Mathematical Modelling

Professor Subhashis Banerjee
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Objective . The objective is to propose a mathematical model describing the construction of orthographic views from a given isometric view or an image of the object and vice-versa.

1. **Constructing orthographic views from isometric view**

   To start with...

   The input is taken from the user and is then converted into a graph which contains all the vertices present in the isometric view. Also, the cutting plane of the object is stored in appropriate format. The set of vertices will contain the co-ordinates of each vertex. Hence each node is a tuple with 3 elements. Furthermore, a corresponding edge set containing all the edges between various vertices of the graph is also stored.

   Mathematical Analysis

   (a) **Adjustment of the co-ordinate system:**
   The given object must be translated and rotated about an axis so as to bring it in accordance with the given cutting plane. Some amount of scaling may also be required for the measurements of the object may sometimes be very large.

      i. **Translation:-**
      Translation is an operation that displaces all the points (and hence, the object) by a fixed distance in a specific direction. If the displacement vector is d then the point P will be moved to

      $$P' = P + d$$

      In matrix form, the equation can be written as:

      $$\mathbf{p'} = \mathbf{p} + \mathbf{d}$$

where,

$$\mathbf{p} = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \mathbf{p'} = \begin{pmatrix} x' \\ y' \\ z' \\ 1 \end{pmatrix} \mathbf{d} = \begin{pmatrix} a \\ b \\ c \\ 1 \end{pmatrix}$$

Thus, the translation matrix for a 3-D space is defined by a 4*4 matrix given below:

$$T = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We have to multiply this translation matrix to the column matrix of co-ordinates with -1 as the last entry to obtain the shifted co-ordinates.

$$\begin{bmatrix} x' \\ y' \\ z' \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ -1 \end{bmatrix} = \begin{bmatrix} x-a \\ y-b \\ z-c \\ -1 \end{bmatrix}$$

Here the column on the left represents the co-ordinates obtained after translating(shifting) the original co-ordinates by a, b, c respectively.

ii. **Rotation:-**
We can also define rotational matrix but the difference here from translation matrix is that we have to define separate matrices for rotation about x, y and z axes. We will assume that the rotation is done anti-clockwise. So there will be 3 rotational matrices as shown below:

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(\theta_x) & sin(\theta_x) & 0 \\ 0 & -sin(\theta_x) & cos(\theta_x) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_y = \begin{bmatrix} cos(\theta_y) & 0 & -sin(\theta_y) & 0 \\ 0 & 1 & 0 & 0 \\ sin(\theta_y) & 0 & cos(\theta_y) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_z = \begin{bmatrix} cos(\theta_z) & sin(\theta_z) & 0 & 0 \\ -sin(\theta_z) & cos(\theta_z) & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here $\theta_x, \theta_y$and θ_z are the angles of rotation about x, y and z axes respectively.

Let the direction cosines of the axis (passing through the origin) be $\alpha_x, \alpha_y, \alpha_z$. Using simple trigonometry, we find that

$$cos\theta_x = \alpha_z/d, sin\theta_x = \alpha_y/d$$

where $d = \sqrt{\alpha_y^2 + \alpha_z^2}$, so, without needing θ_x explicitly, we find

$$R_x = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \alpha_z/d & -\alpha_y/d & 0 \\ 0 & \alpha_y/d & \alpha_z/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We can obtain the new co-ordinates after rotation by multiplying these rotation matrices to the co-ordinate column vector.

$$\begin{bmatrix} x_r \\ y_r \\ z_r \\ -1 \end{bmatrix} = R_x * R_y * R_z * \begin{bmatrix} x \\ y \\ z \\ -1 \end{bmatrix}$$

After applying translation as well, we will get the final co-ordinates.

$$\begin{bmatrix} x_f \\ y_f \\ z_f \\ -1 \end{bmatrix} = T * R_x * R_y * R_z * \begin{bmatrix} x \\ y \\ z \\ -1 \end{bmatrix}$$

iii. **Scaling:-**

Scaling can be applied in any of the three axes independently. If all the scaling factors are equal (along all the three direction cosines), it would be uniform scaling. Uniform scaling is mostly required in this assignment for the dimensions of the given object may be very large.

(b) **Projection on the Orhtogonal Planes:**

Now we have to take projection of every vertex(node) present in the graph on 3 orthogonal planes. This can be done by using projection matrix. Such a matrix for projection in x-y plane is shown below:

$$M_{xy} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If the co-ordinates of any point is multiplied by this matrix then we will get the projection of that vertex on x-y plane as shown below.

$$\begin{pmatrix} x \\ y \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ -1 \end{pmatrix} = \begin{pmatrix} x+0+0 \\ 0+y+0 \\ 0+0+0 \\ 0+0+(-1) \end{pmatrix}$$

Similarly we can define projection matrices for y-z and x-z planes as shown below:

$$M_{yz} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} M_{zx} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now let us take the projection in x-y plane. After multiplying the co-ordinates with the projection matrix we get their projection in the x-y plane. Now, for lines, we can say that those vertices which were connected in the graph before we multiplied them with the projection matrix, will still be connected in their respected projections.

Now we can take the projection on x-y plane by applying the projection matrix corresponding to x-y plane.

$$\begin{bmatrix} x_f \\ y_f \\ 0 \\ -1 \end{bmatrix} = M_{xy} * T * R_x * R_y * R_z * \begin{bmatrix} x \\ y \\ z \\ -1 \end{bmatrix}$$

Similarly, the other 2 projections can be obtained.

$$\begin{bmatrix} 0 \\ y_f \\ z_f \\ -1 \end{bmatrix} = M_{yz} * T * R_x * R_y * R_z * \begin{bmatrix} x \\ y \\ z \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} x_f \\ 0 \\ z_f \\ -1 \end{bmatrix} = M_{zx} * T * R_x * R_y * R_z * \begin{bmatrix} x \\ y \\ z \\ -1 \end{bmatrix}$$

Hence we can obtain all the projections.

2. **Constructing isometric view from orthographic views**

To start with...
The input will be in the form of 3 orthographic views which will be converted into a graph with the vertices as nodes. The set of vertices will contain the co-ordinates of each vertex. Hence each node will contain a tuple consisting of 3 elements. Furthermore, a corresponding edge set containing all the edges between various vertices of the graph is also stored. Also, the point to point correspondence is given in the input.

Mathematical Analysis.

(a) We can obtain the vertices of the required isometric view by using point to point correspondence. Since we have 3 orthographic projections of the given object, we can get all the 3 co-ordinates of the object in the 3-d space.
x and y co-ordinate from the xy projection (front view) and then z co-ordinate from the yz or xz projection (top or side view).

(b) Now, once we get the wire-frame, we can use the fact that between 2 points in an isometric view an edge will exist if and only if there exist an edge between these points in either all of its projections or edge in two of the projections and the points coincide in the third projection.

Method

(a) Firstly we will iterate over all vertices present in the 3 projections to find the 3-D co-ordinate of all the points. From orthographic views we will get 3 tuples of 2 elements each namely (x,y), (y,z) and (z,x). We will find 3 sets such that any 2have atleast 1 co-ordinate same. If such points doesnt exist then the given inputis invalid. If such triplets exist then those triplets will be used as vertices of the3-D model with (x,y,z) as the co-ordinate. By iterating over all points we can findall the vertices of the 3-D view.

(b) We can obtain the vertices of the required isometric view by using point to point correspondence.

(c) After this, we have to construct the edge set. For that we will check edge set from each orthographic view for edge between the 2 vertices in question. We can use the fact that between 2 points in an isometric view an edge will exist if and only if there exist an edge between these points either in all one of its projections or edges in 2 of the projections and in the third projection, the points coincide.

(d) The above steps may result in partially wrong answer because of the inclusion of the hidden edges in the edge set of each orthographic view. So we will draw their orthographic projections and match them with the input.

(e) If the input and the orthographic views matches, then we are good to go. But if they don't match then will remove the extra edges from the solution.

(f) The above step is repeated until we get the correct answer that is consistent with the given projections.
The isometric can be scaled, rotated and scaled as per requirement according to the given view direction.