In-class exercises

Mathematics and Statistics for Economists

August 21, 2024

Exercise 1 For each of the following pairs of sets A and B, state which, if either, is a subset of the other:
(a) A is the set of people living in the USA; B is the set of people living in California.
(b) A is the set of natural numbers divisible by 6; B is the set of natural numbers divisible by 2 .
(c) A is the set of natural numbers divisible by 5; B is the set of natural numbers divisible by 7 .

Exercise 2 Suppose you have a data set consisting of the values of imports and exports for 18 countries.
(a) If an element of the set is considered to be the values of the two variables for one particular country, how many elements does the data set contain? Which $\mathbb{R}^n$ do they belong to?
(b) If an element of the set is considered to be the values of one of the variables for all the countries, how many elements does the data set contain? Which $\mathbb{R}^n$ do they belong to?

Exercise 3 Using the composite function rule, show that

$$\frac{d}{dx}\left(\frac{1}{v}\right)=-\frac{1}{v^2}\frac{dv}{dx}$$

Using this result and the product rule, derive the quotient rule.

Exercise 4 Suppose that a firm's output Q is related to capital input K and labour input L by the production function

$$Q=K^{1/2}L^{1/3}$$

Suppose further that K and L are given by the linear functions

$$K=5+2t, \quad L=2+t$$

Find dQ/dt.

Exercise 5 Given the property of matrix-vector multiplication: if the columns of the matrix A are the vectors $a^1, a^2, \ldots, a^n$, so that A can be written as $\begin{bmatrix} a^1 & a^2 & \ldots & a^n \end{bmatrix}$, then $Ax = x_1a^1 + x_2a^2 + \cdots + x_na^n$. Verify the property when

$$A=\begin{bmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{bmatrix}, \quad x=\begin{bmatrix} -4 \\ 3 \\ 2 \\ -1 \end{bmatrix}$$

Exercise 6 (i) Let

$$\mathbf{a}^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{a}^2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{a}^3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Show that (a) $\mathrm{a}^1, \mathrm{a}^2, \mathrm{a}^3$ are linearly independent; (b) any 3-vector x can be expressed as a linear combination of a^1, a^2, a^3.

(ii) As (i), but with

$$\mathbf{a}^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{a}^2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{a}^3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

[A set of 3-vectors with properties (a) and (b) is called a basis of $\mathbb{R}^3$.]

Exercise 7 Find the critical points of

$$y = x^5(2-x)^4$$

and determine their nature. Find also any points of inflexion and determine the ranges x for which the function (i) convex, (ii) concave.

Exercise 8 The functions f and g are defined as follows:

$$f(x) = x^3 + 1, \quad g(x) = x^4 - 2$$

Find expressions for $f(g(x))$, $g(f(x))$ and their derivatives.

Exercise 9 Consider the equation

$$x^5 - 5x + 2 = 0$$

(i) Show that this equation has exactly three real roots x_1, x_2, x_3, where $x_1 < -1$ and $0 < x_2 < 1 < x_3$. Sketch the the graph.

(ii) Let's suppose that $x_2 = 0.402$ to 3 decimal places. Using your sketch and a small amount of arithmetic (hint: $1.5^4 \approx 5$), show that x_1 is between -2 and -1.5, while x_3 is between 1 and 1.5. Then use Newton's method to approximate x_1 and x_3, each to 3 decimal places.

Exercise 10 Compute the first and second derivatives of each of the following functions:

$$\textbf{(a)} \quad xe^{3x} \quad \textbf{(b)} \quad \frac{x}{e^x} \quad \textbf{(c)} \quad \frac{\ln x}{x}$$