International Finance
Lecture II: An Open Economy with Capital

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Motivation

- This chapter introduces production and physical capital accumulation.
- Doing so will allow us to address two important issues:
  1. For the most commonly used stationary specifications of the shock process—namely, AR(1) specifications—the endowment economy model presented in Chapter 2 fails to predict the observed countercyclicality of the trade balance and the current account.
  2. The assumption that output is an exogenously given stochastic process—maintained throughout Chapter 2—is unsatisfactory if the goal is to understand observed business cycles. As output is perhaps the main variable any theory of the business cycle should aim to explain.
- To allow for a full characterization of the equilibrium dynamics using pen and paper we abstract from depreciation and uncertainty, and assume, as in Chapter 2, that $\beta(1 + r) = 1$.

Intuition

- Allowing for production and capital accumulation might induce the model to predict a counterycyclical trade balance, even for AR(1) shock processes.
- Suppose persistent AR(1) productivity shocks are the main source of uncertainty.
- Then the marginal product of capital is expected to be high **not just in the period of the shock but also in future periods**.
- Thus the economy has an incentive to invest more to take advantage of the higher productivity of capital.
- This increase in domestic demand might be so large that **total domestic demand**, consumption plus investment, rises by more than output, resulting in a countercyclical impact response of the trade balance.

Two Principles

We will derive the following 2 principles:

Principle I: The more persistent productivity shocks are, the more likely an initial deterioration of the trade balance will be.

Principle II: The more pronounced are capital adjustment costs, the smaller will be the initial trade balance deterioration in response to a positive and persistent productivity shock.

3.1 Model

Small open economy, no uncertainty, no depreciation.
Preferences:

$$\sum_{t=0}^{\infty} \beta^t U(c_t) \tag{3.1}$$

Sequential budget constraint of the household:

$$c_t + i_t + (1+r)d_{t-1} = y_t + d_t \tag{3.2}$$

Interpretation: LHS displays the uses of wealth: purchases of consumption goods (c_t); purchases of investment goods (i_t); payment of principal and interest on outstanding debt ($(1+r)d_{t-1}$). RHS displays the sources of wealth: output (y_t) and new debt issuance (d_t).

3.1 Model

Production function:

$$y_t = A_t F(k_t) \tag{3.3}$$

$A_t =$ exogenous and deterministic productivity factor
$F(\cdot) =$ increasing and concave production function which depends on $k_t > 0$. This is physical capital which is determined by the existing stock of capital and investment in $t-1$
Law of motion of capital (we assume no depreciation):

$$k_{t+1} = k_t + i_t \tag{3.4}$$

NPG constraint:

$$\lim_{j\to\infty} \frac{d_{t+j}}{(1+r)^j} \leq 0 \tag{3.5}$$

3.1 Model

Lagrangian of household's problem:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ U(c_t) + \lambda_t \left[A_t F(k_t) + d_t - c_t - (k_{t+1} - k_t) - (1+r) d_{t-1} \right] \right\}.$$

- Note that we we substituted directly $y_t = A_t F(k_t)$ and $i_t = k_{t+1} - k_t$ in the sequential budget constraint
- We could have left y_t and i_t in the sequential budget constraint and added two more constraints with their own Lagrange multipliers

3.1 Model

Lagrangian of household's problem:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ U(c_t) + \lambda_t \left[A_t F(k_t) + d_t - c_t - (k_{t+1} - k_t) - (1+r)d_{t-1} \right] \right\}.$$

The first-order conditions corresponding to c_t, d_t, k_{t+1} (equivalent to i_t. Why no FOC for k_t?), and λ_t, respectively, are

$$U'(c_t) = \lambda_t, \tag{3.6}$$

$$\lambda_t = \beta(1+r)\lambda_{t+1}, \tag{3.7}$$

$$\lambda_t = \beta\lambda_{t+1}[A_{t+1}F'(k_{t+1}) + 1], \tag{3.8}$$

$$A_t F(k_t) + d_t = c_t + k_{t+1} - k_t + (1+r)d_{t-1}. \tag{3.9}$$

Household optimization implies that the NPG constraint holds with equality (transversality condition):

$$\lim_{t\to\infty} \frac{d_t}{(1+r)^t} = 0. \tag{3.10}$$

3.1 Model

Assume that

$$\beta(1 + r) = 1 \tag{*}$$

This assumption together with (3.6) and (3.7) implies that consumption is constant

$$c_{t+1} = c_t \tag{3.11}$$

As we will see shortly, consumption is again determined by non-financial permanent income net of interest on initial debt outstanding.
This assumption also implies that $\lambda_{t+1} = \lambda_t$ (see Equation (3.7))

3.1 Model

Assumption (*) and equilibrium condition (3.8) imply that

$$r = A_{t+1}F'(k_{t+1}) \tag{3.12}$$

This is because λ is constant and we can substitute β with $\frac{1}{1+r}$

- Households invest in physical capital in period t until the expected marginal product of capital in period $t+1$ equals the rate of return on foreign debt.
- It follows from this equilibrium condition that next period's level of physical capital, k_{t+1}, is an increasing function of the future expected level of productivity, A_{t+1}, and a decreasing function of the interest rate r.

$$k_{t+1} = \kappa\left(\frac{A_{t+1}}{r}\right), \tag{3.14}$$

with $\kappa' > 0$.

3.1 Model

To characterize the equilibrium, it will again be convenient to work with the intertemporal budget constraint. Write the sequential budget constraint for period $t+j$:

$$A_{t+j}F(k_{t+j}) + d_{t+j} = c_{t+j} + k_{t+j+1} - k_{t+j} + (1+r)d_{t+j-1}$$

Divide by $(1+r)^j$ and sum for $j=0$ to $j=J$.

$$\sum_{j=0}^{J} \frac{A_{t+j}F(k_{t+j})}{(1+r)^j} + \frac{d_{t+J}}{(1+r)^J} = \sum_{j=0}^{J} \frac{c_{t+j} + k_{t+j+1} - k_{t+j}}{(1+r)^j} + (1+r)d_{t-1}$$

Now use the fact that in eqm consumption is constant over time, (3.11), and rearrange terms

$$c_t \sum_{j=0}^{J} \frac{1}{(1+r)^j} + (1+r)d_{t-1} = \sum_{j=0}^{J} \frac{A_{t+j}F(k_{t+j}) - (k_{t+j+1} - k_{t+j})}{(1+r)^j} + \frac{d_{t+J}}{(1+r)^J}$$

Take limit for $J \to \infty$ and use the transversality condition (3.10) to obtain:

3.1 Model

$$c_t + rd_{t-1} = y_t^p \equiv \frac{r}{1+r} \sum_{j=0}^{\infty} \frac{A_{t+j}F(k_{t+j}) - (k_{t+j+1} - k_{t+j})}{(1+r)^j} \tag{3.13}$$

Interpretation: The right-hand side of (3.13) is the household's nonfinancial permanent income, y_t^p. (It is a natural generalization of a similar expression obtained in the endowment economy, see equation 2.10).

In the present environment, nonfinancial permanent income is given by a weighted average of present and future expected output net of investment expenditure.

Thus, equilibrium condition (3.13) states that each period households allocate their nonfinancial permanent income to consumption and to servicing their debt.

Similar to what we saw for the endowment economy

3.1 Model: Definition of Equilibrium

- A **perfect-foresight equilibrium** is a value c_0 and a sequence $\{k_{t+1}\}_{t=0}^{\infty}$ satisfying (3.13) evaluated at $t = 0$, and (3.12) for all $t \geq 0$, given the initial stock of physical capital, k_0, the initial net external debt position, d_{-1}, and the sequence of productivity $\{A_t\}_{t=0}^{\infty}$.
- This is a system that we can fully characterize with pen and paper. [Obtain eqm values for c_t from (3.11), i_t from (3.4), y_t from (3.3), and d_t from (3.2)]
- Note that k_t for $t > 0$ is a function of the exogenous variable A_t only.
- Thus permanent income, y_t^p, is a function of productivity only and is increasing in present and future values of productivity.

3.1 Model

Trade balance: $tb_t = y_t - c_t - i_t$

To obtain the prediction of a countercyclical trade balance response it is no longer required that consumption increases by more than one-for-one with output.

As long as domestic absorption, $c_t + i_t$, increases by more than output, the model will predict a countercyclical trade balance response.

Next, we study adjustment to permanent and temporary productivity shocks and ask whether the model predicts a countercyclical trade balance response.

Equilibrium with Constant Productivity

Suppose $A_t = \bar{A}$ for all $t \geq 0$ with certainty and $k_0 = \bar{k} \equiv \kappa\left(\frac{\bar{A}}{r}\right)$. These assumptions lead to a steady in which all variables for $t < 0$ are constant:

By (3.14), $k_t = \bar{k}$ for all $t > 0$

By (3.11) and (3.13), $c_t = \bar{c} \equiv -rd_{-1} + \bar{A}F(\bar{k})$

and $d_t = d_{-1}$ for all $t \geq 0$

Output: $y_t = \bar{y} \equiv \bar{A}F(\bar{k})$

Trade balance: $tb_t = \bar{tb} \equiv rd_{-1}$ (investment is zero because of we assumed no depreciation)

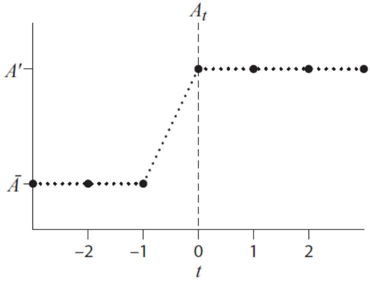
Current account: $ca_t = d_{t-1} - d_t = 0$

Note that $\bar{x}$ denotes the steady-state value of variable x

3.3 Adjustment to a Permanent Unanticipated Increase in Productivity

Experiment: In period 0 it is learned that A_t increases from $\bar{A}$ to $A' > \bar{A}$ for all $t \geq 0$. Prior to period 0, A_t was expected to be $\bar{A}$ forever.

$$A_t = \begin{cases} \bar{A} & \text{for } t \leq -1 \\ A' > \bar{A} & \text{for } t \geq 0 \end{cases} .$$



Adjustment of Capital and Investment

For $t > 0$, by (3.14)

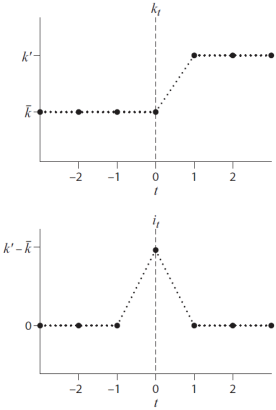
$$k_t = k' \equiv \kappa\left(\frac{A'}{r}\right) > \bar{k}$$

Thus positive investment in period 0 and zero investment thereafter (remember, no depreciation. Thus, if capital stock is constant investment=0).

$$t = 0 : i_0 = k' - \bar{k} > 0$$

$$t > 0 : i_t = 0$$

Adjustment of Capital and Investment

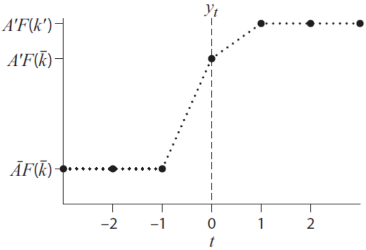


Adjustment of Output

Output increases in period 0 because A_0 rises (remember k_0 is predetermined, as it was decided in t_{-1}) and then again in period 1 because k_1 (decided at t_0) is larger:

$$t = 0 : y_0 = A'F(\bar{k}) > \bar{A}F(\bar{k}) = \bar{y}$$

$$t > 0 : y_t = A'F(k') > A'F(\bar{k}) = y_0 > \bar{y}$$



Adjustment of Consumption

What about consumption?
Intuitively, it should increase. How to show that it does? By (3.11)

$$c_t = c_0$$

for all $t \geq 0$.

Thus, we only need to find c_0.

By (3.13), $c_0 = y_0^p - rd_{-1}$. If permanent income in period 0 rises, so does consumption.

Let's find first the adjustment in y_0^p.

Adjustment of Consumption

Remember that permanent income is:

$$y_t^p \equiv \frac{r}{1+r}\sum_{j=0}^{\infty}\frac{A_{t+j}F(k_{t+j})-(k_{t+j+1}-k_{t+j})}{(1+r)^j}$$

From period 1 on, $i_t = 0$, thus using the definition of y_0^p we have:

$$\begin{aligned}
y_0^p &= \frac{r}{1+r}\left[(A'F(\bar{k})-k'+\bar{k}\right]+\frac{1}{1+r}A'F(k') \\
&= A'F(\bar{k})+\frac{1}{1+r}\left[A'F(k')-A'F(\bar{k})-r(k'-\bar{k})\right] \\
&= A'F(\bar{k})+\frac{1}{1+r}\left[A'F(k')-A'F(\bar{k})-A'F'(k')(k'-\bar{k})\right] \\
&> A'F(\bar{k})(=y_0) \\
&> \bar{A}F(\bar{k})(=y_{-1}^p).
\end{aligned}$$

(Note that we used equation 3.12 to substitute r. Also note that the first inequality follows from the facts that $F(\cdot)$ is increasing and concave and that $k' > \bar{k}$. So: $\frac{F(k')-F(\bar{k})}{k'-\bar{k}} > F'(k')$

Adjustment of Consumption

$$
\begin{aligned}
y_0^p &> A'F(\bar{k})(= y_0) \\
y_0^p &> \bar{A}F(\bar{k})(= y_{-1}^p).
\end{aligned}
$$

* Because in period 0 permanent income exceeds current income, we have that c_0 increases by more than y_0.

$$
c' > A'F(\bar{k}) - r\bar{d} > \bar{A}F(\bar{k}) - r\bar{d} = \bar{c} \tag{1}
$$

* Consumption experiences a once-for-all increase in period 0. In fact, consumption increases more than output

$$
\begin{aligned}
c_0 - c_{-1} &= c' - \bar{c} = c' - (\bar{A}F(\bar{k}) - r\bar{d}) \\
&> (A'F(\bar{k}) - r\bar{d}) - (\bar{A}F(\bar{k}) - r\bar{d}) \\
&= A'F(\bar{k}) - \bar{A}F(\bar{k}) = y_0 - y_{-1} \\
c_0 - c_{-1} &> y_0 - y_{-1}
\end{aligned}
$$

Adjustment of Consumption

* Because in period 0 permanent income exceeds current income, we have that c_0 increases by more than y_0.
* Consumption experiences a once-for-all increase in period 0. In fact, consumption increases more than output
* This by itself—that is, ignoring the increase in i_0—leads to a negative trade balance response in period 0.
* By contrast, in the endowment economy of Chapter 2 a once-and-for-all increase in the endowment leaves the trade balance unchanged.
* The intuition for this result is that the path of output is upward sloping in the economy with capital in response to the permanent shock. Output keeps growing after period 0.
* As a consequences HHs borrow against future income to finance consumption (like with an AR(2) shocks in the previous chapter)

Adjustment of the Trade Balance

* We just established that the trade balance deteriorates in period 0. But what about period 1?
* In period 1, output is higher than in period 0, consumption is the same (Why?), and investment is lower. Thus $tb_1 > tb_0$. The trade balance improves.

For $t > 0$, $tb_t = tb' > tb_0$. Is tb' greater or less than tb_{-1}?
By (3.2), for $t > 0$:

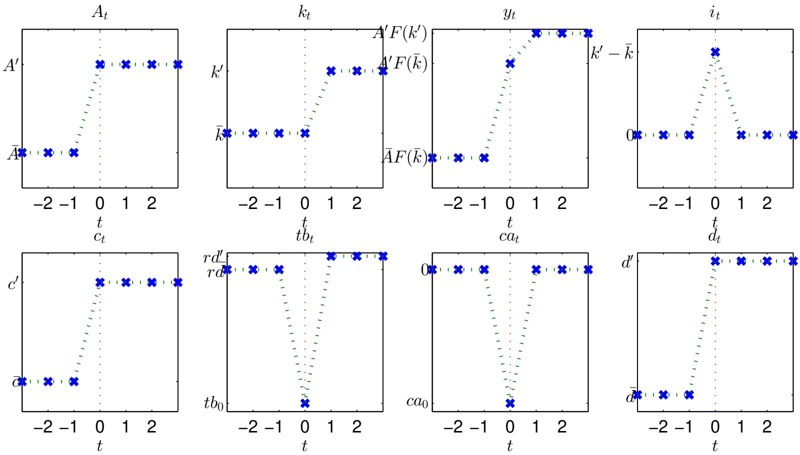
$$d_t = (1 + r)d_{t-1} - tb'$$

This will satisfy (3.10) only if:

$$tb' = rd_0$$

where $d_0 = d_{-1} + y_0^p - y_0 > d_{-1}$.
The new level of debt is permanently higher than it was prior to the productivity shock and therefore the trade balance, which is used to service the interest on the debt, must also be permanently higher.

Summary of Adjustment to Permanent Productivity Shock



Recap

* A new element in this model wrt the endowment economy is that a productivity shock has two effects demand: consumption and investment
* The latter increases because the productivity shock is permanent
* The combination of higher productivity and higher investment leads to an increase in income that lasts for 2 periods.
* Note that we don't have capital adjustment costs. With adjustment costs, investment is spread across periods

3.4 Adjustment to Temporary Productivity Shocks: Income

Experiment: In period 0 it is learned that $A_0 = A' > A_{-1} = \bar{A}$ and that $A_t = \bar{A}$ for all $t > 0$.

$$A_t = \begin{cases} \bar{A} & \text{for } t \leq -1 \\ A' > \bar{A} & \text{for } t = 0 \\ \bar{A} & \text{for } t > 0 \end{cases}$$

By (3.12) ($r = A_{t+1}F'(k_{t+1})$):

$$k_t = \bar{k}; \text{ for all } t > 0$$

By (3.4) ($k_{t+1} = k_t + i_t$):

$$i_t = 0; \text{ for all } t \geq 0$$

By (3.3) ($y_t = A_t F(k_t)$):

$$y_0 = A'F(\bar{k}) > \bar{y}; \quad \text{ and } y_t = \bar{y} = \bar{A}F(\bar{k}); \text{ for all } t > 0$$

The adjustment to a purely temporary shock in the economy with capital is thus the same as the adjustment to a purely temporary **endowment** shock in the economy without capital studied in Chapter 2.

3.4 Adjustment to Temporary Productivity Shocks: Cons.

By (3.11):

$$c_t = c_0 \quad \text{for all } t \geq 0$$

By (3.13)

$$c_0 = -rd_{-1} + \bar{A}F(\bar{k}) + \frac{r}{1+r}\left(A'F(\bar{k}) - \bar{A}F(\bar{k})\right)$$

Recalling that $c_{-1} = -r\bar{d} + \bar{A}F(\bar{k})$ and that $d_{-1} = \bar{d}$ yields

$$c_0 - c_1 = \frac{r}{1+r}\left(A'F(\bar{k}) - \bar{A}F(\bar{k})\right) > 0$$

Thus, consumption increases by only a small fraction $(\frac{r}{1+r})$ of the increase in income.

3.4 Adjustment to Temporary Productivity Shocks: TB & CA

From the definition of the trade balance we have

$$tb_0 - tb_{-1} = (y_0 - y_{-1}) - (c_0 - c_{-1}) - (i_0 - i_{-1}) = \frac{1}{1+r}(y_0 - y_{-1}) > 0$$

$\Rightarrow$ procyclical trade balance adjustment in period 0.
For $t > 0$: c_t, y_t, i_t are all constant. Hence tb_t is also constant. At what level? By same argument as above

$$tb_t = tb' = rd_0; \text{ and } d_t = d_0; \quad \forall t > 0$$

Because c_0 increases by less than y_0 and i_0 is unchanged (at zero), it must be that $d_0 < d_{-1} = \bar{d}$. It follows that

$$tb' < tb_{-1} < tb_0$$

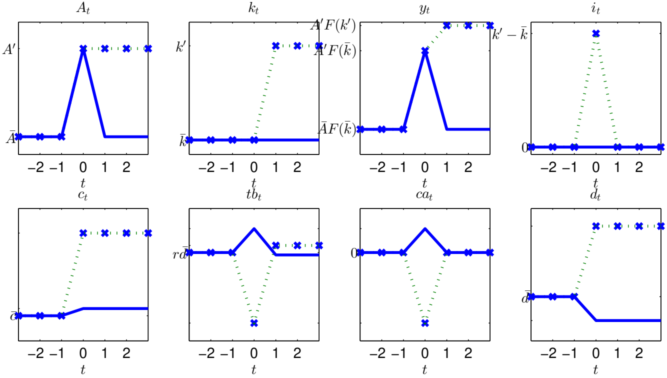
Finally, the adjustment of the current acount is

$$ca_0 - ca_{-1} = tb_0 - tb_{-1} > 0$$

and

$$ca_t = 0; \quad \forall t > 0$$

Adjustment to Temporary and Permanent Productivity Increases



: Temporary Productivity Increase x. . . x. . . x. . . : Permanent Productivity Increase

Principle I: The more persistent productivity shocks are, the more likely an initial deterioration of the trade balance will be.

3.5 Capital Adjustment Costs

- Capital adjustment costs are a standard feature of open economy business cycle models.
- They are used to ensure that the predicted volatility of investment relative to the volatility of output does not exceed the observed one.
- In the presence of adjustment costs, investment will be spread out over a number of periods.
- This will have two consequences for the period 0 adjustment of the trade balance:
    1. The increase in investment in period 0 will be lower.
    2. The increase in permanent income will be lower (because output increases slowly to its new permanently higher level) and therefore the consumption response in period 0 will be lower.
- Both factors contribute to a more muted trade balance response.

Principle II: The more pronounced are capital adjustment costs are, the smaller will be the initial trade balance deterioration in response to a positive and persistent productivity shock.

3.5 Capital Adjustment Costs

$$\text{Capital adjustment costs } = \frac{1}{2}\ \frac{i_t^2}{k_t}$$

* If $i_t = 0$, then adj costs are nil.
* Adj costs are convex in i_t
* These are actual resources lost!
* Slope of adjustment costs: $\frac{\partial \frac{i_t^2}{2k_t}}{\partial i_t} = \frac{i_t}{k_t}$
* In steady state $i_t = 0$, so adjustment costs and marginal adjustment costs are nil in steady state.

3.5 Capital Adjustment Costs

With adjustment costs, the sequential budget constraint becomes:

$$c_t + i_t + \frac{1}{2}\frac{i_t^2}{k_t} + (1+r)d_{t-1} = A_t F(k_t) + d_t \tag{3.16}$$

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ U(c_t) + \lambda_t \left[A_t F(k_t) + d_t - (1+r)d_{t-1} - c_t - i_t - \frac{1}{2}\frac{i_t^2}{k_t} + q_t(k_t + i_t - k_{t+1}) \right] \right\}$$

Note that we didn't substitute i. Instead we include $i_t = k_{t+1} - k_t$ as an additional constraint with its own Lagrange multiplier q_t

3.5 Capital Adjustment Costs

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left\{ U(c_t) + \lambda_t \left[A_t F(k_t) + d_t - (1+r)d_{t-1} - c_t - i_t - \frac{1}{2}\frac{i_t^2}{k_t} + q_t(k_t + i_t - k_{t+1}) \right] \right\}$$

Optimality conditions: (3.4), (3.5), (3.6), and (3.7) hold. Plus (3.16) requires:

$$1 + \frac{i_t}{k_t} = q_t \tag{3.17}$$

This equates the marginal cost of producing a unit of capital (LHS) to the marginal revenue of selling that unit of capital (RHS)

$$\lambda_t q_t = \beta \lambda_{t+1} \left[q_{t+1} + A_{t+1} F'(k_{t+1}) + \frac{1}{2} \left(\frac{i_{t+1}}{k_{t+1}} \right)^2 \right] \tag{3.18}$$

$q_t =$ Tobin's q.

This Lagrange multiplier is the shadow relative price of capital in terms of consumption goods. When $q_t > 1$, we get that $i_t > 0$ and when $q_t < 1$, we get $i_t < 0$

Again assume that $\beta(1+r) = 1$, then (3.18) can be written as:

$$(1+r)q_t = A_{t+1}F'(k_{t+1}) + q_{t+1} + \frac{1}{2}\left(\frac{i_{t+1}}{k_{t+1}}\right)^2 \tag{3.19}$$

Interpretation: Suppose you have q_t units of consumption goods:
LHS is the return if those goods are invested in bonds.
RHS is the return if those goods are invested in physical capital.
Adding one unit to the existing stock costs q_t, it yields $A_{t+1}F'(k_{t+1})$ units of output in the next period (the marginal product of capital), it reduces tomorrow's adjustment costs by $\frac{1}{2}\left(\frac{i_{t+1}}{k_{t+1}}\right)^2$, and can be sold next period at the price q_{t+1}

3.5 Capital Adjustment Costs

As in the case without adjustment costs, we can separate c_t dynamics from k_t or i_t dynamics.

Solving the sequential budget constraint (3.16) forward and using the no-Ponzi-game constraint (3.5) holding with equality yields:

$$c_t = -rd_{t-1} + \frac{r}{1+r} \sum_{j=0}^{\infty} \frac{A_{t+j}F(k_{t+j}) - i_{t+j} - \frac{1}{2}(i_{t+j}^2/k_{t+j})}{(1+r)^j}.$$

3.5 Capital Adjustment Costs

$$c_t = -rd_{t-1} + \frac{r}{1+r} \sum_{j=0}^{\infty} \frac{A_{t+j}F(k_{t+j}) - i_{t+j} - \frac{1}{2}(i_{t+j}^2/k_{t+j})}{(1+r)^j}.$$

This is by now a familiar expression:

* Households split their nonfinancial permanent income, given by the second term on the right-hand side, to service their outstanding debt and to consume.
* The definition of nonfinancial permanent income is adapted to include adjustment costs as one additional component of domestic absorption subtracted from the flow of output.
* The right-hand side of the above expression is known as *permanent income* and is given by the sum of net investment income ($-rd_{t-1}$) and nonfinancial permanent income.

Dynamics of the Capital Stock

Combine (3.4), (3.17), and (3.19), to obtain two first-order, nonlinear difference equations in k_t and q_t:

$$k_{t+1} = q_t k_t \tag{3.20}$$

$$q_t = \frac{A_{t+1} F'(q_t k_t) + (q_{t+1} - 1)^2/2 + q_{t+1}}{1 + r} \tag{3.21}$$

We can also rewrite (3.21) as

$$\frac{(q_{t+1} - 1)^2/2 + q_{t+1}}{1 + r} = q_t - \frac{A_{t+1} F'(q_t k_t)}{1 + r} \tag{***}$$

Steady state solution : (q, k)

Suppose $A_t = \bar{A}$ for all t

By (3.20),

$$q = 1$$

And using this result in (3.21) yields: $1 = \frac{\bar{A}F'(k)+1}{1+r}$. Or:

$$r = \bar{A}F'(k)$$

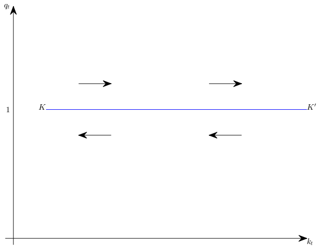
$\Rightarrow$ Adjustment costs play no role for long run values of k and q

But they do play a role for the short-run dynamics, which we will analyze next using a phase diagram

Dynamics of the Capital Stock

Let's plot the locus of pairs (k_t, q_t) such that $k_{t+1} = k_t$. Call it the $\overline{KK'}$ locus. By (3.20) if

$$\begin{aligned} q_t > 1, &\quad k_{t+1} > k_t \\ q_t = 1, &\quad k_{t+1} = k_t \\ q_t < 1, &\quad k_{t+1} < k_t \end{aligned}$$



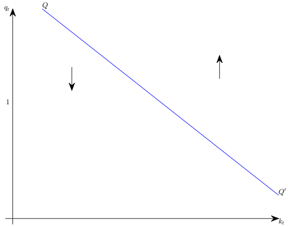
Dynamics of the Capital Stock

Assume that $A_t = \bar{A}$ for all t. Plot the locus of pairs (k_t, q_t) such that $q_{t+1} = q_t$ in a neighborhood around $q_t = 1$ (this is a local analysis).
Call this the $\overline{QQ'}$ locus. By (3.21), the $\overline{QQ'}$ locus is given by

$$rq_t = \bar{A}F'(q_t k_t) + (q_t - 1)^2/2$$

Around $q = 1$ we can ignore the quadratic term and we can show that this is downward sloping

$$\frac{dq_t}{dk_t} = \frac{q_t \bar{A} F''}{r - k_t \bar{A} F''} < 0$$



Dynamics of the Capital Stock

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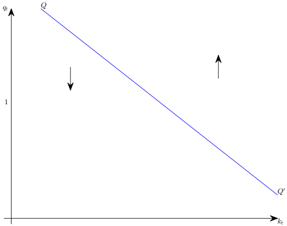
Call this the $\overline{QQ'}$ locus. By (3.21), the $\overline{QQ'}$ locus is given by

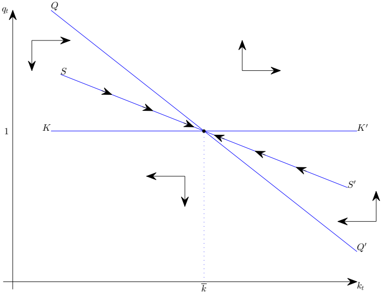
$$rq_t = \bar{A}F'(q_t k_t) + (q_t - 1)^2/2$$

$$\text{If} \quad \begin{cases} (k_t, q_t) \text{ above } \overline{QQ'}, & q_{t+1} > q_t \\ (k_t, q_t) \text{ on } \overline{QQ'}, & q_{t+1} = q_t \\ (k_t, q_t) \text{ below } \overline{QQ'}, & q_{t+1} < q_t \end{cases}$$



3.5 Capital Adjustment Costs

This yields the phase diagram:



* The intersection of $\overline{KK'}$ and $\overline{QQ'}$ is the steady state pair $(k, q) = (\bar{k}, 1)$. The SS of q is clearly one and the SS of k is implicitly determined by $r = \bar{A}F'(\bar{k})$.
* This the same that we obtained in the economy without adjustment costs. Not surprising as in SS adjustment costs are zero
* The locus $\overline{SS'}$ is the saddle path. Given the initial capital stock, k_0, Tobin's q, q_0, jumps to the saddle path, and (k_t, q_t) converge monotonically to $(\bar{k}, 1)$.

3.5 Capital Adjustment Costs

Experiment 1: Adjustment to a temporary productivity shock. $\rightarrow$ identical to the economy without capital adjustment costs, as there is no reason to adjust the capital stock. (results as in Section 3.4).

Experiment 2: Adjustment to a permanent productivity shock.

In period 0 it is learned that A_t increases from $\bar{A}$ to $A' > \bar{A}$ for all $t \geq 0$. Prior to period 0, A_t was expected to be $\bar{A}$ forever.

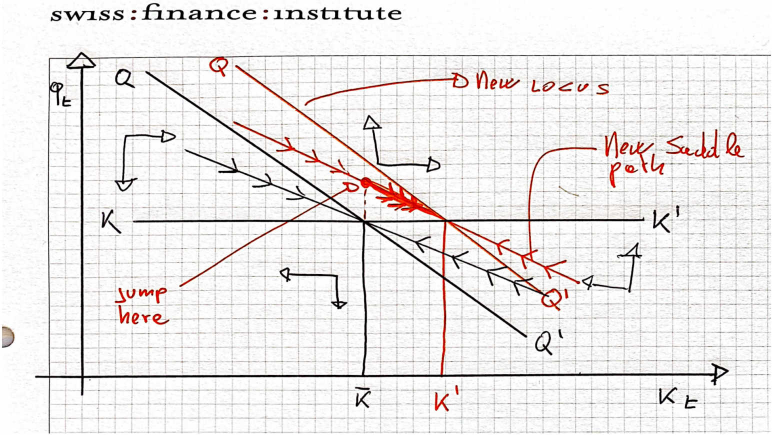
$$A_t = \begin{cases} \bar{A} & \text{for } t \leq -1 \\ A' > \bar{A} & \text{for } t \geq 0 \end{cases} .$$

How can we capture this in the phase diagram?

* The $\overline{KK'}$ locus does not change. But the $\overline{QQ'}$ locus changes.
* The new locus is implicitly given by $rq_t = A'F'(q_t k_t) + (q_t - 1)^2/2$.
* This means that the $\overline{QQ'}$ locus shifts up and to the right. The new steady state is $(k_t, q_t) = (k', 1)$, where k' solves $r = A'F'(k')$.
* The initial capital stock is $k_0 = \bar{k}$, hence $k_0 < k'$.

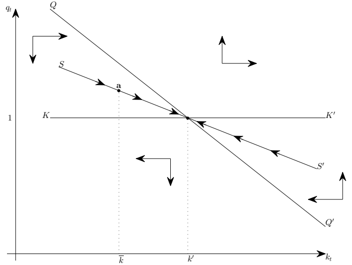
3.5 Capital Adjustment Costs

The dynamics of the capital stock can be read in the graph below.



3.5 Capital Adjustment Costs

The dynamics of the capital stock can be read in the graph below.



* In period 0 the economy jumps to point a, where $q_0 > 1$ and $k_0 = \bar{k}$. That is, capital converges monotonically to k' from below and Tobin's q converges monotonically to 1 from above.
* Investment is positive during the entire transition, but, importantly, $i_0 < k' - \bar{k}$.
* With adjustment costs k can't jump. Only q can jump
* It follows that domestic absorption increases by less on impact in the presence of capital adjustment costs.
* And thus, the deterioration of the trade balance in response to a positive permanent productivity shock is smaller on impact.
* We summarize these results as follows:

Principle II: The more pronounced are capital adjustment costs, the smaller will be the initial trade balance deterioration in response to a positive and persistent productivity shock.

3.5 Capital Adjustment Costs

Thus far, to determine the dynamics in the model with capital adjustment costs we used a phase diagram. The phase diagram is a convenient graphical tool to analyze dynamics qualitatively. Specifically, we used the phase diagram to establish that if k_0 is below steady state, then:

* The model is saddle path stable
* The price of capital converges to its steady state value from above
* Capital converges to its steady state value from below.
* Investment is positive along the entire transition.
* Capital adjustment costs dampen the trade balance deterioration in response to a permanent productivity increase.

There is an alternative method to determine whether the model is saddle path stable and to characterize the adjustment of the economy when k_0 is below its steady state value.

Characterization of Adjustment using a Log-linear approximation (will not do)

We wish to characterize the dynamics of q_t and k_t described by

$$k_{t+1} = q_t k_t \tag{3.20R}$$

$$q_t = \frac{A_{t+1} F'(q_t k_t) + (q_{t+1} - 1)^2/2 + q_{t+1}}{1 + r} \tag{3.21R}$$

$k_t =$ endogenous predetermined variable
$q_t =$ endogenous nonpredetermined variable
$A_t =$ exogenous variable

Characterization of Adjustment using a Log-linear approximation

Consider the dynamics around the steady state associated with $A_t = A' > \bar{A}$ for all $t \geq 0$. The steady state solution to (3.20) and (3.21) is

$$q^{ss} = 1$$

$$k^{ss} = k'$$

where k' is the solution to $r = A'F'(k')$.
Let

$$\hat{q}_t \equiv \ln \frac{q_t}{q^{ss}}$$

$$\hat{k}_t \equiv \ln \frac{k_t}{k^{ss}}$$

Log-linearize (3.20) and (3.21) around the point $(q_t, k_t) = (1, k')$

Characterization of Adjustment using a Log-linear approximation

Take logs of (3.20), then take the total differential:

$$\ln k_{t+1} = \ln k_t + \ln q_t$$
$$(\ln k_{t+1} - \ln k^{ss}) = (\ln k_t - \ln k^{ss}) + (\ln q_t - \ln q^{ss})$$

$$\hat{k}_{t+1} = \hat{q}_t + \hat{k}_t \tag{3.20'}$$

Applying the same steps to (3.21) is a little more complicated. To make the presentation clearer, let $x_{t+1} = A'F'(k_{t+1}) + (q_{t+1} - 1)^2/2 + q_{t+1}$. (Note that $x^{ss} = 1 + r$). With this notation in hand, after takings logs of both sides, (3.21) becomes

$$\ln(1 + r) + \ln q_t = \ln x_{t+1}$$

Take total differential with respect to $\ln q_t$ and $\ln x_{t+1}$

$$\ln q_t - \ln q^{ss} = \ln x_{t+1} - \ln x^{ss}$$
$$\hat{q}_t = \hat{x}_{t+1}$$

Characterization of Adjustment using a Log-linear approximation

To find $\hat{x}_{t+1}$ proceed as follows

$$x_{t+1} = A'F'(k_{t+1}) + (q_{t+1} - 1)^2/2 + q_{t+1}$$
$$\ln x_{t+1} = \ln[A'F'(k_{t+1}) + (q_{t+1} - 1)^2/2 + q_{t+1}]$$

Totally differentiate

$$\ln x_{t+1} - \ln x^{ss} = \frac{1}{[A'F'(k^{ss}) + (q^{ss} - 1)^2/2 + q^{ss}]} \tag{2}$$
$$\times \left(A'F''(k^{ss})k^{ss}(\ln k_{t+1} - \ln k^{ss}) + (q^{ss} - 1)q^{ss}(\ln q_{t+1} - \ln q^{ss}) + q^{ss}(\ln q_{t+1} - \ln q^{ss})\right) \tag{3}$$
$$\hat{x}_{t+1} = \frac{1}{1+r}\left(A'F''(k^{ss})k^{ss}\hat{k}_{t+1} + \hat{q}_{t+1}\right) \tag{4}$$

Let

$$\epsilon_{F'} \equiv -\frac{F''(k^{ss})k^{ss}}{F'(k^{ss})} > 0$$

$$(1+r)\hat{x}_{t+1} = -r\epsilon_{F'}\hat{k}_{t+1} + \hat{q}_{t+1}$$

The log-linearized version of (3.21) then is

$$(1+r)\hat{q}_t = -r\epsilon_{F'}\hat{k}_{t+1} + \hat{q}_{t+1}$$

Characterization of Adjustment using a Log-linear approximation

After some rearranging and substituting we have that the log-linearization of (3.20) and (3.21) around the steady state $(q^{ss}, k^{ss}) = (1, k')$ is

$$\begin{bmatrix} \hat{k}_{t+1} \\ \hat{q}_{t+1} \end{bmatrix} = M \begin{bmatrix} \hat{k}_t \\ \hat{q}_t \end{bmatrix}; \quad M = \begin{bmatrix} 1 & 1 \\ r\epsilon_{F'} & 1 + r + r\epsilon_{F'} \end{bmatrix} \qquad (***)$$

* If we knew the initial values k_0 and q_0 we could trace out the dynamics.
* We do know k_0 as it is an initial condition.
* But we do not know the initial value of Tobin's q, q_0.
* To obtain it, we impose a terminal condition, we require that the economy converges back to the steady state.
* Thus our question becomes, does there exist such a solution and if so, is it unique?

We are interested in solutions such that

$$\lim_{t \to \infty} \begin{bmatrix} \hat{k}_t \\ \hat{q}_t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which says that $k_t \to k^{ss} = k'$ and $q_t \to q^{ss} = 1$.

Characterization of Adjustment using a Log-linear approximation

By (***)

$$\lim_{t\to\infty}\begin{bmatrix}\hat{k}_t\\ \hat{q}_t\end{bmatrix}=\lim_{t\to\infty}M^t\begin{bmatrix}\hat{k}_0\\ \hat{q}_0\end{bmatrix}$$

* If both eigenvalues of $\boldsymbol{M}$ lie outside the unit circle, then no equilibrium converging to the steady state exists.
* If both eigenvalue of $\boldsymbol{M}$ lie inside the unit circle, then for any initial value of q_0, an equilibrium converging to the steady state exists, that is, the equilibrium is locally indeterminate.
* If one eigenvalue of $\boldsymbol{M}$ lies inside the unit circle and one outside, then a unique value for q_0 exists, such that the equilibrium converges to the steady state given some k_0 in the neighborhood of the steady state.

Let λ_1 and λ_2 be the eigenvalues of $\boldsymbol{M}$. Then the equilibrium is locally unique iff

$$|\lambda_1|>1 \quad \text{and} \quad |\lambda_2|<1$$

Characterization of Adjustment using a Log-linear approximation

Is this eigenvalue condition satisfied in our economy? Yes, it is. To see this note, use that in general for any matrix

$$det(M) = \lambda_1 \lambda_2; \quad \text{and} \quad trace(M) = \lambda_1 + \lambda_2$$

In our case,

$$det(M) = 1 + r > 1; \quad \text{and} \quad trace(M) = 1 + 1 + r + r\epsilon_{F'} > 2 + r$$

* From here it follows that both eigenvalues are positive (or have positive real parts) and that at least one is greater than one in modulus.
* In turn this implies that if an equilibrium of the type we are looking for exists, then it would be unique.
* To find whether it exists, let's first consider the case that the eigenvalues are real.
* Make a graph with λ_1 on the x-axis and λ_2 on the y-axis and plot: (1.) $\lambda_2 = \frac{1+r}{\lambda_1}$ and (2.) $\lambda_2 = 2 + r + r\epsilon_{F'} - \lambda_1$.
* These lines must intersect twice in the positive quadrant because (1.) is positive, decreasing, and becomes arbitrarily large as $\lambda_1 \to 0$ from above and converges to zero as $\lambda_1 \to \infty$ and at the same time (2.) is positive and finite for $\lambda_1 = 0$, decreasing, and converges to $-\infty$ as $\lambda_1 \to \infty$.
* The question is: is one intersection at $\lambda_1 < 1$ and the second at $\lambda_1 > 1$? At $\lambda_1 = 1$ (2.) is: $1 + r + r\epsilon_{F'}$, which is greater than 1.
* It then follows that the conditions for uniqueness are satisfied, that is, we have shown that $\lambda_1 > 1$ and $0 < \lambda_2 < 1$.

Characterization of Adjustment using a Log-linear approximation

How to find q_0? Let

$$y_t = \begin{bmatrix} \hat{k}_t \\ \hat{q}_t \end{bmatrix}$$

Premultiply (***) by the left eigenvector of M associated with the unstable eigenvalue, λ_1, denoted v_1

$$v_1 y_{t+1} = v_1 M y_t = \lambda_1 v_1 y_t$$

Let $\tilde{y}_t = v_1 y_t$. Then $\tilde{y}_t = \lambda_1^t \tilde{y}_0$. Because $|\lambda_1| > 1$, $\lim_{t\to\infty} \tilde{y}_t = 0$ only if $\tilde{y}_0 = 0$, that is, if

$$0 = v_1 \begin{bmatrix} \hat{k}_0 \\ \hat{q}_0 \end{bmatrix} = v_1^1 \hat{k}_0 + v_1^2 \hat{q}_0$$

$$\hat{q}_0 = -\frac{v_1^1}{v_1^2} \hat{k}_0 = -(1-\lambda_2)\hat{k}_0$$

The last equality follows from

$$\begin{aligned} \begin{bmatrix} v_1^1 & v_1^2 \end{bmatrix} M =& \lambda_1 \begin{bmatrix} v_1^1 & v_1^2 \end{bmatrix} \\ v_1^1 + v_1^2(1 + r + r\epsilon_{F'}) =& \lambda_1 v_1^2 \\ v_1^1 + v_1^2(trace(M) - 1) =& \lambda_1 v_1^2 \\ v_1^1 + v_1^2(\lambda_1 + \lambda_2 - 1) =& \lambda_1 v_1^2 \\ \frac{v_1^1}{v_1^2} =& (1-\lambda_2) \end{aligned}$$

Characterization of Adjustment using a Log-linear approximation

Now use $\hat{q}_t = -(1-\lambda_2)\hat{k}_t$ in (50) to obtain:

$$\begin{aligned}\hat{k}_{t+1} =& \hat{k}_t + \hat{q}_t \\ =& (1-1+\lambda_2)\hat{k}_t \\ =& \lambda_2 \hat{k}_t.\end{aligned}$$

Summary of dynamics:

$$\begin{aligned}\hat{k}_t =& \lambda_2^t \hat{k}_0 \\ \hat{q}_t =& -(1-\lambda_2)\lambda_2^t \hat{k}_0\end{aligned}$$

* Unique saddle path stable eqm exists locally in the neighborhood around (q^{ss}, k^{ss}).
* The adjustment to a permanent increase in productivity induces capital to converge monotonically from below, Tobin's q to converge monotonically from above.
* Because the increase in capital is spread out of many periods, investment is also positive for many periods and because the total increase in capital is independent of the size of the adjustment cost, it follows that adjustment costs dampen the initial increase in investment, and hence the initial deterioration in the trade balance and the current account. (Principle II)