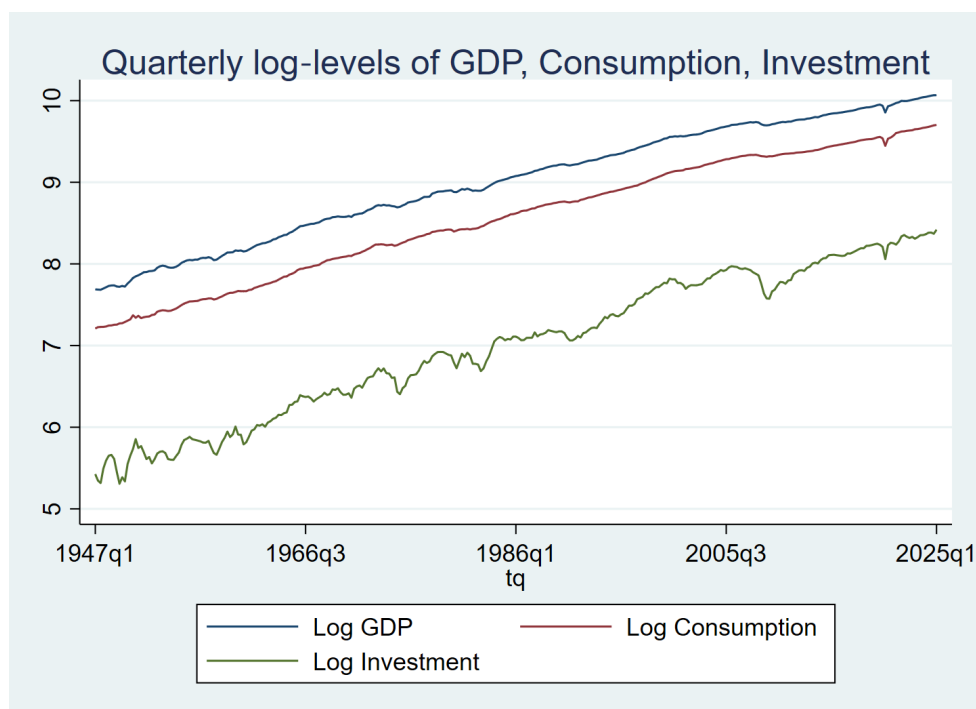# PS4 Solutions
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1 Problem 1

Solution (a).

All three series exhibit a strong, almost linear upward trajectory in log-levels, indicating persistent positive growth over the entire sample. $\ln I_t$ (investment) has the steepest slope, $\ln Y_t$ (GDP) is intermediate, and $\ln C_t$ (consumption) has the gentlest slope. This ordering implies

$$\text{Avg. growth:} \quad I > Y > C.$$

The amplitude of deviations appears somewhat smaller after the mid-1980s (the Great Moderation), especially for GDP and consumption. Investment's trend also smooths after 1984 but remains markedly more volatile than the other two.

- *Investment* shows the largest cyclical swings around its trend (notice the deeper troughs in the early 1980s, the 2008-09 crisis, and the 2020 COVID collapse).

- *GDP* displays moderate deviations, with visible slowdowns at known recessions (1974-75, 1980-82, 2008-09, 2020 Q2).

- *Consumption* is the smoothest, with only mild dips in severe downturns (2008-09, 2020 COVID collapse).

```stata
import delimited using "GDP.csv", varnames(1) clear
gen date = date(observation_date, "YMD")
format date %td
gen tq = qofd(date)
format tq %tq
rename gdpc1 GDP
keep tq GDP
save gdp_temp, replace

import delimited using "RPCE.csv", varnames(1) clear
gen date = date(observation_date, "YMD")
format date %td
gen tq = qofd(date)
format tq %tq
rename pcecc96 PCE
keep tq PCE
save pce_temp, replace

import delimited using "GPDI.csv", varnames(1) clear
gen date = date(observation_date, "YMD")
format date %td
gen tq = qofd(date)
format tq %tq
rename gpdic1 GPDI
keep tq GPDI

merge m:1 tq using gdp_temp
drop _merge
merge m:1 tq using pce_temp
drop _merge

tsset tq, quarterly
gen lnGDP  = ln(GDP)
gen lnPCE  = ln(PCE)
gen lnGPDI = ln(GPDI)

set scheme s2color

tsline lnGDP lnPCE lnGPDI, ///
    legend(order(1 "Log GDP" 2 "Log Consumption" 3 "Log Investment")) ///
    title("Quarterly log-levels of GDP, Consumption, Investment")

```

```
43  graph export "a.png", as(png) replace
44
```

Solution (b).

Let $\mathbf{X} = (\mathbf{1}, \mathbf{t})$ be the $N \times 2$ design matrix, with rows $(1, t)$, and $\mathbf{y} = (y_1, \ldots, y_N)'$. Then the OLS estimator is

$$
\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \frac{1}{S_{tt}} \begin{pmatrix} \sum_t t^2 & -\sum_t t \\ -\sum_t t & \sum_t 1 \end{pmatrix} \begin{pmatrix} \sum_t y_t \\ \sum_t t\, y_t \end{pmatrix},
$$

where

$$
\bar{t} = \frac{1}{N} \sum_{t=1}^{N} t, \quad \bar{y} = \frac{1}{N} \sum_{t=1}^{N} y_t, \quad S_{tt} = \sum_{t=1}^{N} (t - \bar{t})^2.
$$

Equivalently,

$$
\hat{\beta}_2 = \frac{\sum_{t=1}^{N}(t - \bar{t})(y_t - \bar{y})}{\sum_{t=1}^{N}(t - \bar{t})^2}, \quad \hat{\beta}_1 = \bar{y} - \hat{\beta}_2 \bar{t}.
$$

The fitted values and residuals are

$$
\hat{y}_t = \hat{\beta}_1 + \hat{\beta}_2 t, \qquad \hat{u}_t = y_t - \hat{y}_t.
$$

y_t is the log of three variables: GDP, PCE, and GPDI. The estimated models are as below:

Window	Series	$\hat{\beta}_1$	$\hat{\beta}_2$ (per qtr)	Annualized Growth (%)
1965 Q1–2006 Q4	ln GDP	7.8509	0.0077707	3.11
	ln PCE	7.3017	0.0083520	3.34
	ln GPDI	5.4989	0.0100216	4.01
2007 Q1–2019 Q4	ln GDP	8.4997	0.0048888	1.96
	ln PCE	8.0947	0.0049259	1.97
	ln GPDI	5.2006	0.0104206	4.17
2007 Q1–2022 Q2	ln GDP	8.4867	0.0049376	1.98
	ln PCE	8.0394	0.0051370	2.05
	ln GPDI	5.3432	0.0098678	3.95

Table 1: Trend regression estimates and annualized growth rates

```
1  gen t = _n
2
3  * 1) 1965Q1-2006Q4
4  reg lnGDP t if inrange(tq, tq(1965q1), tq(2006q4))
5  predict uhat_GDP_65_06, resid
6  display "GDP 1965-06 annualized % = " _b[t]*4*100
7  reg lnPCE t if inrange(tq, tq(1965q1), tq(2006q4))
8  predict uhat_PCE_65_06, resid
```

```
 9  display "PCE 1965-06 annualized % = " _b[t]*4*100
10  reg lnGPDI t if inrange(tq, tq(1965q1), tq(2006q4))
11  predict uhat_GPDI_65_06, resid
12  display "GPDI 1965-06 annualized % = " _b[t]*4*100
13
14  * 2) 2007Q1-2019Q4
15  reg lnGDP t if inrange(tq, tq(2007q1), tq(2019q4))
16  predict uhat_GDP_07_19, resid
17  display "GDP 2007-19 annualized % = " _b[t]*4*100
18
19  reg lnPCE t if inrange(tq, tq(2007q1), tq(2019q4))
20  predict uhat_PCE_07_19, resid
21  display "PCE 2007-19 annualized % = " _b[t]*4*100
22
23  reg lnGPDI t if inrange(tq, tq(2007q1), tq(2019q4))
24  predict uhat_GPDI_07_19, resid
25  display "GPDI 2007-19 annualized % = " _b[t]*4*100
26
27  * 3) 2007Q1-2022Q2
28  reg lnGDP t if inrange(tq, tq(2007q1), tq(2022q2))
29  predict uhat_GDP_07_22, resid
30  display "GDP 2007-22 annualized % = " _b[t]*4*100
31
32  reg lnPCE t if inrange(tq, tq(2007q1), tq(2022q2))
33  predict uhat_PCE_07_22, resid
34  display "PCE 2007-22 annualized % = " _b[t]*4*100
35
36  reg lnGPDI t if inrange(tq, tq(2007q1), tq(2022q2))
37  predict uhat_GPDI_07_22, resid
38  display "GPDI 2007-22 annualized % = " _b[t]*4*100
39
```

Solution (c).

$\hat{\beta}_2$ measures the *average quarterly* change in $\ln y_t$. The *approximate quarterly growth rate* is $\hat{\beta}_2 \times 100\%$. The *exact annualized growth rate* is

$$\left(e^{4\hat{\beta}_2} - 1\right) \times 100\% \approx 4\hat{\beta}_2 \times 100\%,$$

- **1965-2006 period:** GDP grew at approximately 3.11% per year, PCE at 3.34%, and GPDI at 4.01%. Investment trend outpaced both output and consumption over this long sample.

- **2007-2019 period:** After the financial crisis, GDP and consumption trend growth fell sharply to about 1.96%-1.97% annually, nearly halving. In contrast, investment trend remained robust at 4.17% per year, possibly reflecting policy support or structural shifts.

- **2007-2022 period:** Including the COVID-19 shock, GDP and PCE annualized growth rates stabilized around 2% (1.98% and 2.05%), while investment eased slightly to 3.95% but still outperformed output and consumption. This suggests that even amid extreme cyclical volatility, the long-run investment trend remained comparatively strong.

Solution (d).

The (kth) sample autocorrelation of $\{\hat{u}_t\}$ is

$$\hat{\rho}(k) = \frac{\displaystyle\sum_{t=k+1}^{N} \hat{u}_t \, \hat{u}_{t-k}}{\displaystyle\sum_{t=1}^{N} \hat{u}_t^2}, \qquad k = 0, 1, 2, \ldots.$$

We compute $\hat{\rho}(k)$ for $k = 1, \ldots, K$ (here $K = 8$) in each subsample to assess the persistence of residual cycles.

Table 2: Sample autocorrelations (AC) of detrended residuals by lag

Window	Series	Lag 1	Lag 2	Lag 3	Lag 4	Lag 5	Lag 6	Lag 7	Lag 8
	GDP	0.9207	0.8038	0.6640	0.5231	0.3799	0.2457	0.1218	0.0211
1965 Q1–2006 Q4	PCE	0.9463	0.8724	0.7853	0.6864	0.5883	0.4880	0.3892	0.2972
	GPDI	0.9143	0.8018	0.6805	0.5437	0.4146	0.3171	0.2192	0.1401
	GDP	0.8952	0.7492	0.5887	0.4183	0.2782	0.1379	0.0236	-0.0304
2007 Q1–2019 Q4	PCE	0.9183	0.8167	0.6955	0.5650	0.4382	0.3085	0.2057	0.1304
	GPDI	0.8844	0.7045	0.4978	0.2961	0.1250	-0.0311	-0.1723	-0.2628
	GDP	0.6975	0.5012	0.3325	0.1832	0.0808	-0.0320	-0.0779	-0.0736
2007 Q1–2022 Q2	PCE	0.7061	0.5167	0.3387	0.1636	0.0639	-0.0153	-0.0416	-0.0518
	GPDI	0.8437	0.6556	0.4650	0.2928	0.1289	-0.0343	-0.1699	-0.2362

```
1  corrgram uhat_GDP_65_06 if inrange(tq, tq(1965q1), tq(2006q4)), lags(8)
2  corrgram uhat_PCE_65_06 if inrange(tq, tq(1965q1), tq(2006q4)), lags(8)
3  corrgram uhat_GPDI_65_06 if inrange(tq, tq(1965q1), tq(2006q4)), lags(8)
4
5  corrgram uhat_GDP_07_19 if inrange(tq, tq(2007q1), tq(2019q4)), lags(8)
6  corrgram uhat_PCE_07_19 if inrange(tq, tq(2007q1), tq(2019q4)), lags(8)
7  corrgram uhat_GPDI_07_19 if inrange(tq, tq(2007q1), tq(2019q4)), lags(8)
8
9  corrgram uhat_GDP_07_22 if inrange(tq, tq(2007q1), tq(2022q2)), lags(8)
10 corrgram uhat_PCE_07_22 if inrange(tq, tq(2007q1), tq(2022q2)), lags(8)
11 corrgram uhat_GPDI_07_22 if inrange(tq, tq(2007q1), tq(2022q2)), lags(8)
12
```

Solution (e).
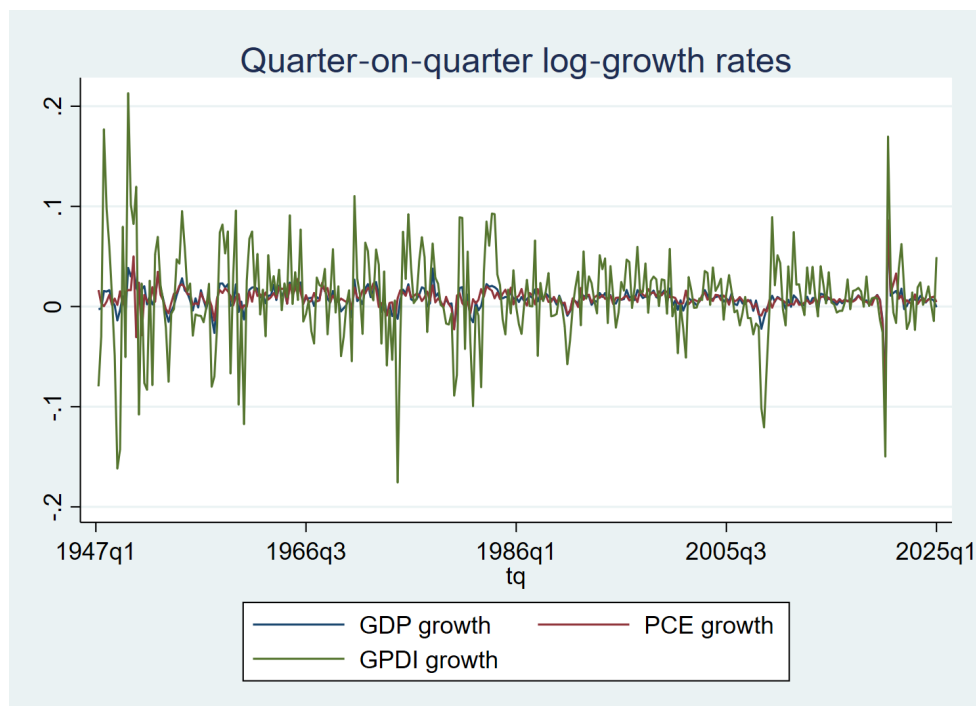
- **Volatility patterns:** Investment (green) shows the largest fluctuations, frequently exceeding ± 0.10 (i.e. $\pm 10\%$ annualized). Real GDP (blue) is smoother yet more volatile than Real Consumption (red), which remains within ± 0.01 per quarter.

- **Business-cycle troughs and peaks:** All three series co-move, with pronounced downturns in 2008 Q4 (Financial Crisis) and especially 2020 Q2 (COVID shock), where Investment plunges below -0.15.

- **Great Moderation vs. COVID era:** From the mid-1980s to 2007, the amplitudes of Real GDP and Real Consumption narrowed (Great Moderation). Post-2007 and dramatically in 2020, their volatility increases again, whereas investment retains structurally high cyclicality throughout.

- **Economic interpretation:** The stability of consumption reflects strong automatic stabilizers and consumption smoothing. Investment's high cyclicality underscores its role as the accelerator in business cycles, amplifying both recessions and recoveries.

```stata
gen dgdp  = D.lnGDP
gen dpce  = D.lnPCE
gen dgpdi = D.lnGPDI

tsline dgdp dpce dgpdi, ///
    legend(order(1 "GDP growth" 2 "Cons growth" 3 "Inv growth")) ///
    title("Quarter-on-quarter log-growth rates")

graph export "e.png", as(png) replace
```

Solution (f).

For each series $g_t \in \{\Delta y_t, \Delta c_t, \Delta i_t\}$ in a given subsample of length M, we compute

$$\bar{g} = \frac{1}{M} \sum_{t=1}^{M} g_t, \qquad s_g = \sqrt{\frac{1}{M-1} \sum_{t=1}^{M} (g_t - \bar{g})^2}.$$

These compare the *average realized growth* $\bar{g}$ with the *trend-based* quarterly slope $\hat{\beta}_2$, and measure cyclical volatility via s_g.

Window	Series	Obs	Mean of $\Delta \ln y$	Std. of $\Delta \ln y$	$\hat{\beta}_2$ (per qtr)
1965 Q2–2006 Q4	GDP	167	0.0080837	0.0083304	0.0077707
	PCE	167	0.0087461	0.0068057	0.0083520
	GPDI	167	0.0104904	0.0394865	0.0100216
2007 Q2–2019 Q4	GDP	51	0.0045828	0.0061137	0.0048888
	PCE	51	0.0045793	0.0043225	0.0049259
	GPDI	51	0.0058344	0.0346509	0.0104206
2007 Q2–2022 Q2	GDP	61	0.0045452	0.0159656	0.0049376
	PCE	61	0.0050521	0.0175900	0.0051370
	GPDI	61	0.0064772	0.0444316	0.0098678

Comparison to part (b):

- *1965 Q2-2006 Q4:* The sample means of $\Delta \ln y$ for GDP (0.00808), PCE (0.00875) and GPDI (0.01049) are almost identical to the estimated quarterly trend slopes $\hat{\beta}_2$ (0.0077707, 0.0083520, 0.0100216). This confirms that, over the long sample, the linear-trend regression accurately captures the average growth rate.

- *2007 Q2-2019 Q4:* GDP and PCE mean growth rates ($\approx$0.00458) lie slightly below their $\hat{\beta}_2$ ($\approx$0.00489 and 0.00493), reflecting that downturns (2008-09 crisis) pull the sample average below the fitted trend. For GPDI, the mean (0.00583) is markedly below its trend slope (0.01042), since investment experienced large negative shocks that the OLS trend, minimizing squared errors, spreads more evenly across the sample.

- *2007 Q2-2022 Q2:* Including the COVID-19 shock further widens the gap: GDP and PCE means (0.00455, 0.00505) remain below their slopes (0.00494, 0.00514), and investment's mean (0.00648) stays well under its slope (0.00987). This again shows that severe downturns pull down the simple average growth below the fitted linear trend.

```
1  * 1965Q2-2006Q4
2  sum dgdp dpce dgpdi if inrange(tq, tq(1965q2), tq(2006q4))
3
```

```
4  * 2007Q2-2019Q4
5  sum dgdp dpce dgpdi if inrange(tq, tq(2007q2), tq(2019q4))
6
7  * 2007Q2-2022Q2
8  sum dgdp dpce dgpdi if inrange(tq, tq(2007q2), tq(2022q2))
9
```

Solution (g).

Window	Series	Mean of $\Delta \ln y$	Std. dev. of $\Delta \ln y$
1947 Q1–1983 Q4	GDP	0.0087338	0.0117128
	PCE	0.0089202	0.0100335
	GPDI	0.0104315	0.0602435
1984 Q1–2006 Q4	GDP	0.0080725	0.0051354
	PCE	0.0086507	0.0048490
	GPDI	0.0107001	0.0267834

- *Mean growth rates:* GDP's average growth rises slightly from 0.8733% to 0.8073%; PCE falls marginally from 0.8920% to 0.8651%; GPDI increases from 1.043% to 1.0700%. Overall, the mean growth rates remain essentially unchanged in magnitude.

- *Volatility:* All three series exhibit a dramatic reduction in standard deviation.

  - GDP's σ falls from 1.17% to 0.51%.

  - PCE's σ falls from 1.00% to 0.48%.

  - GPDI's σ falls from 6.02% to 2.68%.

  This confirms the *Great Moderation* after 1984, while average growth remained stable, business-cycle volatility was substantially dampened.

```
1  * 1947Q2-1983Q4
2  sum dgdp dpce dgpdi if inrange(tq, tq(1947q2), tq(1983q4))
3
4  * 1984Q1-2006Q4
5  sum dgdp dpce dgpdi if inrange(tq, tq(1984q1), tq(2006q4))
6
```

2   Problem 2

Solution (a).

Weak Stationarity: A stochastic process $\{y_t\}$ is weakly stationary if

- its mean is the same at every time period, $\mu_t = \mu \quad \forall t$;

- every autocovariance $\gamma_{h,t} = \text{Cov}(y_t, y_{t-h})$ depends only on the displacement h, not on the time period t, or equivalently, $\gamma_{h,t} = \gamma_h, \quad \forall t$

Strict Stationarity: A process is strictly stationary if for any values $h_1, \ldots, h_k$, the joint distribution of $(y_t, y_{t-h_1}, \ldots, y_{t-h_k})$ depends only on the intervals separating the dates (displacements) $h_1, \ldots, h_k$ and not on the date t itself, i.e.

$$f_{Y_t, Y_{t-h_1}, \ldots, Y_{t-h_k}} = f_{Y_\tau, Y_{\tau-h_1}, \ldots, Y_{\tau-h_k}} \quad \forall \tau.$$

Weak Stationarity

A process $\{y_t\}$ is weakly stationary if

$$\mathbb{E}[y_t] = \mu \quad \forall t, \quad \text{Cov}(y_t, y_{t-k}) = \gamma_k \text{ depends only on } k.$$

Here

$$\mathbb{E}[y_t] = \sum_{j=0}^{3} \psi_j \, \mathbb{E}[u_{t-j}] = 0,$$

so the mean is constant. Next,

$$\text{Cov}(y_t, y_{t-k}) = \mathbb{E}\big[y_t \, y_{t-k}\big] = \sum_{j=0}^{3} \sum_{i=0}^{3} \psi_j \psi_i \, \mathbb{E}\big[u_{t-j} u_{t-k-i}\big].$$

Since $\mathbb{E}[u_s u_r] = 0$ for $s \neq r$, only terms with $t - j = t - k - i$ survive, so $\text{Cov}(y_t, y_{t-k})$ depends on k alone. Hence $\{y_t\}$ is weakly stationary.

Strict Stationarity

If $\{u_t\}$ is strictly stationary (e.g. i.i.d.), then any finite-order linear filter $\sum_{j=0}^{3} \psi_j \, u_{t-j}$ yields a strictly stationary $\{y_t\}$. Thus under i.i.d. innovations, y_t is strictly stationary.

Solution (b).
Define $\gamma_k = \text{Cov}(y_t, y_{t-k})$. With ψ_j as above,

$$\gamma_k = \sum_{j=0}^{3} \sum_{i=0}^{3} \psi_j \psi_i \, E[u_{t-j} u_{t-k-i}] = \sum_{j=0}^{3} \psi_j \, \psi_{j-k},$$

interpreting $\psi_m = 0$ for $m > 3$. From the question, we know that:

$$\sigma^2 = \mathbb{V}[u_t] = \mathbb{E}[u_t^2] - \mathbb{E}[u_t]^2 = \mathbb{E}[u_t^2] = 1$$

Thus:

$$\gamma_0 = \sum_{j=0}^{3} \psi_j^2 = 1^2 + (-2.4)^2 + 0.8^2 + (-0.4)^2 = 7.56,$$

$$\gamma_1 = \psi_0\psi_1 + \psi_1\psi_2 + \psi_2\psi_3 = (1 \cdot (-2.4) + (-2.4) \cdot 0.8 + 0.8 \cdot (-0.4)) = -4.64,$$

$$\gamma_2 = \psi_0\psi_2 + \psi_1\psi_3 = (1 \cdot 0.8 + (-2.4) \cdot (-0.4)) = 1.76,$$

$$\gamma_3 = \psi_0\psi_3 = (1 \cdot (-0.4)) = -0.4,$$

$$\gamma_k = 0 \quad \text{for } |k| \geq 4.$$

Solution (c).

Consider

$$S_T = \sum_{t=1}^{T} y_t, \qquad V_T = \mathbb{V}[S_T] = \sum_{i=1}^{T}\sum_{j=1}^{T} \gamma_{i-j}.$$

Re-index $h = j - i$:

$$V_T = \sum_{h=-(T-1)}^{T-1} (T - |h|)\,\gamma_h.$$

Hence

$$\mathbb{V}\left[\tfrac{1}{\sqrt{T}}S_T\right] = \frac{V_T}{T} = \gamma_0 + 2\sum_{h=1}^{T-1}\left(1 - \tfrac{h}{T}\right)\gamma_h.$$

As $T \to \infty$, $\frac{h}{T} \to 0$ for fixed h, and $\gamma_h = 0$ for $h \geq 4$, so

$$\lim_{T\to\infty} \mathbb{V}\left[\tfrac{1}{\sqrt{T}}S_T\right] = \gamma_0 + 2(\gamma_1 + \gamma_2 + \gamma_3) = 7.56 + 2(-4.64 + 1.76 - 0.4) = 1.$$

Solution (d).

Assume $\{u_t\}$ is i.i.d. with $\mathbb{E}[u_t] = 0$, $\mathbb{V}[u_t] = \sigma^2$.

Strict Stationarity

$x_t = f(u_t, u_{t-4})$ depends on two i.i.d. draws; hence its joint distributions do not change with shifts in t. So $\{x_t\}$ is strictly stationary.

Ergodicity

An i.i.d. sequence is ergodic and any measurable function of it remains ergodic. Thus $\{x_t\}$ is ergodic.

White-Noise Properties

- $\mathbb{E}[x_t] = \mathbb{E}[u_t]\mathbb{E}[u_{t-4}] = 0.$

- For $k \neq 0$,
$$\text{Cov}(x_t, x_{t-k}) = \mathbb{E}[u_t u_{t-4} u_{t-k} u_{t-k-4}] = 0,$$

  since among the four factors at least one is independent with zero mean.

- For $k = 0$,
$$\mathbb{V}[x_t] = \mathbb{E}[u_t^2 u_{t-4}^2] = \sigma^4,$$

  which is constant in t.

Hence $\{x_t\}$ is zero-mean uncorrelated white noise, but not strong white noise, since x_t and x_{t+4} share u_t.

Solution (e).

We have:
$$\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} x_t\right] = \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}[x_t] = 0,$$
and
$$\mathbb{V}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} x_t\right] = \sum_{h=-(T-1)}^{T-1}(T - |h|)\gamma(h)$$

where $\gamma(h) = \text{Cov}(x_t, x_{t-h})$. We know that $\gamma(h) = 0$ for $h \neq 0$ and $\gamma(0) = \mathbb{E}[u_t^2 u_{t-4}^2] = \sigma^4$. Thus:

$$\mathbb{V}\left[\frac{1}{\sqrt{T}}S_T\right] = \frac{1}{T}T\gamma(0) = \sigma^4.$$