Mathematical Analysis

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Maxima and minima

Convexity and concavity

Homogeneous Functions and Euler's Formula

Exponential and logarithmic functions

Approximations

Log-Linearization

Maxima and minima

Critical points

Let f be a differentiable function. The points on the curve $y = f(x)$ where $f'(x) = 0$ are called **critical points**, and the value taken by the function at such a point is called a **critical value**.

There are three kind of critical points :

- **Maximum points** At a maximum point, y has its greatest value in a neighbourhood of the point.
- **Minimum points** At a minimum point, y has its least value in a neighbourhood of the point.
- **Inflexion points** At a inflexion point, $f'(x) = 0$ but $f'(x)$ does not change sign in passing through the point.

Critical points

Four steps to find the critical points of a function $y = f(x)$

Step 1 : find $f'(x)$

Step 2 : find the values of x for which $f'(x) = 0$.

Step 3 : for each such x, find the corresponding value of y.

Step 4 : for each critical point (x^*, y^*), find the sign of $f'(x)$ for values of x that are slightly less than x^* (written $x = x^*-$) and for values for x that are slightly greater than x^* (written $x = x^*+$).

If $f'(x)$ changes from **positive** to **negative** as x changes from x^*- to x^*+, then (x^*, y^*) is a **maximum**.

If $f'(x)$ changes from **negative** to **positive** as x changes from x^*- to x^*+, then (x^*, y^*) is a **minimum**.

Critical points

Exercise 4 Find and classify the critical points of the curve

$$y = x^3 - 9x^2 + 24x + 10$$

Exercise 5 Find and classify the critical points of the curve

$$y = x^4 - 4x^3 + 5$$

The second derivative

Let f be a differentiable function and if f' is also differentiable.

We denote the derivative $f'(x)$ wrt x by $f''(x)$ and is called the **second derivative** of $f(x)$.

The function f is said to be **twice differentiable**.

In the alternative notation, the second derivative is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

and is usually denoted by

$$\frac{d^2y}{dx^2}$$

The second derivative

If $f''(x_0) > 0$, then $f'(x_0)$ is increasing in x_0. This can happen in three ways :

- $f'(x_0)$ is positive and $f'(x)$ becomes more positive as x passes from x_0- to x_0+.
- $f'(x_0)$ is negative and $f'(x)$ becomes less negative as x passes from x_0- to x_0+.
- $f'(x_0) = 0$ and $f'(x)$ goes from negative to positive as x passes from x_0- to x_0+.

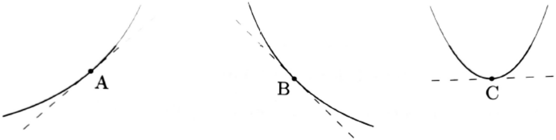


Figure 8.4: **Positive second derivative**

The second derivative

We can classify the critical points using the second derivative.

Step 1 : find $f'(x)$ and $f''(x)$.

Step 2 : find the critical points in the usual way.

Step 3 : for each critical point (x^*, y^*), calculate $f''(x)$. If $f''(x^*) < 0$ then (x^*, y^*) is a maximum. If $f''(x^*) > 0$ then (x^*, y^*) is a minimum.

Exercise 5 Find and classify the critical points of the function

$$y = 2x^3 - 3x^2 - 12x + 9$$

Optimisation

The necessary (1) and sufficient (2) conditions for a **local maximum** are :

(1) If the function f has a local maximum where $x = x^*$, then $f'(x^*) = 0$ and $f''(x^*) \leq 0$.

(2) If $f'(x^*) = 0$ and $f''(x^*) < 0$, the function f has a local maximum where $x = x^*$.

The necessary (1) and sufficient (2) conditions for a **local minimum** are :

(1) If the function f has a local minimum where $x = x^*$, then $f'(x^*) = 0$ and $f''(x^*) \geq 0$.

(2) If $f'(x^*) = 0$ and $f''(x^*) > 0$, the function f has a local minimum where $x = x^*$.

Optimisation

A local maximum point (x^*, y^*) of the curve $y = f(x)$, which has the additional property that

$$y^* \geq f(x) \quad \text{for all } x \text{ in } \mathbb{R}$$

is called a **global maximum point**.

Similarly, a **global minimum point** of the curve $y = f(x)$ is a local minimum point (x^*, y^*) with the additional property that

$$y^* \leq f(x) \quad \text{for all } x \text{ in } \mathbb{R}$$

Optimisation

Two interesting facts.

The function f has a local minimum where $x = x^*$ if and only if the function $-f$ has a local maximum where $x = x^*$, and the same is true for global minima and maxima.

Suppose $g(x) = H(f(x))$, where H is a *strictly increasing* function. If the function f has a global maximum (minimum) point at (x^*, y^*), then the function g has a global maximum (minimum) point at $(x^*, H(y^*))$.

Optimisation

Suppose you want to maximise a function $f(x)$ subject to $x \geq 0$.
Three cases :

If $f'(x^*) = 0$ and $f'(x)$ changes from positive to negative, then we have an **interior local maximum** (Panel A).

If $f'(0) < 0$, we have a **boundary local maximum** (Panel B).

If $f'(0) = 0$ and $f'(x) < 0$ for $x > 0$, we also have a **boundary local maximum** (Panel C).

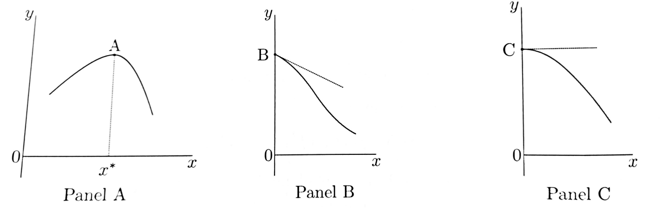


Figure 8.8: Local maxima subject to $x \geq 0$

Optimisation

Exercise 6

(a) Find the critical points of $y = x^3 - 12x^2 + 21x + 1$ and determine their nature.

(b) Write down the coordinates of any local maxima, local minima, global maxima and global minima.

(c) Repeat (b) when the constraint $x \geq 0$ is imposed.

Convexity and concavity

Convexity and concavity

Convex and concave functions are important for two reasons :

(1) They give a class of cases for which it is easy to find global optimas.

(2) They occur frequently in economics.

Convexity

A convex function has the property that a chord joining any two points on its graph lies on or above the graph.

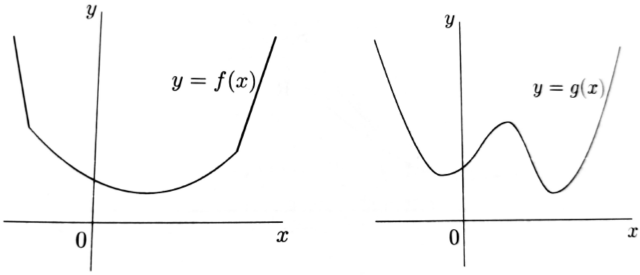


Figure 8.12: f **is a convex function,** g **is not**

Convexity

Algebraically, the function f is **convex** if and only if

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2)$$

for any real numbers x_1, x_2, α such that $0 \leq \alpha \leq 1$.

The function is **strictly convex** if the inequality holds strictly.

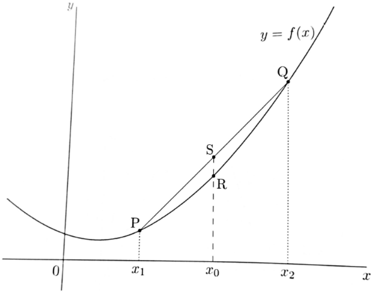


Figure 8.13: f is a convex function; the chord PQ lies above the curve

Convexity

Since f is convex, R cannot lie above S ; hence the slope of the chord PR cannot exceed the slope of the chord PQ.

It follows that :

$$f'(x_1) \leq \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Similarly,by considering what happens to the slope of the straight line through Q and R, we have :

$$f'(x_2) \geq \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Convexity

Important properties of differentiable convex functions :

(1) If f is a differentiable convex function, then
$f(a+h) \geq f(a) + hf'(a)$ for all a and h.

(2) A differentiable function f is convex if and only if
$f'(a) \leq f'(b)$, whenever $a \leq b$.

(3) A twice-differentiable function f is convex if and only if
$f''(x) \geq 0$ for all x.

(4) If f is a differentiable convex function and $f'(x^*) = 0$, then
$(x^*, f(x^*))$ is a global minimum point.

Convexity

Exercise 7 Show that the function

$$y = \frac{2}{1 - x^2} \quad (-1 < x < 1)$$

is convex, find its global minimum point.

Concavity

A concave function has the property that a chord joining any two points on its graph lies *on or below* the graph.

Hence, the **function f is concave if and only if $-f$ is convex**. It follows that,

The function f is **concave** if and only if

$$f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha)f(x_2)$$

for any real numbers x_1, x_2, α such that $0 \leq \alpha \leq 1$.

The function is **strictly concave** if the inequality holds strictly.

Concavity

Important properties of differentiable concave functions :

(1) If a f is a differentiable concave function, then $f(a+h) \leq f(a) + hf'(a)$ for all a and h.

(2) A differentiable function f is concave if and only if $f'(a) \geq f'(b)$, whenever $a \leq b$.

(3) A twice-differentiable function f is concave if and only if $f''(x) \leq 0$ for all x.

(4) If f is a differentiable concave function and $f'(x^*) = 0$, then $(x^*, f(x^*))$ is a global maximum point.

Concavity

Exercise 8 Show that the function

$$y = -x^4 + 4x^3 - 6x^2 + 8x + 3$$

is concave and that its slope is zero where $x = 2$. Deduce the coordinates of the global maximum point.

Quasiconvexity

The function f is **quasiconvex** if and only if

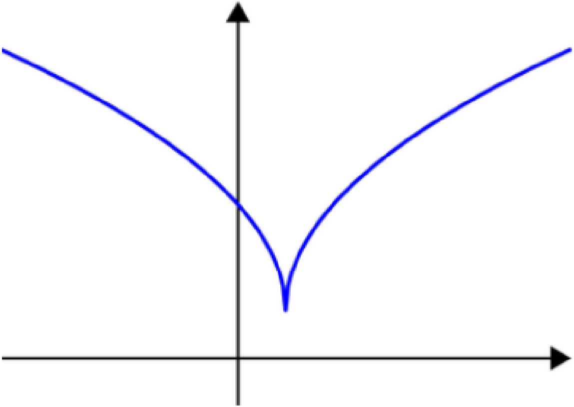
$$f\left(\alpha x + (1-\alpha)x'\right) \leq \mathrm{Max}\left\{f(x), f\left(x'\right)\right\}$$

for any real numbers x_1, x_2, α such that $0 \leq \alpha \leq 1$.

The function is **strictly quasiconvex** if the inequality holds strictly.

Convexity is a stronger property than quasiconvexity.

Quasiconvexity



Quasiconcavity

The **function f is quasiconcave if and only if $-f$ is quasiconvex**. It follows that,

The function f is **quasiconcave** if and only if

$$f\left(\alpha x + (1-\alpha)x'\right) \geq \text{Min}\left\{f(x), f\left(x'\right)\right\}$$

for any real numbers x_1, x_2, α such that $0 \leq \alpha \leq 1$.

The function is **strictly quasiconcave** if the inequality holds strictly.

Homogeneous Functions and Euler's Formula

Homogeneous Functions

Definition A function $f(x_1, \ldots, x_N)$ is **homogeneous of degree r** $r \in \mathbb{Z}$ if

$$f(tx_1, \ldots, tx_N) = t^r f(x_1, \ldots, x_N)$$

If $f(x_1, \ldots, x_N)$ is homogeneous of degree zero, we can write

$$f(1, x_2/x_1, \ldots, x_N/x_1) = f(x_1, \ldots, x_N)$$

If the function is homogeneous of degree 1 then

$$f(1, x_2/x_1, \ldots, x_N/x_1) = (1/x_1) f(x_1, \ldots, x_N)$$

Homogeneous Functions

Theorem : If $f(x_1, \ldots, x_N)$ is homogeneous of degree r, then for any $n = 1, \ldots, N$ the partial derivative function $\partial f(x_1, \ldots, x_N)/\partial x_n$ is homogeneous of degree $r - 1$.

Proof : By the definition of homogeneity

$$f(tx_1, \ldots, tx_N) - t^r f(x_1, \ldots, x_N) = 0$$

Differentiating this expression with respect to x_n gives

$$t\frac{\partial f(tx_1, \ldots, tx_N)}{\partial x_n} - t^r \frac{\partial f(x_1, \ldots, x_N)}{\partial x_n} = 0,$$

so that

$$\frac{\partial f(tx_1, \ldots, tx_N)}{\partial x_n} = t^{r-1}\frac{\partial f(x_1, \ldots, x_N)}{\partial x_n}.$$

Homogeneous Functions

Note that if $f(\cdot)$ is a homogeneous function of any degree then $f(x_1, \ldots, x_N) = f(x_1', \ldots, x_N')$ implies

$$f(tx_1, \ldots, tx_N) = f\left(tx_1', \ldots, tx_N'\right)$$

Euler's Formula

Theorem **Euler's Formula**

Suppose that $f(x_1, \dots, x_N)$ is homogeneous of degree r and differentiable. Then at any $(\bar{x}_1, \dots, \bar{x}_N)$ we have

$$\sum_{n=1}^{N} \frac{\partial f(\bar{x}_1, \dots, \bar{x}_N)}{\partial x_n} \bar{x}_n = rf(\bar{x}_1, \dots, \bar{x}_N),$$

or, in matrix notation, $\nabla f(\bar{x}) \cdot \bar{x} = rf(\bar{x})$.

Euler's Formula

Proof : By definition we have

$$f\left(t\bar{x}_1, \ldots, t\bar{x}_N\right) - t^r f\left(\bar{x}_1, \ldots, \bar{x}_N\right) = 0.$$

Differentiating this expression with respect to t gives

$$\sum_{n=1}^{N} \frac{\partial f\left(t\bar{x}_1, \ldots, t\bar{x}_N\right)}{\partial x_n} \bar{x}_n - r t^{r-1} f\left(\bar{x}_1, \ldots, \bar{x}_N\right) = 0.$$

Evaluating at $t = 1$, we obtain Euler's formula.

Exercises

Exercise : tell the homogeneous degree of the following functions and their partial derivative of x_1, and show the Euler's Formula holds :

(a) $f(x_1, x_2) = x_1/x_2$

(b) $f(x_1, x_2) = (x_1 x_2)^{1/2}$

Exponential and logarithmic functions

The exponential function

Th exponential function is one of the most important functions in mathematics. It has wide applications in economics.

It has five key properties :

(1) $\exp 0 = 1$

(2) $\exp(a + b) = (\exp a) \times (\exp b)$

(3) $\exp(-a) = 1/(\exp a)$

(4) $\exp(ac) = (\exp a)^c$

(5) $\frac{d}{dx}(\exp x) = \exp x$

(6) $\frac{d}{dx}\big(\exp\{u(x)\}\big) = u'(x) \exp\{u(x)\}$ if $u(x)$ is differentiable.

The exponential function

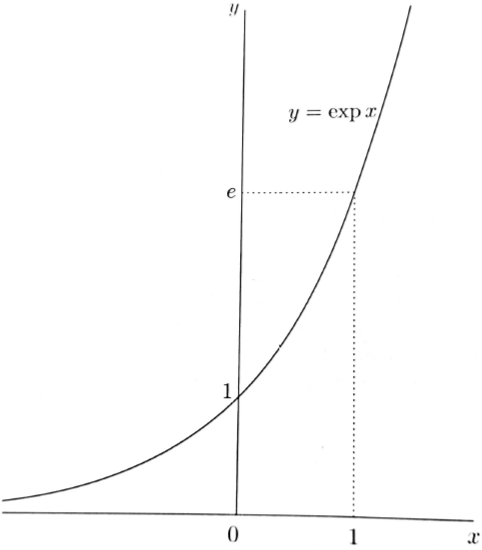


Figure 9.1: **The exponential function**

The exponential function

From the chart, we can also see that :
- $\exp x$ is positive for all real x.

- $\exp x$ is monotonic increasing.

Using **(5)**, we can also show that :

$$\frac{d^2}{dx^2}(\exp x) = \exp x > 0$$

so $\exp x$ is a strictly convex function.

The exponential function

Exercise 9 Differentiate

(a) $e^{2x} + 3e^{-4x}$

(b) xe^{2x}

(c) $1/(1 + e^x)$

The natural logarithm

The natural logarithm is the **inverse of the exponential function**.

Thus,

$$\exp(\ln x) = x \quad \text{for every positive number } x. \tag{1}$$

$$\ln(\exp y) = y \quad \text{for every real number } y. \tag{2}$$

Since 'ln' is the **inverse function** of 'exp',

- the natural logarithm function is monotonic increasing and concave.

- $\ln 1 = 0$.

The natural logarithm

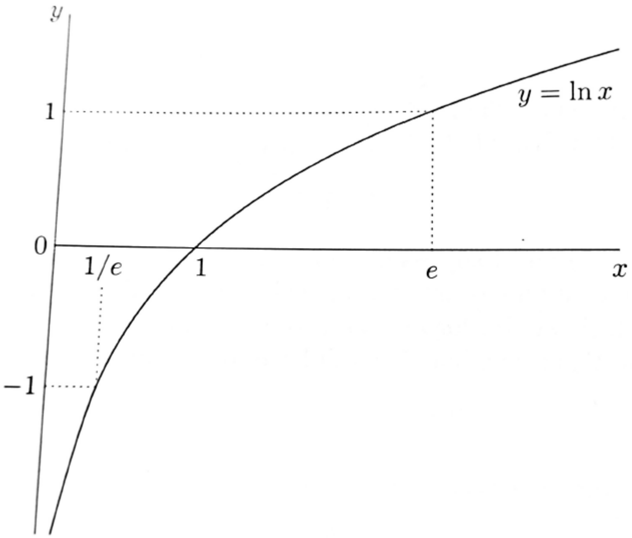


Figure 9.2: **The natural logarithm function**

The natural logarithm

Properties :

(1) $\ln(1/x) = -\ln(x)$

(2) $\ln(ab) = \ln a + \ln b$

(3) $\ln(a/b) = \ln a - \ln b$

(4) $\ln(x^c) = c \ln x$ for any real number c

(5) $\frac{d}{dx}(\ln x) = \frac{1}{x}$

(6) $\frac{d}{dx}\big(\ln u(x)\big) = \frac{u'(x)}{u(x)}$

Finally, a useful approximation in economics :

$$\ln(1 + h) \approx h \quad \text{if } |h| \text{ is small.}$$

The natural logarithm

Exercise 10 Differentiate

(a) $\ln(ax)$

(b) $\ln(x^4 + 1)$

(c) $x \ln x$

Exercise 11 Find the critical points of the function

$$y = x^2 - 2\ln(1 + x^2)$$

and determine their nature.

Approximations

Linear approximation

Recall,

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Which means that if h is close to 0 we have

$$\frac{f(x_0 + h) - f(x_0)}{h} \approx f'(x_0)$$

Therefore we have $f(x + h) - f(x) \approx hf'(x)$ if $|h|$ is small.

This approximation is called the **small increments formula**.

Linear approximation

The **linear approximation** to the curve $y = f(x)$ is the tangent at a given point a. We denote its equation by

$$y = L(x) = f(a) + (x - a)f'(a)$$

When we approximate the curve $y = f(x)$ by its tangent, namely the line $y = L(x)$, we approximate $f(a + h)$ by $L(a + h) = f(a) + hf'(a)$.

Obviously the larger the distance between the two points will be, the less precise will be the approximation.

Linear approximation

Exercise 12 Find the linear approximations to $y = x^3$ at the points where (a) $x = 2$, (b) $x = 3$.

Show in a table the approximate values given by each linear approximation and the true values when $x = 2.1, 2.2, \ldots, 2.9$. Comment.

Newton's method

Suppose we are trying to solve $f(x) = 0$.

We first choose a starting point $x = a$, which is our first approximation of a root of $f(x) = 0$.

We know that a linear approximation to $f(x)$ at point $(a, f(a))$ is $L(x) = f(a) + (x - a)f'(a)$

Then, as a second approximation we choose the value of x such that $L(x) = 0$, we call it b.

So computing $L(b) = 0$ gives us

$$b = a - \frac{f(a)}{f'(a)}$$

Newton's method

We thus have the Newton's method : starting with a first approximation of a to a root of the equation $f(x) = 0$, we choose as our second approximation b. b is indeed a better approximation than a.

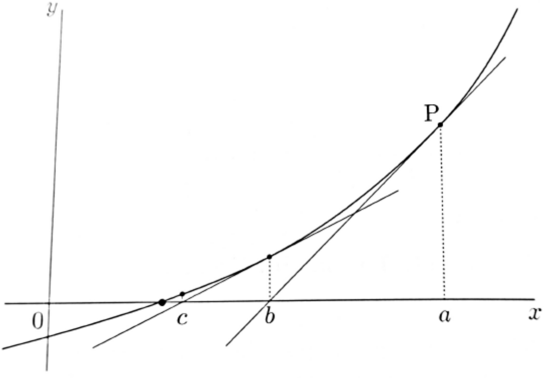


Figure 10.2: **Newton's method**

Newton's method

We can repeat the procedure again applying the "*update function*"

$$U(x) = x - \frac{f(x)}{f'(x)}$$

to every new approximation, until the difference between two successive approximations becomes "*small enough*".

Newton's method

In some cases, depending on the starting point you choose Newton's method fails, and every new approximation given by the update function is further away from the result.

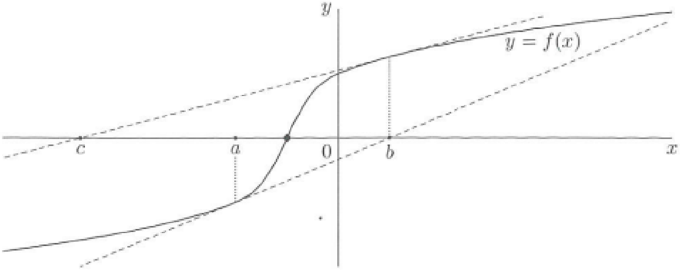


Figure 10.3: A case where Newton's method fails

Newton's method

Exercise 13 Find an approximate solution of the equation, starting from $a = 0.4$

$$x^5 - 5x + 2 = 0$$

Exercise 14 Show that the equation

$$x^7 - 6x + 4 = 0$$

has a root between 0 and 1. Find an initial approximation by ignoring the term x^7. Use Newton's method to find the root correct to 3 decimal places.

The mean value theorem

If f is a differentiable function and a and b are real numbers such that $a < b$, there exists a real number c such that $a < c < b$ and

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

As illustrated below, given the chord AB, there is a point C on the curve such that the tangent at C is parallel to the line-segment AB.

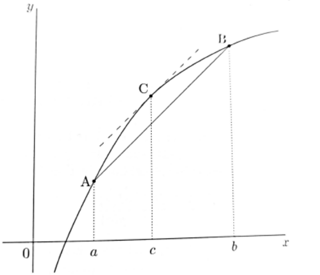


Figure 10.4: The mean value theorem: the tangent at C is parallel to the chord AB

The mean value theorem

Let $h = b - a$. Since c lies between a and b, $c = a + sh$ for some number s such that $0 < s < 1$. Thus,

$$f(a+h) = f(a) + hf'(a+sh), \quad \text{where } 0 < s < 1$$

if $|h|$ is small, we may approximate the term $hf'(a+sh)$ by $hf'(a)$.

Thus,

$$f(a+h) \approx f(a) + hf'(a) \quad \text{if } |h| \text{ is small}$$

We have again the small increment formula.

Quadratic approximation

Second mean value theorem If f is twice differentiable and a and b real numbers such that $a < b$, there exists a real number d such that $a < d < b$ and

$$f(b) = f(a) + (b-a)f'(a) + \frac{1}{2}(b-a)^2 f''(d)$$

As before, let $h = b - a$ and $d = a + th$, with $0 < t < 1$. Thus,

$$f(a+h) = f(a) + hf'(a) + \frac{1}{2}h^2 f''(a+th) \quad \text{where } 0 < t < 1$$

If $|h|$ is small, we may approximate $\frac{1}{2}h^2 f''(a+th)$ by $\frac{1}{2}h^2 f''(a)$. Therefore,

$$f(a+h) \approx f(a) + hf'(a) + \frac{1}{2}h^2 f''(a) \quad \text{if } |h| \text{ is small}$$

Letting $h = x - a$, we get the parabola $y = Q(x)$ as a **quadratic approximation** to the curve $y = f(x)$

$$Q(x) = f(a) + (x-a)f'(a) + \frac{1}{2}(x-a)^2 f''(a)$$

Quadratic approximation

Exercise 15 Find the approximation of $f(x) = x^2 \ln x$ by linear and quadratic functions of x, for x close to 1 (0.95, 1.02, 1.10).

Taylor approximations

We have shown that the first and second mean value theorems enable us to approximate $f(a+h)$, when $|h|$ is small by linear or quadratic functions of h.

Similarly, Taylor's theorem allows us to approximate $f(a+h)$ by higher-degree polynomials.

Taylor's theorem If f is a smooth function and n a positive integer, then

$$f(a+h) = f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{h^n}{n!}f^{(n)}(a+th)$$

If $|h|$ is small, we may approximate $\frac{h^n}{n!}f^{(n)}(a+th)$ by $\frac{h^n}{n!}f^{(n)}(a)$. Therefore,

$$f(a+h) \approx f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\cdots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{h^n}{n!}f^{(n)}(a)$$

Taylor approximations

Exercise 16 Find the Taylor Series for $f(x) = e^{-6x}$ about $x = -4$.
Hint : first find the general term at any x, and then show the general term at $x = -4$.

Log-Linearization

Log-Linearization

The solutions to many discrete time dynamic economic problems take the form of a system of non-linear difference equations.

One way to end up with a solution is what we call the **log-linearization**.

Consider a non linear function $f(x) = \frac{g(x)}{h(x)}$.

We first take natural logs of both sides,

$$\ln f(x) = \ln g(x) - \ln h(x) \tag{3}$$

Log-Linearization

Then we approximate each function using their first order Taylor series expansions around the steady state x^* (here $x - x^* = h$)

$$\ln f(x) = \ln f(x^*) + \frac{f'(x^*)}{f(x^*)}(x - x^*)$$

$$\ln g(x) = \ln g(x^*) + \frac{g'(x^*)}{g(x^*)}(x - x^*)$$

$$\ln h(x) = \ln h(x^*) + \frac{h'(x^*)}{h(x^*)}(x - x^*)$$

If we replace those approximations in (1), rearranging a little bit we get a linear relation of the distance to the steady state x^*

$$\frac{f'(x^*)}{f(x^*)}(x - x^*) = \frac{g'(x^*)}{g(x^*)}(x - x^*) - \frac{h'(x^*)}{h(x^*)}(x - x^*)$$

Log-Linearization

Application Consider a Cobb-Douglas production function

$$y_t = a_t k_t^{\alpha} l_t^{1-\alpha}$$

We first take the natural log of the function

$$\ln y_t = \ln a_t + \alpha \ln k_t + (1-\alpha) \ln l_t$$

Now we take the first order Taylor expansion at the steady state values

$$\begin{aligned} \ln y^* + \frac{1}{y^*}(y_t - y^*) = \ln a^* + \frac{1}{a^*}(a_t - a^*) + \alpha \ln k^* + \frac{\alpha}{k^*}(k_t - k^*) \\ + (1-\alpha) \ln l^* + \frac{1-\alpha}{l^*}(l_t - l^*) \end{aligned}$$

Log-Linearization

Knowing that $\ln y^* = \ln a^* + \alpha \ln k^* + (1-\alpha) \ln l^*$ and defining $\hat{y} = \frac{(y_t - y^*)}{y^*}$ the percentage deviation of x from x^*, we get the following linear relation

$$\hat{y}_t = \hat{a}_t + \alpha \hat{k}_t + (1-\alpha)\hat{l}_t$$

This is again a very useful and commonly used method in economics, in particular for macroeconomics models.

Log-Linearization

Multivariate case First-order Taylor approximations can also be used to convert equations with more than one endogenous variable to log-deviations form (more to come in Lecture 3).

Start with $x_{t+1} = g(x_t, y_t)$ and employ a first-order Taylor approximation at the steady state $x_t = x^*$ and $y_t = y^*$ to get

$$x_{t+1} \approx g(x^*, y^*) + g'_x(x^*, y^*)(x_t - x^*) + g'_y(x^*, y^*)(y_t - y^*)$$

.

Log-Linearization

Exercise 17 Consider the standard capital accumulation equation

$$k_{t+1} = i_t + (1 - \delta)k_t$$

Use log-linearization to approximate the equation around the steady state. *Hint* : at the steady state, $x_{t+1} = x_t = x^*$

Log-Linearization

Uhlig's Approach

Let x_t be our variable of interest around x^*. We first construct $\hat{x}_t$, the log deviation of x_t from x^*

$$\hat{x}_t = \log(x_t) - \log(x^*) = \log\left(\frac{x_t}{x^*}\right)$$

Then

$$x_t = x^* e^{\hat{x}_t}$$

We denote our point of approximation $\hat{x}_0$ and since we are interested in an approximation around the steady state, our point of approximation will be $\hat{x}_0 = 0$.

Log-Linearization

Then, approximating our previous result we get

$$\underbrace{x^* e^{\hat{x}_t}}_{f(\hat{x}_t)} \approx \underbrace{x^* e^{\hat{x}_0}}_{f(\hat{x}_0)} + \underbrace{x^* e^{\hat{x}_0}}_{f'(\hat{x}_0)}(\hat{x}_t - \hat{x}_0) = x^* e^{\hat{x}_0}(1 + \hat{x}_t - \hat{x}_0)$$

Plugging in $\hat{x}_0 = 0$ we get

$$x_t = x^* e^{\hat{x}_t} \approx x^*(1 + \hat{x}_t)$$

Trick we ignore crossed products.

$$x_t y_t \approx x^* y^* (1 + \hat{x}_t)(1 + \hat{y}_t) \approx x^* y^* (1 + \hat{x}_t + \hat{y}_t)$$

Products and powers are easily addressed by this approach.

Log-Linearization

Exercise 18

Linearize $x_t^{\rho} y_t^{1-\sigma} = 1$, around $\hat{x}_0 = 0, \hat{y}_0 = 0$

What about additive terms ?
Linearize the standard macro income identity : $Y_t = C_t + I_t$