Macroeconomics A; EI060

Short problems

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1 Expected labor

Question: The Home and Foreign output relations are given by:

$$Y = \frac{1}{2}\left(\frac{PC}{P_H} + \frac{P^*C^*}{P_H^*}\right) = \frac{\mu}{2}\left(\frac{1}{P_H} + \frac{\mu^*}{\mu P_H^*}\right) = \frac{PC}{2}\left(\frac{1}{P_H} + \frac{1}{\mathcal{E} P_H^*}\right)$$

The price indices are:

$$\begin{aligned} P &= 2\left[P_H\right]^{0.5}\left[P_F\right]^{0.5} \\ P^* &= 2\left[P_H^*\right]^{0.5}\left[P_F^*\right]^{0.5} \end{aligned}$$

The various prices are:

$$\begin{aligned} P_H &= \frac{\theta\kappa}{\theta-1} E\left(\frac{\mu}{Z}\right) \\ P_F^* &= \frac{\theta\kappa}{\theta-1} E\left(\frac{\mu^*}{Z^*}\right) \\ P_F &= \left(\mathcal{E}\right)^{\gamma^*} \frac{\theta\kappa}{\theta-1} E\left(\frac{\mu^*}{Z^*}\left(\mathcal{E}\right)^{1-\gamma^*}\right) \\ P_H^* &= \left(\mathcal{E}\right)^{-\gamma} \frac{\theta\kappa}{\theta-1} E\left(\frac{\mu}{Z}\left(\mathcal{E}\right)^{\gamma-1}\right) \end{aligned}$$

The exchange rate is $\mathcal{E} = \mu/\mu^*$. The technology is $Y = Zl$.

Show that the expected labor is:

$$E(l) = \frac{\theta-1}{\theta\kappa}$$

Answer: The labor is:

$$\begin{aligned} l &= \frac{Y}{Z} \\ l &= \frac{1}{Z}\frac{PC}{2}\left(\frac{1}{P_H} + \frac{1}{\mathcal{E} P_H^*}\right) \end{aligned}$$

$$
\begin{aligned}
l &= \frac{\mu}{2Z}\left(\frac{1}{\frac{\theta\kappa}{\theta-1}E\left(\frac{\mu}{Z}\right)}+\frac{1}{(\mathcal{E})^{1-\gamma}\frac{\theta\kappa}{\theta-1}E\left(\frac{\mu}{Z}(\mathcal{E})^{\gamma-1}\right)}\right)\\
l &= \frac{\mu}{2Z}\left(\frac{1}{E\left(\frac{\mu}{Z}\right)}+\frac{(\mathcal{E})^{\gamma-1}}{E\left(\frac{\mu}{Z}(\mathcal{E})^{\gamma-1}\right)}\right)\frac{\theta-1}{\theta\kappa}\\
l &= \frac{1}{2}\left(\frac{\frac{\mu}{Z}}{E\left(\frac{\mu}{Z}\right)}+\frac{\frac{\mu}{Z}(\mathcal{E})^{\gamma-1}}{E\left(\frac{\mu}{Z}(\mathcal{E})^{\gamma-1}\right)}\right)\frac{\theta-1}{\theta\kappa}
\end{aligned}
$$

We take the expected value:

$$
\begin{aligned}
E(l) &= \frac{1}{2}E\left(\frac{\frac{\mu}{Z}}{E\left(\frac{\mu}{Z}\right)}+\frac{\frac{\mu}{Z}(\mathcal{E})^{\gamma-1}}{E\left(\frac{\mu}{Z}(\mathcal{E})^{\gamma-1}\right)}\right)\frac{\theta-1}{\theta\kappa}\\
E(l) &= \frac{1}{2}\left(\frac{E\left(\frac{\mu}{Z}\right)}{E\left(\frac{\mu}{Z}\right)}+\frac{E\left(\frac{\mu}{Z}(\mathcal{E})^{\gamma-1}\right)}{E\left(\frac{\mu}{Z}(\mathcal{E})^{\gamma-1}\right)}\right)\frac{\theta-1}{\theta\kappa}\\
E(l) &= \frac{1}{2}(1+1)\frac{\theta-1}{\theta\kappa}\\
E(l) &= \frac{\theta-1}{\theta\kappa}
\end{aligned}
$$

2 Expected log consumption

Question: We can show that when prices are flexible (take this as given):

$$
C^{\text{flex}} = \frac{\theta-1}{\theta 2\kappa}(Z)^{0.5}(Z^*)^{0.5}
$$

Show that consumption under sticky prices is:

$$
C = \frac{(\mu)^{1-\frac{\gamma^*}{2}}(\mu^*)^{\frac{\gamma^*}{2}}}{\left[E\left(\frac{\mu}{Z}\right)\right]^{0.5}\left[E\left(\frac{\mu^*}{Z^*}(\mathcal{E})^{1-\gamma^*}\right)\right]^{0.5}}\frac{\theta-1}{2\theta\kappa}
$$

Show that the gap between the expected log consumption and its value under flexible prices is:

$$
\begin{aligned}
E(\ln C)-E\left(\ln C^{\text{flex}}\right) = &\left(1-\frac{\gamma^*}{2}\right)\sum_k \pi_k \ln\mu_k+\frac{\gamma^*}{2}\sum_k \pi_k\ln\mu_k^*\\
&-\frac{1}{2}\sum_k \pi_k \ln Z_k-\frac{1}{2}\sum_k\pi_k\ln Z_k^*\\
&-\frac{1}{2}\ln\left[\sum_k \pi_k\frac{\mu_k}{Z_k}\right]-\frac{1}{2}\ln\left[\sum_k\pi_k\frac{(\mu_k)^{1-\gamma^*}(\mu_k^*)^{\gamma^*}}{Z_k^*}\right]
\end{aligned}
$$

where k is an index of state of nature, and π_k denotes the probability of the state.

Answer: Consumption is given by:

$$
\begin{aligned}
C &= \frac{\mu}{P} \\
C &= \frac{\mu}{2\left[P_H\right]^{0.5}\left[P_F\right]^{0.5}} \\
C &= \frac{\mu}{2\left[\frac{\theta\kappa}{\theta-1}E\left(\frac{\mu}{Z}\right)\right]^{0.5}\left[\left(\mathcal{E}\right)^{\gamma^*}\frac{\theta\kappa}{\theta-1}E\left(\frac{\mu^*}{Z^*}\left(\mathcal{E}\right)^{1-\gamma^*}\right)\right]^{0.5}} \\
C &= \frac{\mu}{\left[E\left(\frac{\mu}{Z}\right)\right]^{0.5}\left[\left(\frac{\mu}{\mu^*}\right)^{\gamma^*}E\left(\frac{\mu^*}{Z^*}\left(\mathcal{E}\right)^{1-\gamma^*}\right)\right]^{0.5}}\frac{\theta-1}{2\theta\kappa} \\
C &= \frac{\left(\mu\right)^{1-\frac{\gamma^*}{2}}\left(\mu^*\right)^{\frac{\gamma^*}{2}}}{\left[E\left(\frac{\mu}{Z}\right)\right]^{0.5}\left[E\left(\frac{\mu^*}{Z^*}\left(\mathcal{E}\right)^{1-\gamma^*}\right)\right]^{0.5}}\frac{\theta-1}{2\theta\kappa}
\end{aligned}
$$

We then write:

$$
\begin{aligned}
E\left(\ln C\right) &= \left(1-\frac{\gamma^*}{2}\right)E\left(\ln\mu\right)+\frac{\gamma^*}{2}E\left(\ln\mu^*\right) \\
&\quad -\frac{1}{2}\ln\left[E\left(\frac{\mu}{Z}\right)\right]-\frac{1}{2}\ln\left[E\left(\frac{\mu^*}{Z^*}\left(\mathcal{E}\right)^{1-\gamma^*}\right)\right]+\ln\frac{\theta-1}{2\theta\kappa} \\
E\left(\ln C\right)-E\left(\ln C^{\text{flex}}\right) &= \left(1-\frac{\gamma^*}{2}\right)E\left(\ln\mu\right)+\frac{\gamma^*}{2}E\left(\ln\mu^*\right) \\
&\quad -\frac{1}{2}\ln\left[E\left(\frac{\mu}{Z}\right)\right]-\frac{1}{2}\ln\left[E\left(\frac{\mu^*}{Z^*}\left(\mathcal{E}\right)^{1-\gamma^*}\right)\right]+\ln\frac{\theta-1}{2\theta\kappa} \\
&\quad -\ln\frac{\theta-1}{2\theta\kappa}-\frac{1}{2}E\left(\ln Z\right)-\frac{1}{2}E\left(\ln Z^*\right) \\
E\left(\ln C\right)-E\left(\ln C^{\text{flex}}\right) &= \left(1-\frac{\gamma^*}{2}\right)E\left(\ln\mu\right)+\frac{\gamma^*}{2}E\left(\ln\mu^*\right) \\
&\quad -\frac{1}{2}E\left(\ln Z\right)-\frac{1}{2}E\left(\ln Z^*\right) \\
&\quad -\frac{1}{2}\ln\left[E\left(\frac{\mu}{Z}\right)\right]-\frac{1}{2}\ln\left[E\left(\frac{\mu^*}{Z^*}\left(\mathcal{E}\right)^{1-\gamma^*}\right)\right] \\
E\left(\ln C\right)-E\left(\ln C^{\text{flex}}\right) &= \left(1-\frac{\gamma^*}{2}\right)\sum_k\pi_k\ln\mu_k+\frac{\gamma^*}{2}\sum_k\pi_k\ln\mu_k^* \\
&\quad -\frac{1}{2}\sum_k\pi_k\ln Z_k-\frac{1}{2}\sum_k\pi_k\ln Z_k^* \\
&\quad -\frac{1}{2}\ln\left[\sum_k\pi_k\frac{\mu_k}{Z_k}\right]-\frac{1}{2}\ln\left[\sum_k\pi_k\frac{\left(\mu_k\right)^{1-\gamma^*}\left(\mu_k^*\right)^{\gamma^*}}{Z_k^*}\right]
\end{aligned}
$$

3 Optimal Home policy

Question: Show that the value of a specific state μ_k that maximizes $E\left(\ln C\right)-E\left(\ln C^{\text{flex}}\right)$ is:

$$
0 = \left(1-\frac{\gamma^*}{2}\right)-\frac{1}{2}\frac{\frac{\mu_k}{Z_k}}{\sum_k\pi_k\frac{\mu_k}{Z_k}}-\frac{1-\gamma^*}{2}\frac{\frac{\left(\mu_k\right)^{1-\gamma^*}\left(\mu_k^*\right)^{\gamma^*}}{Z_k^*}}{\sum_k\pi_k\frac{\left(\mu_k\right)^{1-\gamma^*}\left(\mu_k^*\right)^{\gamma^*}}{Z_k^*}}
$$

We take a log linear approximation around $\mu_k = \mu_k^* = Z_k = Z_k^* = \mu_0 = Z_0 = 1$. For instance:

$$\begin{aligned}
\frac{\mu_k}{Z_k} &= \frac{\mu_0}{Z_0} + \frac{1}{Z_0}(\mu_k - \mu_0) - \frac{\mu_0}{(Z_0)^2}(Z_k - Z_0) \\
\frac{\mu_k}{Z_k} &= \frac{\mu_0}{Z_0}\left[1 + \frac{\mu_k - \mu_0}{\mu_0} - \frac{Z_k - Z_0}{Z_0}\right] \\
\frac{\mu_k}{Z_k} &= \frac{\mu_0}{Z_0}\left[1 + \left[\ln(\mu_k) - \ln(\mu_0)\right] - \left[\ln(Z_k) - \ln(Z_0)\right]\right] \\
\frac{\mu_k}{Z_k} &= \frac{\mu_0}{Z_0}\left[1 + \ln(\mu_k) - \ln(Z_k)\right]
\end{aligned}$$

Note that to a first order, $E[\ln(\mu_k)] = E[\ln(Z_k)] = 0$. Show that with this approximation:

$$\left[1 + (1-\gamma^*)^2\right]\ln(\mu_k) = \ln(Z_k) + (1-\gamma^*)\ln(Z_k^*) - \gamma^*(1-\gamma^*)\ln(\mu_k^*)$$

Answer: We take the derivative as follows:

$$\begin{aligned}
0 &= \frac{\partial E(\ln C) - E\left(\ln C^{\text{flex}}\right)}{\partial \mu_k} \\
0 &= \left(1 - \frac{\gamma^*}{2}\right)\pi_k \frac{1}{\mu_k} \\
&\quad -\frac{1}{2}\frac{1}{\sum_k \pi_k \frac{\mu_k}{Z_k}}\frac{\partial\left(\sum_k \pi_k \frac{\mu_k}{Z_k}\right)}{\partial \mu_k} - \frac{1}{2}\frac{1}{\sum_k \pi_k \frac{(\mu_k)^{1-\gamma^*}(\mu_k^*)^{\gamma^*}}{Z_k^*}}\frac{\partial\left(\sum_k \pi_k \frac{(\mu_k)^{1-\gamma^*}(\mu_k^*)^{\gamma^*}}{Z_k^*}\right)}{\partial \mu_k} \\
0 &= \left(1 - \frac{\gamma^*}{2}\right)\pi_k \frac{1}{\mu_k} - \frac{1}{2}\frac{1}{\sum_k \pi_k \frac{\mu_k}{Z_k}}\pi_k \frac{1}{Z_k} - \frac{1}{2}\frac{1-\gamma^*}{\sum_k \pi_k \frac{(\mu_k)^{1-\gamma^*}(\mu_k^*)^{\gamma^*}}{Z_k^*}}\pi_k \frac{(\mu_k)^{-\gamma^*}(\mu_k^*)^{\gamma^*}}{Z_k^*} \\
0 &= \frac{\pi_k}{\mu_k}\left[\left(1 - \frac{\gamma^*}{2}\right) - \frac{1}{2}\frac{1}{\sum_k \pi_k \frac{\mu_k}{Z_k}}\frac{\mu_k}{Z_k} - \frac{1}{2}\frac{1-\gamma^*}{\sum_k \pi_k \frac{(\mu_k)^{1-\gamma^*}(\mu_k^*)^{\gamma^*}}{Z_k^*}}\frac{(\mu_k)^{1-\gamma^*}(\mu_k^*)^{\gamma^*}}{Z_k^*}\right] \\
0 &= \left(1 - \frac{\gamma^*}{2}\right) - \frac{1}{2}\frac{\frac{\mu_k}{Z_k}}{\sum_k \pi_k \frac{\mu_k}{Z_k}} - \frac{1-\gamma^*}{2}\frac{\frac{(\mu_k)^{1-\gamma^*}(\mu_k^*)^{\gamma^*}}{Z_k^*}}{\sum_k \pi_k \frac{(\mu_k)^{1-\gamma^*}(\mu_k^*)^{\gamma^*}}{Z_k^*}}
\end{aligned}$$

We now take a log approximation of the various elements:

$$\begin{aligned}
\frac{\frac{\mu_k}{Z_k}}{\sum_k \pi_k \frac{\mu_k}{Z_k}} &= \frac{\frac{\mu_0}{Z_0}}{\sum_k \pi_k \frac{\mu_0}{Z_0}} + \frac{\frac{1}{Z_0}}{\sum_k \pi_k \frac{\mu_0}{Z_0}}(\mu_k - \mu_0) - \frac{\frac{\mu_0}{(Z_0)^2}}{\sum_k \pi_k \frac{\mu_0}{Z_0}}(Z_k - Z_0) \\
&\quad - \frac{\frac{\mu_0}{Z_0}}{\left(\sum_k \pi_k \frac{\mu_0}{Z_0}\right)^2}\sum_k \pi_k \left[\frac{1}{Z_0}(\mu_k - \mu_0) - \frac{\mu_0}{(Z_0)^2}(Z_k - Z_0)\right] \\
\frac{\frac{\mu_k}{Z_k}}{\sum_k \pi_k \frac{\mu_k}{Z_k}} &= 1 + \frac{\mu_k - \mu_0}{\mu_0} - \frac{Z_k - Z_0}{Z_0} - \sum_k \pi_k \left[\frac{\mu_k - \mu_0}{\mu_0} - \frac{Z_k - Z_0}{Z_0}\right] \\
\frac{\frac{\mu_k}{Z_k}}{\sum_k \pi_k \frac{\mu_k}{Z_k}} &= 1 + \left[\ln(\mu_k) - \ln(\mu_0)\right] - \left[\ln(Z_k) - \ln(Z_0)\right] - \sum_k \pi_k \left[\left[\ln(\mu_k) - \ln(\mu_0)\right] - \left[\ln(Z_k) - \ln(Z_0)\right]\right] \\
\frac{\frac{\mu_k}{Z_k}}{\sum_k \pi_k \frac{\mu_k}{Z_k}} &= 1 + \ln(\mu_k) - \ln(Z_k)
\end{aligned}$$

Similarly:

$$\begin{aligned}
\frac{\frac{(\mu_k)^{1-\gamma^*}(\mu_k^*)^{\gamma^*}}{Z_k^*}}{\sum_k \pi_k \frac{(\mu_k)^{1-\gamma^*}(\mu_k^*)^{\gamma^*}}{Z_k^*}} &= \frac{\frac{(\mu_0)^{1-\gamma^*}(\mu_0)^{\gamma^*}}{Z_0}}{\sum_k \pi_k \frac{(\mu_0)^{1-\gamma^*}(\mu_0)^{\gamma^*}}{Z_0}} \\
&\quad + \frac{1}{\sum_k \pi_k \frac{(\mu_0)^{1-\gamma^*}(\mu_0)^{\gamma^*}}{Z_0}} \left[\begin{array}{c} (1-\gamma^*) \frac{(\mu_0)^{-\gamma^*}(\mu_0)^{\gamma^*}}{Z_0} (\mu_k - \mu_0) \\ +\gamma^* \frac{(\mu_0)^{1-\gamma^*}(\mu_0)^{\gamma^*-1}}{Z_0} (\mu_k^* - \mu_0) \\ -\frac{(\mu_0)^{1-\gamma^*}(\mu_0)^{\gamma^*}}{(Z_0)^2} (Z_k^* - Z_0) \end{array} \right] \\
&\quad - \frac{\frac{(\mu_0)^{1-\gamma^*}(\mu_0)^{\gamma^*}}{Z_0}}{\left(\sum_k \pi_k \frac{(\mu_0)^{1-\gamma^*}(\mu_0)^{\gamma^*}}{Z_0}\right)^2} \sum_k \pi_k \left[\begin{array}{c} (1-\gamma^*) \frac{(\mu_0)^{-\gamma^*}(\mu_0)^{\gamma^*}}{Z_0} (\mu_k - \mu_0) \\ +\gamma^* \frac{(\mu_0)^{1-\gamma^*}(\mu_0)^{\gamma^*-1}}{Z_0} (\mu_k^* - \mu_0) \\ -\frac{(\mu_0)^{1-\gamma^*}(\mu_0)^{\gamma^*}}{(Z_0)^2} (Z_k^* - Z_0) \end{array} \right] \\
\frac{\frac{(\mu_k)^{1-\gamma^*}(\mu_k^*)^{\gamma^*}}{Z_k^*}}{\sum_k \pi_k \frac{(\mu_k)^{1-\gamma^*}(\mu_k^*)^{\gamma^*}}{Z_k^*}} &= 1 + \quad (1-\gamma^*) \frac{\mu_k - \mu_0}{\mu_0} + \gamma^* \frac{\mu_k^* - \mu_0}{\mu_0} - \frac{Z_k^* - Z_0}{Z_0} \\
&\quad - \sum_k \pi_k \left[\quad (1-\gamma^*) \frac{\mu_k - \mu_0}{\mu_0} + \gamma^* \frac{\mu_k^* - \mu_0}{\mu_0} - \frac{Z_k^* - Z_0}{Z_0} \quad \right] \\
\frac{\frac{(\mu_k)^{1-\gamma^*}(\mu_k^*)^{\gamma^*}}{Z_k^*}}{\sum_k \pi_k \frac{(\mu_k)^{1-\gamma^*}(\mu_k^*)^{\gamma^*}}{Z_k^*}} &= 1 + \quad (1-\gamma^*)\left[\ln(\mu_k) - \ln(\mu_0)\right] + \gamma^* \left[\ln(\mu_k^*) - \ln(\mu_0)\right] - \left[\ln(Z_k^*) - \ln(Z_0)\right] \\
&\quad - \sum_k \pi_k \left[\quad (1-\gamma^*)\left[\ln(\mu_k) - \ln(\mu_0)\right] + \gamma^* \left[\ln(\mu_k^*) - \ln(\mu_0)\right] - \left[\ln(Z_k^*) - \ln(Z_0)\right] \quad \right] \\
\frac{\frac{(\mu_k)^{1-\gamma^*}(\mu_k^*)^{\gamma^*}}{Z_k^*}}{\sum_k \pi_k \frac{(\mu_k)^{1-\gamma^*}(\mu_k^*)^{\gamma^*}}{Z_k^*}} &= 1 + \quad (1-\gamma^*)\ln(\mu_k) + \gamma^* \ln(\mu_k^*) - \ln(Z_k^*)
\end{aligned}$$

Combining these elements, we have:

$$\begin{aligned}
0 &= \left(1 - \frac{\gamma^*}{2}\right) - \frac{1}{2} \frac{\frac{\mu_k}{Z_k}}{\sum_k \pi_k \frac{\mu_k}{Z_k}} - \frac{1-\gamma^*}{2} \frac{\frac{(\mu_k)^{1-\gamma^*}(\mu_k^*)^{\gamma^*}}{Z_k^*}}{\sum_k \pi_k \frac{(\mu_k)^{1-\gamma^*}(\mu_k^*)^{\gamma^*}}{Z_k^*}} \\
0 &= \left(1 - \frac{\gamma^*}{2}\right) - \frac{1}{2}\left[1 + \ln(\mu_k) - \ln(Z_k)\right] \\
&\quad - \frac{1-\gamma^*}{2} \left[\; 1 + (1-\gamma^*)\ln(\mu_k) + \gamma^* \ln(\mu_k^*) - \ln(Z_k^*) \; \right] \\
0 &= -\frac{1}{2}\left[\ln(\mu_k) - \ln(Z_k)\right] - \frac{1-\gamma^*}{2} \left[\; (1-\gamma^*)\ln(\mu_k) + \gamma^* \ln(\mu_k^*) - \ln(Z_k^*) \; \right] \\
0 &= \ln(\mu_k) - \ln(Z_k) + (1-\gamma^*) \left[\; (1-\gamma^*)\ln(\mu_k) + \gamma^* \ln(\mu_k^*) - \ln(Z_k^*) \; \right] \\
0 &= \left[1 + (1-\gamma^*)^2\right] \ln(\mu_k) + \gamma^* (1-\gamma^*) \ln(\mu_k^*) - \ln(Z_k) - (1-\gamma^*)\ln(Z_k^*)
\end{aligned}$$

which implies:

$$\left[1 + (1-\gamma^*)^2\right] \ln(\mu_k) \quad = \quad \ln(Z_k) + (1-\gamma^*)\ln(Z_k^*) - \gamma^* (1-\gamma^*) \ln(\mu_k^*)$$

4 Joint optimal rule

Question: Following similar steps, we can show (take this as given):

$$\left[1+(1-\gamma)^2\right]\ln(\mu_k^*) \quad = \quad (1-\gamma)\ln(Z_k)+\ln(Z_k^*)-\gamma(1-\gamma)\ln(\mu_k)$$

Assuming a symmetric situation ($\gamma=\gamma^*$) show that:

$$\begin{aligned}
\ln(\mu_k)+\ln(\mu_k^*) &= \ln(Z_k)+\ln(Z_k^*) \\
\ln(\mu_k)-\ln(\mu_k^*) &= \frac{\gamma}{1+(1-2\gamma)(1-\gamma)}\left[\ln(Z_k)-\ln(Z_k^*)\right]
\end{aligned}$$

hence:

$$\begin{aligned}
\ln(\mu_k) &= \frac{1-\gamma(1-\gamma)}{1-\gamma(1-\gamma)+(1-\gamma)^2}\ln(Z_k)+\frac{(1-\gamma)^2}{1-\gamma(1-\gamma)+(1-\gamma)^2}\ln(Z_k^*) \\
\ln(\mu_k) &= \frac{1-\gamma(1-\gamma)}{1-\gamma(1-\gamma)+(1-\gamma)^2}\ln(Z_k^*)+\frac{(1-\gamma)^2}{1-\gamma(1-\gamma)+(1-\gamma)^2}\ln(Z_k)
\end{aligned}$$

Answer: The linearized optimality conditions are:

$$\begin{aligned}
\left[1+(1-\gamma)^2\right]\ln(\mu_k) &= \ln(Z_k)+(1-\gamma)\ln(Z_k^*)-\gamma(1-\gamma)\ln(\mu_k^*) \\
\left[1+(1-\gamma)^2\right]\ln(\mu_k^*) &= (1-\gamma)\ln(Z_k)+\ln(Z_k^*)-\gamma(1-\gamma)\ln(\mu_k)
\end{aligned}$$

Take the sum of these conditions:

$$\begin{aligned}
\left[1+(1-\gamma)^2\right]\left[\ln(\mu_k)+\ln(\mu_k^*)\right] &= (2-\gamma)\left[\ln(Z_k)+\ln(Z_k^*)\right]-\gamma(1-\gamma)\left[\ln(\mu_k)+\ln(\mu_k^*)\right] \\
\left[1+(1-\gamma)^2+\gamma(1-\gamma)\right]\left[\ln(\mu_k)+\ln(\mu_k^*)\right] &= (2-\gamma)\left[\ln(Z_k)+\ln(Z_k^*)\right] \\
(2-\gamma)\left[\ln(\mu_k)+\ln(\mu_k^*)\right] &= (2-\gamma)\left[\ln(Z_k)+\ln(Z_k^*)\right] \\
\ln(\mu_k)+\ln(\mu_k^*) &= \ln(Z_k)+\ln(Z_k^*)
\end{aligned}$$

Then take the difference:

$$\begin{aligned}
\left[1+(1-\gamma)^2\right]\left[\ln(\mu_k)-\ln(\mu_k^*)\right] &= \gamma\left[\ln(Z_k)-\ln(Z_k^*)\right]+\gamma(1-\gamma)\left[\ln(\mu_k)-\ln(\mu_k^*)\right] \\
\left[1-\gamma(1-\gamma)+(1-\gamma)^2\right]\left[\ln(\mu_k)-\ln(\mu_k^*)\right] &= \gamma\left[\ln(Z_k)-\ln(Z_k^*)\right] \\
\left[1-\gamma(1-\gamma)+(1-\gamma)^2\right]\left[\ln(\mu_k)-\ln(\mu_k^*)\right] &= \gamma\left[\ln(Z_k)-\ln(Z_k^*)\right] \\
\ln(\mu_k)-\ln(\mu_k^*) &= \frac{\gamma}{1-\gamma(1-\gamma)+(1-\gamma)^2}\left[\ln(Z_k)-\ln(Z_k^*)\right]
\end{aligned}$$

Combining these relations, we get:

$$\begin{aligned}
\ln(\mu_k)+\ln(\mu_k^*) &= \ln(Z_k)+\ln(Z_k^*) \\
\ln(\mu_k)+\ln(\mu_k) &= \ln(Z_k)+\ln(Z_k^*)+\frac{\gamma}{1-\gamma(1-\gamma)+(1-\gamma)^2}\left[\ln(Z_k)-\ln(Z_k^*)\right] \\
2\ln(\mu_k) &= \left(1+\frac{\gamma}{1-\gamma(1-\gamma)+(1-\gamma)^2}\right)\ln(Z_k)+\left(1-\frac{\gamma}{1-\gamma(1-\gamma)+(1-\gamma)^2}\right)\ln(Z_k^*)
\end{aligned}$$

$$
\begin{aligned}
2\ln\left(\mu_k\right) &= \frac{1+\gamma^2+\left(1-\gamma\right)^2}{1-\gamma\left(1-\gamma\right)+\left(1-\gamma\right)^2}\ln\left(Z_k\right)+\frac{2-2\gamma}{1-\gamma\left(1-\gamma\right)+\left(1-\gamma\right)^2}\left(1-\gamma\right)\ln\left(Z_k^*\right) \\
\ln\left(\mu_k\right) &= \frac{1+\gamma^2-\gamma}{1-\gamma\left(1-\gamma\right)+\left(1-\gamma\right)^2}\ln\left(Z_k\right)+\frac{\left(1-\gamma\right)^2}{1-\gamma\left(1-\gamma\right)+\left(1-\gamma\right)^2}\ln\left(Z_k^*\right) \\
\ln\left(\mu_k\right) &= \frac{1-\gamma\left(1-\gamma\right)}{1-\gamma\left(1-\gamma\right)+\left(1-\gamma\right)^2}\ln\left(Z_k\right)+\frac{\left(1-\gamma\right)^2}{1-\gamma\left(1-\gamma\right)+\left(1-\gamma\right)^2}\ln\left(Z_k^*\right)
\end{aligned}
$$

Finally:

$$
\begin{aligned}
\ln\left(\mu_k^*\right) &= \ln\left(\mu_k\right)-\frac{\gamma}{1-\gamma\left(1-\gamma\right)+\left(1-\gamma\right)^2}\left[\ln\left(Z_k\right)-\ln\left(Z_k^*\right)\right] \\
\ln\left(\mu_k^*\right) &= \frac{1-\gamma\left(1-\gamma\right)-\gamma}{1-\gamma\left(1-\gamma\right)+\left(1-\gamma\right)^2}\ln\left(Z_k\right)+\frac{\left(1-\gamma\right)^2+\gamma}{1-\gamma\left(1-\gamma\right)+\left(1-\gamma\right)^2}\ln\left(Z_k^*\right) \\
\ln\left(\mu_k^*\right) &= \frac{\left(1-\gamma\right)^2}{1-\gamma\left(1-\gamma\right)+\left(1-\gamma\right)^2}\ln\left(Z_k\right)+\frac{\gamma^2+1-2\gamma+\gamma}{1-\gamma\left(1-\gamma\right)+\left(1-\gamma\right)^2}\ln\left(Z_k^*\right) \\
\ln\left(\mu_k^*\right) &= \frac{\left(1-\gamma\right)^2}{1-\gamma\left(1-\gamma\right)+\left(1-\gamma\right)^2}\ln\left(Z_k\right)+\frac{1-\gamma\left(1-\gamma\right)}{1-\gamma\left(1-\gamma\right)+\left(1-\gamma\right)^2}\ln\left(Z_k^*\right)
\end{aligned}
$$