Analysis

Théodore Renault

The Graduate Institute of International and Development Studies, Geneva

Differentiation

Maxima and minima

Convexity and concavity

Exponential and logarithmic functions

Approximations

Log-Linearization

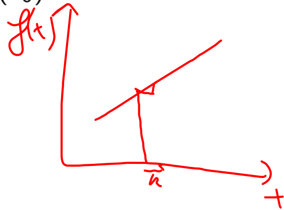
Differentiation

Differentiation

There are as many definitions and uses of derivatives as scientific fields.

In economics, the process of differentiation can simply be seen as **the process of finding the slope of a function**.

The slope of the curve $y = f(x)$ at the point $(x_0, f(x_0))$ is called the **derivative** of $f(x)$ at $x = x_0$ and is denoted $f'(x_0)$.



Analytically,

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

In other words, the derivative of a function with respect to a variable, is the variation of the function due to a change in that variable.

Differentiation

Formally, the derivative f' at any x is :

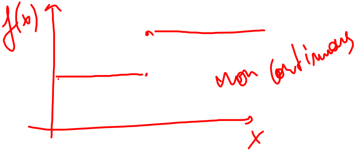
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The process of going from function f to the function f' is called **differentiation**.

Alternative notation :

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Continuity and differentiability



Continuity : we say that $f(x)$ is **continuous** at x_0 is $\lim_{x \to x_0} f(x) = f(x_0)$.

We say that f is a **continuous function** if it continuous at x for *every* x.

Differentiability : a function is said to be **differentiable** at a particular point if the derivative of the function can be found at that point.

Hence, a function which is differentiable at a point must be continuous at that point.

We say that f is a **differentiable function** if it *differentiable* at every point.

Hence, every differentiable function is continuous.

Rules of differentiation

Straight line rule : $\frac{d}{dx}(ax+b)$ if a and b are constants.

$y = ax + b$

$y = e^{x^2}$

Power rule : $\frac{d}{dx}(x^n) = nx^{n-1}$ if n is an integer.

Combination rule : $\frac{d}{dx}(ay+bz) = a\frac{dy}{dx} + b\frac{dz}{dx}$ if a and b are constants.

$(uv)' = u'v + uv'$

Product rule : $\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$.

$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

Quotient rule : $\frac{d}{dx}\left(\frac{u}{v}\right) = \left(v\frac{du}{dx} - u\frac{dv}{dx}\right) \Big/ v^2$.

Composite function rule : If $f(x) = p(q(x))$, then $f'(x) = p'(q(x))q'(x)$.

Let $u = q(x)$ and $y = p(u)$ so that $y = p(q(x)) = f(x)$, then $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Rules of differentiation

Exercise 1 : derive the following functions :

(a) $f(x) = 2x^4 - 7x^{-1}$

(b) $f(x) = \underbrace{(4x-2)}_{u(x)}\underbrace{x^3}_{v(x)}$

(c) $f(x) = \underbrace{(x^4 - 3x^2)}_{u(x)}\underbrace{(5x+1)}_{v(x)}$

(d) $f(x) = \underbrace{(x^2+1)}_{u(x)}/\underbrace{(2x^3+1)}_{v(x)}$

(e) $f(x) = \frac{2x^3-5}{\sqrt{x^2+1}}$ $\frac{u(x)}{v(x)}$

a. $f(x) = 2x^4 - 7x^{-1}$ $\qquad -1x^{-2} = -\frac{1}{x^2}$

$f'(x) = 8x^3 + 7\frac{1}{x^2}$

b. $f(x) = \underbrace{(4x-2)}_{u(x)}\underbrace{x^3}_{v(x)}$

$f'(x) = \underbrace{4}_{u'(x)}\underbrace{x^3}_{v(x)} + \underbrace{(4x-2)}_{u(x)} \cdot \underbrace{3x^2}_{v'(x)}$

$= 16x^3 - 6x^2$

c. $f(x) = (\underbrace{x^4 - 3x^2}_{u(x)})(\underbrace{5x+1}_{v(x)})$

$f'(x) = \underbrace{(4x^3 - 6x)}_{u'(x)}\underbrace{(5x+1)}_{v(x)} + \underbrace{(x^4 - 3x^2)}_{u(x)} \cdot \underbrace{5}_{v'(x)}$

$= 25x^4 - 45x^2 + 4x^3 - 6x$

$$\frac{4x^4 + 2x - 6x^4 - 6x^2}{(2x^3+1)^2} = \frac{-6x^2 - 2x^4 + 2x}{(2x^3+1)^2}$$

e $\quad f(x) = \dfrac{2x^3 - 5}{\sqrt{x^2+1}}$ $\quad u(x)$, $v(x)$

$$\frac{df(x)}{x} = \frac{\underbrace{6x^2}_{u'(x)}\underbrace{(x^2+1)^{1/2}}_{v(x)} - \overbrace{x(x^2+1)^{-1/2}}^{v'(x)}\overbrace{(2x^3-5)}^{u(x)}}{\left(\sqrt{x^2+1}\right)^2}$$

$$= \frac{6x^2(x^2+1)^{1/2} - (x^2+1)^{-1/2}(2x^4 - 5x)}{\left(\sqrt{x^2+1}\right)^2}$$

$p(q(x))$

$p(y) = \sqrt{y}$

$q(x) = x^2+1$

$p'(q(x))\,q'(x)$

Derivatives in Economics

Marginal functions Differentiation is commonly applied to economics via marginal functions.

For instance the **marginal revenue** is defined to be the derivative of the revenue with respect to quantity. If $H(Y)$ denotes the level of imports when national income is Y, then the **marginal propensity to import** is $H'(Y)$.

Another example, let consumption C be related to national income Y by the consumption function

$$C = 10 + 0.7Y - 0.002Y^2$$

.

The marginal propensity to consumption is

$$\frac{dC}{dY} = 0.7 - 0.004Y$$

.

Derivatives in Economics

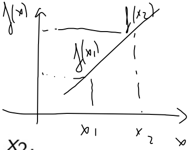
Elasticity Marginal functions give the rate of change of one variable with respect to the other, and the value will depend on the unit in which variables are measured.

It is more useful in economics to have a **unit-free measure** of response. We call that the **elasticity**. It is the ratio of proportional changes in the variables of interest.

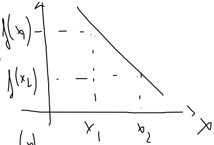
For instance the **price elasticity of demand** is given by

$$\epsilon_p^d = \frac{\text{proportional change in quantity demanded}}{\text{proportional change in price}} = \frac{\Delta q/q}{\Delta p/p} = \frac{p}{q}\frac{dq}{dp}$$

Monotonic functions



f is **strictly increasing** if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$.



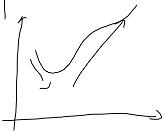
f is **strictly decreasing** if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$.

In either case, f is said to be a **monotonic function**.

g is **non-increasing** if $g(x_1) \geq g(x_2)$ whenever $x_1 < x_2$.

g is **non-decreasing** if $g(x_1) \leq g(x_2)$ whenever $x_1 < x_2$.

In either case, g is said to be a **weakly monotonic function**.



Inverse function

If f is monotonic and the solution of the equation $f(x) = y$ is $x = g(y)$, then g is called the **inverse function** of f.

$y = x^{-1} - 2 \Rightarrow x^{-1} = y+2 \Rightarrow x = 1/(y+2)$

Thus, if $f(x) = x^{-1} - 2$, the inverse function g is given by

$x =$ $g(y) = 1/(y + 2)$.

In general, $f(g(y)) = y$ and $g(f(x)) = x$.

Inverse function rule : Let f be a monotonic function with inverse function g ; for a given x, if f is differentiable at x and $f'(x) \neq 0$, then g is differentiable at $y = f(x)$ and

$$\boxed{g'(y) = \frac{1}{f'(x)}}$$

Inverse function rule

Exercise 2
Find $g'(y)$ for $f(x) = x^{-1} - 2$.

Exercise 3
Let p be the price of a good, x the quantity demanded. Let the demand function for the good be

$$x = f(p) = 5/p^2$$

Find the inverse demand function g and show that the price elasticity of demand can be written

$$\epsilon = \frac{g(x)}{xg'(x)}$$

$$f(x) = x^{-1} - 2$$

$$y = x^{-1} - 2$$

$$y + 2 = x^{-1}$$

$$\boxed{g(y) = \frac{1}{(y+2)}}$$

$$\frac{1}{v}$$

$$g'(y) = \frac{0 \cdot (y+2) - (1 \cdot 1)}{(y+2)^2} \Rightarrow \frac{-1}{(y+2)^2}$$

$$e = \frac{g(x)}{x \cdot g'(x)}$$

$$x = \frac{5}{p^2} \quad , \quad p = \frac{\sqrt{5}}{\sqrt{x}} = g(x)$$

$$g'(x) = \frac{-\sqrt{5}}{2x^{3/2}} = \frac{dq}{dp}$$

$$e = \frac{P}{q} \cdot \frac{dq}{dp}$$

$$e = \frac{P}{x} \cdot \frac{2x^{3/2}}{\sqrt{5}} = -\frac{2\sqrt{x} \cdot x(p)}{\sqrt{5}\, x}$$

$$= \frac{P}{x\, g'(x)} = \frac{g}{x' g(x)}$$

$$e = \frac{\frac{\Delta x}{x}}{\frac{\Delta p}{p}} = \frac{dx}{dp} \frac{p}{x} = f'(p) \cdot \frac{g(x)}{f(p)} = \frac{1}{g'(x)} \cdot \frac{g(x)}{x} = \frac{g(x)}{x\, g'(x)}$$

$$= \frac{p}{\frac{dp}{dx}\, x} = \frac{g(x)}{g'(x)\, x}$$

$$e = \frac{g(x)}{x \cdot g'(x)}$$

$$x = \frac{5}{p^2} \quad , \quad p = \frac{\sqrt{5}}{\sqrt{x}} = g(x)$$

$$g'(x) = \frac{-\sqrt{5}}{2x^{3/2}} = \frac{dq}{dp}$$

$$e = \frac{p}{q} \cdot \frac{dq}{dp}$$

$$e = \frac{p}{x} \cdot \frac{2x^{3/2}}{\sqrt{5}} = -\frac{2\sqrt{x} \cdot x(p)}{\sqrt{5}\, x}$$

$$= \frac{p}{x\, g'(x)} = \frac{g}{x \cdot g(x)}$$

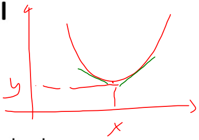
$$e = \frac{\frac{\Delta x}{x}}{\frac{\Delta p}{p}} = \frac{dx}{dp} \frac{p}{x} = f'(p) \cdot \frac{g(x)}{f(p)} = \frac{1}{g'(x)} \cdot \frac{g(x)}{x} = \frac{g(x)}{x\, g'(x)}$$

$$= \frac{p}{\frac{dp}{dx}\, x} = \frac{g(x)}{g'(x)\, x}$$

Maxima and minima

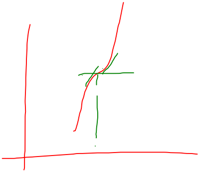
Critical points

Let f be a differentiable function. The points on the curve $y = f(x)$ where $f'(x) = 0$ are called **critical points**, and the value taken by the function at such point a point is called a **critical value**.



There are three kind of critical points :

- **Maximum points** At a maximum point, y has its greatest value in a neighbourhood of the point.
- **Minimum points** At a minimum point, y has its least value in a neighbourhood of the point.
- **Inflexion points** At a inflexion point, $f'(x) = 0$ but $f'(x)$ does not change sign in passing through the point.



Critical points

Four steps to find the critical points of a function $y = f(x)$

Step 1 : find $f'(x)$

Step 2 : find the values of x for which $f'(x) = 0$.

Step 3 : for each such x, find the corresponding value of y.

Step 4 : for each critical point (x^*, y^*), find the sign of $f'(x)$ for values of x that are slightly less than x^* (written $x = x^*-$) and for values for x that are slightly greater than x^* (written $x = x^*+$).

If $f'(x)$ changes from **positive** to **negative** as x changes from x^*- to x^*+, then (x^*, y^*) is a **maximum**.

If $f'(x)$ changes from **negative** to **positive** as x changes from x^*- to x^*+, then (x^*, y^*) is a **minimum**.

Critical points

Exercise 4 Find and classify the critical points of the curve

$$y = x^3 - 9x^2 + 24x + 10$$

Exercise 5 Find and classify the critical points of the curve

$$y = x^4 - 4x^3 + 5$$

Ex 4

$$y = x^3 - 9x^2 + 24x + 10$$

$f(x^*, y^*) = (2, 30)$

$f(x^*, y^*) = (4, 26)$

$$y' = 3x^2 - 18x + 24$$

$$0 = x^2 - 6x + 8 \Rightarrow x = 4, \quad x = 2$$

$x = 4 \rightarrow 0$

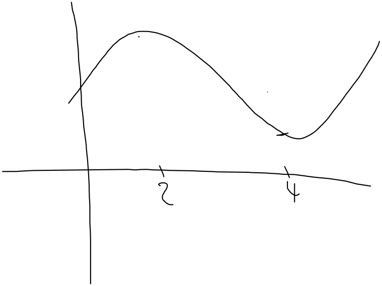
$x = 4.1 \rightarrow +0.63$

$x = 3.9 \rightarrow -0.57$

$x = 2 \rightarrow 0$

$x = 2.1 \rightarrow -0.63$

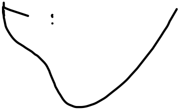
$x = 1.9 \rightarrow +0.57$

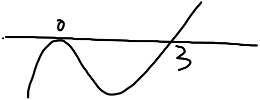


$$y' = 4x^3 - 12x^2$$

$$4x^3 - 12x^2 = 0$$

$$4x^2(x-3) = 0$$





$x = 0$ inflexion $\quad y = 5 \quad (0, 5)$

$x = 3$ min $\quad y = 81 - 108 + 5 = -22 \quad (3, -22)$

The second derivative

Let f be a differentiable function and if f' is also differentiable.

We denote the derivative $f'(x)$ wrt x by $f''(x)$ and is called the **second derivative** of $f(x)$.

The function f is said to be **twice differentiable**.

In the alternative notation, the second derivative is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

and is usually denoted by $\frac{d^2y}{dx^2}$.

The second derivative

If $f''(x_0) > 0$, then $f'(x_0)$ is increasing in x_0. This can happen in three ways :

- $f'(x_0)$ is positive and $f'(x)$ becomes more positive as x passes from x_0- to x_0+.
- $f'(x_0)$ is negative and $f'(x)$ becomes less negative as x passes from x_0- to x_0+.
- $f'(x_0) = 0$ and $f'(x)$ goes from negative to positive as x passes from x_0- to x_0+.

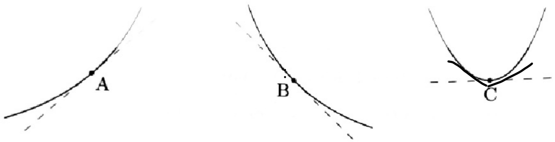


Figure 8.4: **Positive second derivative**

The second derivative

We can classify the critical points using the second derivative.

Step 1 : find $f'(x)$ and $f''(x)$.

Step 2 : find the critical points in the usual way.

Step 3 : for each critical point (x^*, y^*), calculate $f''(x)$. If $f''(x^*) < 0$ then (x^*, y^*) is a maximum. If $f''(x^*) > 0$ then (x^*, y^*) is a minimum.

Exercise 5 Find and classify the critical points of the function

$$y = 2x^3 - 3x^2 - 12x + 9$$

$$y = 2x^3 - 3x^2 - 12x + 9$$

$$y' = 6x^2 - 6x - 12$$

$$y' = x^2 - x - 2 = 0$$

$$x = 2 \quad x = -1$$

$$y'' = 12x - 6$$

$$y''(2) = 2(2) - 1 \overset{+}{=} 3 \text{ (min)}$$

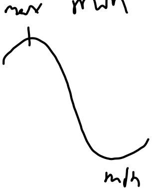
$$y''(-1) = 2(-1) - 1 = -3 \text{ (max)}$$

$$y(2) = -11$$

$$y(-1) = 16$$

$(2, -11)$ $(-1, 16)$

min max



(I) $y = x^4$

$y'(x) = 4x^3$

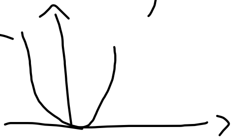
$y''(x) = 12x^2$

$x^*, y^* = (0,0)$

$y'(x) = 4 \cdot (-0.5)^3$

$= -$

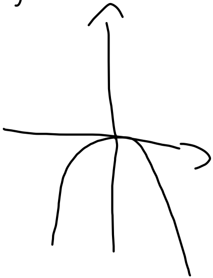
$\ominus \rightarrow \oplus$



(II) $y = -x^4$

$y'(x) = -4x^3$

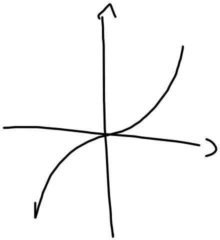
$y''(x) = -12x^2$



(III) $y = x^3$

$y'(x) = 3x^2$

$y''(x) = 6x$



Optimisation

The necessary (1) and sufficient (2) conditions for a **local maximum** are :

(1) If the function f has a local maximum where $x = x^*$, then $f'(x^*) = 0$ and $f''(x^*) \leq 0$.

(2) If $f'(x^*) = 0$ and $f''(x^*) < 0$, the function f has a local maximum where $x = x^*$.

The necessary (1) and sufficient (2) conditions for a **local minimum** are :

(1) If the function f has a local minimum where $x = x^*$, then $f'(x^*) = 0$ and $f''(x^*) \geq 0$.

(2) If $f'(x^*) = 0$ and $f''(x^*) > 0$, the function f has a local minimum where $x = x^*$.

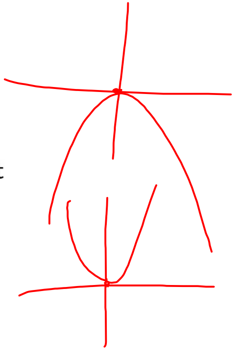
Optimisation

A local maximum point (x^*, y^*) of the curve $y = f(x)$, which has the additional property that

$f(x) = -x^2$

$$f(x^*) = y^* \geq f(x) \quad \text{for all } x \text{ in } \mathbb{R}$$

is called a **global maximum point**.

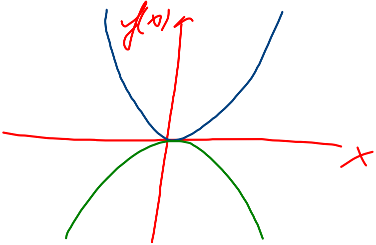


Similarly, a **global minimum point** of the curve $y = f(x)$ is a local minimum point (x^*, y^*) with the additional property that

$$y^* \leq f(x) \quad \text{for all } x \text{ in } \mathbb{R}$$

$f(x) = x^2$

Optimisation



Two interesting facts.

The function f has a local minimum where $x = x^*$ if and only if the function $-f$ has a local maximum where $x = x^*$, and the same is true for global minima and maxima.

Suppose $g(x) = H(f(x))$, where H is a *strictly increasing* function. If the function f has a global maximum (minimum) point at (x^*, y^*), then the function g has a global maximum (minimum) point at $(x^*, H(y^*))$.

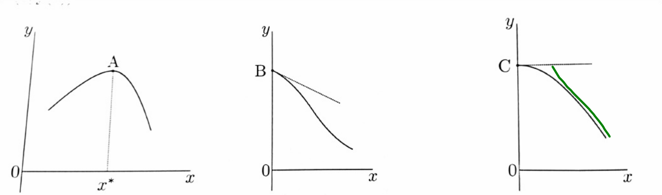
Optimisation

Suppose you want to maximise a function $f(x)$ subject to $x \geq 0$.
Three cases :

If $f'(x^*) = 0$ and $f'(x)$ changes from positive to negative, then we have an **interior local maximum** (Panel A).

If $dy/dx < 0$ at $x = 0$, we have a **boundary local maximum** (Panel B).

If $dy/dx = 0$ at $x = 0$ and $dy/dx < 0$ for $x > 0$, we also have a **boundary local maximum** (Panel C).



Panel A Panel B Panel C

Optimisation

Exercise 6

(a) Find the critical points of $y = x^3 - 12x^2 + 21x + 1$ and determine their nature.

(b) Write down the coordinates of any local maxima, local minima, global maxima and global minima.

(c) Repeat (b) when the constraint $x \geq 0$ is imposed.

$y = x^3 - 12x^2 + 21x + 1$

$y' = 3x^2 - 24x + 21$

$y'' = 6x - 24$

$y' = 0 \Rightarrow 3x^2 - 24x + 21 = 0 \begin{cases} 7 \\ 1 \end{cases}$

$x = 1 \Rightarrow y = 11$

$x = 7 \Rightarrow y = -97$

C.P. $\begin{cases} (1, 11) \\ (7, -97) \end{cases}$

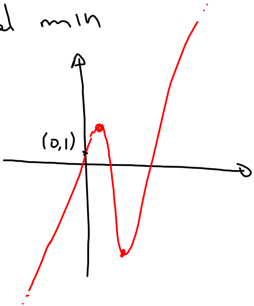
$f''(1) = 6 - 24 = -18 < 0 \Rightarrow$ local max

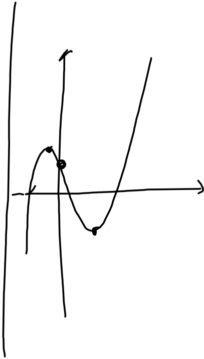
$f''(7) = 42 - 24 = 18 > 0 \Rightarrow$ local min

if $x \geq 0 \Rightarrow (7, -97)$ is the global min

$f'(0) = 3(0)^2 - 24(0) + 21 > 0$

B. LOCAL MIN





Convexity and concavity

Convexity and concavity

Convex and concave functions are important for two reasons :

(1) They give a class of cases for which it is easy to find global optimas.

(2) They occur frequently in economics.

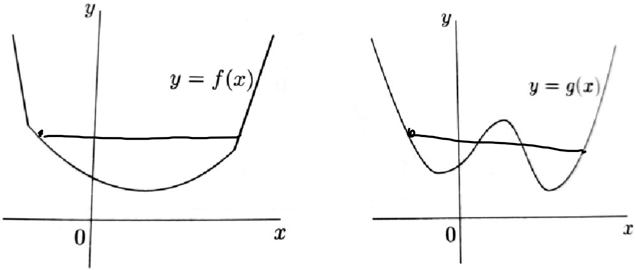


Figure 8.12: f is a convex function, g is not

Convexity

A convex function has the property that a chord joining any two points on its graph lies on or above the graph.

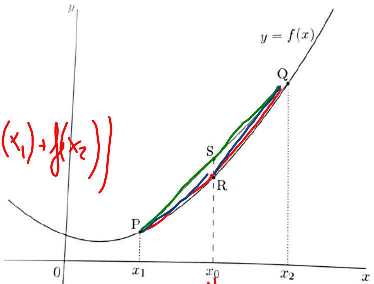
Algebraically, the function f is **convex** if and only if

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$$

for any real numbers x_1, x_2, α such that $0 \leq \alpha \leq 1$.

$\alpha = \frac{1}{2}$

$f\left(\frac{x_1+x_2}{2}\right) \leq \frac{1}{2}\left(f(x_1)+f(x_2)\right)$



$x_0 = \frac{x_1+x_2}{2}$

Figure 8.13: f is a convex function; the chord PQ lies above the curve

Convexity

Since f is convex, R cannot lie above S ; hence the slope of the chord PR cannot exceed the slope of the chord PQ.

It follows that :

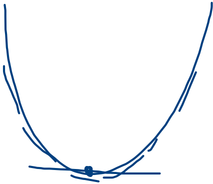
$$f'(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$f(\underbrace{a+h}_{x_2}) \geq f(\underbrace{a}_{x_1}) + h f'(a)$$

Similarly,by considering what happens to the slope of the straight line through Q and R, we have :

$$f'(x_2) \geq \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Convexity



Important properties of differentiable convex functions :

(1) If a f is a differentiable convex function, then $f(a+h) \geq f(a) + hf'(a)$ for all a and h.

(2) A differentiable function f is convex if and only if $f'(a) \leq f'(b)$, whenever $a \leq b$.

(3) A twice-differentiable function f is convex if and only if $f''(x) \geq 0$ for all x.

(4) If f is a differentiable convex function and $f'(x^*) = 0$, then $(x^*, f(x^*))$ is a global minimum point.

Convexity

Exercise 7 Show that the function

$$y = \frac{2}{1 - x^2} \quad (-1 < x < 1)$$

is convex, find its global minimum point.

$$y = \frac{2}{1-x^2} \quad (-1 < x < 1) = 2(1-x^2)^{-1}$$

$$y' = -2(1-x^2)^{-2} \cdot (-2x)$$

$$= \frac{4x}{(1-x^2)^2}$$

$$y'' = \frac{4(1-x^2)^2 - 4x \cdot 2(1-x^2)(-2x)}{(1-x^2)^4}$$

$$= \frac{(4-4x^2)(1-x^2) + 16x^2(1-x^2)}{(1-x^2)^4} = \frac{4+8x^2}{(1-x^2)^3}$$

$$-1 < x < 1 \Rightarrow x^2 < 1 \Rightarrow 1-x^2 > 0$$

$$4+8x^2 > 0$$

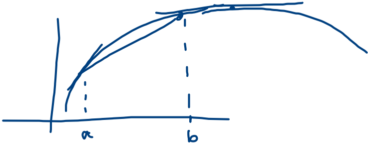
$$\Rightarrow y'' > 0 \text{ for } -1 < x < 1$$

$$f'(x) = \frac{4x}{(1-x^2)^2} = 0 \quad (-1 < x < 1)$$

$$x = 0$$

$$(0, 2)$$

Concavity



A concave function has the property that a chord joining any two points on its graph lies *on or below* the graph.

Hence, the **function f is concave if and only if $-f$ is convex**. It follows that,

The function f is **concave** if and only if

$$f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha)f(x_2)$$

for any real numbers x_1, x_2, α such that $0 \leq \alpha \leq 1$.

Concavity

Important properties of differentiable concave functions :

(1) If a f is a differentiable concave function, then $f(a+h) \leq f(a) + hf'(a)$ for all a and h.

(2) A differentiable function f is concave if and only if $f'(a) \geq f'(b)$, whenever $a \leq b$.

(3) A twice-differentiable function f is concave if and only if $f''(x) \leq 0$ for all x.

(4) If f is a differentiable concave function and $f'(x^*) = 0$, then $(x^*, f(x^*))$ is a global maximum point.

Concavity

Exercise 8 Show that the function

$$y = -x^4 + 4x^3 - 6x^2 + 8x + 3$$

is concave and that its slope is zero where $x = 2$. Deduce the coordinates of the global maximum point.

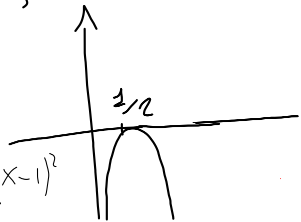
$$y = -x^4 + 4x^3 - 6x^2 + 8x + 3$$

$$y' = -4x^3 + 12x^2 - 12x + 8$$

$$y'' = -12x^2 + 24x - 12$$

$$= -12(x^2 - 2x + 1) = -12(x-1)^2$$

$$-x^2 + 2x - 1 = 0$$



$$x = \frac{1}{2} \quad y'' \leq 0 \ \forall x \in \mathbb{R}$$

$$f'(2) = 0$$

global MAX $(2, 11)$

Exponential and logarithmic functions

The exponential function

Th exponential function is one of the most important functions in mathematics. It has wide applications in economics.

It has five key properties :
(1) $\exp 0 = 1$

(2) $\exp(a + b) = (\exp a) \times (\exp b)$

(3) $\exp(-a) = 1/(\exp a)$

(4) $\exp(ac) = (\exp a)^c$

(5) $\frac{d}{dx}\left(\exp\{\underset{=x}{u(x)}\}\right) = \underset{=1}{u'(x)}\exp\{\underset{=x}{u(x)}\}$ if $u(x)$ is differentiable.

It follows that $\frac{d}{dx}(\exp x) = \exp x$

The exponential function

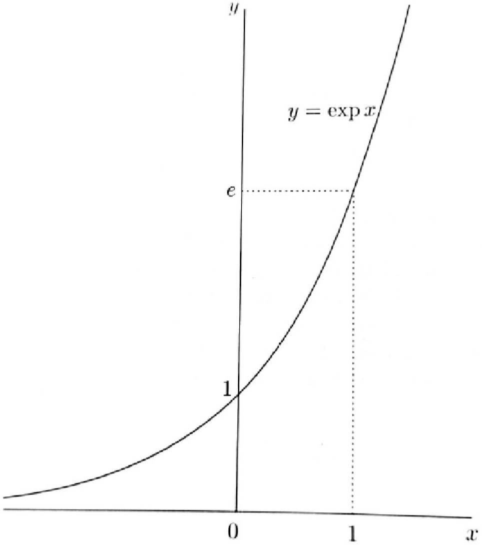


Figure 9.1: **The exponential function**

The exponential function

From the chart, we can also see that :
- $\exp x$ is positive for all real x.

- $\exp x$ is monotonic increasing.

Using **(5)**, we can also show that :

$$\frac{d^2}{dx^2}(\exp x) = \exp x > 0$$

so $\exp x$ is a strictly convex function.

Noting $\exp 1 = e$ and using Property **(4)**, we can also note

$$\exp x = e^x \quad \text{for all } x \in \mathbb{R}$$

The exponential function

$2e^{2x} - 12e^{-4x}$

Exercise 9 Differentiate

(a) $e^{2x} + 3e^{-4x}$

(b) xe^{2x}

(c) $1/(1 + e^x)$

$e^{2x} + x \cdot 2e^{2x} = e^{2x}(1 + 2x)$

$-\dfrac{e^x}{(1+e^x)^2}$

The natural logarithm

The natural logarithm is the **inverse of the exponential function**.

Thus,

$$\begin{aligned} \exp(\ln x) = x \quad & \text{for every positive number } x. \\ \ln(\exp y) = y \quad & \text{for every real number } y. \end{aligned}$$

Since 'ln' is the **inverse function** of 'exp',

- the natural logarithm function is monotonic increasing and concave.

- $\ln 1 = 0$.

The natural logarithm

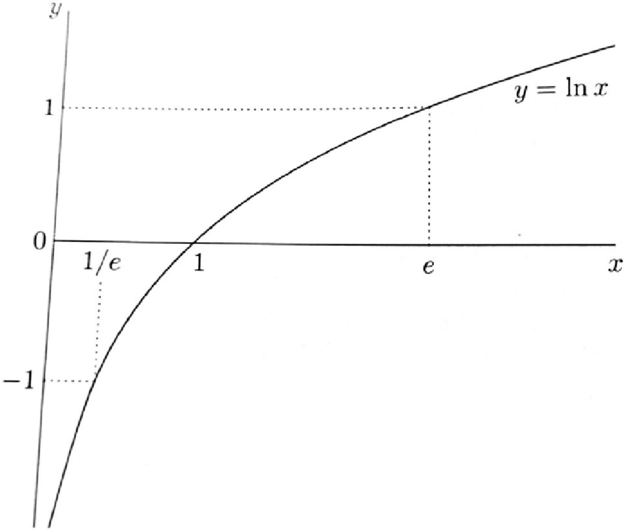


Figure 9.2: The natural logarithm function

The natural logarithm

Properties :

(1) $\ln(1/x) = -\ln(x)$

(2) $\ln(ab) = \ln a + \ln b$

(3) $\ln(a/b) = \ln a - \ln b$

(4) $\ln(x^c) = c \ln x$ for any real number c

(5) $\frac{d}{dx}\big(\ln u(x)\big) = \frac{u'(x)}{u(x)}$

It follows that : $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

Finally, a useful approximation in economics :

$$\ln(1+h) \approx h \quad \text{if } |h| \text{ is small.}$$

The natural logarithm

$\ln(a) + \ln(x)$

Exercise 10 Differentiate

(a) $\ln(ax)$ $\rightarrow f'(x) = \frac{1}{x}$

(b) $\ln(x^4 + 1)$ $\rightarrow f'(x) = \frac{1}{x^4+1} \cdot 4x^3 = \frac{4x^3}{x^4+1}$

(c) $x \ln x$ $\rightarrow f'(x) = \ln x + x \cdot \frac{1}{x} = \ln x + 1$

Exercise 11 Find the critical points of the function

$$y = x^2 - 2\ln(1 + x^2)$$

and determine their nature.

$$y = x^2 - 2\ln(1+x^2)$$

$$y' = \frac{2x^3 - 2x}{x^2+1}$$

$$y'' = \frac{(x^2+1)(6x^2-2) - 4x^2(x^2-1)}{(x^2+1)^2}$$

For critical points

$$\frac{2x(x+1)(x-1)}{1+x^2} = 0$$

$$x = 0,\ 1,\ -1$$

$$y_1(x=0) = 0$$

$$y_2(x=\pm 1) = -0.38 = 1 - 2\ln 2$$

$$y''(0) = \frac{(-2) - 0}{1} = -2 < 0$$

Max

$$y''(1) = \frac{(2)(4) - 0}{4} = 2 > 0$$

Min

$$y''(-1) = \frac{(1+1)(4) - 0}{4} = 2 > 0$$

Min

Approximations

Linear approximation

Recall,

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

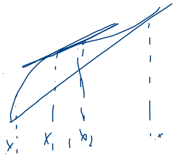
Which means that if h is close to 0 we have

$$\frac{f(x_0 + h) - f(x_0)}{h} \approx f'(x_0)$$

Therefore we have $f(x + h) - f(x) \approx hf'(x)$ if $|h|$ is small.

This approximation is called the **small increments formula**.

Linear approximation



The **linear approximation** to the curve $y = f(x)$ is the tangent at a given point a. We denote its equation by

$$y = L(x) = f(a) + (x - a)f'(a)$$

When we approximate the curve $y = f(x)$ by its tangent, namely the line $y = L(x)$, we approximate $f(a + h)$ by
$L(a + h) = f(a) + hf'(a)$.

Obviously the larger the distance between the two points will be, the less precise will be the approximation.

Linear approximation

Exercise 12 Find the linear approximations to $y = x^3$ at the points where (a) $x = 2$, (b) $x = 3$.

Show in a table the approximate values given by each linear approximation and the true values when $x = 2.1, 2.2, \ldots, 2.9$. Comment.

	x	2.1	2.2	$\ldots$	2.9
$a = 2$	$L(x)$				
$a = 3$	$L(x)$				
	x^3				

$f(x) = x^3$

$f'(x) = 3x^2$

$$L(x) = f(a) + (x-a)f'(a)$$

$$= a^3 + (x-a)\,3a^2$$

$$= a^3 + 3a^2x - 3a^3$$

$L_2(x)$

$L_3($

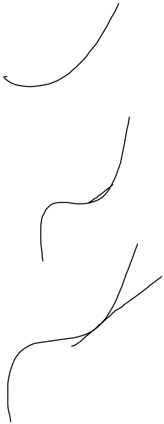
$a=2$ $\quad L_2(x) = 8 + 12x - 24$

$$= 12x - 16$$

$a=3$ $\quad L_3(x) = 27 + 27x - 81$

$$= 27x - 54$$

	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9
$L_2(x)$	9.2								18.8
$L_3(x)$	2.7								24.3
x^3	9.261								24.39



Newton's method

Suppose we are trying to solve $f(x) = 0$.

We first choose a starting point $x = a$, which is our first approximation of a root of $f(x) = 0$.

We know that a linear approximation to $f(x)$ at point $(a, f(a))$ is $L(x) = f(a) + (x - a)f'(a)$ $\quad L(x) = 0 \Rightarrow f(a) + (\underbrace{x}_{b} - a) f'(a) = 0$

Then, as a second approximation we choose the value of x such that $L(x) = 0$, we call it b.

$$b = a - \frac{f(a)}{f'(a)}$$

So computing $L(b) = 0$ gives us

$$b = a - \frac{f(a)}{f'(a)}$$

Newton's method

We thus have the Newton's method : starting with a first approximation of a to a root of the equation $f(x) = 0$, we choose as our second approximation b. b is indeed a better approximation than a.

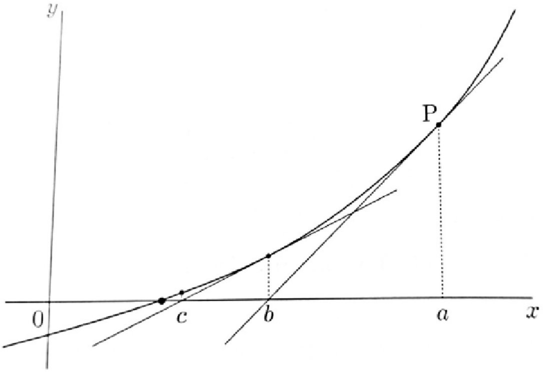


Figure 10.2: **Newton's method**

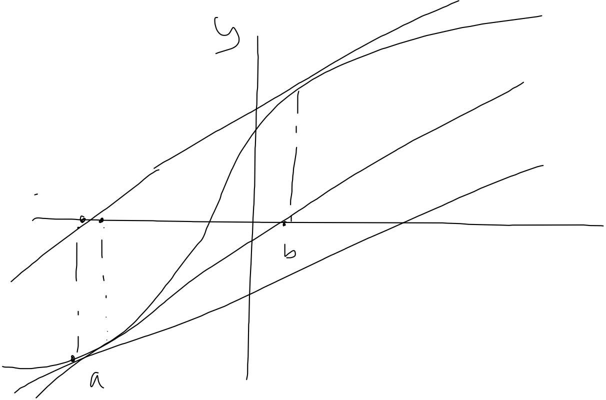
Newton's method

We can repeat the procedure again applying the "*update function*"

$$U(x) = x - \frac{f(x)}{f'(x)}$$

to every new approximation, until the difference between two successive approximations becomes "*small enough*".

N.B. In some cases, depending on the starting point you choose Newton's method fails, and every new approximation given by the update function is further away from the result.


y
a
b

Newton's method

$$u(x) = x - \frac{f(x)}{f'(x)} \quad \underline{\underline{x = 0.4}} \quad 0.40210186$$

Exercise 13 Find an approximate solution of the equation, starting from $a = 0.4$

$$x^5 - 5x + 2 = 0$$

Exercise 14 Show that the equation

$$x^7 - 6x + 4 = 0$$

$$a = \frac{4}{6} = \frac{2}{3} = 0.662$$

has a root between 0 and 1. Find an initial approximation by ignoring the term x^7. Use Newton's method to find the root correct to 3 decimal places.

The mean value of theorem

If f is a differentiable function and a and b are real numbers such that $a < b$, there exists a real number c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

As illustrated below, given the chord AB, there is a point C on the curve such that the tangent at C is parallel to the line-segment AB.

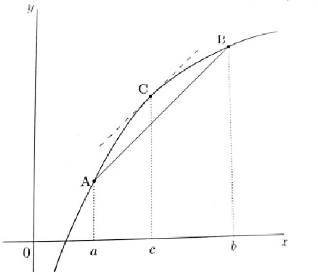


Figure 10.4: The mean value theorem: the tangent at C is parallel to the chord AB

The mean value of theorem

Let a, b, c be as previously and let $h = b - a$. Since c lies between a and b, $c = a + sh$ for some number s such that $0 < s < 1$. Thus,

$$f(a+h) = f(a) + hf'(a+sh), \quad \text{where } 0 < s < 1$$

if $|h|$ is small, we may approximate the term $hf'(a+sh)$ by $hf'(a)$.

Thus,

$$f(a+h) \approx f(a) + hf'(a) \quad \text{if } |h| \text{ is small}$$

We have again the small increment formula.

Quadratic approximation

Second mean value theorem If f is twice differentiable and a and b real numbers such that $a < b$, there exists a real number d such that $a < d < b$ and

$$f(b) = f(a) + (b-a)f'(a) + \frac{1}{2}(b-a)^2 f''(d)$$

As before, let $h = b - a$ and $d = a + th$, with $0 < t < 1$. Thus,

$$f(a+h) = f(a) + hf'(a) + \frac{1}{2}h^2 f''(a+th) \quad \text{where } 0 < t < 1$$

If $|h|$ is small, we may approximate $\frac{1}{2}h^2 f''(a+th)$ by $\frac{1}{2}h^2 f''(a)$. Therefore,

$$f(a+h) \approx f(a) + hf'(a) + \frac{1}{2}h^2 f''(a) \quad \text{if } |h| \text{ is small}$$

Letting $h = x - a$, we get the parabola $y = Q(x)$ as a **quadratic approximation** to the curve $y = f(x)$

$$Q(x) = f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a)$$

Quadratic approximation

$$f'(x) = x + 2x\ln x \qquad f''(x) = 3 + 2\ln x$$

$$L(x): \; f(a) + (x-a)f'(a) \Rightarrow L(x) = f(1) + (x-1)f'(1) = x-1$$

$$Q(x): \; f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 = x - 1 + \frac{3}{2}(x-1)^2$$

Exercise 15 Find the approximation of $f(x) = x^2 \ln x$ by linear and quadratic functions of x, for x close to 1 (0.95, 1.02, 1.10).

x	0.95	1.02	1.10
$L(x)$	-0.05	0.02	0.10
$Q(x)$	-0.04625	0.0206	0.115
$f(x)$	-0.04629	0.0206	0.11533

Taylor approximations

We have shown that the first and second mean value theorems enable us to approximate $f(a+h)$, when $|h|$ is small by linear or quadratic functions of h.

Similarly, Taylor's theorem allows us to approximate $f(a+h)$ by higher-degree polynomials.

$n! = n(n-1)(n-2)\cdots$

Taylor's theorem If f is a smooth function and n a positive integer, then

$2! = 2\times 1$

$3! = 3\times 2\times 1 = ($

$0! = 1$

$\frac{h^3}{3!} f^{(3)}(a)$

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+th)$$

If $|h|$ is small, we may approximate $\frac{h^n}{n!}f^{(n)}(a+th)$ by $\frac{h^n}{n!}f^{(n)}(a)$.

≈ 0

Therefore,

$(x-a)$ $(x-a)^2$

$$f(a+h) \approx f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a)$$

x

Taylor approximations

Exercise 16 Find the Taylor Series for $f(x) = e^{-6x}$ about $x = -4$. *Hint* : first find the general term at any x, and then show the general term at $x = -4$.

$$f(x) = \sum \quad - \quad -$$

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{1}{k!}(x-a)^k f^{(k)}(a) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k$$

where $f^{(k)} = k$-derivative;

$$f(x) = e^{-6x}$$

$$\frac{df}{dx} = f'(x) = -6e^{-6x}$$

$$\frac{d^2f}{(dx)^2} = f''(x) = (-6)\cdot(-6)\cdot e^{-6x} = (-6)^2 e^{-6x}$$

$$\frac{d^kf}{(dx)^k} = f^{(k)}(x) = (-6)^k \cdot e^{-6x}$$

$$f^{(k)}(a) = (-6)^k e^{-6a}$$

$$e^{-6x} \approx \sum_{k=0}^{n} \frac{(x+4)^k}{k!} e^{24} (-6)^k$$

Taylor approximation $a = -4$

$$f(x) = f(a) + \sum_{k=1}^{\infty\, n} \frac{1}{k!}(x-a)^k f^{(k)}(a) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(a)(x-a)^k$$

where $f^{(k)} = k$-derivative;

$$f(x) = e^{-6x}$$

$$\frac{df}{dx} = f'(x) = -6e^{-6x}$$

$$\frac{d^2f}{(dx)^2} = f''(x) = (-6)\cdot(-6) \cdot e^{-6x} = (-6)^2 e^{-6x}$$

$$\frac{d^kf}{(dx)^k} = f^{(k)}(x) = (-6)^k \cdot e^{-6x}$$

$$f^{(k)}(a) = (-6)^k e^{-6\times a}$$

$$e^{-6x} \approx \sum_{k=0}^{n} \frac{(x+4)^k}{k!} e^{24} (-6)^k$$

Taylor approximation $a = -4$

Log-Linearization

Log-Linearization

Then we approximate each function using their first order Taylor series expansions around the steady state x^* (here $x - x^* = h$)

$a = x^*$

$u(x) = u(a) + (x-a)u'(a)$

$u(x) = \ln(f(x))$

$u'(x) = \frac{f'(x)}{f(x)}$ $\quad u'(x^*) = \frac{f'(x^*)}{f(x^*)}$

$$\ln f(x) = \ln f(x^*) + \frac{f'(x^*)}{f(x^*)}(x - x^*)$$

$$\ln g(x) = \ln g(x^*) + \frac{g'(x^*)}{g(x^*)}(x - x^*)$$

$$\ln h(x) = \ln h(x^*) + \frac{h'(x^*)}{h(x^*)}(x - x^*)$$

If we replace those approximations in (1), rearranging a little bit we get a linear relation of the distance to the steady state x^*

$$\frac{f'(x^*)}{f(x^*)}(x - x^*) = \frac{g'(x^*)}{g(x^*)}(x - x^*) - \frac{h'(x^*)}{h(x^*)}(x - x^*)$$

$\ln(f(x)) = \ln(g(x)) - \ln(h(x))$

$\ln f(x^*) + \frac{f'(x^*)}{f(x^*)}(x-x^*) = \ln(g(x^*)) + \frac{g'(x^*)}{g(x^*)}(x-x^*) - \ln(h(x^*)) - \frac{h'(x^*)}{h(x^*)}(x-x^*)$

Log-Linearization

$K_{t+1} = I_t + SK_t$

The solutions to many discrete time dynamic economic problems take the form of a system of non-linear difference equations.

It is often the case for Macroeconomics problems. One way to end up with a solution, or more precisely with a system of linear difference equations, is what we call the **log-linearization**.

Consider a non linear function $f(x) = \frac{g(x)}{h(x)}$.

$\ln(f(x)) = \ln\left(\frac{g(x)}{h(x)}\right) = \ln(g(x)) - \ln(h(x))$

We first take natural logs of both sides,

$x_t = x_{t+1} = x^*$

$$\ln f(x) = \ln g(x) - \ln h(x) \tag{1}$$

$\ln(f(x^*)) = \ln(g(x^*)) - \ln(h(x^*))$

Log-Linearization

Application Consider a Cobb-Douglas production function

$$y_t = a_t k_t^{\alpha} l_t^{1-\alpha}$$

$\ln(y_t) = \ln\left[a_t k_t^{\alpha} l_t^{1-\alpha}\right]$

We first take the natural log of the function

$$\ln y_t = \ln a_t + \alpha \ln k_t + (1-\alpha) \ln l_t$$

$\ln y^* = \ln(a^*) + \alpha \ln k^* + (1-\alpha) \ln l^*$

Now we take the first order Taylor expansion at the steady state values

$$\ln y^* + \frac{1}{y^*}(y_t - y^*) = \ln a^* + \frac{1}{a^*}(a_t - a^*) + \alpha \ln k^* + \frac{\alpha}{k^*}(k_t - k^*)$$
$$+ (1-\alpha) \ln l^* + \frac{1-\alpha}{l^*}(l_t - l^*)$$

Log-Linearization

$$\hat{y}_t = \frac{1}{y^*}(y_t - y^*) = \frac{1}{a^*}(a_t - a^*) + \frac{\alpha}{k^*}(k_t - k^*) + \frac{1-\alpha}{l^*}(l_t - l^*)$$

Knowing that $\ln y^* = \ln a^* + \alpha \ln k^* + (1-\alpha) \ln l^*$ and defining $\hat{x}_t = \frac{x_t - x^*}{x^*}$
$\hat{y}_t = \frac{(y_t - y^*)}{y^*}$ the percentage deviation of x_t (y_t) from x^* (y^*), we get the following linear relation

$$\hat{y}_t = \hat{a}_t + \alpha \hat{k}_t + (1 - \alpha)\hat{l}_t$$

This is again a very useful and commonly used method in economics, in particular for macroeconomics models.

Log-Linearization

$g(x,y) = \ln f(x,y)$

$g_x(x,y) = \frac{f_x(x,y)}{f(x,y)}$

Multivariate case First-order Taylor approximations can also be used to convert equations with more than one endogenous variable to log-deviations form (more to come in Lecture 3).

Start with $x_{t+1} = g(x_t, y_t)$ and employ a first-order Taylor approximation at the steady state $x_t = x^*$ and $y_t = y^*$ to get

$$x_{t+1} \approx g(x^*, y^*) + g'_x(x^*, y^*)(x_t - x^*) + g'_y(x^*, y^*)(y_t - y^*)$$

.

$f(x,y) = 2x + 4y \quad \frac{\partial f}{\partial x} = 2 \quad \frac{\partial f}{\partial y} = 4$

Log-Linearization

Exercise 17 Consider the standard capital accumulation equation

$$k_{t+1} = i_t + (1-\delta)k_t$$

Use log-linearization to approximate the equation around the steady state. *Hint* : at the steady state, $x_{t+1} = x_t = x^*$

$$\ln k_{t+1} = \ln\left(\underbrace{i_t + (1-\delta)k_t}_{f(i_t, k_t)}\right)$$

$$k_{t+1} = i_t + (1-\delta)k_t$$

$$\ln(f(x,y)) \approx \ln(f(x^*,y^*)) + \frac{f_x(x^*,y^*)}{f(x^*,y^*)}(x_t - x^*) + \frac{f_y(x^*,y^*)}{f(x^*,y^*)}(y_t - y^*)$$

$$\ln k_{t+1} = \ln(\underbrace{i_t + (1-\delta)k_t}_{f(i_t,\,k_t)})$$

$$k_{t+1} = i_t + (1-\delta)k_t$$

$$k^* = i^* + (1-\delta)k^* \Rightarrow \delta k^* = i^*$$

$$\Rightarrow k^* = \frac{i^*}{\delta}$$

$$\ln k^* = \ln(i^* + (1-\delta)k^*)$$

$$\cancel{\ln k^*} + \frac{1}{k^*}(k_{t+1} - k^*) = \cancel{\ln(i^* + (1-\delta)k^*)} + \frac{1}{\underbrace{i^* + (1-\delta)k^*}_{k^*}}(i_t - i^*) + \frac{1-\delta}{\underbrace{i^* + (1-\delta)k^*}_{k^*}}(k_t - k^*)$$

$$k^* = \frac{i^*}{\delta}$$

$$\hat{k}_{t+1} = \delta\,\hat{i}_t + (1-\delta)\,\hat{k}_t$$

Log-Linearization

Uhlig's Approach Let x_t be our variable of interest around x^*. We first construct $\hat{x}_t$, the log deviation of x_t from x^*

$$\hat{x}_t = \log(x_t) - \log(x^*) = \log\left(\frac{x_t}{x^*}\right)$$

Then

$$x_t = x^* e^{\hat{x}_t}$$

We denote our point of approximation $\hat{x}_0$ and since we are interested in an approximation around the steady state, our point of approximation will be $\hat{x}_0 = 0$.

Log-Linearization

Then, approximating our previous result we get

$$\underbrace{x^* e^{\hat{x}_t}}_{f(\hat{x}_t)} \approx \underbrace{x^* e^{\hat{x}_0}}_{f(\hat{x}_0)} + \underbrace{x^* e^{\hat{x}_0}}_{f'(\hat{x}_0)}(\hat{x}_t - \hat{x}_0) = x^* e^{\hat{x}_0}(1 + \hat{x}_t - \hat{x}_0)$$

Plugging in $\hat{x}_0 = 0$ we get

$$x_t = x^* e^{\hat{x}_t} \approx x^*(1 + \hat{x}_t)$$

Trick we ignore crossed products.

$$x_t y_t \approx x^* y^*(1 + \hat{x}_t)(1 + \hat{y}_t) \approx x^* y^*(1 + \hat{x}_t + \hat{y}_t)$$

Products and powers are easily addressed by this approach.

Log-Linearization

Exercise 18
Linearize $x_t^{\rho} y_t^{1-\sigma} = 1$, around $\hat{x}_0 = 0, \hat{y}_0 = 0$

What about additive terms ?
Linearize the standard macro income identity : $Y_t = C_t + I_t$