Macroeconomics A; EI056

Short problems

Cédric Tille

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1 Consumption allocation

1.1 Optimal choice of a specific variety

Question: The consumer wants to reach the highest value of the consumption basket C which is a geometric average of consumption of various varieties indexed by j (distributed over the unit interval $j \in [0,1]$). Consumption of variety j is denoted by C_j. The basket is:

$$C = \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}}$$

The consumer minimizes the total spending $\int_0^1 P_j C_j dj$ subject to the basket being at a target value C. Show that the first-order condition with respect to consumption of various varieties indexed by j is (is the multiplier on the constraint):

$$P_j = \lambda \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{1}{\theta-1}} [C_j]^{-\frac{1}{\theta}}$$

Answer: The Lagrangian is:

$$\mathcal{L} = \int_0^1 P_j C_j dj + \lambda \left[C - \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} \right]$$

We take the first-order condition with respect to C_j:

$$\begin{aligned}
0 &= \frac{\partial \mathcal{L}}{\partial C_j} \\
0 &= P_j - \lambda \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}-1} [C_j]^{\frac{\theta-1}{\theta}-1} \\
0 &= P_j - \lambda \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{1}{\theta-1}} [C_j]^{-\frac{1}{\theta}} \\
P_j &= \lambda \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{1}{\theta-1}} [C_j]^{-\frac{1}{\theta}}
\end{aligned}$$

1.2 Total spending

Question: We denote total spending by $PC = \int_0^1 P_j C_j dj$. From the result above, show that: $P = \lambda$.

Answer: We multiply the first-order condition by C_j and take the integral over all varieties:

$$\begin{aligned}
P_j &= \lambda \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{1}{\theta-1}} [C_j]^{-\frac{1}{\theta}} \\
P_j C_j &= \lambda \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{1}{\theta-1}} [C_j]^{-\frac{1}{\theta}+1} \\
P_j C_j &= \lambda \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{1}{\theta-1}} [C_j]^{\frac{\theta-1}{\theta}} \\
\int_0^1 P_j C_j dj &= \int_0^1 \left\{ \lambda \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{1}{\theta-1}} [C_j]^{\frac{\theta-1}{\theta}} \right\} dj \\
\int_0^1 P_j C_j dj &= \lambda \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{1}{\theta-1}} \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right] \\
\int_0^1 P_j C_j dj &= \lambda \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{1}{\theta-1}+1} \\
\int_0^1 P_j C_j dj &= \lambda \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} \\
\int_0^1 P_j C_j dj &= \lambda C \\
PC &= \lambda C \\
P &= \lambda
\end{aligned}$$

1.3 Demand for a specific variety

Question: Using the results so far show that the demand for C_j is:

$$C_j = \left[\frac{P_j}{P} \right]^{-\theta} C$$

Answer: The optimality condition with respect to C_j can be rewritten as:

$$\begin{aligned}
P_j &= \lambda \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{1}{\theta-1}} [C_j]^{-\frac{1}{\theta}} \\
P_j &= P \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}\frac{1}{\theta}} [C_j]^{-\frac{1}{\theta}} \\
P_j &= P \left\{ \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}} dj \right]^{\frac{\theta}{\theta-1}} \right\}^{\frac{1}{\theta}} [C_j]^{-\frac{1}{\theta}}
\end{aligned}$$

$$
\begin{aligned}
P_j &= P\{C\}^{\frac{1}{\theta}}[C_j]^{-\frac{1}{\theta}} \\
[P_j]^{\theta} &= [P]^{\theta} C [C_j]^{-1} \\
C_j &= [P_j]^{-\theta}[P]^{\theta} C \\
C_j &= \left[\frac{P_j}{P}\right]^{-\theta} C
\end{aligned}
$$

1.4 Price index

Question: Using the results and the definition of the consumption basket, show that::

$$
P = \left[\int_0^1 [P_j]^{1-\theta}\, dj\right]^{\frac{1}{1-\theta}}
$$

Answer: We start from the consumption basket and use the demand for specific varieties:

$$
\begin{aligned}
C &= \left[\int_0^1 [C_j]^{\frac{\theta-1}{\theta}}\, dj\right]^{\frac{\theta}{\theta-1}} \\
C &= \left[\int_0^1 \left[\left[\frac{P_j}{P}\right]^{-\theta} C\right]^{\frac{\theta-1}{\theta}} dj\right]^{\frac{\theta}{\theta-1}} \\
C &= \left[\int_0^1 \left[\frac{P_j}{P}\right]^{-\theta\frac{\theta-1}{\theta}} [C]^{\frac{\theta-1}{\theta}}\, dj\right]^{\frac{\theta}{\theta-1}} \\
C &= \left[[C]^{\frac{\theta-1}{\theta}} \int_0^1 \left[\frac{P_j}{P}\right]^{-(\theta-1)} dj\right]^{\frac{\theta}{\theta-1}} \\
C &= C\left[\int_0^1 \left[\frac{P_j}{P}\right]^{-(\theta-1)} dj\right]^{\frac{\theta}{\theta-1}} \\
1 &= \left[\int_0^1 \left[\frac{P_j}{P}\right]^{-(\theta-1)} dj\right]^{\frac{\theta}{\theta-1}} \\
1 &= \int_0^1 \left[\frac{P_j}{P}\right]^{-(\theta-1)} dj \\
1 &= \int_0^1 \left[\frac{P_j}{P}\right]^{1-\theta} dj \\
1 &= \left[\frac{1}{P}\right]^{1-\theta} \int_0^1 [P_j]^{1-\theta}\, dj \\
[P]^{1-\theta} &= \int_0^1 [P_j]^{1-\theta}\, dj \\
P &= \left[\int_0^1 [P_j]^{1-\theta}\, dj\right]^{\frac{1}{1-\theta}}
\end{aligned}
$$

2 Flexible price allocation

2.1 Real marginal cost

Question: Consider the New Keynesian model with all prices flexible. The labor supply and the Euler conditions are:

$$\begin{aligned}\frac{W_t}{P_t}(C_t)^{-\sigma} &= \chi(N_t)^{\eta} \\ \frac{1}{P_t}(C_t)^{-\sigma} &= \beta(1+i_t)E_t\left[\frac{1}{P_{t+1}}(C_{t+1})^{-\sigma}\right]\end{aligned}$$

The firm producing variety j uses the following technology (N_{j,t} is the labor it uses, Z_t is a productivity parameter):

$$Y_{j,t} = Z_t(N_{j,t})^a$$

The firm pays the wage W_t. Show that the real marginal cost is:

$$\Psi_{j,t} = \frac{1}{a}\frac{W_t}{P_t}(Z_t)^{-\frac{1}{a}}(Y_{j,t})^{\frac{1-a}{a}}$$

Answer: The total cost of production is the wage bill $W_tN_{j,t}$. Using the technology, it is written as:

$$W_tN_{j,t} = W_t\left(\frac{Y_{j,t}}{Z_t}\right)^{\frac{1}{a}}$$

To get the marginal cost, we take the derivative with respect to output:

$$\begin{aligned}\Psi_{j,t}^{nominal} &= \frac{\partial}{\partial Y_{j,t}}\left[W_t\left(\frac{Y_{j,t}}{Z_t}\right)^{\frac{1}{a}}\right] \\ \Psi_{j,t}^{nominal} &= \frac{1}{a}W_t\left(\frac{Y_{j,t}}{Z_t}\right)^{\frac{1}{a}-1}\frac{1}{Z_t} \\ \Psi_{j,t}^{nominal} &= \frac{1}{a}W_t(Z_t)^{-\frac{1}{a}}(Y_{j,t})^{\frac{1-a}{a}}\end{aligned}$$

The real marginal cost is simply $\Psi_{j,t} = \Psi_{j,t}^{nominal}/P_t$.

2.2 Optimal price

Question: The firm producing variety j knows that it faces the following demand curve (output of any variety goes solely to consumption:

$$Y_{j,t} = \left[\frac{P_{j,t}}{P_t}\right]^{-\theta}C_t$$

Show that it sets the following real price:

$$\frac{P_{j,t}}{P_t} = \frac{\theta}{\theta-1}\Psi_t$$

Answer: The profits of the firms are:

$$\begin{aligned}
\Pi_{j,t} &= P_{j,t}Y_{j,t} - W_t N_{j,t} \\
\Pi_{j,t} &= P_{j,t}Y_{j,t} - W_t \left(\frac{Y_{j,t}}{Z_t}\right)^{\frac{1}{a}} \\
\Pi_{j,t} &= P_{j,t}\left[\frac{P_{j,t}}{P_t}\right]^{-\theta} C_t - W_t \left(\frac{1}{Z_t}\left[\frac{P_{j,t}}{P_t}\right]^{-\theta} C_t\right)^{\frac{1}{a}} \\
\Pi_{j,t} &= [P_{j,t}]^{1-\theta}\left[\frac{1}{P_t}\right]^{-\theta} C_t - W_t \left(\frac{1}{Z_t}[P_{j,t}]^{-\theta}\left[\frac{1}{P_t}\right]^{-\theta} C_t\right)^{\frac{1}{a}} \\
\Pi_{j,t} &= [P_{j,t}]^{1-\theta}\left[\frac{1}{P_t}\right]^{-\theta} C_t - W_t \left(\frac{1}{Z_t}\left[\frac{1}{P_t}\right]^{-\theta} C_t\right)^{\frac{1}{a}} [P_{j,t}]^{-\frac{\theta}{a}}
\end{aligned}$$

Take the first-order condition with respect to $P_{j,t}$:

$$\begin{aligned}
0 &= \frac{\partial \Pi_{j,t}}{\partial P_{j,t}} \\
0 &= (1-\theta)[P_{j,t}]^{-\theta}\left[\frac{1}{P_t}\right]^{-\theta} C_t - W_t \left(\frac{1}{Z_t}\left[\frac{1}{P_t}\right]^{-\theta} C_t\right)^{\frac{1}{a}}\left(-\frac{\theta}{a}\right)[P_{j,t}]^{-\frac{\theta}{a}-1} \\
0 &= (1-\theta)\left[\frac{P_{j,t}}{P_t}\right]^{-\theta} C_t - W_t \left([P_{j,t}]^{\theta}\frac{1}{Z_t}\left[\frac{P_{j,t}}{P_t}\right]^{-\theta} C_t\right)^{\frac{1}{a}}\left(-\frac{\theta}{a}\right)[P_{j,t}]^{-\frac{\theta}{a}-1} \\
0 &= (1-\theta)\left[\frac{P_{j,t}}{P_t}\right]^{-\theta} C_t - W_t [P_{j,t}]^{\frac{\theta}{a}}\left(\frac{1}{Z_t}\left[\frac{P_{j,t}}{P_t}\right]^{-\theta} C_t\right)^{\frac{1}{a}}\left(-\frac{\theta}{a}\right)[P_{j,t}]^{-\frac{\theta}{a}-1} \\
0 &= (1-\theta)Y_{j,t} - W_t\left(\frac{1}{Z_t}Y_{j,t}\right)^{\frac{1}{a}}\left(-\frac{\theta}{a}\right)[P_{j,t}]^{-1} \\
(\theta-1)Y_{j,t} &= \frac{\theta}{a}W_t\left(\frac{1}{Z_t}Y_{j,t}\right)^{\frac{1}{a}}[P_{j,t}]^{-1} \\
(\theta-1)P_{j,t} &= \frac{\theta}{a}W_t\left(\frac{1}{Z_t}\right)^{\frac{1}{a}}(Y_{j,t})^{\frac{1-a}{a}} \\
(\theta-1)\frac{P_{j,t}}{P_t} &= \theta\frac{1}{a}\frac{W_t}{P_t}\left(\frac{1}{Z_t}\right)^{\frac{1}{a}}(Y_{j,t})^{\frac{1-a}{a}} \\
(\theta-1)\frac{P_{j,t}}{P_t} &= \theta\Psi_{j,t} \\
\frac{P_{j,t}}{P_t} &= \frac{\theta}{\theta-1}\Psi_{j,t}
\end{aligned}$$

2.3 Labor market clearing

Question: As all firms set their prices, they choose the same price ($P_{j,t} = P_t$) and produce the same quantity ($Y_{j,t} = Y_t = Y_t$).

Using the optimal price and the labor supply, show that:

$$Y_t = \left[\frac{a}{\chi}\frac{\theta-1}{\theta}\right]^{\frac{1}{1+\eta+(\sigma-1)a}} (Z_t)^{\frac{1+\eta}{1+\eta+(\sigma-1)a}}$$

which in terms of log deviations from the steady-state (deviations denoted by lower case letters) is:

$$y_t = \frac{1+\eta}{1+\eta+(\sigma-1)\,a} z_t$$

Answer: The optimal price, along with $P_{j,t} = P_t$ implies that:

$$\begin{aligned}
1 &= \frac{\theta}{\theta-1}\Psi_t \\
1 &= \frac{\theta}{\theta-1}\frac{1}{a}\frac{W_t}{P_t}(Z_t)^{-\frac{1}{a}}(Y_t)^{\frac{1-a}{a}}
\end{aligned}$$

The labor supply, along with the technology, implies:

$$\begin{aligned}
\frac{W_t}{P_t}(C_t)^{-\sigma} &= \chi (N_t)^{\eta} \\
\frac{W_t}{P_t}(Y_t)^{-\sigma} &= \chi \left(\left(\frac{Y_t}{Z_t}\right)^{\frac{1}{a}}\right)^{\eta} \\
\frac{W_t}{P_t}(Y_t)^{-\sigma} &= \chi (Y_t)^{\frac{\eta}{a}}\left(\frac{1}{Z_t}\right)^{\frac{\eta}{a}} \\
\frac{W_t}{P_t} &= \chi (Y_t)^{\frac{\eta}{a}+\sigma}\left(\frac{1}{Z_t}\right)^{\frac{\eta}{a}}
\end{aligned}$$

Combining the two relations, we get:

$$\begin{aligned}
1 &= \frac{\theta}{\theta-1}\frac{1}{a}\frac{W_t}{P_t}(Z_t)^{-\frac{1}{a}}(Y_t)^{\frac{1-a}{a}} \\
1 &= \frac{\theta}{\theta-1}\frac{1}{a}\chi (Y_t)^{\frac{\eta}{a}+\sigma}\left(\frac{1}{Z_t}\right)^{\frac{\eta}{a}}(Z_t)^{-\frac{1}{a}}(Y_t)^{\frac{1-a}{a}} \\
1 &= \frac{\theta}{\theta-1}\frac{1}{a}\chi (Z_t)^{-\frac{1+\eta}{a}}(Y_t)^{\frac{1-a}{a}+\frac{\eta}{a}+\sigma} \\
1 &= \frac{\theta}{\theta-1}\frac{1}{a}\chi (Z_t)^{-\frac{1+\eta}{a}}(Y_t)^{\frac{1+\eta}{a}+\sigma-1} \\
(Y_t)^{\frac{1+\eta+(\sigma-1)a}{a}} &= \frac{a}{\chi}\frac{\theta-1}{\theta}(Z_t)^{\frac{1+\eta}{a}} \\
Y_t &= \left[\frac{a}{\chi}\frac{\theta-1}{\theta}\right]^{\frac{1}{1+\eta+(\sigma-1)a}}(Z_t)^{\frac{1+\eta}{1+\eta+(\sigma-1)a}}
\end{aligned}$$

Note that this is linear in logs:

$$ln\,(Y_t) = \frac{1}{1+\eta+(\sigma-1)\,a} ln\left[\frac{a}{\chi}\frac{\theta-1}{\theta}\right] + \frac{1+\eta}{1+\eta+(\sigma-1)\,a} ln\,(Z_t)$$

Hence in log deviations from the steady state we get:

$$y_t = \frac{1+\eta}{1+\eta+(\sigma-1)\,a} z_t$$

2.4 Output dynamics

Question: Show that taking a log approximation of the Euler condition we get:

$$y_t = E_t y_{t+1} - \frac{1}{\sigma} r_t$$

where the real interest rate is (upper bar denotes steady state values):

$$r_t = \frac{i_t - \bar{r}}{1+\bar{r}} - (E_t p_{t+1} - p_t)$$

Answer: The Euler conditions is given by:

$$\frac{1}{P_t}(Y_t)^{-\sigma} = \beta(1+i_t) E_t \left[\frac{1}{P_{t+1}}(Y_{t+1})^{-\sigma}\right]$$

In the steady state the price level is constant, as is output, so the nominal interest rate is equal to the real interest rate and we have (upper bars denote steady state values:

$$1 = \beta(1+\bar{r})$$

The left-hand side of the Euler is approximated as:

$$\begin{aligned}
\frac{1}{P_t}(Y_t)^{-\sigma} &= \frac{1}{\bar{P}}(\bar{Y})^{-\sigma} - \frac{1}{(\bar{P})^2}(\bar{Y})^{-\sigma}(P_t - \bar{P}) - \sigma\frac{1}{\bar{P}}(\bar{Y})^{-\sigma-1}(Y_t - \bar{Y}) \\
\frac{1}{P_t}(Y_t)^{-\sigma} &= \frac{1}{\bar{P}}(\bar{Y})^{-\sigma} - \frac{1}{\bar{P}}(\bar{Y})^{-\sigma}\left(\frac{P_t - \bar{P}}{\bar{P}}\right) - \sigma\frac{1}{\bar{P}}(\bar{Y})^{-\sigma}\left(\frac{Y_t - \bar{Y}}{\bar{Y}}\right) \\
\frac{1}{P_t}(Y_t)^{-\sigma} &= \frac{1}{\bar{P}}(\bar{Y})^{-\sigma}\left[1 - \left(\frac{P_t - \bar{P}}{\bar{P}}\right) - \sigma\left(\frac{Y_t - \bar{Y}}{\bar{Y}}\right)\right] \\
\frac{1}{P_t}(Y_t)^{-\sigma} &= \frac{1}{\bar{P}}(\bar{Y})^{-\sigma}[1 - p_t - \sigma y_t]
\end{aligned}$$

In any specific state of nature at time $t+1$, the right-hand side is approximated as (we drop the state of nature index for brevity):

$$\begin{aligned}
\beta(1+i_t)\frac{1}{P_{t+1}}(Y_{t+1})^{-\sigma} &= \beta(1+\bar{r})\frac{1}{\bar{P}}(\bar{Y})^{-\sigma} + \beta\frac{1}{\bar{P}}(\bar{Y})^{-\sigma}(i_t - \bar{r}) \\
&\quad -\beta(1+\bar{r})\frac{1}{(\bar{P})^2}(\bar{Y})^{-\sigma}(P_{t+1} - \bar{P}) - \beta(1+\bar{r})\sigma\frac{1}{\bar{P}}(\bar{Y})^{-\sigma-1}(Y_{t+1} - \bar{Y}) \\
\beta(1+i_t)\frac{1}{P_{t+1}}(Y_{t+1})^{-\sigma} &= \beta(1+\bar{r})\frac{1}{\bar{P}}(\bar{Y})^{-\sigma} + \beta(1+\bar{r})\frac{1}{\bar{P}}(\bar{Y})^{-\sigma}\frac{i_t - \bar{r}}{1+\bar{r}} \\
&\quad -\frac{1}{\bar{P}}(\bar{Y})^{-\sigma}\left(\frac{P_{t+1} - \bar{P}}{\bar{P}}\right) - \sigma\frac{1}{\bar{P}}(\bar{Y})^{-\sigma}\left(\frac{Y_{t+1} - \bar{Y}}{\bar{Y}}\right) \\
\beta(1+i_t)\frac{1}{P_{t+1}}(Y_{t+1})^{-\sigma} &= \beta(1+\bar{r})\frac{1}{\bar{P}}(\bar{Y})^{-\sigma}\left[1 + \frac{i_t - \bar{r}}{1+\bar{r}} - p_{t+1} - \sigma y_{t+1}\right]
\end{aligned}$$

Combining the two sides (applying expectations on the right side) we get:

$$\begin{aligned}
\frac{1}{P_t}(Y_t)^{-\sigma} &= \beta(1+i_t) E_t \left[\frac{1}{P_{t+1}}(Y_{t+1})^{-\sigma}\right] \\
\frac{1}{\bar{P}}(\bar{Y})^{-\sigma}[1 - p_t - \sigma y_t] &= \beta(1+\bar{r})\frac{1}{\bar{P}}(\bar{Y})^{-\sigma}\left[1 + \frac{i_t - \bar{r}}{1+\bar{r}} - E_t p_{t+1} - \sigma E_t y_{t+1}\right]
\end{aligned}$$

$$
\begin{aligned}
1 - p_t - \sigma y_t &= 1 + \frac{i_t - \bar{r}}{1 + \bar{r}} - E_t p_{t+1} - \sigma E_t y_{t+1} \\
-\sigma y_t &= \frac{i_t - \bar{r}}{1 + \bar{r}} - (E_t p_{t+1} - p_t) - \sigma E_t y_{t+1} \\
-\sigma y_t &= r_t - \sigma E_t y_{t+1} \\
y_t &= E_t y_{t+1} - \frac{1}{\sigma} r_t
\end{aligned}
$$