Derivations of equations in Justiniano and Preston (JAE, 2010)*

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Capital letters denote variables in levels, small ones denote log deviations from steady state. State states are capital letters without time subscript. All references to equations like "eq. (1)" refer to Justiniano and Preston (2010).
The variables are C_t, Y_t, R_t, Q_t, S_t, π_t, $\tilde{P}_{h,t}$, $\tilde{P}_{f,t}$, Y_t^*, π_t^*, R_t^*, A_t. This note should come with Dynare codes that allow to simulate the model at first order (`JP10.mod`) and at second order (`JP10nl.mod`).[1]

1 The equations in levels

Household Euler equation

$$(C_t - hC_{t-1})^{-\sigma} \frac{\exp(\sigma_g \varepsilon_t^g)}{\exp(\sigma_g \varepsilon_{t+1}^g)} = \frac{\beta R_t}{\pi_{t+1}} (C_{t+1} - hC_t)^{-\sigma} \tag{1}$$

From (external habits! i.e. $H_t = hC_{t-1}$)

$$\max_{C_t, N_t, D_t, B_t} E_0 \sum_{t=0}^{\infty} \beta^t \exp(\sigma_g \varepsilon_t^g) \left[\frac{(C_t - H_t)^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\psi}}{1+\psi} \right]$$
$$\text{s.t. } C_t + \frac{D_t}{P_t} + \frac{e_t B_t}{P_t} = \frac{R_{t-1} D_{t-1}}{P_t} + \frac{e_t B_{t-1} R_{t-1}^* \phi(A_t)}{P_t} + \frac{W_t}{P_t} N_t + \frac{\Pi_{h,t} + \Pi_{f,t} + T_t}{P_t}$$

FOCs w.r.t. C_t and D_t give

$$\lambda_t = \exp(\sigma_g \varepsilon_t^g)(C_t - hC_{t-1})^{-\sigma}$$
$$\lambda_t = \frac{\beta R_t}{\pi_{t+1}} \lambda_{t+1},$$

from which the Euler equation follows.

Stochastic discount factor

$$\lambda_t = (C_t - hC_{t-1})^{-\sigma} \exp(\sigma_g \varepsilon_t^g) \tag{2}$$

The FOC w.r.t. C_t above [Eq. (2), corrected for the right exponent.]

Goods market clearing

$$Y_t = \tilde{P}_{h,t}^{-\eta} [(1-\alpha) C_t + Q_t^{\eta} Y_t^*] \tag{3}$$

This requires the following: Start with eq. (13) and insert the equation below (13) as well as the one for $C_{H,t}$ (1), also using $\lambda = \eta$. Then use $\tilde{P}_{h,t} = \frac{P_{H,t}}{P_t}$, $P_{H,t}^* = \frac{P_{H,t}}{\tilde{e}_t}$ and $Q_t = \frac{\tilde{e}_t P_t^*}{P_t}$.

*This note and the codes coming with it have been prepared by myself, based on the authors' paper. Any errors are my own. If you find any, drop me a line under benedikt@bkolb.eu!

[1] This note and the codes are available under www.bkolb.eu/codes/JP10.zip.

Terms of trade

$$S_t = \tilde{P}_{f,t}/\tilde{P}_{h,t} \tag{4}$$

From $S_t = P_{f,t}/P_{h,t}$ as well as $\tilde{P}_{h,t} = P_{h,t}/P_t$ and $\tilde{P}_{f,t} = P_{f,t}/P_t$.

Real exchange rate

$$Q_t = \frac{e_t P_t^*}{P_t} \tag{5}$$

Eq. (17)

Retailer markup

$$\mu_{h,t} = \lambda_t \tilde{P}_{h,t} \frac{[\exp(\varepsilon_t^a)]^{1+\psi}}{\exp(\varepsilon_t^g)} Y_t^{-\psi} \tag{6}$$

The (nominal) price of the (perfectly competitive) wholesale producer is its marginal cost, so W_t/ε_t^a, so that the (real) marginal cost of the retailer is

$$MC_t = \frac{W_t}{P_{h,t}\exp(\varepsilon_t^a)}.$$

Using the household FOC w.r.t. N_t and the aggregate production function,

$$\lambda_t = \exp(\varepsilon_t^g)\frac{P_t N_t^\psi}{W_t}$$
$$Y_t = \exp(\varepsilon_t^a) N_t,$$

we can solve for the marginal cost as

$$MC_t = \frac{P_t N_t^\psi}{P_{h,t}\lambda_t}\frac{\exp(\varepsilon_t^g)}{\exp(\varepsilon_t^a)} = \frac{Y_t^\psi}{\tilde{P}_{h,t}\lambda_t}\frac{\exp(\varepsilon_t^g)}{[\exp(\varepsilon_t^a)]^{1+\psi}}$$

Finally, we define the markup as the inverse of the (real) marginal cost, $\mu_{h,t} = 1/MC_t$, see Galí (2008), p74.

Retailer optimal prices

$$DAC_{h,t}\frac{\tilde{P}_{h,t}}{\tilde{P}_{h,t-1}}\pi_t = \epsilon_h(\mu_{h,t}^{-1} + AC_{h,t}) - (\epsilon_h - 1) + \beta\frac{\lambda_{t+1}}{\lambda_t}\frac{Y_{t+1}}{Y_t} DAC_{h,t+1}\frac{\tilde{P}_{h,t+1}}{\tilde{P}_{h,t}}\pi_{t+1} \tag{7}$$

Retailer owned by households, so same discounting. Buys output from wholesale producer and transforms costlessly into retail good. Monopolistic competition, Rotemberg (1982) style quadratic adjustment costs for changing prices.

$$\max_{P_{h,t}(z)} E_0 \sum_{t=0}^{\infty} \beta^t \frac{\lambda_t}{\lambda_0}\left\{\left[\frac{P_{h,t}(z) - P_{h,t}/\mu_{h,t}}{P_{h,t}} - \frac{\chi_h}{2}\left(\frac{P_{h,t}(z)}{P_{h,t-1}(z)} - 1\right)^2\right]\left(\frac{P_{h,t}(z)}{P_{h,t}}\right)^{-\epsilon_h} Y_t\right\}$$

Take the FOC, then use the fact that $P_{h,t}(z) = P_{h,t}$ in the aggregate and the following two definitions,

$$AC_{h,t} \equiv \frac{\chi_h}{2}\left(\frac{P_{h,t}}{P_{h,t-1}} - 1\right)^2 = \frac{\chi_h}{2}\left(\frac{\tilde{P}_{h,t}}{\tilde{P}_{h,t-1}}\pi_t - 1\right)^2$$
$$DAC_{h,t} \equiv \frac{\partial AC_{h,t}}{\partial \frac{P_{h,t}}{P_{h,t-1}}} = \chi_h\left(\frac{P_{h,t}}{P_{h,t-1}} - 1\right) = \chi_h\left(\frac{\tilde{P}_{h,t}}{\tilde{P}_{h,t-1}}\pi_t - 1\right)$$

to arrive at the above equation.

Retailer price adjustment costs

$$AC_{h,t} = \frac{\chi_h}{2}\left(\frac{P_{h,t}}{P_{h,t-1}}\pi_t - 1\right)^2 \tag{8}$$

From above.

Retailer price adjustment costs (derivative)

$$DAC_{h,t} = \chi_h\left(\frac{P_{h,t}}{P_{h,t-1}}\pi_t - 1\right) \tag{9}$$

From above.

Importers' optimal prices

$$DAC_{f,t}\frac{\tilde{P}_{f,t}}{\tilde{P}_{f,t-1}}\pi_{f,t} = \epsilon_f\left(\frac{Q_t}{\tilde{P}_{f,t}} + AC_{f,t}\right) - (\epsilon_f - 1) + \beta\frac{\lambda_{t+1}}{\lambda_t}\frac{C_{t+1}}{C_t}\left(\frac{\tilde{P}_{f,t+1}}{\tilde{P}_{f,t}}\right)^{1-\eta} DAC_{f,t+1}\pi_{f,t+1} \tag{10}$$

Importer owned by households, so same discounting. Buys output from foreign retailer at price $e_t P_t^*$ and transforms costlessly into import good. Monopolistic competition, Rotemberg (1982) style quadratic adjustment costs for changing prices. So the problem for the importer is given by

$$\max_{P_{f,t}(z)} E_0\sum_{t=0}^{\infty}\beta^t\frac{\lambda_t}{\lambda_0}\left\{\left[\frac{P_{f,t}(z) - e_t P_t^*}{P_{f,t}} - \frac{\chi_f}{2}\left(\frac{P_{f,t}(z)}{P_{f,t-1}(z)} - 1\right)^2\right]\left(\frac{P_{f,t}(z)}{P_{f,t}}\right)^{-\epsilon_f} C_{f,t}\right\}$$

Take the FOC, then use the fact that $P_{f,t}(z) = P_{f,t}$ in the aggregate, the demand for foreign goods,

$$C_{f,t} = \alpha\tilde{P}_{f,t}^{-\eta}C_t$$

and the following two definitions,

$$AC_{f,t} \equiv \frac{\chi_f}{2}\left(\frac{P_{f,t}}{P_{f,t-1}} - 1\right)^2 = \frac{\chi_f}{2}\left(\frac{\tilde{P}_{f,t}}{\tilde{P}_{f,t-1}}\pi_t - 1\right)^2$$

$$DAC_{f,t} \equiv \frac{\partial AC_{f,t}}{\partial\frac{P_{f,t}}{P_{f,t-1}}} = \chi_f\left(\frac{P_{f,t}}{P_{f,t-1}} - 1\right) = \chi_f\left(\frac{\tilde{P}_{f,t}}{\tilde{P}_{f,t-1}}\pi_t - 1\right)$$

to arrive at the above equation.

Importer price adjustment costs

$$AC_{f,t} = \frac{\chi_f}{2}\left(\frac{\tilde{P}_{f,t}}{\tilde{P}_{f,t-1}}\pi_t - 1\right)^2 \tag{11}$$

From above.

Importer price adjustment costs (derivative)

$$DAC_{f,t} = \chi_f\left(\frac{\tilde{P}_{f,t}}{\tilde{P}_{f,t-1}}\pi_t - 1\right) \tag{12}$$

From above.

Domestic CPI

$$1 = (1-\alpha)\tilde{P}_{h,t}^{1-\eta} + \alpha S_t^{1-\eta}\tilde{P}_{h,t}^{1-\eta} \tag{13}$$

From the price identity

$$P_t = \left[(1-\alpha)P_{h,t}^{1-\eta} + \alpha P_{f,t}^{1-\eta}\right]^{\frac{1}{1-\eta}}$$

Uncovered interest rate parity

$$R_t = \frac{Q_{t+1}}{Q_t}\frac{\pi_{t+1}}{\pi_{t+1}^*}R_t^* \exp(-\chi A_t)\exp(\varepsilon_t^p) \tag{14}$$

From household FOCs w.r.t. D_t and B_t are

$$\lambda_t = \frac{\beta R_t}{\pi_{t+1}}\lambda_{t+1}$$
$$\lambda_t = \frac{\beta R_t^*}{\pi_{t+1}}\frac{e_{t+1}}{e_t}\phi(A_{t+1})\lambda_{t+1},$$

which gives

$$R_t = R_t^*\frac{e_{t+1}}{e_t}\phi(A_{t+1})$$

Now simply use $Q_t \equiv e_t P_t^*/P_t$ so that $\frac{e_{t+1}}{e_t} = \frac{Q_{t+1}}{Q_t}\frac{\pi_{t+1}^*}{\pi_{t+1}}$.
Note that the χ in front of the shock ϕ_t on p. 96 is missing in eq. (21).

Flow budget constraint

$$C_t + Y\cdot A_t = \frac{Q_t}{Q_{t-1}}\frac{\pi_t}{\pi_t^*}A_{t-1}R_{t-1}^*\exp(-\chi A_t)\exp(\sigma_p\varepsilon_t^p) + \tilde{P}_{h,t}Y_t + \alpha(\tilde{P}_{f,t} - Q_t)\tilde{P}_{f,t}^{-\eta}C_t \tag{15}$$

From the flow budget constraint on p. 96, with the following alterations:

1. $D_t = 0\ \forall t$: Domestic bonds are in net zero supply, see p. 100 (top)
2. $W_tN_t + \Pi_{h,t} + \Pi_{f,t} = P_{h,t}Y_{h,t} + (P_{f,t} - \tilde{e}_tP_t^*)C_{f,t}$, see p. 97 (top)
3. $T_t = 0$: Lump-sum taxes are not further specified, see p. 96 bottom
4. $A_t = \frac{\tilde{e}_tB_t}{Y\cdot P_t}$, see p. 101 bottom (on p. 96 bottom the timing is wrong!)

Defining $\tilde{P}_{h,t} \equiv P_{h,t}/P_t$, $\tilde{P}_{f,t} \equiv P_{f,t}/P_t$, using $Q_t \equiv \tilde{e}_tP_t^*/P_t$, $C_{f,t} = \alpha\tilde{P}_{f,t}^{-\eta}C_t$ and calling $Y_{h,t} = Y_t$, this gives

$$C_t + Y\cdot A_t = \frac{\tilde{e}_t}{\tilde{e}_{t-1}}\frac{R_{t-1}^*}{\pi_t}A_{t-1}\phi(A_t) + \tilde{P}_{h,t}Y_t + \alpha(\tilde{P}_{f,t} - Q_t)\tilde{P}_{f,t}^{-\eta}C_t$$

We obtain the log-linearised eq. (22) using the following: $C = Y$, $A/Y = 0$, $\tilde{P}_f = \tilde{P}_h = Q = 1$, $(A_t - A) = a$, $R^* = 1/\beta$, $\phi(A) = e^0 = 1$, as well as the log-linear relationships $q_t = \psi_{F,t} + (1-\alpha)s_t$, $\tilde{p}_{h,t} = \tilde{p}_{f,t} - s_t$ and $\tilde{p}_{h,t} = -\alpha s_t$[2].

[2]From the price index in log deviations:

$$0 = (1-\alpha)\tilde{p}_{h,t} + \alpha\tilde{p}_{f,t}$$

Taylor Rule

$$R_t = R_{t-1}^{\psi_r} R^{1-\psi_r} \Pi_t^{\psi_p} (Y_t/Y)^{\psi_Y} \exp(\varepsilon_t^r) \tag{16}$$

Eq. (23), with gross instead of net interest rates and a steady-state of interest rates in the equation.

AR(1) for foreign GDP

$$Y_t^* = \left(Y_{t-1}^*\right)^{\rho_{y*}} \left(Y^*\right)^{1-\rho_{y*}} \exp(\varepsilon_t^{y*}) \tag{17}$$

AR(1) for foreign inflation

$$\Pi_t^* = \left(\Pi_{t-1}^*\right)^{\rho_{\pi*}} \left(\Pi^*\right)^{1-\rho_{\pi*}} \exp(\varepsilon_t^{\pi*}) \tag{18}$$

AR(1) for foreign interest rate

$$R_t^* = \left(R_{t-1}^*\right)^{\rho_{r*}} \left(R^*\right)^{1-\rho_{r*}} \exp(\varepsilon_t^{r*}) \tag{19}$$

2 Errata in Justiniano and Preston (2010)

p96 bottom: The timing in the definition of A_t is wrong, it should read $A_t \equiv \frac{\tilde{e}_t B_t}{Y \cdot P_t}$ (compare to p101 bottom)

p97, eq (2): The exponent should be $-\sigma$, not $1-\sigma$.

p97, eq (4) and (5): P_t and P_{t+1} should be reversed.

p99, third equation: The θ_H should be θ_F and the $P_{H,t}^*(i)$ a $P_{F,t}^*(i)$.

p99, bottom (foreign demand for domestic goods) A vital point not mentioned explicitely is that $\lambda = \eta$ as in Monacelli (2005).

p101, eq (21) If a_t is supposed to be the log-linearised version of A_t (i.e. $a_t = A_t$ as $A = 0$), then the timing is inconsistent with $\phi_t = \exp[-\chi(A_t + \tilde{\phi}_t)]$ on p96 (bottom) – it would have to be a_{t+1} then. Also the χ disappears in front of the shock term – assuming that ϕ_t in in eq (21) is supposed to be the (loglinearised) shock $\tilde{\phi}_t$ on p96. In general, it seems easier to define $\phi(A_t) = \exp[-\chi A_t + \varepsilon_t^{rp}]$ and $A_t \equiv e_t B_t/(Y \cdot P_t)$.

p102, eq (23) For the specified Taylor rule, an additional model variable/equation, $\Delta e_t = \Delta q_t + \pi_t - \pi_t^*$, is required.

References

Galí, J. (2008). *Monetary Policy, Inflation, and the Business Cycle: An Introduction to the New Keynesian Framework.* Princeton University Press.

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Monacelli, T. (2005, December). Monetary policy in a low pass-through environment. *Journal of Money, Credit, and Banking 37*(6), 1047–1066.

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