

PS1 Solutions

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1 Consumption Allocation

Home Country

The consumption basket is given by

$$C_t = C_{T,t}^\gamma C_{N,t}^{1-\gamma}.$$

The traded good is the numéraire (price normalized to 1) and the non-traded good is priced at $P_{N,t}$.

Cost minimization yields the demands:

$$C_{T,t} = \gamma P_t C_t, \quad C_{N,t} = (1 - \gamma) \frac{P_t}{P_{N,t}} C_t,$$

where the consumption price index is

$$P_t = (P_{N,t})^{1-\gamma}.$$

Intuition: An increase in $P_{N,t}$ raises P_t and induces households to adjust the consumption shares accordingly.

Foreign Country

Analogously, the Foreign consumption basket is

$$C_t^* = (C_{T,t}^*)^\gamma (C_{N,t}^*)^{1-\gamma},$$

with

$$C_{T,t}^* = \gamma P_t^* C_t^*, \quad C_{N,t}^* = (1 - \gamma) \frac{P_t^*}{P_{N,t}^*} C_t^*,$$

and

$$P_t^* = (P_{N,t}^*)^{1-\gamma}.$$

2 Market Clearing

Non-Traded Goods Market

For Home, market clearing in non-traded goods is:

$$n C_{N,t} = A_{N,t} (L_{N,t})^{1-\alpha}.$$

Substituting $C_{N,t} = (1 - \gamma) (P_t / P_{N,t}) C_t$ with $P_t = (P_{N,t})^{1-\gamma}$, we obtain:

$$\boxed{n(1 - \gamma) (P_{N,t})^{-\gamma} C_t = A_{N,t} (L_{N,t})^{1-\alpha}.$$

For Foreign:

$$\boxed{(1 - n)(1 - \gamma) (P_{N,t}^*)^{-\gamma} C_t^* = A_{N,t}^* (L_{N,t}^*)^{1-\alpha}.$$

Traded Goods Market

Global market clearing for traded goods is:

$$n C_{T,t} + (1 - n) C_{T,t}^* = A_{T,t} (n - L_{N,t})^{1-\alpha} + A_{T,t}^* ((1 - n) - L_{N,t}^*)^{1-\alpha}.$$

Substituting $C_{T,t} = \gamma P_t C_t$ with $P_t = (P_{N,t})^{1-\gamma}$ (and similarly for Foreign), we have:

$$\boxed{n\gamma (P_{N,t})^{1-\gamma} C_t + (1 - n)\gamma (P_{N,t}^*)^{1-\gamma} C_t^* = A_{T,t} (n - L_{N,t})^{1-\alpha} + A_{T,t}^* ((1 - n) - L_{N,t}^*)^{1-\alpha}.$$

Intuition: Non-traded goods are produced and consumed domestically, while traded goods are balanced across countries.

3 Intertemporal Allocation

Home Optimization

The Home agent maximizes:

$$U_t = \sum_{s=0}^{\infty} (\beta_{H,t+s})^s \ln C_{t+s},$$

subject to the period- t budget constraint:

$$n P_t C_t + n B_{t+1} = A_{T,t} (n - L_{N,t})^{1-\alpha} + P_{N,t} A_{N,t} (L_{N,t})^{1-\alpha} + n(1 + r_t) B_t.$$

The first-order condition gives the Euler equation:

$$\frac{1}{C_t} = \beta_{H,t+1} (1 + r_{C,t+1}) \frac{1}{C_{t+1}},$$

with the consumption-based real return:

$$1 + r_{C,t+1} = (1 + r_{t+1}) \left(\frac{P_t}{P_{t+1}} \right) = (1 + r_{t+1}) \left(\frac{P_{N,t}}{P_{N,t+1}} \right)^{1-\gamma}.$$

Thus,

$$C_{t+1} = \beta_{H,t+1} (1 + r_{C,t+1}) C_t.$$

Also, since $C_{T,t} = \gamma P_t C_t$:

$$C_{T,t+1} = \beta_{H,t+1} (1 + r_{t+1}) C_{T,t}.$$

Foreign Optimization

Similarly, for the Foreign country:

$$C_{t+1}^* = \beta_{F,t+1} (1 + r_{C,t+1}^*) C_t^*, \quad C_{T,t+1}^* = \beta_{F,t+1} (1 + r_{t+1}) C_{T,t}^*,$$

with

$$1 + r_{C,t+1}^* = (1 + r_{t+1}) \left(\frac{P_{N,t}^*}{P_{N,t+1}^*} \right)^{1-\gamma}.$$

Intuition: Households equate the marginal utility cost of consuming today versus tomorrow, with intertemporal decisions affected by relative price changes.

4 Labor Allocation

Home Country

Firms allocate labor such that:

$$(1 - \alpha) A_{T,t} (n - L_{N,t})^{-\alpha} = P_{N,t} (1 - \alpha) A_{N,t} (L_{N,t})^{-\alpha}.$$

Canceling $(1 - \alpha)$ and rearranging yields:

$$A_{T,t} (n - L_{N,t})^{-\alpha} = P_{N,t} A_{N,t} (L_{N,t})^{-\alpha}.$$

Foreign Country

Similarly, for Foreign:

$$A_{T,t}^* ((1-n) - L_{N,t}^*)^{-\alpha} = P_{N,t}^* A_{N,t}^* (L_{N,t}^*)^{-\alpha}.$$

Intuition: Labor is allocated until the marginal value products in the traded and non-traded sectors are equalized (after accounting for relative prices).

5 Resource Constraints and the Real Exchange Rate

Resource Constraints

The Home resource constraint is given by:

$$n\gamma (P_{N,t})^{1-\gamma} C_t + n B_{t+1} = A_{T,t} (n - L_{N,t})^{1-\alpha} + n(1 + r_t)B_t.$$

Similarly, for Foreign:

$$(1-n)\gamma (P_{N,t}^*)^{1-\gamma} C_t^* - n B_{t+1} = A_{T,t}^* ((1-n) - L_{N,t}^*)^{1-\alpha} - n(1 + r_t)B_t.$$

Real Exchange Rate

Define the real exchange rate as:

$$Q_t \equiv \frac{P_t^*}{P_t} = \left(\frac{P_{N,t}^*}{P_{N,t}} \right)^{1-\gamma}.$$

A higher Q_t implies that Foreign's consumption basket is relatively more expensive, equivalent to a real depreciation for Home.

6 Steady State

Assume a symmetric steady state with no cross-country assets ($B_0 = 0$) and constant discount factor β_0 . Calibration is chosen as:

$$A_{N,0} = A_{T,0} \left(\frac{1-\gamma}{\gamma} \right)^\alpha, \quad A_{T,0}^* = A_{T,0} \left(\frac{1-n}{n} \right)^\alpha, \quad A_{N,0}^* = A_{T,0} \left(\frac{1-n}{n} \right)^\alpha \left(\frac{1-\gamma}{\gamma} \right)^\alpha.$$

Then:

- **Interest Rate:** $1 = \beta_0(1 + r_0)$, so

$$r_0 = \frac{1 - \beta_0}{\beta_0}.$$

- **Non-Traded Prices:** $P_{N,0} = P_{N,0}^* = 1$.
- **Labor Allocation:** From the intratemporal condition,

$$\frac{L_{N,0}}{n - L_{N,0}} = \frac{1 - \gamma}{\gamma} \Rightarrow L_{N,0} = n(1 - \gamma),$$

and similarly,

$$L_{N,0}^* = (1 - n)(1 - \gamma).$$

- **Consumption:** One can show that

$$C_0 = (A_{T,0}^\gamma A_{N,0}^{1-\gamma}) [n^\gamma (1 - \gamma)^{1-\gamma}]^{-\alpha},$$

with a similar expression for C_0^* so that $C_0 = C_0^*$.

Intuition: In the steady state, relative prices and allocations are balanced; country size differences aside, both countries have identical marginal conditions.

7 Log-Linear Approximation

Let the “hat” denote log-deviations from the steady state, e.g. $\hat{C}_t = \frac{C_t - C_0}{C_0}$.

Key Linearized Equations

Non-Traded Goods Market (Home):

$$-\gamma \hat{P}_{N,t} + \hat{C}_t = \hat{A}_{N,t} + (1 - \alpha) \hat{L}_{N,t}. \quad (7a)$$

For Foreign:

$$-\gamma \hat{P}_{N,t}^* + \hat{C}_t^* = \hat{A}_{N,t}^* + (1 - \alpha) \hat{L}_{N,t}^*.$$

Resource Constraint (Home):

$$(1 - \gamma) \hat{P}_{N,t} + \hat{C}_t + \hat{B}_{t+1} = \hat{A}_{T,t} - \frac{(1 - \alpha)(1 - \gamma)}{\gamma} \hat{L}_{N,t} + \frac{1}{\beta_0} \hat{B}_t. \quad (7b)$$

Euler Equation (Home):

$$\hat{C}_{t+1} - \hat{C}_t = (1 - \gamma) (\hat{P}_{N,t} - \hat{P}_{N,t+1}) + \hat{\beta}_{H,t+1} + \beta_0 \hat{r}_{C,t+1}. \quad (7c)$$

Labor Allocation (Home):

$$\hat{A}_{T,t} + \frac{\alpha}{\gamma} \hat{L}_{N,t} = \hat{P}_{N,t} + \hat{A}_{N,t}. \quad (7d)$$

Real Exchange Rate:

$$\hat{Q}_t = (1 - \gamma)(\hat{P}_{N,t}^* - \hat{P}_{N,t}).$$

Intuition: These equations capture the first-order responses of consumption, labor, and relative prices to shocks.

8 Worldwide Equilibrium

Define world aggregates as population-weighted averages (e.g., $\hat{C}_t^W = n\hat{C}_t + (1 - n)\hat{C}_t^*$). Then, from the above log-linearized equations one can show:

- $\hat{L}_{N,t}^W = 0$, i.e., aggregate non-traded labor remains fixed.
- The world non-traded price satisfies

$$\hat{P}_{N,t}^W = \hat{A}_{T,t}^W - \hat{A}_{N,t}^W.$$

- World consumption is given by

$$\hat{C}_t^W = \gamma \hat{A}_{T,t}^W + (1 - \gamma) \hat{A}_{N,t}^W.$$

- The world Euler equation becomes

$$\beta_0 \hat{r}_{t+1} = -\hat{\beta}_{t+1}^W + \left(\hat{A}_{T,t+1}^W - \hat{A}_{T,t}^W \right).$$

Intuition: World aggregates respond only to symmetric shocks, with the real interest rate driven by global productivity changes and aggregate patience.

9 Cross-Country Differences

Define differences as (Home minus Foreign) for a variable x by $\tilde{x}_t = \hat{x}_t - \hat{x}_t^*$. Then:

Non-Traded Goods Market (Difference):

$$\frac{\gamma}{1 - \gamma} \hat{Q}_t + \tilde{C}_t = \tilde{A}_{N,t} + (1 - \alpha) \tilde{L}_{N,t}. \quad (9a)$$

Resource Constraints (Difference):

$$-\hat{Q}_t + \tilde{C}_t + \frac{\hat{B}_{t+1}}{1 - n} = \tilde{A}_{T,t} - \frac{(1 - \alpha)(1 - \gamma)}{\gamma} \tilde{L}_{N,t}. \quad (9b)$$

Euler Equation (Difference):

$$\tilde{C}_{t+1} - \tilde{C}_t = (1 - \gamma) \left[(\hat{P}_{N,t} - \hat{P}_{N,t}^*) - (\hat{P}_{N,t+1} - \hat{P}_{N,t+1}^*) \right] + (\hat{\beta}_{H,t+1} - \hat{\beta}_{F,t+1}) + \beta_0 (\hat{r}_{C,t+1} - \hat{r}_{C,t+1}^*). \quad (9c)$$

Labor Allocation (Difference):

$$\frac{\alpha}{\gamma} \tilde{L}_{N,t} = -\frac{1}{1 - \gamma} \hat{Q}_t - \tilde{A}_{N,t}. \quad (9d)$$

Intuition: These equations link cross-country differences in consumption, labor, and the real exchange rate to differences in productivity and intertemporal preferences.

10 Long-Run Allocation (Period $t + 1$)

Assume that from $t + 1$ onward the economy reaches a new steady state with no further discount factor shocks ($\hat{\beta}_{H,t+2} = \hat{\beta}_{F,t+2} = 0$). Taking the cross-country asset position \hat{B}_{t+1} as given, one can show:

$$\hat{Q}_{t+1} = -(1 - \gamma) \left[(\hat{A}_{T,t+1} - \hat{A}_{T,t+1}^*) - (\hat{A}_{N,t+1} - \hat{A}_{N,t+1}^*) \right] - \frac{\alpha(1 - \gamma)}{1 - \beta_0} \frac{1}{\beta_0} \frac{\hat{B}_{t+1}}{1 - n},$$

$$\tilde{L}_{N,t+1} = \frac{\gamma}{1 - \beta_0} \frac{1}{\beta_0} \frac{\hat{B}_{t+1}}{1 - n},$$

$$\tilde{C}_{t+1} = \gamma(\hat{A}_{T,t+1} - \hat{A}_{T,t+1}^*) + (1 - \gamma)(\hat{A}_{N,t+1} - \hat{A}_{N,t+1}^*) + \frac{\gamma}{1 - \beta_0} \frac{1}{\beta_0} \frac{\hat{B}_{t+1}}{1 - n}.$$

Interpretation:

- A positive \hat{B}_{t+1} (Home wealthier) implies higher relative consumption and a lower \hat{Q}_{t+1} (Home's goods become relatively more expensive).
- Permanent productivity differences affect steady state consumption and prices directly.

11 Short-Run Allocation (Period t)

Assume initially $\hat{B}_t = 0$. Solving the system (starting with the labor and non-traded market conditions, then using the resource constraint and Euler equations) yields:

$$\frac{\hat{B}_{t+1}}{1 - n} = \frac{\beta_0}{\gamma + \alpha(1 - \gamma)} \left[(\hat{\beta}_{H,t+1} - \hat{\beta}_{F,t+1}) - (\hat{A}_{T,t+1} - \hat{A}_{T,t+1}^*) + (\hat{A}_{T,t} - \hat{A}_{T,t}^*) \right], \quad (11a)$$

$$\tilde{C}_t = \gamma(\hat{A}_{T,t} - \hat{A}_{T,t}^*) + (1 - \gamma)(\hat{A}_{N,t} - \hat{A}_{N,t}^*) - \frac{\gamma\beta_0}{\gamma + \alpha(1 - \gamma)} \left[(\hat{\beta}_{H,t+1} - \hat{\beta}_{F,t+1}) - (\hat{A}_{T,t+1} - \hat{A}_{T,t+1}^*) + (\hat{A}_{T,t} - \hat{A}_{T,t}^*) \right], \quad (11b)$$

$$\hat{Q}_t = -(1-\gamma)(\hat{A}_{T,t} - \hat{A}_{T,t}^*) + (1-\gamma)(\hat{A}_{N,t} - \hat{A}_{N,t}^*) + \frac{(1-\gamma)\alpha\beta_0}{\gamma + \alpha(1-\gamma)} \left[(\hat{\beta}_{H,t+1} - \hat{\beta}_{F,t+1}) - (\hat{A}_{T,t+1} - \hat{A}_{T,t+1}^*) + (\hat{A}_{T,t} - \hat{A}_{T,t}^*) \right] \quad (11c)$$

$$\tilde{L}_{N,t} = -\frac{\gamma\beta_0}{\gamma + \alpha(1-\gamma)} \left[(\hat{\beta}_{H,t+1} - \hat{\beta}_{F,t+1}) - (\hat{A}_{T,t+1} - \hat{A}_{T,t+1}^*) + (\hat{A}_{T,t} - \hat{A}_{T,t}^*) \right]. \quad (11d)$$

Interpretation:

- A temporary increase in Home patience (i.e. $\hat{\beta}_{H,t+1} - \hat{\beta}_{F,t+1} > 0$) leads to $\hat{B}_{t+1} > 0$ (Home runs a current account surplus), lower relative consumption, and a real exchange rate movement consistent with a trade surplus.
- Temporary traded or non-traded productivity shocks affect the current account and labor allocation differently.
- For permanent shocks ($\hat{A}_{T,t} = \hat{A}_{T,t+1}$), intertemporal balance is restored with $\hat{B}_{t+1} = 0$ and immediate adjustment to the new steady state.

12 Summary of Key Economic Insights

- **Consumption and Prices:** The structure of the consumption basket implies that a rise in the non-traded good price $P_{N,t}$ increases the overall consumption price P_t and shifts the consumption mix.
- **Market Clearing:** Non-traded goods are produced and consumed domestically, whereas traded goods are allocated internationally, linking domestic production choices to the real exchange rate.
- **Intertemporal Choices:** The Euler equations show that higher patience or higher real returns lead to deferred consumption. Cross-country differences in patience drive current account imbalances.
- **Labor Allocation:** Labor is reallocated across sectors until the marginal value products are equalized; productivity shifts affect both output composition and relative prices.
- **Steady State and Log-Linearization:** In a symmetric steady state, relative prices and allocations are balanced. Log-linearization permits analysis of small shocks and their propagation.
- **Short-Run vs. Long-Run Dynamics:** Temporary shocks generate current account imbalances and short-run reallocation, while permanent shocks adjust consumption and prices directly with no asset accumulation.
- **Wealth Effects:** A positive net asset position (Home wealthier) implies higher steady-state consumption and a relatively stronger (appreciated) currency.