Macroeconomics A; EI060

Technical appendix: uncertainty

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1 Optimization under uncertainty

1.1 States of nature and optimal consumption path

Two period model, where the agent maximizes an expected utility:

$$u(C_1) + \beta E u(C_2)$$

Uncertainty regarding period 2. We can be in various states of nature k, of probability $\pi(k)$. Consumption in a state is denoted by $C_2(k)$. The utility is the:

$$u(C_1) + \beta \sum_k \pi(k) u(C_2(k))$$

Can invest in bond giving and interest rate r. There is one budget constraint for period 1 and for for each state in period 2:

$$\begin{aligned} C_1 + B_2 &= Y_1 \\ C_2(k) &= (1+r) B_2 + Y_2(k) \end{aligned}$$

The Lagrangian is:

$$\begin{aligned} \mathcal{L} = & \; u(C_1) + \beta \sum_k \pi(k) u(C_2(k)) \\ & + \lambda_1 [Y_1 - C_1 - B_2] \\ & + \sum_k \lambda_2(k) [(1+r) B_2 + Y_2(k) - C_2(k)] \end{aligned}$$

The optimality conditions for C_1, $C_2(k)$ and B_2 are:

$$0 = u'(C_1) - \lambda_1$$

$$
\begin{aligned}
0 &= \beta\pi(k)\,u'(C_2(k)) - \lambda_2(k) \\
0 &= -\lambda_1 + \sum_k \lambda_2(k)(1+r)
\end{aligned}
$$

Combining, we get the Euler condition:

$$
\begin{aligned}
\lambda_1 &= \sum_k \lambda_2(k)(1+r) \\
u'(C_1) &= \sum_k (1+r)\,\beta\pi(k)\,u'(C_2(k)) \\
u'(C_1) &= \beta(1+r)\sum_k \pi(k)\,u'(C_2(k)) \\
u'(C_1) &= \beta(1+r)\,E[u'(C_2)]
\end{aligned}
$$

1.2 Simple case of precautionnary savings

There are two states of nature with equal probability. In one $Y_2(k) = Y_1 + \Delta$, and in the other $Y_2(k) = Y_1 - \Delta$. We consider $\beta(1+r) = 1$. The Euler condition is:

$$
u'(Y_1 - B_2) = \frac{1}{2}\left[u'((1+r)B_2 + Y_1 + \Delta) + u'((1+r)B_2 + Y_1 - \Delta)\right]
$$

If we take a log utility:

$$
\begin{aligned}
\frac{1}{Y_1 - B_2} &= \frac{1}{2}\left[\frac{1}{(1+r)B_2 + Y_1 + \Delta} + \frac{1}{(1+r)B_2 + Y_1 - \Delta}\right] \\
\frac{1}{Y_1 - B_2} &= \frac{1}{2}\left[\frac{(1+r)B_2 + Y_1 - \Delta + (1+r)B_2 + Y_1 + \Delta}{[(1+r)B_2 + Y_1 + \Delta]\,[(1+r)B_2 + Y_1 - \Delta]}\right] \\
\frac{1}{Y_1 - B_2} &= \frac{(1+r)B_2 + Y_1}{[(1+r)B_2 + Y_1]^2 - [\Delta]^2}
\end{aligned}
$$

Note the following derivatives:

$$
\begin{aligned}
\frac{\partial}{\partial B_2}\left[\frac{1}{Y_1 - B_2}\right] &= \frac{1}{[Y_1 - B_2]^2} > 0 \\
\frac{\partial}{\partial B_2}\left[\frac{(1+r)B_2 + Y_1}{[(1+r)B_2 + Y_1]^2 - [\Delta]^2}\right] &= -(1+r)\,\frac{[(1+r)B_2 + Y_1]^2 + [\Delta]^2}{\left[[(1+r)B_2 + Y_1]^2 - [\Delta]^2\right]^2} < 0 \\
\frac{\partial}{\partial \Delta}\left[\frac{(1+r)B_2 + Y_1}{[(1+r)B_2 + Y_1]^2 - [\Delta]^2}\right] &= \frac{(1+r)B_2 + Y_1}{\left[[(1+r)B_2 + Y_1]^2 - [\Delta]^2\right]^2}2\Delta
\end{aligned}
$$

The last is 0 when $\Delta = 0$, but positive when $\Delta > 0$.

Absent uncertainty ($\Delta = 0$) we have:

$$
\begin{aligned}
\frac{1}{Y_1 - B_2} &= \frac{1}{(1+r)B_2 + Y_1} \\
Y_1 - B_2 &= (1+r)B_2 + Y_1
\end{aligned}
$$

$$0 = (2+r) B_2$$

hence $B_2 = 0$. With uncertainty $(\Delta = 0)$ the right hand-side of the Euler gets larger, hence it must be that $\frac{1}{Y_1 - B_2}$ gets larger, i.e. $B_2 > 0$.

1.3 Case of certainty equivalence

Consider that the utility is:

$$\begin{aligned} u(C) &= C - \frac{a}{2} C^2 \\ u'(C) &= 1 - aC \end{aligned}$$

The Euler condition is then:

$$\begin{aligned} 1 - aC_1 &= \beta (1+r) E[1 - aC_2] \\ -C_1 &= \frac{\beta (1+r) - 1}{a} - \beta (1+r) EC_2 \\ EC_2 &= \frac{\beta (1+r) - 1}{a\beta (1+r)} + \frac{1}{\beta (1+r)} C_1 \end{aligned}$$

If $\beta (1+r) = 1$ this simplifies to $EC_2 = C_1$.

2 Complete asset markets

2.1 Budget constraints and Euler conditions

Enrich the two period model. In addition to the bond, the consumer can purchase contingent securities. A state k security pays off 1 unit if that state occurs, and zero otherwise. The cost of the security is $p_1(k) / (1+r)$.

Complete financial market: there is a security for each possible state (Arrow-Debreu set of complete contingent securities). The budget constraints are:

$$\begin{aligned} C_1 + \sum_k \frac{p_1(k)}{1+r} B_2(k) + B_2 &= Y_1 \\ C_2(k) &= (1+r) B_2 + B_2(k) + Y_2(k) \end{aligned}$$

Note that buying the bond guarantees a payoff, as does buying the same amount of all securities. The cost of the two strategies must be then the same:

$$\begin{aligned} \sum_k \frac{p_1(k)}{1+r} &= \frac{1}{1+r} \\ \sum_k p_1(k) &= 1 \end{aligned}$$

The Lagrangian is:

$$
\begin{aligned}
\mathcal{L} &= u(C_1) + \beta \sum_k \pi(k)\, u(C_2(k)) \\
&\quad + \lambda_1 \left[Y_1 - C_1 - B_2 - \sum_k \frac{p_1(k)}{1+r} B_2(k) \right] \\
&\quad + \sum_k \lambda_2(k) \left[(1+r) B_2 + B_2(k) + Y_2(k) - C_2(k) \right]
\end{aligned}
$$

The optimality conditions for C_1, $C_2(k)$, $B_2(k)$, and B_2 are:

$$
\begin{aligned}
0 &= u'(C_1) - \lambda_1 \\
0 &= \beta \pi(k)\, u'(C_2(k)) - \lambda_2(k) \\
0 &= -\lambda_1 \frac{p_1(k)}{1+r} + \lambda_2(k) \\
0 &= -\lambda_1 + \sum_k \lambda_2(k)(1+r)
\end{aligned}
$$

Combining these, we get two Euler conditions. The one for the bond is:

$$
\begin{aligned}
\lambda_1 &= \sum_k \lambda_2(k)(1+r) \\
u'(C_1) &= \sum_k \pi(k)\, \beta u'(C_2(k))(1+r) \\
u'(C_1) &= \beta(1+r) E\left[u'(C_2)\right]
\end{aligned}
$$

The one for a state k security is:

$$
\begin{aligned}
\lambda_1 \frac{p_1(k)}{1+r} &= \lambda_2(k) \\
u'(C_1) \frac{p_1(k)}{1+r} &= \beta \pi(k)\, u'(C_2(k)) \\
\frac{p_1(k)}{1+r} &= \pi(k) \frac{\beta u'(C_2(k))}{u'(C_1)}
\end{aligned}
$$

Note that:

$$
1 = (1+r) E\left[\frac{\beta u'(C_2)}{u'(C_1)} \right]
$$

Combining the two Eulers, the expected discounted excess return of an asset is zero:

$$
E\left[\frac{\beta u'(C_2)}{u'(C_1)} (1+r) \right] = E\left[\frac{\beta u'(C_2)}{u'(C_1)} \frac{1}{p_1/(1+r)} \right]
$$

Note that summing up the budget constraints implies that consumption is equal to output when added up over time, and pricing the future consumptions and outputs with the state contingent security prices:

$$\begin{aligned}
C_1 + \sum_k \frac{p_1(k)}{1+r} C_2(k) + \sum_k \frac{p_1(k)}{1+r} B_2(k) + B_2 &= Y_1 + \sum_k \frac{p_1(k)}{1+r} \left((1+r) B_2 + B_2(k) + Y_2(k)\right) \\
C_1 + \sum_k \frac{p_1(k)}{1+r} C_2(k) + B_2 &= Y_1 + B_2 \sum_k p_1(k) + \sum_k \frac{p_1(k)}{1+r} (Y_2(k)) \\
C_1 + \sum_k \frac{p_1(k)}{1+r} C_2(k) &= Y_1 + \sum_k \frac{p_1(k)}{1+r} (Y_2(k))
\end{aligned}$$

2.2 Gain from international financial trade

In autarky, consumption is equal to output. If we have complete securities, the Euler with each of them gives the autarky asset price:

$$\frac{p_1^A(k)}{1+r^A} = \pi(k) \frac{\beta u'(Y_2(k))}{u'(Y_1)}$$

When agents can trade financial assets, it must be that the resulting consumption allocation is not affordable at the autarky asset prices, otherwise there would be no gains from trade:

$$\begin{aligned}
C_1^{\text{asset trade}} + \sum_k \frac{p_1(k)}{1+r} C_2^{\text{asset trade}}(k) &= Y_1 + \sum_k \frac{p_1(k)}{1+r} Y_2(k) \\
C_1^{\text{asset trade}} + \sum_k \frac{p_1^A(k)}{1+r^A} C_2^{\text{asset trade}}(k) &\geq Y_1 + \sum_k \frac{p_1^A(k)}{1+r^A} Y_2(k)
\end{aligned}$$

Taking the difference:

$$\begin{aligned}
\sum_k \left[\frac{p_1^A(k)}{1+r^A} - \frac{p_1(k)}{1+r}\right] C_2^{\text{asset trade}}(k) &\geq \sum_k \left[\frac{p_1^A(k)}{1+r^A} - \frac{p_1(k)}{1+r}\right] Y_2(k) \\
\sum_k \left[\frac{p_1^A(k)}{1+r^A} - \frac{p_1(k)}{1+r}\right] \left[C_2^{\text{asset trade}}(k) - Y_2(k)\right] &\geq 0
\end{aligned}$$

The country is a net buyer of securities in states of nature where such securities are expensive (i.e. consumption is valuable) under autarky.

2.3 General equilibrium

Introduce a foreign economy, denoted by F. The Home Euler conditions and the Foreign Euler conditions are:

$$\begin{aligned}
\frac{1}{1+r} &= E\left[\frac{\beta u'\left(C_2^H\right)}{u'\left(C_1^H\right)}\right] = E\left[\frac{\beta u'\left(C_2^F\right)}{u'\left(C_1^F\right)}\right] \\
\frac{p_1(k)}{1+r} &= \pi(k) \frac{\beta u'\left(C_2^H(k)\right)}{u'\left(C_1^H\right)} = \pi(k) \frac{\beta u'\left(C_2^F(k)\right)}{u'\left(C_1^F\right)}
\end{aligned}$$

The Euler for the state contingent securities imply that the ratios of marginal utilities are equalized across states:

$$\frac{u'\left(C_2^H(k)\right)}{u'\left(C_2^F(k)\right)} = \frac{u'\left(C_1^H\right)}{u'\left(C_1^F\right)}$$

With a CRRA utility $u(C) = \frac{(C)^{1-\sigma}}{1-\sigma}$. The Euler conditions imply:

$$\begin{aligned}
\frac{p_1(k)}{1+r} &= \beta\pi(k)\left(\frac{C_2^H(k)}{C_1^H}\right)^{-\sigma} = \beta\pi(k)\left(\frac{C_2^F(k)}{C_1^F}\right)^{-\sigma} \\
\frac{C_2^H(k)}{C_1^H} &= \frac{C_2^F(k)}{C_1^F} = \left[\frac{\beta(1+r)}{p_1(k)}\pi(k)\right]^{\frac{1}{\sigma}}
\end{aligned}$$

The clearing of the good market are:

$$\begin{aligned}
C_1^H + C_1^F &= Y_1^H + Y_1^F = Y_1^W \\
C_2^H(k) + C_2^F(k) &= Y_2^H(k) + Y_2^F(k) = Y_2^W(k)
\end{aligned}$$

Combining with the Euler:

$$\begin{aligned}
C_2^H(k) + C_2^F(k) &= Y_2^W(k) \\
\left[\frac{\beta(1+r)}{p_1(k)}\pi(k)\right]^{\frac{1}{\sigma}}\left(C_1^H + C_1^F\right) &= Y_2^W(k) \\
\left[\frac{\beta(1+r)}{p_1(k)}\pi(k)\right]^{\frac{1}{\sigma}} Y_1^W &= Y_2^W(k) \\
\frac{p(k)}{1+r} &= \beta\pi(k)\left[\frac{Y_2^W(k)}{Y_1^W}\right]^{-\sigma}
\end{aligned}$$

The price of a security paying off in state k reflects the relative likelihood and scarcity of output in that state

Add up across state to solve for the interest rate, which reflects output growth:

$$\begin{aligned}
\sum_k \frac{p(k)}{1+r} &= \beta\sum_k \pi(k)\left[\frac{Y_2^W(k)}{Y_1^W}\right]^{-\sigma} \\
\frac{1}{1+r}\sum_k p(k) &= \beta E\left[\frac{Y_2^W}{Y_1^W}\right]^{-\sigma} \\
\frac{1}{1+r} &= \beta E\left[\frac{Y_2^W}{Y_1^W}\right]^{-\sigma} \\
\beta(1+r) &= \frac{\left[Y_1^W\right]^{-\sigma}}{E\left[\left(Y_2^W\right)^{-\sigma}\right]}
\end{aligned}$$

The cross-country ratios of consumption are equalized across states. Using the clearing of good market, each country consumes the same share of world output:

$$\begin{aligned}\frac{C_2^H(k)}{C_2^F(k)} &= \frac{C_2^H(g)}{C_2^F(g)} \\ \frac{C_2^H(k)}{C_2^H(g)} &= \frac{C_2^F(k)}{C_2^F(g)} = \frac{Y_2^W(k)}{Y_2^W(g)}\end{aligned}$$

2.4 In nominal terms

To bring the exchange rate into the picture, we consider some state-contingent assets that pay off in Home currency, and others that pay off in Foreign currency.

S is the nominal exchange rate, expressed in terms of units of Home currency per unit of Foreign currency, so a higher value is a weaker Home currency. The CPI faced by the Home agent is P^H (Home currency) and the one by the Foreign agent is P^F (Foreign currency).

At time t agents can buy state contingent securities in two currencies. A security paying off one unit of Home currency in state k_{t+s} at time $t+s$ costs $p_t^H(k_{t+s})$ units of Home currency. A security paying off one unit of Foreign currency in state k_{t+s} at time $t+s$ costs $p_t^F(k_{t+s})$ units of Foreign currency.

From the point of view of time t the Home agent maximizes:

$$u\left(C_t^H\right) + \sum_{s=1}^{\infty} \beta^s \left[\sum_{k_{t+s}} \pi(k_{t+s})\, u\left(C_{t+s}^H(k_{t+s})\right)\right]$$

The budget constraints are (focusing on the first two periods):

$$\begin{aligned}P_t^H C_t^H &= P_t^H Y_t^H - \sum_{s=1}^{\infty}\sum_{k_{t+s}} \frac{p_t^H(k_{t+s})}{1+r} B_t^H(k_{t+s}) - \sum_{s=1}^{\infty}\sum_{k_{t+s}} \frac{S_t p_t^F(k_{t+s})}{1+r} B_t^F(k_{t+s}) - B_{t+1}^H \\ P_{t+1}^H(k_{t+1})\, C_{t+1}^H(k_{t+1}) &= (1+r) B_{t+1} + B_t^H(k_{t+1}) + S_{t+1} B_t^F(k_{t+1}) + P_{t+1}^H(k_{t+1})\, Y_{t+1}^H(k_{t+1})\end{aligned}$$

The Lagrangian is:

$$\begin{aligned}\mathcal{L}_t &= u\left(C_t^H\right) + \sum_{s=1}^{\infty} \beta^s \left[\sum_{k_{t+s}} \pi(k_{t+s})\, u\left(C_{t+s}^H(k_{t+s})\right)\right] \\ &\quad + \lambda_t \left[P_t^H Y_t^H - P_t^H C_t^H - \sum_{s=1}^{\infty}\sum_{k_{t+s}} \frac{p_t(k_{t+s})}{1+r} B_t^H(k_{t+s}) - \sum_{s=1}^{\infty}\sum_{k_{t+s}} \frac{S_t p_t^F(k_{t+s})}{1+r} B_t^F(k_{t+s}) - B_{t+1}^H\right] \\ &\quad + \sum_{k_{t+1}} \lambda_{t+1}(k_{t+1}) \left[\begin{array}{c}(1+r) B_{t+1} + B_t^H(k_{t+1}) + S_{t+1}(k_{t+1})\, B_t^F(k_{t+1}) \\ + P_{t+1}^H(k_{t+1})\, Y_{t+1}^H(k_{t+1}) - P_{t+1}^H(k_{t+1})\, C_{t+1}^H(k_{t+1})\end{array}\right]\end{aligned}$$

The optimality conditions for C_t^H, $C_{t+1}^H(k_{t+1})$, $B_t^H(k_{t+1})$, $B_t^F(k_{t+1})$ and B_{t+1} are:

$$\begin{aligned}0 &= u'\left(C_t^H\right) - \lambda_t P_t^H \\ 0 &= \beta\pi(k_{t+1})\, u'\left(C_{t+1}^H(k_{t+1})\right) - \lambda_{t+1}(k_{t+1})\, P_{t+1}^H(k_{t+1})\end{aligned}$$

$$\begin{aligned}
0 &= -\lambda_t \frac{p_t(k_{t+1})}{1+r} + \lambda_{t+1}(k_{t+1}) \\
0 &= -\lambda_t \frac{S_t p_t^F(k_{t+1})}{1+r} + \lambda_{t+1}(k_{t+1}) S_{t+1}(k_{t+1}) \\
0 &= -\lambda_t + \sum_{k_{t+1}} \lambda_{t+1}(k_{t+1})(1+r)
\end{aligned}$$

Combining these, we get the Euler conditions:

$$\begin{aligned}
\frac{p_t(k_{t+1})}{1+r} &= \pi(k_{t+1}) \frac{\beta u'\left(C_{t+1}^H(k_{t+1})\right)/P_{t+1}^H(k_{t+1})}{u'\left(C_t^H\right)/P_t^H} \\
\frac{S_t p_t^F(k_{t+1})}{1+r} &= S_{t+1}(k_{t+1})\pi(k_{t+1}) \frac{\beta u'\left(C_{t+1}^H(k_{t+1})\right)/P_{t+1}^H(k_{t+1})}{u'\left(C_t^H\right)/P_t^H} \\
1 &= (1+r)\sum_{k_{t+1}} \pi(k_{t+1}) \frac{\beta u'\left(C_{t+1}^H(k_{t+1})\right)/P_{t+1}^H(k_{t+1})}{u'\left(C_t^H\right)/P_t^H}
\end{aligned}$$

We derive the Euler conditions for the Foreign agent following similar steps:

$$\begin{aligned}
\frac{p_t(k_{t+1})}{1+r}\frac{1}{S_t} &= \frac{1}{S_{t+1}(k_{t+1})}\pi(k_{t+1}) \frac{\beta u'\left(C_{t+1}^F(k_{t+1})\right)/P_{t+1}^F(k_{t+1})}{u'\left(C_t^F\right)/P_t^F} \\
\frac{p_t^F(k_{t+1})}{1+r} &= \pi(k_{t+1}) \frac{\beta u'\left(C_{t+1}^F(k_{t+1})\right)/P_{t+1}^F(k_{t+1})}{u'\left(C_t^F\right)/P_t^F} \\
\frac{1}{S_t} &= (1+r)\sum_{k_{t+1}} \frac{1}{S_{t+1}(k_{t+1})}\pi(k_{t+1}) \frac{\beta u'\left(C_{t+1}^F(k_{t+1})\right)/P_{t+1}^F(k_{t+1})}{u'\left(C_t^F\right)/P_t^F}
\end{aligned}$$

Combining the Euler conditions for state contingent securities, we get:

$$\begin{aligned}
\frac{u'\left(C_{t+1}^H(k_{t+1})\right)/P_{t+1}^H(k_{t+1})}{u'\left(C_t^H\right)/P_t^H} &= \frac{S_t}{S_{t+1}(k_{t+1})}\frac{u'\left(C_{t+1}^F(k_{t+1})\right)/P_{t+1}^F(k_{t+1})}{u'\left(C_t^F\right)/P_t^F} \\
\frac{u'\left(C_{t+1}^H(k_{t+1})\right)/P_{t+1}^H(k_{t+1})}{u'\left(C_{t+1}^F(k_{t+1})\right)/P_{t+1}^F(k_{t+1})} &= \frac{S_t}{S_{t+1}(k_{t+1})}\frac{u'\left(C_t^H\right)/P_t^H}{u'\left(C_t^F\right)/P_t^F} \\
\frac{u'\left(C_{t+1}^H(k_{t+1})\right)}{u'\left(C_{t+1}^F(k_{t+1})\right)}\frac{S_{t+1}(k_{t+1})P_{t+1}^F(k_{t+1})}{P_{t+1}^H(k_{t+1})} &= \frac{u'\left(C_t^H\right)}{u'\left(C_t^F\right)}\frac{S_t P_t^F}{P_t^H}
\end{aligned}$$

Hence, in any state of nature:

$$\frac{u'\left(C_{t+1}^H\right)}{u'\left(C_{t+1}^F\right)}\frac{S_{t+1}P_{t+1}^F}{P_{t+1}^H} = \frac{u'\left(C_t^H\right)}{u'\left(C_t^F\right)}\frac{S_t P_t^F}{P_t^H} = \Gamma_t$$

3 International portfolio choice

3.1 Budget constraints

Consider a two period model with w countries, H and F. Agents can invest in a bond, and in equity in the two countries. Equity is a claim on the country's endowment.

The expected utility for the Home and Foreign agents are:

$$u\left(C_1^H\right)+\beta E u\left(C_2^H\right) \qquad ; \qquad u\left(C_1^F\right)+\beta E u\left(C_2^F\right)$$

The Home household purchases x_2^{HH} units of the Home equity, with a price V_1^H. Each unit pays of the endowment Y_2^H. She also purchases x_2^{HF} units of the Foreign equity, with a price V_1^F. Each unit pays of the endowment Y_2^F. The Foreign household purchases $x_2^{FH}$and x_2^{FF}units of equity. The budget constraints of the Home households are:

$$\begin{aligned} C_1^H &= Y_1^H+V_1^H-B_2^H-x_2^{HH}V_1^H-x_2^{HF}V_1^F \\ C_2^H(k) &= (1+r)B_2^H+x_2^{HH}Y_2^H(k)+x_2^{HF}Y_2^F(k) \end{aligned}$$

The constraints for the Foreign household are:

$$\begin{aligned} C_1^F &= Y_1^F+V_1^F-B_2^F-x_2^{FH}V_1^H-x_2^{FF}V_1^F \\ C_2^F(k) &= (1+r)B_2^F+x_2^{FH}Y_2^H(k)+x_2^{FF}Y_2^F(k) \end{aligned}$$

3.2 Optimization

Consider a two period model with w countries, H and F. Agents can invest in a bond, and in equity in the two countries. Equity is a claim on the country's endowment.

The Lagrangian is:

$$\begin{aligned} \mathcal{L} &= u\left(C_1^H\right)+\beta\sum_k \pi(k)\,u\left(C_2^H(k)\right) \\ &\quad +\lambda_1\left[Y_1^H-C_1^H-B_2^H-x_2^{HH}V_1^H-x_2^{HF}V_1^F\right] \\ &\quad +\sum_k \lambda_2(k)\left[(1+r)B_2^H+x_2^{HH}Y_2^H(k)+x_2^{HF}Y_2^F(k)-C_2^H(k)\right] \end{aligned}$$

The optimality conditions for C_1^H, $C_2^H(k)$, B_2^H, x_2^{HH}, and x_2^{HF} are:

$$\begin{aligned} 0 &= u'\left(C_1^H\right)-\lambda_1 \\ 0 &= \pi(k)\,\beta u'\left(C_2^H(k)\right)-\lambda_2(k) \\ 0 &= -\lambda_1+\sum_k \lambda_2(k)(1+r) \\ 0 &= -\lambda_1 V_1^H+\sum_k \lambda_2(k)\,Y_2^H(k) \\ 0 &= -\lambda_1 V_1^F+\sum_k \lambda_2(k)\,Y_2^F(k) \end{aligned}$$

Combining these, we get three Euler conditions. The one for the bond is:

$$\lambda_1 = \sum_k \lambda_2(k)(1+r)$$

$$
\begin{aligned}
u'\left(C_1^H\right) &= \sum_k \pi\left(k\right)\beta u'\left(C_2^H\left(k\right)\right)\left(1+r\right) \\
u'\left(C_1^H\right) &= \beta\left(1+r\right)E\left[u'\left(C_2^H\right)\right]
\end{aligned}
$$

The one for Home equity is:

$$
\begin{aligned}
\lambda_1 V_1^H &= \sum_k \lambda_2\left(k\right)Y_2^H\left(k\right) \\
u'\left(C_1^H\right)V_1^H &= \sum_k \pi\left(k\right)\beta u'\left(C_2^H\left(k\right)\right)Y_2^H\left(k\right) \\
u'\left(C_1^H\right) &= \beta\sum_k \pi\left(k\right)u'\left(C_2^H\left(k\right)\right)\frac{Y_2^H\left(k\right)}{V_1^H} \\
u'\left(C_1^H\right) &= \beta E\left[u'\left(C_2^H\right)\frac{Y_2^H}{V_1^H}\right]
\end{aligned}
$$

The one for Foreign equity is:

$$
\begin{aligned}
\lambda_1 V_1^F &= \sum_k \lambda_2\left(k\right)Y_2^F\left(k\right) \\
u'\left(C_1^H\right)V_1^F &= \sum_k \pi\left(k\right)\beta u'\left(C_2^H\left(k\right)\right)Y_2^F\left(k\right) \\
u'\left(C_1^H\right) &= \beta E\left[u'\left(C_2^H\right)\frac{Y_2^F}{V_1^F}\right]
\end{aligned}
$$

Combining the Eulers, the expected discounted excess return of an asset is zero:

$$
E\left[\frac{\beta u'\left(C_2^H\right)}{u'\left(C_1^H\right)}\left(1+r\right)\right] = E\left[\frac{\beta u'\left(C_2^H\right)}{u'\left(C_1^H\right)}\frac{Y_2^H}{V_1^H}\right] = E\left[\frac{\beta u'\left(C_2^H\right)}{u'\left(C_1^H\right)}\frac{Y_2^F}{V_1^F}\right]
$$

hence:

$$
\begin{aligned}
0 &= E\left[\frac{\beta u'\left(C_2^H\right)}{u'\left(C_1^H\right)}\left(\frac{Y_2^H}{V_1^H}-\left(1+r\right)\right)\right] \\
0 &= E\left[\frac{\beta u'\left(C_2^H\right)}{u'\left(C_1^H\right)}\left(\frac{Y_2^F}{V_1^F}-\left(1+r\right)\right)\right]
\end{aligned}
$$

3.3 Asset pricing

The last two Euler give a partial solution for the asset prices:

$$
V_1^H = E\left[\frac{\beta u'\left(C_2^H\right)}{u'\left(C_1^H\right)}Y_2^H\right] \qquad ; \qquad V_1^F = E\left[\frac{\beta u'\left(C_2^H\right)}{u'\left(C_1^H\right)}Y_2^F\right]
$$

As $E(ab) = E(a)E(b) + Cov(a,b)$ we write:

$$V_1^F = E\left[\frac{\beta u'(C_2^H)}{u'(C_1^H)}\right] E\left[Y_2^F\right] + Cov\left[\frac{\beta u'(C_2^H)}{u'(C_1^H)}, Y_2^F\right]$$

3.4 Optimal portfolio and asset market clearing

Consider that the utility is:

$$u(C) = \frac{(C)^{1-\sigma}}{1-\sigma}$$

The Euler conditions become:

$$\begin{aligned} 1 &= (1+r)E\left[\beta\left(\frac{C_2^H}{C_1^H}\right)^{\sigma}\right] \\ 1 &= E\left[\beta\left(\frac{C_2^H}{C_1^H}\right)^{\sigma}\frac{Y_2^H}{V_1^H}\right] \\ 1 &= E\left[\beta\left(\frac{C_2^H}{C_1^H}\right)^{\sigma}\frac{Y_2^F}{V_1^F}\right] \end{aligned}$$

One can show that the utility implies that Home and Foreign households allocate the same share of their savings (difference between output and asset endowment and consumption) to specific assets.

$$\begin{aligned} \frac{x_2^{HH}V_1^H}{V_1^H + Y_1^H - C_1^H} &= \frac{x_2^{FH}V_1^H}{V_1^F + Y_1^F - C_1^F} \\ \frac{x_2^{HH}V_1^H}{V_1^H + Y_1^H - \mu_1^H Y_1^W} &= \frac{x_2^{FH}V_1^H}{V_1^F + Y_1^F - \mu_1^F Y_1^W} \end{aligned}$$

where μ is the country's share of global consumption (equal to global output). Similarly for the Foreign equity:

$$\frac{x_2^{HF}V_1^F}{V_1^H + Y_1^H - \mu_1^H Y_1^W} = \frac{x_2^{FF}V_1^F}{V_1^F + Y_1^F - \mu_1^F Y_1^W}$$

The clearing of asset markets implies:

$$\begin{aligned} x_2^{HH} + x_2^{FH} &= 1 \\ x_2^{HF} + x_2^{FF} &= 1 \\ B_2^H + B_2^F &= 0 \end{aligned}$$

As the two investors have the same asset allocation, and bonds are in zero net supply, we get $B_2^H = 0$.

This implies:

$$\frac{x_2^{HH}V_1^H}{V_1^H + Y_1^H - \mu_1^H Y_1^W} = \frac{x_2^{FH}V_1^H}{V_1^F + Y_1^F - \mu_1^F Y_1^W}$$

$$\begin{aligned}
\frac{x_2^{HH}V_1^H}{V_1^H+Y_1^H-\mu_1^H Y_1^W} &= \frac{\left(1-x_2^{HH}\right)V_1^H}{V_1^F+Y_1^F-\mu_1^F Y_1^W} \\
\frac{x_2^{HH}}{1-x_2^{HH}} &= \frac{V_1^H+Y_1^H-\mu_1^H Y_1^W}{V_1^F+Y_1^F-\mu_1^F Y_1^W} \\
\frac{x_2^{HH}}{1-x_2^{HH}} &= \frac{x_2^{HF}}{1-x_2^{HF}} \\
x_2^{HH} &= x_2^{HF}
\end{aligned}$$

Therefore, the Home investor accounts for the same share of all equity investments: $x_2^{HH} = x_2^{HF} = x_2^H$.

3.5 Consumption and current account

The second period budget constraint implies:

$$\begin{aligned}
C_2^H(k) &= x_2^{HH}Y_2^H(k) + x_2^{HF}Y_2^F(k) \\
C_2^H(k) &= x_2^H\left(Y_2^H(k)+Y_2^F(k)\right) \\
C_2^H(k) &= x_2^H Y_2^W(k) \\
C_2^H(k) &= \mu_2^H Y_2^W(k)
\end{aligned}$$

Note that μ_2^H is set in period 1. The Euler for Home equity for the two investors are:

$$V_1^H = E\left[\beta\left(\frac{C_2^H}{C_1^H}\right)^{\sigma} Y_2^H\right] \qquad ; \qquad V_1^H = E\left[\beta\left(\frac{C_2^F}{C_1^F}\right)^{\sigma} Y_2^H\right]$$

which implies:

$$\begin{aligned}
E\left[\beta\left(\frac{C_2^H}{C_1^H}\right)^{\sigma} Y_2^H\right] &= E\left[\beta\left(\frac{C_2^F}{C_1^F}\right)^{\sigma} Y_2^H\right] \\
E\left[\left(\frac{\mu_2^H Y_2^W}{\mu_1^H Y_1^W}\right)^{\sigma} Y_2^H\right] &= E\left[\left(\frac{\left(1-\mu_2^H\right)Y_2^W}{\left(1-\mu_2^F\right)Y_1^W}\right)^{\sigma} Y_2^H\right] \\
\left(\frac{\mu_2^H}{\mu_1^H}\right)^{\sigma} E\left[\left(\frac{Y_2^W}{Y_1^W}\right)^{\sigma} Y_2^H\right] &= \left(\frac{1-\mu_2^H}{1-\mu_1^H}\right)^{\sigma} E\left[\left(\frac{Y_2^W}{Y_1^W}\right)^{\sigma} Y_2^H\right] \\
\frac{\mu_2^H}{\mu_1^H} &= \frac{1-\mu_2^H}{1-\mu_1^H} \\
\mu_2^H - \mu_1^H\mu_2^H &= \mu_1^H - \mu_1^H\mu_2^H \\
\mu_2^H &= \mu_1^H
\end{aligned}$$

The Home first period budget constraint then implies:

$$\begin{aligned}
C_1^H &= Y_1^H + V_1^H - B_2^H - x_2^{HH}V_1^H - x_2^{HF}V_1^F \\
\mu^H Y_1^W &= Y_1^H + V_1^H - x^H\left(V_1^H+V_1^F\right) \\
\mu^H Y_1^W &= Y_1^H + V_1^H - \mu^H\left(V_1^H+V_1^F\right)
\end{aligned}$$

$$\begin{aligned}
\mu^H\left(Y_1^W+V_1^W\right) &= Y_1^H+V_1^H \\
\mu^H &= \frac{Y_1^H+V_1^H}{Y_1^W+V_1^W}
\end{aligned}$$

The current account is then:

$$\begin{aligned}
CA_1 &= Y_1^H-C_1^H \\
CA_1 &= Y_1^H-\frac{Y_1^H+V_1^H}{Y_1^W+V_1^W}Y_1^W \\
CA_1 &= Y_1^H-\frac{Y_1^H}{Y_1^W+V_1^W}Y_1^W-\frac{V_1^H}{Y_1^W+V_1^W}Y_1^W \\
CA_1 &= \frac{Y_1^W+V_1^W-Y_1^W}{Y_1^W+V_1^W}Y_1^H-\frac{Y_1^W}{Y_1^W+V_1^W}V_1^H \\
CA_1 &= \frac{V_1^W}{Y_1^W+V_1^W}Y_1^H-\frac{Y_1^W}{Y_1^W+V_1^W}V_1^H
\end{aligned}$$

From the Euler, the asset prices are:The Euler for Home equity for the two investors are:

$$\begin{aligned}
V_1^H &= E\left[\beta\left(\frac{C_2^H}{C_1^H}\right)^\sigma Y_2^H\right] \\
V_1^H &= E\left[\beta\left(\frac{\mu_2^H Y_2^W}{\mu_1^H Y_1^W}\right)^\sigma Y_2^H\right] \\
V_1^H &= E\left[\beta\left(\frac{Y_2^W}{Y_1^W}\right)^\sigma Y_2^H\right]
\end{aligned}$$

Similarly:

$$V_1^F = E\left[\beta\left(\frac{Y_2^W}{Y_1^W}\right)^\sigma Y_2^F\right]$$