Macroeconomics A; EI060

Short problems

Cédric Tille

Class of March 19, 2025

1 Maximum debt under default risk

Question: Consider a two period model. The country starts period 2 with a debt d_1 and gets a stochastic output y_2 uniformly distributed over the interval $\left[0, y_2^H\right]$.

The probability that $y_2 < \alpha$ is $\frac{\alpha}{y_2^H}$.

If the country does not default it repays $(1 + r^s)\, d_1$. In case of default, the borrower loses ϕy_2 and the lender gets nothing. The probability of default is:

$$\begin{aligned} \pi &= Prob\left[y_2 < \frac{(1+r^s)\, d_1}{\phi}\right] \\ \pi &= \frac{(1+r^s)\, d_1}{\phi y_2^H} \end{aligned}$$

The lender is risk neutral and requires an expected return $1 + r$. Show that:

$$1 + r^s = \frac{1+r}{1 - (1+r^s)\left(\frac{d_1}{\phi y_2^H}\right)}$$

What is the probability of default and r^sif $d_1 \leq 0$?

Show that the maximum debt that the economy can borrow is:

$$d_1^{high} = \frac{\phi y_2^H}{4\,(1+r)}$$

This is subtle, so proceed as follows:

1. The expression above has a left-hand side $1 + r^s$ and a more complex right hand side, call it a function $G(1 + r^s)$.

2. Show that $G' > 0$ and $G'' > 0$, $G(0) = 1 + r$. At $1 + r^s = 0$, how do the left and right-hand side compare (which is larger)?

3. The maximmum debt is when the two side of the equations are tangent, i.e. have the same level and slope. The equality of slopes gives $(1 + r^s) = 2\,(1 + r)$, and the equality of levels gives d_1^{high}.

Answer: The arbitrage by the lender implies:

$$
\begin{aligned}
1+r &= (1-\pi)(1+r^s) \\
1+r &= \left(1-\frac{(1+r^s)\,d_1}{\phi y_2^H}\right)(1+r^s) \\
1+r^s &= \frac{1+r}{1-(1+r^s)\left(\frac{d_1}{\phi y_2^H}\right)} \\
1+r^s &= G(1+r^s)
\end{aligned}
$$

If $d_1 \leq 0$, the country is a creditor and thus defaults makes no sense. It gets the world return on its investment, $r^s = r$.

Otherwise, the country is a borrower and default may happen.

1. The left and right-hand sides $1+r^s$ are and $G(1+r^s)$.
2. We can show:

$$
G(0) = \frac{1+r}{1-0} = 0
$$

So at $1+r^s = 0$ the right-hand side is larger than the left-hand side. We can also show:

$$
\begin{aligned}
G(1+r^s) &= (1+r)\left[1-(1+r^s)\left(\frac{d_1}{\phi y_2^H}\right)\right]^{-1} \\
G'(1+r^s) &= (1+r)\left[1-(1+r^s)\left(\frac{d_1}{\phi y_2^H}\right)\right]^{-2}\left(\frac{d_1}{\phi y_2^H}\right) > 0 \\
G''(1+r^s) &= 2(1+r)\left[1-(1+r^s)\left(\frac{d_1}{\phi y_2^H}\right)\right]^{-3}\left(\frac{d_1}{\phi y_2^H}\right)^2 > 0
\end{aligned}
$$

So $G(1+r^s)$ is a convex function.

3. An increase in d_1 shift the function G up. At some point, d_1 is so high that the right-hand side is always larger than the left-hand side and there is no solution. The highest possible value of d_1 is such that the two sides of the equation have the same level and slope. The equality of slopes implies:

$$
\begin{aligned}
1 &= G'(1+r^s) \\
1 &= (1+r)\left[1-(1+r^s)\left(\frac{d_1}{\phi y_2^H}\right)\right]^{-2}\left(\frac{d_1}{\phi y_2^H}\right)
\end{aligned}
$$

The equality of levels implies:

$$
\begin{aligned}
1+r^s &= (1+r)\left[1-(1+r^s)\left(\frac{d_1}{\phi y_2^H}\right)\right]^{-1} \\
(1+r^s)\left[1-(1+r^s)\left(\frac{d_1}{\phi y_2^H}\right)\right] &= (1+r) \\
-(1+r^s)\left(\frac{d_1}{\phi y_2^H}\right) &= \frac{(1+r)}{(1+r^s)} - 1 \\
\left(\frac{d_1}{\phi y_2^H}\right) &= \frac{1}{(1+r^s)}\left[1-\frac{(1+r)}{(1+r^s)}\right]
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
1 &= (1+r)\left[1-(1+r^s)\left(\frac{d_1}{\phi y_2^H}\right)\right]^{-2}\left(\frac{d_1}{\phi y_2^H}\right) \\
1 &= (1+r)\left[\frac{(1+r)}{(1+r^s)}\right]^{-2}\frac{1}{(1+r^s)}\left[1-\frac{(1+r)}{(1+r^s)}\right] \\
1 &= \left[\frac{(1+r)}{(1+r^s)}\right]^{-1}\left[1-\frac{(1+r)}{(1+r^s)}\right] \\
\frac{(1+r)}{(1+r^s)} &= 1-\frac{(1+r)}{(1+r^s)} \\
\frac{(1+r)}{(1+r^s)} &= \frac{1}{2} \\
(1+r^s) &= 2(1+r)
\end{aligned}
$$

The equality of levels then implies:

$$
\begin{aligned}
1+r^s &= G(1+r^s) \\
1+r^s &= \frac{1+r}{1-(1+r^s)\left(\frac{d_1}{\phi y_2^H}\right)} \\
2(1+r) &= \frac{1+r}{1-2(1+r)\left(\frac{d_1}{\phi y_2^H}\right)} \\
2-4(1+r)\left(\frac{d_1}{\phi y_2^H}\right) &= 1 \\
4(1+r)\left(\frac{d_1}{\phi y_2^H}\right) &= 1 \\
\frac{d_1}{\phi y_2^H} &= \frac{1}{4(1+r)} \\
d_1 &= \frac{\phi y_2^H}{4(1+r)}
\end{aligned}
$$

2 Endogenous interest rate

Question: Take the non-linear expression in $1+r^s$:

$$
1+r^s = \frac{1+r}{1-(1+r^s)\left(\frac{d_1}{\phi y_2^H}\right)}
$$

Show that it is a quadratic polynomial:

$$
0 = \frac{d_1}{\phi y_2^H}(1+r^s)^2-(1+r^s)+(1+r)
$$

Show that the solution is:

$$
1+r^s = 2(1+r)\frac{d_1^{high}}{d_1}\left(1-\sqrt{1-\frac{d_1}{d_1^{high}}}\right)
$$

What is the interest rate if $d_1 = d_1^{high}$?

Note: there is another solution with a higher interest rate, but it is unstable.

Answer: Re-arrange the terms in the quadratic expression:

$$\begin{aligned}
1+r^s &= \frac{1+r}{1-(1+r^s)\left(\frac{d_1}{\phi y_2^H}\right)} \\
(1+r^s)-(1+r^s)^2\left(\frac{d_1}{\phi y_2^H}\right) &= (1+r) \\
0 &= \frac{d_1}{\phi y_2^H}(1+r^s)^2-(1+r^s)+(1+r)
\end{aligned}$$

Use the standard solution for a quadratic root, focusing on the lowest term:

$$\begin{aligned}
1+r^s &= \frac{1-\sqrt{1-4(1+r)\frac{d_1}{\phi y_2^H}}}{2\frac{d_1}{\phi y_2^H}} \\
1+r^s &= \frac{1-\sqrt{1-4(1+r)\frac{d_1}{d_1^{high}}\frac{d_1^{high}}{\phi y_2^H}}}{2\frac{d_1}{d_1^{high}}\frac{d_1^{high}}{\phi y_2^H}} \\
1+r^s &= \frac{1-\sqrt{1-4(1+r)\frac{d_1}{d_1^{high}}\frac{1}{\phi y_2^H}\frac{\phi y_2^H}{4(1+r)}}}{2\frac{d_1}{d_1^{high}}\frac{1}{\phi y_2^H}\frac{\phi y_2^H}{4(1+r)}} \\
1+r^s &= \frac{1-\sqrt{1-\frac{d_1}{d_1^{high}}}}{\frac{d_1}{d_1^{high}}\frac{1}{2(1+r)}} \\
1+r^s &= 2(1+r)\frac{d_1^{high}}{d_1}\left(1-\sqrt{1-\frac{d_1}{d_1^{high}}}\right)
\end{aligned}$$

If $d_1 = d_1^{high}$ we get:

$$\begin{aligned}
1+r^s &= 2(1+r)\frac{d_1^{high}}{d_1}\left(1-\sqrt{1-\frac{d_1}{d_1^{high}}}\right) \\
1+r^s &= 2(1+r)\left(1-\sqrt{1-1}\right) \\
1+r^s &= 2(1+r)
\end{aligned}$$

Which is what we found in the previous question when the economy is at the maximal debt level.

3 Intertemporal allocation

Question: The borrower maximizes a linear utility, where $r < \delta$:

$$U = c_1 + \frac{1}{1+\delta}Ec_2$$

The budget constraints are:

$$c_1 = y_1 + d_1$$

$$\begin{aligned} c_2^{ND} &= y_2 - (1+r^s)\, d_1 \\ c_2^{D} &= (1-\phi)\, y_2 \end{aligned}$$

Some useful properties that you can take as given:

$$\begin{aligned} \pi E(y_2 \mid D) + (1-\pi) E(y_2 \mid ND) &= E(y_2) = \frac{y_2^H}{2} \\ E(y_2 \mid D) &= \frac{\pi y_2^H}{2} \end{aligned}$$

Show that the utility is:

$$U = Y_1 + \frac{\delta - r}{1+\delta} d_1 + \frac{1}{1+\delta} E(y_2) - \frac{1}{1+\delta} \pi \phi E(y_2 \mid D)$$

Using the expected values of output, show that:

$$U = y_1 + \frac{\delta - r}{1+\delta} d_1 + \frac{1}{1+\delta} \frac{y_2^H}{2} \left(1 - \phi \pi^2\right)$$

Show that the optimal debt level is such that (recall that π is a function of the debt):

$$\delta - r = \phi \pi y_2^H \frac{\partial \pi}{\partial d_1}$$

Answer: Using the budget constraints, the utility is written as:

$$\begin{aligned} U &= c_1 + \frac{1}{1+\delta} E c_2 \\ U &= c_1 + \frac{1}{1+\delta} \left[\pi E\left(c_2^D \mid D\right) + (1-\pi) E\left(c_2^{ND} \mid ND\right)\right] \\ U &= y_1 + d_1 + \frac{\pi}{1+\delta} E\left[(1-\phi)\, y_2 \mid D\right] \\ &\quad + \frac{(1-\pi)}{1+\delta} E\left[(y_2 - (1+r^s)\, d_1) \mid ND\right] \\ U &= y_1 + d_1 + \frac{\pi}{1+\delta} (1-\phi) E\left[y_2 \mid D\right] \\ &\quad + \frac{(1-\pi)}{1+\delta} E\left[(y) \mid ND\right] - \frac{(1-\pi)}{1+\delta} E\left[((1+r^s)\, d_1) \mid ND\right] \\ U &= y_1 + d_1 + \frac{1}{1+\delta} \left(\pi E\left[y_2 \mid D\right] + (1-\pi) E\left[y_2 \mid ND\right]\right) \\ &\quad - \frac{\pi}{1+\delta} \phi E\left[y_2 \mid D\right] - \frac{(1-\pi)}{1+\delta} E\left[((1+r^s)\, d_1) \mid ND\right] \\ U &= y_1 + d_1 + \frac{1}{1+\delta} E(y_2) \\ &\quad - \frac{\pi}{1+\delta} \phi E\left[y_2 \mid D\right] - \frac{(1-\pi)}{1+\delta} (1+r^s)\, d_1 \\ U &= y_1 + \left[1 - \frac{(1-\pi)}{1+\delta} (1+r^s)\right] d_1 + \frac{1}{1+\delta} E(y_2) - \frac{\pi}{1+\delta} \phi E\left[y_2 \mid D\right] \end{aligned}$$

Use $1 + r = (1-\pi)(1+r^s)$ to write:

$$U = y_1 + \left[1 - \frac{(1-\pi)}{1+\delta} (1+r^s)\right] d_1 + \frac{1}{1+\delta} E(y_2) - \frac{\pi}{1+\delta} \phi E\left[y_2 \mid D\right]$$

$$
\begin{aligned}
U &= y_1 + \left[1 - \frac{1+r}{1+\delta}\right] d_1 + \frac{1}{1+\delta} E\left(y_2\right) - \frac{\pi}{1+\delta}\phi E\left[y_2 \mid D\right] \\
U &= y_1 + \frac{\delta - r}{1+\delta} d_1 + \frac{1}{1+\delta} E\left(y_2\right) - \frac{\pi}{1+\delta}\phi E\left[y_2 \mid D\right]
\end{aligned}
$$

Using the expressions for expected output, we get:

$$
\begin{aligned}
U &= y_1 + \frac{\delta - r}{1+\delta} d_1 + \frac{1}{1+\delta} E\left(y_2\right) - \frac{\pi}{1+\delta}\phi E\left[y_2 \mid D\right] \\
U &= y_1 + \frac{\delta - r}{1+\delta} d_1 + \frac{1}{1+\delta}\frac{y_2^H}{2} - \frac{\pi}{1+\delta}\phi\frac{\pi y_2^H}{2} \\
U &= y_1 + \frac{\delta - r}{1+\delta} d_1 + \frac{1}{1+\delta}\left(1 - \phi\pi^2\right)\frac{y_2^H}{2}
\end{aligned}
$$

Taking the first-order condition with respect to d_1 we get:

$$
\begin{aligned}
0 &= \frac{\delta - r}{1+\delta} + \frac{1}{1+\delta}\left(-2\phi\pi\frac{\partial\pi}{\partial d_1}\right)\frac{y_2^H}{2} \\
0 &= \delta - r - \phi\pi\frac{\partial\pi}{\partial d_1} y_2^H \\
\delta - r &= \phi\pi y_2^H \frac{\partial\pi}{\partial d_1}
\end{aligned}
$$

4 Marginal impact of debt on default

Question: The probability of default, and the relation between the risk-free rate and risky rate are:

$$
\begin{aligned}
\pi &= \frac{\left(1+r^s\right) d_1}{\phi y_2^H} \\
1 + r &= \left(1 - \pi\right)\left(1 + r^s\right)
\end{aligned}
$$

Differentiate these two relations with respect to the risky interest rate, the probability of default, and the debt/GDP ratio to get:

$$
\begin{aligned}
d\pi &= d\left(1+r^s\right)\left(\frac{d_1}{\phi y_2^H}\right) + \left(1+r^s\right) d\left(\frac{d_1}{\phi y_2^H}\right) \\
\left(1+r^s\right) d\pi &= \left(1-\pi\right) d\left(1+r^s\right)
\end{aligned}
$$

Combine these to show:

$$
\frac{\partial\pi}{\partial d_1} = \frac{\left(1+r^s\right)}{1 - \frac{1+r^s}{1-\pi}\frac{d_1}{\phi y_2^H}}\frac{1}{\phi y_2^H}
$$

Answer: The differentiation of the probability of default is:

$$
\begin{aligned}
\pi &= \left(1+r^s\right)\left(\frac{d_1}{\phi y_2^H}\right) \\
d\pi &= \left(\frac{d_1}{\phi y_2^H}\right) d\left(1+r^s\right) + \left(1+r^s\right) d\left(\frac{d_1}{\phi y_2^H}\right)
\end{aligned}
$$

The differentiation of the relation between interest rates is (r is given):

$$\begin{aligned}(1+r) &= (1-\pi)(1+r^s) \\ 0 &= (1-\pi)\, d(1+r^s) - (1+r^s)\, d\pi \\ (1+r^s)\, d\pi &= (1-\pi)\, d(1+r^s)\end{aligned}$$

We combine them as follows:

$$\begin{aligned}d\pi &= \left(\frac{d_1}{\phi y_2^H}\right) d(1+r^s) + (1+r^s)\, d\left(\frac{d_1}{\phi y_2^H}\right) \\ d\pi &= \left(\frac{d_1}{\phi y_2^H}\right) \frac{1+r^s}{1-\pi} d\pi + (1+r^s)\, d\left(\frac{d_1}{\phi y_2^H}\right) \\ \left[1 - \frac{1+r^s}{1-\pi}\frac{d_1}{\phi y_2^H}\right] d\pi &= (1+r^s)\, d\left(\frac{d_1}{\phi y_2^H}\right) \\ d\pi &= \frac{(1+r^s)}{1-\frac{1+r^s}{1-\pi}\frac{d_1}{\phi y_2^H}} d\left(\frac{d_1}{\phi y_2^H}\right) \\ d\pi &= \frac{(1+r^s)}{1-\frac{1+r^s}{1-\pi}\frac{d_1}{\phi y_2^H}} \frac{1}{\phi y_2^H} d(d_1) \\ \frac{\partial \pi}{\partial d_1} &= \frac{(1+r^s)}{1-\frac{1+r^s}{1-\pi}\frac{d_1}{\phi y_2^H}} \frac{1}{\phi y_2^H}\end{aligned}$$

5 Overall solution

Question: The system is given by the probability of default, the relation between the risk-free rate and the risky rate, and the optimality condition above:

$$\begin{aligned}\pi &= \frac{(1+r^s)\, d_1}{\phi y_2^H} \\ 1+r &= (1-\pi)(1+r^s) \\ \delta - r &= \phi \pi y_2^H \frac{\partial \pi}{\partial d_1}\end{aligned}$$

Using our results, show that:

$$\begin{aligned}\pi &= \frac{\delta - r}{1+2\delta - r} \\ 1+r^s &= \frac{1+r}{1+\delta}(1+2\delta - r) \\ \frac{d_1}{\phi y_2^H} &= \frac{(1+\delta)(\delta - r)}{(1+r)(1+2\delta - r)^2}\end{aligned}$$

Answer: We first use the marginal effect of debt on the default risk to rewrite the system as:

$$\begin{aligned}\pi &= \frac{(1+r^s)\, d_1}{\phi y_2^H} \\ 1+r &= (1-\pi)(1+r^s) \\ \delta - r &= \pi \frac{(1+r^s)}{1-\frac{1+r^s}{1-\pi}\frac{d_1}{\phi y_2^H}}\end{aligned}$$

We use the second equation to substitute out for $1+r^s$ in the right-hand side of the last relation, and use the first relation to substitute out for $\frac{(1+r^s)d_1}{\phi y_2^H}$:

$$\begin{aligned}
\delta - r &= \pi \frac{(1+r^s)}{1 - \frac{1+r^s}{1-\pi}\frac{d_1}{\phi y_2^H}} \\
(\delta - r)\left[1 - \frac{1+r^s}{1-\pi}\frac{d_1}{\phi y_2^H}\right] &= \pi (1+r^s) \\
(\delta - r)\left[1 - \frac{1}{1-\pi}\pi\right] &= \pi \frac{1+r}{1-\pi} \\
(\delta - r)\left[1 - \frac{1}{1-\pi}\pi\right] &= \pi \frac{1+r}{1-\pi} \\
(\delta - r)(1-2\pi) &= \pi(1+r) \\
(\delta - r) &= \pi(1+r+2\delta-2r) \\
(\delta - r) &= \pi(1+2\delta-r) \\
\pi &= \frac{\delta - r}{1+2\delta-r}
\end{aligned}$$

The relation between the risk-free rate and the risky rate becomes:

$$\begin{aligned}
1+r &= (1-\pi)(1+r^s) \\
1+r &= \left(1 - \frac{\delta - r}{1+2\delta-r}\right)(1+r^s) \\
1+r &= \frac{1+\delta}{1+2\delta-r}(1+r^s) \\
1+r^s &= \frac{1+r}{1+\delta}(1+2\delta-r)
\end{aligned}$$

Finally, the definition of the probability of default implies:

$$\begin{aligned}
\pi &= \frac{(1+r^s)d_1}{\phi y_2^H} \\
\frac{\delta - r}{1+2\delta-r} &= \frac{1+r}{1+\rho}(1+2\rho-r)\frac{d_1}{\phi y_2^H} \\
\frac{1+r}{1+\delta}\frac{d_1}{\phi y_2^H} &= \frac{\delta - r}{(1+2\delta-r)^2} \\
\frac{d_1}{\phi y_2^H} &= \frac{(1+\delta)(\delta - r)}{(1+r)(1+2\delta-r)^2}
\end{aligned}$$