Report: Programming Project Part 2

How to get the perfect cake!

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November 19, 2017

1 Problem

The theoretical problem we want to solve is the heat equation, which is given by:

$$\frac{\partial u}{\partial t} = \nabla(\alpha \nabla u) \tag{1}$$

$$u(\vec{x}, t)|_{\partial\Omega^D} = u^D \tag{2}$$

$$\left(\frac{\partial u(\vec{x}, t)}{\partial \nu}\right)\bigg|_{\partial\Omega^N} = g \tag{3}$$

$$u(\vec{x}, t)|_{t=0} = u_0(\vec{x}) \tag{4}$$

Where:

- $u(\vec{x}, t)$ is the heat function.
- α is a constant denoting the thermal diffusivitie.
- Ω is the domain of the problem.
- $\partial\Omega^D$ is the part of the boundary with Dirichlet boundary-conditions.
- $\partial\Omega^N$ is the part of the boundary with Neumann boundary-conditions.
- ν is the normal vector on the Neumann-boundary.

2 Setting up the System

In this section we find a relaxation of the problem stated in section 1. To do so we first find a weak formulation of it. Multiplying both sides of (1) with an arbitrary test function $v \in V := H^1(\Omega)$ and integrating over the domain, we get:

$$\int_\Omega \frac{\partial u}{\partial t} v = \int_\Omega (\nabla(\alpha \nabla u))v$$

Integrating the right hand side by parts will result in:

$$\int_\Omega \frac{\partial u}{\partial t} v = -\int_\Omega \alpha(\nabla u) \cdot (\nabla v) + \int_{\partial\Omega} \alpha v (\nabla u) \cdot \nu$$

Using

$$(\nabla u) \cdot \nu = \frac{\partial u}{\partial \nu} = g$$

leads to:

$$\int_\Omega \frac{\partial u}{\partial t} v = -\int_\Omega \alpha (\nabla u) \cdot (\nabla v) + \int_{\partial\Omega^N} \alpha g v + \int_{\partial\Omega^D} \alpha v (\nabla u) \cdot \nu \tag{5}$$

To solve the weak formulation (5) numerically we discretise our domain Ω. Our notation will follow [QQ09]. Also we will not state every step in detail, if you wish, to get deeper insights in the theory behind it, we also recommend reading [QQ09].

Let $\mathcal{T}_h$ be a set of non overlapping tetrahedrons covering Ω with and $\mathcal{N} = N_1, \ldots N_{n_h}$ the nodes of this mesh. As the theory about this is not new, and not closely related, to our problem, we don't want to go into detail about this. The approximated domain is then $\Omega_h := \bigcup_{K \in \mathcal{T}_h} K$.

As an approximation of the functions in V, we now search for functions in $X_h := \{v_h \in C^0(\overline{\Omega}_h) : v_h|_K \text{ linear } \forall K \in \mathcal{T}_h\}$, which are the continuous functions on Ω_h, that are piecewise linear on each tetrahedron.

A basis for this space is given by the characteristic Lagrangian functions $\phi_j \in X_h, j = 1, \ldots n_h$, with $\phi_j(N_i) = \delta_{ij}$. So we can write every $v_h \in X_h$ in the following way:

$$v_h(x) = \sum_{j=1}^{j=n_h} v_j \phi_j.$$

On this space equation (5) is equivalent to the following, as v:

$$\begin{aligned}\int_\Omega \sum_i \frac{\partial u_i}{\partial t} \phi_i \phi_j = &-\int_\Omega \sum_i \alpha u_i (\nabla \phi_i) \cdot (\nabla \phi_j) \\ &+ \int_{\partial\Omega^N} \alpha g \phi_j \\ &+ \int_{\partial\Omega^D} \sum_i \alpha u_i (\nabla \phi_i) \cdot \nu \phi_j \quad \forall j.\end{aligned} \tag{6}$$

Having homogeneous Dirichlet conditions (i.e. $u^D = 0$), we only have to search on the subspace $V_h := \mathring{X}_h := \{v_h \in X_h : v_h|_{\partial\Omega_h^D} = 0\}$. Let wlog be the last indices $n_D, \ldots n_h$, the indices of the nodes on the Dirichlet boundary. As $v_h \in V_h$ leads to $v_j = 0, \forall j \geq n_D$, (6) becomes:

$$\begin{aligned}\int_\Omega \sum_i \frac{\partial u_i}{\partial t} \phi_i \phi_j = &-\int_\Omega \sum_i \alpha u_i (\nabla \phi_i) \cdot (\nabla \phi_j) \\ &+ \int_{\partial\Omega^N} \alpha g \phi_j \qquad\qquad \forall j < n_D.\end{aligned} \tag{7}$$

We can reduce the non-homogeneous case to the homogeneous one, by introducing a lifting $R_g \in X_h$ as follows:

$$R_g(x) := \sum_{i=n_D}^{n_h} d_i \phi_i(x),$$

where $d_i := u^D(N_i)$. With the homogeneous solution

$$\mathring{u} := \sum_{i=1}^{n_D-1} u_i \phi_i(x),$$

the final solution is given by

$$u = \mathring{u} + R_g. \tag{8}$$

To find this homogeneous solution, we insert (8) in (7) and get:

$$\begin{aligned} \int_\Omega \sum_{i=1}^{n_D-1} \frac{\partial u_i}{\partial t} \phi_i \phi_j = &- \int_\Omega \sum_{i=1}^{n_D-1} \alpha u_i (\nabla \phi_i) \cdot (\nabla \phi_j) \\ &+ \int_{\partial\Omega^N} \alpha g \phi_j \\ &- \int_\Omega \sum_{i=n_D}^{n_h} \left(\dot{d}_i \phi_i \phi_j + d_i (\nabla \phi_i) \cdot (\nabla \phi_j) \right) \quad \forall j, \end{aligned} \tag{9}$$

with $\dot{d}_i := \frac{\partial u^D}{\partial t}(N_i)$.

We now define the matrices M, A and the vectors N, D by:

$$\begin{aligned} M_{ij} &:= \int_\Omega \phi_i \phi_j \\ A_{ij} &:= \int_\Omega \alpha (\nabla \phi_i) \cdot (\nabla \phi_j) \\ N_j &:= \int_{\partial\Omega^N} \alpha g \phi_j \\ D_j &:= \int_\Omega \sum_{k=n_D}^{n_h} \left(\dot{d}_k \phi_k \phi_j + d_k (\nabla \phi_k) \cdot (\nabla \phi_j) \right), \end{aligned}$$

with $i, j = 1, \dots (n_D - 1)$. Using these we get the the differential equation:

$$M \frac{\partial u}{\partial t} = -Au + N - D. \tag{10}$$

We will later see how to solve this.

References

[QQ09] Alfio Quarteroni and Silvia Quarteroni. *Numerical models for differential problems*, volume 2. Springer, 2009.