Game Theory

Information, Knowledge and Beliefs

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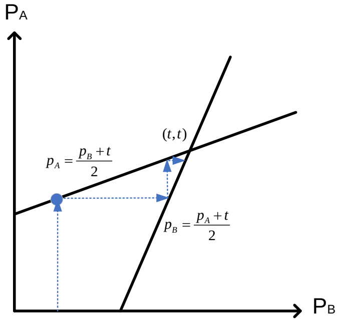
Review

In previous lectures, when solving the Nash equilibrium by using the "intersection" of the best responses, we assume that the best responses of both parties are "common knowledge."



1. Starting from an arbitrary p_B^1, A's best response is $p_A^1 = \frac{p_B^1 + t}{2}$
2. B **knows** that A will charge $p_A^1 = \frac{p_B^1 + t}{2}$, and then responds by charging $p_B^2 = \frac{p_A^1 + t}{2}$.
3. A **knows** that what B is thinking in step 2 (A knows that B knows), and hence responds by charging $p_A^2 = \frac{p_B^2 + t}{2}$.
4. B **knows** that what A is thinking in step 3 (B knows that A knows that B knows), and then responds by charging $p_B^3 = \frac{p_A^2 + t}{2}$.
5. ...

Eventually, the intersection becomes **common knowledge**, i.e., A knows what B will respond, and B knows what A will respond to what B will respond, and A knows what B will respond to what A will respond to what B will respond, ...

Outline

We explore such "high-order beliefs" in this lecture

- Examples
- Definitions of "information" and "knowledge"
- Common knowledge (共同知识)
- Theorem: People cannot agree to disagree (Aumann, 1976).
  在一定假设下，人们不可能"和而不同"

Reference: Ch5 in Osborne & Rubinstein, A course in game theory, MIT press, 1994

Example 1: Blue-eyed islanders

- There is an island upon which a tribe resides. The tribe consists of 1000 people, with various eye colors. Of the 1000 islanders, it turns out that 100 of them have blue eyes and 900 of them have brown eyes, although the islanders are not initially aware of these statistics. Their religion forbids them to know their own eye color; thus, each resident can (and does) see the eye colors of all other residents, but has no way of discovering his or her own. If a tribesperson does discover his or her own eye color, then their religion compels them to commit ritual suicide at noon the following day in the village square for all to witness. All the tribespeople are highly logical and devout, and they all know that each other is also highly logical and devout (and they all know that they all know that each other is highly logical and devout, and so forth).

- One day, a blue-eyed foreigner visits to the island and wins the complete trust of the tribe. However, not knowing the customs, the foreigner makes the mistake of mentioning eye color in his address, remarking "how unusual it is to see another blue-eyed person like myself in this region of the world."
- What is the outcome of the announcement?

Claim

Suppose that the tribe had n blue-eyed people for some positive integer n. Then n days after the traveller's address, all n blue-eyed people commit suicide.

Proof

- When $n = 1$, the only blue-eyed person realizes that the traveler is referring to him or her (because all the other 999 are known to be brown). Hence this guy commits suicide the next day.
- When $n = 2$, each blue-eyed person observes 1 blue and 998 brown. If the blue-eyed guy doesn't commits suicide the next day, then both know that they are blue-eyed. And commit suicide the day after the next day.
- Therefore, when $n = 100$, nobody commits suicide after 99 days, and all the blue-eyed person commit suicide after 100 days.

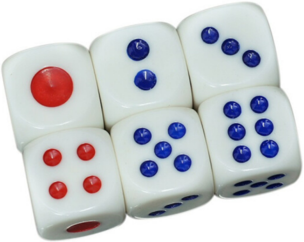
Example 2: In-class game

Let's play it in class!

Modelling Knowledge

- Information function:
  - A set of all possible states: Ω
  - Each state: $\omega \in \Omega$
  - Information function $P(\omega)$
  - When the state is ω, the decision-maker knows only that the state is in the set $P(\omega)$: the true state could be any state in $P(\omega)$ but not any state outside $P(\omega)$.
- Two conditions
  1. $\omega \in P(\omega)$
  2. if $\omega' \in P(\omega)$, then $P(\omega') = P(\omega)$
- An information function $P(\omega)$ is partitional (分割)

Example: A Dice



The host tosses a dice. The set of all possible states is $\Omega = \{1, 2, 3, 4, 5, 6\}$. Alice locates near the host and is able to see each number clearly. Bob locates far away from the host, and is able to identify the color but not the exact number.

- $P_{\text{Alice}} : \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$
- $P_{\text{Bob}} : \underbrace{\{1, 4\}}_{\text{red}}, \underbrace{\{2, 3, 5, 6\}}_{\text{blue}}$

The Knowledge Function

Definition

*A set of states (a subset of Ω) is an **event** E.*

Definition

A decision maker for whom $P(\omega) \subseteq E$ knows, in the state ω, that some state in the event E has occurred. We say in the state ω the decision-maker knows E. The knowledge function is

$$K(E) = \{\omega \in \Omega | P(\omega) \subseteq E\}$$

Example

Alice: $\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}$. Bob: $\{1,4\},\{2,3,5,6\}$.
Consider the event $E = \{4\}$.

- For Alice, because $\{4\} \subseteq E$, hence $K_{\text{Alice}}(E) = \{4\}$, i.e., Alice knows E when the host shows 4.
- For Bob, $\{1,4\} \nsubseteq E$ and $\{2,3,5,6\} \nsubseteq E$, hence $K_{\text{Bob}}(E) = \varnothing$. Bob does not know E whatever showed by the host.

Consider event $F = \{1,2,3,4\}$, i.e., the number is no more than 4.

- For Bob, $\{1,4\} \subseteq E$ while $\{2,3,5,6\} \nsubseteq E$, hence $K_{\text{Bob}}(E) = \{1,4\}$. That is, Bob knows E ("the number is no greater than 4) only when the host shows red colors.

Notes

- “知道 E”，“不知道 E”与“是否确认事件 E 发生”的区别
  - 有人问我：你是否“知道”张三？
  - 如果我从没听说过张三，在你问我这句话之前，我“不知道”张三，但你说出来“张三”这个概念后，我就“知道”了张三的“存在”，所以准确的回答是：你问我前我不知道，你问之后我就知道了（虽然我不认识张三）。
- “井底之蛙”不知道井口外部的世界 (unawareness)
- 一个人掉到井里，他“知道”外部的世界各种“可能性”，但除了眼睛能看到的井口，井口外部所发生的状况他无法确认
  - “井底之蛙”没见过井外的世界，所以“不知道”此刻是否有一只兔子路过（没有“兔子”的概念）
  - 掉到井里的某个人，他也“不知道”此时井口外是否有一只兔子路过（无法确认这个事件是否发生），但他知道存在这种可能性。
- 本课中所谓的“know”不讨论“井底之蛙”的情况，表示“确认某事发生”。

Example

$P : \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5\}, \{\omega_6\}, \{\omega_7, \omega_8\}$. Consider two events: $E = \{\omega_4, \omega_5, \omega_6, \omega_7\}$, $F = \{\omega_2, \omega_3, \omega_4, \omega_5\}$.

- $K(E) = \{\omega_5, \omega_6\}$, i.e., the agent knows E upon observing ω_5 or ω_6.
- $K(F) = \{\omega_3, \omega_4, \omega_5\}$, i.e., the agent knows F upon observing ω_3, ω_4 or ω_5.
- At state ω_2, the agent does not know F, although $\omega_2 \in F$, because he/she cannot distinguish ω_1 and ω_2.

Properties of Knowledge Function

Theorem (Axiom of awareness/omniscience)

$K(\Omega) = \Omega$

Theorem

If E implies F then if an agent knows E then he/she must know F. That is,

$$E \subseteq F \Rightarrow K(E) \subseteq K(F)$$

- Example: $P : \{1, 4\}, \{2, 3, 5, 6\}$, $E = \{1, 4\}$ and $F = \{1, 2, 3, 4\}$
- $K(E) = \{1, 4\}$, $K(F) = \{1, 4\}$, and $K(E) \subseteq K(F)$.
- If the number is red (E), then the number must be no greater than 4 (F). If Bob sees "red," then he knows that the host shows a number that is no greater than 4.

$$K(E) \cap K(F) = K(E \cap F)$$

The agent knows that both E and F occur, iff he/she knows E and he/she knows F.

- Example: $P_2 : \{1,4\}, \{2,3,5,6\}$, $E = \{1,4\}$ and $F = \{2,4,6\}$
- $K_2(E) = \{1,4\}$ and $K_2(F) = \varnothing$, hence $K_2(E) \cap K_2(F) = \varnothing$. Meanwhile, $E \cap F = \{4\}$, and $K_2(E \cap F) = K_2(\{4\}) = \varnothing$. The event $E \cap F = \{4\}$ means that "if the number is 4, then it must be a red even number." Because number 4 is not identifiable, hence there is no way for Bob to confirm whether the event "a red even number" occurs.
- But for Alice, $P_1 : \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$, $K_1(E) = \{1,4\}$ and $K_1(F) = \{2,4,6\} \Rightarrow K_1(E) \cap K_1(F) = \{4\}$. And $K(E \cap F) = K(\{4\}) = \{4\}$. When the host shows 4, Alice knows both "the number is red" and "it is an even number," and Alice knows that "the number is a red even number."

Axiom of Knowledge

$$K(E) \subseteq E$$

If an agent knows that E occurs, then the event E occurs.

- Example: $P : \{1, 4\}, \{2, 3, 5, 6\}$ and the event is $E = \{1, 4, 5\}$.
- $K(E) = \{1, 4\}$, i.e., as long as Bob sees "red," then he knows E.
- $K(E) = \{1, 4\} \subseteq \{1, 4, 5\} = E$, i.e., as long as Bob sees "red," then he knows E, then the event E must be true.

Axiom of Transparency

$$K(E) \subseteq K(K(E))$$

If an agent knows E, then he/she knows that he/she knows E.

- Example: $P : \{1,4\},\{2,3,5,6\}$ and the event is $E = \{1,4,5\}$.
- $K(E) = \{1,4\}$, i.e., as long as Bob sees "red," then he knows E.
- $K(K(E)) = K(\{1,4\}) = \{1,4\}$, i.e., as long as Bob sees "red," then he knows that he knows E.

Axiom of Wisdom

$$\Omega \backslash K(E) \subseteq K(\Omega \backslash K(E))$$

"Be aware of what I don't know."

- It's not what a man don't know that makes him a fool, but what he does know that ain't so (Josh Billings, 19th century American humorist).
- "不知为不知，是知也"
- In the example $P : \{1,4\}, \{2,3,5,6\}$ and the event is $E = \{4\}$,
  - $K(E) = \varnothing \Rightarrow \Omega \backslash K(E) = \Omega = \{1,2,3,4,5,6\}$, i.e., Bob does not know "the number is 4" for any number showed.
  - $K(\Omega \backslash K(E)) = K(\{1,2,3,4,5,6\}) = \{1,2,3,4,5,6\}$, i.e., whatever the host shows, Bob knows that he does not know "the number is 4."

'I can't believe *that*!' said Alice.

'Can't you?' the Queen said in a pitying tone. 'Try again: draw a long breath, and shut your eyes.'

Alice laughed. 'There's no use trying,' she said: 'one *can't* believe impossible things.'

'I daresay you haven't had much practice,' said the Queen. 'When I was your age, I always did it for half-an-hour a day. Why, sometimes I've believed as many as six impossible things before breakfast.'

Common Knowledge （共同知识）

Definition

An event E is ***mutual knowledge*** *（共有知识） between 1 and 2 at state ω, if $\omega \in K_1(E)$ and $\omega \in K_2(E)$.*

Definition

An event E is ***common knowledge*** *(共同知识) between 1 and 2 at state ω, if ω is a member of every set in the* ***infinite*** *sequence $K_1(E)$, $K_2(E)$, $K_1(K_2(E))$, $K_2(K_1(E))$, ...*

- $K_i(E)$: player i knows E
- $K_i(K_j(E))$: player i knows that player j knows E
- $K_i(K_j(K_i(E)))$: player i knows that player j knows that player i knows E.

Example

$$\text{Alice: } \{1\},\{2\},\{3\},\{4\},\{5\},\{6\}$$
$$\text{Bob: } \{1,4\},\{2,3,5,6\}$$

Is event $E = \{4\}$ common knowledge at state 4?

- $K_1(E) = \{4\}$; $K_2(E) = \varnothing$; $K_1(K_2(E)) = K_1(\varnothing) = \varnothing$. Hence E is not common knowledge.

Is event $F = \{1,4\}$ (the number is red) common knowledge at state 4?

- $K_1(F) = \{1,4\}$, $K_2(F) = \{1,4\}$; $K_1(K_2(F)) = K_1(F) = \{1,4\}$, $K_2(K_1(F)) = K_2(F) = \{1,4\}$, ... Hence F is common knowledge.

Continued:

Alice: $\{1\},\{2\},\{3\},\{4\},\{5\},\{6\}$

Bob: $\{1,4\},\{2,3,5,6\}$

Is event $G=\{1,3,4,5,6\}$ common knowledge at state 4? (the host shows the side with number 4). The event G means "the number is not 2.

- $K_1(G)=\{1,3,4,5,6\}$, $K_2(G)=\{1,4\}$. Bob can confirm that the number is not 2 only upon observing "red."
- $K_1(K_2(G))=K_1(\{1,4\})=\{1,4\}$: Alice knows that Bob knows that the number is not 2 only when the host shows "red."
- $K_2(K_1(G))=K_2(\{1,3,4,5,6\})=\{1,4\}$: Bob knows that Alice knows that the number is not 2 only when the host shows "red."
- $K_1(K_2(K_1(G)))=K_1(\{1,4\})=\{1,4\}$,
  $K_2(K_1(K_2(G)))=K_2(\{1,4\})=\{1,4\}$, ... Hence
  $G=\{$the number is not 2$\}$ is common knowledge, as long as the host shows the side with a red number.

Example

- Two players:

$$P_1 : \{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}$$
$$P_2 : \{\omega_1\}, \{\omega_2\}, \{\omega_3, \omega_4\}$$

  Is event $E = \{\omega_3, \omega_4\}$ common knowledge at state ω_4?
  - $K_1(E) = \{\omega_4\}$; $K_2(E) = \{\omega_3, \omega_4\}$.
  - $K_1(K_2(E)) = \{\omega_4\}$, but $K_2(K_1(E)) = \varnothing$.
- Is event $F = \{\omega_2, \omega_3, \omega_4\}$ common knowledge at state ω_4?
  - $K_1(F) = \{\omega_2, \omega_3, \omega_4\} = F$, $K_2(F) = F$
  - $K_1(K_2(F)) = F$, $K_2(K_1(F)) = F$,...
- Self-evident （自明） of F: for all $\omega \in F$ we have $P_i(\omega) \in F$.

Example

If E is not self-evident, then E is not common knowledge.

- Two players

$$P_1 : \{\omega_1, \omega_2\}, \{\omega_3, \omega_4, \omega_5\}, \{\omega_6\}$$
$$P_2 : \{\omega_1\}, \{\omega_2, \omega_3, \omega_4\}, \{\omega_5\}, \{\omega_6\}$$

- Event $E = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ is not common knowledge at any state
    - $K_1(E) = \{\omega_1, \omega_2\}$, $K_2(E) = \{\omega_1, \omega_2, \omega_3, \omega_4\} = E$
    - $K_1(K_2(E)) = \{\omega_1, \omega_2\}$, $K_2(K_1(E)) = \{\omega_1\}$
    - $K_1(K_2(K_1(E))) = \varnothing$, $K_2(K_1(K_2(E))) = \{\omega_1\}$
    - $K_1(K_2(K_1(K_2(E)))) = \varnothing$, $K_2(K_1(K_2(K_1(E)))) = \varnothing$.
- Instead, the event $F = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ is self-evident, and is common knowledge.

Can People Agree to Disagree (和而不同)?

- Can it be common knowledge between two agents that agent 1 assigns probability η_1 to an event and agent 2 assigns probability $\eta_2 \neq \eta_1$ to the same event? Aumann (1976) shows that if the agents have the same prior beliefs and their information functions are partitional, then the answer is negative.

Theorem

Assume the same prior beliefs. If each individual's information function is partitional and it is ***common knowledge*** *between 1 and 2 in some state $\omega^* \in \Omega$ that agent 1 assigns (posterior) probability η_1 to event E and agent 2 assigns posterior probability η_2 to E, then $\eta_1 = \eta_2$.*

- People cannot agree to disagree!

The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2005

Robert J. Aumann
Thomas C. Schelling

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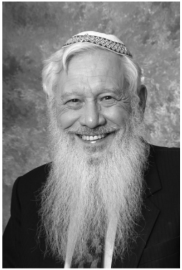

Photo: D. Porges

Robert J. Aumann
The Sveriges Riksbank Prize in Economic Sciences in Memory of Alfred Nobel 2005

Born: 8 June 1930, Frankfurt-on-the-Main, Germany

Affiliation at the time of the award: University of Jerusalem, Center for RationalityHebrew, Jerusalem, Israel

Prize motivation: "for having enhanced our understanding of conflict and cooperation through game-theory analysis."

Example: 猜女婿

老太太和老爷爷有个女儿，他们知道（且 common knowledge）未来的女婿可能是以下四位中 (1,2,3,4) 的其中一个，且同等可能性

1. 北京人，大款
2. 北京人，非大款
3. 外地人（中国人），非大款
4. 外国人，大款

- 老太太跟踪过女儿，通过背影发现不是外国人（排除 4）。
- 老爷爷接过未来女婿打来的电话，通过口音发现是北京话（排除 3 和 4）。
- 假设老爷爷知道“老太太跟踪过女儿”这个事，知道老太太看到的是中国人或外国人，但具体不知道老太太看到的是中国人这一组还是外国人这一组。
- 假设老太太知道“老爷爷接过电话”这个事，知道老爷爷听到的是北京话或非北京话，但具体不知道老爷爷听到的是北京话这一组还是非北京话这一组。
- 假设二老在吵架冷战，相互不透露具体信息，但关于对方所持有的信息结构，假设双方都知道对方知道对方知道⋯⋯

有一天，一位朋友告诉二老，他们未来的女婿是“大款”。老太太和老爷爷同时汇报这位朋友的话“有几成准”。

1. **第一轮汇报：老太太看到的是中国人，这 3 个中国人中，只有一个是有钱人，所以老太太说“这位朋友说话靠谱的概率是 1/3”。同理，老爷爷确认是北京人的一组，其中一位是有钱人，所以汇报的概率是 1/2。**

   当老爷爷说出来 1/2 后，他的话没有任何信息量。因为老太太并不能区分老爷爷说的 1/2 指的是北京人的这一组，还是非北京人的这一组。但是，当老太太说出来 1/3 后，老爷爷就能排除外国人。

2. **第二轮汇报：由于老爷爷的话没有信息量，所以老太太依然汇报 1/3；由于老太太第一轮汇报的 1/3 排除了外国人，所以老爷爷第二次汇报的概率依然是 1/2。**

   当老爷爷第二次的汇报依然坚持了 1/2 后，此时的 1/2 对老太太而言就具有了信息量：老太太心想，自己汇报 1/3 后，老爷爷就能排除外国人，假如老爷爷接的电话听到的不是北京话，那只剩“外地非大款”了，那么他第二次汇报的概率应该是 0，而他第二次汇报的概率不是 0，所以老爷爷听到的电话一定是北京话的那一组。

3. **所以老太太也排除 3，最后改口说概率是 1/2。至此两人都同意“这位朋友说的话有五成准”。**

Modelling "Agree to Disagree"

- Four candidates: $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ with prior probability $1/4$ each.
- The information function of the two players:
    - Grandma: $P_1 : \{\omega_1, \omega_2, \omega_3\}, \{\omega_4\}$;
    - Grandpa: $P_2 : \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}$
- Event $E = \{\omega_1, \omega_4\}$.

$P_1 : \underbrace{\{\omega_1, \omega_2, \omega_3\}}_{I_{11}}, \underbrace{\{\omega_4\}}_{I_{12}}$, where I_{11} is observed;

$P_2 : \underbrace{\{\omega_1, \omega_2\}}_{I_{21}}, \underbrace{\{\omega_3, \omega_4\}}_{I_{22}}$, where I_{21} is observed.

The event is $E = \{\omega_1, \omega_4\}$

- Initially, the prior probability for "E is true" of player 1 is

$$\eta_1 = \Pr(E|I_{11}) = \frac{\Pr(EI_{11})}{\Pr(I_{11})} = \frac{\Pr(\{\omega_1\})}{\Pr(\{\omega_1, \omega_2, \omega_3\})} = \frac{1/4}{3/4} = \frac{1}{3}$$

  For player 2, the probability is

$$\eta_2 = \Pr(E|I_{21}) = \frac{\Pr(EI_{21})}{\Pr(I_{21})} = \frac{\Pr(\{\omega_1\})}{\Pr(\{\omega_1, \omega_2\})} = \frac{1/4}{2/4} = \frac{1}{2}.$$

- After player 2 announces $\eta_2 = \frac{1}{2}$ for the first time, such signal is uninformative and player 1 does not learn anything (so she reports $\eta_1' = \frac{1}{3}$ again). Because from the view of player 1:

$$\underbrace{\Pr(E|I_{21}) = \frac{\Pr(EI_{21})}{\Pr(I_{21})}}_{\text{player 2 observes } I_{21}} = \frac{1}{2} = \underbrace{\Pr(E|I_{22}) = \frac{\Pr(EI_{22})}{\Pr(I_{22})}}_{\text{player 2 observes }_{22}}$$

$P_1 : \underbrace{\{\omega_1, \omega_2, \omega_3\}}_{I_{11}}, \underbrace{\{\omega_4\}}_{I_{12}}$, $P_2 : \underbrace{\{\omega_1, \omega_2\}}_{I_{21}}, \underbrace{\{\omega_3, \omega_4\}}_{I_{22}}$. The event is $E = \{\omega_1, \omega_4\}$

- After player 1 reports $\eta_1 = \frac{1}{3}$ for the first time, player 2 knows that ω_4 is not possible. Eliminating ω_4, player 2 updates his information sets:

$$P_2' : \underbrace{\{\omega_1, \omega_2\}}_{I_{21}'}, \underbrace{\{\omega_3, \cancel{\omega_4}\}}_{I_{22}'}$$

  and claims that the posterior probability is

$$\eta_2' = \Pr(E|I_{21}') = \frac{\Pr(EI_{21}')}{\Pr(I_{21}')} = \frac{\Pr(\{\omega_1\})}{\Pr(\{\omega_1, \omega_2\})} = \frac{1}{2}$$

  "again."

$P_2 : \underbrace{\{\omega_1, \omega_2\}}_{I_{21}}, \underbrace{\{\omega_3, \omega_4\}}_{I_{22}} \xrightarrow{\text{updates to}} P_2' : \underbrace{\{\omega_1, \omega_2\}}_{I_{21}'}, \underbrace{\{\omega_3, \cancel{\omega_4}\}}_{I_{22}'}$. The event is $E = \{\omega_1, \omega_4\}$.

- After player 2 reports $\eta_2' = \frac{1}{2}$ again, such information is informative. To see this, consider what player 1 will think
  1. Suppose that player 2 initially observes I_{22}. After I said "$\eta_1 = \frac{1}{3}$," player 2 should update his P_2 to P_2' by eliminating ω_4. Then player 2 should report
  $$\Pr(E|I_{22}') = \frac{\Pr(EI_{22}')}{\Pr(I_{22}')} = \frac{\Pr(\varnothing)}{\Pr\{\omega_3\}} = 0.$$
  2. But player 2 actually reports $\eta_2' = \frac{1}{2}$ rather than 0. Therefore, player 2 observes $I_{21} = \{\omega_1, \omega_2\}$ at the first round.
- Therefore, ω_3 is not possible, and player 1 updates her information sets:

$$P_1 : \underbrace{\{\omega_1, \omega_2, \omega_3\}}_{I_{11}}, \underbrace{\{\omega_4\}}_{I_{12}} \xrightarrow{\text{updates to}} P_1' : \underbrace{\{\omega_1, \omega_2, \cancel{\omega_3}\}}_{I_{11}'}, \underbrace{\{\omega_4\}}_{I_{12}'}$$

Now the information structure becomes $P_1' : \underbrace{\{\omega_1, \omega_2, \cancel{\omega_3}\}}_{I_{11}'}, \underbrace{\{\omega_4\}}_{I_{12}'}$

and $P_2' : \underbrace{\{\omega_1, \omega_2\}}_{I_{21}'}, \underbrace{\{\omega_3, \cancel{\omega_4}\}}_{I_{22}'}$. The event is $E = \{\omega_1, \omega_4\}$

- After player 1 found player 2 claims $\eta_2' = \frac{1}{2}$ at the second round and updated her beliefs, she reports the posterior:

$$\eta_1'' = \Pr(EI_{11}') = \frac{\Pr(EI_{11}')}{\Pr(I_{11}')} = \frac{\Pr(\{\omega_1\})}{\Pr(\{\omega_1, \omega_2\})} = \frac{1/4}{2/4} = \frac{1}{2}.$$

- Eventually, they agree that the probability of "the claim $E = \{\omega_1, \omega_4\}$ is true" is $\eta = \frac{1}{2}$. That is, for the first round, they report $(\eta_1, \eta_2) = \left(\frac{1}{3}, \frac{1}{2}\right)$; then, they report $(\eta_1', \eta_2') = \left(\frac{1}{3}, \frac{1}{2}\right)$; then, they report $(\eta_1'', \eta_2'') = \left(\frac{1}{2}, \frac{1}{2}\right)$, reaching to an agreement.

Sometimes "by reaching an agreement, we get closer to the truth"

Assume $P_1 : \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$ and $P_2 : \{1, 4\}, \{2, 3, 5, 6\}$. The true state is 4. Consider the probability of the claim $E = \{$even number, i.e., $2, 4, 6\}$

- $\eta_1 = P(E|4) = \frac{P(4)}{P(4)} = 1$. $\eta_2 = P(E|\{1, 4\}) = \frac{P(4)}{P(\{1,4\})} = \frac{1}{2}$.
- $\eta_1 = 1 \Rightarrow$ player 1 observes $\{2\}, \{4\}, \{6\}$ so $\{1\}$ is not possible. Then $P_2' : \{\cancel{1}, 4\}, \{2, 3, 4, 5\}$ and then $\eta_2' = 1$.
- After player 1 says "$\eta_1 = 1$," player 2 gets closer to the truth.

It is also possible that "we reach a more biased agreement"

Assume $P_1 : \{1,2,3\}, \{4,5,6\}$ and $P_2 : \{1,2\}, \{3,4,5,6\}$. The true state is 5. Consider the probability of the claim $E = \{4,6\}$. The claim is not true.

- $\eta_1 = P(E|\{4,5,6\}) = \frac{P(\{4,6\})}{P(\{4,5,6\})} = \frac{2}{3}$.
  $\eta_2 = P(E|\{3,4,5,6\}) = \frac{P(\{4,6\})}{P(\{3,4,5,6\})} = \frac{1}{2}$. Player 2 is relatively closer to the truth.
- Since $\eta_1 = \frac{2}{3}$, player 1 observes $\{4,5,6\}$ rather than $\{1,2,3\}$, hence 3 is not possible. $P_2' : \{1,2,\}, \{\cancel{3},4,5,6\}$, and $\eta_2' = P(E|\{4,5,6\}) = \frac{2}{3}$.
- They agree that "the probability of E is true" is $\frac{2}{3}$, which is higher than the prior probability $\eta_2 = \frac{1}{2}$ of player 2. Player 2 becomes more inclined to believe a false claim.

Application: No-Trade Theorem?

- A father tells his two sons that he has placed $\$10^n$ in one envelop and $\$10^{n+1}$ in another, where n is chosen with equal probability among integers between 1 and 5. The two sons completely believe their father. Each son is randomly given an envelope.
  - If the father privately asks each son whether he would be willing to pay \$1 to switch envelopes, what would the sons say?
  - If the father calls both his sons in together and ask them if they are willing to make the deal, what would they say?

- Private: "Yes" unless the one who gets 10^6.

$$10^5 = 100,000 < \left(\frac{1}{5}\right)\left(10 + 10^2 + 10^3 + 10^4 + 10^6\right) = 202,222.$$

- Public:
  - Suppose one gets 10^1 and his brother gets 10^k ($k > 1$). The one who gets 10^1 will definitely say "yes." After observing "yes," his brother will say "no."
  - If one gets 10^2. After the first-round, if his brother does not say "yes," then 10^2 is the smallest envelop. Then he will say "yes." Then, his brother will say "no."
  - ...
  - No trade.