# Macroeconomics A; EI056

# Technical appendix: time consistency and policy rules

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## 1 General setup

The objective of the central bank is to minimize a quadratic loss function in output and inflation (unlike Walsh, we work with the function quadratic in both arguments):

$$V = \frac{1}{2}\left[\lambda\left(y - y^*\right)^2 + \left(\pi - \pi^*\right)^2\right]$$

where $y^*$ and $\pi^*$ are the ideal output and inflation. For simplicity set $\pi^* = 0$.

The economy is characterized by an AS relation ($\pi^e$ is expected inflation):

$$y = y^n + a\left(\pi - \pi^e\right) + e$$

where $e$ is a shock of expected value zero ($Ee = 0$). The central bank's ideal output is higher than the natural output:

$$k = y^* - y^n > 0$$

The loss function can thus be written as:

$$V = \frac{1}{2}\left[\lambda\left(y - y^n - k\right)^2 + \pi^2\right] \tag{1}$$

Inflation depends on the monetary policy $m$ (we simplify from Walsh by writing it as $m$ instead of $\Delta m$) and a shock $v$ of expected value zero ($Ev = 0$):

$$\pi = m + v$$

$m$ can be set after the shock $e$ is realized. We assume that the shocks are uncorrelated, so $E\left(ev\right) = 0$.

## 2 Equilibrium under discretion

The central bank chooses $m$ once $e$ is realized and expectations $\pi^e$ have been set. The expected objective (1) is then:

$$
\begin{aligned}
EV &= \frac{1}{2}\left[\lambda E\left[\left(y-y^n-k\right)^2\right]+E\left(\pi^2\right)\right] \\
EV &= \frac{1}{2}\left[\lambda E\left[\left(a\left(\pi-\pi^e\right)+e-k\right)^2\right]+E\left(\pi^2\right)\right] \\
EV &= \frac{1}{2}\left[\lambda E\left[\left(a\left(m+v-\pi^e\right)+e-k\right)^2\right]+E\left[\left(m+v\right)^2\right]\right]
\end{aligned}
$$

where only $v$ is unknown at the time of the decision. The optimality condition is:

$$
\begin{aligned}
0 &= \frac{\partial EV}{\partial m} \\
0 &= \lambda E\left[\left(a\left(m+v-\pi^e\right)+e-k\right)a\right]+E\left(m+v\right) \\
0 &= \lambda\left(a\left(m+Ev-\pi^e\right)+e-k\right)a+\left(m+Ev\right) \\
0 &= \lambda a\left(a\left(m-\pi^e\right)+e-k\right)+m \\
0 &= \lambda a\left(a\left(-\pi^e\right)+e-k\right)+\left(1+\lambda a^2\right)m \\
\left(1+\lambda a^2\right)m &= \lambda a\left(a\pi^e-e+k\right) \\
m &= \frac{\lambda a^2}{1+\lambda a^2}\pi^e-\frac{\lambda a}{1+\lambda a^2}e+\frac{\lambda a}{1+\lambda a^2}k
\end{aligned}
$$

Actual inflation is then:

$$
\begin{aligned}
\pi &= m+v \\
\pi &= \frac{\lambda a^2}{1+\lambda a^2}\pi^e-\frac{\lambda a}{1+\lambda a^2}e+\frac{\lambda a}{1+\lambda a^2}k+v
\end{aligned}
$$

Agents understand this results at the time of forming their expectations. Taking expectations of this relation, we get ($\pi^e = E\pi$):

$$
\begin{aligned}
\pi^e &= \frac{\lambda a^2}{1+\lambda a^2}\pi^e+\frac{\lambda a}{1+\lambda a^2}k \\
\frac{1}{1+\lambda a^2}\pi^e &= \frac{\lambda a}{1+\lambda a^2}k \\
\pi^e &= \lambda a k > 0
\end{aligned}
$$

The monetary stance is thus:

$$
\begin{aligned}
m &= \frac{\lambda a^2}{1+\lambda a^2}\pi^e-\frac{\lambda a}{1+\lambda a^2}e+\frac{\lambda a}{1+\lambda a^2}k \\
m &= \frac{\lambda a^2}{1+\lambda a^2}\lambda a k-\frac{\lambda a}{1+\lambda a^2}e+\frac{\lambda a}{1+\lambda a^2}k \\
m &= \lambda a k-\frac{\lambda a}{1+\lambda a^2}e \qquad (2)
\end{aligned}
$$

The inflation is thus:

$$\pi = \lambda ak - \frac{\lambda a}{1+\lambda a^2}e + v \tag{3}$$

Its variance is:

$$E\left[(\pi-\pi^e)^2\right] = E\left[\left(-\frac{\lambda a}{1+\lambda a^2}e+v\right)^2\right] = \left(\frac{\lambda a}{1+\lambda a^2}\right)^2 E\left(e^2\right) + E\left(v^2\right)$$

The output is:

$$\begin{aligned}
y &= y^n + a\left(\pi - \pi^e\right) + e \\
y &= y^n + a\left(\lambda ak - \frac{\lambda a}{1+\lambda a^2}e + v - \lambda ak\right) + e \\
y &= y^n + \frac{1}{1+\lambda a^2}e + av \qquad (4)
\end{aligned}$$

Its variance is:

$$E\left[(y-y^n)^2\right] = \left(\frac{1}{1+\lambda a^2}\right)^2 E\left(e^2\right) + a^2 E\left(v^2\right)$$

The expected loss function, taken from the very initial point (i.e. before all shocks are realized) is:

$$\begin{aligned}
EV^{disc} &= \frac{1}{2}E\left[\lambda\left(y-y^n-k\right)^2 + \pi^2\right] \\
EV^{disc} &= \frac{1}{2}E\left[\lambda\left(\frac{1}{1+\lambda a^2}e+av-k\right)^2 + \left(\lambda ak - \frac{\lambda a}{1+\lambda a^2}e+v\right)^2\right] \\
EV^{disc} &= \frac{1}{2}\left[\begin{array}{c} \lambda k^2 + \lambda E\left[\left(\frac{1}{1+\lambda a^2}e+av\right)^2\right] - 2\lambda kE\left(\frac{1}{1+\lambda a^2}e+av\right) \\ +\left(\lambda ak\right)^2 + E\left[\left(-\frac{\lambda a}{1+\lambda a^2}e+v\right)^2\right] + 2\lambda akE\left(-\frac{\lambda a}{1+\lambda a^2}e+v\right) \end{array}\right] \\
EV^{disc} &= \frac{1}{2}\left[\begin{array}{c} \lambda k^2 + \lambda\left(\frac{1}{1+\lambda a^2}\right)^2 E\left(e^2\right) + \lambda a^2 E\left(v^2\right) \\ +\left(\lambda a\right)^2 k^2 + \left(\frac{\lambda a}{1+\lambda a^2}\right)^2 E\left(e^2\right) + E\left(v^2\right) \end{array}\right] \\
EV^{disc} &= \frac{1}{2}\left[\left(1+\lambda a^2\right)\lambda k^2 + \frac{\lambda}{1+\lambda a^2}E\left(e^2\right) + \left(1+\lambda a^2\right)E\left(v^2\right)\right] \qquad (5)
\end{aligned}$$

# 3 Equilibrium under strict commitment

The central bank now announces a value for $m$ that is will deliver no matter what. Expected inflation is then:

$$\pi^e = m + Ev = m$$

Output is then independent from the monetary stance:

$$\begin{aligned} y &= y^n + a(\pi - \pi^e) + e \\ y &= y^n + a(m + v - m) + e \\ y &= y^n + av + e \end{aligned} \tag{6}$$

The expected loss function, taken from the very initial point (i.e. before all shocks are realized) is:

$$\begin{aligned} EV &= \frac{1}{2} E\left[\lambda (y - y^n - k)^2 + \pi^2\right] \\ EV &= \frac{1}{2}\left[\lambda E\left[(av + e - k)^2\right] + E\left[(m + v)^2\right]\right] \end{aligned}$$

The optimality condition is:

$$\begin{aligned} 0 &= \frac{\partial EV}{\partial m} \\ 0 &= E(m + v) \\ 0 &= m + Ev \\ 0 &= m \end{aligned} \tag{7}$$

The expected inflation is then zero. The inflation is thus $\pi = v$, and its variance is:

$$E\left[(\pi - \pi^e)^2\right] = E\left(v^2\right)$$

The variance of output is:

$$E\left[(y - y^n)^2\right] = E\left(e^2\right) + a^2 E\left(v^2\right)$$

The expected loss function, taken from the very initial point (i.e. before all shocks are realized) is:

$$\begin{aligned} EV^{strict} &= \frac{1}{2} E\left[\lambda (y - y^n - k)^2 + \pi^2\right] \\ EV^{strict} &= \frac{1}{2} E\left[\lambda (av + e - k)^2 + v^2\right] \\ EV^{strict} &= \frac{1}{2}\left[\lambda k^2 + \lambda E\left[(av + e)^2\right] - 2\lambda k E(av + e) + E\left(v^2\right)\right] \\ EV^{strict} &= \frac{1}{2}\left[\lambda k^2 + \lambda a^2 E\left(v^2\right) + \lambda E\left(e^2\right) + E\left(v^2\right)\right] \\ EV^{strict} &= \frac{1}{2}\left[\lambda k^2 + \lambda E\left(e^2\right) + \left(1 + \lambda a^2\right) E\left(v^2\right)\right] \end{aligned} \tag{8}$$

Comparing (5) and (8), we see that the first term in $k^2$ is higher under discretion. The second terms in $E\left(e^2\right)$ is smaller under discretion. The final term in $E\left(v^2\right)$ is identical in both cases.

# 4 Equilibrium under flexible commitment

Consider now that the central bank commits not in term of a level of $m$ but in terms of a rule linking $m$ to shocks:

$$m = b_0 + b_1 e$$

$b_0$ and $b_1$ are known by the public ex-ante. Expected inflation is then:

$$\begin{aligned}
\pi^e &= Em + Ev \\
\pi^e &= b_0 + b_1 Ee + Ev \\
\pi^e &= b_0
\end{aligned}$$

Output is then given by:

$$\begin{aligned}
y &= y^n + a(\pi - \pi^e) + e \\
y &= y^n + a(m + v - b_0) + e \\
y &= y^n + a(b_1 e + v) + e \\
y &= y^n + av + (1 + ab_1) e
\end{aligned}$$

The expected loss function, taken from the very initial point (i.e. before all shocks are realized) is:

$$\begin{aligned}
EV &= \frac{1}{2} E\left[\lambda (y - y^n - k)^2 + \pi^2\right] \\
EV &= \frac{1}{2} E\left[\lambda (av + (1 + ab_1) e - k)^2 + (b_0 + b_1 e + v)^2\right]
\end{aligned}$$

The optimality condition with respect to $b_0$ is:

$$\begin{aligned}
0 &= \frac{\partial EV}{\partial b_0} \\
0 &= E(b_0 + b_1 e + v) \\
0 &= b_0 + b_1 Ee + Ev \\
0 &= b_0
\end{aligned}$$

Expected inflation is thus zero, as under a strict commitment.

The optimality condition with respect to $b_1$ is:

$$\begin{aligned}
0 &= \frac{\partial EV}{\partial b_1} \\
0 &= E\left[\lambda (av + (1 + ab_1) e - k) ae + e(b_0 + b_1 e + v)\right] \\
0 &= \lambda a \left(aEev + (1 + ab_1) E\left(e^2\right) - kEe\right) + b_1 E\left(e^2\right) + E(ev) \\
0 &= \lambda a (1 + ab_1) E\left(e^2\right) + b_1 E\left(e^2\right) \\
0 &= \lambda a + \left(1 + \lambda a^2\right) b_1 \\
b_1 &= -\frac{\lambda a}{1 + \lambda a^2}
\end{aligned}$$

The monetary policy rule is thus:

$$m = -\frac{\lambda a}{1 + \lambda a^2} e \tag{9}$$

which is the second part of the rule under discretion (2). Inflation is then:

$$\pi = -\frac{\lambda a}{1 + \lambda a^2} e + v \tag{10}$$

which corresponds to the last two terms of inflation under discretion (3). The variance of inflation is:

$$E\left(\pi^2\right) = \left(\frac{\lambda a}{1 + \lambda a^2}\right)^2 E\left(e^2\right) + E\left(v^2\right)$$

Output is equal to:

$$\begin{aligned} y &= y^n + av + (1 + ab_1) e \\ y &= y^n + av + \left(1 - \frac{\lambda a^2}{1 + \lambda a^2}\right) e \\ y &= y^n + av + \frac{1}{1 + \lambda a^2} e \qquad (11) \end{aligned}$$

which is identical to output under discretion (4). The allocation under a flexible commitment is thus identical to the one under discretion, except for the bias in inflation under discretion (the term in $k$ in (3)).

The expected loss function, taken from the very initial point (i.e. before all shocks are realized) is:

$$\begin{aligned} EV^{flexible} &= \frac{1}{2} E\left[\lambda\left(y - y^n - k\right)^2 + \pi^2\right] \\ EV^{flexible} &= \frac{1}{2} E\left[\lambda\left(av + \frac{1}{1 + \lambda a^2} e - k\right)^2 + \left(-\frac{\lambda a}{1 + \lambda a^2} e + v\right)^2\right] \\ EV^{flexible} &= \frac{1}{2}\left[\begin{array}{c} \lambda k^2 + \lambda E\left[\left(\frac{1}{1+\lambda a^2} e + av\right)^2\right] - 2\lambda k E\left(\frac{1}{1+\lambda a^2} e + av\right) \\ + E\left[\left(-\frac{\lambda a}{1+\lambda a^2} e + v\right)^2\right] \end{array}\right] \\ EV^{flexible} &= \frac{1}{2}\left[\begin{array}{c} \lambda k^2 + \lambda\left(\frac{1}{1+\lambda a^2}\right)^2 E\left(e^2\right) + \lambda a^2 E\left(v^2\right) \\ + \left(\frac{\lambda a}{1+\lambda a^2}\right)^2 E\left(e^2\right) + E\left(v^2\right) \end{array}\right] \\ EV^{flexible} &= \frac{1}{2}\left[\lambda k^2 + \frac{\lambda}{1 + \lambda a^2} E\left(e^2\right) + \left(1 + \lambda a^2\right) E\left(v^2\right)\right] \qquad (12) \end{aligned}$$

This is identical to the loss function under discretion (5), except for the first term in $k$ which is smaller under the flexible rule.

# 5 Delegation

Monetary policy is delegated to a central bank whose loss function puts a greater weight on inflation than (1), i.e. $\delta > 0$:

$$V = \frac{1}{2}\left[\lambda\left(y - y^n - k\right)^2 + (1+\delta)\,\pi^2\right]$$

The central banker conducts policy under discretion, so the solution is as above, except that $\lambda$ is replaced by $\lambda/(1+\delta)$. (3) and (4) are then:

$$\begin{aligned}
\pi &= \frac{a\lambda}{1+\delta}k - \frac{\frac{\lambda}{1+\delta}a}{1+\frac{\lambda}{1+\delta}a^2}e + v = \frac{a\lambda}{1+\delta}k - \frac{a\lambda}{1+\delta+a^2\lambda}e + v \\
y &= y^n + \frac{1}{1+\frac{\lambda}{1+\delta}a^2}e + av = y^n + \frac{1+\delta}{1+\delta+a^2\lambda}e + av
\end{aligned}$$

The variances of inflation is lower when $\delta$ is positive, and the variance of output is higher when $\delta$ is positive.

The expected loss function of the society, taken from the very initial point (i.e. before all shocks are realized) is:

$$\begin{aligned}
EV^{disc} &= \frac{1}{2}E\left[\lambda\left(y - y^n - k\right)^2 + \pi^2\right] \\
EV^{disc} &= \frac{1}{2}E\left[\lambda\left(\frac{1+\delta}{1+\delta+a^2\lambda}e + av - k\right)^2 + \left(\frac{a\lambda}{1+\delta}k - \frac{a\lambda}{1+\delta+a^2\lambda}e + v\right)^2\right] \\
EV^{disc} &= \frac{1}{2}\left[\begin{array}{c} \lambda k^2 + \lambda E\left[\left(\frac{1+\delta}{1+\delta+a^2\lambda}e + av\right)^2\right] - 2\lambda k E\left(\frac{1+\delta}{1+\delta+a^2\lambda}e + av\right) \\ +\left(\frac{a\lambda}{1+\delta}k\right)^2 + E\left[\left(-\frac{a\lambda}{1+\delta+a^2\lambda}e + v\right)^2\right] + 2\frac{a\lambda}{1+\delta}kE\left(-\frac{a\lambda}{1+\delta+a^2\lambda}e + v\right) \end{array}\right] \\
EV^{disc} &= \frac{1}{2}\left[\begin{array}{c} \lambda k^2 + \lambda\left(\frac{1+\delta}{1+\delta+a^2\lambda}\right)^2 E\left(e^2\right) + \lambda a^2 E\left(v\right)^2 \\ +\left(\frac{a\lambda}{1+\delta}\right)^2 k^2 + \left(\frac{a\lambda}{1+\delta+a^2\lambda}\right)^2 E\left(e^2\right) + E\left(v^2\right) \end{array}\right] \\
EV^{disc} &= \frac{1}{2}\left[\left(1 + \lambda\left(\frac{a}{1+\delta}\right)^2\right)\lambda k^2 + \lambda\frac{(1+\delta)^2 + \lambda a^2}{\left(1+\delta+a^2\lambda\right)^2}E\left(e^2\right) + \left(1+\lambda a^2\right)E\left(v^2\right)\right]
\end{aligned}$$

The optimal extent of delegation sets $\delta$ to minimize this:

$$
\begin{aligned}
0 &= \frac{\partial EV^{disc}}{\partial \delta} \\
0 &= \frac{1}{2}\left[-2\lambda a^2\left(\frac{1}{1+\delta}\right)^3\right]\lambda k^2+\frac{1}{2}\left[2\frac{1+\delta}{\left(1+\delta+a^2\lambda\right)^2}-2\frac{(1+\delta)^2+\lambda a^2}{\left(1+\delta+a^2\lambda\right)^3}\right]\lambda E\left(e^2\right) \\
0 &= -\lambda^2 a^2\left(\frac{1}{1+\delta}\right)^3 k^2+\left[\frac{1+\delta}{\left(1+\delta+a^2\lambda\right)^2}-\frac{(1+\delta)^2+\lambda a^2}{\left(1+\delta+a^2\lambda\right)^3}\right]E\left(e^2\right) \\
0 &= -\lambda^2 a^2\left(\frac{1}{1+\delta}\right)^3 k^2+\frac{(1+\delta)\left(1+\delta+a^2\lambda\right)-(1+\delta)^2-\lambda a^2}{\left(1+\delta+a^2\lambda\right)^3}E\left(e^2\right) \\
0 &= -\lambda^2 a^2\left(\frac{1}{1+\delta}\right)^3 k^2+\frac{\delta\lambda a^2}{\left(1+\delta+a^2\lambda\right)^3}E\left(e^2\right) \\
\lambda\left(\frac{1}{1+\delta}\right)^3 k^2 &= \frac{\delta}{\left(1+\delta+a^2\lambda\right)^3}E\left(e^2\right) \\
k^2 &= \frac{\delta}{\lambda}\left(\frac{1+\delta}{1+\delta+a^2\lambda}\right)^3 E\left(e^2\right) \\
\delta &= \lambda\frac{k^2}{E\left(e^2\right)}\left(\frac{1+\delta+a^2\lambda}{1+\delta}\right)^3 = g(\delta)
\end{aligned}
$$

The left-hand side is clearly increasing in $\delta$, being equal to zero when $\delta = 0$ and going to infinity when $\delta$ goes to infinity. The right-hand side is decreasing in $\delta$, positive when $\delta = 0$ and going to a finite value when $\delta$ goes to infinity. The equation therefore implies a unique solution for a value of $\delta$ that is positive.

$$
\begin{aligned}
\frac{\partial g(\delta)}{\partial \delta} &= 3\lambda\frac{k^2}{E\left(e^2\right)}\left(\frac{1+\delta+a^2\lambda}{1+\delta}\right)^2\frac{(1+\delta)-\left(1+\delta+a^2\lambda\right)}{(1+\delta)^2} \\
&= 3\lambda\frac{k^2}{E\left(e^2\right)}\left(\frac{1+\delta+a^2\lambda}{1+\delta}\right)^2\frac{-a^2\lambda}{(1+\delta)^2} < 0 \\
g(0) &= \lambda\frac{k^2}{E\left(e^2\right)}\left(1+a^2\lambda\right)^3 > 0 \\
\lim_{g\to\infty} g(\delta) &= \lambda\frac{k^2}{E\left(e^2\right)}
\end{aligned}
$$