PS7

1 Solution

1. b is determined by the steady-state level of the nominal interest rate, which is in general a policy variable.

2. The Central Bank's problem is:

$$\min_{i_t} \frac{1}{2} x_t^2$$

subject to:

$$x_t = \mathbb{E}_t x_{t+1} - \sigma(i_t - \mathbb{E}_t \pi_{t+1} - \hat{r}_t), \quad i_t \geq -b$$

Under discretion, at time $t+1$ we'll be back in the steady state, so $\mathbb{E}_t x_{t+1} = \mathbb{E}_t \pi_{t+1} = 0$. If $\hat{r}_t \geq -b$, then setting $i_t = \hat{r}_t$ achieves the unconstrained optimum of $x_t = 0$ (i.e., what minimizes the objective function, even if there wasn't any constraint), and is therefore also the constrained optimum. If $\hat{r}_t < -b$, we're constrained by the ZLB, and then the ZLB is going to bind. Hence, $i_t = -b$ and $x_t = \sigma b + \sigma \hat{r}_t < 0$. The disciplined way to show this is by using the Kuhn-Tucker necessary conditions for the solution to an optimization problem with inequality constraints. But it's intuitive that it's going to bind (objective function convex etc).

3. For $\hat{r} < -b$ and the shock lasts for two periods. Hence, $E_{t+1} r_{t+2} = E_{t+1} x_{t+2} = E_{t+1} \pi_{t+2} = 0$. As above, the ZLB is going to bind in the second period so $i_{t+1} = -b$ and $x_{t+1} = \sigma b + \sigma \hat{r}$ and hence from PC $\pi_{t+1} = k\sigma(b + \hat{r})$:

So we solve:

$$\min_{i_t} \frac{1}{2} x_t^2$$

subject to:

$$x_t = \sigma(b + \hat{r}) - \sigma(i_t - \kappa\sigma(b + \hat{r}) - \hat{r}_t) = (1 + \kappa\sigma)\sigma(b + \hat{r}) - \sigma(i_t - \hat{r}_t)$$

$$i_t \geq -b$$

If the ZLB does not bind, then the optimal solution would be (plugging IS into the objective function):

$$\min_{i_t} \frac{1}{2} \left[(1 + \kappa\sigma)\sigma(b + \hat{r}) - \sigma(i_t - \hat{r}_t)\right]^2$$

which has FOC:

$$-\sigma \left[(1 + \kappa\sigma)\sigma(b + \hat{r}) - \sigma(i_t - \hat{r}_t)\right] = 0$$

$$\implies i_t = (1+\kappa\sigma)(b+\hat{r}) + \hat{r}$$

Does this satisfy the ZLB? No it doesn't:

$$i_t = (1+\kappa\sigma)(b+\hat{r}) + \hat{r} < \hat{r} < -b$$

Therefore, the ZLB will bind again, and $i_t = -b$. The output gap becomes:

$$x_t^* = \sigma(b+\hat{r}) + \sigma(b + \kappa\sigma(b+\hat{r}) + \hat{r}) = 2\sigma(b+\hat{r}) + \kappa\sigma^2(b+\hat{r}) < 0$$

4. After time $t+1$, the effect of the shock is gone, and we're back in the steady state. Hence, the optimal policy with two-period commitment solves:

$$\min_{i_t, i_{t+1}} L_t + L_{t+1}$$

subject to:

$$\begin{aligned} x_t &= E_t x_{t+1} - \sigma(i_t - \mathbb{E}_t\pi_{t+1} - \hat{r}), \\ x_{t+1} &= -\sigma i_{t+1}, \\ \pi_{t+1} &= \kappa x_{t+1}, \\ i_t &\geq -b. \end{aligned}$$

(note that here I've used the fact that $\hat{r}_{t+1} = 0$ and that expectations of period $t+2$ variables are zero). The PC at time t is not a constraint because π_t doesn't matter anywhere here. We can simplify this problem to:

$$\min_{i_t, i_{t+1}} \frac{1}{2}x_t^2 + \frac{1}{2}(\sigma i_{t+1})^2$$

subject to:

$$\begin{aligned} x_t &= -(1+\kappa\sigma)\sigma i_{t+1} - \sigma(i_t - \hat{r}), \\ i_t &\geq -b. \end{aligned}$$

Again I'm going to compare the solution without the ZLB to the solution with the ZLB binding. In the first case:

$$\min_{i_t, i_{t+1}} \frac{1}{2}\left((1+\kappa\sigma)\sigma i_{t+1} + \sigma(i_t - \hat{r})\right)^2 + \frac{1}{2}(\sigma i_{t+1})^2$$

with FOCs:

$$\begin{aligned} \big((1+\kappa\sigma)\sigma i_{t+1} + \sigma(i_t - \hat{r})\big)(1+\kappa\sigma)\sigma + \sigma i_{t+1} &= 0, \\ \big((1+\kappa\sigma)\sigma i_{t+1} + \sigma(i_t - \hat{r})\big)\sigma &= 0. \end{aligned}$$

Rewriting the second as:

$$-(1+\kappa\sigma)i_{t+1} + \hat{r} = i_t.$$

and plug into the first one to get:

$$((1+\kappa\sigma)\sigma i_{t+1} + \sigma(-(1+\kappa\sigma)i_{t+1} + \hat{r} - \hat{r}))(1+\kappa\sigma)\sigma + \sigma i_{t+1} = 0$$

and simplify to get:

$$i_{t+1} = 0$$

and hence:

$$\hat{r} = i_t$$

Is this feasible? No. So again the solution must be the constrained one:

$$i_t = -b$$

and hence the problem becomes:

$$\min_{i_t, i_{t+1}} \frac{1}{2}x_t^2 + \frac{1}{2}(\sigma i_{t+1})^2$$

subject to:

$$\begin{aligned} x_t = -(1+\kappa\sigma)\sigma i_{t+1} - \sigma(i_t - \hat{r}), \\ i_t = -b. \end{aligned}$$

and therefore:

$$\min_{i_{t+1}} \frac{1}{2}((1+\kappa\sigma)\sigma i_{t+1} - \sigma(b+\hat{r}))^2 + \frac{1}{2}(\sigma i_{t+1})^2$$

with the FOC implying that:

$$((1+\kappa\sigma)\sigma i_{t+1} - \sigma(b+\hat{r}))(1+\kappa\sigma) + (\sigma i_{t+1}) = 0$$

$$i_{t+1} = \frac{1+\kappa\sigma}{1+(1+\kappa\sigma)^2}(b+\hat{r}) < 0$$

Note that by assumption this satisfies the ZLB. What's the objective function value (i.e. total loss)? Under commitment:

$$\begin{aligned} L_t + L_{t+1} &= \frac{1}{2}(-(1+\kappa\sigma)\sigma i_{t+1} - \sigma(-b-\hat{r}))^2 + \frac{1}{2}(\sigma i_{t+1})^2 \\ &= \frac{1}{2}\sigma^2\left\{(-(1+\kappa\sigma)i_{t+1} + (b+\hat{r}))^2 + (i_{t+1})^2\right\} \\ &= \frac{1}{2}\sigma^2\left\{\left[(1+\kappa\sigma)^2+1\right](i_{t+1})^2 + (b+\hat{r})^2 - 2(b+\hat{r})(1+\kappa\sigma)i_{t+1}\right\} \\ &= \frac{1}{2}\sigma^2\left\{\frac{1+2(1+\kappa\sigma)^2}{1+(1+\kappa\sigma)^2}(b+\hat{r})^2 - 2\frac{(1+\kappa\sigma)^2}{1+(1+\kappa\sigma)^2}(b+\hat{r})^2\right\} \\ &= \frac{1}{2}\frac{\sigma^2}{1+(1+\kappa\sigma)^2}(b+\hat{r})^2 \end{aligned}$$

Under discretion (answer to 2):

$$L_t + L_{t+1} = L_t = \frac{1}{2}x_t^2 = \frac{1}{2}\sigma^2(b+\hat{r}_t)^2$$

Note that the fact that $(1+\kappa\sigma)^2 > 0$ implies that the total loss under commitment is lower than under discretion (as expected). By spreading the stabilization over time, the CB is able to create a more beneficial outcome. Note that this is the case despite hitting the ZLB in period 1 in both settings: the fact that under commitment the CB can affect $\mathbb{E}_t x_{t+1} + \sigma \mathbb{E}_t \pi_{t+1}$ means that it has an "additional tool" (i.e., an additional term in the IS curve) besides i_t at their disposal to help keep x_t closer to zero.

To answer the question about long-term interest rates, note that under discretion $\mathbb{E}_t i_{t+1} = 0$, whereas under commitment $\mathbb{E}_t i_{t+1} < 0$ (i.e., time $t+1$ interest rates are lower than in the steady state). By the expectations hypothesis:

$$i_t^{(2)} = \frac{1}{2}\left(i_t^{(1)} + \mathbb{E}_t i_{t+1}^{(1)}\right)$$

hence under commitment long-term bond yields at time t are going to be lower than under discretion.