Linear Algebra

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System of linear equations

Matrix algebra

Some specific matrices

Partitioned matrices

Rank, inverse and determinant of a matrix

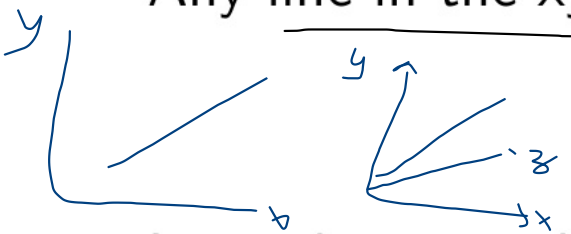
Eigenvalues and eigenvectors

Kronecker product

System of linear equations

System of linear equations

Any line in the xy-plane can be represented as a *Linear Equation* :



$$a_1 x + a_2 y = b$$

It can be generalized to a *linear equation* in the n **variables** $x_1, x_2, ..., x_n$:

$$a_1 x_1 + a_2 x_2 + ... + a_n x_n = b$$

with $a_1, a_2, ..., a_n$ and b real constants, and called the **parameters**.

N.B. : It does not involve any products or roots of variables. All variables occur only to the first power and do not appear as arguments for trigonometric, logarithmic or exponential functions. Very important when we consider econometric models !

$$x^2 + xy - \sin x + \sqrt{y} = 3 \quad \text{is } \textbf{not} \text{ linear !}$$

System of linear equations

Objective : solving linear systems of m equations in n unknowns

m equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

n variables

A **solution** of this system is n-tuple of real numbers $x_1, x_2, ..., x_n$, which satisfies each of the m equations.

System of linear equations

We are interested in the three following questions :

1) Does a solution exist ?

2) How many solutions are there ?

3) Is there an efficient algorithm that computes actual solutions ?

Three ways of solving such systems :

1) substitution,

2) elimination of variables, and

3) matrix methods.

Substitution

Methodology :

- Solve one equation of the system for one variable, say x_n, in terms of the other variables in that equation.
- Substitute this expression for x_n into the other $m-1$ equations.
- The result is a new system of $m-1$ equations in the $n-1$ unknowns $x_1, x_2, \ldots, x_{n-1}$.
- Etc.
- Proceed until you reach a system with just a single equation.
- Finally, use the earlier expressions of one variable in terms of the others to find all the x_i's.

Straightforward but cumbersome !

Substitution

Example.

$$\begin{array}{rcrcr} x & - & 2y & = & 8 \\ 3x & + & y & = & 3 \end{array} \Rightarrow 3(8+2y)+y=3 \Rightarrow y=-3 \Rightarrow x=2$$

Exercise 1 Solve the following system by substitution :

$$\begin{array}{rcrcrcr} x & - & 3y & + & 6z & = & -1 \\ 2x & - & 5y & + & 10z & = & 0 \\ 3x & - & 8y & + & 17z & = & 1 \end{array}$$

Substitution

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$x - 3y + 6z = -1 \rightarrow x = 3y - 6z - 1$

$2x - 5y + 10z = 0 \rightarrow 2(3y - 6z - 1) - 5y + 10z = 0$

$3x - 8y + 17z = 1 \rightarrow 3(3y - 6z - 1) - 8y + 17z = 1$

1) $y - 2z - 2 = 0$

2) $y - z = 4 \rightarrow y = z + 4$

$z + 4 - 2z - 2 = 0$

$-z + 2 = 0$

$z = 2, \; y = 6$

$x = 18 - 12 - 1 = 5$

$$\begin{pmatrix} 5 \\ 6 \\ 2 \end{pmatrix}$$

Elimination of variables

Methodology :

- Use the coefficient of x_1 in the first equation to eliminate the x_1 term from all the equations *below* it.
- To do this, add proper multiples of the first equation to each of the succeeding equations.
- Now eliminate the next variable x_2 from the last $m - 1$ equations just as before.
- Continue eliminating variables until you reach the last equation.
- The resulting simplified system can then easily be solved by substitution.

Elimination of variables

Example.

$$\begin{array}{rcrcr} x & - & 2y & = & 8 \\ 3x & + & y & = & 3 \end{array} \Rightarrow \begin{array}{rcrcr} -3x & + & 6y & = & -24 \\ 3x & + & y & = & 3 \end{array} \Rightarrow y = -3 \Rightarrow x = 2$$

Exercise 2 Solve the following system by elimination :

$$\begin{array}{rcrcrcr} x & - & 3y & + & 6z & = & -1 \\ 2x & - & 5y & + & 10z & = & 0 \\ 3x & - & 8y & + & 17z & = & 1 \end{array}$$

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$$\mathrm{I}\quad x-3y+6z=-1 \xrightarrow{\cdot(-2)} -2x+6y-12z=+2$$

$$\mathrm{II}\quad 2x-5y+10z=0 \qquad \xrightarrow{\cdot(-3)} -3x+9y-18z=+3$$

$$\mathrm{III}\quad 3x-8y+17z=1$$

$\mathrm{I}+\mathrm{II}$

$$\begin{aligned} -2x+6y-12z&=2 \\ 2x-5y+10z&=0 \\ \hline y-2z&=2 \end{aligned}$$

$\mathrm{I}+\mathrm{III}$

$$y-z=4$$

$$4+z=2+2z$$

$$z=2,\ y=6,\ x=5$$

Representation in matrix form

The following system

$$\left|\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}\right.$$

can be written in a matrix form as $AX = b$, with

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

What is a matrix ?

- A **matrix** is a rectangular array of numbers called **entries**.
- The dimension of a matrix is written **n x m**, where **n** is the number of row and **m** the number of columns.
- A matrix I whose diagonal elements are 1s and whose nondiagonal elements are all 0.s is called the **identity matrix**.
- A matrix whose elements is 0 is called the **zero matrix**.

$$I = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

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Representation in matrix form

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

We can add on the column b to A to obtain :

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

$\hat{A}$ is called the **augmented matrix**.

Elementary row operations

We can solve a system of linear equations by performing some elementary row operations on its augmented matrix.

The **elementary row operations** are :

- Multiply a row (an equation) through by a nonzero constant.
- Interchange two rows (equations).
- Add a multiple of one row (equation) to another row (equation).

$\Rightarrow$ The aim is to get to the **reduced row-echelon form** of the augmented matrix.

Elementary row operations

Definition. A row of a matrix is said to have k **leading zeros** if the first k elements of the row are all zeros and the $(k+1)$th element of the row is not zero. A matrix is in **row echelon form** if each row has more leading zeros than the row preceding it.

Example :

$$C = \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 6 \end{pmatrix} \quad D = \begin{pmatrix} 2 & 3 \\ 0 & 6 \\ 0 & 0 \end{pmatrix} \quad E = \begin{pmatrix} 1 & 5 & 2 \\ 2 & 0 & 1 \end{pmatrix}$$

C and D are in row echelon form. E is not in row echelon form.

The process of getting the row echelon form is called the **Gaussian elimination process**.

Elementary row operations

To better solve the system, one might want to use row operations to reduce the matrix even further.

To do so :

- First, multiply each row of the row echelon form by the reciprocal of the **pivot** (i.e. the first nonzero entry in each row) in that row and create a new matrix all of whose pivots are 1s.
- Use these pivots (starting with the last row) to turn each nonzeron entry *above* it (in the same column) into a zero.

Definition. A row echelon matrix in which each pivot is a 1 and in which each column containing a pivot contains no other nonzero entries is said to be in **reduced form**. The process of getting the reduced row echelon form is called the **Gauss-Jordan elimination.**

Elementary row operations

Example : find the reduced row echelon form of F

$$F = \begin{array}{c} \overbrace{\qquad\qquad\qquad}^{A} \quad b \\ \begin{pmatrix} 1 & -0.4 & -0.3 & 130 \\ 0 & \boxed{0.8} & -0.2 & 100 \\ 0 & 0 & \boxed{0.7} & 210 \end{pmatrix} \end{array}$$

Multiply the ~~first~~ second row by $1/0.8$ and the third row by $1/0.7$ to get :

$R_1 \leftarrow R_1 + 0.3R_3$

$R_2 \leftarrow R_2 + 0.25R_3$

$$F = \begin{pmatrix} 1 & -0.4 & -0.3 & 130 \\ 0 & 1 & -0.25 & 125 \\ 0 & 0 & 1 & 300 \end{pmatrix}$$

Elementary row operations

Use the pivot in row 3 to first add 0.25 times row 3 to row 2 and 0.3 times row 3 to row 1 :

$R_1 \leftarrow R_1 + 0.4 R_2$

$$F = \begin{pmatrix} 1 & -0.4 & 0 & 220 \\ 0 & 1 & 0 & 200 \\ 0 & 0 & 1 & 300 \end{pmatrix}$$

Use pivot in row 2 to eliminate the nonzero entry above it by adding 0.4 times row 2 to row 1 :

$$F = \begin{pmatrix} 1 & 0 & 0 & 300 \\ 0 & 1 & 0 & 200 \\ 0 & 0 & 1 & 300 \end{pmatrix}$$

and we get the solution : $x_1 = 300$, $x_2 = 200$, $x_3 = 300$.

Gaussian elimination

Exercise 3 Solve the following system by Gaussian-Jordan elimination :

$$\begin{cases} x + y + z &= 0 \\ 12x + 2y - 3z &= 5 \\ 3x + 4y + z &= -4 \end{cases}$$

$$x + y + z = 0$$
$$12x + 2y - 3z = 5$$
$$3x + 4y + z = -4$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 12 & 2 & -3 & 5 \\ 3 & 4 & 1 & -4 \end{array}\right]$$

$R_2 \to R_2 - 4R_3$; $R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -14 & -7 & 21 \\ 0 & 1 & 2 & -4 \end{array}\right]$$

$R_2 \to R_2 / -14$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/2 & -3/2 \\ 0 & 1 & -2 & -4 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/2 & -3/2 \\ 0 & 0 & -5/2 & -5/2 \end{array}\right]$$

$R_3 \to R_3 / (-2/5)$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 1/2 & -3/2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_2 \to R_2 - (R_3)/2 \quad ; \quad R_1 \to R_1 - R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$R_1 \to R_1 - R_2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$(x, y, z) = (1, -2, 1)$

Matrix algebra

Operations on matrices

Size The size of a matrix is indicated by the number of its rows and the number of its columns. A matrix with n rows and m columns is a $n \times m$ matrix.

Addition If A and B are any two matrices **of the same size**, the sum $A + B$ is the matrix obtained by adding together the corresponding entries in the two matrices.
If we have :

$$A = \begin{pmatrix} 7 & 12 \\ 12 & 28 \\ -5 & -1 \end{pmatrix} B = \begin{pmatrix} -2 & 13 \\ 6 & -8 \\ 5 & -4 \end{pmatrix} C = \begin{pmatrix} 7 & 12 & -11 \\ 12 & 28 & 2 \\ -5 & -1 & 7 \end{pmatrix}$$

Then

$$A + B = \begin{pmatrix} 5 & 25 \\ 18 & 20 \\ 0 & -5 \end{pmatrix}$$

while $A + C$ and $B + C$ are **undefined** !

Operations on matrices

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If we have :

$$\underset{(3\times 2)}{A} = \begin{pmatrix} 7 & 12 \\ 12 & 28 \\ -5 & -1 \end{pmatrix} \underset{(3\times 2)}{B} = \begin{pmatrix} -2 & 13 \\ 6 & -8 \\ 5 & -4 \end{pmatrix} \underset{(3\times 3)}{C} = \begin{pmatrix} 7 & 12 & -11 \\ 12 & 28 & 2 \\ -5 & -1 & 7 \end{pmatrix}$$

Then

$$\underset{(3\times 2)}{A + B} = \begin{pmatrix} 5 & 25 \\ 18 & 20 \\ 0 & -5 \end{pmatrix}$$

while $A + C$ and $B + C$ are **undefined** !

Operations on matrices

Multiply by a scalar If A is any matrix and c is any scalar, then the **product** cA is the matrix obtained by multiplying **each entry** of A by c.
If we have :

$$A = \begin{pmatrix} 7 & 12 \\ 12 & 28 \\ -5 & -1 \end{pmatrix} \Rightarrow \quad (-2)A = \begin{pmatrix} -14 & -24 \\ -24 & -56 \\ 10 & 2 \end{pmatrix}$$

(3x2)

Operations on matrices

Product The product of an n x r matrix A and an r x m matrix B is an n x m matrix AB. We **need** :

$$\textit{number of columns of } A = \textit{number of rows of } B$$

$$\underset{(2\times 3)}{A} = \begin{pmatrix} 3 & 1 & 4 \\ 2 & 0 & 5 \end{pmatrix} \overset{(3\times 4)}{B} = \begin{pmatrix} 3 & 1 & 4 & 2 \\ 1 & 0 & 6 & 3 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

The dimensions are the key ! Here, BA is **undefined** !

The product AB is constructed as follow : the entry in row i and column j of the matrix AB is the sum of the product of the elements of the row i of matrix A by the column j of matrix B.

In other words : the (i, j)th entry of the product AB is $\boxed{\sum_{h=1}^{r} a_{ih} b_{hj}}$.

Operations on matrices

So matrix AB will be an 2 x 4 matrix :

$$AB = \begin{pmatrix} 3 & 1 & 4 \\ 2 & 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 & 4 & 2 \\ 1 & 0 & 6 & 3 \\ 2 & 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} - & - & - & - \\ - & - & 23 & - \end{pmatrix}$$

$(3 \times 3) + (1 \times 1) + (4 \times 2)$

$$(4 \cdot 2) + (6 \cdot 0) + (3 \cdot 5) = 23$$

$$AB = \begin{pmatrix} 3 & 1 & 4 \\ 2 & 0 & 5 \end{pmatrix} \cdot \begin{pmatrix} 3 & 1 & 4 & 2 \\ 1 & 0 & 6 & 3 \\ 2 & 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} - & 7 & - & - \\ - & - & 23 & - \end{pmatrix}$$

$$(1 \cdot 3) + (0 \cdot 1) + (1 \cdot 4) = 7$$

We keep doing that for the rest of the entries and we get :

$$AB = \begin{pmatrix} 18 & 7 & 30 & 25 \\ 16 & 7 & 23 & 24 \end{pmatrix}$$

Laws of Matrix Algebra

- $A + B = B + A$ *Commutative law for addition*
- $A + (B + C) = (A + B) + C$ *Associative law for addition*
- $A(BC) = (AB)C$ *Associative law for multiplication*
- $A(B + C) = AB + AC$ *Distributive law*
- $(B - C)A = BA - CA$ *Distributive law*
- $(a + b)C = aC + bC$

Operations on matrices

Exercise 4

- Consider the matrices

$$A = \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$$

  Find and compare AB and BA.

- Consider the matrices

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 4 & 3 \\ 2 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$$

  Show the *associative law* for matrix multiplication.

Operations on matrices

Exercise 4

- Consider the matrices

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  Show the *associative law* for matrix multiplication.

$A \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix} \quad B \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \qquad \underline{2} \times \underline{2} \cdot \underline{2} \times \underline{2}$

$\underline{AB} \begin{pmatrix} \underline{-1+0} & \underline{-2+0} \\ \underline{2+3} & \underline{4+0} \end{pmatrix} \Rightarrow \underline{\begin{pmatrix} -1 & -2 \\ 11 & 4 \end{pmatrix}}$

$BA \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} \underline{-1+4} & \underline{0+6} \\ \underline{-3+0} & \underline{0+0} \end{pmatrix} \Rightarrow \underline{\begin{pmatrix} 3 & 6 \\ -3 & 0 \end{pmatrix}}$

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}_{(3\times 2)} \qquad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}_{(2\times 2)} \qquad C = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}_{2\times 2}$$

$$A\cdot(BC) = (AB)\cdot C \quad \boxed{\text{to prove}}$$

$$BC = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}_{(2\times 2)} \cdot \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}_{(2\times 2)}$$

$$= \begin{bmatrix} 4+6 & 4\times 0+9 \\ 2+2 & 2\times 0+3 \end{bmatrix} = \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix}_{2\times 2}$$

$$A.(BC) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}_{(3\times 2)} \begin{bmatrix} 10 & 9 \\ 4 & 3 \end{bmatrix}_{(2\times 2)}$$

$$= \begin{bmatrix} 10+8 & 9+6 \\ 30+16 & 27+12 \\ 0+4 & 0+3 \end{bmatrix} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}_{(3\times 2)}$$

$$AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{bmatrix}_{(3\times 2)} \cdot \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}_{(2\times 2)}$$

$$AB = \begin{bmatrix} 4+4 & 3+2 \\ 12+8 & 9+4 \\ 0+2 & 0+1 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix}_{(3\times 2)}$$

$$(A.B).C = \begin{bmatrix} 8 & 5 \\ 20 & 13 \\ 2 & 1 \end{bmatrix}_{(3\times 2)} \cdot \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}_{(2\times 2)}$$

$$= \begin{bmatrix} 8+10 & 0+15 \\ 20+26 & 39 \\ 2+2 & 3 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 18 & 15 \\ 46 & 39 \\ 4 & 3 \end{bmatrix}_{(3\times 2)}$$

$$\therefore A.(BC) = (AB).C$$

The transpose

Consider any $n \times m$ matrix A. The transpose of A, denoted A' or A^T, is defined as the $m \times n$ matrix obtained by interchanging the rows and columns of A.

$$\underset{3\times 2}{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 0 & 1 \end{pmatrix} \Rightarrow \underset{2\times 3}{A'} = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 4 & 1 \end{pmatrix}$$

Rules for transposition

- $(A')' = A$
- $(A+B)' = A' + B'$
- $(\alpha A)' = \alpha A'$
- $(AB)' = B'A'$
- $(ABC)' = C'B'A'$

A $(m \times n)$ $\quad$ B $(n \times m)$ $\quad$ AB $(n \times m)$

A' $(n \times m)$ $\quad$ B' $(m \times n)$ $\quad$ $B'A'$ $(m \times m)$

Some specific matrices

A matrix full of zeros is called a **zero matrix** and will be denoted by $0_{(n \times m)}$

$$\underset{3\times 2}{0} \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \underset{2\times 2}{0} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \underset{(1\times 5)}{0} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Most of the time, the size of the zero matrix is obvious, however, if necessary, we can write $0_{n \times m}$.

Some properties of the zero matrix

- $A + 0 = 0 + A = A$
- $A - A = 0$
- $0 - A = -A$
- $A0 = 0 \quad ; \quad 0A = 0$

Some specific matrices

Another very important matrix is called the **identity matrix**. It is a squared matrix, with 1's on the main diagonal and 0's off the main diagonal, such as

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$$

An $n \times n$ identity matrix is denoted by I_n. If A an $n \times m$ matrix, then $AI_m = A$ and $I_nA = A$

Example

If we consider the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

We need identity matrices of different size if we have AI_m or I_nA.

Some specific matrices

A matrix is said **square** if # rows n = # columns m $n = m$

A square matrix is said **symmetric** if $A' = A$

A matrix is **orthogonal** if $AA' = A'A = I$

A matrix is **idempotent** if $AA = A$

Exercise 5 If X is an arbitrary $n \times m$ matrix, show that XX' and $X'X$ are both symmetric.

$$X = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \quad (2\times 3) \qquad X' = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix} \quad (3\times 2) \quad = (2\times 2)$$

$$XX' = \begin{bmatrix} a^2+b^2+c^2 & ad+be+cf \\ da+eb+fc & d^2+e^2+f^2 \end{bmatrix}$$

$$X' = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}_{(3\times 2)} \quad X = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}_{(2\times 3)} = (3\times 3)$$

$$X'X \begin{bmatrix} a^2+d^2 & \underline{ab+de} & \underline{ac+df} \\ \underline{ba+ed} & b^2+e^2 & \underline{bc+ef} \\ \underline{ca+fd} & \underline{cb+fe} & c^2+f^2 \end{bmatrix}$$

$XX' \Rightarrow (XX')' = XX'$

$n \times m \quad m \times n \Rightarrow n \times n \qquad (XX')' = (X')'X' = XX'$

$(X'X)' = X'(X')' = X'X$

$m \times m$

$(AB)' = B'A'$

$(A')' = A$

Partitioned matrices

Given the 3×5 matrix

$$\underset{(3\times 5)}{A} = \begin{pmatrix} 2 & 4 & 6 & 8 & 0 \\ 1 & 3 & 5 & 7 & 9 \\ 4 & 0 & 4 & 0 & 4 \end{pmatrix}$$

can be partitioned in four 'blocks' A_1, A_2, A_3, A_4 by writing

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$$

where

$$A_1 = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 8 & 0 \\ 7 & 9 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 4 & 0 & 4 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 4 \end{pmatrix}$$

Partitioned matrices

$$X = \left(\begin{array}{cccc|cccc} x_{11} & \cdots & x_{1n} & & 1 & 0 & \cdots & 0 \\ x_{21} & \cdots & x_{2n} & & 0 & 1 & 0 \cdots & 0 \\ \hline 1 & 0 & \cdots & 0 & & & & \\ 0 & 0 & \cdots & 0 & y_{11} & \cdots & & y_{1m} \\ & & & 1 & & & & \end{array}\right) \quad Z = \left(\begin{array}{c|c} \begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \end{matrix} & y \\ \hline \square & \square \end{array}\right)$$

We can multiply two partitioned matrices block by block as in ordinary matrix multiplication.

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} P & Q \\ R & S \end{pmatrix} = \begin{pmatrix} AP + BR & AQ + BS \\ CP + DR & CQ + DS \end{pmatrix}$$

Exercise 6 Let A and B be 4×4 matrices partitioned as follows

$$\underset{(4\times 6)}{A} = \begin{pmatrix} \underset{2\times 2}{A_1} & \underset{2\times 2}{O} \\ \underset{2\times 2}{O} & \underset{2\times 2}{A_2} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 & O \\ O & B_2 \end{pmatrix}$$

where A_1, A_2, B_1, B_2 are 2×2 matrices. Find AB.

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \qquad B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$$

$$AB = \begin{pmatrix} A_1 B_1 + 0 & 0 \\ 0 & A_2 B_2 + 0 \end{pmatrix}$$

Rank, inverse and determinant of a matrix

Rank of a matrix

The **rank** of an $n \times m$ matrix A, denoted $r(A)$, is the highest number of columns that are linearly independent.

In other words it is the number of nonzero rows in its row echelon form.

When all the vectors are linearly independent the matrix is said to be of **full rank** ($r(A) = min(n, m)$).

Exercise 7

Find the rank of the following matrix

$$A = \begin{array}{c} \begin{array}{ccc} v_1 & v_2 & v_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{pmatrix} \end{array}$$

$v_1 - v_2 = v_3$

$$\begin{pmatrix} 1 & 0 & 1 \\ -2 & -3 & 1 \\ 3 & 3 & 0 \end{pmatrix} \quad \begin{matrix} R_2 \leftarrow R_2 + 2R_1 \\ R_3 \leftarrow R_3 - 3R_1 \end{matrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{pmatrix} \quad R_3 \leftarrow R_3 - R_2$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \quad R(2)$$

$$v_3 = \lambda v_1 + \gamma v_2$$

$$A = \begin{array}{c} \begin{matrix} v_1 & v_2 & v_3 \end{matrix} \\ \begin{pmatrix} | & | & | \\ | & | & | \\ | & | & | \end{pmatrix} \end{array}$$

Rank of a matrix

Some properties :

- $r(A) \leq min(n, m)$
- $r(A) = r(A') = r(A'A) = r(AA')$
- $r(AB) \leq min(r(A), r(B))$
- $r(AB) = r(A)$ if B is of full rank
- $r(A + B) \leq r(A) + r(B)$
- $r(A) = 0$ if and only if A is a *zero matrix*

A square matrix B $(n \times n)$ is

- **Non-singular** if it is of full rank. $r(B) = n$
- **Singular** if $r(B) < n$.

Inverse

Definition Let A be a $n \times n$ square matrix ($\mathbf{M}_n$). The matrix B in $\mathbf{M}_n$ is an inverse for A if :

$$AB = BA = I_n$$

The matrix B, denoted as A^{-1}, is the **inverse** of A.

If an $n \times n$ matrix A is invertible, then it is nonsingular.

If an $n \times n$ matrix A is nonsingular, then it is invertible.

Exercise 8

- Show that $B = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix}$ is an inverse of $A = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix}$.

$$\underset{2\times 2}{B} = \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \quad \underset{2\times 2}{A} = \begin{pmatrix} 2 & -5 \\ -1 & 3 \end{pmatrix} \quad AB = BA$$

$$BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

$$AB \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

Inverse

Let A and B be invertible $n \times n$ matrices. Then :

- I is invertible, and $I^{-1} = I$
- A^{-1} is invertible, $(A^{-1})^{-1} = A$
- AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$
- The transpose A' is invertible, and $(A')^{-1} = (A^{-1})'$
- $(cA)^{-1} = c^{-1}A^{-1} = \frac{1}{c} \cdot A^{-1}$ whenever c is a number $\neq 0$

How to invert a matrix

Let's start with the special case of a 2 x 2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

If $ad - bc \neq 0$, then

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\begin{matrix} a & b & 1 & 0 \\ c & d & 0 & 1 \end{matrix}$$

Ex

You can verify that $AA^{-1} = I_2$ and $A^{-1}A = I_2$.

To find the inverse of an invertible matrix A whose size is greater than 2 x 2, we should start with the matrix $(\,A \quad I\,)$, and reduce it by Gaussian elimination to the form $(\,I \quad B\,)$ where $B = A^{-1}$.

$A^{-1}A \quad A^{-1}I$

IA^{-1}

$$AA^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{ad-bc}{ad-bc} & \frac{-ab+ab}{ad-bc} \\ \frac{cd-cd}{ad-bc} & \frac{-bc+ad}{ad-bc} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

How to invert a matrix

The invert of a matrix A, can also be used to solve a system of linear equations $AX = b$. Indeed we have

$$\begin{aligned} & AX = b \\ \Rightarrow \quad & A^{-1}AX = A^{-1}b \\ \Rightarrow \quad & X = A^{-1}b \end{aligned}$$

If a $n \times n$ matrix A is invertible, then it is nonsingular, and the unique solution to the system of linear equations $AX = b$ is $X = A^{-1}b$.

How to invert a matrix

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{pmatrix} \quad D \; B$$

Method : Start with the matrix $(\,A \quad I\,)$, reduce it by Gauss-Jordan elimination to the form $(\,D \quad B\,)$ where D is a diagonal matrix with no diagonal entry equal to zero. A^{-1} is then found by dividing each row of B by the corresponding diagonal entry of D.

Exercise 9

Invert the matrix

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 1 & 1 \\ 3 & 1 & 2 \end{pmatrix} \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$$

$$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \quad \boxed{}$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad R_3 \leftarrow R_3 - R_2$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 2 & -2 \\ 3 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \xrightarrow[R_2 \leftarrow R_2 - R_3]{R_1 \leftarrow R_1 - 2R_3} \left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & -1 \\ 3 & 1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad R_1 \leftarrow R_1 - R_2$$

$$\left[\begin{array}{ccc|ccc} -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 3 & 2 & -4 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad R_2 \leftarrow R_2 + 3R_1$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 3 & 2 & -4 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right] \quad R_1 \leftarrow -R_1$$

Determinant

The determinant is a real-valued function of a square matrix.

It is used to *determine* whether or not a given matrix is nonsingular.

A square matrix A is **invertible**, and therefore **non-singular**, if and only if $det(A) \neq 0$

How to find the determinant

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$\det A \neq 0$

- Let's start again with the special case of a 2 x 2 matrix. If we have $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $det(A) = |A| = ad - bc$

- In the case of a triangular matrix, the determinant is equal to the product of the elements of the diagonal.

$$B = \begin{pmatrix} a_{11} & 0 & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 \\ a_{13} & a_{23} & a_{33} & 0 \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix} \Rightarrow |B| = a_{11} \cdot a_{22} \cdot a_{33} \cdot a_{44}$$

N.B. If you change the matrix by applying any row operation, its determinant will also change!

Cofactor expansion

Let A is a $n \times n$ matrix. Let A_{ij} be the $(n-1) \times (n-1)$ submatrix obtained by deleting row i and column j from A. Then, the scalar

$$M_{ij} \equiv \det A_{ij}$$

is called the (i,j)th **minor** of A and the scalar

$$C_{i,j} \equiv (-1)^{i+j} M_{ij}$$

is called the (i,j)th **cofactor** of A.

$$\underset{3\times 3}{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$A_{11} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

$$M_{11} = a_{22}a_{33} - a_{23}a_{32}$$

For a 3×3 matrix A, we denote C its matrix of cofactors :

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

With

$$C_{11} = (-1)^2 \cdot \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \qquad C_{23} = (-1)^5 \cdot \det \begin{pmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{pmatrix}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ \vdots & & & a_{24} \\ \vdots & & & a_{34} \\ \cdots & \cdots & \cdots & a_{44} \end{pmatrix}$$

$$M_{14} = \det\Big($$

Cofactor expansion

The **determinant** of a 3×3 matrix A is given by

$$
\begin{aligned}
|A| =& a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + a_{13} \cdot C_{13} \\
=& a_{11} \cdot M_{11} - a_{12} \cdot M_{12} + a_{13} \cdot M_{13} \\
=& a_{11} \cdot \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12} \cdot \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13} \cdot \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}
\end{aligned}
$$

The **determinant** of a $n \times n$ matrix A is given by

$$
\begin{aligned}
|A| =& a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + \cdots + a_{1n} \cdot C_{1n} \\
=& a_{11} \cdot M_{11} - a_{12} \cdot M_{12} + \cdots + (-1)^{n+1} a_{1n} \cdot M_{1n}
\end{aligned}
$$

Cofactor expansion

We used the first row to compute the determinant of the matrix A. However, we can use any row or column to compute the determinant of a matrix.

The determinant of the 3×3 matrix A is therefore :

$$\begin{aligned} |A| &= a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + a_{13} \cdot C_{13} \\ &= a_{12} \cdot C_{12} + a_{22} \cdot C_{22} + a_{32} \cdot C_{32} \\ &\vdots \end{aligned}$$

Exercise 10

Give the determinant of the following matrix by using (1) the first row and (2) the third column.

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

i) First row:

$$|A| = 3C_{11} + 1 \cdot C_{12} + \cancel{0 \cdot C_{13}}^{\,0}$$

$$C_{11} = (-1)^{1+1} \det \begin{bmatrix} -4 & 3 \\ 4 & -2 \end{bmatrix}$$

$$= 1 \cdot (8-12) = -4$$

$$C_{12} = (-1)^{1+2} \det \begin{bmatrix} -2 & 3 \\ 5 & -2 \end{bmatrix}$$

$$= -(4-15)$$

$$= 11$$

$$|A| = 3 \times (-4) + 1 \cdot (11)$$

$$= -12 + 11 = -1$$

ii) third column:

$$|A| = \cancel{0 \cdot C_{13}}^{\,0} + 3 \cdot C_{23} + (-2) \cdot C_{33}$$

$$C_{23} = (-1)^{2+3} \det \begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix}$$

$$= -(12-5)$$

$$= -7$$

$$C_{33} = (-1)^{3+3} \det \begin{bmatrix} 3 & 1 \\ -2 & -4 \end{bmatrix} = -12+2$$

$$= -10$$

$$\therefore |A| = 3(-7) + (-2)(-10)$$

$$= -21 + 20$$

$$= -1$$

Adjoint matrix

The $n \times n$ matrix whose (i, j)th entry is C_{ji}, the (j, i)th cofactor of A (i.e. **the transpose of the matrix of cofactors**), is called the **adjoint of** A and is written $adj(A)$.

$$C = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

$$\begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix}$$

If A is an invertible matrix, then

$$A^{-1} = \frac{1}{det(A)} \cdot adj(A)$$

Exercise 11

Find the inverse of matrix A using the cofactors method.

$$A = \begin{pmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{pmatrix}$$

$$A = \begin{bmatrix} 3 & 1 & 0 \\ -2 & -4 & 3 \\ 5 & 4 & -2 \end{bmatrix}$$

$$C_A = \begin{bmatrix} +(8-12) & -(4-15) & +(-8+20) \\ -(-2-0) & +(-6-0) & -(12-5) \\ +(3-0) & -(9-0) & +(-12+2) \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 11 & 12 \\ 2 & -6 & -7 \\ 3 & -9 & -10 \end{bmatrix}$$

$$\text{adj.}A = \begin{bmatrix} -4 & 2 & 3 \\ 11 & -6 & -9 \\ 12 & -7 & -10 \end{bmatrix}$$

$$\therefore \frac{1}{|A|} \cdot \text{adj.}A = A^{-1} = \begin{bmatrix} 4 & -2 & -3 \\ -11 & 6 & 9 \\ -12 & 7 & 10 \end{bmatrix}$$

Cramer's Rule

Let A be a nonsingular matrix. Then, the unique solution $x = (x_1, \ldots, x_n)$ of the $n \times n$ system $Ax = b$ is

$$x_1 = \frac{det(B_1)}{det(A)}, \quad x_2 = \frac{det(B_2)}{det(A)}, \ldots, x_n = \frac{det(B_n)}{det(A)}$$

where B_i is the matrix A with the right-hand side b replacing the ith column of A.

Example for a 3×3 system : $B_1 = \begin{pmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}$.

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{x} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}_{b}$$

$$B_1 = \begin{pmatrix} b_1 & a_{12} & a_3 \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{pmatrix}$$

Exercise 12

Use Cramer's Rule to solve

$$5x_1 + x_2 = 3$$
$$2x_1 - x_2 = 4$$

$$\underbrace{\begin{bmatrix} 5 & 1 \\ 2 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 3 \\ 4 \end{bmatrix}}_{b}$$

$$\det(A) = -5 - 2 = -7$$

$$x_1 = \frac{-7}{-7} = 1$$

$$B_1 = \begin{bmatrix} 3 & 1 \\ 4 & -1 \end{bmatrix} \qquad \det(B_1) = -3 - 4 = -7$$

$$B_2 = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix} \qquad \det(B_2) = 20 - 6 = 14$$

$$x_2 = \frac{14}{-7} = -2$$

Some properties

Let A be a square matrix. Then

- $\det(A) = det(A^T)$
- $\det(AB) = det(A)det(B)$
- $\det(A + B) \neq \det A + \det B$ in general.

A matrix A is **non-singular** if and only if :

- $det(A) \neq 0$
- A is invertible.
- A is of full rank.
- AX=b has a unique solution.
- The rows and columns of A are linearly independent.

The **trace** of a square matrix is the sum of its diagonal entries.

$$\text{trace } A = a_{11} + a_{22} + \cdots + a_{kk}$$

Eigenvalues and eigenvectors

Eigenvalues and eigenvectors

Eigenvalues and eigenvectors play a central role in many aspects of economic theory.

In particular, the signs of the eigenvalues determine the stability of equilibria in *nonlinear* dynamic models.

Diagonalisable matrices

If $\mathbf{A}$ is a square matrix, then

$$\mathbf{A}^2 = \mathbf{A}\mathbf{A}, \quad \mathbf{A}^3 = \mathbf{A}^2\mathbf{A}, \quad \mathbf{A}^4 = \mathbf{A}^3\mathbf{A}$$

By the associative law of matrix multiplication, $\mathbf{A}^{r+s} = \mathbf{A}^r\mathbf{A}^s$ for any pair of natural number r,s.

$$\begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix} = \begin{pmatrix} b^2 & 0 \\ 0 & c^2 \end{pmatrix}$$

It is really easy to calculate powers of a diagonal matrix. If $\mathbf{A} = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}$ then

$$\mathbf{A}^2 = \begin{pmatrix} b^2 & 0 \\ 0 & c^2 \end{pmatrix}, \quad \mathbf{A}^3 = \begin{pmatrix} b^3 & 0 \\ 0 & c^3 \end{pmatrix}$$

Diagonalisable matrices

To generalise the example, we denote $\text{diag}(d_1, \ldots, d_n)$ the diagonal matrix whose diagonal entries are $d_1, \ldots, d_n$.

Powers of diagonal matrices obey the following rule :
if $\mathbf{D} = \text{diag}(d_1, \ldots, d_n)$ then $\mathbf{D}^k = \text{diag}(d_1^k, \ldots, d_n^k)$ for $k = 1, 2, \ldots$

We say that square matrix $\mathbf{A}$ is **diagonalisable** (sometimes called d-matrix) if and only if

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$$

for some invertible matrix $\mathbf{S}$ and some diagonal matrix $\mathbf{D}$.

Diagonalisable matrices

If this is so, then

$$\mathbf{S}\ \mathbf{D}\ \mathbf{S}^{-1} = \mathbf{S}\ \mathbf{S}^{-1}\mathbf{A}\ \mathbf{S}\ \mathbf{S}^{-1} = \mathbf{I}\ \mathbf{A}\ \mathbf{I} = \mathbf{A}$$

It follows that

$$\mathbf{A}^2 = (\mathbf{SDS}^{-1})(\mathbf{SDS}^{-1}) = \mathbf{S}(\mathbf{DID})\mathbf{S}^{-1} = \mathbf{SD}^2\mathbf{S}^{-1}$$

We can generalize, if $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{D}$ then $\mathbf{A}^k = \mathbf{SD}^k\mathbf{S}^{-1}$, for $k = 1, 2, \ldots$.

The questions arise :

1) How can we tell if a given matrix is diagonalisable ?

2) and if it is, how do we find $\mathbf{S}$ and $\mathbf{D}$?

To answer these questions, we introduce the concepts of **eigenvalue** and **eigenvector**.

Diagonalisable matrices

If this is so, then

$$\mathbf{S}\ \mathbf{D}\ \mathbf{S}^{-1} = \mathbf{S}\ \mathbf{S}^{-1}\mathbf{A}\ \mathbf{S}\ \mathbf{S}^{-1} = \mathbf{I}\ \mathbf{A}\ \mathbf{I} = \mathbf{A}$$

It follows that

$$\mathbf{A}^2 = (\mathbf{S}\mathbf{D}\mathbf{S}^{-1})(\mathbf{S}\mathbf{D}\mathbf{S}^{-1}) = \mathbf{S}(\mathbf{D}\mathbf{I}\mathbf{D})\mathbf{S}^{-1} = \mathbf{S}\mathbf{D}^2\mathbf{S}^{-1}$$

We can generalize, if $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$ then $\mathbf{A}^k = \mathbf{S}\mathbf{D}^k\mathbf{S}^{-1}$, for $k = 1, 2, \ldots$.

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Diagonalisable matrices

If this is so, then

$$\mathbf{S}\ \mathbf{D}\ \mathbf{S}^{-1} = \mathbf{S}\ \mathbf{S}^{-1}\mathbf{A}\ \mathbf{S}\ \mathbf{S}^{-1} = \mathbf{I}\ \mathbf{A}\ \mathbf{I} = \mathbf{A}$$

It follows that

$$\mathbf{A}^2 = (\mathbf{S}\mathbf{D}\mathbf{S}^{-1})(\mathbf{S}\mathbf{D}\mathbf{S}^{-1}) = \mathbf{S}(\mathbf{D}\mathbf{I}\mathbf{D})\mathbf{S}^{-1} = \mathbf{S}\mathbf{D}^2\mathbf{S}^{-1}$$

We can generalize, if $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$ then $\mathbf{A}^k = \mathbf{S}\mathbf{D}^k\mathbf{S}^{-1}$, for $k = 1, 2, \ldots$.

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Diagonalisable matrices

If this is so, then

$$\underset{S^{-1}AS}{\mathbf{S}\ \mathbf{D}\ \mathbf{S}^{-1}} = \underbrace{\mathbf{S}\ \mathbf{S}^{-1}}_{I}\mathbf{A}\ \underbrace{\mathbf{S}\ \mathbf{S}^{-1}}_{I} = \mathbf{I}\ \mathbf{A}\ \mathbf{I} = \mathbf{A}$$

It follows that

$$\mathbf{A}^2 = (\mathbf{S}\mathbf{D}\mathbf{S}^{-1})(\mathbf{S}\mathbf{D}\mathbf{S}^{-1}) = \mathbf{S}(\mathbf{D}\mathbf{I}\mathbf{D})\mathbf{S}^{-1} = \mathbf{S}\mathbf{D}^2\mathbf{S}^{-1}$$

We can generalize, if $\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{D}$ then $\mathbf{A}^k = \mathbf{S}\mathbf{D}^k\mathbf{S}^{-1}$, for $k = 1, 2, \ldots$.

The questions arise :

1) How can we tell if a given matrix is diagonalisable ?
2) and if it is, how do we find $\mathbf{S}$ and $\mathbf{D}$?

To answer these questions, we introduce the concepts of **eigenvalue** and **eigenvector**.

Eigenvalues and eigenvectors

Definition Let $\mathbf{A}$ be a square matrix, λ a scalar. We say that λ is an eigenvalue of $\mathbf{A}$ if there exists a vector $\mathbf{x}$ such that $\mathbf{x} \neq 0$ and

$$\boxed{\mathbf{A}\mathbf{x} = \lambda\mathbf{x}}$$

$\mathbf{x}$ is called an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ.

Another way of writing the equation $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ is

$$(\lambda\mathbf{I} - \mathbf{A})\mathbf{x} = 0$$

$$\begin{pmatrix} - & - \\ c_1 & c_2 \cdots c_3 \cdots c_n \\ - & - \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \vdots \end{pmatrix}$$

Thus, λ is an eigenvalue of $\mathbf{A}$ if and only if $\lambda\mathbf{I} - \mathbf{A}$ is a **singular matrix**.

$$x_1 c_1 + x_2 c_2 + \cdots x_n c_n = 0$$

Eigenvalues and eigenvectors

Therefore, λ is an eigenvalue of $\mathbf{A}$ if and only if

$$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

For a $n \times n$ matrix $\mathbf{A}$, $\det(\lambda\mathbf{I} - \mathbf{A})$ is a nth order polynomial in the variable λ, called the **characteristic polynomial** of $\mathbf{A}$.

A nth order polynomial has at most n roots. Therefore, a 2×2 matrix has at most 2 eigenvalues ; an $n \times n$ matrix has at most n eigenvalues.

$a\lambda^2 + b\lambda + c = 0$

Eigenvalues and eigenvectors

Example : consider the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$$

Since

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{pmatrix}$$

Then

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det\begin{pmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{pmatrix} = \lambda^2 - 3\lambda + 2 = -\lambda(3-\lambda) + 2$$

The characteristic polynomial of $\mathbf{A}$ is $\lambda^2 - 3\lambda + 2 = 0$

$\Rightarrow$ So the eigenvalues of $\mathbf{A}$ are $\lambda = 1$ and $\lambda = 2$.

Eigenvalues and eigenvectors

By definition we have $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ so we can rewrite $(\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0$ as

$$\begin{pmatrix} 3-\lambda & 2 \\ -1 & -\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For $\lambda = 1$

$$\begin{pmatrix} 2 & 2 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We can easily see that $x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$. Taking for example $x_1 = 1$, we can conclude that one eigenvector is $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ -x_1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. But another candidate could be $x_1 = -1$, and one eigenvector is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Thus the eigenvectors of $\mathbf{A}$ corresponding to $\lambda = 1$ are the **nonzero vectors of the form** $\mathbf{x} = \begin{pmatrix} k \\ -k \end{pmatrix} = k \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ for $k \in \mathbb{R}$.

$\lambda = 2$

$$A - \lambda I = \begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ -1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 + 2x_2 = 0$$

$$x_1 = -\frac{1}{2} x_2 \implies x = \begin{pmatrix} k \\ -0.5k \end{pmatrix} = k \begin{pmatrix} 1 \\ -0.5 \end{pmatrix}$$

for $k \in \mathbb{R}$

$\lambda_1 = 2 \; ; \; \lambda_2 = -1$

$\lambda = 2 \Rightarrow A - \lambda I = \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}$

$x_1 = x_2 \Rightarrow X = \begin{pmatrix} k \\ k \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ for $k \in \mathbb{R}$

$\lambda = 1 \Rightarrow A - \lambda I = \begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix}$

$\Rightarrow 4x_1 - x_2 = 0$

$x_1 = \frac{1}{4} x_2 \Rightarrow X = k \begin{pmatrix} 0.25 \\ 1 \end{pmatrix}$ for $k \in \mathbb{R}$

Eigenvalues and eigenvectors

Exercise 13

Find the eigenvalues and eigenvectors of $A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$.

$$\det(A - \lambda I) = 0$$

$$\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (3-\lambda)(-2-\lambda) - (4)(-1) = \lambda^2 - \lambda - 2$$

$$\lambda^2 - \lambda - 2 = 0 \quad \begin{cases} \lambda = 2 \\ \lambda = -1 \end{cases}$$

Eigenvalues and eigenvectors

The following propositions connect eigenvalues and eigenvectors with diagonalisability.

Proposition 1 An $n \times n$ matrix $\mathbf{A}$ is diagonalisable if and only if it has n linearly independent eigenvectors.

Proposition 2 If $\mathbf{x}^1, \ldots, \mathbf{x}^k$ are eigenvectors corresponding to k different eigenvalues of the $n \times n$ matrix $\mathbf{A}$, then $\mathbf{x}^1, \ldots, \mathbf{x}^k$ are linearly independent.

Proposition 3 An $n \times n$ matrix $\mathbf{A}$ is diagonalisable if it has n different eigenvalues.

Eigenvalues and eigenvectors

To prove **Proposition 1**, first suppose the $n \times n$ matrix $\mathbf{A}$ has n linearly independent eigenvectors $\mathbf{x}^1, \ldots, \mathbf{x}^n$, corresponding to eigenvalues $\lambda_1, \ldots, \lambda_n$ respectively. Then,

$$\begin{array}{cc} & x_1 \quad x_2 \cdots x_n \\ & \begin{pmatrix} x_1\lambda_1 & & \\ & \ddots & \\ & & \lambda_n x_n \end{pmatrix} \end{array}$$

$$\mathbf{A}\mathbf{x}^j = \lambda_j \mathbf{x}^j \quad \text{for } j = 1, \ldots, n$$

Define the two $n \times n$ matrices

$$\mathbf{S} = [\mathbf{x}^1 \ldots \mathbf{x}^n], \quad \mathbf{D} = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

Since $\mathbf{x}^1, \ldots, \mathbf{x}^n$ are linearly independent, $\mathbf{S}$ is invertible. Therefore,

$$S^{-1}\mathbf{AS} = \underset{I}{S^{-1}S}\mathbf{D}$$

and we have $\mathbf{S}^{-1}\mathbf{AS} = \mathbf{D}$, so $\mathbf{A}$ is a d-matrix.

Eigenvalues and eigenvectors

We now prove **Proposition 2** in the special case $k = 3$.

Let $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ be eigenvectors of $\mathbf{A}$ corresponding to three distinct eigenvalues λ, μ, ν, thus

$$\mathbf{A}\mathbf{x} = \lambda\mathbf{x}, \quad \mathbf{A}\mathbf{y} = \mu\mathbf{y}, \quad \mathbf{A}\mathbf{z} = \nu\mathbf{z} \quad (\textbf{Eq.1})$$

and none of $\mathbf{x}, \mathbf{y}, \mathbf{z}$ is 0. Let α, β, γ be scalars such that

$$\alpha\mathbf{x} + \beta\mathbf{y} + \gamma\mathbf{z} = 0 \quad (\textbf{Eq.2})$$

We wish to show that $\alpha = \beta = \gamma = 0$ to prove that $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$ are linearly independent.

Recall that $\mathbf{b}^1, \mathbf{b}^2, \ldots, \mathbf{b}^k$ are **linearly independent** if none of the vectors can be expressed as a linear combination of the others.

Eigenvalues and eigenvectors

Pre-multiplying **Eq.2** by $\lambda\mathbf{I} - \mathbf{A}$ and using **Eq.1**, we see that

$$\beta(\lambda - \mu)\mathbf{y} + \gamma(\lambda - \nu)\mathbf{z} = 0 \quad (\textbf{Eq.3})$$

Premultiplying **Eq.3** by $\mu\mathbf{I} - \mathbf{A}$, and using **Eq.1**, we see that

$$\gamma(\lambda - nu)(\mu - \nu)\mathbf{z} = 0$$

This implies that $\gamma = 0$, $\beta = 0$ (**Eq.3**) and $\alpha = 0$ (**Eq.2**).

Proposition 3 is therefore a sufficient condition for **A** to be a d-matrix.

$$Ax = \lambda x \qquad Ay = \mu y \qquad Az = \nu z$$

$$(\underbrace{\lambda I - A})x = 0 \qquad (\mu I - A)y = 0 \qquad (\nu I - A)z = 0$$

$$\alpha x + \beta y + \gamma z = 0$$

$$(\lambda I - A)(\alpha x + \beta y + \gamma z) = 0$$

$$\underbrace{\alpha(\lambda - IA)x}_{=0} + \beta(\lambda I - A)y + \gamma(\lambda I - A)z = 0$$

$$\beta\lambda y - \beta \underbrace{Ay}_{\mu y} + \gamma\lambda z - \gamma \underbrace{Az}_{\nu z} = 0$$

$$\boxed{\beta(\lambda - \mu)y + \gamma(\lambda - \nu)z = 0}$$

$$\underbrace{(\mu I - A)\beta(\lambda - \mu)y}_{=0} + (\mu I - A)\gamma(\lambda - \nu)z = 0$$

$$\mu\gamma(\lambda - \nu)z - \gamma(\lambda - \nu)\underbrace{Az}_{\nu z} = 0$$

$$\mu\gamma(\lambda - \nu)z - \gamma(\lambda - \nu)\nu z = 0$$

$$\gamma(\lambda - \nu)(\mu - \nu)z = 0$$

$$\Rightarrow \gamma = 0$$

$$\Rightarrow \beta = 0$$

$$\Rightarrow \alpha = 0$$

Eigenvalues and eigenvectors

Exercise 14

Show that $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$ is a d-matrix, and find $\mathbf{A}^k$ for any positive integer k. Hint : keep in the form $\mathbf{A}^k = \mathbf{S}\mathbf{D}^k\mathbf{S}^{-1}$

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$$

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 3-\lambda & -1 \\ 4 & -2-\lambda \end{vmatrix}$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0$$

$$\lambda_1 = 2$$

$$\lambda_2 = -1$$

$$D^k = \begin{pmatrix} 2^k & 0 \\ 0 & (-1)^k \end{pmatrix}$$

$$\lambda_1 : \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 : \begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}$$

$$S^{-1} = \frac{1}{3} \begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix}$$

$$A^2 = SD\cancel{S^{-1}S}DS^{-1}$$

$$A^k = SD^kS^{-1}$$

Eigenvalues and eigenvectors

Let **A** be a **symmetric matrix**. Then, **A** is

- *positive definite* if and only if all its eigenvalues > 0
- *positive semidefinite* if and only if all its eigenvalues are ≥ 0
- *negative definite* if and only if all it eigenvalues are < 0
- *negative semidefinite* if and only if all it eigenvalues are ≤ 0
- *indefinite* if and only if A has a positive eigenvalue and a negative eigenvalue.

Some left properties

- **A** and $\mathbf{A}^T$ have the same eigenvalues but not necessarily the same eigenvectors.
- If λ is an eigenvalue of **A**, then $\frac{1}{\lambda}$ is an eigenvalue of $\mathbf{A}^{-1}$.

Let **A** be a $k \times k$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_k$. Then,

- $\lambda_1 + \lambda_2 + \cdots + \lambda_k =$ trace of **A**
- $\lambda_1 \times \lambda_2 \times \ldots \lambda_k = \det \mathbf{A}$.

Exercise 15 Compute the eigenvalues and eigenvectors of the following 3×3 matrix. Determine its definiteness.

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 3 & 0 & 2 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.25 & -0.3 \\ -0.3 & 0.5 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.25 & -0.3 \\ -0.3 & 0.5 \end{pmatrix}$$

$$\det(B - \lambda I) = \det \begin{pmatrix} 0.25-\lambda & -0.3 \\ -0.3 & 0.5-\lambda \end{pmatrix}$$

$$= \lambda^2 - 0.75\lambda + 0.035 = 0.7 \times 0.05$$

$$= (\lambda - 0.7)\cdot(\lambda - 0.05)$$

$\lambda_1 = 0.7$, $\lambda_2 = 0.05$

positive definite

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 3 & 0 & 2 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$A - \lambda I = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 3 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & 5-\lambda & 0 \\ 3 & 0 & 2-\lambda \end{bmatrix}$$

$$\underbrace{(5-\lambda)}_{5}\left((1-\lambda)(2-\lambda) - 6\right) = 0$$

$$\det(A - \lambda I) = (1-\lambda)\left\{(5-\lambda)(2-\lambda) - 0\right\} + 2\left\{0 - 3(5-\lambda)\right\}$$

$$= -11\lambda - 20 + 8\lambda^2 - \lambda^3$$

$$\det(A-\lambda I)=0$$

$$-\lambda^3+8\lambda^2-11\lambda-20=0$$

$$\Rightarrow \lambda^3-8\lambda^2+11\lambda+20=0$$

For $\lambda=-1$ the above expression is 0

$$\lambda^2(\lambda+1)-9\lambda(\lambda+1)+20(\lambda+1)=0$$

$$\Rightarrow (\lambda+1)(\lambda^2-9\lambda+20)=0$$

$$\lambda^2 - (5+4)\lambda + 20 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda - 4\lambda + 20 = 0$$

$$\Rightarrow \lambda(\lambda - 5) - 4(\lambda - 5) = 0$$

$$\therefore \lambda = -1 \text{ or } \lambda = 5 \text{ or } \lambda = 4$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

For $\lambda = -1$

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 6 & 0 \\ 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 2x_1 + 2x_3 = 0$

$\Rightarrow x_1 = -x_3$

and $6x_2 = 0 \Rightarrow x_2 = 0$

$\therefore$ for $\lambda = -1$, eigen vector : $\begin{bmatrix} z \\ 0 \\ -z \end{bmatrix}$, $z \in R$

For $\lambda = 5$

$$\begin{bmatrix} -4 & 0 & 2 \\ 0 & 0 & 0 \\ 3 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow -4x_1 + 2x_3 = 0$ —— (1)

$\Rightarrow 2x_1 = x_3$

$x_2 \in \mathbb{R}$

$3x_1 - 3x_3 = 0$ —— (2)

From (1) and (2) we get

$x_1 = x_3 = 0$

For 5, eigenvector : $\begin{bmatrix} 0 \\ k \\ 0 \end{bmatrix}$, $k \in R$

For $\lambda = 4$

$$\begin{bmatrix} -3 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore -3x_1 + 2x_3 = 0$ — (1)

$x_2 = 0$ — (2)

$3x_1 - 2x_3 = 0$ — (3)

$\therefore x_1 = \frac{2x_3}{3}$, $x_2 = 0$

For 4, eigenvector : $\begin{bmatrix} 2p \\ 0 \\ 3p \end{bmatrix}$, $p \in R$

Kronecker product

Kronecker product

iota

Given A a matrix of dimensions n x m and B another matrix of dimension p x q. The Kronecker product between A and B, denoted as $A \otimes B$, is a matrix of dimension np x mq

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{pmatrix}$$

$$\iota_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\underset{(2\times 2)}{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

$$\underset{2\times 1}{B} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\underset{(4\times 2)}{A \otimes B} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 4 \\ 6 & 8 \end{pmatrix}$$

Exercise 16

Let e' be defined as $e' = (1 \;\; 1 \;\; 0)$, compute the following products

- $I_2 \otimes e$; $e \otimes I_2$; $I_2 \otimes e'$
- $(e \otimes \iota_2 \iota_2')$, $(e' \otimes \iota_2' \iota_2)$
  Note : ι corresponds to a vector formed by ones

(i) $I_2 \otimes e$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} \otimes \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{3\times 1}$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$e \otimes I_2$

$$= \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{3\times 1} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$I_2 \otimes e'$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2} \otimes \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}_{1\times 3}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$l_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad l_2' = \begin{bmatrix} 1 & 1 \end{bmatrix} \qquad l_2 l_2' = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

2×1 $\qquad$ 1×2 $\qquad$ 2×2

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix} \; 2$$

$$e \otimes l_2 l_2' = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

3×1 $\qquad$ 2×2 $\qquad$ 6×2

$$l_2' l_2 = 2$$

$$e' \otimes l' l_2 = e' \otimes 2 = e' = [2\ 2\ 0]$$

Kronecker product

Some properties

- $A \otimes B \neq B \otimes A$
- $A \otimes B \otimes C = (A \otimes B) \otimes C = A \otimes (B \otimes C)$
- $(A + B) \otimes (C + D) = A \otimes C + A \otimes D + B \otimes C + B \otimes D$
- $(A \otimes B)(C \otimes D) = AC \otimes BD$
- $(A \otimes B)' = A' \otimes B'$
- $tr(A \otimes B) = tr(A) \cdot tr(B)$, if A and B square
- $r(A \otimes B) = r(A) \cdot r(B)$, if A and B square
- $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$, if A and B non-singular
- $|A \otimes B| = |A|^p |B|^m$, where A is m x m and B is p x p
- $\alpha \otimes A = \alpha A$, when α is a scalar
- $x' \otimes y = yx' = y \otimes x$, where x and y are two vectors