Macroeconomics A: Problem Set 1

Geneva Graduate Institute — Fall 2024

1 The AK Model

Consider the production function $Y = AK + BL$, where A and B are positive constants.

(a) Is this production function neoclassical?

The production function $Y = AK + BL$ is not neoclassical because it does not satisfy the condition of diminishing marginal returns. Both capital K and labor L have constant marginal products, which are A and B respectively, violating one of the key neoclassical conditions.
Let's check the 3 conditions to have a neoclassical production function.

- CRS condition is satisfied

$$F(\lambda K, \lambda L) \; = \; A(\lambda K) + B(\lambda L) = \lambda(AK + BL) = \lambda F(K, L)$$

- Positive and diminishing marginal returns to K and L.

$$\frac{\partial F}{\partial K} = \; A > 0 \quad ; \quad \frac{\partial F}{\partial L} = \; B > 0 \quad ; \frac{\partial^2 F}{\partial^2 K} = 0 \quad ; \frac{\partial^2 F}{\partial^2 L} \; = 0$$

  The condition of diminishing marginal returns is not satisfied.

- You can also check that the Inada conditions are not satisfied.

(b) Output per person as a function of capital per person

Let k be the capital per person. Then, the production function per person is:

$$y = Ak + B$$

Marginal Product of k: The marginal product of capital per person is constant at A.
Average Product of k: The average product of capital per person is:

$$\text{AP}_k = \frac{Ak + B}{k} = A + \frac{B}{k}$$

The average product of capital decreases as K increases due to the $\frac{B}{K}$ term.

(c) Fundamental equation of the Solow model

The fundamental equation for capital accumulation in the Solow model, considering depreciation rate δ and population growth rate n, is:

$$\Delta K = sf(K, L) - \delta K$$

In per capita terms:

$$\frac{\Delta K}{L} = \frac{sf(K, L)}{L} - \frac{\delta K}{L} = sf(k) - \delta k$$

since we cannot directly write the change in capital stock per unit of labor, we need to do a little trick with the term on the lhs to account also for population growth. Mathematically, it is not true that $\frac{\Delta K}{L} \neq \Delta \left(\frac{K}{L} \right)$
We know that, for small t, Δk approximates the derivative of capital per unit of labor.

$$\Delta k = \Delta \frac{K}{L} = \frac{\Delta K L - k \Delta L}{L^2} = \frac{\Delta K}{L} - \frac{K}{L} \frac{\Delta L}{L}$$

$$\Delta k = \frac{\Delta K}{L} - kn \quad ; \frac{\Delta K}{L} = sf(k) - \delta k$$

Now it's straightforward to see that:

$$\Delta k = sf(k) - (\delta + n)k$$

This equation highlights the net investment after accounting for the new capital needed to keep up with population growth and to replace depreciated capital.
The term $(n + \delta)k$ represents the capital needed to maintain the per capita capital stock in the face of increasing population and wearing out of existing capital.

(d) Steady State of per capita capital and endogenous growth

Again the fundamental equation of the Solow model:

$$\Delta k \;=\; sf(k) - (\delta + n)k$$

In steady state (i.e. $\Delta k = 0$), we can compare investment net of capital depreciation and population growth

$$sf(k) = (\delta + n)k \longrightarrow sAk + sB = (\delta + n)k$$

If $sA < (\delta + n)$ then the model will feature a steady state where there will be no growth in per capita capital. On the contrary if $sA > (\delta + n)$ the model will display endogenous growth. (Try to think through it graphically by plotting $m(k) = sf(k) = s(Ak + B)$ and $q(k) = (\delta + n)k$.

(e) Capital, output and consumption per capita growth with endogenous growth

In the case of endogenous growth the growth rate of the capital stock is equal to:

$$\frac{\Delta K(t)}{K(t)} = \frac{k_{t+1}L_{t+1} - k_t L_t}{k_t L_t} = \frac{k_{t+1} - k_t}{k_t} + \frac{L_{t+1} - L_t}{L_t}$$

where

$$\frac{\Delta K(t)}{K(t)} = s\frac{f(k)}{k} - (\delta + n) + n = s\Big(A + \frac{B}{k}\Big) - \delta$$

The growth rate is positive but decreasing over time as the capital stock grows and it asymptotically reach, as $k \to \infty$, a constant growth rate of $[sA - (\delta)] > 0$.

The growth rate of output per capita can be written as:

$$\frac{\Delta y_t}{y_t} = \frac{y_{t+1} - y_t}{y_t}.$$

Substituting $y(t) = Ak(t) + B$, we get:

$$\frac{\Delta y_t}{y_t} = \frac{Ak_{t+1} + B - (Ak_t + B)}{Ak_t + B} = \frac{A\Delta k_t}{Ak_t + B}.$$

You can see that for large values of k_t the growth rate in output per capita becomes exactly the same as the growth rate in capital per capita.
Similarly, looking at consumption growth per capita:

$$\frac{\Delta c(t)}{c(t)} = \frac{(1-s)\,(Ak(t+1) + B) - (1-s)\,(Ak(t) + B)}{(1-s)\,(Ak(t) + B)}.$$

Factor out $(1 - s)$ from the numerator and denominator:

$$\frac{\Delta c(t)}{c(t)} = \frac{A\,(k(t+1) - k(t))}{Ak(t) + B}.$$

This simplifies to:

$$\frac{\Delta c(t)}{c(t)} = \frac{A\Delta k(t)}{Ak(t) + B}.$$

(f) Long-run growth rate

In the long-run B does not affect the growth rate of this economy and the growth rate of the economy is $sA - (\delta + n) = 0.4 * 1 - (0.08 + 0.02) = 0.3$

Exercise 2

Part (a) Assuming $C(t) = cY(t)$ is not very reasonable since it implies that consumption for a given level of aggregate income would be independent of government spending. Since government spending is financed by taxes, it is more reasonable to assume that higher government spending would reduce consumption to some extent.

As an alternative, we may assume that consumers follow the rule of consuming a constant share of their after-tax income, captured by the functional form:

$$C(t) = c(Y(t) - G(t))$$

Using $G(t) = \sigma Y(t)$, this functional form is also equivalent to:

$$C(t) = (c - c\sigma)Y(t)$$

In Part (b), we assume a more general consumption rule:

$$C(t) = (c - \lambda\sigma)Y(t)$$

with the parameter $\lambda \in [0, 1]$ controlling the response of consumption to increased taxes. The case $\lambda = 0$ corresponds to the extreme case of no response, $\lambda = c$ corresponds to a constant after-tax savings rule, and $\lambda \in [0, 1]$ correspond to other alternatives.

Part (b) The aggregate capital stock in the economy accumulates according to:

$$K(t+1) = I(t) + (1-\delta)K(t)$$

$$= Y(t) - C(t) - G(t) + (1-\delta)K(t)$$
$$= (1 - c - \sigma(1-\lambda))Y(t) + (1-\delta)K(t)$$

where the last line uses $C(t) = (c - \lambda\sigma)Y(t)$ and $G(t) = \sigma Y(t)$. Let $f(k) \equiv \frac{Y(t)}{L} = F(K, 1, A)$ and assume, for simplicity, that there is no population growth. Then dividing the equation by L, we have:

$$k(t+1) = (1 - c - \sigma(1-\lambda))f(k(t)) + (1-\delta)k(t)$$

Given $k(0)$, the preceding equation characterizes the whole equilibrium sequence for the capital-labor ratio $\{k_\sigma(t)\}_{t=0}^{\infty}$ in this model, where we use the subscript σ to refer to the economy with parameter σ for government spending.

We claim that with higher government spending and the same initial $k(0)$, the effective capital-labor ratio would be lower at all $t > 0$, that is:

$$k_\sigma(t) > k_{\sigma'}(t) \quad \text{for all } t, \text{ where } \sigma < \sigma'$$

To prove this claim by induction, note that it is true for $t = 1$, and suppose it is true for some $t \geq 1$. Then, we have:

$$k_\sigma(t+1) = (1 - c - \sigma(1-\lambda))f(k_\sigma(t)) + (1-\delta)k_\sigma(t)$$

$$> (1 - c - \sigma(1-\lambda))f(k_{\sigma'}(t)) + (1-\delta)k_{\sigma'}(t)$$
$$> (1 - c - \sigma'(1-\lambda))f(k_{\sigma'}(t)) + (1-\delta)k_{\sigma'}(t) = k_{\sigma'}(t+1)$$

where the second line uses the induction hypothesis ($k_\sigma(t) > k_(\sigma')(t)$) and the fact that $f(k)$ is increasing in k, and the third line uses $\sigma' > \sigma$. This proves our claim by induction. Intuitively, higher government spending reduces net income and savings in the economy and depresses the equilibrium capital-labor ratio in the Solow growth model.

As in the baseline Solow model, the capital-labor ratio in this economy converges to a unique positive steady-state level k^* characterized by:

$$\frac{f(k_\sigma)}{k_\sigma} = \frac{\delta}{1 - c - \sigma(1-\lambda)}$$

The unique solution k^* is decreasing in σ and increasing in λ since $f(k)/k$ is a decreasing function of k. In the economy with higher government spending (higher σ), the capital-labor ratio is lower at all times, and in particular, is also

lower at the steady state. Also, the more individuals reduce their consumption in response to government spending and taxes (higher λ), the more they save, the higher the capital-labor ratio at all times and, in particular, the higher the steady-state capital-labor ratio.
Think also about it mathematically and graphically :

$$\frac{d}{d\sigma}\frac{\delta}{1-c-\sigma(1-\lambda)} = \frac{\delta(1-\lambda)}{[1-c-\sigma(1-\lambda)]^2} > 0$$

Thus an increase in government spending shifts up the constant line $\frac{\delta}{1-c-\sigma(1-\lambda)}$ and capital must decrease to compensate. Intuitively, a portion of output that would otherwise be saved and invested in capital accumulation is instead redirected to government spending.
Try to do the same for λ

Part (c) In this case, the dynamic equation of capital becomes:

$$k(t+1) = [1-c-(1-(1-\lambda)\sigma+\phi\sigma]f(k(t)) + (1-\delta)k(t)$$

Again, in steady state:

$$\frac{f(k^*)}{k^*} = \frac{\delta}{1-c-\sigma(1-\lambda-\phi)}$$

Since $f(k)/k$ is decreasing in k. With respect to σ, it can be seen that k^* is increasing in σ if $\phi > 1-\lambda$ and decreasing in σ if $\phi < 1-\lambda$.

$$\frac{d}{d\sigma}\frac{\delta}{1-c-\sigma(1-\lambda-\phi)} = \frac{\delta(1-\lambda-\phi)}{[1-c-\sigma(1-\lambda-\phi)]^2}$$

In words, when the share of public investment in government spending (i.e. ϕ) is sufficiently high, in particular higher than the reduction of individuals' savings in response to higher taxes, the steady-state capital-labor ratio will increase as a result of increased government spending. This prediction is not too reasonable since it obtains when the government has a relatively high propensity to save from the tax receipts (high ϕ) and when public consumption falls relatively more in response to taxes (high λ), both of which are not too realistic assumptions.

An alternative is to assume that public investment (such as infrastructure investment) will increase the productivity of the economy. Let us posit a production function $F(K, L, \phi G, A)$, which is increasing in public investment ϕG, and assume, as an extreme case, that F has constant returns to scale in K, L, and public investment ϕG. With this assumption, doubling all the capital (e.g. factories) and the labor force in the economy results in two times the output only if the government also doubles the amount of roads and other necessary public infrastructure. Define:

$$f(k, \phi g) = F(k, 1, \phi g, A)$$

where $g = G/L$. Then, the steady-state capital-labor ratio k^* and government spending per capita g^* are solved by the system of equations:

$$\frac{f(k^*, \phi g^*)}{k^*} = \frac{\delta}{1 - s - \sigma(1 - \lambda)}$$

$$g^* = \sigma f(k^*, \phi g^*)$$

The second equation defines an implicit function $g^*(k^*)$ for government spending in terms of the capital-labor ratio, which can be plugged into the first equation from which k^* can be solved. In this model, k^* is increasing in σ for some choice of parameters. Since some infrastructure is necessary for production, output per capita is 0 when public investment per capita is 0, which implies that k^* is increasing in σ in a neighborhood of $\sigma = 0$. Intuitively, when public infrastructure increases the productivity of the economy, increased government spending might increase the steady-state capital-labor ratio.

Exercise 3: The Green Solow Model

a) Emissions in steady state

From the dynamic equation of capital we have:

$$\Delta k_{t+1} = s(1-\theta)k_t^\alpha - (\delta + n)k_t$$

Now you can think at the solution graphically. $sf(k)$ is monotonically increasing and with $f''(k) < 0$. $(\delta + n)k$ is the usual line we've seen. You can check that the steady state exists and it is unique.

$$s(1-\theta)k^{\alpha *} = (\delta + n)k^* \longrightarrow k^* = \left[\frac{(\delta + n)}{s(1-\theta)}\right]^{\frac{1}{\alpha - 1}}$$

.

Let's now see what happens to emissions E in steady state. We know tha $E_t = a(\theta)\Omega_t Y_t$. Let's now compute the growth rate of emissions in the steady state.

$$g_e = \frac{\Delta E_{t+1}}{E_t} = \frac{a(\theta)\Omega_{t+1}Y_{t+1}}{a(\theta)\Omega_t Y_t} - 1 = \frac{1}{1+g_a}\frac{K_{t+1}^{\alpha}L_{t+1}^{1-\alpha}}{K_t^{\alpha}L_t^{1-\alpha}} = \frac{L_{t+1}}{L_t}\frac{1}{1+g_a}\frac{k_{t+1}^{\alpha}}{k_t^{\alpha}} - 1 = \frac{1+n}{1+g_a} - 1$$

Where the last equality comes from canceling out k in steady state k^* and from population growth. This expression can be approximated to:

$$g_E \approx 1 + n - g_a - 1 = n - g_a$$

Of course, during the *transition* to the steady state, pollution may be growing either faster or slower than this. Now, let's **define 'sustainable' growth** to be a situation where $g_E \leq 0$. That is, pollution remains bounded.

b) Environmental Kuznets Curve

(Hints: (1) Derive expressions for $\Delta k_{t+1}/k_t$ and $\Delta E_{t+1}/E_t$ as functions of k_t during the transition to the steady state, (2) The growth rate of $\Omega_t k_t^{\alpha}$ can be approximated by $\Delta\Omega_t/\Omega_t + \alpha\Delta k_{t+1}/k_t$.

From eq. (2), we have the following expression for the (percentage) change in k_t during the transition to the steady state:

$$\frac{\Delta k_{t+1}}{k_t} = s(1-\theta)k_t^{\alpha-1} - (\delta + n)$$

Note that this is a declining function of k, which is positive for small values of k, and negative for large values of k. The steady state occurs when it crosses the horizontal axis. Next, by definition, we have the following expression for the (percentage) change in pollution emissions during the transition to the steady state:

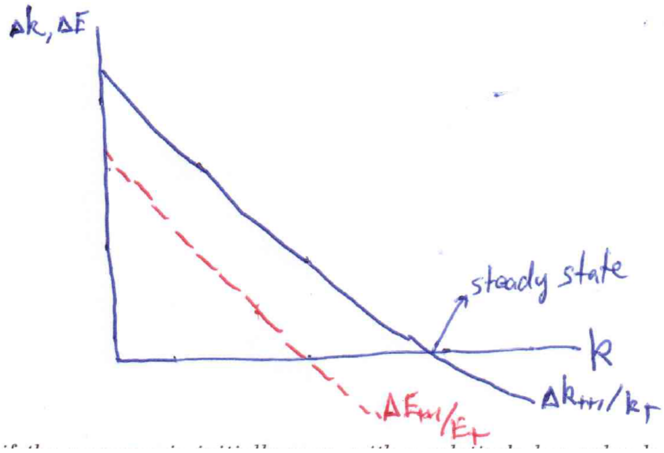
$$\frac{\Delta E_{t+1}}{E_t} = \frac{\Omega_{t+1}Y_{t+1} - \Omega_t Y_t}{\Omega_t Y_t}$$

Remembering that the growth rate of a product of two variables can be approximated, in discrete time, as the sum of the growth rates of the two variables:

$$\frac{\Delta E_{t+1}}{E_t} = \frac{\Delta\Omega_t}{\Omega_t} + \frac{\Delta Y_t}{Y_t}$$

Starting from the growth in output:

$$\frac{\Delta Y_t}{Y_t} = \alpha\frac{\Delta k_t}{k_t} + n = \alpha[s(1-\theta)k^{\alpha-1} - (\delta+n)] + n$$



and the growth in pollution abatement net of population growth is $\frac{\Delta\Omega_t}{\Omega_t} = -g_a$ you have that:

$$\frac{\Delta E_{t+1}}{E_t} = \alpha\left[s(1-\theta)k_t^{\alpha-1} - (\delta+n)\right] + n - g_a$$

Note that this has the same basic shape as the capital accumulation function, but it crosses the horizontal axis **before** it. This is because of the fact that when the economy is *sustainable*, $n < g_a$. In sum, we can describe the paths of capital and pollution in the picture above:

Now, notice that if the economy is initially poor, with a relatively low value k, then during the transition both output and pollution increase as the economy develops. Then, since the $\frac{\Delta E_{t+1}}{E_t}$ line hits zero first, pollution begins to decline, while per capita income continues to grow. This would produce an Environmental Kuznets Curve (EKC).

However, if the economy starts out relatively rich, meaning that its initial k is to the right of the point where $\frac{\Delta E_{t+1}}{E_t} = 0$ but to the left of the steady state, the economy continues to grow, but pollution declines monotonically. In this case, there would be no EKC.

The intuition here is that due to diminishing returns, output growth is faster in poor countries. At the same time, since pollution growth is just proportional to output growth, pollution emissions grow rapidly in poor countries. The good news is that *productivity* in pollution abatement is constant, and independent of the level of income. Eventually, productivity growth in pollution abatement comes to dominate the output effect, since output growth declines over time.

c) Increase in the abatement effort

How does an increase in abatement effort (i.e., an increase in θ) affect the time path of pollution? Explain intuitively, and relate your conclusions to how a standard Solow model reacts to an increase in the savings rate. *(Hint: You do not need to solve for anything. Just sketch out a time path.)*

In this model, changes in 'abatement effort', $a(\theta)$, are just like changes in the saving rate in the regular Solow model. They produce temporary changes in the growth rate of pollution, but not long-lasting changes. Those are determined entirely by the underlying growth rates of output and abatement productivity. The picture would look just like the economy's response to a (permanent) change in the savings rate. There would be a sudden drop in the growth rate of pollution, but it would eventually climb back to its long-run balanced growth path.