Final exam

Mathematics and Statistics for Economists

September 9, 2021

Exercise 1 Inverse the following matrix:

$$A = \begin{pmatrix} 4 & 5 & -2 \\ -2 & -2 & 1 \\ -1 & -1 & 1 \end{pmatrix}$$

Answer:

$$B = A^{-1} = \begin{pmatrix} -1 & -3 & 1 \\ 1 & 2 & 0 \\ 0 & -1 & 2 \end{pmatrix}$$

Exercise 2 Let A and B be the following square, invertible matrices of the same order:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 3 & 1 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Solve for X in the following equation:

$$X^T A = B$$

Answer

$$X = (A^{-1})^T B^T = \begin{pmatrix} 6 & 1 & 1 \\ -9 & -2 & -1 \\ -2 & 0 & 0 \end{pmatrix}$$

Exercise 3 Given $\mathbf{A} = \begin{pmatrix} 6 & 6 \\ 6 & -3 \end{pmatrix}$

(a) Find the eigenvalues and determine the sign of the definiteness of $\mathbf{A}$.

(b) Find the eigenvectors.

Answer

(a) $\lambda_1 = 9$, $\lambda_2 = -6$, $\mathbf{A}$ is indefinite.

(b) $\lambda_1 = 9 \Rightarrow x_1 = 2x_2 \Rightarrow v_1 = k \begin{pmatrix} 2 \\ 1 \end{pmatrix}, k \in \mathbb{R}$, $\lambda_2 = -6 \Rightarrow x_1 = -2x_2 \Rightarrow v_2 = k \begin{pmatrix} 1 \\ -2 \end{pmatrix}, k \in \mathbb{R}$

Exercise 4 Log-linearize the standard consumption Euler equation that emerges from household optimization problems:

$$(\frac{c_{t+1}}{c_t})^{\sigma} = \beta(1+r_t)$$

where σ is is the coefficient of relative risk aversion. *Hint*: recall that at the steady state, $x_t = x_{t+1} = x^*$. **Once you get to the final equation**, note that is is common to leave $\hat{r}_t$ in absolute (as opposed to percentage deviations) as r_t is already a percent. Hence, $\hat{r}_t = (r_t - r^*)$. Note also that $\frac{1}{1+r^*} \approx 1$.

Answer (see PDF on Moodle)

$$\hat{c}_{t+1} - \hat{c}_t = \frac{1}{\sigma}\hat{r}_t$$

Exercise 5 Let $f(x,y) = \frac{1}{x} + xe^{-y}$, for $x \neq 0$. Find the gradient and the Hessian.

Answer

$$\nabla f(x,y) = \begin{pmatrix} -\frac{1}{x^2} + e^{-y} \\ -xe^{-y} \end{pmatrix}$$
$$H = \begin{pmatrix} \frac{2}{x^3} & -e^{-y} \\ -e^{-y} & xe^{-y} \end{pmatrix}$$

Exercise 6 Find and classify the critical points of the following functions:

(a) $f(x,y) = x^2 - y^2 + xy$

(Bonus) **(b)** $f(x,y) = xe^{-x}(y^2 - 4y)$

Answer

a) $\nabla f(x,y) = (2x+y, -2y+x)'$. Solution is $(0,0)$. $H = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$, which is indefinite. Hence, $(0,0)$ is a saddle point.

b) $\nabla f(x,y) = e^{-}x\big((1-x)(y^2-4y), x(2y-4)\big)'$.

Critical points are solutions that solve:

$$(1-x)(y^2-4y) = 0$$
$$x(2y-4) = 0$$

that is $(0,0)$, $(0,4)$ and $(1,2)$.

$H = e^{-x}\begin{pmatrix} (x-2)(y^2-4y) & (1-x)(2y-4) \\ (1-x)(2y-4) & 2x \end{pmatrix}$. At the critical points, $Hf(0,0)$ is indefinite, $Hf(0,4)$ is indefinite and $Hf(1,2)$ is positive definite. Hence $(0,0)$, $(0,4)$ are saddle points and $(1,2)$ is a local minimum.

Exercise 7 Solve the problem

$$\max_{x,y} \quad xy \quad \text{subject to } x^2 + y^2 \leq 1$$

Answer

$L(x,y,\lambda) = xy - \lambda(x^2 + y^2 - 1)$. $\partial L/\partial x = y - 2\lambda x = 0$, $\partial L/\partial y = x - 2\lambda y = 0$ and $\lambda(x^2 + y^2 - 1) = 0$.
We have $\lambda = \frac{y}{2x} = \frac{x}{2y}$ or $x^2 = y^2$.
If $\lambda = 0$, then $x = y = 0$, is a candidate.
If $\lambda \neq 0$, then $x^2 + y^2 - 1 = 0$ and therefore $x^2 = y^2 = 1/2$, or $x = \pm 1/\sqrt{2}, y = \pm 1/\sqrt{2}$. In addition to the $(0,0)$ candidate, we have the four following: $(1/\sqrt{2}, 1/\sqrt{2})$ (with $\lambda = 1/2$), $(-1/\sqrt{2}, -1/\sqrt{2})$ (with $\lambda = 1/2$), $(1/\sqrt{2}, -1/\sqrt{2})$ (with $\lambda = -1/2$) and $(-1/\sqrt{2}, 1/\sqrt{2})$ (with $\lambda = -1/2$). Plugging in the original equation, we find that $(1/\sqrt{2}, 1/\sqrt{2})$, $(-1/\sqrt{2}, -1/\sqrt{2})$ are solutions of the problem.

Exercise 8 Consider a monopolist that produces two goods in quantities x and y. Solve the following maximisation problem and indicate the optimal quantities x^*, y^* and the maximal profit.

$$\max_{x,y} \quad \Pi(x,y) = \frac{1}{10}(-80 + 24x + 78y - 3x^2 - 3xy - 4y^2) \quad \text{subject to } x \geq 0 \text{ and } y \geq 0$$

Answer

$\partial \Pi/\partial x = \frac{1}{10}(24 - 6x - 3y)$ and $\partial \Pi/\partial y = \frac{1}{10}(78 - 3x - 8y)$. Solutions of this is $x = \frac{14}{13}$ and $y = \frac{132}{13}$, which violates the constraints. Hence $x = 0$. When $x = 0$, $\partial \Pi/\partial y = 0 \Rightarrow y = 9.75$. Maximal profit is $30.025 \approx 30$.

Exercise 9 Let X be a continuous random variable with pdf

$$f_X(x) = \begin{cases} \frac{x+2}{18}, & -2 < x < 4 \\ 0, & \text{otherwise} \end{cases}$$

(a) Find $E(X)$

(b) Find $E(X^2)$

Answer

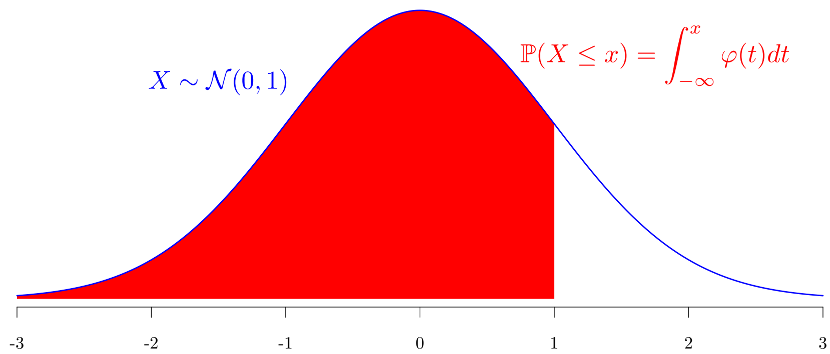
a) $E(X) = 2$.
b) $E(X^2) = 6$.

Exercise 10 If $X \sim \mathcal{N}(75, 100)$, find $P(X < 60)$ and $P(70 < X < 100)$.

Answer

$P(X < 60) = P(Z < -1.5) = 1 - P(Z < 1.5) = 1 - 0.933 = 0.0668$

$P(70 < X < 100) = P(X < 100) - P(X < 70) = P(Z < 2.5) - P(Z < -0.5) = P(Z < 2.5) - (1 - P(Z < 0.5)) = 0.9937 - (1 - 0.691) = 0.685.$



x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990