Macroeconomics A; EI056

Short problems

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1 Intertemporal budget constraint

1.1 Household's flow constraint (a)

Question: At time t the household consumes c_t, saves in capital, $k_{t+1} - k_t$ (saving is the change in an asset holding between the beginning of a period and the end of the period (also beginning of next period)) and in government bonds $b_{t+1} - b_t$. The income is the wage w_t, net of taxes τ_t, plus the return r_t on assets (capital and bonds):

$$w_t - \tau_t + r_t (k_t + b_t) = c_t + (k_{t+1} + b_{t+1}) - (k_t + b_t)$$

Show that:

$$(k_t + b_t) = R_{t,t} [c_t - (w_t - \tau_t)] + R_{t,t+1} [c_{t+1} - (w_{t+1} - \tau_{t+1})] + R_{t,t+1} (k_{t+2} + b_{t+2})$$

where:

$$R_{t,t} = \frac{1}{1 + r_t} \quad , \quad R_{t,t+1} = \frac{1}{1 + r_t} \frac{1}{1 + r_{t+1}}$$

Answer: We start by putting current asset on the left-hand side:

$$\begin{aligned}
w_t - \tau_t + r_t (k_t + b_t) &= c_t + (k_{t+1} + b_{t+1}) - (k_t + b_t) \\
(1 + r_t)(k_t + b_t) &= c_t - (w_t - \tau_t) + (k_{t+1} + b_{t+1}) \\
(k_t + b_t) &= \frac{1}{(1 + r_t)} [c_t - (w_t - \tau_t)] + \frac{1}{(1 + r_t)} (k_{t+1} + b_{t+1})
\end{aligned}$$

Do the same for $k_{t+1} + b_{t+1}$, and plug it in:

$$\begin{aligned}
(k_t + b_t) &= \frac{1}{(1 + r_t)} [c_t - (w_t - \tau_t)] + \frac{1}{(1 + r_t)} (k_{t+1} + b_{t+1}) \\
(k_t + b_t) &= \frac{1}{(1 + r_t)} [c_t - (w_t - \tau_t)] + \frac{1}{(1 + r_t)} \left[\frac{1}{(1 + r_{t+1})} [c_{t+1} - (w_{t+1} - \tau_{t+1})] + \frac{1}{(1 + r_{t+1})} (k_{t+2} + b_{t+2}) \right] \\
(k_t + b_t) &= \frac{1}{(1 + r_t)} [c_t - (w_t - \tau_t)] + \frac{1}{(1 + r_t)} \left[\frac{1}{(1 + r_{t+1})} [c_{t+1} - (w_{t+1} - \tau_{t+1})] \right] \\
&\quad + \frac{1}{(1 + r_t)} \frac{1}{(1 + r_{t+1})} (k_{t+2} + b_{t+2}) \\
(k_t + b_t) &= R_{t,t} [c_t - (w_t - \tau_t)] + R_{t,t+1} [c_{t+1} - (w_{t+1} - \tau_{t+1})] + R_{t,t+1} (k_{t+2} + b_{t+2})
\end{aligned}$$

1.2 Household's flow constraint (b)

Question: Show that:

$$(k_t + b_t) = \sum_{s=0}^{\infty} R_{t,t+s} \left[c_{t+s} - (w_{t+s} - \tau_{t+s})\right] + \lim_{k\to\infty} R_{t,t+k} \left(k_{t+k+1} + b_{t+k+1}\right)$$

where $R_{t,t+s} = \frac{1}{1+r_t} \frac{1}{1+r_{t+1}} ... \frac{1}{1+r_{t+s}} = \prod_{i=0}^{s} \frac{1}{1+r_{t+i}}$.

We assume that the transversality condition holds: $\lim_{k\to\infty} R_{t,t+k} \left(k_{t+k+1} + b_{t+k+1}\right) = 0$.

Do changes in tax matter for consumption? Does government debt b_t matter for consumption?

Answer: We simply repeat the iterations:

$$\begin{aligned}
(k_t + b_t) &= R_{t,t} \left[c_t - (w_t - \tau_t)\right] + R_{t,t+1} \left[c_{t+1} - (w_{t+1} - \tau_{t+1})\right] + R_{t,t+1} \left(k_{t+2} + b_{t+2}\right) \\
(k_t + b_t) &= R_{t,t} \left[c_t - (w_t - \tau_t)\right] + R_{t,t+1} \left[c_{t+1} - (w_{t+1} - \tau_{t+1})\right] \\
&\quad + R_{t,t+1} \frac{1}{(1+r_{t+2})} \left[c_{t+2} - (w_{t+2} - \tau_{t+2})\right] + R_{t,t+1} \frac{1}{(1+r_{t+2})} \left(k_{t+3} + b_{t+3}\right) \\
(k_t + b_t) &= R_{t,t} \left[c_t - (w_t - \tau_t)\right] + R_{t,t+1} \left[c_{t+1} - (w_{t+1} - \tau_{t+1})\right] + R_{t,t+2} \left[c_{t+3} - (w_{t+3} - \tau_{t+3})\right] + R_{t,t+2} \left(k_{t+3} + b_{t+3}\right) \\
(k_t + b_t) &= \sum_{s=0}^{t+2} R_{t,t+s} \left[c_{t+s} - (w_{t+s} - \tau_{t+s})\right] + R_{t,t+2} \left(k_{t+3} + b_{t+3}\right) \\
(k_t + b_t) &= \sum_{s=0}^{t+3} R_{t,t+s} \left[c_{t+s} - (w_{t+s} - \tau_{t+s})\right] + R_{t,t+3} \left(k_{t+4} + b_{t+4}\right) \\
(k_t + b_t) &= \sum_{s=0}^{t+k} R_{t,t+s} \left[c_{t+s} - (w_{t+s} - \tau_{t+s})\right] + R_{t,t+k} \left(k_{t+k+1} + b_{t+k+1}\right)
\end{aligned}$$

We take the limit of k going to infinity:

$$(k_t + b_t) = \sum_{s=0}^{\infty} R_{t,t+s} \left[c_{t+s} - (w_{t+s} - \tau_{t+s})\right] + \lim_{k\to\infty} R_{t,t+k} \left(k_{t+k+1} + b_{t+k+1}\right)$$

Changes in taxes matter to the extent that they affect the net present value of taxes:

$$\sum_{s=0}^{\infty} R_{t,t+s} \tau_{t+s}$$

Government debt matter. A higher value of b_t is associated - everything else equal - to higher consumption. This is because it represents a higher wealth of the household.

1.3 Government's constraint

Question: In period t the government spends g_t and pays the interest on its debt. This is financed by taxes and borrowing:

$$\tau_t + b_{t+1} - b_t = g_t + r_t b_t$$

Show that:

$$b_t = \sum_{s=0}^{\infty} R_{t,t+s} \left(\tau_{t+s} - g_{t+s}\right)$$

Answer: We follow the same steps as for the household:

$$
\begin{aligned}
b_t &= \frac{1}{(1+r_t)}(\tau_t - g_t) + \frac{1}{(1+r_t)} b_{t+1} \\
b_t &= \frac{1}{(1+r_t)}(\tau_t - g_t) + \frac{1}{(1+r_t)}\frac{1}{(1+r_{t+1})}(\tau_{t+1} - g_{t+1}) + \frac{1}{(1+r_t)}\frac{1}{(1+r_{t+1})} b_{t+2} \\
b_t &= R_{t,t}(\tau_t - g_t) + R_{t,t+1}(\tau_{t+1} - g_{t+1}) + R_{t,t+1} b_{t+2}
\end{aligned}
$$

Iterating forward we get:

$$
\begin{aligned}
b_t &= R_{t,t}(\tau_t - g_t) + R_{t,t+1}(\tau_{t+1} - g_{t+1}) + R_{t,t+1} b_{t+2} \\
b_t &= R_{t,t}(\tau_t - g_t) + R_{t,t+1}(\tau_{t+1} - g_{t+1}) + R_{t,t+1}(\tau_{t+2} - g_{t+2}) + R_{t,t+2} b_{t+3} \\
b_t &= \sum_{s=0}^{t+2} R_{t,t+s}(\tau_{t+s} - g_{t+s}) + R_{t,t+2} b_{t+3} \\
b_t &= \sum_{s=0}^{t+3} R_{t,t+s}(\tau_{t+s} - g_{t+s}) + R_{t,t+3} b_{t+4} \\
b_t &= \sum_{s=0}^{t+k} R_{t,t+s}(\tau_{t+s} - g_{t+s}) + R_{t,t+k} b_{t+k+1}
\end{aligned}
$$

Taking the limit for k going to infinity:

$$
b_t = \sum_{s=0}^{\infty} R_{t,t+s}(\tau_{t+s} - g_{t+s}) + \lim_{k\to\infty} R_{t,t+k}(b_{t+k+1})
$$

We apply the transversality condition $\lim_{k\to\infty} R_{t,t+k}(b_{t+k+1}) = 0$ to get:

$$
b_t = \sum_{s=0}^{\infty} R_{t,t+s}(\tau_{t+s} - g_{t+s})
$$

1.4 Aggregate constraint

Question: Combining the steps so far, do taxes matter for consumption? Does government debt b_t matter for consumption?

Answer: We combine the budget constraints of the household and that of the government:

$$
\begin{aligned}
(k_t + b_t) &= \sum_{s=0}^{\infty} R_{t,t+s}[c_{t+s} - (w_{t+s} - \tau_{t+s})] \\
(k_t + b_t) &= \sum_{s=0}^{\infty} R_{t,t+s}[c_{t+s} - w_{t+s}] + \sum_{s=0}^{\infty} R_{t,t+s}\tau_{t+s} \\
(k_t + b_t) &= \sum_{s=0}^{\infty} R_{t,t+s}[c_{t+s} - w_{t+s}] + \sum_{s=0}^{\infty} R_{t,t+s}(\tau_{t+s} - g_{t+s} + g_{t+s}) \\
(k_t + b_t) &= \sum_{s=0}^{\infty} R_{t,t+s}[c_{t+s} - w_{t+s}] + \sum_{s=0}^{\infty} R_{t,t+s} g_{t+s} + \sum_{s=0}^{\infty} R_{t,t+s}(\tau_{t+s} - g_{t+s})
\end{aligned}
$$

$$
\begin{aligned}
(k_t + b_t) &= \sum_{s=0}^{\infty} R_{t,t+s}\left[c_{t+s} - (w_{t+s} - g_{t+s})\right] + \sum_{s=0}^{\infty} R_{t,t+s}\left(\tau_{t+s} - g_{t+s}\right) \\
(k_t + b_t) &= \sum_{s=0}^{\infty} R_{t,t+s}\left[c_{t+s} - (w_{t+s} - g_{t+s})\right] + b_t \\
k_t &= \sum_{s=0}^{\infty} R_{t,t+s}\left[c_{t+s} - (w_{t+s} - g_{t+s})\right]
\end{aligned}
$$

Consumption only reflects the net present value of government spending $\sum_{s=0}^{\infty} R_{t,t+s} g_{t+s}$. Taxes per se don't matter. A higher net present value of taxes means that the government can fund more spending. By doing so, it takes takes real resources away from the household (for given GDP). This is a real transfer that matters. But what truly matters is the government spending, not the taxes.

Initial government debt does not matter. The government's budget constraint shows that it repaid through the net present value of surpluses. A higher debt thus requires the government to increase taxes today or in the future. The fact that the debt is a wealth for the household is exactly offsets by the corresponding net present value of taxes.

2 From individual to average measures

2.1 Individual choice

Question: Take the model where each agent lives for two periods. Agent borne at time t maximizes utility over consumption when young at time t, $c_{1,t}$, and when old at time $t+1$, $c_{2,t+1}$:

$$
U_t = ln\left(c_{1,t}\right) + \frac{1}{1+\rho} ln\left(c_{2,t+1}\right)
$$

Consider that there is an endowment y_t when young, and that the agent can store goods with an exogenous return r. For simplicity, we abstract from taxes.

Show that:

$$
c_{2,t+1} = \frac{1+r}{1+\rho} c_{1,t}
$$

Show that optimal consumption is:

$$
\begin{aligned}
c_{1,t} &= \frac{1+\rho}{2+\rho} y_t \\
c_{2,t+1} &= \frac{1+r}{2+\rho} y_t \\
s_{1,t} &= \frac{1}{2+\rho} y_t
\end{aligned}
$$

Answer: In the solution, we consider the general utility:

$$
U_t = \frac{\left(c_{1,t}\right)^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{\left(c_{2,t+1}\right)^{1-\theta}}{1-\theta}
$$

The log utility corresponds to $\theta = 1$.

The agent born at time t faces the following budget constraints, with $s_{1,t}$ indicating the amount stored:

$$\begin{aligned} c_{1,t} + s_{1,t} &= y_t \\ c_{2,t+1} &= (1+r)\, s_{1,t} \end{aligned}$$

We combine the get the intertemporal constraint:

$$\begin{aligned} c_{1,t} + s_{1,t} &= y_t \\ c_{1,t} + \frac{c_{2,t+1}}{1+r} &= y_t \\ \frac{c_{2,t+1}}{1+r} &= y_t - c_{1,t} \\ c_{2,t+1} &= -(1+r)\, c_{1,t} + (1+r)\, y_t \end{aligned}$$

The utility is then written as:

$$U_t = \frac{(c_{1,t})^{1-\theta}}{1-\theta} + \frac{1}{1+\rho} \frac{(-(1+r)\, c_{1,t} + (1+r)\, y_t)^{1-\theta}}{1-\theta}$$

The first-order condition with respect to $c_{1,t}$ is:

$$\begin{aligned} 0 &= (c_{1,t})^{-\theta} - (c_{2,t+1})^{-\theta} \frac{1+r}{1+\rho} \\ (c_{1,t})^{-\theta} &= (c_{2,t+1})^{-\theta} \frac{1+r}{1+\rho} \\ (c_{2,t+1})^{\theta} &= (c_{1,t})^{\theta} \frac{1+r}{1+\rho} \\ c_{2,t+1} &= c_{1,t} \left(\frac{1+r}{1+\rho} \right)^{1/\theta} \end{aligned}$$

Using this in the budget constraint we get:

$$\begin{aligned} c_{2,t+1} &= -(1+r)\, c_{1,t} + (1+r)\, y_t \\ c_{1,t} \left(\frac{1+r}{1+\rho} \right)^{1/\theta} &= -(1+r)\, c_{1,t} + (1+r)\, y_t \\ c_{1,t} \left(\frac{1+r}{1+\rho} \right)^{1/\theta} \frac{1}{1+r} &= -c_{1,t} + y_t \\ c_{1,t} \left(1 + \left(\frac{1+r}{1+\rho} \right)^{1/\theta} \frac{1}{1+r} \right) &= y_t \\ c_{1,t} \left((1+\rho)^{\frac{1}{\theta}} + (1+r)^{\frac{1}{\theta}} \frac{1}{1+r} \right) &= (1+\rho)^{\frac{1}{\theta}}\, y_t \\ c_{1,t} \left((1+\rho)^{\frac{1}{\theta}} + (1+r)^{\frac{1-\theta}{\theta}} \right) &= (1+\rho)^{\frac{1}{\theta}}\, y_t \\ c_{1,t} &= \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}} + (1+r)^{\frac{1-\theta}{\theta}}} y_t \end{aligned}$$

The consumption when old is then:

$$c_{2,t+1} = c_{1,t} \left(\frac{1+r}{1+\rho} \right)^{1/\theta}$$

$$
\begin{aligned}
c_{2,t+1} &= \left(\frac{1+r}{1+\rho}\right)^{1/\theta} \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}} y_t \\
c_{2,t+1} &= \frac{(1+r)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}} y_t
\end{aligned}
$$

The storage is:

$$
\begin{aligned}
s_{1,t} &= y_t - c_{1,t} \\
s_{1,t} &= y_t - \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}} y_t \\
s_{1,t} &= \frac{(1+r)^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}} y_t
\end{aligned}
$$

2.2 Per capita consumption

Question: Consider size of successive generations grows at a rate $1+n$.
Show that per capita consumption at time t is:

$$
c_t^{\text{capita}} = \frac{1+n}{2+n} c_{1,t} + \frac{1}{2+n} c_{2,t}
$$

Answer: At time t there are N_t young agents and N_{t-1}old ones. The total consumption is then:

$$
\begin{aligned}
C_t &= N_t c_{1,t} + N_{t-1} c_{2,t} \\
C_t &= N_{t-1}\left((1+n)\, c_{1,t} + c_{2,t}\right)
\end{aligned}
$$

The total population size is $N_t + N_{t-1} = N_{t-1}\left((1+n)+1\right) = N_{t-1}(2+n)$. The per capita consumption is:

$$
\begin{aligned}
c_t^{\text{capita}} &= \frac{N_{t-1}\left((1+n)\, c_{1,t} + c_{2,t}\right)}{N_t + N_{t-1}} \\
c_t^{\text{capita}} &= \frac{N_{t-1}\left((1+n)\, c_{1,t} + c_{2,t}\right)}{N_{t-1}(2+n)} \\
c_t^{\text{capita}} &= \frac{1+n}{2+n} c_{1,t} + \frac{1}{2+n} c_{2,t}
\end{aligned}
$$

2.3 Consumption and saving ratio

Question: We assume that endowment grows at a rate g: $y_t = (1+g)\, y_{t-1}$. Show that the aggregate consumption to output ratio is:

$$
\frac{c_t^{\text{capita}}}{y_t^{\text{capita}}} = \frac{1+\rho}{2+\rho} + \frac{1}{(1+n)(1+g)} \frac{1+r}{2+\rho}
$$

and that the ratio of net saving to output is:

$$\frac{s_t^{\text{capita}}}{y_t^{\text{capita}}} = \frac{(1+n)(1+g)-1}{(1+n)(1+g)}\frac{1}{2+\rho}$$

where s_t^{capita} denotes net savings, $N_t s_{1,t} - N_{t-1}s_{1,t-1}$, scaled by endowment per capita y_t^{capita}.

How do the per-capit consomption/output and saving/output compare to the corresponding ratios for a young agent?

Answer: Ouptut per capita is:

$$\begin{aligned}
y_t^{\text{capita}} &= \frac{N_t y_t}{N_t + N_{t-1}} \\
y_t^{\text{capita}} &= \frac{N_{t-1}(1+n)y_t}{N_{t-1}(2+n)} \\
y_t^{\text{capita}} &= \frac{1+n}{2+n}y_t
\end{aligned}$$

Using the individual choice rules, consumption per capita is:

$$\begin{aligned}
c_t^{\text{capita}} &= \frac{1+n}{2+n}c_{1,t} + \frac{1}{2+n}c_{2,t} \\
c_t^{\text{capita}} &= \frac{1+n}{2+n}\frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}}y_t + \frac{1}{2+n}\frac{(1+r)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}}y_{t-1} \\
c_t^{\text{capita}} &= \frac{1+n}{2+n}\frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}}y_t + \frac{1}{2+n}\frac{(1+r)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}}\frac{1}{1+g}y_t \\
c_t^{\text{capita}} &= \left[(1+n)(1+g)\frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}} + \frac{(1+r)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}}\right]\frac{1}{2+n}\frac{1}{1+g}y_t
\end{aligned}$$

Savings per capita are, taking account of the dissaving by the old agents:

$$\begin{aligned}
s_t^{\text{capita}} &= \frac{N_t s_{1,t} - N_{t-1}s_{1,t-1}}{N_t + N_{t-1}} \\
s_t^{\text{capita}} &= \frac{N_t y_t - N_{t-1}y_{t-1}}{N_t + N_{t-1}}\frac{(1+r)^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}} \\
s_t^{\text{capita}} &= \frac{(1+n)y_t - y_{t-1}}{2+n}\frac{(1+r)^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}} \\
s_t^{\text{capita}} &= \frac{(1+n)y_t - \frac{1}{1+g}y_t}{2+n}\frac{(1+r)^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}} \\
s_t^{\text{capita}} &= \frac{(1+n)(1+g)-1}{2+n}\frac{(1+r)^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}}\frac{1}{1+g}y_t
\end{aligned}$$

We now take ratio to output, The ratio for consumption is:

$$\frac{c_t^{\text{capita}}}{y_t^{\text{capita}}} = \frac{\left[(1+n)(1+g)\frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}} + \frac{(1+r)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}}\right]\frac{1}{2+n}\frac{1}{1+g}y_t}{\frac{1+n}{2+n}y_t}$$

$$
\begin{aligned}
\frac{c_t^{\text{capita}}}{y_t^{\text{capita}}} &= \frac{\left[(1+n)(1+g)\frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}}+\frac{(1+r)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}}\right]\frac{1}{1+g}}{1+n} \\
\frac{c_t^{\text{capita}}}{y_t^{\text{capita}}} &= \frac{(1+n)(1+g)\frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}}+\frac{(1+r)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}}}{(1+n)(1+g)} \\
\frac{c_t^{\text{capita}}}{y_t^{\text{capita}}} &= \frac{(1+\rho)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}}+\frac{1}{(1+n)(1+g)}\frac{(1+r)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}} \\
\frac{c_t^{\text{capita}}}{y_t^{\text{capita}}} &= \frac{c_{1,t}}{y_t}+\frac{1}{(1+n)(1+g)}\frac{(1+r)^{\frac{1}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}}
\end{aligned}
$$

The per capita ratio is thus larger than the ratio for young agents. This simply reflects the fact that old agents consume, but get no endowment. Also, even if the ratio for young agents is not sensitive to the interest rate, the aggregate one is as the return r affects the consumption of old agents.

The ratio of net savings to output is:

$$
\begin{aligned}
\frac{s_t^{\text{capita}}}{y_t^{\text{capita}}} &= \frac{\frac{(1+n)(1+g)-1}{2+n}\frac{(1+r)^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}}\frac{1}{1+g}y_t}{\frac{1+n}{2+n}y_t} \\
\frac{s_t^{\text{capita}}}{y_t^{\text{capita}}} &= \frac{(1+n)(1+g)-1}{(1+n)(1+g)}\frac{(1+r)^{\frac{1-\theta}{\theta}}}{(1+\rho)^{\frac{1}{\theta}}+(1+r)^{\frac{1-\theta}{\theta}}} \\
\frac{s_t^{\text{capita}}}{y_t^{\text{capita}}} &= \frac{(1+n)(1+g)-1}{(1+n)(1+g)}\frac{s_{1,t}}{y_t}
\end{aligned}
$$

The per capita ratio is proportional to the ratio for young agents, and is positive as long as the size of gross savings grows either because there are more young agents as time goes by, or each young agent gets a larger endowment.