Macroeconomics A; EI056

Short problems

Cédric Tille

Class of October 17, 2023

1 Utility functions

1.1 Derivatives

Question: We consider two standard utility functions in macroeconomics

The first is the so called CRRA (constant relative risk aversion):

$$U_{CRRA}(c) = \frac{c^{1-\theta}}{1-\theta}$$

The second is the CARA (constant absolute risk aversion), more standard in finance:

$$U_{CARA}(c) = -\exp[-\chi c]$$

Compute the first and second derivatives.

Answer: The derivatives of the CRRA are:

$$\begin{aligned} U'_{CRRA}(c) &= c^{-\theta} \\ U''_{CRRA}(c) &= -\theta c^{-\theta-1} \end{aligned}$$

The derivatives of the CARA are:

$$\begin{aligned} U'_{CARA}(c) &= \chi \exp[-\chi c] \\ U''_{CARA}(c) &= -\chi^2 \exp[-\chi c] \end{aligned}$$

1.2 Elasticity

Question: The absolute risk aversion of a function is $-U''/U'$, while the relative risk aversion is $-cU''/U'$.

Compute these for CARA and CRRA.

Can you see why macroeconomists prefer CRRA, while finance economists like CARA?

Answer: For the CRRA we get:

$$\begin{aligned} -\frac{U''_{CRRA}(c)}{U'_{CRRA}(c)} &= -\frac{-\theta c^{-\theta-1}}{c^{-\theta}} \\ -\frac{U''_{CRRA}(c)}{U'_{CRRA}(c)} &= \theta c^{-1} \end{aligned}$$

and:

$$\begin{aligned}
-c\frac{U''_{CRRA}(c)}{U'_{CRRA}(c)} &= -c\frac{-\theta c^{-\theta-1}}{c^{-\theta}} \\
-c\frac{U''_{CRRA}(c)}{U'_{CRRA}(c)} &= c\theta c^{-1} \\
-c\frac{U''_{CRRA}(c)}{U'_{CRRA}(c)} &= \theta
\end{aligned}$$

Turning to the CARA, we have:

$$\begin{aligned}
-\frac{U''_{CARA}(c)}{U'_{CARA}(c)} &= -\frac{-\chi^2 \exp[-\chi c]}{\chi \exp[-\chi c]} \\
-\frac{U''_{CARA}(c)}{U'_{CARA}(c)} &= \chi
\end{aligned}$$

and:

$$\begin{aligned}
-c\frac{U''_{CARA}(c)}{U'_{CARA}(c)} &= -c\frac{-\chi^2 \exp[-\chi c]}{\chi \exp[-\chi c]} \\
-c\frac{U''_{CARA}(c)}{U'_{CARA}(c)} &= c\chi
\end{aligned}$$

In the macroeconomic model, the Euler condition shows that the response of the growth rate of consumption to the real interest rate is a function of the *relative* risk aversion $-cU''/U'$. The higher the relative risk aversion, the lower the sensitivity.

With CRRA this is a constant parameter, and thus the sensitivity is not a function of the level of consumption. That means that the properties of the economy do not change when the economy grows. As long-run growth is a key fact, we want an economy whose behavior is stable along the growth path.

With CARA the risk aversion increases with consumption, and thus consumption becomes less sensitive to the real interest when it grows. This is interesting if you study a cross-section of households with different incomes (and different consumption level) at a given time. Richer consumers behave differently than poorer ones.

2 Dynamics of consumption and capital

2.1 Steady state

Question: Consider the Ramsey model seen in class. The production function is a function of capital (scaled by labor) as in the Solow model: $y_t = (k_t)^\alpha$

The overall model boils down to the Euler condition and the budget constraint (capital dynamics):

$$\begin{aligned}
\frac{c_{t+1}}{c_t} &= \left(\frac{1+\alpha(k_{t+1})^{\alpha-1}-\delta}{1+\rho}\right)^{\frac{1}{\theta}} \\
(1+n)(k_{t+1}-k_t) &= (k_t)^\alpha - (n+\delta)k_t - c_t
\end{aligned}$$

Show that the steady state is:

$$k^* = c^*\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}} \quad ; \quad c^* = \left[\frac{\rho+\delta}{\alpha} - (n+\delta)\right]\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}$$

Show that the real interest rate is $r^* = \rho + \delta$.

Answer: In the steady state, the variables are constant. The Euler condition implies:

$$\begin{aligned}
\frac{c^*}{c^*} &= \left(\frac{1+\alpha\left(k^*\right)^{\alpha-1}-\delta}{1+\rho}\right)^{\frac{1}{\theta}} \\
1 &= \left(\frac{1+\alpha\left(k^*\right)^{\alpha-1}-\delta}{1+\rho}\right)^{\frac{1}{\theta}} \\
1 &= \frac{1+\alpha\left(k^*\right)^{\alpha-1}-\delta}{1+\rho} \\
1+\rho &= 1+\alpha\left(k^*\right)^{\alpha-1}-\delta \\
\rho+\delta &= \alpha\left(k^*\right)^{\alpha-1} \\
\left(k^*\right)^{1-\alpha} &= \frac{\alpha}{\rho+\delta} \\
k^* &= \left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}
\end{aligned}$$

The budget constraint is then:

$$\begin{aligned}
(1+n)\left(k^*-k^*\right) &= \left(k^*\right)^{\alpha}-(n+\delta)\,k^*-c^* \\
0 &= \left(k^*\right)^{\alpha}-(n+\delta)\,k^*-c^* \\
c^* &= \left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha}{1-\alpha}}-(n+\delta)\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}} \\
c^* &= \left[\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha-1}{1-\alpha}}-(n+\delta)\right]\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}} \\
c^* &= \left[\frac{\rho+\delta}{\alpha}-(n+\delta)\right]\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}
\end{aligned}$$

The real interest rate is the marginal product of capital:

$$\begin{aligned}
r^* &= \alpha\left(k^*\right)^{\alpha-1} \\
r^* &= \alpha\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha-1}{1-\alpha}} \\
r^* &= \alpha\frac{\rho+\delta}{\alpha} \\
r^* &= \rho+\delta
\end{aligned}$$

2.2 Linearization of the Euler

Question: Show that the Euler condition is linearized as (where $\hat{x}_t = \left(x_t - x^*\right)/x^*$):

$$\hat{c}_{t+1}-\hat{c}_t = -(1-\alpha)\,\frac{1}{\theta}\frac{r^*}{1+r^*-\delta}\hat{k}_{t+1}$$

Answer: The Euler is given by:

$$\frac{c_{t+1}}{c_t} = \left(\frac{1+\alpha\left(k_{t+1}\right)^{\alpha-1}-\delta}{1+\rho}\right)^{\frac{1}{\theta}}$$

We start with a linear expansion of the left-hand side:

$$\begin{aligned}
\frac{c_{t+1}}{c_t} &= \frac{c^*}{c^*} + \frac{1}{c^*}\left(c_{t+1}-c^*\right) - \frac{c^*}{\left(c^*\right)^2}\left(c_t - c^*\right) \\
\frac{c_{t+1}}{c_t} &= \frac{c^*}{c^*} + \frac{c_{t+1}-c^*}{c^*} - \frac{c^*}{c^*}\frac{c_t - c^*}{c^*} \\
\frac{c_{t+1}}{c_t} &= 1 + \hat{c}_{t+1} - \hat{c}_t
\end{aligned}$$

The right-hand side is:

$$\begin{aligned}
\left(\frac{1+\alpha\left(k_{t+1}\right)^{\alpha-1}-\delta}{1+\rho}\right)^{\frac{1}{\theta}} &= \left(\frac{1+\alpha\left(k^*\right)^{\alpha-1}-\delta}{1+\rho}\right)^{\frac{1}{\theta}} \\
&\quad + \frac{1}{\theta}\left(\frac{1+\alpha\left(k^*\right)^{\alpha-1}-\delta}{1+\rho}\right)^{\frac{1}{\theta}-1} \frac{\alpha\left(\alpha-1\right)\left(k^*\right)^{\alpha-2}}{1+\rho}\left(k_{t+1}-k^*\right) \\
\left(\frac{1+\alpha\left(k_{t+1}\right)^{\alpha-1}-\delta}{1+\rho}\right)^{\frac{1}{\theta}} &= \left(\frac{1+\alpha\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha-1}{1-\alpha}}-\delta}{1+\rho}\right)^{\frac{1}{\theta}} \\
&\quad + \frac{1}{\theta}\left(\frac{1+\alpha\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha-1}{1-\alpha}}-\delta}{1+\rho}\right)^{\frac{1}{\theta}-1} \frac{\alpha\left(\alpha-1\right)\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha-1}{1-\alpha}}}{1+\rho}\frac{k_{t+1}-k^*}{k^*} \\
\left(\frac{1+\alpha\left(k_{t+1}\right)^{\alpha-1}-\delta}{1+\rho}\right)^{\frac{1}{\theta}} &= \left(\frac{1+\rho+\delta-\delta}{1+\rho}\right)^{\frac{1}{\theta}} + \frac{1}{\theta}\left(\frac{1+\rho+\delta-\delta}{1+\rho}\right)^{\frac{1}{\theta}-1}\frac{\left(\alpha-1\right)\left(\rho+\delta\right)}{1+\rho}\hat{k}_{t+1} \\
\left(\frac{1+\alpha\left(k_{t+1}\right)^{\alpha-1}-\delta}{1+\rho}\right)^{\frac{1}{\theta}} &= \left(1\right)^{\frac{1}{\theta}} + \frac{1}{\theta}\left(1\right)^{\frac{1}{\theta}-1}\frac{\left(\alpha-1\right)\left(\rho+\delta\right)}{1+\rho}\hat{k}_{t+1} \\
\left(\frac{1+\alpha\left(k_{t+1}\right)^{\alpha-1}-\delta}{1+\rho}\right)^{\frac{1}{\theta}} &= 1 - \frac{1}{\theta}\frac{\left(1-\alpha\right)\left(\rho+\delta\right)}{1+\rho}\hat{k}_{t+1} \\
\left(\frac{1+\alpha\left(k_{t+1}\right)^{\alpha-1}-\delta}{1+\rho}\right)^{\frac{1}{\theta}} &= 1 - \frac{1}{\theta}\frac{\left(1-\alpha\right)r^*}{1+r^*-\delta}\hat{k}_{t+1} \\
\left(\frac{1+\alpha\left(k_{t+1}\right)^{\alpha-1}-\delta}{1+\rho}\right)^{\frac{1}{\theta}} &= 1 - \left(1-\alpha\right)\frac{1}{\theta}\frac{r^*}{1+r^*-\delta}\hat{k}_{t+1}
\end{aligned}$$

Putting everything together, we get:

$$\frac{c_{t+1}}{c_t} = \left(\frac{1+\alpha\left(k_{t+1}\right)^{\alpha-1}-\delta}{1+\rho}\right)^{\frac{1}{\theta}}$$

$$\begin{aligned}
1+\hat{c}_{t+1}-\hat{c}_t &= 1-(1-\alpha)\frac{1}{\theta}\frac{r^*}{1+r^*-\delta}\hat{k}_{t+1} \\
\hat{c}_{t+1}-\hat{c}_t &= -(1-\alpha)\frac{1}{\theta}\frac{r^*}{1+r^*-\delta}\hat{k}_{t+1}
\end{aligned}$$

2.3 Linearization of the budget constraint

Question: Show that the budget constraint is linearized as:

$$\hat{k}_{t+1}=\frac{1+r^*-\delta}{1+n}\hat{k}_t-\frac{1}{\alpha}\frac{r^*-\alpha(n+\delta)}{1+n}\hat{c}_t$$

Answer: The budget constraint is:

$$(1+n)(k_{t+1}-k_t)=(k_t)^\alpha-(n+\delta)k_t-c_t$$

We start with the approximation of the left-hand side:

$$\begin{aligned}
(1+n)(k_{t+1}-k_t) &= (1+n)(k^*-k^*)+(1+n)(k_{t+1}-k^*)-(1+n)(k_t-k^*) \\
(1+n)(k_{t+1}-k_t) &= (1+n)k^*\frac{k_{t+1}-k^*}{k^*}-(1+n)k^*\frac{k_t-k^*}{k^*} \\
(1+n)(k_{t+1}-k_t) &= (1+n)k^*\left(\hat{k}_{t+1}-\hat{k}_t\right)
\end{aligned}$$

The right-hand side is approximated as follows:

$$\begin{aligned}
(k_t)^\alpha-(n+\delta)k_t-c_t &= (k^*)^\alpha-(n+\delta)k^*-c^*+\alpha(k^*)^{\alpha-1}(k_t-k^*) \\
&\quad -(n+\delta)(k_t-k^*)-(c_t-c^*) \\
(k_t)^\alpha-(n+\delta)k_t-c_t &= (k^*)^\alpha-(n+\delta)k^*-c^* \\
&\quad +\alpha(k^*)^\alpha\frac{k_t-k^*}{k^*}-(n+\delta)k^*\frac{k_t-k^*}{k^*}-c^*\frac{c_t-c^*}{c^*} \\
(k_t)^\alpha-(n+\delta)k_t-c_t &= (k^*)^\alpha-(n+\delta)k^*-\left[(k^*)^\alpha-(n+\delta)k^*\right] \\
&\quad +\alpha(k^*)^\alpha\hat{k}_t-(n+\delta)k^*\hat{k}_t-c^*\hat{c}_t \\
(k_t)^\alpha-(n+\delta)k_t-c_t &= \alpha(k^*)^\alpha\hat{k}_t-(n+\delta)k^*\hat{k}_t-c^*\hat{c}_t
\end{aligned}$$

Putting everything together, we write:

$$\begin{aligned}
(1+n)(k_{t+1}-k_t) &= (k_t)^\alpha-(n+\delta)k_t-c_t \\
(1+n)k^*\left(\hat{k}_{t+1}-\hat{k}_t\right) &= \alpha(k^*)^\alpha\hat{k}_t-(n+\delta)k^*\hat{k}_t-c^*\hat{c}_t \\
(1+n)\left(\hat{k}_{t+1}-\hat{k}_t\right) &= \alpha(k^*)^{\alpha-1}\hat{k}_t-(n+\delta)\hat{k}_t-\frac{c^*}{k^*}\hat{c}_t \\
(1+n)\left(\hat{k}_{t+1}-\hat{k}_t\right) &= \alpha\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{\alpha-1}{1-\alpha}}\hat{k}_t-(n+\delta)\hat{k}_t-\frac{\left[\frac{\rho+\delta}{\alpha}-(n+\delta)\right]\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}}{\left(\frac{\alpha}{\rho+\delta}\right)^{\frac{1}{1-\alpha}}}\hat{c}_t \\
(1+n)\left(\hat{k}_{t+1}-\hat{k}_t\right) &= (\rho+\delta)\hat{k}_t-(n+\delta)\hat{k}_t-\left[\frac{\rho+\delta}{\alpha}-(n+\delta)\right]\hat{c}_t \\
(1+n)\left(\hat{k}_{t+1}-\hat{k}_t\right) &= r^*\hat{k}_t-(n+\delta)\hat{k}_t-\left[\frac{r^*}{\alpha}-(n+\delta)\right]\hat{c}_t \\
(1+n)\hat{k}_t &= \left[r^*-(n+\delta)+(1+n)\right]\hat{k}_t-\left[\frac{r^*}{\alpha}-(n+\delta)\right]\hat{c}_t
\end{aligned}$$

$$
\begin{aligned}
(1+n)\,\hat{k}_t &= [1+r^*-\delta]\,\hat{k}_t - \frac{1}{\alpha}\left[r^*-\alpha\,(n+\delta)\right]\hat{c}_t \\
\hat{k}_{t+1} &= \frac{1+r^*-\delta}{1+n}\hat{k}_t - \frac{1}{\alpha}\frac{r^*-\alpha\,(n+\delta)}{1+n}\hat{c}_t
\end{aligned}
$$

2.4 Undetermined coefficients: consumption

Question: The form of the linearized model is that current consumption and future capital are both linear function of current capital:

$$
\hat{c}_t = \eta_{ck}\hat{k}_t \qquad ; \qquad \hat{k}_{t+1} = \eta_{kk}\hat{k}_t
$$

where η_{ck} and η_{kk} are coefficients to compute.

Using the budget constraint, shows that:

$$
\eta_{ck} = \frac{\alpha\,(1+r^*-\delta)}{r^*-\alpha\,(n+\delta)} - \frac{\alpha\,(1+n)\,\eta_{kk}}{r^*-\alpha\,(n+\delta)}
$$

Answer: The budget constaint is:

$$
\hat{k}_{t+1} = \frac{1+r^*-\delta}{1+n}\hat{k}_t - \frac{1}{\alpha}\frac{r^*-\alpha\,(n+\delta)}{1+n}\hat{c}_t
$$

Using the undetermined coefficients, we write it as:

$$
\begin{aligned}
\eta_{kk}\hat{k}_t &= \frac{1+r^*-\delta}{1+n}\hat{k}_t - \frac{1}{\alpha}\frac{r^*-\alpha\,(n+\delta)}{1+n}\eta_{ck}\hat{k}_t \\
\eta_{kk} &= \frac{1+r^*-\delta}{1+n} - \frac{1}{\alpha}\frac{r^*-\alpha\,(n+\delta)}{1+n}\eta_{ck} \\
\frac{1}{\alpha}\frac{r^*-\alpha\,(n+\delta)}{1+n}\eta_{ck} &= \frac{1+r^*-\delta}{1+n} - \eta_{kk} \\
\frac{1}{\alpha}\frac{r^*-\alpha\,(n+\delta)}{1+n}\eta_{ck} &= \frac{1+r^*-\delta}{1+n} - \eta_{kk} \\
\frac{1}{\alpha}\left(r^*-\alpha\,(n+\delta)\right)\eta_{ck} &= (1+r^*-\delta) - (1+n)\,\eta_{kk} \\
\eta_{ck} &= \alpha\frac{1+r^*-\delta-(1+n)\,\eta_{kk}}{r^*-\alpha\,(n+\delta)}
\end{aligned}
$$

2.5 Undetermined coefficients: capital

Question: Using the result for η_{ck} and the Euler condition, shows that:

$$
\begin{aligned}
0 = & -\alpha\,(1+r^*-\delta)\,(1+n)\,(\eta_{kk})^2 \\
& + \left[\begin{array}{c} \alpha\,(1+r^*-\delta)\,(1+n) + \alpha\,(1+r^*-\delta)^2 \\ +\left(r^*-\alpha\,(n+\delta)\right)\frac{1}{\theta}\,(1-\alpha)\,r^* \end{array}\right]\eta_{kk} \\
& -\alpha\,(1+r^*-\delta)^2
\end{aligned}
$$

What are the values of the two solutions for η_{kk}? Don't compute them, but instead think of whether they are positive, negative, larger or smaller than 1 in absolute value.

Which solution makes economic sense?

Answer: The Euler equation is rewritten using the undetermined coefficients:

$$
\begin{aligned}
\hat{c}_{t+1}-\hat{c}_t &= -(1-\alpha)\frac{1}{\theta}\frac{r^*}{1+r^*-\delta}\hat{k}_{t+1} \\
\eta_{ck}\left(\hat{k}_{t+1}-\hat{k}_t\right) &= -(1-\alpha)\frac{1}{\theta}\frac{r^*}{1+r^*-\delta}\hat{k}_{t+1} \\
\eta_{ck}(\eta_{kk}-1)\hat{k}_t &= -(1-\alpha)\frac{1}{\theta}\frac{r^*}{1+r^*-\delta}\eta_{kk}\hat{k}_t \\
\eta_{ck}(\eta_{kk}-1) &= -(1-\alpha)\frac{1}{\theta}\frac{r^*}{1+r^*-\delta}\eta_{kk} \\
0 &= \eta_{ck}(\eta_{kk}-1)+(1-\alpha)\frac{1}{\theta}\frac{r^*}{1+r^*-\delta}\eta_{kk}
\end{aligned}
$$

Using the result for η_{ck} this becomes:

$$
\begin{aligned}
0 &= \alpha\frac{1+r^*-\delta-(1+n)\eta_{kk}}{r^*-\alpha(n+\delta)}(\eta_{kk}-1)+(1-\alpha)\frac{1}{\theta}\frac{r^*}{1+r^*-\delta}\eta_{kk} \\
0 &= \alpha(1+r^*-\delta)(1+r^*-\delta-(1+n)\eta_{kk})(\eta_{kk}-1)+(r^*-\alpha(n+\delta))\frac{1}{\theta}(1-\alpha)r^*\eta_{kk} \\
0 &= \alpha(1+r^*-\delta)(1+r^*-\delta-(1+n)\eta_{kk})\eta_{kk} \\
&\quad -\alpha(1+r^*-\delta)(1+r^*-\delta-(1+n)\eta_{kk})+(r^*-\alpha(n+\delta))\frac{1}{\theta}(1-\alpha)r^*\eta_{kk} \\
0 &= -\alpha(1+r^*-\delta)(1+n)(\eta_{kk})^2+\alpha(1+r^*-\delta)\alpha(1+r^*-\delta)\eta_{kk} \\
&\quad +\alpha(1+r^*-\delta)(1+n)\eta_{kk}+(r^*-\alpha(n+\delta))\frac{1}{\theta}(1-\alpha)r^*\eta_{kk} \\
&\quad -\alpha(1+r^*-\delta)(1+r^*-\delta) \\
0 &= -\alpha(1+r^*-\delta)(1+n)(\eta_{kk})^2 \\
&\quad +\left[\begin{array}{c}\alpha(1+r^*-\delta)(1+n)+\alpha(1+r^*-\delta)^2 \\ +(r^*-\alpha(n+\delta))\frac{1}{\theta}(1-\alpha)r^*\end{array}\right]\eta_{kk} \\
&\quad -\alpha(1+r^*-\delta)^2
\end{aligned}
$$

This is of the form $0=Pol(\eta_{kk})$ where $Pol(\eta_{kk})$ is the quadratic polynomial. Note that:

$$
\begin{aligned}
lim_{\eta_{kk}\to-\infty}Pol(\eta_{kk}) &= -\alpha(1+r^*-\delta)(1+n)(-\infty)^2<0 \\
Pol(\eta_{kk}=0) &= -\alpha(1+r^*-\delta)^2<0 \\
Pol(\eta_{kk}=1) &= (r^*-\alpha(n+\delta))\frac{1}{\theta}(1-\alpha)r^*>0 \\
lim_{\eta_{kk}\to+\infty}Pol(\eta_{kk}) &= -\alpha(1+r^*-\delta)(1+n)(+\infty)^2<0
\end{aligned}
$$

So, if we graph the polynomial as a function of η_{kk} we start with negative values when η_{kk} is a large negative number, remain at a negative value when $\eta_{kk}=0$, cross to a positive value somewhere in the range $\eta_{kk}\in(0,1)$ to reach a positive value when $\eta_{kk}=1$, and then cross to a negative value somewhere in the range $\eta_{kk}>1$ to get to negative values when η_{kk} is a large positive number.

There are thus two values of η_{kk} for which $Pol(\eta_{kk})=0$. One is $\eta_{kk}>1$, but this implies an explosive path for capital. The other is $0<\eta_{kk}<1$ which leads to a stable system.