Demystifying DSGE Models

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Lab Session (Part 1)

- To start, use the ***DynareHandout2025.mod*** programme to reproduce the results shown in the Handout
- If you are wondering where this came from, the basis is the following assumed functions

$$E_t\left[\sum_{i=0}^{\infty}\beta^i\left(\frac{C_{t+i}^{1-\eta}}{1-\eta}-\zeta N_{t+i}\right)\right]$$

$$Y_t = C_t + I_t = A_t K_t^{\alpha} N_t^{1-\alpha}$$

$$K_{t+1} = I_t + (1-\delta)K_t$$

- They yield the ...

- ***Lagrangian*** (after substituting in law of motion for capital):

$Y_{t+i} - C_{t+i} - I_{t+i}$

$$\mathcal{L} = E_t \left[\sum_{i=0}^{\infty} \beta^i \left\{ \boxed{\frac{C_{t+i}^{1-\eta}}{1-\eta} - \zeta N_{t+i}} + \lambda_t \left(A_{t+i} K_{t+i}^{\alpha} N_{t+i}^{1-\alpha} - C_{t+i} - K_{t+i+1} + (1-\delta) K_{t+i} \right) \right\} \right]$$

- ***Trick***: following is the ***only*** part of this infinite sum which contains variables at times {t} or {t+1}

$$\frac{C_t^{1-\eta}}{1-\eta} - \zeta N_t + \lambda_t \left(A_t K_t^{\alpha} N_t^{1-\alpha} - C_t - K_{t+1} + (1-\delta) K_t \right)$$
$$+\beta E_t \lambda_{t+1} \left(A_{t+1} K_{t+1}^{\alpha} N_{t+1}^{1-\alpha} - C_{t+1} - K_{t+2} + (1-\delta) K_{t+1} \right)$$

- Now do the math to get the FOCs:

$$\frac{C_t^{1-\eta}}{1-\eta} - \zeta N_t + \lambda_t \left(A_t K_t^{\alpha} N_t^{1-\alpha} - C_t - K_{t+1} + (1-\delta) K_t \right)$$
$$+\beta E_t \lambda_{t+1} \left(A_{t+1} K_{t+1}^{\alpha} N_{t+1}^{1-\alpha} - C_{t+1} - K_{t+2} + (1-\delta) K_{t+1} \right)$$

$$\frac{\partial \mathcal{L}}{\partial C_t} : \quad C_t^{-\eta} - \lambda_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} : \beta E_t \left(\alpha \frac{Y_{t+1}}{K_{t+1}} \lambda_{t+1} + (1-\delta) \lambda_{t+1} \right) - \lambda_t = 0$$

$$\frac{\partial \mathcal{L}}{\partial N_t} : \quad -\zeta + (1-\alpha) \lambda_t \frac{Y_t}{N_t} = 0 \longrightarrow \frac{Y_t}{N_t} = \frac{\zeta}{(1-\alpha)\lambda_t}$$

- Defining ***gross interest rate*** as marginal value of an additional unit of capital, we have

$$R_{t+1} = \alpha \frac{Y_{t+1}}{K_{t+1}} + (1-\delta)$$

- Hence (from $\frac{\partial \mathcal{L}}{\partial K_{t+1}}$)

$$\lambda_t = \beta E_t \left(\lambda_{t+1} R_{t+1} \right)$$

$$C_t^{-\eta} = \beta E_t \left(C_{t+1}^{-\eta} R_{t+1} \right)$$

$$\longrightarrow \beta E_t \left(\left(\frac{C_t}{C_{t+1}} \right)^{\eta} R_{t+1} \right) = 1$$

- ➔ ***Nonlinear*** model:

$$Y_t = C_t + I_t$$

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$$

$$K_t = I_t + (1-\delta)K_t$$

$$1 = \beta E_t \left(\left(\frac{C_t}{C_{t+1}} \right)^{\eta} R_{t+1} \right)$$

$$\frac{Y_t}{N_t} = \frac{\zeta}{1-\alpha} C_t^{\eta}$$

$$R_t = \alpha \frac{Y_t}{K_t} + 1 - \delta$$

A_t is the TFP process, assumed as usual to be AR(1)

- which, when ***log-linearised***, ➔ model as presented in Handout

```
% Model used in Dynare Handout
% Endogenous variables
var
    y       ${y}$               (long_name='Output')
    c       ${c}$               (long_name='Consumption')
    i       ${i}$               (long_name='Investment')
    k       ${k}$               (long_name='Capital stock')
    n       ${n}$               (long_name='Hours worked')
    r       ${r}$               (long_name='Nominal interest rate')
    a       ${\varepsilon_a}$   (long_name='Productivity process')
;
% Changing the timing convention
predetermined_variables k;
% Shocks
varexo e;
```

```
% Parameters of the model
parameters
alpha    ${\alpha}$         (long_name='capital share')
rho      ${\rho}$           (long_name='persistence of TFP shock')
beta     ${\beta}$          (long_name='discount factor')
delta    ${\delta}$         (long_name='depreciation rate')
eta      ${\eta}$           (long_name='risk aversion')
Yss_Kss ${\frac{Yss}{Kss}}$ (long_name='SS output-capital ratio')
Iss_Kss  ${\frac{Iss}{Kss}}$ (long_name='SS investment-capital ratio')
Iss_Yss  ${\frac{Iss}{Yss}}$ (long_name='SS investment-output ratio')
Css_Yss ${\frac{Css}{Yss}}$ (long_name='SS consumption-output ratio')
Rss      ${Rss}$            (long_name='SS return on capital')
;
```

```
% Calibration
alpha = 0.33;
rho   = 0.95;
beta  = 0.99;
delta = .015;
eta   = 1;
Rss = 1/beta;
Yss_Kss = ((1/beta)+delta-1)/alpha;
Iss_Kss = delta;
Iss_Yss = (Iss_Kss)/(Yss_Kss);
Css_Yss = 1 - (Iss_Yss);
```

```
% Dynamic Equations
model;
[name='Resource Constraint']
            y = (Css_Yss)*c + (Iss_Yss)*i;
[name='Production Function']
   y = a + alpha*k + (1 - alpha)*n;
[name='Law of motion of capital']
            k(+1) = (Iss_Kss)*i + (1 - delta)*k;
[name='Labour FOC']
   n = y - eta*c;
[name='Euler equation']
   c = c(+1) - (1/eta)*r(+1);
[name='Real interest rate/firm FOC capital']
            r = (alpha*(Yss_Kss)/Rss)*(y - k);
[name='Exogenous TFP process']
   a = rho*a(-1) + e;
end;
```

```
% Steady-State
resid;
steady;
% Check Blanchard-Kahn conditions
check;
% Shocks of the model
% In Dynare, these shocks are Normal with zero mean and standard error given by user here
shocks;
var e;
stderr 0.1; % ==> variance = .01 (ie, 1% in a log-linearised model)
end;
% Stochastic Simulation
stoch_simul(Tex,irf=100,order=1) y, c, i, k, n, r, a;
```

Dynare will print the ***residuals*** of the static equations, the ***steady-state*** values for all the variables, the results of the ***Blanchard-Kahn*** check, and a ***summary*** of the model variables:

Residuals of the static equations:

Equation number 1 : 0 : Resource Constraint
Equation number 2 : 0 : Production Function
Equation number 3 : 0 : Law of motion of capital
Equation number 4 : 0 : Labour FOC
Equation number 5 : 0 : Euler equation
Equation number 6 : 0 : real interest rate/firm FOC capital
Equation number 7 : 0 : exogenous TFP process

The equation residuals are all **0**, so the model is internally consistent

STEADY-STATE RESULTS:

y	**0**
c	**0**
i	**0**
k	**0**
n	**0**
r	**0**
a	**0**

In this model, all variables are expressed as deviations from their steady-state values, so the steady-states of the ***transformed*** (***model)*** variables are by definition ***zero***

```
EIGENVALUES:
         Modulus                  Real            Imaginary

            0.95                  0.95                    0
           0.952                 0.952                    0
           1.061                 1.061                    0
       1.132e+16            -1.132e+16                    0

There are 2 eigenvalue(s) larger than 1 in modulus
for 2 forward-looking variable(s)

The rank condition is verified.

MODEL SUMMARY

  Number of variables:          7
  Number of stochastic shocks:  1
  Number of state variables:    2
  Number of jumpers:            2
  Number of static variables:   3
```

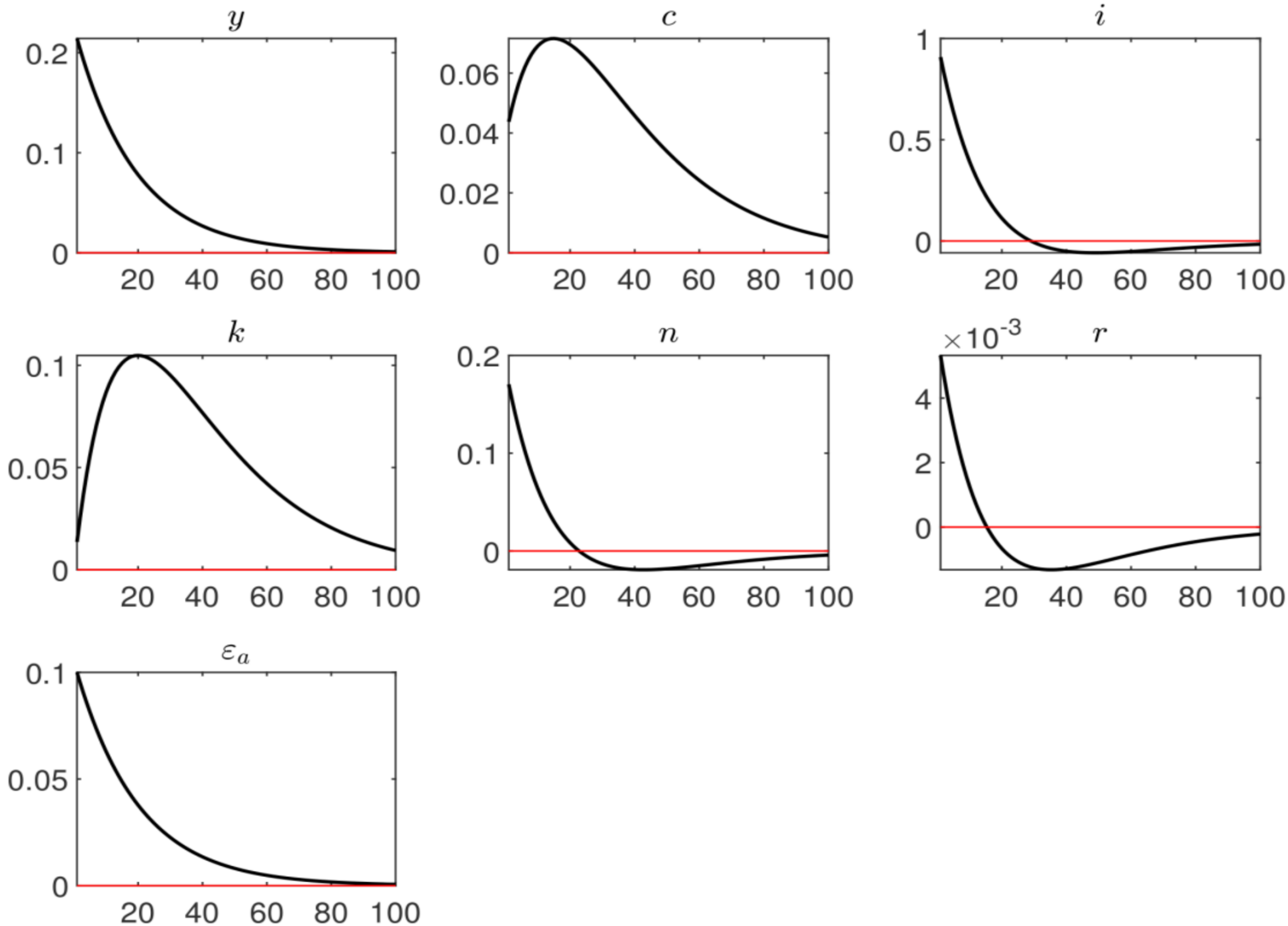
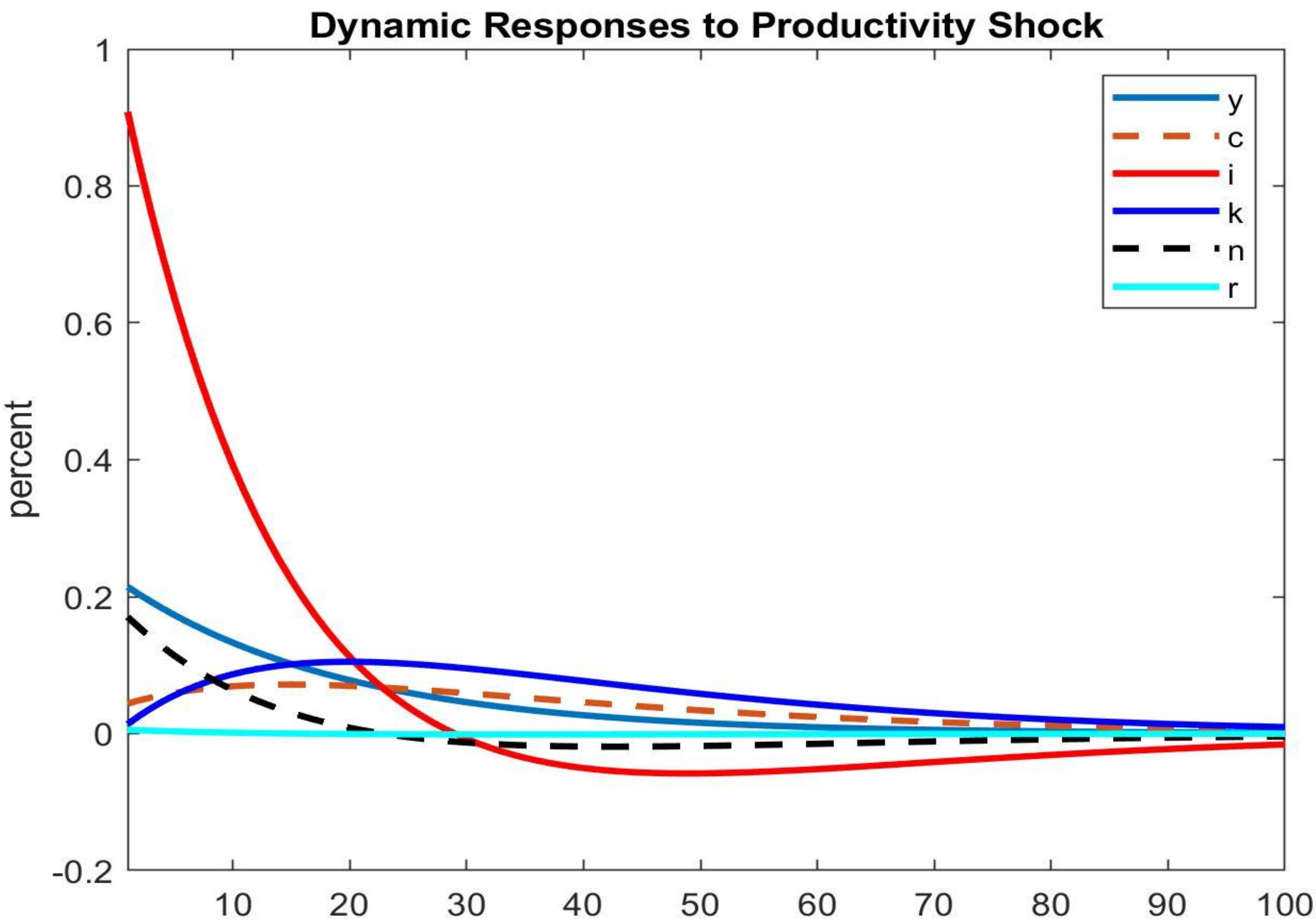


Figure 1: Impulse response functions (orthogonalized shock to e).



Dynamic Responses to Productivity Shock
percent
1
0.8
0.6
0.4
0.2
0
-0.2
10
20
30
40
50
60
70
80
90
100
y
c
i
k
n
r

Lab Session (Part 2)

- Let us set up a ***simple RBC*** with a slight twist:
  - add ***oil*** as a factor of production
- As usual, economy is inhabited by many infinitely lived agents who optimize their decisions to maximise a given utility function
- Each agent derives utility from two elements: ***consumption*** (C_t) and ***leisure*** (L_t) ➔

$$\max_{C_t, L_t} E_t \sum_{t=0}^{\infty} \beta^t \left[\boxed{\log(C_t)} + \gamma \boxed{\log(L_t)} \right], \quad \beta > 0$$

- where utility function is here simplified to ***double log***

- Households have an endowment of time, each distributing that endowment between hours of ***work*** (H_t) and ***leisure*** (L_t): $H_t + L_t = 1$
- ➔ Rewrite utility maximization problem to use ***hours of work*** instead of leisure:

$$\max_{C_t, H_t} E_t \sum_{t=0}^{\infty} \beta^t \left[\log(C_t) + \gamma \boxed{\log(1 - H_t)}\right]$$

- ***Budget constraint:*** Households own economy's capital, so their ***income*** comes from ***wages*** (W_t) received for their working hours (H_t) and from ***returns*** (R_t) on capital (K_t) ➔ $W_t H_t + R_t K_t$

- Households spend their income in ***consumption*** (C_t) and ***saving*** (S_t), the saving being invested in capital ➔

$$C_t + S_t = W_t H_t + R_t K_t$$

- To simplify analysis, assume households transform ***savings*** into ***investments*** instantly, without any costs ➔ $S_t = I_t$
- Law of motion of capital is (as usual)

$$K_{t+1} = I_t + (1 - \delta)K_t$$

- ➔ Household dynamic ***maximisation*** problem:

$$\max_{C_t, H_t} E_t \sum_{t=0}^{\infty} \beta^t \left[\log(C_t) + \gamma \log(1 - H_t)\right]$$

$$\text{s.t. } C_t + S_t = W_t H_t + R_t K_t$$

- ➔ ***Lagrangian***:

$$\max_{C_t, H_t, K_{t+1}} \mathcal{L} = E_t \sum_{t=0}^{\infty} \beta^t \{[\log(C_t) + \gamma \log(1 - H_t)]$$

$$-\lambda_t [C_t + K_{t+1} - W_t H_t - (1 + R_t - \delta) K_t]\}$$

Recall, $S_t = I_t$ and (from law of motion for capital) $I_t = K_{t+1} - (1 - \delta)K_t$

$$\max_{C_t, H_t, K_{t+1}} \mathcal{L} = E_t \sum_{t=0}^{\infty} \beta^t \{[\log(C_t) + \gamma \log(1 - H_t)]$$
$$-\lambda_t [C_t + K_{t+1} - W_t H_t - (1 + R_t - \delta) K_t]\}$$

- ➔ first-order conditions:

$$\frac{\partial \mathcal{L}}{\partial C_t} = \beta^t \left[\frac{1}{C_t} - \lambda_t \right] = 0$$
$$\frac{\partial \mathcal{L}}{\partial H_t} = \beta^t \left[-\frac{\gamma}{1 - H_t} + \lambda_t W_t \right] = 0$$
$$\frac{\partial \mathcal{L}}{\partial K_{t+1}} = \beta^{t+1} \lambda_{t+1} (1 + R_{t+1} - \delta) - \beta^t \lambda_t = 0$$

- Solving for ***labour supply*** ➔

$$H_t = \frac{W_t - \gamma C_t}{W_t}$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial K_{t+1}} = \beta^{t+1} \lambda_{t+1} (1 + R_{t+1} - \delta) - \beta^t \lambda_t = 0}$$

From FOC-1, $\lambda_t = 1/C_t$

Substituting this into FOC-3 ➔

$\beta\beta^t (1/C_{t+1}) (1 + R_{t+1} - \delta) - \beta^t (1/C_t) = 0$

Factor out and cancel β^t ➔

$(1/C_t) = \beta (1/C_{t+1}) (1 + R_{t+1} - \delta)$ ➔

$C_{t+1}/C_t = \beta (1 + R_{t+1} - \delta)$ **QED**

- Solving for ***Euler Equation*** ➔

$$\frac{C_{t+1}}{C_t} = \beta [R_{t+1} + 1 - \delta]$$

- What about economy's ***firms***?
- Their ***production function*** is now (with ***oil O_t***)

$$Y_t = A_t K_t^{\alpha} H_t^{\theta} O_t^{1-\alpha-\theta}$$

- Firms will ***maximize their profits*** (= output – costs) where costs involve
  - ***return*** they pay on capital (R_t)
  - ***wages*** (W_t) they pay to workers
  - price of ***oil*** (Q_t) they pay (per barrel of oil) to oil producers
- Thus, their costs are:

$$W_t H_t + R_t K_t + Q_t O_t$$

- ➔ Firm ***maximisation*** problem:

$$\max \Pi = A_t K_t^{\alpha} H_t^{\theta} O_t^{1-\alpha-\theta} - W_t H_t - R_t K_t - Q_t O_t$$

$$\boxed{\max \Pi = A_t K_t^{\alpha} H_t^{\theta} O_t^{1-\alpha-\theta} - W_t H_t - R_t K_t - Q_t O_t}$$

- ➔ first order conditions of firm's maximisation problem:

$$\frac{\partial \Pi}{\partial K_t} = K_t^{-1} \alpha \overset{Y_t}{\overbrace{\boxed{A_t K_t^{\alpha} H_t^{\theta} O_t^{1-\alpha-\theta}}}} - R_t = 0$$

$$\frac{\partial \Pi}{\partial H_t} = H_t^{-1} \theta \boxed{A_t K_t^{\alpha} H_t^{\theta} O_t^{1-\alpha-\theta}} - W_t = 0$$

$$\frac{\partial \Pi}{\partial O_t} = O_t^{-1} (1 - \alpha - \theta) \boxed{A_t K_t^{\alpha} H_t^{\theta} O_t^{1-\alpha-\theta}} - Q_t = 0$$

- Solving FOCs for prices of factors of production ➔

$$R_t = \alpha \frac{Y_t}{K_t}$$

$$W_t = \theta \frac{Y_t}{H_t}$$

$$Q_t = (1 - \alpha - \theta) \frac{Y_t}{O_t}$$

- What about ***oil***? In principle, it should also have a ***production function***
- Here, for simplicity, ***assume exogenous***: $O_t = mZ_t$

$$\ln(Z_t) = \rho_Z \ln(Z_{t-1}) + u_t, \quad u_t \sim N\left(0, \sigma^2\right)$$

- ➔ model:

$$
\begin{aligned}
Y_t &= C_t + I_t \\
K_{t+1} &= I_t + (1-\delta)K_t \\
\frac{C_{t+1}}{C_t} &= \beta\left[R_{t+1} + 1 - \delta\right] \\
Y_t &= A_t K_t^{\alpha} H_t^{\theta} O_t^{1-\alpha-\theta} \\
H_t &= \frac{W_t - \gamma C_t}{W_t} \\
R_t &= \alpha \frac{Y_t}{K_t} \\
W_t &= \theta \frac{Y_t}{H_t} \\
Q_t &= (1-\alpha-\theta)\frac{Y_t}{O_t} \\
O_t &= m Z_t \\
\ln(A_t) &= \rho_A \ln(A_{t-1}) + e_t \\
\ln(Z_t) &= \rho_Z \ln(Z_{t-1}) + u_t
\end{aligned}
$$

- ***Calibration*** implies assigning values to all parameters corresponding to actual data in our model economy ➔ needed to solve model
- Set of parameters is $\Omega = \{\alpha, \theta, \beta, \delta, \gamma, m, \rho_A, \rho_Z\}$
- Usually, these values can be sourced or derived from statistical sources or other research papers where authors have already estimated them
- In our example, we will calibrate for the U.S.

- Where do these values come from?
- α: represents ***capital share*** in Cobb-Douglas production function, calculated from National Accounts database.
- In most developed economies, the value ranges from 0.25 to 0.35; here, we set α to be 0.32
- θ: represents ***labour share*** in Cobb-Douglas production function, derived from National Accounts database
- Without oil, $\theta = 1 - \alpha$; here we set $\theta = 0.64$

- *$(1 - \alpha - \theta)$*: represents ***oil (or energy) share*** in Cobb-Douglas production function; here 0.06
- ***β***: ***discount factor*** indicates how much households discount the future
- If the value of β were one, agents would value future consumption/leisure as much as present consumption/leisure
- For quarterly data, value often used in literature is 0.96-0.99; here we use 0.97
- ***δ***: represents ***depreciation rate*** of capital
- Value generally assigned to quarterly data ranges from 0.0125 - 0.06; here, we set δ to 0.06

- ***γ***: represents ***leisure share*** parameter
- Households typically assign twenty percent of their time to work and eighty percent to "leisure"
- Here, γ is set to 0.4 to target hours worked = 0.2 in a steady state
- ***m***: represents oil endowment, or ***oil reserves***, here set to 0.05
- $\boldsymbol{\rho_A}$ and $\boldsymbol{\rho_Z}$: represent the ***persistence of productivity shock*** in the firms' production function and oil producers respectively
- Values are set to 0.95 as commonly in literature

- ***Dynare*** model equations are congruent to those of model shown earlier:

```
%Dynamic Equations
model;
y = c + i;
k(+1) = i +(1-delta)*k;
c(+1)/c = beta*(r(+1)+ (1 - delta));
y = a*(k^alpha)*(h^theta)*o^(1-alpha-theta);
h = (w - gamma*c)/w;
r = alpha*(y/k);
w = (theta)*(y/h);
q = (1-alpha-theta)*(y/o);
o = m*z;
log(a)= rhoa*log(a(-1))+ e;
log(z)= rhoz*log(z(-1))+ u;
end;
```

$$Y_t = C_t + I_t$$
$$K_{t+1} = I_t + (1-\delta)K_t$$
$$\frac{C_{t+1}}{C_t} = \beta\left[R_{t+1} + 1 - \delta\right]$$
$$Y_t = A_t K_t^{\alpha} H_t^{\theta} O_t^{1-\alpha-\theta}$$
$$H_t = \frac{W_t - \gamma C_t}{W_t}$$
$$R_t = \alpha \frac{Y_t}{K_t}$$
$$W_t = \theta \frac{Y_t}{H_t}$$
$$Q_t = (1-\alpha-\theta)\frac{Y_t}{O_t}$$
$$O_t = m Z_t$$
$$\ln(A_t) = \rho_A \ln(A_{t-1}) + e_t$$
$$\ln(Z_t) = \rho_Z \ln(Z_{t-1}) + u_t$$

- ***Initial*** part of **Dynare** mod-file:

```
%Endogenous Variables
var y, c, i, k, h, w, r, a, z, o, q;

%Changing the timing convention
predetermined_variables k;

%Exogenous Variables
varexo e u;

%Parameters of the model
parameters alpha, beta, theta, delta ,gamma, m, rhoa, rhoz;
alpha = 0.32;
beta = 0.97;
theta = 0.64;
delta = 0.06;
gamma = 0.4;
m = 0.05;
rhoa = 0.95;
rhoz = 0.95;
```

- ***Final*** part of Dynare mod-file:

```
%Steady-State
steady;

%Check Blanchard-Kahn conditions
check;

%Shocks of the model
shocks;
var e; stderr 0.01;
var u; stderr 0.01;
end;

%Stochastic Simulation
stoch_simul y c i k h r w o q;
```

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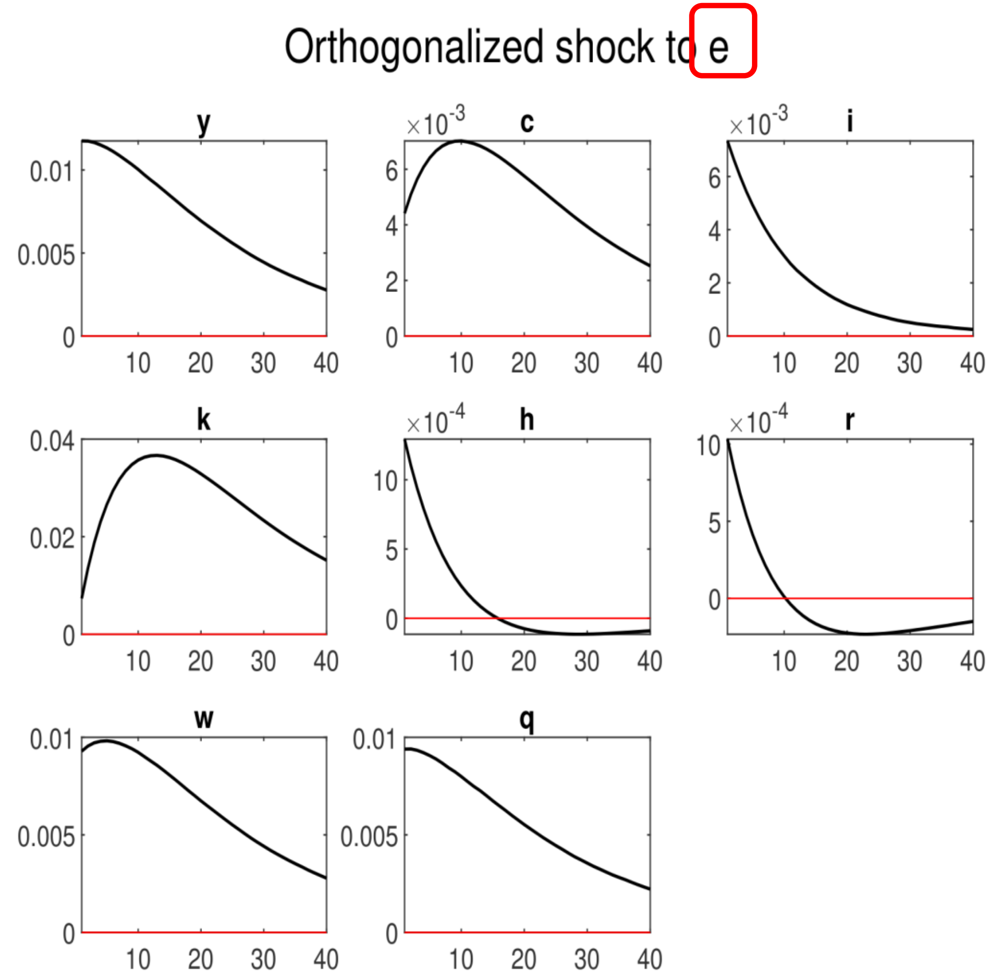
- ***Why*** does model fail?
- Notice error message:
- “Impossible to find the steady state (the sum of square residuals of the static equations is 3.1881). Either the model doesn't have a steady state, there are an infinity of steady states, or the guess values are too far from the solution”
- ➔ need some initial (“guess”) values

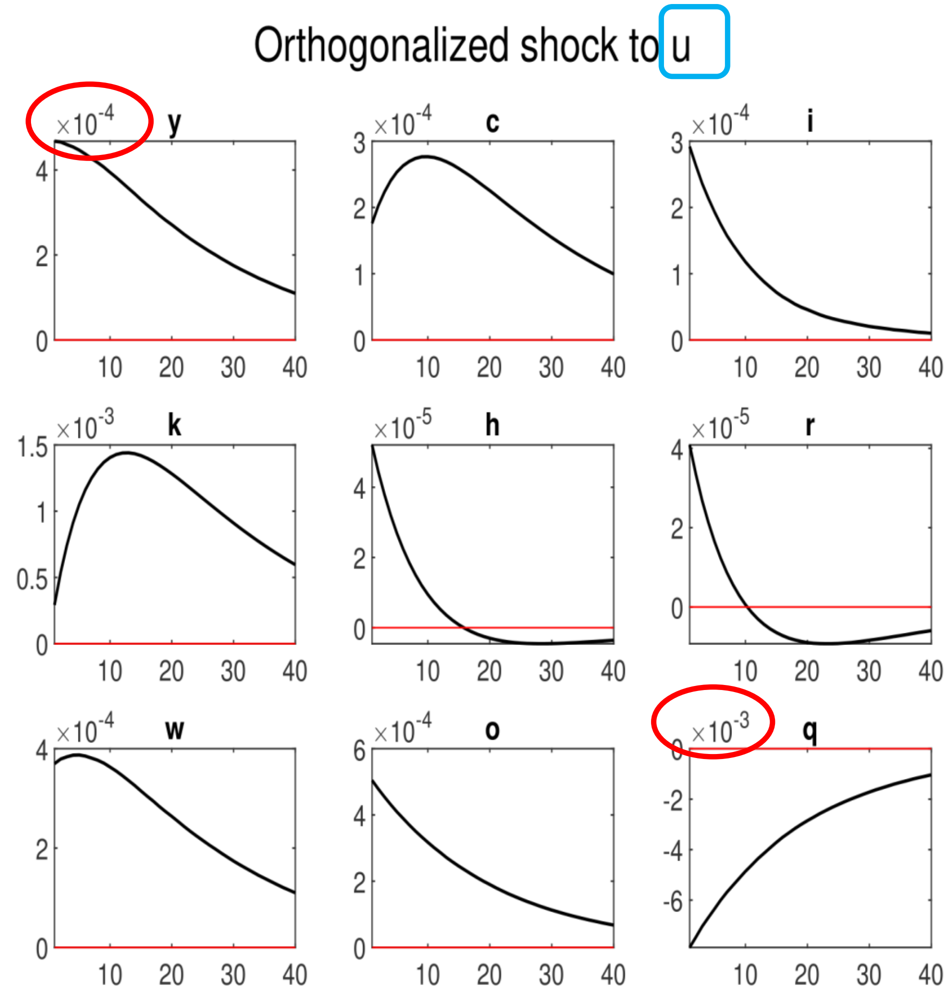
- ***Initial values*** part of **Dynare** mod-file:

```
%Initial Values
initval;
y = 1;
c = 0.8;
i = 0.2;
k = 3.5;
h = 0.2;
r = 0.05;
w = 1.3;
a = 1;
z = 1;
e = 0;
u = 0;
o = z*m;
q = (1-alpha-theta)*y/o;
end;
```

- ***And now model works***
- Look at IRFs
- First shock is a ***demand shock*** (e_t) in productivity of firms
- As productivity of firms increases, they will initially ***demand more inputs*** (oil, labour and capital) to produce more output
- As usual, when there is an increase in demand for a product, there is an ***increase in its price***, as seen for r, w and q
- Note that as oil is ***exogenous***, it does not appear



- In second IRF, we see how model's endogenous variables respond to a (positive) ***shock in oil supply*** (u_t)
- As expected, an ***increase in supply*** of oil results in an initial ***decrease in price*** of oil (q_t)
- It also encourages ***greater output*** and thus consumption and investment, but ***scale*** is a factor of ten ***smaller*** than for oil price



- ***Variance decomposition*** shows how much of the model's variation is due to each specific shock
- In our example, we can see that ***demand shocks*** explain around 70% of variation in oil prices, while ***supply shocks*** explain almost 30% of variation in oil prices
- Note that ***oil supply*** (o_t) variation due to demand shocks is 0, because oil supply is ***exogenous*** and consequently, oil supply will only respond to changes in ***productivity*** of oil production

VARIANCE DECOMPOSITION

	e	u
y	99.84	0.16
...		
o	0.00	100.00
q	70.83	29.17