PS4 Solutions

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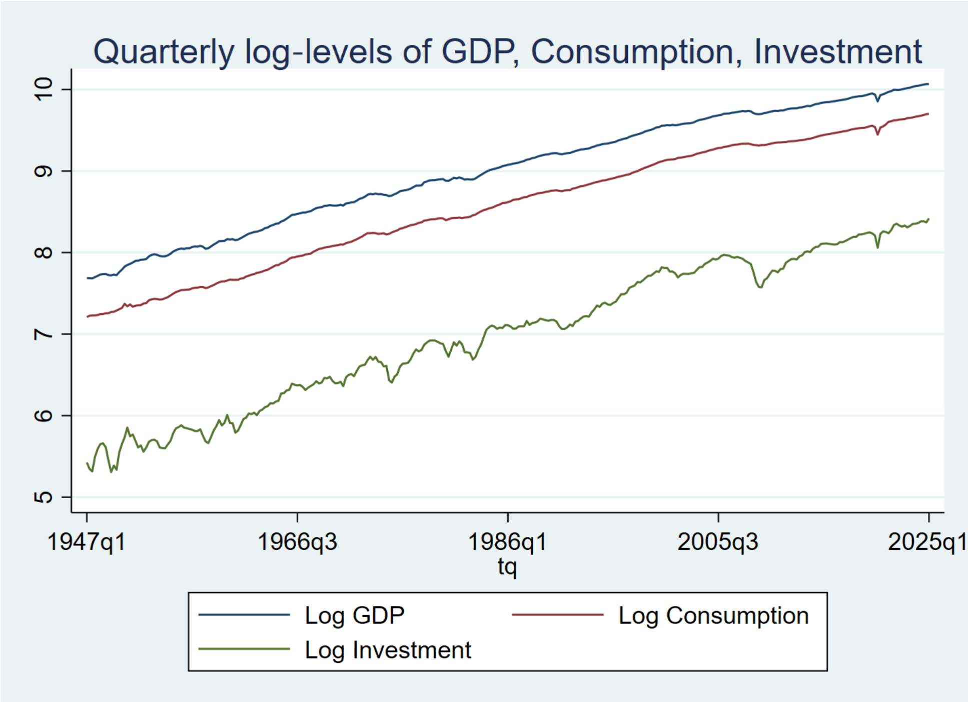
1 Problem 1

Solution (a).



Solution (b).
Let $\mathbf{X} = (\mathbf{1}, \mathbf{t})$ be the $N \times 2$ design matrix, with rows $(1, t)$, and $\mathbf{y} = (y_1, \ldots, y_N)'$. Then the OLS estimator is

$$\begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \end{pmatrix} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} = \frac{1}{S_{tt}} \begin{pmatrix} \sum_t t^2 & -\sum_t t \\ -\sum_t t & \sum_t 1 \end{pmatrix} \begin{pmatrix} \sum_t y_t \\ \sum_t t\, y_t \end{pmatrix},$$

where

$$\bar{t} = \frac{1}{N}\sum_{t=1}^{N} t, \quad \bar{y} = \frac{1}{N}\sum_{t=1}^{N} y_t, \quad S_{tt} = \sum_{t=1}^{N}(t - \bar{t})^2.$$

Equivalently,

$$\hat{\beta}_2 = \frac{\sum_{t=1}^{N}(t - \bar{t})(y_t - \bar{y})}{\sum_{t=1}^{N}(t - \bar{t})^2}, \quad \hat{\beta}_1 = \bar{y} - \hat{\beta}_2\,\bar{t}.$$

The fitted values and residuals are

$$\hat{y}_t = \hat{\beta}_1 + \hat{\beta}_2 \, t, \qquad \hat{u}_t = y_t - \hat{y}_t.$$

y_t is the log of three variables: GDP, PCE, and GPDI. The estimated models are as below:

Window	Series	$\hat{\beta}_1$	$\hat{\beta}_2$ (per qtr)	Annualized Growth (%)
1965 Q1–2006 Q4	ln GDP	7.8509	0.0077707	3.11
	ln PCE	7.3017	0.0083520	3.34
	ln GPDI	5.4989	0.0100216	4.01
2007 Q1–2019 Q4	ln GDP	8.4997	0.0048888	1.96
	ln PCE	8.0947	0.0049259	1.97
	ln GPDI	5.2006	0.0104206	4.17
2007 Q1–2022 Q2	ln GDP	8.4867	0.0049376	1.98
	ln PCE	8.0394	0.0051370	2.05
	ln GPDI	5.3432	0.0098678	3.95

Table 1: Trend regression estimates and annualized growth rates

Solution (c).
$\hat{\beta}_2$ measures the *average quarterly* change in $\ln y_t$.
The *approximate quarterly growth rate* is $\hat{\beta}_2 \times 100\%$.
The *exact annualized growth rate* is

$$\left(e^{4\hat{\beta}_2} - 1\right) \times 100\% \approx 4\hat{\beta}_2 \times 100\%,$$

- **1965–2006:** GDP grows at about 0.78% per quarter ⇒ 3.11% per year; Consumption: 0.84% qtr ⇒ 3.34% yr; Investment: 1.00% qtr ⇒ 4.01% yr.
- **2007–2019:** GDP/Consumption trend roughly halves to 0.49% qtr ⇒ 1.96% yr. Investment trend remains high, 1.04% qtr ⇒ 4.17% yr.
- **2007–2022:** GDP/Consumption trend stays near 2.05% annualized; Investment trend slightly eases to 3.95%.

GDP and PCE grow at a similar rate, but investment grow faster, at about twice the rate of GDP and PCE.

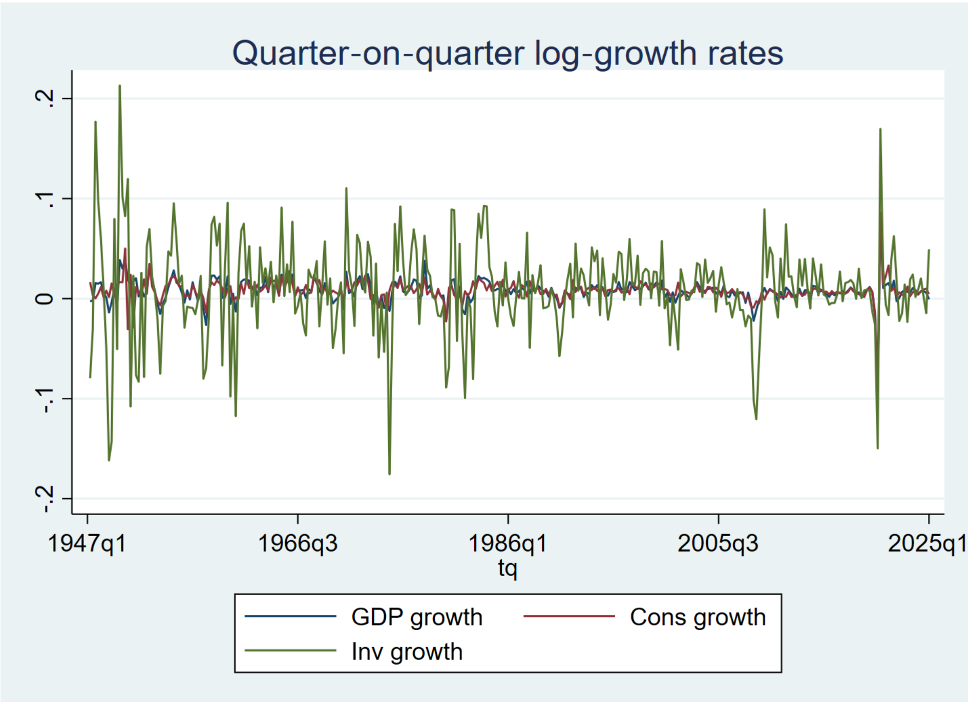
Solution (d).

The (kth) sample autocorrelation of $\{\hat{u}_t\}$ is

$$\hat{\rho}(k) = \frac{\sum_{t=k+1}^{N} \hat{u}_t \, \hat{u}_{t-k}}{\sum_{t=1}^{N} \hat{u}_t^2}, \qquad k = 0, 1, 2, \ldots.$$

We compute $\hat{\rho}(k)$ for $k = 1, \ldots, K$ (here $K = 8$) in each subsample to assess the persistence of residual cycles.

Solution (e).



Solution (f).

For each series $g_t \in \{\Delta y_t, \Delta c_t, \Delta i_t\}$ in a given subsample of length M, we compute

$$\bar{g} = \frac{1}{M} \sum_{t=1}^{M} g_t, \qquad s_g = \sqrt{\frac{1}{M-1} \sum_{t=1}^{M} \left(g_t - \bar{g}\right)^2}.$$

These compare the *average realized growth* $\bar{g}$ with the *trend-based* quarterly slope $\hat{\beta}_2$, and measure cyclical volatility via s_g.

Comparison to part (b):

- *1965 Q2-2006 Q4:* The sample means of $\Delta \ln y$ for GDP (0.00799), PCE (0.00867) and GPDI (0.01001) are almost identical to the estimated quarterly trend slopes $\hat{\beta}_2$ (0.0077707, 0.0083520, 0.0100216). This confirms that, over the long sample, the linear-trend regression accurately captures the average growth rate.

Window	Series	Obs	Mean of $\Delta \ln y$	Std. of $\Delta \ln y$	$\hat{\beta}_2$ (per qtr)
1965 Q2–2006 Q4	GDP	167	0.0079889	0.0082641	0.0077707
	PCE	167	0.0086664	0.0067471	0.0083520
	GPDI	167	0.0100079	0.0391054	0.0100216
2007 Q2–2019 Q4	GDP	51	0.0045828	0.0061137	0.0048888
	PCE	51	0.0045793	0.0043225	0.0049259
	GPDI	51	0.0058344	0.0346509	0.0104206
2007 Q2–2022 Q2	GDP	61	0.0045452	0.0159656	0.0049376
	PCE	61	0.0050521	0.0175900	0.0051370
	GPDI	61	0.0064772	0.0444316	0.0098678

- *2007 Q2-2019 Q4:* GDP and PCE mean growth rates (≈0.00458) lie slightly below their $\hat{\beta}_2$ (≈0.00489 and 0.00493), reflecting that downturns (2008-09 crisis) pull the sample average below the fitted trend. For GPDI, the mean (0.00583) is markedly below its trend slope (0.01042), since investment experienced large negative shocks that the OLS trend, minimizing squared errors, spreads more evenly across the sample.

- *2007 Q2-2022 Q2:* Including the COVID-19 shock further widens the gap: GDP and PCE means (0.00455, 0.00505) remain below their slopes (0.00494, 0.00514), and investment's mean (0.00648) stays well under its slope (0.00987). This again shows that severe cyclical downturns pull down the simple average growth below the fitted linear trend.

Solution (g).

Window	Series	Mean of $\Delta \ln y$	Std. dev. of $\Delta \ln y$
1965 Q2–1983 Q4	GDP	0.0078863	0.0109888
	PCE	0.0086857	0.0085560
	GPDI	0.0091587	0.0504674
1984 Q1–2006 Q4	GDP	0.0080725	0.0051354
	PCE	0.0086507	0.0048490
	GPDI	0.0107001	0.0267834

- *Mean growth rates:* GDPs average growth rises slightly from 0.7886% to 0.8073%; PCE falls marginally from 0.8686% to 0.8651%; GPDI increases from 0.9159% to 1.0700%. Overall, the mean growth rates remain essentially unchanged in magnitude.

- *Volatility:* All three series exhibit a dramatic reduction in standard deviation.
  - GDP's σ falls from 1.10% to 0.51%.

  - PCE's σ falls from 0.86% to 0.48%.
  - GPDI's σ falls from 5.05% to 2.68%.

  This confirms the *Great Moderation*after 1984, while average growth remained stable, business-cycle volatility was substantially dampened.

2 Problem 2

Solution (a).

Weak Stationarity

A process $\{y_t\}$ is weakly stationary if

$$\mathbb{E}[y_t] = \mu \quad \forall t, \quad \mathrm{Cov}(y_t, y_{t-k}) = \gamma_k \text{ depends only on } k.$$

Here

$$\mathbb{E}[y_t] = \sum_{j=0}^{3} \psi_j \, \mathbb{E}[u_{t-j}] = 0,$$

so the mean is constant. Next,

$$\mathrm{Cov}(y_t, y_{t-k}) = \mathbb{E}\big[y_t \, y_{t-k}\big] = \sum_{j=0}^{3} \sum_{i=0}^{3} \psi_j \psi_i \, \mathbb{E}\big[u_{t-j} u_{t-k-i}\big].$$

Since $\mathbb{E}[u_s u_r] = 0$ for $s \neq r$, only terms with $t - j = t - k - i$ survive, so $\mathrm{Cov}(y_t, y_{t-k})$ depends on k alone. Hence $\{y_t\}$ is weakly stationary.

Strict Stationarity

If $\{u_t\}$ is strictly stationary (e.g. i.i.d.), then any finite-order linear filter $\sum_{j=0}^{3} \psi_j \, u_{t-j}$ yields a strictly stationary $\{y_t\}$. Thus under i.i.d. innovations, y_t is strictly stationary.

Solution (b).
Define $\gamma_k = \mathrm{Cov}(y_t, y_{t-k})$. With ψ_j as above,

$$\gamma_k = \sum_{j=0}^{3} \sum_{i=0}^{3} \psi_j \psi_i \, E[u_{t-j} u_{t-k-i}] = \sum_{j=0}^{3} \psi_j \, \psi_{j-k},$$

interpreting $\psi_m = 0$ for $m < 0$ or $m > 3$.

Thus:

$$
\begin{aligned}
\gamma_0 &= \sum_{j=0}^{3} \psi_j^2 = 1^2 + (-2.4)^2 + 0.8^2 + (-0.4)^2 = 7.56, \\
\gamma_1 &= \psi_0\psi_1 + \psi_1\psi_2 + \psi_2\psi_3 = 1 \cdot (-2.4) + (-2.4) \cdot 0.8 + 0.8 \cdot (-0.4) = -4.64, \\
\gamma_2 &= \psi_0\psi_2 + \psi_1\psi_3 = 1 \cdot 0.8 + (-2.4) \cdot (-0.4) = 1.76, \\
\gamma_3 &= \psi_0\psi_3 = 1 \cdot (-0.4) = -0.4, \\
\gamma_k &= 0 \quad \text{for } |k| \geq 4.
\end{aligned}
$$

The autocorrelation is $\rho_k = \gamma_k/\gamma_0$.

Solution (c).
Consider

$$S_T = \sum_{t=1}^{T} y_t, \qquad V_T = \mathbb{V}[S_T] = \sum_{i=1}^{T}\sum_{j=1}^{T} \gamma_{i-j}.$$

Re-index $h = j - i$:

$$V_T = \sum_{h=-(T-1)}^{T-1} (T - |h|)\, \gamma_h.$$

Hence

$$\mathbb{V}\left[\tfrac{1}{\sqrt{T}} S_T\right] = \frac{V_T}{T} = \gamma_0 + 2\sum_{h=1}^{T-1}\Big(1 - \tfrac{h}{T}\Big)\gamma_h.$$

As $T \to \infty$, $\frac{h}{T} \to 0$ for fixed h, and $\gamma_h = 0$ for $h \geq 4$, so

$$\lim_{T\to\infty} \mathbb{V}\left[\tfrac{1}{\sqrt{T}} S_T\right] = \gamma_0 + 2(\gamma_1 + \gamma_2 + \gamma_3) = 7.56 + 2(-4.64 + 1.76 - 0.4) = 1.$$

Solution (d).
Assume $\{u_t\}$ is i.i.d. with $\mathbb{E}[u_t] = 0$, $\mathbb{V}[u_t] = \sigma^2$.

Strict Stationarity

$x_t = f(u_t, u_{t-4})$ depends on two i.i.d. draws; hence its joint distributions do not change with shifts in t. So $\{x_t\}$ is strictly stationary.

Ergodicity

An i.i.d. sequence is ergodic and any measurable function of it remains ergodic. Thus $\{x_t\}$ is ergodic.

White-Noise Properties

- $\mathbb{E}[x_t] = \mathbb{E}[u_t]\mathbb{E}[u_{t-4}] = 0$.

- For $k \neq 0$,
$$\mathrm{Cov}(x_t, x_{t-k}) = \mathbb{E}[u_t u_{t-4} u_{t-k} u_{t-k-4}] = 0,$$
since among the four factors at least one is independent with zero mean.

Hence $\{x_t\}$ is zero-mean uncorrelated white noise, but not i.i.d. (since x_t and x_{t+4} share u_t).

Solution (e).
We have:
$$\mathbb{E}\left[\tfrac{1}{T}\sum_{t=1}^{T} x_t\right] = \tfrac{1}{T}\sum_{t=1}^{T}\mathbb{E}[x_t] = 0,$$
and since $\mathrm{Cov}(x_i, x_j) = 0$ for $i \neq j$,
$$\mathbb{V}\left[\tfrac{1}{\sqrt{T}}\sum_{t=1}^{T} x_t\right] = \frac{1}{T}\sum_{t=1}^{T}\mathbb{V}[x_t] = \sigma^4.$$