Report: Programming Project Part 2

How to get the perfect cake!

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Outline

In this project, we modelled the baking process of a cake in the oven. Starting with the heat equation (section 1), we will derive a time wise differential equation with the help of finite elements in section 2. ...

1 Problem

The theoretical problem we want to solve is the heat equation, which is given by:

$$\frac{\partial u}{\partial t} = \nabla(\alpha \nabla u) \tag{1}$$

$$u(\vec{x}, t)|_{\partial\Omega^D} = u^D \tag{2}$$

$$\left(\frac{\partial u(\vec{x}, t)}{\partial \nu}\right)\bigg|_{\partial\Omega^N} = g \tag{3}$$

$$u(\vec{x}, t)|_{t=0} = u_0(\vec{x}) \tag{4}$$

Where:

- $u(\vec{x}, t)$ is the heat function.
- α is a constant denoting the thermal diffusivitie.
- Ω is the domain of the problem.
- $\partial\Omega^D$ is the part of the boundary with Dirichlet boundary-conditions.
- $\partial\Omega^N$ is the part of the boundary with Neumann boundary-conditions.
- ν is the normal vector on the Neumann-boundary.

2 Setting up the System

In this section we find a relaxation of the problem stated in section 1. To do so we first find a weak formulation of it. Multiplying both sides of (1) with an arbitrary test function $v \in V := H^1(\Omega)$ and integrating over the domain, we get:

$$\int_\Omega \frac{\partial u}{\partial t} v = \int_\Omega (\nabla(\alpha \nabla u)) v$$

Integrating the right hand side by parts will result in:

$$\int_\Omega \frac{\partial u}{\partial t} v = -\int_\Omega \alpha (\nabla u) \cdot (\nabla v) + \int_{\partial\Omega} \alpha v (\nabla u) \cdot \nu$$

Using

$$(\nabla u) \cdot \nu = \frac{\partial u}{\partial \nu} = g$$

leads to:

$$\int_\Omega \frac{\partial u}{\partial t} v = -\int_\Omega \alpha (\nabla u) \cdot (\nabla v) + \int_{\partial\Omega^N} \alpha g v + \int_{\partial\Omega^D} \alpha v (\nabla u) \cdot \nu \tag{5}$$

To solve the weak formulation (5) numerically we discretise our domain Ω. Our notation will follow [QQ09]. Also we will not state every step in detail, if you wish, to get deeper insights in the theory behind it, we also recommend reading [QQ09].

Let $\mathcal{T}_h$ be a set of non overlapping tetrahedrons covering Ω with and $\mathcal{N} = N_1, \dots N_{n_h}$ the nodes of this mesh. As the theory about this is not new, and not closely related, to our problem, we don't want to go into detail about this. The approximated domain is then $\Omega_h := \bigcup_{K \in \mathcal{T}_h} K$.

As an approximation of the functions in V, we now search for functions in $X_h := \{v_h \in C^0(\overline{\Omega}_h) : v_h|_K \text{ linear } \forall K \in \mathcal{T}_h\}$, which are the continuous functions on Ω_h, that are piecewise linear on each tetrahedron.

A basis for this space is given by the characteristic Lagrangian functions $\phi_j \in X_h, j = 1, \dots n_h$, with $\phi_j(N_i) = \delta_{ij}$. So we can write every $v_h \in X_h$ in the following way:

$$v_h(x) = \sum_{j=1}^{j=n_h} v_j \phi_j.$$

On this space equation (5) is equivalent to the following, as v:

$$\begin{aligned} \int_\Omega \sum_i \frac{\partial u_i}{\partial t} \phi_i \phi_j = &-\int_\Omega \sum_i \alpha u_i (\nabla \phi_i) \cdot (\nabla \phi_j) \\ &+ \int_{\partial\Omega^N} \alpha g \phi_j \\ &+ \int_{\partial\Omega^D} \sum_i \alpha u_i (\nabla \phi_i) \cdot \nu \phi_j \quad \forall j. \end{aligned} \tag{6}$$

Having homogeneous Dirichlet conditions (i.e. $u^D = 0$), we only have to search on the subspace $V_h := \mathring{X}_h := \{v_h \in X_h : v_h|_{\partial\Omega_h^D} = 0\}$. Let wlog be the last indices $n_D, \dots n_h$, the indices of the nodes on the Dirichlet boundary. As $v_h \in V_h$ leads to $v_j = 0, \forall j \geq n_D$, (6) becomes:

$$\begin{aligned} \int_\Omega \sum_i \frac{\partial u_i}{\partial t} \phi_i \phi_j = &- \int_\Omega \sum_i \alpha u_i (\nabla \phi_i) \cdot (\nabla \phi_j) \\ &+ \int_{\partial \Omega^N} \alpha g \phi_j \qquad\qquad \forall j < n_D. \end{aligned} \tag{7}$$

We can reduce the non-homogeneous case to the homogeneous one, by introducing a lifting $R_g \in X_h$ as follows:

$$R_g(x) := \sum_{i=n_D}^{n_h} d_i \phi_i(x),$$

where $d_i := u^D(N_i)$. With the homogeneous solution

$$\mathring{u} := \sum_{i=1}^{n_D - 1} u_i \phi_i(x),$$

the final solution is given by

$$u = \mathring{u} + R_g. \tag{8}$$

To find this homogeneous solution, we insert (8) in (7) and get:

$$\begin{aligned} \int_\Omega \sum_{i=1}^{n_D-1} \frac{\partial u_i}{\partial t} \phi_i \phi_j = &- \int_\Omega \sum_{i=1}^{n_D-1} \alpha u_i (\nabla \phi_i) \cdot (\nabla \phi_j) \\ &+ \int_{\partial \Omega^N} \alpha g \phi_j \\ &- \int_\Omega \sum_{i=n_D}^{n_h} \left(\dot{d}_i \phi_i \phi_j + \alpha d_i (\nabla \phi_i) \cdot (\nabla \phi_j) \right) \quad \forall j, \end{aligned} \tag{9}$$

with $\dot{d}_i := \frac{\partial u^D}{\partial t}(N_i)$.

We now define the matrices M, A and the vectors N, D by:

$$\begin{aligned} M_{ij} &:= \int_\Omega \phi_i \phi_j \\ A_{ij} &:= \int_\Omega \alpha (\nabla \phi_i) \cdot (\nabla \phi_j) \\ N_j &:= \int_{\partial \Omega^N} \alpha g \phi_j \\ D_j &:= \int_\Omega \sum_{k=n_D}^{n_h} \left(\dot{d}_k \phi_k \phi_j + \alpha d_k (\nabla \phi_k) \cdot (\nabla \phi_j) \right), \end{aligned}$$

with $i, j = 1, \dots (n_D - 1)$. Using these we get the the differential equation:

$$M \frac{\partial u}{\partial t} = -Au + N - D. \tag{10}$$

We will later see how to solve this.

3 Code

Everything we derived in the previous sections has been implemented in matlab.The program is able to simulate the heat equation, for any given tetrahization. It can handle Neumann and Dirichlet conditions, which can depend on place and time. Also the thermal diffusivity α can be given for every single element separately.

4 Modelling the cake

To model the baking process, we now only need a few more ingredients. First we have to know the thermal properties of cake dough and the baking form, which we will consider to be out of aluminium. For the dough [BSMC99] found out, that the thermal diffusivity of cupcake dough, which should be close enough to our cake dough is $\alpha_{cake} = 1.02 \times 10^{-7}\text{m/s}^2 - 1.698 \times 10^{-7}\text{m/s}^2$. For the thermal diffusivity of aluminium the appendix of [Kot06] gives us $\alpha_{Al} = 9.444 \times 10^{-5}\text{m/s}^2$.

Second we need the mesh, which models. This was given by the task and can be seen in figure 1, where the yellow part identifies an aluminium stick, which will help to heat up the cake from the inside. For the radius we chose 15cm, as we wanted to have a big cake, which had a realistic chance, to get baked in a reasonable time.

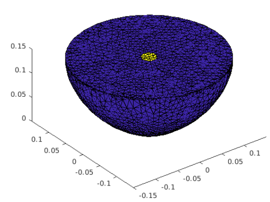

Figure 1: Mesh of the cake

5 Results

References

[BSMC99] OD Baik, SS Sablani, M Marcotte, and F Castaigne. Modeling the thermal properties of a cup cake during baking. *Journal of Food Science*, 64(2):295–299, 1999.

[Kot06] CP Kothandaraman. *Fundamentals of heat and mass transfer*. New Age International, revised third edition, 2006.

[QQ09] Alfio Quarteroni and Silvia Quarteroni. *Numerical models for differential problems*, volume 2. Springer, 2009.