Macroeconomics B: EI060

Problem set 2

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The assignment asks you to solve three questions, each structured in a serie of sub-questions that build on each other.

1 First generation crisis model

1.1 Consumption under a peg

We consider a model of balance of payment crisis, without an initial fiscal deficit.

The model is analyzed under continuous time. The household in the small open economy maximizes the utility of consumption

$$\int_0^{\infty} \exp\left[-\beta t\right] \frac{(c_t)^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} dt$$

where β is the discount rate and $\sigma > 1$.

There is a constant endowment y of the world traded good. The household can lend and borrow at an interest rate $r = \beta$. The government taxes consumption at a rate θ. The household holds real balances because of a cash in advance constraint: $m_t = \alpha c_t$, where m are the real balances (i.e. adjusted by the price level). The flow budget constraint is:

$$\dot{a}_t = r a_t + y - c_t (1 + \theta) - i_t m_t$$

where i is the nominal interest rate, a denotes foreign assets, and $\dot{a}_t$ is the change of foreign assets (recall that in continuous time $\dot{a}_t$ is the equivalent of $a_{t+1} - a_t$ in discrete time). The intertemporal budget constraint of the consumer is then (take this as given):

$$a_0 + \frac{y}{r} = \int_0^{\infty} \exp\left[-rt\right] \left[c_t (1 + \theta) + i_t m_t\right] dt$$

The interest parity links the rate of exchange rate depreciation ε_t and the interest rates:

$$i_t = r + \varepsilon_t$$

Optimal consumption is given by the Euler condition (λ is the multiplier on the resource constraint, take this as given):

$$(c_t)^{-\frac{1}{\sigma}} = \lambda (1 + \theta + \alpha i_t)$$

Show that if the exchange rate is permanently fixed, consumption is:

$$\tilde{c} = \frac{ra_0 + y}{1 + \theta + \alpha r}$$

1.2 Unsustainable peg

The government must finance a constant real expenditure g. We denote the government's foreign reserves by h. The foreign reserves increase thanks to the primary surplus, $s_t^p = \theta c_t - g$, money creation (which can be zero) and any depreciation ε of the exchange rate:

$$\dot{h}_t = rh_t + s_t^p + \dot{m}_t + \varepsilon_t m_t$$

where $\dot{h}_t$ is the change in reserves.

The exchange rate peg could break at time T, and we allow for a discrete jump in reserves and nominal balances at that time ($T-$ indexes the value of variables just before the jump):

$$h_T - h_{T-} = \frac{M_T - M_{T-}}{E_T}$$

The intertemporal government budget constraint is then (take this as given):

$$\frac{g}{r} = h_0 + \int_0^{\infty} e^{-rt} \left[\theta c_t + \dot{m}_t + \varepsilon_t m_t\right] dt + e^{-rT} \left[m_T - m_{T-}\right]$$

We assume that government spending is high, in the sense that it exceeds the following threshold:

$$g > rh_0 + \frac{\theta (ra_0 + y)}{1 + \theta + \alpha r}$$

Show that the exchange rate peg cannot hold forever.

1.3 Consumption and depreciation pre- and post-break

The government abandons the peg at a date T. Before that date, consumption and real balances are constant at c_1 and m_1. After date T, the exchange rate depreciates at a constant rate $\varepsilon > 0$, and consumption and real balances are constant at c_2 and m_2.

We assume that the initial consumption is such that the primary fiscal surplus is zero: $\theta c_1 = g$. Using the consumption Euler equations, show that:

$$\left(\frac{c_1}{c_2}\right)^{\frac{1}{\sigma}} = \frac{1+\theta+\alpha\left(r+\varepsilon\right)}{1+\theta+\alpha r} > 1$$

Using the budget constraint and the cash in advance, shows that:

$$\varepsilon = \frac{\theta}{\alpha}\left(\frac{c_1}{c_2}-1\right)$$

What are the equilibrium values of c_1/c_2 and ε? You don't have to compute them, but show that they are above 1 and 0, respectively.

1.4 Dynamics of reserves and assets

Before the peg breaks, use the dynamics of reserves to show:

$$\dot{h}_t = r h_t$$

Show that the dynamics of overall assets are:

$$\dot{a}_t + \dot{h}_t = r\left(a_t + h_t\right) + y - g\frac{1+\theta+r\alpha}{\theta}$$

Evaluate these at $t = 0$. What are the dynamics of reserves, $\dot{h}_0$, and of overall assets, $\dot{a}_0 + \dot{h}_0$ – which corresponds to the current account balance.

If you were to observe macroeconomic data before the crisis, could you foresee a crisis coming?

1.5 Timing of break

Recall that the intertemporal budget constraint is:

$$\frac{g}{r} = h_0 + \int_0^{\infty} e^{-rt}\left[\theta c_t + \dot{m}_t + \varepsilon_t m_t\right]dt + e^{-rT}\left[m_T - m_{T-}\right]$$

Using the solution for consumption before and after the break of the peg, show that:

$$T = \frac{1}{r}\ln\left(\frac{m_{T-} - m_T}{h_0}\right)$$

2 Choice of policy regime

2.1 Constant money

Consider a small open economy with the uncovered interest parity (where we normalize the foreign interest rate to zero),the aggregate supply linking output to unexpected inflation (the steady state output is normalized to zero), the aggregate demand reflecting the real exchange rate (the foreign price level and the steady state real exchange rate are normalized to zero) and a shock ε, and the money demand reflecting output, the interest rate, and a shock υ:

$$\begin{aligned} i_{t+1} &= \mathbb{E}_t e_{t+1} - e_t \\ y_t &= \theta\left(p_t - \mathbb{E}_{t-1} p_t\right) \\ y_t &= \delta\left(e_t - p_t\right) + \varepsilon_t \\ m_t - p_t &= -\eta i_{t+1} + \phi y_t + \upsilon_t \end{aligned}$$

where $e_t > 0$ indicates that the nominal exchange rate is depreciated relative to the steady state.

The shocks ε_t and υ_t have zero expected value, variances equal to σ_ε^2 and σ_υ^2, and are uncorrelated.

The central bank aims at minimizing the variance of output, $\mathbb{E}_{t-1}\left(y_t\right)^2$.

Consider a constant money supply rule where $m_t = \overline{m}$. Show that:

$$\begin{aligned} p_t &= \overline{m} + \frac{\eta\varepsilon_t - \delta\upsilon_t}{\eta\left(\theta+\delta\right) + \left(1+\phi\theta\right)\delta} \\ e_t &= \overline{m} - \frac{\left(1+\phi\theta\right)\varepsilon_t + \left(\theta+\delta\right)\upsilon_t}{\eta\left(\theta+\delta\right) + \left(1+\phi\theta\right)\delta} \\ y_t &= \frac{\theta\eta\varepsilon_t - \theta\delta\upsilon_t}{\eta\left(\theta+\delta\right) + \left(1+\phi\theta\right)\delta} \end{aligned}$$

Compute the variance of output, $\mathbb{E}_{t-1}\left(y_t\right)^2$, which you'll need later.

2.2 Exchange rate peg

Consider that the central bank pegs the exchange rate at $\overline{e} = \overline{m}$, adjusting m_t as needed around $\overline{m}$.

Show that:

$$\begin{aligned} p_t &= \overline{m} + \frac{\varepsilon_t}{\theta+\delta} \\ m_t &= \overline{m} + \frac{1+\phi\theta}{\theta+\delta}\varepsilon_t + \upsilon_t \\ y_t &= \frac{\theta}{\theta+\delta}\varepsilon_t \end{aligned}$$

Compute the variance of output, $\mathbb{E}_{t-1}\left(y_t\right)^2$, which you'll need later.

2.3 Regime choice

Compare the volatility of output under the exchange rate peg and under the constant money supply.

Show that a peg is clearly preferred if $\sigma_\varepsilon^2/\sigma_\upsilon^2$ goes towards zero. Interpret the result.

2.4 Optimal rule

Consider that instead of stabilizing the money supply or the exchange rate, the central bank sets a Taylor rule where money reacts to movements in the exchange rate, with a more expansionnary money supply when the exchange rate is below the long run target (i.e. appreciated):

$$m_t = \overline{m} + \Phi\left(\overline{e} - e_t\right)$$

Notice that the expected exchange rate is always $\overline{e}$. Using the money market, show that:

$$\overline{m} - p_t = -\left(\eta + \Phi\right)\left(\overline{e} - e_t\right) + \phi y_t + \upsilon_t$$

Notice that this looks similar to the model with constant money supply, except for the coefficient $\eta + \Phi$ instead of η. Show that:

$$\begin{aligned}
p_t &= \overline{m} + \frac{\left(\eta + \Phi\right)\varepsilon_t - \delta\upsilon_t}{\left(\eta + \Phi\right)\left(\theta + \delta\right) + \left(1 + \phi\theta\right)\delta} \\
e_t &= \overline{m} - \frac{\left(1 + \phi\theta\right)\varepsilon_t + \left(\theta + \delta\right)\upsilon_t}{\left(\eta + \Phi\right)\left(\theta + \delta\right) + \left(1 + \phi\theta\right)\delta} \\
y_t &= \frac{\theta\left(\eta + \Phi\right)\varepsilon_t - \theta\delta\upsilon_t}{\left(\eta + \Phi\right)\left(\theta + \delta\right) + \left(1 + \phi\theta\right)\delta}
\end{aligned}$$

Show that the optimal value of Φ is:

$$\Phi = \frac{\left(\theta + \delta\right)\delta}{1 + \phi\theta}\frac{\sigma_\upsilon^2}{\sigma_\varepsilon^2} - \eta$$

Discuss the intuition for the case where $\sigma_\varepsilon^2/\sigma_\upsilon^2 \to 0$, as well as the opposite case where $\sigma_\upsilon^2/\sigma_\varepsilon^2 \to 0$ (does the central bank limits exchange rate movements in that second case?).

3 Taxation of debt

3.1 Decentralized and centralized choice

Consider a two period small open economy where the household has a linear utility in consumption:

$$U = C_1 + \frac{1}{1+\delta} C_2$$

Borrowing is cheap compared to the household's time preference, $r < \delta$.

The economy gets an endowment Y_2 in period 2, and can finance initial consumption only through a debt D_1, so $C_1 = D_1$. It cannot borrow at a rate r but instead at a rate r^s that includes a premium which is increasing with the level of debt:

$$r^s = r + \alpha D_1$$

We consider two allocations. Under the decentralized one, the household takes the interest rate r^s as given. Under the planner allocation, the planner takes account of the impact of the debt on r^s. Show that the optimal debt levels are:

$$\begin{aligned} D_1^{decentralized} &= \frac{\delta - r}{\alpha} \\ D_1^{planner} &= \frac{\delta - r}{2\alpha} \end{aligned}$$

What is the interest rate under the two allocations? Explain the intuition behind the results.

3.2 Taxes

Borrowing can be taxed to induce the household to borrow the same amount as the planner. We consider two tax regimes.

- In the first one, the tax rate on borrowing depends on the debt level: $\tau^{variable} = \gamma D_1$. The household still takes the tax rate as a given in the optimization).
- In the second one, there is a flat tax rate τ^{flat}.

Compute the tax rates under the two regimes (i.e. compute γ and τ^{flat}), in order to get the same amount of debt as the planner.

Is one regime likely to be more feasible (think of the parameters that the governments would have to estimate to calibrate the tax)?