Production Functions

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Production Functions

A production function is often represented as :

$$Y = f(K, L, \text{ other inputs })$$

where

- Output Y : the goods or services produced.
- Capital K : physical assets like machinery, buildings, and equipment.
- Labor L : human effort, including both physical and intellectual labor.
- Other inputs : raw materials, land, energy, technology, etc.

Elasticity of Substitution

The **elasticity of substitution** captures the responsiveness of the ratio of inputs to changes in their relative productivity (= relative price in perfect competition market).

$$\sigma = \frac{d\left(\frac{K}{L}\right) / \left(\frac{K}{L}\right)}{d\left(\frac{MP_L}{MP_K}\right) / \left(\frac{MP_L}{MP_K}\right)} = \frac{d\ln\left(\frac{K}{L}\right)}{d\ln\left(\frac{MP_L}{MP_K}\right)}$$

It measures how easily one input can be substituted for another input in the production prosess.

Constant Elasticity of Substitution (CES) function

The CES production function is typically expressed as :

$$Y = A\left(\alpha K^{\rho} + (1-\alpha)L^{\rho}\right)^{\frac{1}{\rho}}$$

where

- Distribution parameter $\alpha \in (0,1)$: relative importance of capital versus labor.
- ρ : a parameter related to the elasticity of substitution.

Define

$$\sigma = \frac{1}{1-\rho}$$

σ is the elasticity of substitution between capital and labor.

Constant Elasticity of Substitution (CES) function

The CES function

$$Y = A\left(\alpha X_1^{\rho} + (1-\alpha)X_2^{\rho}\right)^{\frac{1}{\rho}}$$

can be converted into other production functions by setting specific values of σ (or ρ).

- $\sigma = 1$ $(\rho = 0)$:CES to Cobb-Douglas $Y = AX_1^{\alpha}X_2^{1-\alpha}$
- $\sigma = 0$ $(\rho = \infty)$:CES to Leontief $Y = A \cdot \min\left(X_1, X_2\right)$
- $\sigma = \infty$ $(\rho = 1)$:CES to Perfect Substitutes $Y = A\left(\alpha X_1 + (1-\alpha)X_2\right)$

CES to CD

Rewrite the CES function

$$Y = A\exp\left(\frac{1}{\rho}\ln\left[\alpha X_1^\rho + (1-\alpha)X_2^\rho\right]\right)$$

Expand $\ln\left[\alpha X_1^\rho + (1-\alpha)X_2^\rho\right]$ around $\rho = 0$

$$\ln\left[\alpha X_1^\rho + (1-\alpha)X_2^\rho\right] \approx \rho\ln\left(X_1^\alpha X_2^{1-\alpha}\right)$$

Substituting this into the CES function gives :

$$Y = A\exp\left(\ln\left(X_1^\alpha X_2^{1-\alpha}\right)\right) = AX_1^\alpha X_2^{1-\alpha}$$

This is the Cobb-Douglas production function.

CES to Leontief

Without loss of generality, assume $X_1 \geq X_2 \Rightarrow X_1^\rho \geq X_2^\rho$. We also have $X_1, X_2 > 0$. Then we verify that the following inequality holds :

$$\alpha X_1^\rho \leq \alpha X_1^\rho + (1-\alpha) X_2^\rho \leq X_1^\rho$$

$$\alpha^{\frac{1}{\rho}} X_1 \leq [\alpha X_1^\rho + (1-\alpha) X_2^\rho]^{\frac{1}{\rho}} \leq X_1$$

We have

$$\lim_{\rho \to \infty} \alpha^{\frac{1}{\rho}} X_1 = X_1$$

which sandwiches the middle term to X_1 So

$$\lim_{\rho \to \infty} Y = AX_1 = A \cdot \min(X_1, X_2)$$

This is the Leontief function, which implies that inputs are perfect complements.

CES to Perfect Substitutes

If $\rho = 1$, the CES function simplifies to :

$$Y = A\left[\alpha X_1 + (1-\alpha)X_2\right]$$

This is the perfect substitutes function, where inputs are perfect substitutes, meaning one input can be completely replaced by the other at a constant rate.

Production Frontiers