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MA 2302: Introduction to Probability and Statistics

Some Discrete Probability Distributions

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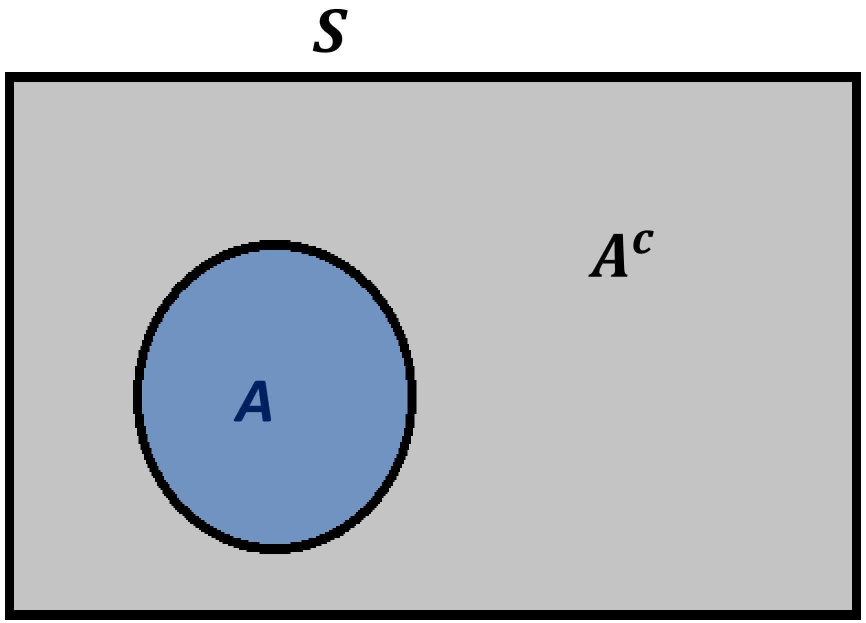
Discrete distributions

1. The Binomial Distribution

Consider a random experiment which results in a sample space S. Let $p = P(A)$, $0 < p < 1$ and let $q = P(A^c)$. Call the occurrence of the event A a success and its nonoccurrence (that is the occurrence of A^c a failure. Repeat the experiment n times independently. Let X be the number of times success occurred.



If the experiment consisting of drawing 10 cards from a pack of 52 cards (wor), A is the events that 4 cards are red, then $p = \frac{\binom{26}{4}\binom{26}{6}}{\binom{52}{10}}$ and if this experiment is repeated n times and X is the number of times 4 red cards are drawn. Then the distribution of X is binomial.

In tossing a biased coin n times let X be the number of times head is obtained. In this case also the distribution of X is binomial.

Discrete distributions $(\boldsymbol{S}, \boldsymbol{F}, \boldsymbol{S}, \boldsymbol{S}, \boldsymbol{F}, \boldsymbol{F}, \boldsymbol{S}, \boldsymbol{F}, \cdots, \boldsymbol{S})$

1. The Binomial Distribution

If in a trial (random experiment) only two thing can happen i.e. a success or a failure (in an exam pass or fail) and if such a trial is repeated independently, then such trials are known as Bernoulli trials. Observe that, in all the Bernoulli trial, the probability of success p remain fixed. If X counts the number of successes in n Bernoulli trials. Observe that if the sample space for one trial $\{S, F\}$, then the sample space for the n trials is $\{S, F\}^n$ an n-tuple whose each component is either an S or an F. There are $\binom{n}{k}$ points of the sample space consisting of k successes and hence $n - k$ failures and probability of each such sample point is $p^k q^{n-k}, k = 0,1,2, \ldots, n$. Hence

$$\Pr\{X = k\} = \binom{n}{k} p^k q^{n-k}, k = 0,1,2, \ldots, n.$$

Thus, the pmf of a binomial distribution is

$$f(x) = \binom{n}{x} p^x q^{n-x}, x = 0,1,2, \ldots, n.$$

Since the probabilities $f(0), f(1), \ldots, f(n)$ are successive terms in the binomial expansion of $(q + p)^n$, the name binomial distribution is justified.

Discrete distributions

Example 1: A bag contains 20 LED bulbs of equal size and shape out of which 8 are 17 W each and 12 are 23W each. Five bulbs are chosen (with replacement). Find the probability that out of these 5 bulbs (a) there is at least two 17 W bulbs, (b) more 23W bulbs than 17W bulbs.

Ans. Let X be the number 17W LED bulbs in the sample. Because of WR policy, the trials are independent. Here $n = 5,\ p = P(17W) = \frac{8}{20} = \frac{2}{5}$, hence $q = \frac{3}{5}$. Now

$$f(x) = \Pr\{X = x\} = \binom{n}{x} p^x q^{n-x} = \binom{5}{x} \frac{2^x \times 3^{5-x}}{5^5}, x = 0,1, \dots, 5.$$

(a) Pr{at least two 17 W bulbs in the sample} $= \Pr\{X \geq 2\} = 1 - \Pr\{X \leq 1\} = 1 - f(0) - f(1) = 1 - \frac{3^5+5\times2\times3^4}{5^5} = 1 - 0.38696 = 0.61304.$

(b) Pr{more 23W bulbs than 17W bulbs} $= \Pr\{X \leq 2\} = f(0) + f(1) + f(2)$

$$= \frac{3^5+5\times2\times3^4+10\times2^2\times3^3}{5^5} = 0.68256.$$

Discrete distributions

Example 2: In a precision bombing attack, there is a 50% chance that any one bomb will strike the target. Two direct hits are required to destroy the target completely. How many bombs must be dropped so as to achieve a 99% or better chance of completely destroying the target?

Ans. Let n be the number of bombs to be dropped and X, the number of hits. Given that $p = \Pr\{\text{single bomb hitting the target}\} = \frac{1}{2}$. Then $X \sim B(n, 1/2)$. Hence,

$$f(x) = \Pr\{\text{Prob. of } x \text{ hits out of } n \text{ bombs to be dropped}\}$$

$$= \binom{n}{x}\left(\frac{1}{2}\right)^{x}\left(1-\frac{1}{2}\right)^{n-x} = \frac{\binom{n}{x}}{2^n}, x = 0,1,2, \dots, n.$$

We need to find the minimum value of n such that

$$\Pr\{X \geq 2\} \geq 0.99 \Rightarrow \Pr\{X \leq 1\} = f(0) + f(1) \leq 0.01$$

$$\Rightarrow \frac{n+1}{2^n} \leq 0.01 \Rightarrow \frac{2^n}{n+1} \geq 100 \Rightarrow n \geq 11.$$

Hence 11 bombs must be dropped.

(**Note**: The notation $X \sim B(n, p)$ means that the distribution of the random variable is binomial with number of trials n and probability of success in each trial p)

Discrete distributions

Mean and variance: If $X \sim B(n, p)$, then

$$\mu = \sum_{x} x\, f(x) = \sum_{x=0}^{n} x\, f(x) = \sum_{x=1}^{n} x \binom{n}{x} p^x q^{n-x}$$

$$= np \sum_{x=1}^{n} \binom{n-1}{x-1} p^{x-1} q^{n-x} = np \sum_{y=0}^{n-1} \binom{n-1}{y} p^y q^{(n-1)-y}$$

Take y=x-1

$$= np(q+p)^{n-1} = np.$$

$$E(X^2) = \sum_{x} x^2 f(x) = \sum_{x=0}^{n} \{x(x-1) + x\} f(x)$$

$$= \sum_{x=0}^{n} x(x-1)\, f(x) + \sum_{x=0}^{n} x\, f(x) = \sum_{x=2}^{n} x(x-1)\, f(x) + np$$

y=x-2

$$= \sum_{x=2}^{n} x(x-1) \binom{n}{x} p^x q^{n-x} + np = n(n-1)p^2 \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{y=0}^{n-2} \binom{n-2}{y} p^y q^{(n-2)-y} + np$$

$$= n(n-1)p^2(q+p)^{n-2} + np = n(n-1)p^2 + np$$

Now,

$$\sigma^2 = E(X^2) - \mu^2 = n(n-1)p^2 + np - (np)^2$$

$$= np - np^2 = np(1-p) = npq$$

Assignment: Using similar method, and the formula $\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2{\mu_1'}^3$, prove that $\mu_3 = npq(q-p)$.

Moment generating function

$$G(t) = E(e^{tX}) = \sum_x e^{tx} f(x) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x} = (q+pe^t)^n.$$

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Thus, the moment generating function is $G(t) = (q + pe^t)^n$. Observe that $G(0) = 1$,

$$G'(t) = n(q + pe^t)^{n-1} \times pe^t = npe^t(q + pe^t)^{n-1}$$

and hence, $\mu_1' = E(X) = G'(0) = np(q + p)^{n-1} = np$. Now,

$$G''(t) = npe^t \times (n-1)(q + pe^t)^{n-2} \times pe^t + npe^t(q + pe^t)^{n-1}$$

Hence,

$$\mu_2' = E(X^2) = G''(0) = n(n-1)p^2 + np$$

and now the variance can be calculated as usual.

Discrete distributions

The hypergeometric distribution

Consider the following example we have discussed earlier with minor modification:

A bag contains N LED bulbs of equal size and shape out of which M are 17 W each and rest are 23W each. n bulbs are chosen at random. What is the probability that x of n bulbs chosen are 17 W?

To address this problem, let X denote the random variable which counts the number of 17 W LED bulbs in the sample. We cannot conclude any thing about the distribution of X unless we know how the bulbs are chosen, with replacement (WR) or WOR. If bulbs are drawn WR, then we can use the idea of Example 1 and answer as follows:

$$f(x) = \Pr\{X = x\} = \binom{n}{x} p^x q^{n-x}, x = 0,1,2,\ldots,n$$

where $p = \frac{M}{N}$ and $q = 1 - p = \frac{N-M}{N}$. Here, clearly, the distribution of X is binomial with parameters n and p.

Now, what will happen if the bulbs are drawn WOR???

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If the bulbs are chosen WOR, then by basic combinatorial principle, the number of ways of choosing n bulbs such that x are 17 W and rest are 23 W is given by $\binom{M}{x}\binom{N-M}{n-x}$ and the total number of choosing n bulbs out of N is given by $\binom{N}{n}$. Hence, by the basic definition of probability

$$f(x) = \Pr\{X = x\} = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}, x = 0,1,2, \dots, n.$$

However, our assumption here is that $n \leq \min\{M, N - M\}$ so that one can choose all 17 W or all 23 W LED bulbs.

The total probability must be 1. Hence,

$$\sum_{x=0}^{n} \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} = 1.$$

Or, equivalently

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Or, equivalently

$$\sum_{x=0}^{n}\binom{M}{x}\binom{N-M}{n-x}=\binom{N}{n}.$$

Why? Try equating coefficients of t^n from both sides of

$$(1+t)^M\cdot(1+t)^{N-M}=(1+t)^N.$$

The distribution of the total probability according to the rule of previous slide is known as a hypergeometric distribution. We answer the question of Ex 1 subject to WOR.

Example 3. A bag contains 20 LED bulbs of equal size and shape out of which 8 are 17 W each and 12 are 23W each. Five bulbs are chosen (~~with~~ **without replacement**). Find the probability that out of these 5 bulbs (a) there is at least two 17 W bulbs, (b) more 23W bulbs than 17W bulbs.

Discrete distributions

Ans. Here $N = 20, M = 8, n = 5$ and as usual X is the number of 17 W LED bulbs in the sample. Thus,

$$f(x) = \Pr\{X = x\} = \frac{\binom{8}{x}\binom{12}{5-x}}{\binom{20}{5}}, x = 0, 1, 2, 3, 4, 5.$$

(a) $\Pr\{\text{at least two 17 W bulbs in the sample}\} = \Pr\{X \geq 2\} = 1 - \Pr\{X \leq 1\}$

$$= 1 - f(0) - f(1) = 1 - \frac{\binom{12}{5}}{\binom{20}{5}} - \frac{8 \times \binom{12}{4}}{\binom{20}{5}} = 0.6935$$

(b) $\Pr\{\text{more 23W bulbs than 17W bulbs}\} = \Pr\{X \leq 2\} = f(0) + f(1) + f(2)$

$$= \frac{\binom{12}{5}}{\binom{20}{5}} + \frac{8\times\binom{12}{4}}{\binom{20}{5}} + \frac{\binom{8}{2}\binom{12}{3}}{\binom{20}{5}} = 0.7038$$

Discrete distributions

Mean and variance of hypergeometric distribution:

$$\mu = \sum_x x\, f(x) = \sum_{x=0}^{n} x\, f(x) = \sum_{x=1}^{n} x \cdot \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

$$= \frac{M}{\binom{N}{n}} \sum_{x=1}^{n} \binom{M-1}{x-1}\binom{N-M}{n-x} = \frac{M}{\binom{N}{n}} \sum_{y=0}^{n-1} \binom{M-1}{y}\binom{N-M}{(n-1)-y}$$

$$= \frac{M}{\binom{N}{n}} \cdot \binom{N-1}{n-1} = \frac{nM}{N}\ .$$

$$E(X^2) = \sum_x x^2 f(x) = \sum_{x=0}^{n} \{x(x-1) + x\} f(x)$$

$$= \sum_{x=0}^{n} x(x-1)\, f(x) + \sum_{x=0}^{n} x\, f(x) = \sum_{x=2}^{n} x(x-1)\, f(x) + \frac{nM}{N}$$

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$$= \sum_{x=2}^{n} x(x-1) \cdot \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} + \frac{nM}{N}$$

$$= \frac{M(M-1)}{\binom{N}{n}} \sum_{x=2}^{n} \binom{M-2}{x-2}\binom{N-M}{n-x} + \frac{nM}{N}$$

$$= \frac{M(M-1)}{\binom{N}{n}} \sum_{y=0}^{n-2} \binom{M-2}{y}\binom{N-M}{(n-2)-y} + \frac{nM}{N}$$

$$= \frac{M(M-1)}{\binom{N}{n}} \cdot \binom{N-2}{n-2} + \frac{nM}{N}$$

Now

$$\sigma^2 = E(X^2) - \mu^2 = \frac{NM(N-M)(N-n)}{N^2(N-1)} .$$

Discrete distributions

In a hypergeometric distribution with parameter N, M and n, assume that both M and $N-M$ are very large such that $\frac{M}{N} \to p$ and hence $\frac{N-M}{N} \to 1-p=q$. Then

$$f(x) = \Pr\{X = x\} = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}}$$

$$= \frac{n!}{x!\,(n-x)!} \times \frac{M(M-1)\cdots(M-x+1) \times (N-M)(N-M-1)\cdots(N-M-n+x+1)}{N(N-1)(N-2)\cdots(N-n+1)}$$

$$= \binom{n}{x} \times \frac{M(M-1)\cdots(M-x+1)}{N(N-1)\cdots(N-x+1)} \times \frac{(N-M)(N-M-1)\cdots(N-M-n+x-1)}{(N-x)(N-x-1)\cdots\cancel{(N-n)}} \text{ (N-n+1)}$$

$$= \binom{n}{x} \cdot \frac{M}{N} \cdot \frac{M-1}{N-1} \cdot \cdots \cdot \frac{M-x+1}{N-x+1} \times \frac{N-M}{N-x} \cdot \frac{N-M-1}{N-x-1} \cdot \cdots \cdot \frac{N-M-n+x+1}{N-n+1}$$

$\to \binom{n}{x} p^x q^{n-x}$, (limiting distribution of hypergeometric is binomial)

which is nothing but the pmf of binomial distribution. Hence, when sampling from a large population, is does not matter whether you sample with or without replacement.

Discrete distributions

Geometric distribution

Once again consider a sequence of Bernoulli trials with probability of success p in each trial. Let X be the number of trials till the first success is achieved. Consider the following example:

Example 4: Let p be the probability of Amar's passing a test of multiple choice type questions in one attempt. If he only guesses with no knowledge of correct answers, what is the probability that he will pass the test in k-th attempt?

Ans. Let X be the number of attempts Amar needs for passing the test. He passes for the first time in the k-th test means that he fails in the previous $k-1$ tests. Hence,

$$\Pr\{X = k\} = P(FFFF \dots \dots FFS) = q^{k-1}p, k = 1,2,3, \dots$$

Thus, in sequence of Bernoulli trials, the probability of first success in the x-th trial is equal to

$$f(x) = q^{x-1}p, x = 1,2,3, \dots$$

Notice that $\sum_{x=1}^{\infty} f(x) = p(1 + q + q^2 + \cdots) = p(1-q)^{-1} = 1$ and the distribution of the total probability 1 according to the above rule is known as a geometric distribution. Successive terms of this distribution forms a geometric progression, justifying the name.

Discrete distributions

The mean is given by

$$\mu = E(X) = \sum_{x} x\, f(x) = \sum_{x=1}^{\infty} x\, q^{x-1} p = p \sum_{x=1}^{\infty} x\, q^{x-1} = p \sum_{x=1}^{\infty} \frac{\partial}{\partial q}(q^{x})$$

$$= p \frac{\partial}{\partial q}\left[\sum_{x=1}^{\infty} q^{x}\right] = p \frac{\partial}{\partial q}\left[\frac{1}{1-q} - 1\right] = \frac{p}{(1-q)^{2}} = \frac{1}{p}.$$

$$E(X^{2}) = \sum_{x} x^{2} f(x) = \sum_{x=1}^{\infty} \{x(x-1)+x\} f(x) = \sum_{x=1}^{\infty} x(x-1)\, q^{x-1} p + \sum_{x=1}^{\infty} x\, f(x)$$

$$= pq \sum_{x=1}^{\infty} x(x-1)\, q^{x-2} + \frac{1}{p} = pq \sum_{x=1}^{\infty} \frac{\partial^{2}}{\partial q^{2}}(q^{x}) + \frac{1}{p} = pq \frac{\partial^{2}}{\partial q^{2}}\left[\sum_{x=1}^{\infty} q^{x}\right] + \frac{1}{p}$$

$$= pq \frac{\partial^{2}}{\partial q^{2}}\left[\frac{1}{1-q} - 1\right] + \frac{1}{p} = \frac{2pq}{(1-q)^{3}} + \frac{1}{p} = \frac{2q}{p^{2}} + \frac{1}{p}\ .$$

Hence, the variance is given by

$$\sigma^{2} = E(X^{2}) - \mu^{2} = \frac{2q}{P^{2}} + \frac{1}{p} - \frac{1}{p^{2}} = \frac{q}{p^{2}}.$$

Discrete distributions

The Poisson distribution

In a binomial distribution, assume that the number of trials n is very large, the probability of success in each trial p is very small in such a way that the mean np is not large. Mathematically,

$$n \to \infty, p \to 0 \text{ s.t. } np \to \lambda.$$

Under this limiting condition,

$$\begin{aligned}
\binom{n}{x} p^x q^{n-x} &= \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!} \cdot p^x \cdot \frac{(1-p)^n}{q^x} \\
&= \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)n^x}{x!} \cdot p^x \cdot \frac{(1-p)^n}{q^x} \\
&= \frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)}{x!} \cdot (np)^x \cdot \frac{\left(1-\frac{np}{n}\right)^n}{q^x} \to \frac{e^{-\lambda}\lambda^x}{x!}
\end{aligned}$$

since, under the above limiting condition, $\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right) \to 1, (np)^x \to \lambda^x$, $q^x \to 1$, $\left(1-\frac{np}{n}\right)^n \to e^{-\lambda}$. (if $x_n \to x$, then $\left(1+\frac{x_n}{n}\right)^n \to e^x$).

Discrete distributions

The pmf of the Poisson distribution is therefore

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \qquad x = 0,1,2,\ldots$$

Observe that $\sum_{x=0}^{\infty} f(x) = e^{-\lambda}\sum_{x=0}^{\infty}\frac{\lambda^x}{x!} = e^{-\lambda}\cdot e^{\lambda} = 1$. Since the Poisson distribution is obtained as a limiting case of binomial distribution under the assumption that $n \to \infty$ and $p \to 0$, it is clear that we can approximate the binomial distribution to Poisson distribution when the event what is called success is a rare one.

Example 5: Suppose that 1.5% of steel rods made by a machine are defective, the defectives occurring at random during production. If the rods are packaged 200 per box with the guarantee that not more than 5 rods per box are defective, what percentage of boxes will fail to meet the guarantee? What is the probability that a box will contain at least one defective rod?

Ans. Let X be the number of defective rods in a box. Then, as per question $X \sim B(200, 0.015)$. Since $n = 200$ is large, $p = 0.015$ is small and $np = 3$ is not large, the distribution of X can be approximated by Poisson with $\lambda = 3$.

Discrete distributions

Hence, the pmf of X is

$$f(x) = \Pr\{X = x\} = \Pr\{x \text{ defective rods in a box}\} = \frac{e^{-3}3^x}{x!}, \ \ x = 0,1,2,\ldots$$

and

$\Pr\{\text{a box will not meet the guarantee}\}$

$$= \Pr\{X > 5\} = 1 - \Pr\{X \le 5\} = 1 - [f(0) + f(1) + f(2) + f(3) + f(4) + f(5)]$$

$$= 1 - e^{-3}\left(1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \frac{3^5}{5!}\right) = 1 - 0.8041 = 0.1959.$$

Thus, about 19.6% boxes will fail to meet the guarantee.

$\Pr\{\text{a box will contain at least one defective rod}\}$

$$= \Pr\{X \ge 1\} = 1 - \Pr\{X = 0\} = 1 - f(0) = 1 - e^{-3} = 0.9502.$$

Discrete distributions

Moments of Poisson distribution

$$\mu = E(X) = \sum_{x} x\, f(x) = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^{x}}{x!} = e^{-\lambda} \sum_{x=1}^{\infty} x \cdot \frac{\lambda^{x}}{x!}$$

$$= \lambda e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} = \lambda e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^{y}}{y!} = \lambda e^{-\lambda} e^{\lambda} = \lambda.$$

$$E(X^2) = \sum_{x} x^2\, f(x) = \sum_{x=0}^{\infty} \{x(x-1) + x\}\, f(x)$$

$$= \sum_{x=2}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^{x}}{x!} + \sum_{x=0}^{\infty} x\, f(x)$$

$$= \lambda^2 e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda = \lambda^2 e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^{y}}{y!} + \lambda$$

$$= \lambda^2 e^{-\lambda} e^{\lambda} + \lambda = \lambda^2 + \lambda.$$

Discrete distributions

Hence, the variance is given by

$$\sigma^2 = E(X^2) - \mu^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

Hence, in case of Poisson distribution, the mean and variance are same.

Exercise for you: Prove that $\mu_3 = \lambda$ and $\mu_4 = \lambda + 3\lambda^2$.

Hence,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{1}{\lambda}, \beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1}{\lambda},$$

$$\gamma_1 = \sqrt{\beta_1} = \frac{1}{\sqrt{\lambda}}, \gamma_2 = \beta_2 - 3 = \frac{1}{\lambda}.$$

Observe that $\beta_1 \to 0$ and $\beta_2 \to 3$ as $\lambda \to \infty$, and hence the Poisson distribution approach to normal distribution as $\lambda \to \infty$.

Discrete distributions

Moment generating function of Poisson distribution

$$G(t) = E(e^{tX}) = \sum_x e^{tx} f(x) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\left(\lambda e^t\right)^x}{x!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}.$$

Hence,

$$\log G(t) = \lambda(e^t - 1) \Rightarrow \frac{G'(t)}{G(t)} = \lambda e^t \Rightarrow G'(t) = \lambda e^t G(t).$$

$$\Rightarrow \mu = G'(0) = \lambda G(0) = \lambda.$$

Furthermore,

$$G'(t) = \lambda e^t G(t) \Rightarrow G''(t) = \lambda e^t G(t) + \lambda e^t G'(t)$$

Thus,

$$E(X^2) = G''\ (0) = \lambda\{G(0) + G'(0)\} = \lambda(1 + \lambda) = \lambda^2 + \lambda$$

and so on. You can calculate $E(X^3)$ and $E(X^4)$ in a similar manner and then μ_3 and μ_4.

Discrete distributions

Example 6: Suppose that a telephone switchboard of some company on the average handles 300 calls per hour, and that the board can make at most 10 connections per minute. Using the Poisson distribution, estimate the probability that the board will be overtaxed during a given minute.

Ans. Let X be the number of calls arriving during a given minute. Observe that mean of X is 5 and its distribution is Poisson. Hence,

$$f(x) = \Pr\{x \text{ calls duting a minute}\} = \frac{e^{-5}5^{x}}{x!}, x = 0, 1, 2, 3, \dots.$$

$$\Pr\{\text{The board will be overtaxed}\} = \Pr\{X > 10\} = 1 - \Pr\{X \leq 10\}$$

$$= 1 - \sum_{x=0}^{10} f(x) = 1 - e^{-5}\left(1 + 5 + \frac{5^2}{2!} + \frac{5^3}{3!} + \cdots + \frac{5^{10}}{10!}\right) \text{ (you simplify)}$$

Example 7: In 1910, E. Rutherford and H. Geiger showed experimentally that the number of alpha particles emitted per second in a radioactive process is a random variable X having a Poisson distribution. If X has mean 0.5, what is the probability of observing two or more particles during any given second? *You solve*.

Discrete distributions

Example 8: Suppose that a certain type of magnetic tape contains, on the average, 2 defects per 100 meters. What is the probability that a roll of tape 300 meters long will contain (a) no defects, (b) not more than 3 defects? Use Poisson distribution. *Discussion is class*

Hints: Let X be the number of defects in a 300 m long tap. Then $X \sim P(6)$. Find $\Pr\{X = 0\}$, $\Pr\{X \leq 3\}$

Example 9: If a ticket office can serve at most 4 customers per minute and the average number of customers is 120 per hour, what is the probability that during a given minute customers will have to wait? (Use the Poisson distribution) *Discussion is class*

Hints: Let X be the number of customers arriving per minute. Then $X \sim P(2)$. Find $\Pr\{X > 4\}$

Example 10: Suppose that in the production of 60-ohm radio resistors, non-defective items are those that have a resistance between 58 and 62 ohms and the probability of a resistor's being defective is 0.02. The resistors are sold in lots of 200, with the guarantee that all resistors are non-defective. What is the probability that a given lot will violate this guarantee? (Use the Poisson distribution.) *Discussion is class*

Hints: Let X be the no. of defective resistors in a box. Then $X \sim B(200, 0.02) \rightarrow \text{Poisson}$ with mean 4.

Discrete distributions

Example 11: A process of manufacturing screws is checked every hour by inspecting n screws selected at random from that hour's production. If one or more screws are defective, the process is halted and carefully examined. How large should n be if the manufacturer wants the probability to be about 95% that the process will be halted when 10% of the screws being produced are defective? (Assume independence of the quality of any screw from that of the other screws.)

Ans. Out of n screws examined every hour, let X be the number of defective screws in the sample. The manufacturer wants to halt the process when at least one out of n screws are defective. When $p = 10\% = 0.1$, $X \sim B(n, 0.1)$.

We want that

$$\Pr\{X \geq 1\} = 0.95 \Rightarrow 1 - f(0) = 0.95 \Rightarrow f(0) = 0.05$$

$$\Rightarrow 0.9^n = 0.05 \Rightarrow n \approx 28.$$

Discrete distributions

Multinomial distribution

Consider a random experiment (say throwing a die) leading to a sample space S and $A_1, A_2, \dots, A_k$ be k mutually exclusive events defined on S such that $S = A_1 \cup A_2 \cup \cdots \cup A_k$, $P(A_i) = p_i > 0, i = 1,2, \dots, k$. Repeat this experiment n times independently. Let X_i be the number times A_i occurs, $i = 1,2, \dots, k$. Then $X_1 + X_2 + \cdots + X_k = n$. The total number of points in the revised sample space is equal to k^n. If an element of the sample space contains $x_1\ A_1s, x_2\ \ A_2s, \dots, x_k\ \ A_ks$, then probability of that point is $p_1^{x_1}p_2^{x_2}\cdots p_k^{x_k}$ and number of such points are

$$\binom{n}{x_1}\binom{n-x_1}{x_2}\binom{n-x_1-x_2}{x_3}\cdots\binom{n-x_1-x_2-\cdots x_{n-1}}{x_k} = \frac{n!}{x_1!\,x_2!\cdots x_k!}$$

where $x_1 + x_2 + \cdots + x_k = n$. Hence

$$\Pr\{X_1 = x_1, X_2 = x_2, \dots, X_k = x_k\} = f(x_1, x_2, \dots, x_k) = \frac{n!}{x_1!\,x_2!\cdots x_k!} \cdot p_1^{x_1}p_2^{x_2}\cdots p_k^{x_k}.$$

Discrete distributions

Example 12: What is the probability of drawing 4 white, 3 black and 2 red balls from a box consisting of 10 white, 8 black and 7 red balls if 9 balls are drawn from it with replacement?

Ans. Let X_1, X_2 and X_3 be the number of white, black and red balls drawn. Observe that

$n = 9, p_1 = \frac{2}{5}, p_2 = \frac{8}{25}, p_3 = \frac{7}{25}$.

$$\Pr\{X_1 = 4, X_2 = 3, X_3 = 2\} = \frac{9!}{4!\,3!\,2!} \cdot \left(\frac{2}{5}\right)^4 \left(\frac{8}{25}\right)^3 \left(\frac{7}{25}\right)^2 .$$

You simplify and answer in four place correct decimals.