Macroeconomics A

Lecture 2 - Neoclassical and Endogenous Growth

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The Neoclassical Growth Model

(aka “Ramsey” or “Ramsey-Cass-Koopmans” model)

The Neoclassical Growth Model

- The same basic environment as Solow model without the assumption of the constant exogenous saving rate
- Important not just as a microfoundation for Solow, but also
  - Dynamic optimization techniques
  - Foundation for modern business cycle models (session 3/4 onwards)
- Unique final good, which is used both as consumption and capital good and is the numeraire (i.e. price $= 1$)
- Representative neoclassical firm with

$$Y = F(K(t), L(t), Z(t))$$

  where $F(\cdot)$ is a neoclassical production function (for now: $Z(t) = Z, L(t) = L$)

- Representative household with preferences

$$U = \sum_{t=0}^{\infty} \beta^t u(C(t)),$$

where u is a concave and strictly increasing period utility function, and $\beta < 1$ is the time discount factor
- Households have a fixed per-period labor endowment $L(t)$, which they supply inelastically to the market.
- Markets:
  - Spot market for the final good (price $= 1$)
  - Spot market for labor (price $= w(t)$)
  - Asset market (see below)
- All markets are perfectly competitive.

The households' budget constraint

- Households' budget constraint

$$C(t) + K(t+1) = (1 + r(t))K(t) + w(t)L(t)$$

  where K denotes the capital holdings and $r(t)$ is the rate of return on lending capital to firms
- Could alternatively have bonds being issued by firms (then HH hold bonds instead of capital $\rightarrow$ isomorphic)
- Since no risk (perfect foresight) here: all other assets in HH's portfolio must yield the same return (otherwise they would not be all held)
- Hence, the above is no loss of generality
- Again, capital depreciates at rate δ

Equilibrium of NGM

A competitive equilibrium of the NGM is a sequence of factor prices $\{w(t), R(t)\}$, quantities $\{C(t), K(t), L(t)\}_t$ and interest rates $\{r(t)\}_t$ such that

- Firms maximize profits (as in Solow)
- Consumers maximize utility subject to the budget constraint
- Capital, labor, and goods markets clear

Characterizing the Equilibrium: Firms

- Firms' side is exactly as in the Solow model
- Because firms rent capital and labor, they solve

$$\max_{K,L} F(K,L) - w(t)L - R(t)K$$

- As before:

$$w(t) = F_L(K,L) = F_L(k,1)$$

$$R(t) = F_K(K,L) = F_K(k,1)$$

Characterizing the Equilibrium: Households

- What is the return on renting out K units of capital (e.g. machines)?
  - For renting out K units, you get $R(t)K$ units of the final good as compensation from firms
  - But δK units depreciate during the period
- Hence:

$$\text{Return} = r(t) = \frac{R(t)L - \delta K}{K} = R(t) - \delta$$

which is the interest rate that households face

Characterizing the Equilibrium: Households

- Consumption expenditures have to be utility maximizing, i.e. they are given by

$$\{C(t)\}_t = \arg \max_{\{C(t),K(t+1)\}_t} \sum_{t=0}^{\infty} \beta^t u(C(t))$$

subject to the flow budget constraint: $\forall t$:

$$C(t) + K(t+1) = (1 + r(t))K(t) + w(t)L(t)$$

and the No-Ponzi condition

$$\lim_{t\to\infty} \left\{ K(t) \left[\prod_{s=1}^{t-1} \left(\frac{1}{1 + r(s)} \right) \right] \right\} \geq 0$$

- No-Ponzi: limit of the PDV of debt has to be nonnegative (you cannot die in debt)

Solving the consumer's problem

- Consider first the finite-dimensional problem (with a fixed horizon T). The Lagrangian is

$$\mathcal{L} = \sum_{t=0}^{T} \beta^t u(C(t)) - \lambda(t) \left[C(t) + K(t+1) - (1+r(t))K(t) - w(t)L\right] - \mu(T+1)K(T+1)$$

- Choice variables: $\{C(t), K(t+1)\}_{t=0}^{T}$
- First-order optimality conditions:

$$\frac{\partial \mathcal{L}}{\partial C(t)} = \beta^t u'(C(t)) - \lambda(t) = 0 \quad (1)$$

$$\frac{\partial \mathcal{L}}{\partial K(t+1)} = -\lambda(t) + \lambda(t+1)(1 + r(t+1)) = 0 \quad (2)$$

$$\frac{\partial \mathcal{L}}{\partial K(T+1)} = -\lambda(T) + \mu(T+1) = 0 \quad (3)$$

$$K(T+1)\mu(T+1) = 0 \quad (4)$$

- The last equation is the complementary slackness condition (by Inada we don't need this for $\lambda(t)$)

First-order conditions for the finite-dimensional problem

- First two equations give you the *Euler equation*

$$\forall t : \quad u'(C(t)) = \beta(1 + r(t+1))u'(C(t+1))$$

  One of the most important equations in macroeconomics.

- Last two equations give:

$$0 = K(T+1)\mu(T+1) = -K(T+1)\beta^T u'(C(T))$$

  As long as you have positive marginal utility $u'(C(T)) > 0$, you should not keep resources behind ($\rightarrow K(T+1) = 0$).

- The Euler equations determine the *slope* of the consumption profile; the terminal condition determines the *level*.

Back to the infinite-horizon model

- With an infinite horizon ($T \to \infty$), the Euler equations still determine the relative consumption levels:

$$\forall t : \quad u'(C(t)) = \beta(1 + r(t+1))u'(C(t+1))$$

- The terminal condition becomes what is called the *transversality condition*, and is the limit of the terminal condition:

$$\lim_{T\to\infty} K(T+1)\beta^T u'(C(T)) = \tag{5}$$

$$\lim_{T\to\infty} K(T+1)\beta^{T+1} u'(C(T+1))(1 + r(T+1)) = 0 \tag{6}$$

- Hence:

$$\lim_{t\to\infty} \beta^t u'(C(t))(1 + r(t))K(t) = 0$$

- With concave utility and Inada-type conditions on $u(\cdot)$, the Euler equations and the transversality condition are also sufficient to characterize the optimal consumption allocation.

Transversality vs No-Ponzi condition

- We have two conditions that involve "$\lim_{t\to\infty}$":

$$\lim_{t\to\infty} \beta^t u'(C(t))(1+r(t))K(t) = 0 \tag{7}$$

$$\lim_{t\to\infty} \left\{ K(t) \left[\prod_{s=1}^{t-1} \left(\frac{1}{1+r(s)} \right) \right] \right\} \geq 0 \tag{8}$$

- The transversality condition (7) is a condition for optimal behavior
- The No-Ponzi condition (8) is a condition to make sure that the flow budget constraints are consistent with a lifetime budget constraint (i.e. aggregated across periods)
- (7) says that you should not hold assets as long as you value them
- (8) says that you cannot hold too much debt
- Often (7) implies that (8) is satisfied (e.g. if $1+r=\beta^{-1}$). But not always.
- In principle, conditions have nothing to do with each other. (7) is about an optimal path and (8) is about the budget constraint (i.e. does not depend on preferences).

Gathering the Equilibrium conditions

- From households:

$$u'(C(t)) = \beta(1 + r(t))u'(C(t+1))$$

$$\lim_{t\to\infty} \beta^t u'(C(t))(1 + r(t))K(t) = 0$$

- From firms:

$$R(t) = F_K(K(t), L)$$

$$w(t) = F_L(K(t), L)$$

- Also:

$$r(t) = R(t) - \delta = F_K(K(t), L) - \delta$$

- Finally, the budget constraint is

$$C(t) + K(t+1) = (1 + r(t))K(t) + w(t)L(t)$$

- From the budget constraint and CRS follows the economy's resource constraint:

$$C(t)+K(t+1) = (1+R_K(K, L)-\delta)K(t)+w(t)L(t) = (1-\delta)K(t)+Y(t)$$

Characterizing the solution

- The solution satisfies:

$$
\begin{aligned}
C(t) + K(t+1) &= (1-\delta)K(t) + F(K(t), L) && (9)\\
u'(C(t)) &= \beta(1 + F_K(K(t+1), L))u'(C(t+1)) && (10)\\
0 &= \lim_{t\to\infty} \beta^t u'(C(t))(1 + r(t))K(t) = 0 && (11)\\
K(0) &= K_0 && (12)
\end{aligned}
$$

- For any $C(0)$, the above system gives a trajectory of $\{C(t), K(t)\}_{t=1}^{\infty}$.
- The optimal $C(0)$ is then the one where the sequence of consumption satisfies the transversality condition

Steady state

- Again, look at a steady state, where consumption and capital is constant and equal to (C^*, K^*)
- From the Euler equation:

$$u'(C^*) = \beta(1 + F_K(K^*, L))u'(C^*)$$

  hence

$$1 + r^* = 1 + F_K(k^*, 1) - \delta = 1/\beta$$

  which fully determines k^*.
- Note: this means that k^* does *not* depend on $u(\cdot)$!
- Level of per-capita consumption is then given by the economy's resource constraint:

$$C^* + K^* = (1 - \delta)K^* + F(K^*, L)$$

  hence

$$c^* = F(k^*, 1) - \delta k^*$$

Steady state

- Steady-state capital-labor ratio:

$$1 = \beta(1 + F_K(k^*, 1) - \delta)$$

hence

$$F_K(k^*, 1) = \delta + \left(\frac{1}{\beta} - 1\right)$$

- In the Solow model the golden-rule level of capital (which maximizes steady-state consumption) satisfies

$$F'_K(k_S^{GR}, 1) = \delta$$

hence

$$k^*_{NGM} < k_S^{GR}$$

- Intuition?

NGM in continuous time

- Many textbooks teach the NGM in continuous time
  - Algebra a bit simpler (in particular with growth in L, A)
  - Continuous-time dynamic optimization techniques
  - Allows for nice visualization of the solution
- Personal opinion: discrete time version much more important, since it serves as the foundation for all modern business cycle models (sessions 3-7)

But for completeness...

NGM in continuous time

- Representative household supplies L to the market. L is assumed to start at one and grow at rate n:

$$L(t) = e^{nt}$$

- Household chooses consumption/saving to maximize

$$U = \int_0^\infty e^{-\rho t} e^{nt} u(c(t))dt = \int_0^\infty e^{(n-\rho)t} u(c(t))dt$$

- Intertemporal budget constraint: as in discrete time case, but take limit of $\Delta K \to 0$:

$$\dot{K}(t) = w(t)L(t) + (1 + r(t))K(t) - C(t)$$

- Write in per-capita units: $k(t) = K(t)/L(t)$:

$$\dot{k}(t) = w(t) + (1 + r(t))k(t) - c(t) - nk(t)$$

- No-Ponzi condition: PDV of debt has to be nonnegative:

$$\lim_{t\to\infty} \left\{ k(t) \times \exp\left[- \int_0^t (r(v) - n)dv \right] \right\} \geq 0$$

Solving this

Standard problem in Optimal Control

1. Set up the *Hamiltonian*:

$$H(c,k,\mu) = e^{-(\rho-n)t}u(c(t)) + \mu(t)[w(t) + (1+r(t)-n)k(t) - c(t)]$$

2. The necessary optimality conditions are

$$\begin{aligned} H_c(c,k,\mu) &= 0 \\ H_k(c,k,\mu) &= -\dot{\mu}(t) \\ H_\mu(c,k,\mu) &= \dot{k}(t) \\ \lim_{t\to\infty} \mu(t)k(t) &= 0 \end{aligned}$$

Hence:

$$\begin{aligned} e^{-(\rho-n)t}u'(c(t)) - \mu(t) &= 0 \\ \mu(t)(1+r(t)-n) &= -\dot{\mu}(t) \end{aligned}$$

Plugging the first into the second, we get

$$
\begin{aligned}
&e^{-(\rho-n)t}u'(c(t))(1+r(t)-n) = \\
&\qquad -e^{-(\rho-n)t}u''(c(t))\dot{c}(t) + (\rho-n)e^{-(\rho-n)t}u'(c(t))
\end{aligned}
$$

and rearranging, we get the Euler equation in continuous time:

$$\sigma(c)(1+r(t)-\rho) = \frac{\dot{c}(t)}{c(t)}$$

where

$$\sigma(c) = -\frac{u'(c)}{u''(c)c}$$

is the intertemporal elasticity of substitution (IES).

- Consumption grows if $1 + r(t) > \rho$
- How much it grows (for given $1 + r(t)$, ρ) depends on the IES.

The solution in continuous time

$$
\begin{aligned}
\frac{\dot{c}(t)}{c(t)} &= \sigma(c)(1 + r(t) - \rho) \\
\dot{k}(t) &= w(t) + (1 + r(t))k(t) - c(t) - nk(t) \\
\lim_{t\to\infty} \mu(t)k(t) &= 0 \\
k(0) &= k_0
\end{aligned}
$$

The other side of the model is exactly as before:

$$
\begin{aligned}
w(t) &= F_L(K, L) \\
1 + r(t) &= F_K(K, L) - \delta
\end{aligned}
$$

Define $y = Y/L = f(k)$.
Any production function with CRS satisfies:

$$Y = F(K, L) = F_K(K, L)K + F_L(K, L)L$$

and hence

$$y = \frac{Y}{L} = \frac{F_K(K, L)K + F_L(K, L)L}{L} = F_K(K, L)k + F_L(K, L)$$

so plugging this alongside the firm's first-order conditions into the law of motion for $k(t)$, we get

$$\begin{aligned} \dot{k}(t) &= w(t) + (1 + r(t))k(t) - c(t) - nk(t) \\ &= f(k) - (n + \delta)k(t) - c(t) \end{aligned}$$

Compare to Solow, where $y(t) - c(t) = sf(k)$.
Similarly, plug firm FOC into the Euler equation to get:

$$\frac{\dot{c}(t)}{c(t)} = \sigma(c)(f'(k) - \rho - \delta)$$

Transitional dynamics in the continuous-time NGM

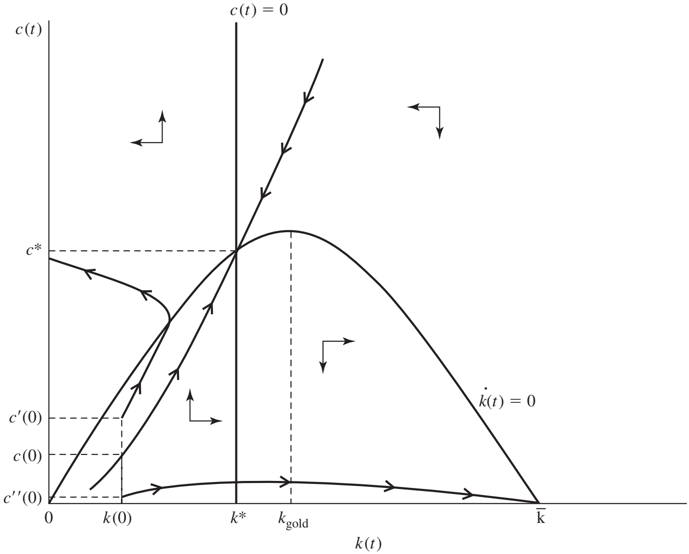

FIGURE 8.1 Transitional dynamics in the baseline neoclassical growth model.

Similarly to the Solow model, we converge to a steady-state where $k = K/L = k^*$ is constant. No long-term growth.

"Augmented" neoclassical growth model has exogenous labor-augmenting technical change: labor productivity T grows at some exogenous rate g, where

$$Y = F(K, T \times L)$$

- Same algebra as here: economy converges to a steady-state where

$$\hat{k} = \frac{K}{TL}, \qquad \hat{y} = \frac{Y}{TL}$$

  are constant.
- Then per-capita income grows at the (exogenous) growth rate g.

See augmented Solow model from the first lecture.

Neoclassical Growth Model

- In the NGM, the dynamic evolution of the economy is driven by capital accumulation
- In turn, capital accumulation is driven by the private incentives to save
  - No life-cycle motives for saving!
- Likewise, the assumption of a representative household is not always appropriate
  - Households typically do not have an infinite planning horizon
  - New households appear over time, old ones cease to exist
- Economic interaction between different generations: decisions made by previous generations will affect the environment faced by the new generations (e.g. bequests)
- *Overlapping generations models* model the different generations explicitly
  - allow for life-cycle considerations in savings decision
  - allows us to study intergenerational policies

Endogenous Growth Models

Motivation

- Empirical literature: at least 50% of GDP/capita differences across countries is due to productivity differences, which can be due to
  - technological differences
  - misallocation of resources
  - quality of institutions
  - ...
- Focus now on the determinants of technological change, the sequence of $\{A(t)\}_t$

From NGM to Endogenous Growth

- Key challenge: if capital is accumulated, and it has decreasing returns $\Rightarrow$ perpetual accumulation and high marginal products are not possible
- How can we keep marginal product high?
- Cheap solution: AK models, where the MPK is constant
- Alternative: increasing returns as a whole. Romer (1980): increasing returns are a result of **ideas** being **non-rival**
  - rival good: my consumption precludes you from consuming it
  - non-rival good: all of us con consume it simultaneously without affecting others
- Rivalry is a technological characteristic, whereas *excludability* is an institutional characteristic:
  - Non-excludable good: impossible to prevent it from being consumed/used by other agents

Ideas and increasing returns

- Non-rivalry naturally suggests increasing returns to scale:
  - Cost of producing a movie is high and a one-time investment
  - Once you produced it: constant returns to scale in DVD production
  - $\Rightarrow$ IRS in "movies on DVD"
  - Movie itself is non-rivalrous!
  - Online piracy suggests that it may even not be (perfectly) excludable
- Hence, natural structure

$$Y(t) = F(A(t), K(t), L(t))$$

with

$$F(A(t), \lambda K(t), \lambda L(t)) = \lambda F(A(t), K(t), L(t))$$
$$F(\lambda A(t), \lambda K(t), \lambda L(t)) > \lambda F(A(t), K(t), L(t))$$

  - Constant returns in rivalrous factors
  - IRS in *all* factors

Perfect competition does not work

- Take the economy as a whole in a perfect competition setup with the previous production function
- Then the profit in the economy is zero:

$$\Pi = F(A(t), K(t), L(t)) - w(t)L(t) - R(t)K(t) = 0$$

  as in Solow/Ramsey.
- So how do we pay for the fixed cost of investment in ideas? Who pays the research labs?
- Two solutions:
  - Ideas accumulate as a by-product of production $\rightarrow$ nobody has to pay (Romer 1986, Lucas 1988, learning-by-doing)
  - Give producers some market power $\rightarrow$ positive ex-post profits $\rightarrow$ can pay for research (Romer, 1990)

Models of R&D

- Most simple models of endogenous technological change are those in which R&D expands the variety of inputs or machines used in production (e.g. Romer 1990)
  $\rightarrow$ due to the increasing number of varieties, the division of labor (specialization) increases
- Alternative: R&D improves the quality of inputs or machines used in production
- **product innovation**: increases the number of inputs or the quality of inputs
- **process innovation:** reduces the cost of existing products

Sketch of Romer (1990) expanding varieties model

Three types of decision makers in the model:

1. Households: maximize utility, subject to the usual budget constraint
2. Final good producers: hire labor and purchase intermediate input $\rightarrow$ combine them to produce the final output, sold at unit price
3. R&D firms
   - Devote resources to invent new intermediate inputs (stage 1)
   - Once a product has been invented, the innovating firm obtains a perpetual patent ($\rightarrow$ allows the firm to sell at the price it chooses)
   - price is chosen to maximize profits (stage 2)

Sketch of Romer (1990) expanding varieties model

Predictions:

- Higher willingness to save (lower rate of discounting ρ or higher elasticity of intertemporal substitution) $\Rightarrow$ higher supply of capital $\Rightarrow$ more resources invested into R&D $\Rightarrow$ higher rate of technological progress $\Rightarrow$ higher growth rate in the steady state
- Higher "success rate" in R&D $\Rightarrow$ higher steady state growth
- Larger population $\Rightarrow$ more resources available for R&D $\Rightarrow$ higher growth rate

The fact that the new "idea" can be used in a nonrival manner is key for the R&D output affecting productivity (or more generally, the cost of production).

Schumpeterian endogenous growth models

Subsequent work used this mechanism to talk about the "industrial organization" of the model (Aghion-Howitt 1992)

- Stronger IP protection incentivizes firms to do R&D
- But IP protection also causes a markup distortion: patents give market power
- And innovation potentially displaces an incumbent ("creative destruction"), which again somewhat lowers the returns to R&D.
- If factor reallocation is not working well, this can have additional costs.

The problem with endogenous growth theory

At the heart of endogenous growth theories are mechanisms that are incredibly hard to observe and measure:

- In the case of Romer it all relies on the R&D production function (and in Aghion-Howitt also on market structure/competition)
- R&D is a process where we observe neither inputs nor outputs well.
  - Little of research output gets patented, and much of what gets patented is not economically valuable
  - Difficult to distinguish R&D input from production inputs with available data
- Market structure/conduct: even more difficult to observe (see: antitrust/IO).

In the end, EGT use metaphors ("production of ideas") that render the theory not falsifiable. And if it's not falsifiable, it's not scientific (Popper, 1934).