Macroeconomics A; EI060

Technical appendix: defautl

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1 Default with non-contingent assets (Harms VI.3.1-VI.3.3)

1.1 Varying income

1.1.1 Budget constraint and optimization

Consider a small open economy with only one traded good. It gets an endowment Y_t in period t. It can lend and borrow at a world interest rate r. The budget constraint is:

$$B_{t+1} + C_t = Y_t + (1+r) B_t$$

We iterate forward and use the transversality constraint:

$$\begin{aligned}
B_t &= -\frac{1}{1+r}(Y_t - C_t) + \frac{1}{1+r} B_{t+1} \\
B_t &= -\frac{1}{1+r}(Y_t - C_t) - \left(\frac{1}{1+r}\right)^2 (Y_t - C_t) + \left(\frac{1}{1+r}\right)^2 B_{t+2} \\
B_t &= -\sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t+1} (Y_s - C_s) \\
B_t &= -\frac{1}{1+r} \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} (Y_s - C_s) \\
\sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} C_s &= (1+r) B_t + \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} Y_s
\end{aligned}$$

Consider that the agent maximizes an intertemporal log utility:

$$U_t = \sum_{s=t}^{\infty} (\beta)^{s-t} \, ln(C_s)$$

The Euler condition implies:

$$\begin{aligned}
\frac{1}{C_t} &= \beta\left(1+r\right)\frac{1}{C_{t+1}} \\
C_{t+1} &= \beta\left(1+r\right)C_t \\
C_{t+h} &= \left[\beta\left(1+r\right)\right]^h C_t \\
C_s &= \left[\beta\left(1+r\right)\right]^{s-t} C_t
\end{aligned}$$

Assuming that $\beta\left(1+r\right)=1$ we get $C_s = C_t$, hence the budget constraint is:

$$\begin{aligned}
\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} C_s &= \left(1+r\right)B_t + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} Y_s \\
C_t\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} &= \left(1+r\right)B_t + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} Y_s \\
C_t\frac{1}{1-\frac{1}{1+r}} &= \left(1+r\right)B_t + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} Y_s \\
C_t\frac{1+r}{r} &= \left(1+r\right)B_t + \sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} Y_s \\
C_t &= rB_t + \frac{r}{1+r}\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} Y_s
\end{aligned}$$

1.1.2 Consumption in the absence of default

Consider that the endowment process is:

$$\begin{aligned}
Y+\Delta &= Y_t = Y_{t+2} = Y_{t+4} = ... \\
Y-\Delta &= Y_{t-1} = Y_{t+1} = Y_{t+3} = ...
\end{aligned}$$

This implies:

$$\begin{aligned}
\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} Y_s &= \left(Y+\Delta\right)\left[1+\left(\frac{1}{1+r}\right)^2+\left(\frac{1}{1+r}\right)^4+...\right] \\
&\quad +\left(Y-\Delta\right)\left[\frac{1}{1+r}+\left(\frac{1}{1+r}\right)^3+\left(\frac{1}{1+r}\right)^5+...\right] \\
\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} Y_s &= \left(\left(Y+\Delta\right)+\left(Y-\Delta\right)\frac{1}{1+r}\right)\left[1+\left(\frac{1}{1+r}\right)^2+\left(\frac{1}{1+r}\right)^4+...\right] \\
\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} Y_s &= \left(\left(Y+\Delta\right)+\left(Y-\Delta\right)\frac{1}{1+r}\right)\left[1+\left(\left(\frac{1}{1+r}\right)^2\right)^1+\left(\left(\frac{1}{1+r}\right)^2\right)^2+...\right] \\
\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} Y_s &= \left(\left(Y+\Delta\right)+\left(Y-\Delta\right)\frac{1}{1+r}\right)\frac{1}{1-\left(\frac{1}{1+r}\right)^2}
\end{aligned}$$

$$
\begin{aligned}
\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}Y_s &= \left((Y+\Delta)+(Y-\Delta)\frac{1}{1+r}\right)\frac{(1+r)^2}{(1+r)^2-1}\\
\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}Y_s &= Y\left(1+\frac{1}{1+r}\right)\frac{(1+r)^2}{(1+r)^2-1}+\Delta\left(1-\frac{1}{1+r}\right)\frac{(1+r)^2}{(1+r)^2-1}\\
\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}Y_s &= Y\frac{2+r}{1+r}\frac{(1+r)^2}{(1+r)^2-1}+\Delta\frac{r}{1+r}\frac{(1+r)^2}{(1+r)^2-1}\\
\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}Y_s &= Y\frac{(2+r)(1+r)}{1+2r+r^2-1}+\Delta\frac{r}{1+r}\frac{(1+r)^2}{(1+r)^2-1}\\
\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}Y_s &= Y\frac{1}{r}\frac{(2+r)(1+r)}{2+r}+\Delta\frac{r(1+r)}{(1+r)^2-1}\\
\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}Y_s &= \frac{1+r}{r}Y+\frac{r(1+r)}{(1+r)^2-1}\Delta
\end{aligned}
$$

The consumption in the absence of default is then:

$$
\begin{aligned}
C_t^{ND} &= rB_t+\frac{r}{1+r}\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}Y_s\\
C_t^{ND} &= rB_t+Y+\frac{r^2}{(1+r)^2-1}\Delta\\
C_t^{ND} &= rB_t+Y+\frac{r^2}{1+2r+r^2-1}\Delta\\
C_t^{ND} &= rB_t+Y+\frac{r}{2+r}\Delta
\end{aligned}
$$

Which gives the utility:

$$
\begin{aligned}
U_t^{ND} &= \sum_{s=t}^{\infty}(\beta)^{s-t}\,ln\left(C_s^{ND}\right)\\
U_t^{ND} &= ln\left(C_t^{ND}\right)\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}\\
U_t^{ND} &= \frac{1+r}{r}n\left(rB_t+Y+\frac{r}{2+r}\Delta\right)
\end{aligned}
$$

The flow budget constraint gives the path of assets:

$$
\begin{aligned}
B_{t+1} &= Y_t-C_t+(1+r)B_t\\
&= \frac{2}{2+r}\Delta+B_t\\
B_{t+2} &= Y_{t+1}-C_t+(1+r)B_{t+1}\\
&= B_t\\
B_{t+2} &= Y_{t+2}-C_t+(1+r)B_{t+1}\\
&= \frac{2}{2+r}\Delta+B_t
\end{aligned}
$$

$$
\begin{aligned}
B_{t+3} &= Y_{t+2} - C_t + (1+r) B_{t+2} \\
&= B_t
\end{aligned}
$$

1.1.3 Consumption with of default

Under default, the negative B_t is brought to zero, but subsequent consumption is equal to endowment.

$$
\begin{aligned}
Y + \Delta &= C_t^D = C_{t+2}^D = C_{t+4}^D = ... \\
Y - \Delta &= C_{t-1}^D = C_{t+1}^D = C_{t+3}^D = ...
\end{aligned}
$$

The utility is then:

$$
\begin{aligned}
\sum_{s=t}^{\infty} (\beta)^{s-t} ln\left(C_s^D\right) &= \sum_{s=t}^{\infty} (\beta)^{s-t} ln\left(Y_s\right) \\
\sum_{s=t}^{\infty} (\beta)^{s-t} ln\left(C_s^D\right) &= \sum_{s=t}^{\infty} \left(\frac{1}{1+r}\right)^{s-t} ln\left(Y_s\right) \\
\sum_{s=t}^{\infty} (\beta)^{s-t} ln\left(C_s^D\right) &= ln\left(Y+\Delta\right)\left[1 + \left(\frac{1}{1+r}\right)^2 + \left(\frac{1}{1+r}\right)^4 + ...\right] \\
&\quad + ln\left(Y-\Delta\right)\left[\frac{1}{1+r} + \left(\frac{1}{1+r}\right)^3 + \left(\frac{1}{1+r}\right)^5 + ...\right] \\
\sum_{s=t}^{\infty} (\beta)^{s-t} ln\left(C_s^D\right) &= \left(ln\left(Y+\Delta\right) + ln\left(Y-\Delta\right)\frac{1}{1+r}\right)\left[1 + \left(\frac{1}{1+r}\right)^2 + \left(\frac{1}{1+r}\right)^4 + ...\right] \\
\sum_{s=t}^{\infty} (\beta)^{s-t} ln\left(C_s^D\right) &= \left(ln\left(Y+\Delta\right) + ln\left(Y-\Delta\right)\frac{1}{1+r}\right)\frac{1}{1-\left(\frac{1}{1+r}\right)^2} \\
\sum_{s=t}^{\infty} (\beta)^{s-t} ln\left(C_s^D\right) &= \left(ln\left(Y+\Delta\right) + ln\left(Y-\Delta\right)\frac{1}{1+r}\right)\frac{(1+r)^2}{(1+r)^2-1} \\
\sum_{s=t}^{\infty} (\beta)^{s-t} ln\left(C_s^D\right) &= \left(ln\left(Y+\Delta\right) + ln\left(Y-\Delta\right)\frac{1}{1+r}\right)\frac{(1+r)^2}{r\left(2+r\right)}
\end{aligned}
$$

1.2 Direct default cost

Consider the same model as above, but without income fluctuations ($\Delta = 0$). Default however reduces output to γY for the future periods, where $\gamma < 1$ (output remains equal to Y at the time of default).

If the country does not default, it consumes:

$$
C_t^{ND} = rB_t + Y
$$

If it defaults, it consumes:

$$
\begin{aligned}
C_t^D &= \frac{r}{1+r}\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}Y_s \\
C_t^D &= \frac{r}{1+r}Y_t+\frac{r}{1+r}\sum_{s=t+1}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}Y_s \\
C_t^D &= \frac{r}{1+r}Y+\frac{r}{1+r}\sum_{s=t+1}^{\infty}\left(\frac{1}{1+r}\right)^{s-t}\gamma Y \\
C_t^D &= \frac{r}{1+r}Y+\frac{r}{1+r}\gamma Y\frac{1}{1+r}\sum_{s=t}^{\infty}\left(\frac{1}{1+r}\right)^{s-t} \\
C_t^D &= \frac{r}{1+r}Y+\frac{r}{1+r}\gamma Y\frac{1}{1+r}\frac{1}{1-\frac{1}{1+r}} \\
C_t^D &= \frac{r}{1+r}Y+\gamma Y\frac{1}{1+r} \\
C_t^D &= \frac{\gamma+r}{1+r}Y
\end{aligned}
$$

Default gives a higher utility if:

$$
\begin{aligned}
ln\left(\frac{\gamma+r}{1+r}Y\right) &> ln\left(rB_t+Y\right) \\
\frac{\gamma+r}{1+r}Y &> rB_t+Y \\
(\gamma+r)\,Y &> (1+r)\,rB_t+(1+r)\,Y \\
\gamma Y &> (1+r)\,rB_t+Y \\
(1+r)\,r\,(-B_t) &> (1-\gamma)\,Y \\
\frac{-B_t}{Y} &> \frac{1-\gamma}{r\,(1+r)}
\end{aligned}
$$

1.3 Risk premium and probability of default (Vegh 2.4, esp. 2.4.3)

1.3.1 Risk premia

The lender requires an expected return $1+r$. The debt carries an actual rate $1+r^s$ which the lender gets with probability $1-\pi$. Otherwise, there is default and the lenders get a fraction z of the repayment ($1-z$ is the haircut):

$$
\begin{aligned}
1+r &= (1-\pi)\,(1+r^s)+pz\,(1+r^s) \\
1+r &= (1-\pi\,(1-z))\,(1+r^s)
\end{aligned}
$$

Assume $z=0$, hence $1+r=(1-\pi)\,(1+r^s)$.

Consider a two period model, where second period output is uniformly distributed on the range $y_2\in\left[0,y_2^H\right]$. The probability that $y_2<\alpha$ is $\frac{\alpha}{y_2^H}$.

If the borrower repays the debt d_1 with interest, she pays $(1+r^s)\,d_1$. If she defaults, there is a cost ϕy_2 paid by the borrower, that does not go to a lender (i.e. a true resource cost). Default is optimal if output is low:

$$d_1 > \frac{\phi y_2}{1+r^s}$$

Denotes by y_2^* as the highest output below which there is default: $d_1 = \phi y_2^* (1+r^s)$. The probability of default increases with the value of the debt:

$$\begin{aligned}
\pi &= Prob\left[y_2 < y_2^*\right] \\
\pi &= \frac{y_2^*}{y_2^H} \\
\pi &= \frac{(1+r^s)\, d_1}{\phi y_2^H}
\end{aligned}$$

The returns are:

$$\begin{aligned}
1+r &= (1-\pi)(1+r^s) \\
1+r &= \left(1 - \frac{(1+r^s)\, d_1}{\phi y_2^H}\right)(1+r^s) \\
\frac{r^s - r}{1+r^s} &= \frac{(1+r^s)\, d_1}{\phi y_2^H}
\end{aligned}$$

This is quadratic in $r^s > r$. The solution is:

$$\begin{aligned}
0 &= -\frac{d_1}{\phi y_2^H}(1+r^s)^2 + (1+r^s) - (1+r) \\
1+r^s &= \frac{-1+\sqrt{1-4(1+r)\frac{d_1}{\phi y_2^H}}}{-2\frac{d_1}{\phi y_2^H}} \\
1+r^s &= 2(1+r)\frac{d_1^{high}}{d_1}\left(1-\sqrt{1-\frac{d_1}{d_1^{high}}}\right)
\end{aligned}$$

where $d_1^{high} = \phi y_2^H / (4(1+r))$. If $0 < d_1 < d_1^{high}$, the interest rate increases with d_1. The debt cannot exceed d_1^{high}, and which point $r^S = 1+2r$.

The derivative is:

$$\begin{aligned}
-(1+r^s)^{-2}(dr^s) &= -\frac{d_1}{\phi y_2^H} dr^s - \frac{(1+r^s)}{\phi y_2^H}(dd_1) \\
(1+r^s)^{-1}\frac{(dr^s)}{1+r^s} &= \frac{(1+r^s)\, d_1}{\phi y_2^H}\left(\frac{dr^s}{1+r^s} + \frac{(dd_1)}{d_1}\right) \\
\frac{(dr^s)}{1+r^s} &= \frac{(1+r^s)^2 d_1}{\phi y_2^H}\left(\frac{dr^s}{1+r^s} + \frac{(dd_1)}{d_1}\right) \\
\left(1 - \frac{(1+r^s)^2 D_2}{\phi Y_2^H}\right)\frac{(dr^s)}{1+r^s} &= \frac{(1+r^s)^2 d_1}{\phi y_2^H}\frac{(dd_1)}{d_1} \\
(1-(r^s - r))\frac{(dr^s)}{1+r^s} &= (r^s - r)\frac{(dd_1)}{d_1}
\end{aligned}$$

$$\frac{(dr^s)}{1+r^s} = \frac{r^s - r}{1-(r^s-r)}\frac{(dd_1)}{d_1}$$

Which is positive, $\frac{\partial r^s}{\partial D_2} > 0$. Note that there are two possible equilibria, but the high interest rate one is unstable.

1.3.2 Optimal allocation

Consider that the borrower maximizes a linear utility function, for simplicity: $U = c_1 + \frac{1}{1+\delta} E c_2$. We assume that there is an incentive to borrow: $r < \delta$.

The budget constraints in the first period, and second period, depending on default or not, are:

$$\begin{aligned} c_1 &= y_1 + d_1 \\ c_2^{ND} &= y_2 - (1+r^s)\, d_1 \\ c_2^{D} &= (1-\phi)\, y_2 \end{aligned}$$

The expected utility can be written as:

$$U = y_1 + d_1 + \frac{1}{1+\delta}\left[\pi (1-\phi) E(y_2 \mid D) + (1-\pi)\left(E(y_2 \mid ND) - (1+r^s)\, d_1\right)\right]$$

where output expectations are computed conditional on being on the default space or not. Note that: $\pi E(y_2 \mid D) + (1-\pi) E(y_2 \mid ND) = E(y_2)$, which implies:

$$\begin{aligned} U &= y_1 + \left[1 - \frac{1-\pi}{1+\delta}(1+r^s)\right] d_1 \\ &\quad + \frac{1}{1+\delta}\left[\pi E(y_2 \mid D) + (1-\pi)\left(E(y_2 \mid ND)\right)\right] \\ &\quad - \frac{1}{1+\delta}\pi\phi E(y_2 \mid D) \\ U &= Y_1 + \frac{\delta - r}{1+\delta} d_1 + \frac{1}{1+\delta} E(y_2) - \frac{1}{1+\delta}\pi\phi E(y_2 \mid D) \end{aligned}$$

where $\frac{\pi}{1+\delta}\phi E(y_2 \mid D)$ is the loss of resources to default.

Form the output process: $E(y_2) = 0.5 y_2^H$.

$$\begin{aligned} E(y_2) &= \int_0^{y_2^H} y_2 \frac{1}{y_2^H} dy_2 \\ E(y_2) &= \frac{1}{y_2^H}\int_0^{y_2^*} y_2 dy_2 \\ E(y_2) &= \frac{1}{y_2^H}\left(\frac{\left(y_2^H\right)^2}{2} - \frac{(0)^2}{2}\right) \\ E(y_2) &= \frac{1}{y_2^H}\frac{\left(y_2^H\right)^2}{2} \\ E(y_2) &= \frac{y_2^H}{2} \end{aligned}$$

Conditional on defaulting, output is uniformly distributed between 0 and y_2^*, with probability $1/y_2^*$. Therefore

$$\begin{aligned}
E(y_2 \mid D) &= \int_0^{y_2^*} y_2 \frac{1}{y_2^*} dy_2 \\
E(y_2 \mid D) &= \frac{1}{y_2^*} \int_0^{y_2^*} y_2 dy_2 \\
E(y_2 \mid D) &= \frac{1}{y_2^*} \left(\frac{(y_2^*)^2}{2} - \frac{(0)^2}{2} \right) \\
E(y_2 \mid D) &= \frac{1}{y_2^*} \frac{(y_2^*)^2}{2} \\
E(y_2 \mid D) &= \frac{y_2^*}{2} \\
E(y_2 \mid D) &= \pi \frac{y_2^H}{2}
\end{aligned}$$

The utility is then:

$$\begin{aligned}
U &= y_1 + \frac{\delta - r}{1+\delta} d_1 + \frac{1}{1+\delta} E(y_2) - \frac{\pi}{1+\delta} \phi E(y_2 \mid D) \\
U &= y_1 + \frac{\delta - r}{1+\delta} d_1 + \frac{1}{1+\delta} \frac{y_2^H}{2} \left(1 - \phi \pi^2\right)
\end{aligned}$$

The optimality condition with respect to d_1 is:

$$\begin{aligned}
0 &= \frac{\delta - r}{1+\delta} - \frac{1}{1+\delta} \frac{y_2^H}{2} 2\phi\pi \frac{\partial \pi}{\partial d_1} \\
\frac{\delta - r}{1+\delta} &= \frac{1}{1+\delta} \phi \pi y_2^H \frac{\partial \pi}{\partial d_1} \\
\delta - r &= \phi \pi y_2^H \frac{\partial \pi}{\partial d_1}
\end{aligned}$$

The system is given by the probability of default, the relation between the risk-free rate and risky rate, and the optimality condition above:

$$\begin{aligned}
\pi &= \frac{(1+r^s) d_1}{\phi y_2^H} \\
1 + r &= (1-\pi)(1+r^s) \\
\delta - r &= \phi \pi y_2^H \frac{\partial \pi}{\partial d_1}
\end{aligned}$$

From differentiating the first two relations, we can compute the derivative of the probability of default:

$$\begin{aligned}
d\pi &= d(1+r^s) \left(\frac{d_1}{\phi y_2^H} \right) + (1+r^s) d \left(\frac{d_1}{\phi y_2^H} \right) \\
(1+r^s) d\pi &= (1-\pi) d(1+r^s)
\end{aligned}$$

which implies:

$$\begin{aligned}
d\pi &= \frac{1+r^s}{1-\pi}\frac{d_1}{\phi y_2^H}d\pi + (1+r^s)\, d\left(\frac{d_1}{\phi y_2^H}\right) \\
\left[1 - \frac{1+r^s}{1-\pi}\frac{d_1}{\phi y_2^H}\right] d\pi &= (1+r^s)\, d\left(\frac{d_1}{\phi y_2^H}\right) \\
\frac{\partial \pi}{\partial d_1} &= \frac{(1+r^s)}{1-\frac{1+r^s}{1-\pi}\frac{d_1}{\phi y_2^H}}\frac{1}{\phi y_2^H}
\end{aligned}$$

We then write:

$$\begin{aligned}
\delta - r &= \pi\frac{(1+r^s)}{1-\frac{1+r^s}{1-\pi}\frac{d_1}{\phi y_2^H}} \\
(\delta - r)\left[1 - \frac{1+r^s}{1-\pi}\frac{d_1}{\phi y_2^H}\right] &= \pi(1+r^s)
\end{aligned}$$

so the system becomes:

$$\begin{aligned}
\pi &= \frac{(1+r^s)\, d_1}{\phi y_2^H} \\
1+r &= (1-\pi)(1+r^s) \\
(\delta - r)\left[1 - \frac{1+r^s}{1-\pi}\frac{d_1}{\phi y_2^H}\right] &= \pi(1+r^s)
\end{aligned}$$

Using the second relation to substitute out for $1+r^s$, we write:

$$\begin{aligned}
(\delta - r)\left[1 - \frac{1+r^s}{1-\pi}\frac{d_1}{\phi y_2^H}\right] &= \pi(1+r^s) \\
(\delta - r)\left[1 - \frac{\pi}{1-\pi}\right] &= \pi\frac{1+r}{1-\pi} \\
(\delta - r)(1-2\pi) &= \pi(1+r) \\
(\delta - r) &= \pi(1+r+2\delta-2r) \\
(\delta - r) &= \pi(1+2\delta-r) \\
\pi &= \frac{\delta - r}{1+2\delta-r}
\end{aligned}$$

This implies:

$$\begin{aligned}
1+r &= (1-\pi)(1+r^s) \\
1+r &= \left(1 - \frac{\delta - r}{1+2\delta-r}\right)(1+r^s) \\
1+r &= \frac{1+\delta}{1+2\delta-r}(1+r^s) \\
1+r^s &= \frac{1+r}{1+\delta}(1+2\delta-r)
\end{aligned}$$

and:

$$\begin{aligned}
\pi &= \frac{(1+r^s)\,d_1}{\phi y_2^H} \\
\frac{\delta - r}{1+2\delta - r} &= \frac{1+r}{1+\rho}(1+2\rho - r)\frac{d_1}{\phi y_2^H} \\
\frac{1+r}{1+\delta}\frac{d_1}{\phi y_2^H} &= \frac{\delta - r}{(1+2\delta - r)^2} \\
\frac{d_1}{\phi y_2^H} &= \frac{(1+\delta)(\delta - r)}{(1+r)(1+2\delta - r)^2}
\end{aligned}$$

The rate of recovery ϕ only affects the debt level, but has no impact on the probability of default and the interest rate. A higher impatience (higher ρ) raises the interest rate, the probability of default, and the amount of debt.

2 Default with contingent payments (OR 6.1.1.1-6.1.1.4)

2.1 Setup of the model

Consider a two period small economy model. The representative agent consumes only period 2, but trades in financial assets in period 1 to maximize expected utility $U = Eu(C_2)$.

The endowment in period 2 is uncertain: $Y_2 = Y + \epsilon$, where ϵ is a shock of expected value zero and distributed over a $[\epsilon_-, \epsilon_+]$ range.

Financial contract with a risk neutral foreign insurer. The agent pays a state contingent amount $P(\epsilon)$ in period 2 (a negative amount is a payment from the insurer), so consumption is $C_2(\epsilon) = Y + \epsilon - P(\epsilon)$.

The risk neutral insurer has expected profits equal to zero: $0 = EP(\epsilon)$.

2.2 Full enforcement

Consider that the insurance contract can always be fully enforced.

The payment function $P(\epsilon)$ is set to maximize expected utility subject to the constraint that the insurance makes zero expected profits $0 = EP(\epsilon)$. The Lagrangian is:

$$\begin{aligned}
\mathcal{L} &= \sum_i \pi(\epsilon_i)\,u(Y + \epsilon_i - P(\epsilon_i)) \\
&\quad + \mu \sum_i \pi(\epsilon_i)\,P(\epsilon_i)
\end{aligned}$$

The first-order condition with respect to $P(\epsilon_i)$ is:

$$\begin{aligned}
\mu\pi(\epsilon_i) &= \pi(\epsilon_i)\,u'(Y + \epsilon_i - P(\epsilon_i)) \\
\mu &= u'(Y + \epsilon_i - P(\epsilon_i))
\end{aligned}$$

This implies that $Y + \epsilon - P(\epsilon)$ is constant, hence $P(\epsilon) = \epsilon$ and $C_2(\epsilon) = Y$, i.e. full insurance.

2.3 Default and partial insurance

The agent always plays along when $P(\epsilon) < 0$, but can decide not to pay when he owes something (i.e. $P(\epsilon) > 0$).

If the agent chooses no to pay, the insurer can only seize a share η of the agent's resources and get: $\eta(Y+\epsilon)$. Therefore, the payment cannot exceed that amount: $P(\epsilon_i) \leq \eta(Y+\epsilon_i)$.

The Lagrangian now includes an inequality constraint:

$$\begin{aligned} L &= \sum_i \pi(\epsilon_i) u(Y + \epsilon_i - P(\epsilon_i)) \\ &\quad + \mu \sum_i \pi(\epsilon_i) P(\epsilon_i) \\ &\quad - \sum_i \lambda(\epsilon_i) [P(\epsilon_i) - \eta(Y+\epsilon_i)] \end{aligned}$$

The first-order condition with respect to $P(\epsilon_i)$ is:

$$\begin{aligned} 0 &= -\pi(\epsilon_i) u'(Y+\epsilon_i - P(\epsilon_i)) + \mu\pi(\epsilon_i) - \lambda(\epsilon_i) \\ u'(Y+\epsilon_i - P(\epsilon_i)) &= \mu - \frac{\lambda(\epsilon_i)}{\pi(\epsilon_i)} \end{aligned}$$

and $0 = \lambda(\epsilon_i)[P(\epsilon_i) - \eta(Y+\epsilon_i)]$.

Payment is not a problem when the shock ϵ_i is below a threshold e (to be computed). The constraint is not binding, $\lambda(\epsilon_i) = 0$, and we have a constant $Y + \epsilon_i - P(\epsilon_i)$. The solution is of the form $P(\epsilon_i) = P_0 + \epsilon_i$ and $C(\epsilon_i) = Y + \epsilon_i - P(\epsilon_i) = Y - P_0$.

If ϵ_i is above the threshold e, the constraint is binding and $P(\epsilon_i) = \eta(Y+\epsilon_i)$, and $C(\epsilon_i) = (1-\eta)(Y+\epsilon_i)$.

When $\epsilon_i = e$, the constraint is just binding:

$$\begin{aligned} P_0 + e &= \eta(Y+e) \\ P_0 &= \eta(Y+e) - e \end{aligned}$$

There is partial insurance. The payment is a flatter function of the state of the world ($\partial P(\epsilon_i)/\partial\epsilon_i < 1$) when the constraint is binding:

$$\begin{aligned} P(\epsilon_i) &= \eta(Y+e) + (\epsilon_i - e) \qquad \text{if} \qquad \epsilon_i < e \\ P(\epsilon_i) &= \eta(Y+e) + \eta(\epsilon_i - e) \qquad \text{if} \qquad \epsilon_i > e \end{aligned}$$

Consumption is fully insured only in the low income states, and is partially insured in the high income states where the constraint is binding:

$$\begin{aligned} C_2(\epsilon_i) &= (1-\eta)(Y+e) \qquad \text{if} \qquad \epsilon_i < e \\ C_2(\epsilon_i) &= (1-\eta)(Y+\epsilon_i) \qquad \text{if} \qquad \epsilon_i > e \end{aligned}$$

The threshold e is obtained from the zero-profit condition $0 = EP(\epsilon)$. Take a uniform distribution over $[\epsilon_-, \epsilon_+] = [-\epsilon_+, \epsilon_+]$, so the density is $1/(2\epsilon_+)$. This implies:

$$
\begin{aligned}
0 &= EP(\epsilon) \\
0 &= \int_{-\epsilon_+}^{\epsilon_+} \frac{1}{2\epsilon_+} P(\epsilon)\, d\epsilon \\
0 &= \int_{-\epsilon_+}^{e} \frac{1}{2\epsilon_+} P(\epsilon)\, d\epsilon + \int_{e}^{\epsilon_+} \frac{1}{2\epsilon_+} P(\epsilon)\, d\epsilon \\
0 &= \int_{-\epsilon_+}^{e} \frac{1}{2\epsilon_+} \left[\eta(Y+e) + (\epsilon - e)\right] d\epsilon + \int_{e}^{\epsilon_+} \frac{1}{2\epsilon_+} \left[\eta(Y+e) + \eta(\epsilon - e)\right] d\epsilon \\
0 &= \int_{-\epsilon_+}^{\epsilon_+} \frac{1}{2\epsilon_+} \eta(Y+e)\, d\epsilon + \int_{-\epsilon_+}^{e} \frac{1}{2\epsilon_+} (\epsilon - e)\, d\epsilon + \int_{e}^{\epsilon_+} \frac{1}{2\epsilon_+} \eta(\epsilon - e)\, d\epsilon \\
0 &= \frac{1}{2\epsilon_+} \eta(Y+e) \int_{-\epsilon_+}^{\epsilon_+} d\epsilon + \frac{1}{2\epsilon_+} \int_{-\epsilon_+}^{e} \epsilon_i d\epsilon - \frac{e}{2\epsilon_+} \int_{-\epsilon_+}^{e} d\epsilon + \frac{\eta}{2\epsilon_+} \int_{e}^{\epsilon_+} \epsilon_i d\epsilon - \frac{\eta e}{2\epsilon_+} \int_{e}^{\epsilon_+} d\epsilon \\
0 &= \frac{1}{2\epsilon_+} \eta(Y+e)(\epsilon_+ + \epsilon_+) + \frac{1}{2\epsilon_+} \left(\frac{(e)^2}{2} - \frac{(\epsilon_+)^2}{2} \right) - \frac{e}{2\epsilon_+}(e + \epsilon_+) + \frac{\eta}{2\epsilon_+} \left(\frac{(\epsilon_+)^2}{2} - \frac{(e)^2}{2} \right) - \frac{\eta e}{2\epsilon_+}(\epsilon_+ - e) \\
0 &= \eta(Y+e) 2\epsilon_+ + \frac{(e)^2}{2} - \frac{(\epsilon_+)^2}{2} - e(e + \epsilon_+) + \eta \left(\frac{(\epsilon_+)^2}{2} - \frac{(e)^2}{2} \right) - \eta e(\epsilon_+ - e) \\
0 &= \eta(Y+e) 2\epsilon_+ - \frac{(e)^2}{2} - \frac{(\epsilon_+)^2}{2} + \eta \frac{(\epsilon_+)^2}{2} + \eta \frac{(e)^2}{2} - (1+\eta) e\epsilon_+ \\
0 &= \eta Y 2\epsilon_+ - (1-\eta) \frac{(e)^2}{2} - (1-\eta) \frac{(\epsilon_+)^2}{2} - (1-\eta) e\epsilon_+ \\
0 &= (1-\eta) \frac{(\epsilon_+)^2}{2} - \eta Y 2\epsilon_+ + (1-\eta) \frac{(e)^2}{2} + (1-\eta) e\epsilon_+ \\
e &= \frac{-(1-\eta)\epsilon_+ \pm \sqrt{(1-\eta)^2 (\epsilon_+)^2 - 2(1-\eta)\left((1-\eta) \frac{(\epsilon_+)^2}{2} - \eta Y 2\epsilon_+ \right)}}{(1-\eta)} \\
e &= -\epsilon_+ + 2\sqrt{\frac{\eta}{1-\eta} Y \epsilon_+}
\end{aligned}
$$

The threshold e is increasing in η and Y. If $\eta = 0$ no insurance takes place as $e = -\epsilon_+$ and $P(\epsilon_i) = 0$.

2.4 Introducing savings

Consider now that the borrower can accumulate savings abroad that can be used as collateral.

Two period model, with income Y in period 1. Maximizes utility over two periods: $U = u(C_1) + \beta E u(C_2)$. Invest and borrow in a bond paying r, with $\beta(1+r) = 1$. The budget constraints are:

$$
\begin{aligned}
C_1 + B_1 &= Y \\
C_2(\epsilon_i) &= (1+r) B_1 + Y + \epsilon_i - P(\epsilon_i)
\end{aligned}
$$

The full-commitment solution delivers full insurance: $C_1 = C_2 = Y$.

In case of default the insurer can seize savings abroad in addition to the share η of resource.

The incentive compatibility constraint to avoid default is:

$$P(\epsilon) \leq \eta(Y+\epsilon) + (1+r)(Y - C_1)$$

The optimal allocation maximizes the Lagrangian:

$$\begin{aligned}\mathcal{L} &= u(C_1) + \beta \sum_i \pi(\epsilon_i)\, u\left[Y + \epsilon_i - P(\epsilon_i) + (1+r)(Y - C_1)\right] \\ &\quad - \sum_i \lambda(\epsilon_i)\left[P(\epsilon_i) - \eta(Y+\epsilon_i) - (1+r)(Y - C_1)\right] \\ &\quad + \mu \sum_i \pi(\epsilon_i)\, P(\epsilon_i)\end{aligned}$$

The first-order condition with respect to C_1 and $P(\epsilon_i)$ are:

$$\begin{aligned}u'(C_1) &= Eu'(C_2) + \frac{1}{\beta}\sum_i \lambda(\epsilon_i) \\ \mu &= \beta u'(C_2(\epsilon_i)) + \frac{\lambda(\epsilon_i)}{\pi(\epsilon_i)}\end{aligned}$$

which implies that $u'(C_1) = \mu/\beta$.

When income is low ($\epsilon < e$), the incentive constraint is not binding ($\lambda(\epsilon) = 0$), and consumption is stabilized: $u'(C_1) = u'(C_2(\epsilon)) \Rightarrow C_2(\epsilon) = C_1$. This implies a payment $P(\epsilon) = \epsilon + (2+r)(Y - C_1)$.

When income is high ($\epsilon > e$) the constraint is binding: $P(\epsilon) = \eta(Y+\epsilon) + (1+r)(Y - C_1)$. Consumption is only partially insured: $C_2(\epsilon) = (1-\eta)(Y+\epsilon)$.

When $\epsilon = e$, the constraint is just binding. This gives consumption in the first period: $C_1 = (1-\eta)(Y+e)$.

e is computed from the condition that $0 = EP(\epsilon)$. Even if no output can be seized ($\eta = 0$) we get $e > -\epsilon_+$.

3 Moral hazard (OR 6.4.1)

3.1 Frictionless optimum

Consider a small country with two periods. The economy receives an endowment Y_1 in period 1. Consumption only takes place in period 2, with a linear utility $U = C_2$ for simplicity.

The initial endowment can be invested in two ways. The first is a safe foreign investment with return r. The second is a risky technology where investment I gives output Z with probability $\pi(I)$ and zero otherwise. Higher investment raises the probability of success, and the first unit of investment has a high marginal return: $\pi' > 0$, $\pi'' < 0$, $AZ\pi'(0) > 1 + r$.

In period 1 the country borrows D (at a cost discussed below), invests I and lends L at world interest rate:

$$L + I = Y_1 + D$$

In the second period, expected consumption is given by:

$$\begin{aligned} EC_2 &= (1+r)(L-D) + \pi(I) Z \\ EC_2 &= (1+r)(Y_1 - I) + \pi(I) Z \end{aligned}$$

The optimal investment is:

$$\begin{aligned} 0 &= -(1+r) + \pi'\left(\tilde{I}\right) Z \\ Z\pi'\left(\tilde{I}\right) &= 1+r \end{aligned}$$

If the borrower can commit to investing $\tilde{I}$ and to repay the debt (according to the contract terms), she chooses $L = 0$ and $D = \tilde{I} - Y_1$. The investment is fully driven by $Z/(1+r)$. Endowment Y_1 does not matter as it can be smoothed by borrowing.

The repayment in case of success is P, and zero in case of failure, such that the lender gets the expected return:

$$P\pi\left(\tilde{I}\right) = (1+r)\left(\tilde{I} - Y_1\right)$$

3.2 Information friction

With asymmetric information, the lender observes output Y_1 and Z, debt D, but not where the money is invested (I or L). The borrower chooses the portfolio I and L, once D and P are set (we assume no repayment in case of failure). There would be no problem if P can be indexed to I.

The consumption of the borrower is ($L = Y_1 + D - I$):

$$\begin{aligned} C_2 &= Z - P + (1+r)(Y_1 + D - I) \qquad \text{if successful} \\ C_2 &= (1+r)(Y_1 + D - I) \qquad \text{if not} \end{aligned}$$

Expected consumption is then:

$$EC_2 = \pi(I)(Z-P) + (1+r)(Y_1 + D - I)$$

This is maximized with respect to I at:

$$\pi'(I)(Z-P) = 1+r$$

As $Z\pi'\left(\tilde{I}\right) = 1+r$ and $P > 0$ in the first best, this condition does not hold for the first-best investment level $\tilde{I}$ and P.

$$\begin{aligned}
Z\pi'\left(\tilde{I}\right) &> \pi'\left(\tilde{I}\right)(Z-P) \\
1+r &> \pi'\left(\tilde{I}\right)(Z-P) \\
\pi'(I) &> \pi'\left(\tilde{I}\right) \\
I &< \tilde{I}
\end{aligned}$$

The borrower prefers to invest some of the debt back on the world markets (L is not seen by the lender), which she can keep if things go wrong. The optimality gives P as a decreasing function of I ($P=0$ when $I=\tilde{I}$), the incentive compatibility condition:

$$\begin{aligned}
\pi'(I)(Z-P) &= 1+r \\
Z-P &= \frac{1+r}{\pi'(I)} \\
P &= Z-\frac{1+r}{\pi'(I)}
\end{aligned}$$

The lender requires to break even in expected terms: $P\pi(I)=(1+r)(I-Y_1)$. In equilibrium we should have $L=0$ as otherwise the borrower would have an extra cost (in the end all inefficiencies are paid by the borrower). Differentiating, we get:

$$\begin{aligned}
P\pi'(I)\,dI+\pi(I)\,dP &= (1+r)\,dI \\
\frac{dP}{dI} &= \frac{(1+r)-P\pi'(I)}{\pi(I)} \\
\frac{dP}{dI} &= (1+r)\frac{\pi(I)-(I-Y_1)\pi'(I)}{[\pi(I)]^2}
\end{aligned}$$

This gives P as an increasing function of I, with $P=0$ when $I=Y_1<\tilde{I}$. In equilibrium, we get $I<\tilde{I}$ at the intersection of the lines.

$$\begin{aligned}
\pi'(I)(Z-P) &= 1+r \\
\pi'(I)\left(Z-\frac{(1+r)(I-Y_1)}{\pi(I)}\right) &= 1+r
\end{aligned}$$

and:

$$P = \frac{(1+r)(I-Y_1)}{\pi(I)}$$

A higher Y_1 reduces the need to borrow, hence P, which reduces $\pi'(I)$, leading to higher investment.