NATIONAL INSTITUTE OF TECHNOLOGY ROURKELA

End Semester Examination, 2018
Subject: Discrete Structures **(No. of pages: 2)** **Full Marks: 50**
Subject Code: CS2003

Answer all questions from Section 1 and 10 questions from Section 2.
Mere answer without justification will not fetch any mark.

Section 1 - each question carries 2 marks

1. Show that $\exists x P(x) \wedge \exists x Q(x)$ and $\exists x (P(x) \wedge Q(x))$ are not logically equivalent.

2. Let a, b be two real numbers. Their arithmetic mean is $\frac{a+b}{2}$ and their geometric mean is $\sqrt{ab}$. Prove (or disprove) that the geometric mean is strictly more than the arithmetic mean.

3. Check whether the two graphs in Figure 1 are isomorphic? If so, give the isomorphism.

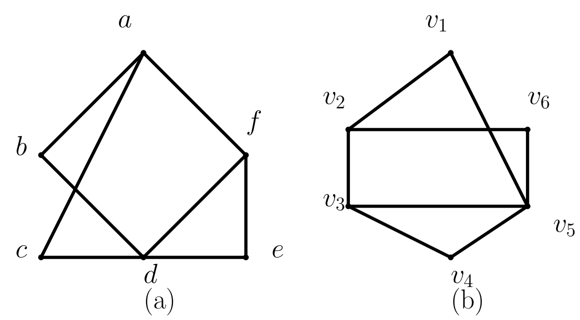

Figure 1: Isomorphism of graphs in (a) and (b)

4. Find a Eulerian circuit, a Eulerian path and a Hamiltonian circuit (if exists) in graphs given in in Figure 2.

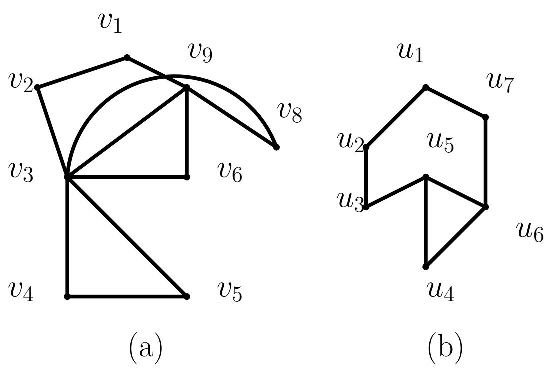

Figure 2: Euler circuits and paths and Hamiltonian circuits in graphs in (a) and (b)

5. Determine the Chromatic number of the graphs given in Figure 3.

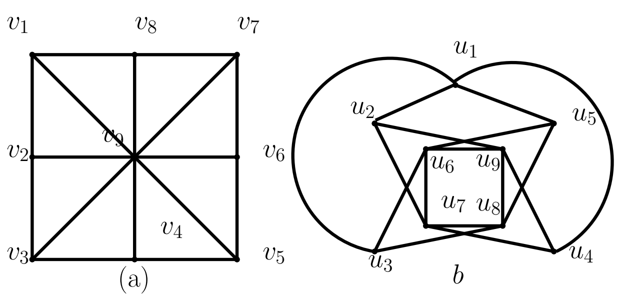

Figure 3: Chromatic number of graphs given in (a) and (b)

6. Show that $\mathbf{n \log n}$ is $\mathbf{O(\log n!)}$ (Hint: $\mathbf{n! > n(n-1)\ldots\lceil\frac{n}{2}\rceil}$).

7. Let $\mathbf{A = \{a, b, c\}}$. Find the smallest and largest cardinality partial order relation on the set A.

8. Let $\mathbf{R}$ be a relation on $\mathbb{Z}$ such that $\mathbf{(a, b) \in R}$ if and only if $\mathbf{a - b}$ is an even integer. Is R and equivalence relation.

9. Let $\lfloor \mathbf{x} \rfloor$ denotes the largest integer less than or equal to $\mathbf{x}$. Show that
(i) $\mathbf{\binom{n}{\lfloor \frac{n}{2} \rfloor} \geq 2^n/n}$, (ii) $\mathbf{\binom{n}{k} \leq n^k/2^{k-1}}$.

10. Construct a connected and a disconnected graph with the following degree sequence. 4,4,4,3,3,2,2

Section 2 - each question carries 3 marks

11. Let $\mathbf{f_n = f_{n-1} + f_{n-2}}$, $\mathbf{f_0 = 0}$, $\mathbf{f_1 = 1}$. Let $\mathbf{A}$ denote the matrix $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. Let $\mathbf{A^n}$ denote $\mathbf{A \times A \times \ldots A}$ ($\mathbf{n}$ times), where $\times$ is matrix multiplication. Prove that

$$\mathbf{A^{n-1}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix}.$$

12. Prove that the product of any three consecutive integers is divisible by 6.

13. A bit string consisting of 0's and 1's is a palindrome if its reversal is same as the original string. For instance, 000111 is not a palindrome where as 1010101 and 101101 are palindromes. How many bit strings of length n are palindromes.

14. A DNA sequence consists of a string of four bases, namely, A, T, G and C. How many distinct DNA sequences of length $\mathbf{n}$ are possible with (i) only A's and T's but no G's and C's.
(ii) exactly 2 G's.
(iii) at least $\mathbf{n/2}$ C's.
(iv) number of A's + number of C's=number of T's + number of G's.

15. How many numbers should be chosen from a set $S = \{1, \ldots, n\}$ such that there exists a pair of numbers whose sum is $\frac{n}{2}$.

16. Prove the identity $\mathbf{\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}}$, whenever n, r, and k are nonnegative integers with $r \leq k \leq n$, (i) using a combinatorial argument.
(ii) using an argument based on the formula for the number of r-combinations of a set with n elements.

17. How many solutions are there to the inequality $\mathbf{x_1 + x_2 + x_3 \leq 11}$, where x_1, x_2, and x_3 are nonnegative integers?

18. Let total number of ways a board of $3 \times n$ dominos can be filled with 3×1 tiles be denoted as f_n. Find a recurrence relation for f_n. Find the initial conditions and find a closed form formula.

19. For any connected bipartite planar graph on $\mathbf{n}$ vertices and $\mathbf{e}$ edges, prove that $\mathbf{e \leq 2n - 4}$ (Hint: use the fact that for connected planar graphs on $\mathbf{n}$ vertices, $\mathbf{f}$ faces and $\mathbf{e}$ edges, $\mathbf{n - e + f = 2}$).

20. Given a graph $\mathbf{G(V, E)}$, an edge coloring of the edges of $\mathbf{G}$ is a mapping $\mathbf{f : E \to \{1, 2, \ldots, k\}}$. A subgraph $\mathbf{H(V', E')}$ of $\mathbf{G}$, $\mathbf{V' \subseteq V}$, $\mathbf{E' \subseteq E}$, is called monochromatic under $\mathbf{f}$ if every edge in $\mathbf{E}$ has the same color. Show that K_5, the complete graph on 5 vertices, has a coloring of edges using two colors such that there is no monochromatic K_3.

21. Answer these questions for the poset $(\{\{\mathbf{1}\}, \{\mathbf{2}\}, \{\mathbf{4}\}, \{\mathbf{1, 2}\}, \{\mathbf{1, 4}\}, \{\mathbf{2, 4}\}, \{\mathbf{3, 4}\}, \{\mathbf{1, 3, 4}\}, \{\mathbf{2, 3, 4}\}\}, \subseteq)$.
(i) Find the maximal elements.
(ii) Is there a greatest element?
(iii) Find all upper bounds of $\{\{\mathbf{2}\}, \{\mathbf{4}\}\}$.
(iv) Find the least upper bound of $\{\{\mathbf{2}\}, \{\mathbf{4}\}\}$, if it exists.
(v) Find all lower bounds of $\{\{\mathbf{1, 3, 4}\}, \{\mathbf{2, 3, 4}\}\}$.
(vi) Find the greatest lower bound of $\{\{\mathbf{1, 3, 4}\}, \{\mathbf{2, 3, 4}\}\}$, if it exists.

22. Define a semigroup, monoid, group, ring and field.

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