Intermediate Microeconomics

Preference and Utility

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Outline

- Preference
- Utility
  - Indifference curve
  - Marginal rate of substitution
  - Convexity and concavity
  - Utility functions
- Budget

Preference (偏好)

Axioms of rational choice

- Completeness (完备性)
  1. A is preferred to B: $A \succ B$
  2. B is preferred to A: $B \succ A$
  3. A and B are equally attractive: $A \sim B$

  (1)+(3): $A \succeq B$; (2)+(3): $B \succeq A$.
- Transitivity (传递性): "A is preferred to B" and "B is preferred to C" implies "A is preferred to C"
- Continuity (连续性): "A is preferred to B," then situations suitably "close to" A must also be preferred to B.

From Preference to Utility (效用)

Utility is an order-preserving transformation.

- $A \succ B \Leftrightarrow U(A) > U(B)$
- $A \succeq B \Leftrightarrow U(A) \geq U(B)$
- Let $U(x)$ be increasing in x and $f(\cdot)$ is an increasing function. A monotonic transformation $f\left(U(x)\right)$ is increasing in x:

$$\frac{d}{dx} f\left(U(x)\right) = f'\left(U(x)\right) U'(x) > 0.$$

If $A \succ B \Leftrightarrow U(A) > U(B)$, then a monotonic transformation preserves the order: $f\left(U(A)\right) > f\left(U(B)\right)$.

Utility function (效用函数)

Definition

Individual's preferences are assumed to be represented by a utility function of the form

$$U(x_1, x_2, ..., x_n)$$

where $x_1, x_2, ..., x_n$ *are the quantities of each of* n *goods that might be consumed in a period.*

Example

Cobb-Douglas utility: $U(x, y) = x^a y^b$, *where* $a, b > 0$. *E.g.,* $U(x, y) = x^{1/2} y^{1/2}$

Indifference Curve (无差异曲线)

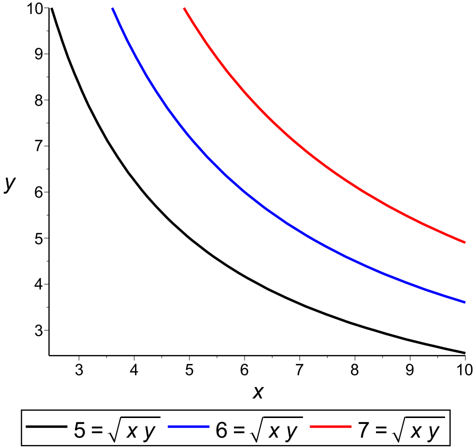
Definition

An indifference curve (or, in many dimensions, an indifference surface) shows a set of consumption bundles about which the individual is indifferent. That is, the bundles all provide the same level of utility.

For two goods (x, y), how to draw indifferent curve(s) in a x-y plane?

- Let x (resp., y) be the label of the horizontal (resp., vertical) axis;
- Given $U(x, y)$, fixing a particular level of utility $u_0 = U(x, y)$, plot y as a function of x.
- Similarly, for another utility level, e.g., $u_1 = U(x, y)$, plot y as a function of x.

Example: $U(x,y) = x^{1/2}y^{1/2}$



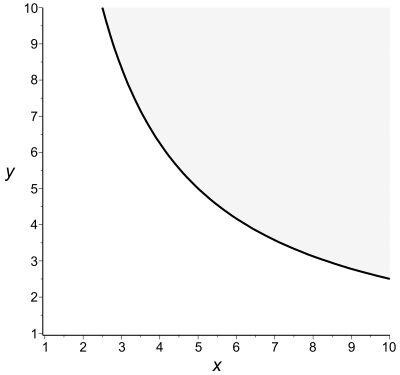
The Shape of an Indifference Curve

Example: $U(x,y) = \sqrt{xy}$. Plot the indifference curve evaluated at utility level $u_1 = 5, u_2 = 6, u_3 = 7$.

- $u_i = \sqrt{xy} \Leftrightarrow u_i^2 = xy \Rightarrow y = \frac{u_i^2}{x}$.
- $y = \frac{5^2}{x}$, $y = \frac{6^2}{x}$, $y = \frac{7^2}{x}$

Some features:

- $\frac{dy}{dx} = -\frac{u_i^2}{x^2} < 0$: downward sloping.
- $\frac{d^2y}{dx^2} = 2\frac{u_i^2}{x^3} > 0$: as x increases, slope: steeper $\rightarrow$ flatter
- $\frac{du_i}{dy} > 0$: fixing x, a greater y gives more utility (the curve moves up). $\frac{du_i}{dx} > 0$: fixing y, a greater x gives more utility (the curve moves to the right).



- The point $(5, 5)$ locates on the indifference curve $5 = \sqrt{xy}$
- Points in the shaded region are preferred to $(5, 5)$
- Points in the non-shaded region are worse than $(5, 5)$.
- All the other points along the indifference curve $5 = \sqrt{xy}$ and the point $(5, 5)$ are equally attractive.

Marginal Rate of Substitution (边际替代率, MRS)

- Marginal rate of substitution: Evaluated at a fixed utility level, how many units of y is going to be given up, in exchange for an additional unit of x?

$$MRS = -\frac{dy}{dx}\bigg|_{U=\text{constant}}$$

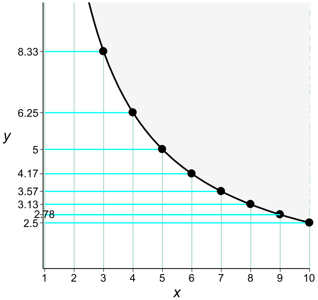
Alternative definition: the |slope| of an indifference curve.

- Equivalently, fixing $u_0 =$ constant, total differentiate $U(x, y) = u_0$:

$$dU(x,y) = U'_x dx + U'_y dy = du_0 = 0$$

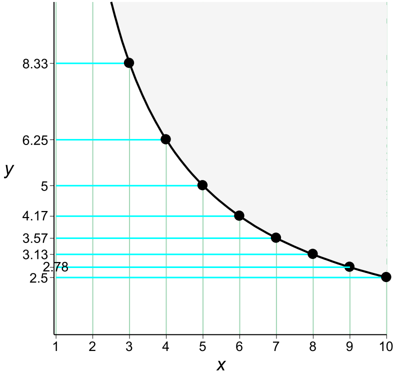
$$\Rightarrow MRS = -\frac{dy}{dx}\bigg|_{U=\text{constant}} = \frac{U'_x}{U'_y} = \frac{MU(x)}{MU(y)}$$

An example for diminishing MRS



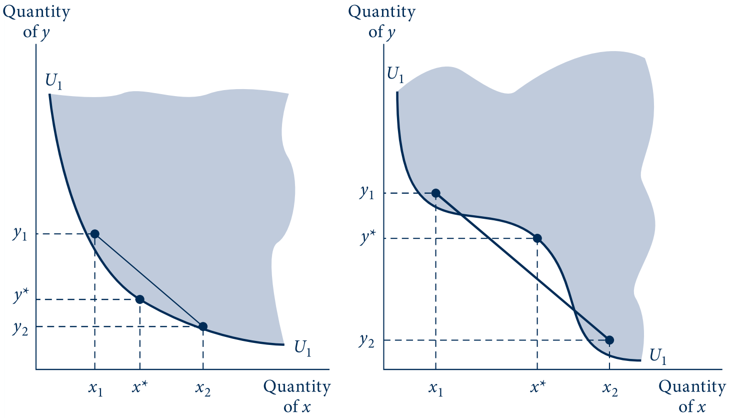
- A "reasonable" assumption of individual's preference is "diminishing marginal utility"
- If currently you have 3 units of x and 8.33 units of y in hand with utility $\sqrt{xy} = 5$, you are willing to give up $8.33 - 6.25 = 2.08$ units of y in exchange for an additional x (from 3 to 4) to keep your utility unchanged (at 5).

Diminishing MRS



- If currently you already have 9 units of x and only 2.78 units of y in hand with utility $\sqrt{xy} = 5$, you are willing to give up $2.78 - 2.5 = 0.28$ units of y in exchange for an additional x (from 9 to 10) to keep your utility unchanged (at 5).

Convexity (凸性)



An equivalent condition for "diminishing marginal rate of substitution" is that the set of points that are preferred to a given indifference curve, is "convex."

Convex Set (凸集合)

- Geometrically, in a convex set, any points in a line segment that connects any two points within the set, should not be located outside the set. Otherwise, it is not a convex set.
- The former definition of a convex set is:

Definition

A set $\mathcal{S} = \{x, y | U(x, y) \geq u\}$ *is convex if* $\forall (x_1, y_1), (x_2, y_2) \in \mathcal{S}$, *the element* $(\lambda x_1 + (1-\lambda) x_2, \lambda y_1 + (1-\lambda) y_2) \in \mathcal{S}$, *where* $\lambda \in (0, 1)$.

Example

汉字“凸”不是个凸集合。

Example

$\mathcal{S} = \{x, y | U(x,y) = \sqrt{xy} \geq u\}$ *is a convex set.*

- Pick two points $A = (x_1, y_1)$ and $B = (x_2, y_2)$ such that $U(x_1, y_1) = U(x_2, y_2) = u$ and let $x_1 < x_2$.
- $\sqrt{x_1 y_1} = \sqrt{x_2 y_2} = u \Rightarrow x_1 y_1 = x_2 y_2 = u^2$.
- $A \in \mathcal{S}$ and $B \in \mathcal{S}$. We need to show that for $\lambda \in (0,1)$, $U\left(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2\right) \geq u$:
  $\sqrt{\left(\lambda x_1 + (1-\lambda)x_2\right)\left(\lambda y_1 + (1-\lambda)y_2\right)} =$
  $\sqrt{\lambda^2 x_1 y_1 + (1-\lambda)^2 \underbrace{x_2 y_2}_{=x_1 y_1} + \lambda(1-\lambda)\underbrace{(x_1 y_2 + x_2 y_1)}_{\geq 2\sqrt{x_1 y_2 x_2 y_1}}} \geq$
  $\sqrt{\lambda^2 u^2 + (1-\lambda)^2 u^2 + \lambda(1-\lambda)2u^2} = u$

Diminishing MRS and Quasi-Concave Functions

- A function $f(\cdot)$ is quasi-concave (拟凹) if the upper contour sets of the function are convex sets, e.g., $\{\boldsymbol{x} \in R^n | f(\boldsymbol{x}) \geq a\}$ are convex for all a.
  - the set of all points for which such a function takes on a value greater than any specific constant is a convex set (i.e., any two points in the set can be joined by a line contained completely within the set).
- A function with two variables $f(x_1, x_2)$ is quasi-concave if

$$f''_{11}(f'_2)^2 - 2f''_{12}f'_1f'_2 + f''_{22}(f'_1)^2 < 0.$$

- Diminishing MRS $\Leftrightarrow$ Quasi-concave $U(x, y)$.

DMRS $\Leftrightarrow$ Quasi-Concave Utility Function

- Consider an indifference curve, $U(x,y)=u_0$, where y is a function of x.
- $MRS = -\frac{dy}{dx}\Big|_{u_0} = \frac{U'_x(x,y(x))}{U'_y(x,y(x))}$
- The question is: under what conditions, MRS is decreasing in x or $\frac{dMRS}{dx} < 0$?

$$
\begin{aligned}
\frac{dMRS}{dx} &= \frac{d}{dx}\left(\frac{U'_x(x,y(x))}{U'_y(x,y(x))}\right) \\
=&\frac{U'_y\left(U''_{xx}+U''_{xy}\frac{dy}{dx}\right)-U'_x\left(U''_{yx}+U''_{yy}\frac{dy}{dx}\right)}{(U'_y)^2} \\
=&\frac{U'_y\left(U''_{xx}-U''_{xy}\frac{U'_x}{U'_y}\right)-U'_x\left(U''_{yx}-U''_{yy}\frac{U'_x}{U'_y}\right)}{U_y^2} \\
=&\frac{(U'_y)^2U''_{xx}-2U'_xU'_yU''_{xy}+(U'_x)^2U''_{yy}}{(U'_y)^3} < 0 \\
\Leftrightarrow&(U'_y)^2U''_{xx}-2U'_xU'_yU''_{xy}+(U'_x)^2U''_{yy} < 0.
\end{aligned}
$$

Example

$U = x^{1/2}y^{1/2}$ is quasi-concave.

- $U'_x = \frac{1}{2}x^{-1/2}y^{1/2}$, $U''_{xx} = -\frac{1}{4}x^{-3/2}y^{1/2}$, $U'_y = \frac{1}{2}x^{1/2}y^{-1/2}$, $U''_{yy} = \frac{1}{4}x^{1/2}y^{-3/2}$, $U''_{xy} = \frac{1}{4}x^{-1/2}y^{-1/2}$
- $(U'_y)^2U''_{xx} - 2U'_xU'_yU''_{xy} + (U'_x)^2U''_{yy} < 0$

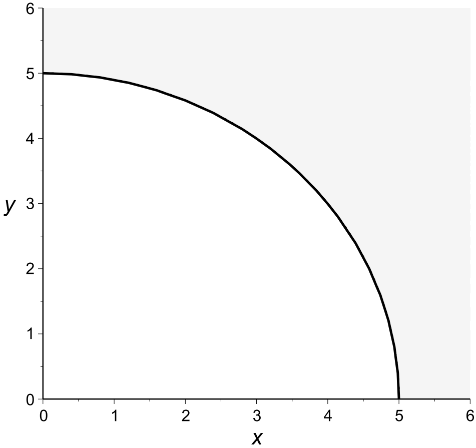
Example

The indifference curve $u_0 = x^{1/2}y^{1/2}$ exhibits diminishing MRS.

- $MRS = \frac{U'_x}{U'_y} = \frac{\frac{1}{2}x^{-1/2}y^{1/2}}{\frac{1}{2}x^{1/2}y^{-1/2}} = \frac{y(x)}{x}$
- $u_0 = x^{1/2}y^{1/2} \Rightarrow u_0^2 = xy \Rightarrow y(x) = \frac{u_0^2}{x}$
- $MRS = \frac{y(x)}{x} = \frac{\frac{u_0^2}{x}}{x} = \frac{u_0^2}{x^2}$
- $\frac{dMRS}{dx} < 0.$

Example (Practice)

The utility function $U(x, y) = \sqrt{x^2 + y^2}$ *is not quasi-concave, and the MRS along the indifference curve* $u_0 = \sqrt{x^2 + y^2}$ *is not decreasing in* x*.*



Second-order Conditions and Concave Functions

- For a differentiable function with only one variable, assume that $f(\cdot)$ achieves a maximum at x^*:
  - First-order condition (一阶条件, FOC): $f'(x^*) = 0$
  - Second-order condition (二阶条件, SOC): $f''(x^*) \leq 0$.
- A function of one variable is concave if (凹函数)

$$f(\lambda x_1 + (1-\lambda)x_2) \geq \lambda f(x_1) + (1-\lambda)f(x_2)$$

  for all x_1 and x_2 and all λ such that $0 \leq \lambda \leq 1$.
  Strictly concave if "$>$".
- If $f(\cdot)$ is differentiable, then $f(\cdot)$ is concave if and only if $f''(\cdot) \leq 0$.

Concavity of a Function with Two Variables

- In consumer theory, we typically deal with a utility function that consists of two variables, i.e., $U(x, y)$.
- How to solve a maximum/minimum of a function $f(x, y)$ with two variables?
  - FOC: $f'_x = 0$ and $f'_y = 0$
  - Second order derivatives: f''_{xx}, f''_{yy} and f''_{xy}
  - How can we define the SOCs are "negative" or "positive?"
- Recall the concepts of "negative (semi) definite" or "positive (semi) definite" learned in linear algebra.

Second-order Conditions for a Maximum

- $\max_{x,y} f(x,y)$
- FOC: $f'_x(x^*, y^*) = 0$ and $f'_y(x^*, y^*) = 0$
- List the second-order derivatives in the following matrix form

$$H = \begin{bmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{bmatrix}$$

  We call H the Hessian matrix.
- If (x^*, y^*) maximize $f(x,y)$, it requires that H is negative semidefinite (负半定)
  - All eigenvalues (特征值) of H are non-positive.
  - The determinants of the principal minors (主子式的行列式) alternate in signs, starting with "$-$" for the first minor.
- If $f''_{11} \leq 0$ and $\det(H) = f''_{11} f''_{22} - (f''_{12})^2 \geq 0$, then H is negative semidefinite, and then (x^*, y^*) maximizes $f(x,y)$.

Second-order Conditions for a Minimum

- $\min_{x,y} f(x,y)$
- FOC: $f'_x(x^*,y^*) = 0$ and $f'_y(x^*,y^*) = 0$
- List the second-order derivatives in the following matrix form

$$H = \begin{bmatrix} f''_{11} & f''_{12} \\ f''_{21} & f''_{22} \end{bmatrix}$$

- If (x^*,y^*) minimizes $f(x,y)$, it requires that H is positive semidefinite (正半定)
  - All eigenvalues (特征值) of H are non-negative.
  - Determinants of all principal minors (主子式的行列式) are non-negative.
- If $f''_{11} \geq 0$ and $\det(H) = f''_{11}f''_{22} - (f''_{12})^2 \geq 0$, then H is positive semidefinite, and hence (x^*,y^*) minimizes $f(x,y)$.

Example: Concave Utility Function

- A utility function $U(x, y)$ is said to be concave if

$$U''_{xx} < 0,\ U''_{xx}U''_{yy} - (U''_{xy})^2 > 0$$

- The associated Hessian

$$H = \left[\begin{array}{cc} U''_{xx} & U''_{xy} \\ U''_{yx} & U''_{yy} \end{array} \right]$$

is negative semidefinite.

Diminishing MRS and Concave Utility

- Notice that concave is "stronger" than quasi-concave.
  - If $U(x, y)$ is concave, then it must be quasi-concave.
  - If $U(x, y)$ is quasi-concave, then it might not necessarily be concave.
- Diminishing $MRS \Leftrightarrow U(x, y)$ is quasi-concave.

Example

$U = \ln(x) + \ln(y)$ *is concave, and exhibits diminishing* MRS.

- Diminishing MRS: an indifference curve $u_0 = \ln(xy) \Rightarrow y(x) = \frac{e^{u_0}}{x}$

$$\frac{dMRS}{dx} = \frac{d}{dx}\left(\frac{U'_x}{U'_y}\right) = \frac{d}{dx}\left(\frac{\frac{1}{x}}{\frac{1}{y(x)}}\right) = \frac{d}{dx}\left(\frac{e^{u_0}}{x^2}\right) < 0$$

- Concavity: $U''_{xx} = -\frac{1}{x^2}$, $U''_{xy} = 0$ and $U''_{yy} = -\frac{1}{y^2}$.

$$U''_{xx} < 0,\ U''_{xx}U''_{yy} - (U''_{xy})^2 = \frac{1}{x^2y^2} > 0.$$

Example: Diminishing $MRS \nRightarrow$ Concave Utility

Example

$U(x, y) = x^2y^2$ exhibits diminishing MRS, but is not concave.

- Diminishing MRS: an indifference curve $u_0 = x^2y^2 \Rightarrow y(x) = \frac{\sqrt{u_0}}{x}$

$$\frac{dMRS}{dx} = \frac{d}{dx}\left(\frac{U'_x}{U'_y}\right) = \frac{d}{dx}\left(\frac{2xy^2}{2x^2y}\right) = \frac{d}{dx}\left(\frac{y(x)}{x}\right) = \frac{d}{dx}\left(\frac{\sqrt{u_0}}{x^2}\right) < 0.$$

- Check concavity: $U''_{xx} = 2y^2$, $U''_{yy} = 2x^2$, $U''_{xy} = 4xy$.

$$H = \begin{bmatrix} 2y^2 & 4xy \\ 4xy & 2x^2 \end{bmatrix}$$

where the determinants of the principal minors are: $2y^2 > 0$, $4x^2y^2 - 16x^2y^2 < 0$. H is not negative semi-definite, and hence $U(x, y)$ is not concave.

Quasi-linear Utility

- A frequently used utility function is quasi-linear utility (拟线性效用), where x is a particular good that is of our interest and y is money.

$$U(x, \text{numéraire}) = u(x) + y$$

- $U'_x = u'(x)$ and $U'_y = 1$, and hence $MRS = \frac{u'(x)}{1} = u'(x)$ depends only on the marginal utility of x.
- For a diminishing MRS, it requires that

$$\frac{dMRS}{dx} = u''(x) < 0 \Leftrightarrow u(x) \text{ is a concave function (with only one variable)}$$

- $u''(x) < 0$ implies diminishing marginal utility of x (边际效用递减)
- We can confirm that $u(x) + y$ is concave by checking the Hessian: $U''_{xx} = u''(x) < 0$, $U''_{xy} = 0$ and $U''_{yy} = 0$, hence

$$U''_{xx} < 0, \ U''_{xx}U''_{yy} - (U''_{xy})^2 = 0.$$

Cobb-Douglas Utility (柯布-道格拉斯)

Example

$U(x,y) = x^a y^b$, assuming $a > 0$ and $b > 0$. Check the parametric conditions of a, b such that (1) diminishing MRS and (2) concavity are satisfied.

- For an indifference curve $u_0 = x^a y^b$,

$$u_0^{\frac{1}{b}} = x^{\frac{a}{b}} y \Rightarrow y(x) = u_0^{\frac{1}{b}} x^{-\frac{a}{b}}$$

For MRS,

$$\frac{dMRS}{dx} = \frac{d}{dx}\left(\frac{ax^{a-1}y^b}{bx^a y^{b-1}}\right) = \frac{d}{dx}\left(\frac{a}{b}\frac{y(x)}{x}\right) = \frac{d}{dx}\left(\frac{a}{b}u_0^{\frac{1}{b}} x^{-\frac{a+b}{b}}\right) < 0$$

for any positive a and b.

Now check the concavity of $U(x, y) = x^a y^b$

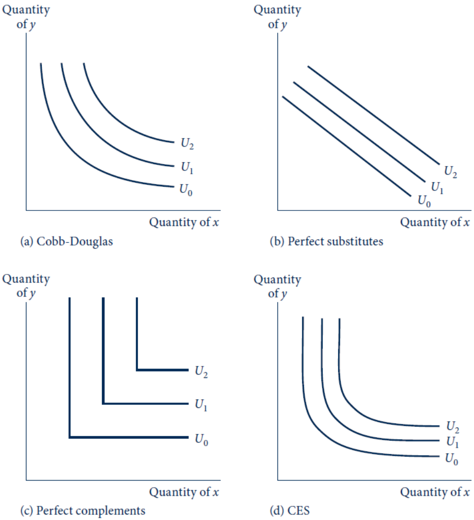
- Second-order derivatives: $U''_{xx} = a(a-1)x^{a-2}y^b$, $U''_{yy} = b(b-1)x^a y^{b-2}$, $U''_{xy} = abx^{a-1}y^{b-1}$

$$H = \begin{bmatrix} a(a-1)x^{a-2}y^b & abx^{a-1}y^{b-1} \\ abx^{a-1}y^{b-1} & b(b-1)x^a y^{b-2} \end{bmatrix}$$

  H is negative definite (concave U) provided that:
  $a(a-1)x^{a-2}y^b < 0 \Leftrightarrow 0 < a < 1$,
  $ab(a-1)(b-1)x^{2a-2}y^{2b-2} - a^2b^2x^{2a-2}y^{2b-2} =$
  $ab(1-a-b)x^{2a-2}y^{2b-2} > 0 \Leftrightarrow ab(1-a-b) > 0$.

- In other words, $U(x, y)$ is concave when $a + b < 1$, i.e., a subset (sufficient condition) for quasi-concavity and diminishing MRS.

Utility Classes that are Typically Used

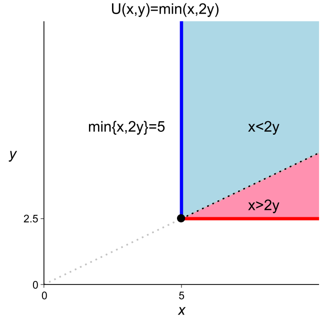


Perfect Substitutes

- $U(x, y) = ax + by$, i.e., U is linear in both x and y
- MRS is a constant: $MRS = \frac{U'_x}{U'_y} = \frac{a}{b}$
- The indifference curve is plotted by solving $y = -\frac{a}{b}x + \frac{u_0}{b}$, fixing a particular utility level at u_0.
- Slope: $-\frac{a}{b} = -MRS$; intercept: $\frac{u_0}{b}$

Perfect Complements

- $U(x, y) = \min\{ax, by\}$. Let's consider $a = 1$ and $b = 2$.
- The utility level is determined by the minimum of x and $2y$
- Fixing a particular utility level $u_0 = 5$:
  - If $x > 2y \Leftrightarrow y < \frac{1}{2}x$, then $u_0 = 5 = 2y \Rightarrow y = \frac{5}{2}$.
  - If $x < 2y \Leftrightarrow y > \frac{1}{2}x$, then $u_0 = 5 = x \Rightarrow x = 5$
  - If $x = 2y \Leftrightarrow y = \frac{1}{2}x$, then $u_0 = 5 = x = 2y \Rightarrow x = 5, y = \frac{5}{2}$.



CES Utility (常替代弹性)

Constant elasticity of substitution: $U(x,y) = (ax^\rho + by^\rho)^{\frac{1}{\rho}}$

- Cobb-Douglas is a special case for $\rho \to 0$ and $a + b = 1$:

$$\ln(U) = \frac{\ln(ax^\rho + by^\rho)}{\rho}$$
$$\lim_{\rho\to 0} \ln(U) = \lim_{\rho\to 0} \frac{ax^\rho \ln(x) + by^\rho \ln(y)}{ax^\rho + by^\rho} = a\ln(x) + b\ln(y)$$
$$\ln(U) = \ln(x^a y^b) \Leftrightarrow U = x^a y^b$$

- Perfect Substitute is a special case for $\rho = 1$.
- If $\rho \to -\infty$
  - $x = y \Leftrightarrow U = (a+b)^{1/\rho} x \to x$
  - $x < y \Leftrightarrow U = \left[x^\rho \left(a + b\left(\frac{y}{x}\right)^\rho\right)\right]^{\frac{1}{\rho}} \to x$
  - similarly, $x > y \Leftrightarrow U = \left[y^\rho \left(a\left(\frac{x}{y}\right)^\rho + b\right)\right]^{\frac{1}{\rho}} \to y$

    Therefore, $\lim_{\rho\to-\infty} U = \min\{x,y\}$.

Budget Set (预算集)

Definition

Let I be the endowed numéraire (e.g., money or income). A budget set for two goods (x, y) is given by

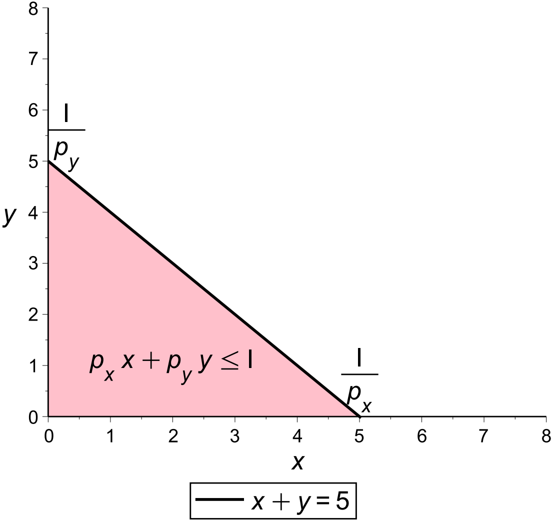
$$\mathcal{B} = \{x, y | p_x x + p_y y \leq I\},$$

where p_x and p_y is the price of good x and y, respectively.

Plot the budget "line" (预算线):

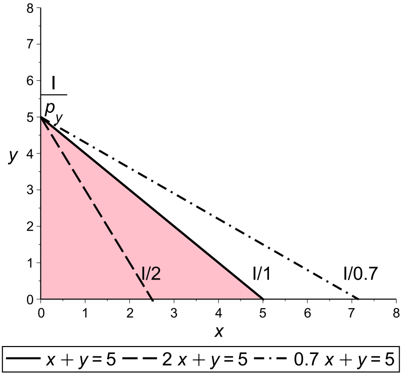
- $p_x x + p_y y = I$: the boundary of the budget set
- $y = -\frac{p_x}{p_y} x + \frac{I}{p_y}$: solve y as a function of x
- slope: $-\frac{p_x}{p_y}$; intercept (at y axis): $\frac{I}{p_y}$; evaluated at $y = 0$, $x = \frac{I}{p_x}$.

Example: $p_x = p_y = 1, I = 5$



Rotation due to a Change in Price

- If p_x increases from 1 to 2 (more expensive), the budget line rotates inward: $y = 0 \Rightarrow \frac{I}{p_x} = \frac{I}{2} < \frac{I}{1}$
- If p_x decreases from 1 to 0.7 (cheaper), the budget line rotates outward: $y = 0 \Rightarrow \frac{I}{p_x} = \frac{I}{0.7} > \frac{I}{1}$



Shift due to a Change in Income

- If I increases from 5 to 6 (richer), the budget line shifts outward: $\frac{I}{p_{x,y}} = \frac{6}{1} > \frac{5}{1}$
- If I decreases from 5 to 3 (poorer), the budget line shifts inward: $\frac{I}{p_{x,y}} = \frac{3}{1} < \frac{5}{1}$