value of living in every location in every time period $\left\{V_{it}\right\}_{i\in S, t\in\mathbb{N}}$, and the option value of living in every location in every time period $\left\{|\Pi_{it}\right\}_{i\in S, t\in\mathbb{N}}$ such that:

1. (Cross-sectional equilibrium) In every time period $t \in \mathbb{N}$, given the distribution of population $\{L_{it}\}_{i \in S}$, the (period) welfare is determined by consumer optimization, markets clearing, and trade balance, i.e.:

$$W_{it}^{\sigma\tilde{\sigma}} L_{it}^{\tilde{\sigma}(1-\alpha(\sigma-1)-\sigma\beta)} = \sum_j \tau_{ij}^{1-\sigma} \left(\frac{\tau_i^B}{\tau_i^A} \frac{\tau_j^A}{\tau_j^B} \right)^{\frac{\sigma}{\sigma-1}\tilde{\sigma}} \left(\bar{A}_i \bar{u}_j \right)^{(\sigma-1)\tilde{\sigma}} \left(\bar{u}_i \bar{A}_j \right)^{\tilde{\sigma}\sigma} W_{jt}^{(1-\sigma)\tilde{\sigma}} L_{jt}^{\tilde{\sigma}(1+\beta(\sigma-1)+\sigma\alpha)}, \tag{16}$$

2. (Value function) In every time period $t \in \mathbb{N}$ and location $i \in S$, the present discounted value of living in that location and time period is equal to the period welfare times the (discounted) option value of being in that period, i.e.:

$$V_{it} = W_{it} \Pi_{it+1}^{\delta} \tag{17}$$

3. (Option value) In every time period $t \in \mathbb{N}$ and location $i \in S$, the option value of being in that location and time period is equal to the expected value of migrating to all other locations, i.e.:

$$\Pi_{it}^{\theta} = \sum_j \mu_{ij}^{-\theta} V_{jt}^{\theta} \tag{18}$$

4. (Law of motion) In every time period $t \in \mathbb{N}$ and location $i \in S$, the population is equal to the number of people who have migrated to that location from all other locations in the previous

$$L_{it} V_{it}^{-\theta} = \sum_j \mu_{ji}^{-\theta} \Pi_{jt}^{-\theta} L_{jt-1}, \tag{19}$$

   This equilibrium summarizes the law of motion of the variables for any time period t. Transition dynamics could be generated by changes in the various parameters of the model such as the country productivities, or the bilateral costs.

3 Characterization of the steady state

While we are not (yet) aware of the characteristics of such a model during its transition path, we can characterize the steady state equilibrium. In the steady state, equations (16)-(19) become:

$$W_i^{\sigma\tilde{\sigma}} L_i^{\tilde{\sigma}(1-\alpha(\sigma-1)-\sigma\beta)} = \sum_j \tau_{ij}^{1-\sigma} \left(\frac{\tau_i^B}{\tau_i^A} \frac{\tau_j^A}{\tau_j^B} \right)^{\frac{\sigma}{\sigma-1}\tilde{\sigma}} \left(\bar{A}_i \bar{u}_j \right)^{(\sigma-1)\tilde{\sigma}} \left(\bar{u}_i \bar{A}_j \right)^{\tilde{\sigma}\sigma} W_j^{(1-\sigma)\tilde{\sigma}} L_j^{\tilde{\sigma}(1+\beta(\sigma-1)+\sigma\alpha)} \tag{20}$$

$$V_i = W_i \Pi_i^{\delta} \tag{21}$$

$$\Pi_i^{\theta} \equiv \sum_j \mu_{ij}^{-\theta} V_j^{\theta} \tag{22}$$

$$L_i V_i^{-\theta} = \sum_j \mu_{ji}^{-\theta} \Pi_j^{-\theta} L_j \tag{23}$$

Much like the Assumption 1 (quasi-symmetry of trade costs) allowed us to simplify the equilibrium spatial distribution of economic activity in each period, Assumption 2 (quasi-symmetry of migration costs) allows us to simplify the steady state relationship between the distribution of labor, the value function of residing in each location, and option value of remaining in that location. We summarize this result in the following proposition:

Proposition 4. *If migration costs are quasi-symmetric, then for any vector of steady-state option values $\{\Pi_j\}$ and populations $\{L_j\}$, the equilibrium vector of value functions $\{V_j\}$ that satisfies equations (22) and (23) must satisfy:*

$$\left(\Pi_i V_i \frac{\mu_i^A}{\mu_i^B}\right)^\theta = \phi L_i, \tag{24}$$

for all $i \in S$ and for some $\phi > 0$.

Proof. See Appendix. $\square$

By combining equation (24) with (21), we can write the option value Π_i as a function of period welfare W_i and the labor supply L_i:

$$\begin{aligned} V_i &= W_i \Pi_i^\delta \iff \\ \Pi_i &= \phi^{\frac{1}{\theta(1+\delta)}} L_i^{\frac{1}{\theta(1+\delta)}} W_i^{-\frac{1}{1+\delta}} \left(\frac{\mu_i^B}{\mu_i^A}\right)^{\frac{1}{1+\delta}} \end{aligned} \tag{25}$$

Finally, we can substitute equations (24) and (25) into equation (22) to get a single equation that only depends on the population and the (period) welfare:

$$\begin{aligned} \left(\phi^{\frac{1}{\theta(1+\delta)}} L_i^{\frac{1}{\theta(1+\delta)}} W_i^{-\frac{1}{1+\delta}} \left(\frac{\mu_i^B}{\mu_i^A}\right)^{\frac{1}{1+\delta}}\right)^\theta &= \phi^\theta \sum_j \mu_{ij}^{-\theta} L_j \left(\phi^{\frac{1}{\theta(1+\delta)}} L_j^{\frac{1}{\theta(1+\delta)}} W_j^{-\frac{1}{1+\delta}} \left(\frac{\mu_j^B}{\mu_j^A}\right)^{\frac{1}{1+\delta}}\right)^{-\theta} \iff \\ W_i^{-\frac{\theta}{1+\delta}} L_i^{\frac{1}{1+\delta}} &= \phi^{\theta - \frac{2}{1+\delta}} \sum_j \mu_{ij}^{-\theta} \left(\frac{\mu_i^A \mu_j^A}{\mu_i^B \mu_j^B}\right)^{\frac{\theta}{1+\delta}} L_j^{\frac{\delta}{1+\delta}} W_j^{\frac{\theta}{1+\delta}} \end{aligned} \tag{26}$$

Hence, we have successfully simplified the steady state equilibrium into two non-linear equations (equations (16) and (26)) that are solely a function of the steady state distribution of people and the steady state distribution of welfare:

$$W_i^{\sigma\tilde{\sigma}} L_i^{\tilde{\sigma}(1-\alpha(\sigma-1)-\sigma\beta)} = \sum_j \tau_{ij}^{1-\sigma} \left(\frac{\tau_i^B}{\tau_i^A}\frac{\tau_j^A}{\tau_j^B}\right)^{\frac{\sigma}{\sigma-1}\tilde{\sigma}} \left(\bar{A}_i \bar{u}_j\right)^{(\sigma-1)\tilde{\sigma}} \left(\bar{u}_i \bar{A}_j\right)^{\tilde{\sigma}\sigma} W_j^{(1-\sigma)\tilde{\sigma}} L_j^{\tilde{\sigma}(1+\beta(\sigma-1)+\sigma\alpha)} \tag{27}$$

$$W_i^{-\frac{\theta}{1+\delta}} L_i^{\frac{1}{1+\delta}} = \phi^{\theta - \frac{2}{1+\delta}} \sum_j \mu_{ij}^{-\theta} \left(\frac{\mu_i^A \mu_j^A}{\mu_i^B \mu_j^B}\right)^{\frac{\theta}{1+\delta}} L_j^{\frac{\delta}{1+\delta}} W_j^{\frac{\theta}{1+\delta}}. \tag{28}$$

3.1 Existence and Uniqueness

Equations (27) and (28) are an example of the mathematical system of nonlinear integral equations that show up in general equilibrium gravity models with intermediate goods in production, and their properties are characterized in Allen, Arkolakis, and Takahashi (2014). As a result, we can prove the following theorem characterizing the existence and uniqueness of a steady state equilibrium:

Theorem 5. *Consider any set of strictly positive and finite productivities* $\{\bar{A}_i\}$*, amenities* $\{\bar{u}_i\}$*, quasi-symmetric trade costs* $\{\tau_{ij}\}$*, quasi-symmetric migration costs* $\{\mu_{ij}\}$*, and model parameters* $\{\alpha, \beta, \sigma, \theta\}$*. Then:*

(i) As long as $\alpha(\sigma-1)+\sigma\left(\beta-\frac{1}{\theta}\right) \neq 1$ *and* $\sigma \neq \frac{1}{2}$*, there exists a steady state equilibrium.*

(ii) If $1-\alpha(\sigma-1)-\sigma\beta > \max\left\{0, \frac{\theta\sigma}{\delta}\right\}$ *and* $1+\beta(\sigma-1)+\alpha\sigma < \frac{\sigma-1}{\theta}$*, the steady state equilibrium is unique.*

Proof. We can write this in a manner to show that this system of equations is similar to the one for general equilibrium gravity models. Define:

$$
\begin{aligned}
x_i &\equiv W_i^{\sigma\tilde{\sigma}} L_i^{\tilde{\sigma}(1-\alpha(\sigma-1)-\sigma\beta)} \\
y_i &\equiv W_i^{-\frac{\theta}{1+\beta}} L_i^{\frac{1}{1+\beta}} \\
T_{ij} &\equiv \tau_{ij}^{1-\sigma} \left(\frac{\tau_i^B}{\tau_i^A}\frac{\tau_j^A}{\tau_j^B}\right)^{\frac{\sigma}{\sigma-1}\tilde{\sigma}} \left(\bar{A}_i\bar{u}_j\right)^{(\sigma-1)\tilde{\sigma}} \left(\bar{u}_i\bar{A}_j\right)^{\tilde{\sigma}\sigma} \\
M_{ij} &\equiv \mu_{ij}^{-\theta}\left(\frac{\mu_i^A\mu_j^A}{\mu_i^B\mu_j^B}\right)^{\frac{\theta}{1+\delta}} \\
\lambda_a &= 1 \\
\lambda_b &\equiv \phi^{\theta-\frac{2}{1+\delta}}.
\end{aligned}
$$

Note that we have:

$$
\begin{pmatrix} \ln x_i \\ \ln y_i \end{pmatrix} = \begin{pmatrix} \sigma\tilde{\sigma} & \tilde{\sigma}(1-\alpha(\sigma-1)-\sigma\beta) \\ -\frac{\theta}{1+\delta} & \frac{1}{1+\delta} \end{pmatrix} \begin{pmatrix} \ln W_i \\ \ln L_i \end{pmatrix}
$$

As long as the matrix $\begin{pmatrix} \sigma\tilde{\sigma} & \tilde{\sigma}(1-\alpha(\sigma-1)-\sigma\beta) \\ -\frac{\theta}{1+\delta} & \frac{1}{1+\delta} \end{pmatrix}$ is invertible, then we can re-write equations (27) and (28) as:

$$
x_i = \lambda_a \sum_j T_{ij} x_j^a y_j^b \tag{29}
$$

$$
y_i = \lambda_b \sum_j M_{ij} x_j^c y_j^d, \tag{30}
$$

where:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} (1-\sigma)\tilde{\sigma} & \tilde{\sigma}(1+\beta(\sigma-1)+\sigma\alpha) \\ \frac{\delta}{1+\delta} & \frac{\theta}{1+\delta} \end{pmatrix} \begin{pmatrix} \sigma\tilde{\sigma} & \tilde{\sigma}(1-\alpha(\sigma-1)-\sigma\beta) \\ -\frac{\theta}{1+\delta} & \frac{1}{1+\delta} \end{pmatrix}^{-1} \implies$$

$$\begin{aligned}
a &= \frac{(1+\beta(\sigma-1)+\alpha\sigma)+\frac{1-\sigma}{\theta}}{(1-\alpha(\sigma-1)-\sigma\beta)+\frac{\sigma}{\theta}} \\
b &= \frac{(\sigma-1)(1+\delta)(1+\alpha)}{\theta(1-\alpha(\sigma-1)-\sigma\beta)+\sigma} \\
c &= \frac{\frac{\delta}{\theta}+\theta}{\tilde{\sigma}(\delta+1)\left(1-\alpha(\sigma-1)-\sigma\beta+\frac{\sigma}{\theta}\right)} \\
d &= -\frac{\delta(1-\alpha(\sigma-1)-\sigma\beta)-\theta\sigma}{\theta(1-\alpha(\sigma-1)-\sigma\beta)+\sigma}.
\end{aligned}$$

Note that as long as $\sigma \neq \frac{1}{2}$, equation (27) is not homogeneous of degree zero in $\{W_i\}$, which means that welfare can always be scaled in order to ensure $\lambda_a = 1$.

For part (i) (existence), Allen, Arkolakis, and Takahashi (2014) prove the existence of any system of equations of the form of (29) and (30). Hence, a sufficient condition for existence of a solution to equations (27) and (28) is the invertibility of $\begin{pmatrix} \sigma\tilde{\sigma} & \tilde{\sigma}(1-\alpha(\sigma-1)-\sigma\beta) \\ -\frac{\theta}{1+\beta} & \frac{1}{1+\beta} \end{pmatrix}$, which occurs if and only if:

$$\begin{aligned}
\begin{vmatrix} \sigma\tilde{\sigma} & \tilde{\sigma}(1-\alpha(\sigma-1)-\sigma\beta) \\ -\frac{\theta}{1+\delta} & \frac{1}{1+\delta} \end{vmatrix} &\neq 0 \iff \\
\left(\frac{\sigma\tilde{\sigma}}{1+\delta}\right) + \frac{\theta\tilde{\sigma}(1-\alpha(\sigma-1)-\sigma\beta)}{1+\delta} &\neq 0 \iff \\
1 \neq \alpha(\sigma-1)+\sigma\left(\beta-\frac{1}{\theta}\right)&.
\end{aligned}$$

For part (ii) (uniqueness), Allen, Arkolakis, and Takahashi (2014) prove the uniqueness of any system of equations of the form of (29) and (30) as long as $a < 0 < c$ and $d < 0 < b$, $c \in [0,1]$, and $b \in [0,1]$. Note that for $\sigma > 1$ and $(1-\alpha(\sigma-1)-\sigma\beta) > 0$, b and c are both positive. For these cases, we require that:

$$\begin{aligned}
& a < 0 \iff \\
& \theta(1+\beta(\sigma-1)+\alpha\sigma) < \sigma - 1
\end{aligned}$$

and:

$$\begin{aligned}
& d < 0 \iff \\
& \frac{\theta\sigma}{\delta} < 1-\alpha(\sigma-1)-\sigma\beta
\end{aligned}$$

Hence, if $1-\alpha(\sigma-1)-\sigma\beta > \max\left\{0, \frac{\theta\sigma}{\delta}\right\}$ and $1+\beta(\sigma-1)+\alpha\sigma < \frac{\sigma-1}{\theta}$, the steady state equilibrium is unique, as claimed. $\square$

Note that condition (i) for existence is identical to the one given in Allen and Arkolakis (2014) (for $\tilde{\beta} = \beta - \frac{1}{\theta}$, which as discussed there is the isomorphism to allow for idiosyncratic preferences). Hence, the inclusion of migration frictions and dynamic optimization does not affect the existence of an equilibrium. The conditions for uniqueness, however, are affected by the presence of the migration frictions and the dynamic optimization, as in Allen and Arkolakis (2014), uniqueness held whenever $\left|\frac{1+\beta(\sigma-1)+\alpha\sigma}{1-\alpha(\sigma-1)-\sigma\beta}\right| \leq 1$.

3.2 The equilibrium distribution of economic activity

We can also analytically characterize the distribution of economic activity as a function of the underlying model parameters. To see this, we start with equation (6) and substitute out wages using equation (7) and then replace welfare using equation (25):

$$
\begin{aligned}
\kappa \tau_i^B P_i^{\sigma-1} w_i L_i &= A_i^{\sigma-1} w_i^{1-\sigma} \tau_i^A \iff \\
\kappa \tau_i^B P_i^{\sigma-1} \left(\frac{W_i P_i}{\bar{u}_i L_i^{\beta}}\right)^{\sigma} L_i &= \bar{A}_i^{\sigma-1} \tau_i^A L_i^{\alpha(\sigma-1)} \iff \\
\left(\phi^{\frac{1}{\theta}} L_i^{\frac{1}{\theta}} \Pi_i^{-(1+\delta)} \left(\frac{\mu_i^B}{\mu_i^A}\right)\right)^{\sigma} L_i^{1-\alpha(\sigma-1)-\sigma\beta} &= \bar{A}_i^{\sigma-1} \frac{\tau_i^A}{\tau_i^B} P_i^{1-2\sigma} \bar{u}_i^{\sigma} \frac{1}{\kappa} \iff \\
L_i^{1-\alpha(\sigma-1)-\sigma\beta+\frac{\sigma}{\theta}} &= \frac{1}{\kappa \phi^{\frac{\sigma}{\theta}}} \frac{\tau_i^A}{\tau_i^B} \left(\frac{\mu_i^A}{\mu_i^B}\right)^{\sigma} \bar{A}_i^{\sigma-1} \bar{u}_i^{\sigma} P_i^{1-2\sigma} \Pi_i^{\sigma(1+\delta)},
\end{aligned}
$$

or equivalently:

$$
\gamma_1 \ln L_i = C_1 + \ln\left(\frac{\tau_i^A}{\tau_i^B}\right) + \sigma \ln\left(\frac{\mu_i^A}{\mu_i^B}\right) + (\sigma-1)\ln \bar{A}_i + \sigma \ln \bar{u}_i - (2\sigma-1)\ln P_i + \sigma(1+\delta)\ln \Pi_i, \tag{31}
$$

where $C_1 \equiv -\ln\left(\kappa \phi^{\frac{\sigma}{\theta}}\right)$ and $\gamma_1 \equiv 1-\alpha(\sigma-1)-\sigma\left(\beta-\frac{1}{\theta}\right)$. for some $\kappa > 0$ (which is determined by the normalization of wages). Note that $\gamma_1 > 0$ as long as the spillovers α and β are not sufficiently strong to generate a "black hole equilibrium." As a result, equation (31) says that all else equal, the population of a location will be higher the greater its productivity $\bar{A}_i$ or its amenity $\bar{u}_i$, the lower its price index P_i (which is a sufficient statistic for a location's trade costs), or the higher its option value of migrating Π_i (which is a sufficient statistic for a location's migration costs). Moreover, if trade or migration cost asymmetries will increase the population of a location if it is relatively less costly to import (or immigrate) than export (or emigrate). As in Allen and Arkolakis (2014), the the local population is more responsive to the underlying geography the greater the strength of the spillovers, as measured by γ_1.

4 Conclusion

This short paper shows how it is possible to allow for the costly movement of both goods and people across space, thereby nesting both quantitative trade models and quantitative economic geography models as special cases. By placing a restriction on the form of bilateral trade and migration frictions (namely that they are "quasi-symmetric"), we show that the steady-state equilibrium can be expressed as a familiar mathematical system, allowing us to characterize the properties of the equilibrium and show how geography, trade costs, and migration costs shape the distribution of economic activity across space.

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5 Appendix

5.1 Proof of Proposition 4.

Proof. By dividing equation (22) by (23) we have:

$$\begin{aligned}
\frac{\Pi_i^\theta V_i^\theta}{L_i} &= \sum_j \frac{\mu_{ij}^{-\theta}}{\sum_k \mu_{ki}^{-\theta} \Pi_k^{-\theta} L_k} V_j^\theta \iff \\
\frac{\Pi_i^\theta V_i^\theta}{L_i} \left(\frac{\mu_i^A}{\mu_i^B}\right)^\theta &= \sum_j \frac{\left(\mu_{ij}^S\right)^{-\theta}}{\sum_k \left(\mu_{ki}^S\right)^{-\theta} \left(\mu_k^A\right)^{-\theta} \Pi_k^{-\theta} L_k} V_j^\theta \left(\mu_j^B\right)^{-\theta} \iff \\
\frac{\Pi_i^\theta V_i^\theta}{L_i} \left(\frac{\mu_i^A}{\mu_i^B}\right)^\theta &= \sum_j \frac{\left(\mu_{ij}^S\right)^{-\theta} \left(\mu_j^A\right)^{-\theta} \Pi_j^{-\theta} L_j}{\sum_k \left(\mu_{ki}^S\right)^{-\theta} \left(\mu_k^A\right)^{-\theta} \Pi_k^{-\theta} L_k} \frac{\Pi_j^\theta V_j^\theta}{L_j} \left(\frac{\mu_j^A}{\mu_j^B}\right)^\theta \iff \\
\phi_i &= \lambda \sum_j F_{ij} \phi_j,
\end{aligned}$$

where $\phi_i \equiv \frac{\Pi_i^\theta V_i^\theta}{L_i} \left(\frac{\mu_i^A}{\mu_i^B}\right)^\theta$, $F_{ij} \equiv \frac{\left(\mu_{ij}^S\right)^{-\theta} \left(\mu_j^A\right)^{-\theta} \Pi_j^{-\theta} L_j}{\sum_k \left(\mu_{ki}^S\right)^{-\theta} \left(\mu_k^A\right)^{-\theta} \Pi_k^{-\theta} L_k}$, and $\lambda = 1$. From the Perron-Frobenius theorem, for any $\{\Pi_i\} > 0$ and $\{L_j\} > 0$, we have $F_{ij} > 0$, so that there exists a unique (to-scale) vector $\{\phi_i\}$ and characteristic value $\lambda > 0$. Hence, if we find an answer, we know it is the unique answer. Let us guess that $\lambda = 1$ and $\phi_i = \phi$. Then we have:

$$\begin{aligned}
\phi &= \sum_j F_{ij} \phi \iff \\
1 &= \sum_j F_{ij} \iff \\
1 &= \sum_j \frac{\left(\mu_{ij}^S\right)^{-\theta} \left(\mu_j^A\right)^{-\theta} \Pi_j^{-\theta} L_j}{\sum_k \left(\mu_{ki}^S\right)^{-\theta} \left(\mu_k^A\right)^{-\theta} \Pi_k^{-\theta} L_k} \iff \\
\sum_j \left(\mu_{ji}^S\right)^{-\theta} \left(\mu_j^A\right)^{-\theta} \Pi_j^{-\theta} L_j &= \sum_j \left(\mu_{ij}^S\right)^{-\theta} \left(\mu_j^A\right)^{-\theta} \Pi_j^{-\theta} L_j,
\end{aligned}$$

which holds from Assumption 2 because $\mu_{ji}^S = \mu_{ij}^S$. Hence:

$$\begin{aligned}
\frac{\Pi_i^\theta V_i^\theta}{L_i} \left(\frac{\mu_i^A}{\mu_i^B}\right)^\theta &= \phi \iff \\
\left(\Pi_i V_i \frac{\mu_i^A}{\mu_i^B}\right)^\theta &= \phi L_i,
\end{aligned}$$

as required. ∎