Geneva Graduate Institute, Econometrics II

Problem Set 4 Solutions

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1 Problem 1

This excercise is inspired by McConnell, Margaret and Gabriel Perez-Quiros (2000): “Output Fluctuations in the United States: What has changed since the early 1980’s?" *American Economic Review*, 90(5), 1464-76.

Download some aggregate time series in quarterly frequency from FRED: Real GDP (GDPC1), Real Personal Consumption Expenditure (PCECC96), Real Gross Private Domestic Investment (GPDIC1).

1. Take logs of GDP, Consumption, and Investment. Plot the three series (you can generate the plots directly in FRED). What are the most striking features?

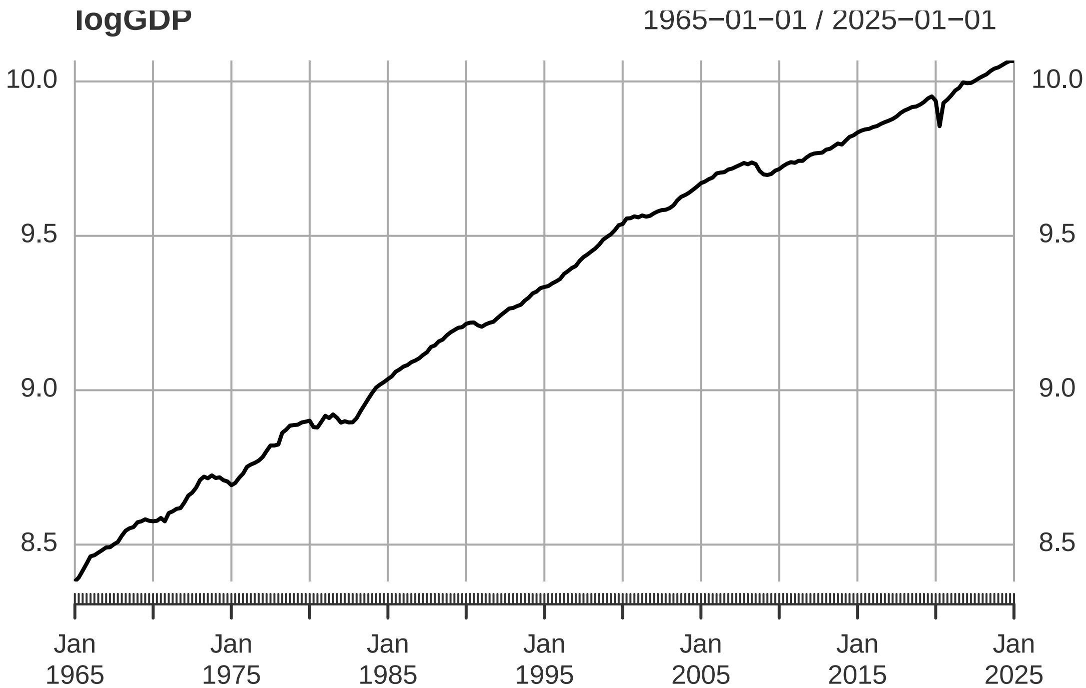
Solution: We use the following variables:

- Real Gross Domestic Product (GDPC1): https://fred.stlouisfed.org/series/GDPC1
- Real Personal Consumption Expenditures (PCECC96): https://fred.stlouisfed.org/series/PCECC96
- Real Gross Private Domestic Investment (GPDIC1): https://fred.stlouisfed.org/series/GPDIC1

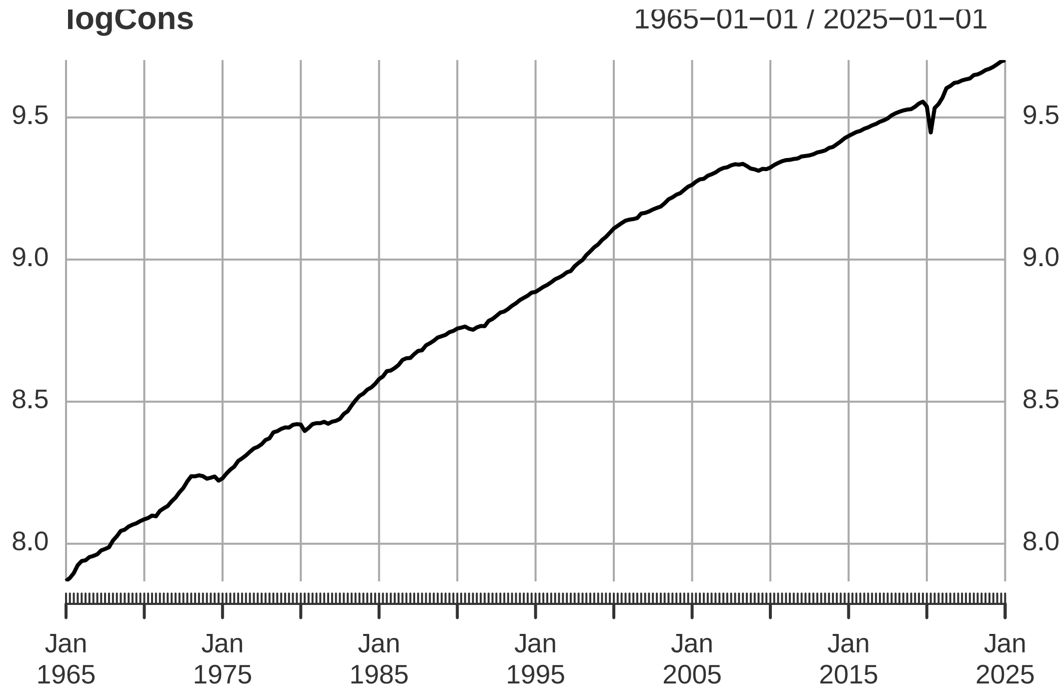
This code below reads the data and plots the series for the time period 1965:1-2022:2.

```
# Download series:

GDP <- getSymbols("GDPC1", src = "FRED", auto.assign = FALSE)
Cons <- getSymbols("PCECC96", src = "FRED", auto.assign = FALSE)
NFI <- getSymbols("GPDIC1", src = "FRED", auto.assign = FALSE)


# Define desired date range:

date.start <- "1965-01-01"
date.end <- "2025-03-01"
mytimerange <- function(x) {
    x[paste(date.start, date.end, sep = "/")]
}
```

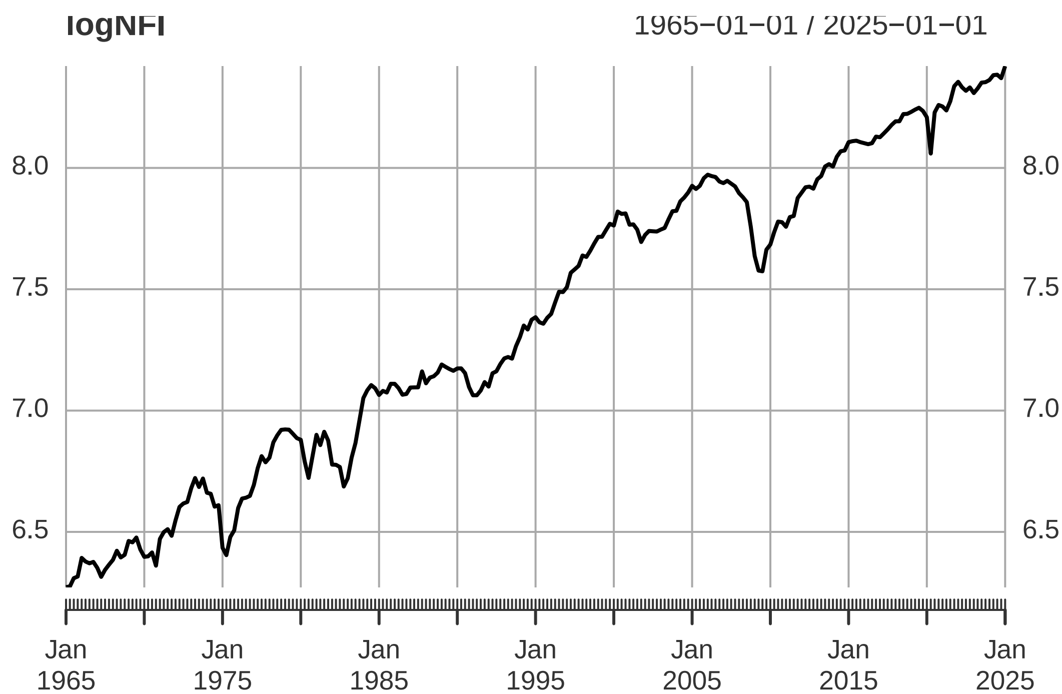
```r
# Take logs and take date range defined above:

logGDP <- log(mytimerange(GDP))
logCons <- log(mytimerange(Cons))
logNFI <- log(mytimerange(NFI))


# Plot series:

plot(logGDP)
```




```r
plot(logCons)
```




```r
plot(logNFI)
```



2. For each series (in logs), estimate a model of the form

$$y_t = \beta_1 + \beta_2 t + u_t$$

using OLS, based on the samples 1965:Q1 to 2006:Q4, 2007:Q1 to 2019:Q4, and 2007:Q1 to 2022:Q2.

Solution: The estimated coefficients for the sample 1965:1 - 2006:4 are printed below. To be concise, I will just print the results for this sample, but you can easily obtain the results for other samples by changing "date.start" and "date.end" in the code below. Note how including the Covid-19 episode changes the estimates.

```
# Define desired date range:

date.start <- "1965-01-01"
date.end <- "2006-12-01"
mytimerange <- function(x){ x[paste(date.start,date.end,sep="/")] }


# Take logs and take date range defined above:

logGDP <- log(mytimerange(GDP))
logCons <- log(mytimerange(Cons))
logNFI <- log(mytimerange(NFI))


# Estimate models:

GDPest <- lm(logGDP~seq(1,length(logGDP),by=1))
names(GDPest$coefficients)=c("beta1_GDP","beta2_GDP")

Consest <- lm(logCons~seq(1,length(logCons),by=1))
names(Consest$coefficients)=c("beta1_Cons","beta2_Cons")

NFIest <- lm(logNFI~seq(1,length(logNFI),by=1))
names(NFIest$coefficients)=c("beta1_NFI","beta2_NFI")


# Print coefficients:

GDPest$coefficients
```

```
##   beta1_GDP   beta2_GDP
## 8.410386598 0.007770652
```

```
Consest$coefficients
```

```
##  beta1_Cons  beta2_Cons
## 7.903089677 0.008351974
```

```
NFIest$coefficients
```

```
##  beta1_NFI  beta2_NFI
## 6.22048140 0.01002159
```

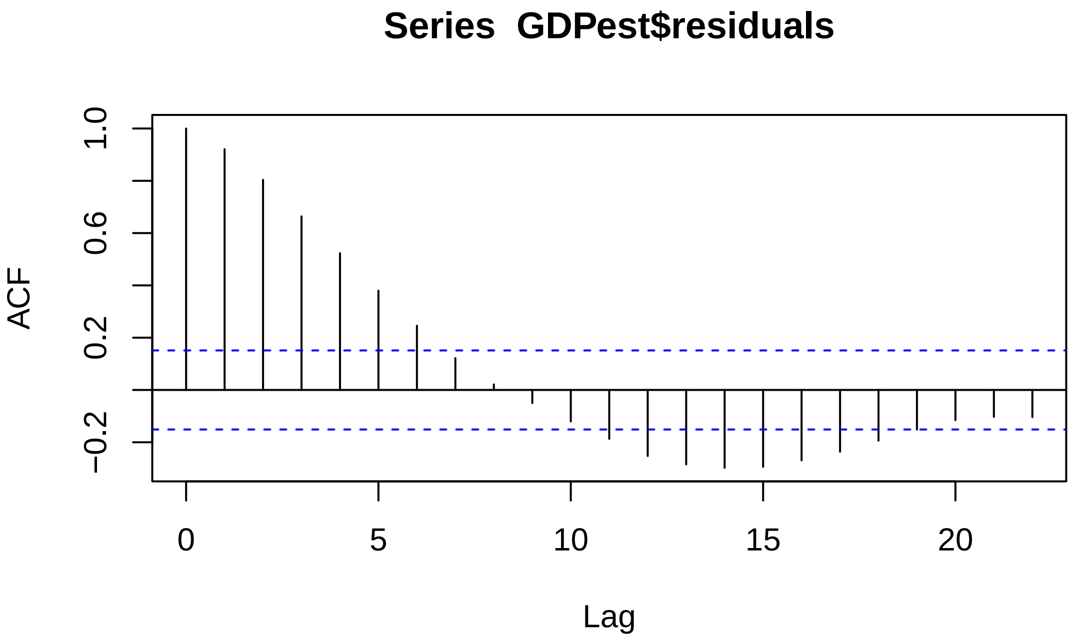
3. According to your estimates, what are the annualized average growth rates (in percent) of GDP, consumption, and investment? Are these series growing, approximately, at the same rate?

Solution: The annualized growth rate is $\log(y_t) - log(y_{t-4})$. According to our model, it corresponds to

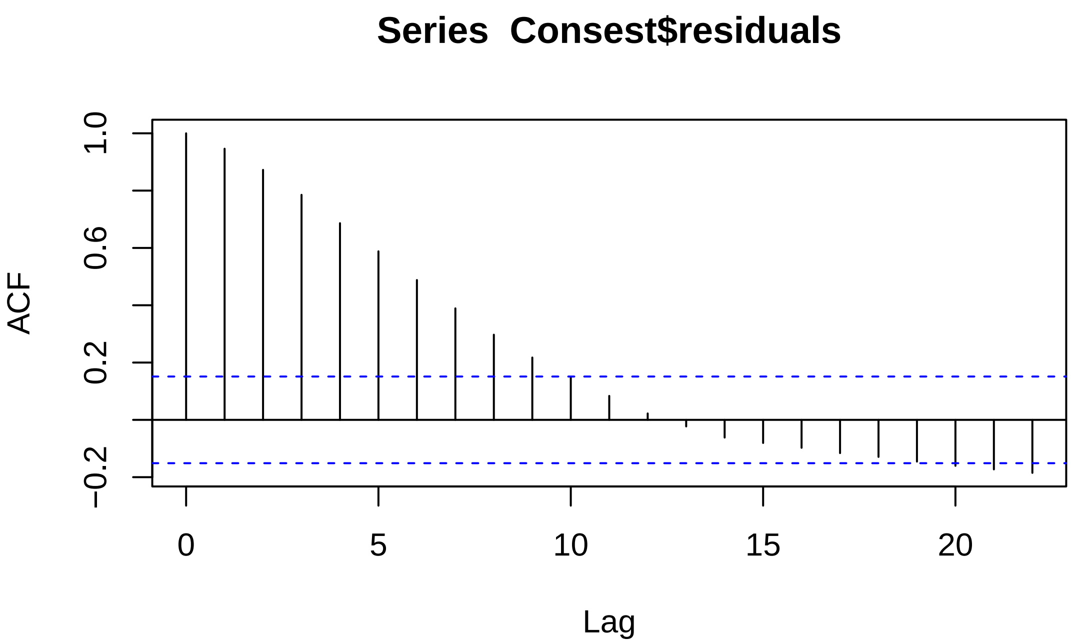
$400 * \beta_2$. The estimated growth rates are $g_{GDP} = 3.108261$, $g_{Cons} = 3.3407897$ and $g_{NFI} = 4.008636$. While GDP and consumption grow at very similar rates, the growth rate of investment is higher

4. For each subsample, compute sample autocorrelation functions for the deviations of output, consumption, and investment (the $\hat{u}_t$'s) from their estimated deterministic trend.

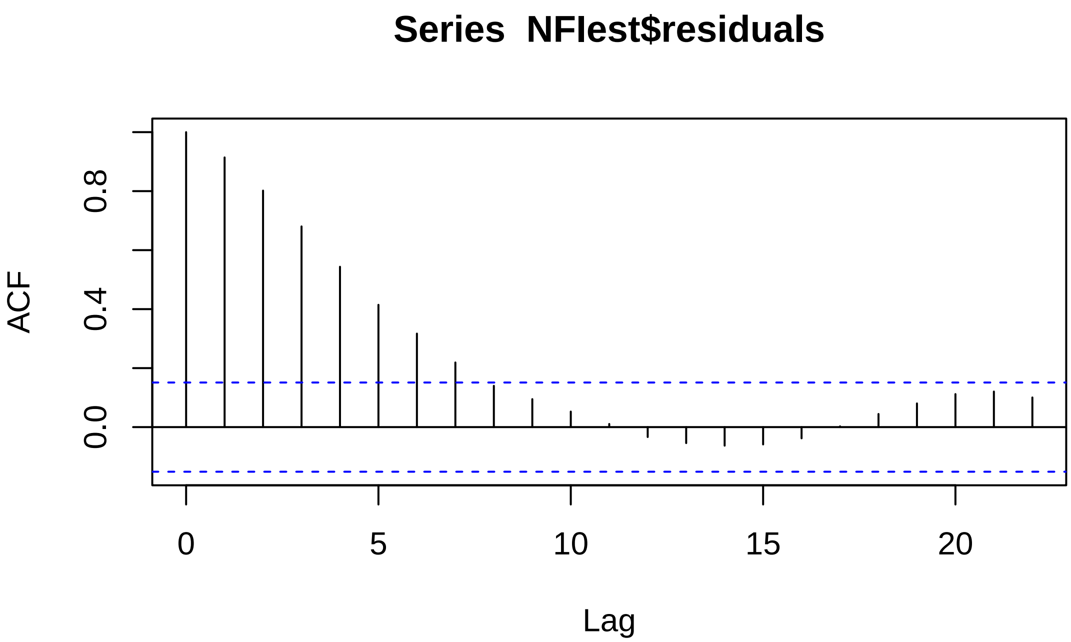
```
acf(GDPest$residuals)
```




```
acf(Consest$residuals)
```

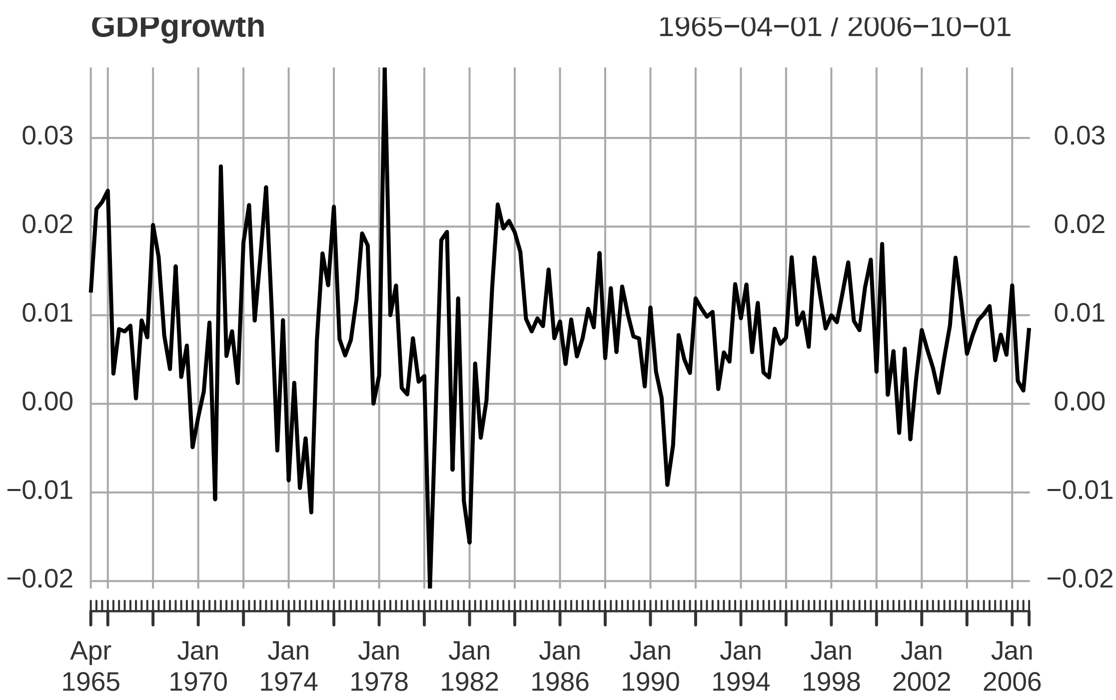


```r
acf(NFIest$residuals)
```

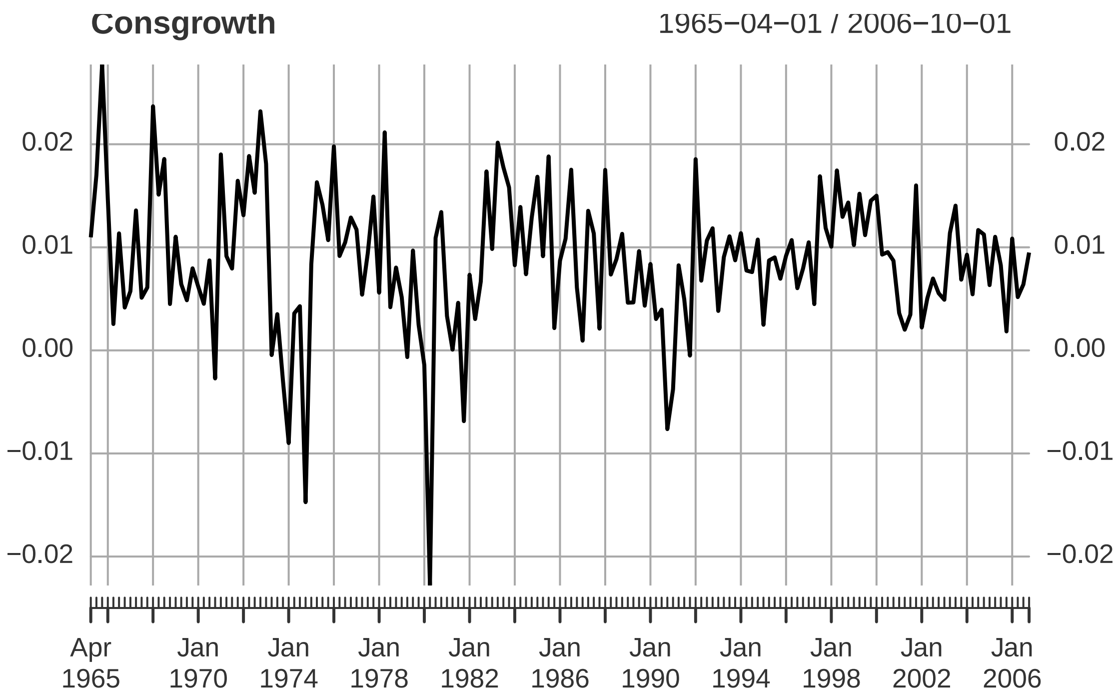


5. Now compute quarter-on-quarter growth rates of these three series as $\ln y_t - \ln y_{t-1}$, plot the growth rates. What are the most striking features?

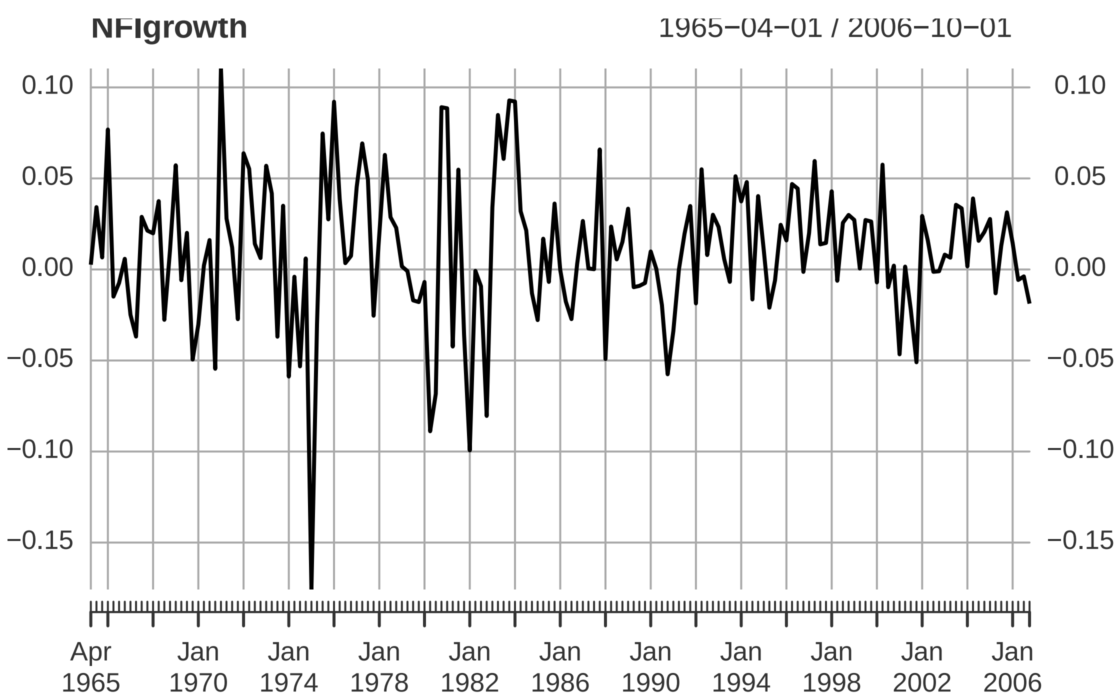
```r
GDPgrowth <- (logGDP - lag(logGDP))[-1]
Consgrowth <- (logCons - lag(logCons))[-1]
NFIgrowth <- (logNFI - lag(logNFI))[-1]

plot(GDPgrowth)
```



```r
plot(Consgrowth)
```




```r
plot(NFIgrowth)
```



6. Compute the sample means of the growth rates for each of the above subsamples. Compare the growth-rate results to (ii). Also compute the sample standard deviations for the growth rates.

```r
mean(GDPgrowth)
```

```
## [1] 0.007988902
```

```r
mean(Consgrowth)
```

```
## [1] 0.008666419
```

```r
mean(NFIgrowth)
```

```
## [1] 0.0100079
```

```r
sd(GDPgrowth)
```

```
## [1] 0.008264141
```

```r
sd(Consgrowth)
```

```
## [1] 0.0067471
```

```r
sd(NFIgrowth)
```

```
## [1] 0.03910535
```

We get pretty much the same growth rates as the ones obtained using the fitted deterministic trend model.

7. Repeat the analysis in (vi) for the subsamples "before 1984", "between 1984 and 2006". Did the means and the volatility of the series change?

Solution: Again, to be concise, I will not print results here in this document. By changing "date.start" and "date.end" you can obtain the results I refer to below.

Before 1970, the growth rates of GDP and consumption were higher and the standard deviation of growth rates was lower.

Analyzing the interval 1970 - 1982, one sees that the growth rates of GDP and in particular those of consumption decreased considerably, and that their standard deviations were higher.

Finally, analyzing only the sample after 1982 one obtains similar results as indicated in the above analysis using the whole sample. This period is referred to as "the Great Moderation" because of the decrease in GDP and consumption volatility post-1982.

Importantly, the above results pertaining to standard deviations of growth rates are contingent on not including the recent Covid-recession. Including it substantially increases the standard deviation in the post-1982 period (and whole sample since 1960) so that the period 1970 - 1982 actually gives a lower standard deviation.

2 Problem 2

Consider the MA(3) process

$$y_t = (1 - 2.4L + 0.8L^2 - 0.4L^3)u_t \ ,$$

where L denotes the lag operator and

$$\mathbb{E}[u_t u_\tau] = \begin{cases} 1 & \text{if } t = \tau \\ 0 & \text{otherwise} \end{cases} .$$

1. Define Weak and Strict Stationarity. Is the process y_t weakly stationary? Is it strictly stationary? You may impose additional conditions on the u_t's.

Solution: Weak stationarity (WS) means that the mean and autocovariances do not depend on time t. Strict stationarity (SS) means that joint CDFs for sets of observations depend only on displacement between the included observations, not time (i.e. not on their time subscripts).

Under the given assumptions, y_t is WS, since its mean and autocovariance do not depend on time (the mean is zero, and for the ACF see below). However, y_t is not necessarily SS. We know u_t is a white noise process but it might have higher order dependencies which would render the process of y_t not SS. One assumption that guarantees SS is assuming u_t is Gaussian. A Gaussian White Noise process is SS since higher order moments of a Gaussian distribution are determined by its first two moments. Then, an MA(q) process with an SS disturbance process also becomes SS.

2. Calculate the autocovariance function of y_t.

Solution: Under covariance stationarity, we can multiply both sides with y_{t+h} and take expectations to get $E[y_t y_{t+h}]$ which equals $\gamma(h)$ since y_t is a zero-mean process.

$$\mathbb{E}[y_t y_{t+h}] = E\big[[u_t - 2.4u_{t-1} + 0.8u_{t-2} - 0.4u_{t-3}][u_{t+h} - 2.4u_{t+h-1} + 0.8u_{t+h-2} - 0.4u_{t+h-3}]\big]$$

Using $\mathbb{E}[u_t u_\tau] = 1$ for $t = \tau$ and $= 0$ for $t \neq \tau$, we can get

$$\begin{aligned}
\gamma(0) &= \mathbb{V}[y_t] = 1 + \theta_1^2 + \theta_2^2 + \theta_3^2 \\
\gamma(1) &= \theta_1 + \theta_2\theta_1 + \theta_3\theta_2 \\
\gamma(2) &= \theta_2 + \theta_1\theta_3 \\
\gamma(3) &= \theta_3 \\
\gamma(h) &= 0 \;\; \forall h > 3 \\
\gamma(h) &= \gamma(-h)\ ,
\end{aligned}$$

where in our application $(\theta_1, \theta_2, \theta_3) = (-2.4, 0.8, -0.4)$.

3. Calculate $\mathbb{V}\left[\dfrac{1}{\sqrt{T}} \sum_{t=1}^{T} y_t\right]$.

Solution:

$$
\begin{aligned}
\mathbb{V}[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} y_t] &= \frac{1}{T}\mathbb{E}[\left[\sum_{t=1}^{T} y_t\right]^2] \\
&= \frac{1}{T}\mathbb{E}[(y_1 + y_2 + ... + y_T) * (y_1 + y_2 + ... + y_T)] \\
&= \frac{1}{T}2\left[\frac{T}{2}\gamma(0) + (T-1)\gamma(1) + ... + \gamma(T-1)\right] \\
&= \frac{1}{T}2\left[\frac{T}{2}\gamma(0) + (T-1)\gamma(1) + (T-2)\gamma(2) + (T-3)\gamma(3)\right] \\
&= 1 + \frac{4.64}{T}
\end{aligned}
$$

4. Suppose $\{u_t\}$ is i.i.d. with $\mathbb{E}[u_t] = 0$ and $\mathbb{V}[u_t] = \sigma^2$, and define $x_t = u_t u_{t-4}$. Is the process x_t strictly stationary? Is x_t ergodic? Is x_t a White Noise process?

Solution: Since u_t is SS and ergodic and $f(x, y) = xy$ is a measurable function, $\{x_t\}$ is also strictly stationary.

Yes, x_t is a WN process: we have

$$
\begin{aligned}
\mathbb{E}[x_t] &= \mathbb{E}[u_t u_{t-4}] \\
&= \mathbb{E}[u_t]\mathbb{E}[u_{t-4}] \\
&= 0 \ ,
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{V}[x_t] &= \mathbb{E}[x_t^2] - E[x_t]^2 \\
&= \mathbb{E}[x_t^2] \\
&= \mathbb{E}[u_t^2 u_{t-4}^2] \\
&= \mathbb{E}[u_t^2]\mathbb{E}[u_{t-4}^2] \\
&= \sigma^4 \ ,
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma(h) &= \mathbb{E}[x_t x_{t-h}] \\
&= \mathbb{E}[u_t u_{t-4} u_{t-h} u_{t-h-4}] \\
&= \mathbb{E}[u_t]\mathbb{E}[u_{t-4} u_{t-h} u_{t-h-4}] \\
&= 0 \ ,
\end{aligned}
$$

for all $h > 0$.

5. Calculate $\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} x_t\right]$, and $\mathbb{V}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} x_t\right]$.

Solution:

$$
\begin{aligned}
\mathbb{E}[\frac{1}{T}\sum_{t=1}^{T} x_t] &= \sum_{t=1}^{T} \frac{1}{T}\mathbb{E}[x_t] \\
&= \mathbb{E}[x_t] = 0
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{V}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} x_t\right) &= \mathbb{E}\left(\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} x_t\right]^2\right) - \left[\mathbb{E}\left(\frac{1}{\sqrt{T}}\sum_{t=1}^{T} x_t\right)\right]^2 \\
&= \mathbb{E}\left(\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} x_t\right]^2\right) \\
&= \frac{1}{T}\left[\mathbb{E}\left(\sum_{t=1}^{T} x_t^2\right) + \mathbb{E}\left(\sum_{t=h+1}^{T} x_t x_{t-h}\right)\right] \\
&= \frac{1}{T}\sum_{t=1}^{T} \mathbb{E}x_t^2 = \sigma^4 \ .
\end{aligned}
$$