Macroeconomics A; EI060

Technical appendix: Currency crises, foreign exchange intervention

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1 Currency crisis: first generation (Harms X.3.2)

1.1 Money demand and supply

Consider the monetary model of the exchange rate, which consists of the money demand (the textbook considers that it reflects GDP instead of consumption), uncovered interest rate parity, and purchasing power parity:

$$\begin{aligned} i_{t+1}^{H} &= i_{t+1}^{F} + \mathbb{E}_t\left(e_{t+1}\right) - e_t \\ m_t - p_t &= \phi y_t - \lambda i_{t+1}^{H} \\ p_t &= e_t + p_t^F \end{aligned}$$

The money demand is rewritten as (assuming that $p_t^F + \phi y_t - \lambda i_{t+1}^F = 0$ for simplicity):

$$\begin{aligned} m_t - p_t &= \phi y_t - \lambda i_{t+1}^{H} \\ m_t - e_t - p_t^F &= \phi y_t - \lambda\left(i_{t+1}^{F} + \mathbb{E}_t\left(e_{t+1}\right) - e_t\right) \\ m_t - e_t &= -\lambda\left(\mathbb{E}_t\left(e_{t+1}\right) - e_t\right) + p_t^F + \phi y_t - \lambda i_{t+1}^F \\ m_t - e_t &= -\lambda\left(\mathbb{E}_t\left(e_{t+1}\right) - e_t\right) \end{aligned}$$

Initially, the country has a pegged exchange rate with money supply set at $\overline{m}$, which implies: $e_t = e_{t+1} = \overline{m}$.

Money is the liability side of the central bank's balance sheet. The assets consist of domestic currency bonds D_t and foreign currency bonds (reserves) R_t:

$$\begin{aligned} \overline{M} &= D_t + R_t \\ \overline{m} &= \ln\left(D_t + R_t\right) \end{aligned}$$

The holdings of domestic bonds increase at a rateμ: $\ln(D_{t+1}) = \ln(D_t) + \mu$.

1.2 Dynamics of reserves

1.2.1 Before and after the peg breaks

As long as the peg holds, reserves decrease as the holdings of domestic bonds increase. When reserves are exhausted, the money stock is only backed by domestic currency bonds and increased at the rate μ.

It is useful to compute the shadow exchange rate, which is the exchange rate that would prevail at time t if the central bank only held domestic bonds. In such a case, the money supply grows at a rate μ, and the money demand in two successive periods are (we drop the expectation sign as the model is under perfect foresight):

$$\begin{aligned} m_t - e_t &= -\lambda(e_{t+1} - e_t) \\ m_t + \mu - e_{t+1} &= -\lambda(e_{t+2} - e_{t+1}) \end{aligned}$$

The exchange rate grows at a constant rate, hence $e_{t+1} - e_t = e_{t+2} - e_{t+1}$, which implies:

$$\begin{aligned} m_t - e_t &= m_t + \mu - e_{t+1} \\ e_{t+1} - e_t &= \mu \end{aligned}$$

Hence:

$$\begin{aligned} m_t - e_t &= -\lambda(e_{t+1} - e_t) \\ m_t - e_t &= -\lambda\mu \\ e_t &= m_t + \lambda\mu \\ e_t &= \ln(D_t) + \lambda\mu \end{aligned}$$

When reserves are exhausted, the exchange rate follows this relation. Until then the exchange rate is $e_t = \overline{m}$.

1.2.2 Breaking of the peg

Until $T+1$ included the exchange rate is pegged, and in period T, the exchange rate moves to a float:

$$\begin{aligned} e_{T-1} &= \overline{m} \\ e_T &= \ln(D_T) + \lambda\mu \end{aligned}$$

Can it be that the money supply is $\overline{m}$ at time T? This would imply$e_T = \overline{m} + \lambda\mu$. But then $e_T - e_{T-1} = \lambda\mu$, so there would be a perfectly predictable exchange rate jump between the two periods, which is not consistent with the interest parity.

The interest parity rules out any predictable jump, hence it must be that:

$$
\begin{aligned}
e_{T-1} &= e_T \\
\overline{m} &= \ln(D_T) + \lambda\mu \\
\ln(D_T) &= \overline{m} - \lambda\mu < \overline{m}
\end{aligned}
$$

The money supply must jump down at time T, which can only takes the form of a discrete drop in reserves.

To compute the time of crisis, we start from period :

$$
\begin{aligned}
\overline{m} &= \ln(D_T) + \lambda\mu \\
\overline{m} &= \ln(D_0) + T\mu + \lambda\mu \\
\ln(D_0 + R_0) &= \ln(D_0) + T\mu + \lambda\mu \\
T\mu &= \ln(D_0 + R_0) - \ln(D_0) - \lambda\mu \\
T\mu &= \ln\left(1 + \frac{R_0}{D_0}\right) - \lambda\mu \\
T &= \frac{1}{\mu}\ln\left(1 + \frac{R_0}{D_0}\right) - \lambda
\end{aligned}
$$

2 Currency crisis: second generation (Harms X.3.3)

2.1 Output gap and loss function

The economy is characterized by a Phillips curve where the output gap x reflects deviations between realized inflation π and expected inflation π^{expected}. With purchasing power parity, inflation reflects the depreciation of the domesrtic currency $\pi = \Delta e$:

$$
x_t = \theta\left(\Delta e_t - \Delta e_t^{\text{expected}}\right)
$$

The central minimizes a loss function which reflects :

1. Volatility of the output gap from a reference point: $(x_t - \overline{x})^2$.
2. Volatility of the exchange rate: $(\Delta e_t)^2$.
3. A cost of letting the exchange rate float, i.e. abandoning the peg: $C(\Delta e_t)$. For simplicity, we assume that only a deviation is costly, with a given cost c if $\Delta e_t > 0$

The loss function is:

$$
\begin{aligned}
L &= \frac{1}{2}\left\{\phi(x_t - \overline{x})^2 + (\Delta e_t)^2\right\} + c_{\Delta e_t > 0} \\
L &= \frac{1}{2}\left\{\phi\left(\theta\left(\Delta e_t - \Delta e_t^{\text{expected}}\right) - \overline{x}\right)^2 + (\Delta e_t)^2\right\} + c_{\Delta e_t > 0}
\end{aligned}
$$

where ϕ reflects the weight put on the output gap volatility, relative to the one put on inflation volatility.

2.2 Outcome under peg or float

If the central bank abandons the peg, it sets Δe_t to minimize the loss, given $\Delta e_t^{\text{expected}}$. The optimal condition is:

$$\begin{aligned}
0 &= \frac{\partial L}{\partial\left(\Delta e_t\right)} \\
0 &= \phi\theta\left(\theta\left(\Delta e_t - \Delta e_t^{\text{expected}}\right) - \overline{x}\right) + \left(\Delta e_t\right) \\
0 &= \phi\theta^2\left(\Delta e_t - \Delta e_t^{\text{expected}}\right) - \phi\theta\overline{x} + \left(\Delta e_t\right) \\
0 &= \phi\theta^2\left(\Delta e_t\right) - \phi\theta^2\left(\Delta e_t^{\text{expected}}\right) - \phi\theta\overline{x} + \left(\Delta e_t\right) \\
\left(1+\phi\theta^2\right)\left(\Delta e_t\right) &= \phi\theta^2\left(\Delta e_t^{\text{expected}}\right) + \phi\theta\overline{x} \\
\Delta e_t &= \frac{\phi\theta}{1+\phi\theta^2}\left(\theta\Delta e_t^{\text{expected}} + \overline{x}\right)
\end{aligned}$$

The output gap is then:

$$\begin{aligned}
x_t &= \theta\left(\Delta e_t - \Delta e_t^{\text{expected}}\right) \\
x_t &= \theta\left(\frac{\phi\theta^2}{1+\phi\theta^2}\Delta e_t^{\text{expected}} - \Delta e_t^{\text{expected}} + \frac{\phi\theta}{1+\phi\theta^2}\overline{x}\right) \\
x_t &= \theta\left(\frac{\phi\theta}{1+\phi\theta^2}\overline{x} - \frac{1}{1+\phi\theta^2}\Delta e_t^{\text{expected}}\right)
\end{aligned}$$

The loss function with the optimal floating exchange rate is:

$$
\begin{aligned}
L^{\text{float}} &= \frac{1}{2}\left\{\phi\left(\theta\left(\Delta e_t - \Delta e_t^{\text{expected}}\right) - \overline{x}\right)^2 + (\Delta e_t)^2\right\} + c \\
L^{\text{float}} &= \frac{1}{2}\left\{\phi\left(\theta\left(\frac{\phi\theta}{1+\phi\theta^2}\overline{x} - \frac{1}{1+\phi\theta^2}\Delta e_t^{\text{expected}}\right) - \overline{x}\right)^2 + \left(\frac{\phi\theta}{1+\phi\theta^2}\left(\theta\Delta e_t^{\text{expected}} + \overline{x}\right)\right)^2\right\} + c \\
L^{\text{float}} &= \frac{1}{2}\left\{\phi\left(\frac{\phi\theta^2}{1+\phi\theta^2}\overline{x} - \frac{\theta}{1+\phi\theta^2}\Delta e_t^{\text{expected}} - \overline{x}\right)^2 + \left(\frac{\phi\theta}{1+\phi\theta^2}\left(\theta\Delta e_t^{\text{expected}} + \overline{x}\right)\right)^2\right\} + c \\
L^{\text{float}} &= \frac{1}{2}\left\{\phi\left(-\frac{1}{1+\phi\theta^2}\overline{x} - \frac{\theta}{1+\phi\theta^2}\Delta e_t^{\text{expected}}\right)^2 + \left(\frac{\phi\theta}{1+\phi\theta^2}\left(\theta\Delta e_t^{\text{expected}} + \overline{x}\right)\right)^2\right\} + c \\
L^{\text{float}} &= \frac{1}{2}\left\{\phi\left(\frac{1}{1+\phi\theta^2}\right)^2\left(\overline{x} + \theta\Delta e_t^{\text{expected}}\right)^2 + \left(\frac{\phi\theta}{1+\phi\theta^2}\left(\theta\Delta e_t^{\text{expected}} + \overline{x}\right)\right)^2\right\} + c \\
L^{\text{float}} &= \frac{1}{2}\left\{\phi + (\phi\theta)^2\right\}\left(\frac{1}{1+\phi\theta^2}\right)^2\left(\overline{x} + \theta\Delta e_t^{\text{expected}}\right)^2 + c \\
L^{\text{float}} &= \frac{1}{2}\phi\left(1+\phi\theta^2\right)\left(\frac{1}{1+\phi\theta^2}\right)^2\left(\overline{x} + \theta\Delta e_t^{\text{expected}}\right)^2 + c \\
L^{\text{float}} &= \frac{\phi}{2\left(1+\phi\theta^2\right)}\left(\overline{x} + \theta\Delta e_t^{\text{expected}}\right)^2 + c
\end{aligned}
$$

The loss function under the peg ($\Delta e_t = 0$) is:

$$
\begin{aligned}
L^{\text{peg}} &= \frac{1}{2}\left\{\phi\left(\theta\left(-\Delta e_t^{\text{expected}}\right) - \overline{x}\right)^2 + (0)^2\right\} \\
L^{\text{peg}} &= \frac{\phi}{2}\left(\overline{x} + \theta\Delta e_t^{\text{expected}}\right)^2
\end{aligned}
$$

2.3 Exchange rate equilibrium

The central bank abandons the peg if the fixed cost is below a critical value:

$$
\begin{aligned}
L^{\text{float}} &< L^{\text{peg}} \\
\frac{\phi}{2\left(1+\phi\theta^2\right)}\left(\overline{x} + \theta\Delta e_t^{\text{expected}}\right)^2 + c &< \frac{\phi}{2}\left(\overline{x} + \theta\Delta e_t^{\text{expected}}\right)^2 \\
c &< \left(1 - \frac{1}{1+\phi\theta^2}\right)\frac{\phi}{2}\left(\overline{x} + \theta\Delta e_t^{\text{expected}}\right)^2 \\
c &< \frac{(\phi\theta)^2}{2\left(1+\phi\theta^2\right)}\left(\overline{x} + \theta\Delta e_t^{\text{expected}}\right)^2 \\
c &< c^{\text{critical}}
\end{aligned}
$$

The critical cost is higher (i.e. the central is more inclined to abandon the peg) if the reference output gap is high, or the expected depreciation is high.

If the fixed cost is equal to the critical value, the central bank is indifferent. This gives a threshold value for the expected depreciation:

$$\begin{aligned}
c &= \frac{(\phi\theta)^2}{2(1+\phi\theta^2)}\left(\overline{x}+\theta\Delta\overline{e}_t^{\text{expected}}\right)^2 \\
2\left(1+\phi\theta^2\right)c &= (\phi\theta)^2(\overline{x})^2+2(\phi\theta)^2\overline{x}\theta\Delta\overline{e}_t^{\text{expected}}+(\phi\theta)^2(\theta)^2\left(\Delta\overline{e}_t^{\text{expected}}\right)^2 \\
0 &= \left[(\phi\theta)^2(\overline{x})^2-2\left(1+\phi\theta^2\right)c\right]+2(\phi\theta)^2\overline{x}\theta\Delta\overline{e}_t^{\text{expected}}+(\phi\theta)^2(\theta)^2\left(\Delta\overline{e}_t^{\text{expected}}\right)^2 \\
\Delta\overline{e}_t^{\text{expected}} &= \frac{-2(\phi\theta)^2\overline{x}\theta+\sqrt{\left[2(\phi\theta)^2\overline{x}\theta\right]^2-4(\phi\theta)^2(\theta)^2\left[(\phi\theta)^2(\overline{x})^2-2\left(1+\phi\theta^2\right)c\right]}}{2(\phi\theta)^2(\theta)^2} \\
\Delta\overline{e}_t^{\text{expected}} &= \frac{-2(\phi\theta)^2\overline{x}\theta+\sqrt{8(\phi\theta)^2(\theta)^2(1+\phi\theta^2)c}}{2(\phi\theta)^2(\theta)^2} \\
\Delta\overline{e}_t^{\text{expected}} &= \frac{-\phi\overline{x}\theta+\sqrt{2(1+\phi\theta^2)c}}{\phi(\theta)^2} \\
\Delta\overline{e}_t^{\text{expected}} &= \frac{\sqrt{2(1+\phi\theta^2)c}}{\phi(\theta)^2}-\frac{\overline{x}}{\theta}
\end{aligned}$$

As we focus on a depreciation, $\Delta\overline{e}_t^{\text{expected}}$ is equal to this term only if it is positive, and otherwise is equal to zero (note that another root is $\Delta\overline{e}_t^{\text{expected}}=-\frac{\sqrt{2(1+\phi\theta^2)c}}{\phi(\theta)^2}-\frac{\overline{x}}{\theta}$, which is clearly negative).

If $\Delta e_t^{\text{expected}}<\Delta\overline{e}_t^{\text{expected}}$ the peg is maintained, and $\Delta e_t^{\text{expected}}=\Delta\overline{e}_t^{\text{expected}}=0$ is an equilibrium. If $\Delta e_t^{\text{expected}}>\Delta\overline{e}_t^{\text{expected}}$ the peg is abandoned and the exchange rate movement (which is expected) is given by the minimization of the loss function:

$$\begin{aligned}
\Delta e_t &= \frac{\phi\theta}{1+\phi\theta^2}\left(\theta\Delta e_t^{\text{expected}}+\overline{x}\right) \\
\Delta e_t &= \frac{\phi\theta}{1+\phi\theta^2}\left(\theta\Delta e_t+\overline{x}\right) \\
\left(1+\phi\theta^2\right)\Delta e_t &= \phi\theta^2\Delta e_t+\phi\theta\overline{x} \\
\Delta e_t &= \phi\theta\overline{x}
\end{aligned}$$

Note that if $\Delta\overline{e}_t^{\text{expected}}>\phi\theta\overline{x}$, then the only equilibrium is an abandonment of the peg. Multiple equilibria can occur if:

$$\begin{aligned}
&0<\Delta\overline{e}_t^{\text{expected}}<\phi\theta\overline{x} \\
&0<\frac{\sqrt{2(1+\phi\theta^2)c}}{\phi(\theta)^2}-\frac{\overline{x}}{\theta}<\phi\theta\overline{x} \\
&0<\frac{\sqrt{2(1+\phi\theta^2)c}}{\phi\theta}-\overline{x}<\phi\theta^2\overline{x} \\
&\overline{x}<\frac{\sqrt{2(1+\phi\theta^2)c}}{\phi\theta}<\left(1+\phi\theta^2\right)\overline{x}
\end{aligned}$$

3 Currency crisis: third generation (Harms X.3.4)

3.1 Bank net worth

Consider a two-period small open economy with uncovered interest rate parity between the expected return of domestic currency and foreign currency bonds:

$$1+i_1^H = \left(1+i_1^F\right)\frac{E_2^{\text{expected}}}{E_1}$$

Banks play a central role. In period t a bank i earns a revenue $Y_t^{H,i}$ in domestic currency an a revenue $Y_t^{F,i}$ in foreign currency. It makes payments on liabilities in domestic and foreign currencies equal to $D_t^{H,i}$ and $D_t^{F,i}$ respectively. The net worth of the bank is the expected discounted value of its profits:

$$V_1^{H,i} = \left(Y_1^{H,i}-D_1^{H,i}\right)+E_1\left(Y_1^{F,i}-D_1^{F,i}\right)+\frac{1}{1+i_1^H}\left[\left(Y_2^{H,i}-D_2^{H,i}\right)+E_2^{\text{expected}}\left(Y_2^{F,i}-D_2^{F,i}\right)\right]$$

The foreign currency value of this net worth is:

$$\begin{aligned}
\widetilde{V}_1^{H,i} &= \frac{V_1^{H,i}}{E_1}\\
\widetilde{V}_1^{H,i} &= \frac{1}{E_1}\left(Y_1^{H,i}-D_1^{H,i}\right)+\left(Y_1^{F,i}-D_1^{F,i}\right)\\
&\quad+\frac{1}{1+i_1^H}\frac{E_2^{\text{expected}}}{E_1}\left[\frac{1}{E_2^{\text{expected}}}\left(Y_2^{H,i}-D_2^{H,i}\right)+\left(Y_2^{F,i}-D_2^{F,i}\right)\right]\\
\widetilde{V}_1^{H,i} &= \frac{\left(1+i_1^H\right)}{\left(1+i_1^F\right)E_2^{\text{expected}}}\left(Y_1^{H,i}-D_1^{H,i}\right)+\left(Y_1^{F,i}-D_1^{F,i}\right)\\
&\quad+\frac{1}{1+i_1^H}\frac{E_2^{\text{expected}}}{E_1}\frac{1}{E_2^{\text{expected}}}\left(Y_2^{H,i}-D_2^{H,i}\right)+\frac{1}{1+i_1^F}\left(Y_2^{F,i}-D_2^{F,i}\right)\\
\widetilde{V}_1^{H,i} &= \frac{\left(1+i_1^H\right)}{\left(1+i_1^F\right)E_2^{\text{expected}}}\left(Y_1^{H,i}-D_1^{H,i}\right)+\left(Y_1^{F,i}-D_1^{F,i}\right)\\
&\quad+\frac{1}{\left(1+i_1^F\right)E_2^{\text{expected}}}\left(Y_2^{H,i}-D_2^{H,i}\right)+\frac{1}{1+i_1^F}\left(Y_2^{F,i}-D_2^{F,i}\right)\\
\widetilde{V}_1^{H,i} &= \frac{\left(1+i_1^H\right)\left(Y_1^{H,i}-D_1^{H,i}\right)+\left(Y_2^{H,i}-D_2^{H,i}\right)}{\left(1+i_1^F\right)E_2^{\text{expected}}}+\left(Y_1^{F,i}-D_1^{F,i}\right)+\frac{1}{1+i_1^F}\left(Y_2^{F,i}-D_2^{F,i}\right)
\end{aligned}$$

3.2 Bank net worth

We simplify by abstracting from domestic currency liabilities ($D_1^{H,i}=D_2^{H,i}=0$) and initial period domestic currency revenue ($Y_1^{H,i}=0$).

We also consider a maturity mismatch with revenue being backloaded ($Y_1^{F,i}-D_1^{F,i}<0$), and a currency mismatch with a negative value of foreign currency discounted earnings:

$$0 \quad > \quad E_1 \left(Y_1^{F,i} - D_1^{F,i} \right) + \frac{1}{1+i_1^H} E_2^{\text{expected}} \left(Y_2^{F,i} - D_2^{F,i} \right)$$

The foreign currency net worth is:

$$\widetilde{V}_1^{H,i} \quad = \quad \frac{Y_2^{H,i}}{\left(1+i_1^F\right) E_2^{\text{expected}}} + \left(Y_1^{F,i} - D_1^{F,i} \right) + \frac{1}{1+i_1^F} \left(Y_2^{F,i} - D_2^{F,i} \right)$$

It is positive if:

$$\frac{Y_2^{H,i}}{\left(1+i_1^F\right) E_2^{\text{expected}}} \quad > \quad \left(D_1^{F,i} - Y_1^{F,i} \right) + \frac{1}{1+i_1^F} \left(D_2^{F,i} - Y_2^{F,i} \right)$$

3.3 Multiple equilibria

For simplification, consider that initially $E_1 = E_2^{\text{expected}} = 1$ and $\widetilde{V}_1^{H,i} > 0$. An expected depreciation ($E_2^{\text{expected}} > 1$) reduces $\widetilde{V}_1^{H,i}$ by lowering the foreign currency value of future domestic earnings $Y_2^{H,i}$.

Denote by $E_{2,i}^{\text{expected}}$ the expected exchange rate at which $\widetilde{V}_1^{H,i} = 0$. We assume that banks are heterogeneous and the cumulative distribution of $E_{2,i}^{\text{expected}}$ is given by the function F. The share of insolvent banks as a function of the expected exchange rate is $n = F\left(E_2^{\text{expected}}\right)$. The value of expected exchange rate at which all banks are insolvent, $\hat{E}_2^{\text{expected}}$, is defined by $1 = F\left(\hat{E}_2^{\text{expected}}\right)$.

The actual exchange rate is an increasing function $E_2 = G(n)$ of the fraction of failing banks, as bailing them out entails a monetary expansion. As both $E_2 = G(n)$ and $n = F\left(E_2^{\text{expected}}\right)$ are positive relations between n and the exchange rate, multiple equilibria where $E_2 = E_2^{\text{expected}}$ are possible.

4 Sterilized foreign exchange rate interventions (Harms VIII.6.2)

4.1 Equilibrium in the absence of shock

The current account is matched by the financial account, which we split between private capital flows and foreign reserve accumulation:

$$CA_t \quad = \quad FA_t^{NR} + \Delta R_{t+1}$$

The private financial account reflects deviations from uncovered interest parity:

$$FA_t^{NR} \quad = \quad \alpha \left(i_t^F - i_t^H + \mathbb{E}_t e_{t+1} - e_t \right)$$

The case where UIP always holds corresponds to an infinite value for α.

The current account reflects the exchange rate, with a weak currency boosting exports:

$$CA_t = \gamma e_t$$

Monetary policy takes the form of the interest rate following the foreign one, reacting to the exchange rate, and subjected to a shift ξ_t:

$$i_t^H = i_t^F + \delta e_t + \xi_t$$

The policy maker also chooses the level of reserves: ΔR_{t+1}.

The balance of payments is written as:

$$\begin{aligned}
CA_t &= FA_t^{NR} + \Delta R_{t+1} \\
\gamma e_t &= \alpha\left(i_t^F - i_t^H + \mathbb{E}_t e_{t+1} - e_t\right) + \Delta R_{t+1} \\
\gamma e_t &= \alpha\left(i_t^F - i_t^F - \delta e_t - \xi_t + \mathbb{E}_t e_{t+1} - e_t\right) + \Delta R_{t+1} \\
\gamma e_t &= \alpha\left(-\left(1+\delta\right) e_t - \xi_t + \mathbb{E}_t e_{t+1}\right) + \Delta R_{t+1} \\
\left(\alpha\left(1+\delta\right)+\gamma\right) e_t &= \alpha\left(-\xi_t + \mathbb{E}_t e_{t+1}\right) + \Delta R_{t+1} \\
\left(\alpha\left(1+\delta\right)+\gamma\right) e_t &= \alpha\mathbb{E}_t e_{t+1} - \alpha\xi_t + \Delta R_{t+1} \\
e_t &= \frac{\alpha\mathbb{E}_t e_{t+1} - \alpha\xi_t + \Delta R_{t+1}}{\alpha\left(1+\delta\right)+\gamma}
\end{aligned}$$

If UIP holds ($\alpha \to \infty$) reserves do not matter:

$$\begin{aligned}
e_t &= \frac{\mathbb{E}_t e_{t+1} - \xi_t - \frac{1}{\alpha}\Delta R_{t+1}}{1+\delta+\frac{\gamma}{\alpha}} \\
e_t &= \frac{\mathbb{E}_t e_{t+1} - \xi_t}{1+\delta}
\end{aligned}$$

We assume that ξ_t and ΔR_{t+1} are one-off shifts with no further expected movements.. this implies that $\mathbb{E}_t e_{t+1} = 0$ and $e_t = \frac{-\alpha\xi_t + \Delta R_{t+1}}{\alpha(1+\delta)+\gamma}$.

4.2 Introducing shocks and policy rules

The private financial account now reflects a shock in addition to the deviations from uncovered interest parity:

$$FA_t^{NR} = \alpha\left(i_t^F - i_t^H + \mathbb{E}_t e_{t+1} - e_t\right) - z_t$$

The shock follows an autoregressive process $z_t = \phi z_{t-1} + \varepsilon_t$ where $0 < \phi < 1$ and $\mathbb{E}_{t-1}\varepsilon_t = 0$.

The current account also includes an autoregressive shock in addition to the exchange rate:

$$CA_t = \gamma e_t + \zeta_t$$

where $\zeta_t = \varphi\zeta_{t-1} + \nu_t$, $0 < \varphi < 1$ and $\mathbb{E}_{t-1}\nu_t = 0$.

The monetary policy rule is as before, with ξ_t now being an autoregressive shock:

$$i_t^H = i_t^F + \delta e_t + \xi_t$$

where $\xi_t = \psi\xi_{t-1} + \iota_t$, $0 < \psi < 1$ and $\mathbb{E}_{t-1}\iota_t = 0$.

Reserves are set through a rule reacting to shocks in capital flows ($0 < \theta < 1$):

$$\Delta R_{t+1} = \theta z_t$$

4.3 Exchange rate

The balance of payments is written as:

$$\begin{aligned}
CA_t &= FA_t^{NR} + \Delta R_{t+1} \\
\gamma e_t + \zeta_t &= \alpha\left(i_t^F - i_t^H + \mathbb{E}_t e_{t+1} - e_t\right) - z_t + \theta z_t \\
\gamma e_t + \zeta_t &= \alpha\left(i_t^F - i_t^F - \delta e_t - \xi_t + \mathbb{E}_t e_{t+1} - e_t\right) - z_t + \theta z_t \\
\gamma e_t + \zeta_t &= \alpha\left(-\left(1+\delta\right)e_t - \xi_t + \mathbb{E}_t e_{t+1}\right) - \left(1-\theta\right)z_t \\
\left(\alpha\left(1+\delta\right) + \gamma\right)e_t + \zeta_t &= \alpha\left(-\xi_t + \mathbb{E}_t e_{t+1}\right) - \left(1-\theta\right)z_t \\
\left(\alpha\left(1+\delta\right) + \gamma\right)e_t &= \alpha\mathbb{E}_t e_{t+1} - \alpha\xi_t - \zeta_t - \left(1-\theta\right)z_t \\
e_t &= \frac{\alpha\mathbb{E}_t e_{t+1} - \alpha\xi_t - \zeta_t - \left(1-\theta\right)z_t}{\alpha\left(1+\delta\right) + \gamma}
\end{aligned}$$

If UIP holds ($\alpha \to \infty$), the shocks ζ_t and z_t do not play a role, and neither does reserve accumulation (θ does not enter):

$$\begin{aligned}
e_t &= \frac{\mathbb{E}_t e_{t+1} - \xi_t - \frac{\zeta_t + (1-\theta)z_t}{\alpha}}{1 + \delta + \frac{\gamma}{\alpha}} \\
e_t &= \frac{\mathbb{E}_t e_{t+1} - \xi_t}{1 + \delta}
\end{aligned}$$

We solve for the exchange rate by iterating forward and using the transversality condition:

$$\begin{aligned}
e_t &= -\frac{\alpha\xi_t + \zeta_t + (1-\theta) z_t}{\alpha(1+\delta)+\gamma} + \frac{\alpha}{\alpha(1+\delta)+\gamma}\mathbb{E}_t e_{t+1} \\
e_t &= -\frac{\alpha\xi_t + \zeta_t + (1-\theta) z_t}{\alpha(1+\delta)+\gamma} \\
&\quad -\frac{\alpha}{\alpha(1+\delta)+\gamma}\mathbb{E}_t\left(\frac{\alpha\xi_t + \zeta_t + (1-\theta) z_t}{\alpha(1+\delta)+\gamma}\right) + \left(\frac{\alpha}{\alpha(1+\delta)+\gamma}\right)^2 \mathbb{E}_t e_{t+2} \\
e_t &= -\frac{\alpha\xi_t + \zeta_t + (1-\theta) z_t}{\alpha(1+\delta)+\gamma} - \frac{\alpha}{\alpha(1+\delta)+\gamma}\mathbb{E}_t\left(\frac{\alpha\xi_t + \zeta_t + (1-\theta) z_t}{\alpha(1+\delta)+\gamma}\right) \\
&\quad -\left(\frac{\alpha}{\alpha(1+\delta)+\gamma}\right)^2 \mathbb{E}_t\left(\frac{\alpha\xi_{t+2} + \zeta_{t+2} + (1-\theta) z_{t+2}}{\alpha(1+\delta)+\gamma}\right) + \left(\frac{\alpha}{\alpha(1+\delta)+\gamma}\right)^3 \mathbb{E}_t e_{t+3} \\
e_t &= -\sum_{s=t}^{\infty}\left(\frac{\alpha}{\alpha(1+\delta)+\gamma}\right)^{s-t} \mathbb{E}_t\left(\frac{\alpha\xi_s + \zeta_s + (1-\theta) z_s}{\alpha(1+\delta)+\gamma}\right) \\
e_t &= -\sum_{s=t}^{\infty}\left(\frac{\alpha}{\alpha(1+\delta)+\gamma}\right)^{s-t} \frac{\alpha\psi^{s-t}\xi_t + \varphi^{s-t}\zeta_t + (1-\theta)\phi^{s-t} z_t}{\alpha(1+\delta)+\gamma}
\end{aligned}$$

Splitting across the shocks, we get:

$$\begin{aligned}
e_t &= -\frac{\alpha\xi_t}{\alpha(1+\delta)+\gamma}\sum_{s=t}^{\infty}\left(\frac{\alpha\psi}{\alpha(1+\delta)+\gamma}\right)^{s-t} \\
&\quad -\frac{\zeta_t}{\alpha(1+\delta)+\gamma}\sum_{s=t}^{\infty}\left(\frac{\alpha\varphi}{\alpha(1+\delta)+\gamma}\right)^{s-t} \\
&\quad -\frac{(1-\theta) z_t}{\alpha(1+\delta)+\gamma}\sum_{s=t}^{\infty}\left(\frac{\alpha\phi}{\alpha(1+\delta)+\gamma}\right)^{s-t} \\
e_t &= -\frac{\alpha\xi_t}{\alpha(1+\delta)+\gamma}\frac{1}{1-\frac{\alpha\psi}{\alpha(1+\delta)+\gamma}} \\
&\quad -\frac{\zeta_t}{\alpha(1+\delta)+\gamma}\frac{1}{1-\frac{\alpha\varphi}{\alpha(1+\delta)+\gamma}} \\
&\quad -\frac{(1-\theta) z_t}{\alpha(1+\delta)+\gamma}\frac{1}{1-\frac{\alpha\phi}{\alpha(1+\delta)+\gamma}} \\
e_t &= -\frac{\alpha}{\alpha(1+\delta-\psi)+\gamma}\xi_t - \frac{1}{\alpha(1+\delta-\varphi)+\gamma}\zeta_t - \frac{1-\theta}{\alpha(1+\delta-\phi)+\gamma} z_t
\end{aligned}$$