# Mathematical Induction

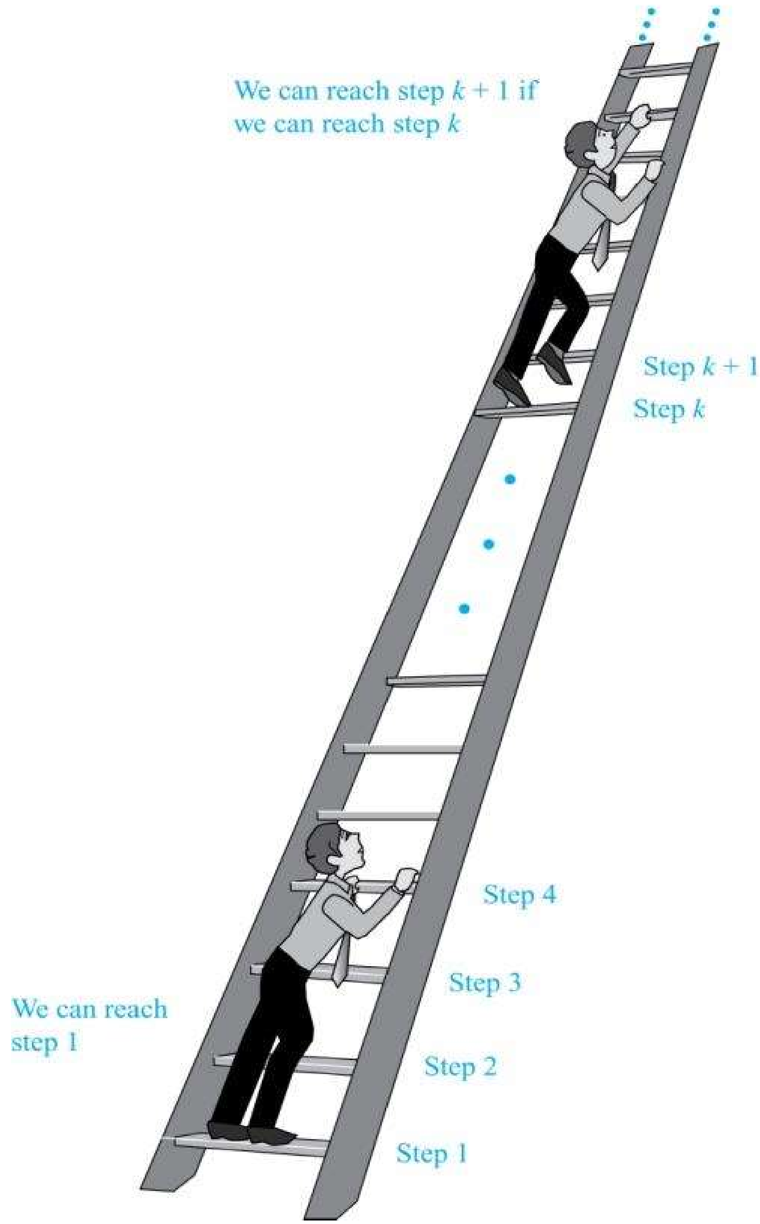
Section 5.1

# Section Summary

- Mathematical Induction
- Examples of Proof by Mathematical Induction
- Guidelines for Proofs by Mathematical Induction

# Climbing an Infinite Ladder

- **Suppose we have an infinite ladder:**
  - **We can reach the first rung of the ladder.**
  - **If we can reach a particular rung of the ladder, then we can reach the next rung.**
- **Can we reach every step on the ladder?**



# Principle of Mathematical Induction

- **Principle of Mathematical Induction: To prove that P(n) is true for all positive integers n, we complete these steps:**
  - **Basis Step: Show that P(1) is true.**
  - **Inductive Step: Show that $P(k) \rightarrow P(k + 1)$ is true for all positive integers k.**
- **To complete the inductive step, assuming the inductive hypothesis that P(k) holds for an arbitrary integer k, show that must P(k + 1) be true.**

# Principle of Mathematical Induction

- **Climbing an Infinite Ladder Example:**
  - **BASIS STEP: By (1), we can reach rung 1.**
  - **INDUCTIVE STEP: Assume the inductive hypothesis that we can reach rung k. Then by (2), we can reach rung k + 1.**
- **Hence, $P(k) \rightarrow P(k + 1)$ is true for all positive integers k. We can reach every rung on the ladder.**

# Important Points

- Mathematical induction can be expressed as the rule of inference

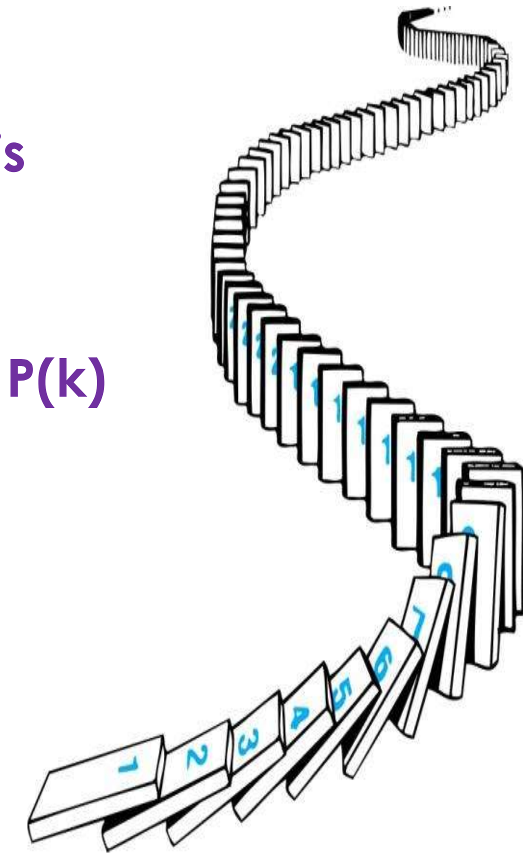
$$\boxed{(P(1) \land \forall k\,(P(k) \rightarrow P(k+1))) \rightarrow \forall n\, P(n),}$$

where the domain is the set of positive integers.

- In a proof by mathematical induction, we don't assume that P(k) is true for all positive integers! We show that if **we assume that P(k) is true, then P(k + 1) must also be true**.
- Proofs by mathematical induction do not always start at the integer 1. In such a case, the basis step begins at a **starting point b** where b is an integer.
- Mathematical induction is valid because of the **well ordering** property

# How Mathematical Induction Works

- Consider an infinite sequence of dominoes, labeled 1,2,3, …, where each domino is standing.
  - Let P(n) be the proposition that the $n^{th}$ domino is knocked over. (A domino is a rectangular piece that is one square by two squares).
  - know that the first domino is knocked down, i.e., P(1) is true.
  - We also know that if whenever the $k^{th}$ domino is knocked over, it knocks over the $(k + 1)^{st}$ domino, i.e, $P(k) \to P(k + 1)$ is true for all positive integers k.
  - Hence, all dominos are knocked over.
  - P(n) is true for all positive integers n.



# Examples

- **Example: Show that:** $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ **for all positive integers.**
- **Solution:**
  - **BASIS STEP:** P(1) is true since 1(1 + 1)/2 = 1.
  - **INDUCTIVE STEP:** Assume true for P(k).
    - The inductive hypothesis is $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$
    - Under this assumption,

$$\begin{aligned} 1 + 2 + \ldots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \end{aligned}$$

  - **Hence**, we have shown that P(k + 1) follows from P(k). Therefore the sum of the first n positive integers is $\frac{n(n+1)}{2}$

# Examples

- **Example: Use mathematical induction to show that**

$$1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$$

  **for all nonnegative integers *n***

- **Solution:**
- **P(n):** $1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1$ **for all nonnegative integers *n***
  - **BASIS STEP: P(0) is true since $2^0 = 1 = 2^1 - 1$. This completes the basis step.**
  - **INDUCTIVE STEP: assume that P(k) is true for an arbitrary nonnegative integer k**
  - $1 + 2 + 2^2 + \cdots + 2^k = 2^{k+1} - 1.$
  - **show that assume that P(k) is true, then P(k + 1) is also true.**
  - $1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = 2^{(k+1)+1} - 1 = 2^{k+2} - 1$
  - **Under the assumption of P(k), we see that**
  - $1 + 2 + 2^2 + \cdots + 2^k + 2^{k+1} = (1 + 2 + 2^2 + \cdots + 2^k) + 2^{k+1} = (2^{k+1} - 1) + 2^{k+1} = 2 \cdot 2^{k+1} - 1 = 2^{k+2} - 1.$
  - **Because we have completed the basis step and the inductive step, by mathematical induction we know that P(n) is true for all nonnegative integers n. That is, $1 + 2 + \cdots + 2^n = 2^{n+1} - 1$ for all nonnegative integers n.**

# Examples

- Example: Conjecture and prove correct a formula for the sum of the first n positive odd integers. Then prove your conjecture.
- Solution:
  - We have:
    - $1 = 1$,
    - $1 + 3 = 4$,
    - $1 + 3 + 5 = 9$,
    - $1 + 3 + 5 + 7 = 16$,
    - $1 + 3 + 5 + 7 + 9 = 25$.
  - We can conjecture that the sum of the first n positive odd integers is $n^2$,
  - $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

# Examples

- $P(n): 1 + 3 + 5 + \cdots + (2n - 1) = n^2$.
  - **BASIS STEP**: $P(1)$ is true since $1^2 = 1$.
  - **INDUCTIVE STEP**: $P(k) \rightarrow P(k + 1)$ for every positive integer k.
    - Assume the inductive hypothesis holds and then show that P(k) holds has well.

**Inductive Hypothesis:** $1 + 3 + 5 + \cdots + (2k - 1) = k^2$

    - So, assuming P(k), it follows that:

$$\begin{aligned} 1 + 3 + 5 + \cdots + (2k - 1) + (2k + 1) &= [1 + 3 + 5 + \cdots + (2k - 1)] + (2k + 1) \\ &= k^2 + (2k + 1) \\ &= k^2 + 2k + 1 \\ &= (k + 1)^2 \end{aligned}$$

  - **Hence**, we have shown that $P(k + 1)$ follows from P(k). Therefore the sum of the first n positive odd integers is $n^2$.

# Examples

- Example: Use mathematical induction to prove that $2^n < n!$, for every integer $n \geq 4$.
- Solution:
- Let P(n) be the proposition that $2^n < n!$.
  - BASIS STEP: P(4) is true since $2^4 = 16 < 4! = 24$.
  - INDUCTIVE STEP: Assume P(k) holds, i.e., $2^k < k!$ for an arbitrary integer $k \geq 4$.
  - To show that $P(k + 1)$ holds:

$$\begin{aligned} 2^{k+1} &= 2 \cdot 2^k \\ &< 2 \cdot k! \qquad \textit{(by the inductive hypothesis)} \\ &< (k + 1)k! \\ &= (k + 1)! \end{aligned}$$

  - Therefore, $2^n < n!$ holds, for every integer $n \geq 4$.

# Examples

- Example: Use mathematical induction to show that if $S$ is a finite set with $n$ elements, where $n$ is a nonnegative integer, then $S$ has $2^n$ subsets.
- Solution:
- $P(n)$ be the proposition that a set with $n$ elements has $2^n$ subsets.
  - Basis Step: $P(0)$ is true, because the empty set has only itself as a subset and $2^0 = 1$.
  - Inductive Step: Assume $P(k)$ is true for an arbitrary nonnegative integer $k$. Inductive Hypothesis: For an arbitrary nonnegative integer $k$, every set with $k$ elements has $2^k$ subsets.
    - Let $T$ be a set with $k + 1$ elements. Then $T = S \cup \{a\}$, where $a \in T$ and $S = T - \{a\}$.
    - For each subset $X$ of $S$, there are exactly two subsets of $T$, i.e., $X$ and $X \cup \{a\}$.
    - By the inductive hypothesis $S$ has $2^k$ subsets. Since there are two subsets of $T$ for each subset of $S$, the number of subsets of $T$ is $2 \cdot 2^k = 2^{k+1}$ .
  - Because we have completed the basis step and the inductive step, by mathematical induction if $S$ is a finite set with $n$ elements, where $n$ is a nonnegative integer, then $S$ has $2^n$ subsets.

# Remark

- Note : When we show that the inductive step is true, we do not show P(k+1) is true.

  Instead, we show the conditional statement

  $$P(k) \rightarrow P(k+1) \text{ is true.}$$

  This allows us to use P(k) as the premise, and gives us an easier way to show P(k+1)

- Once basis step and inductive step are proven, by mathematical induction, $\forall n\, P(n)$ is true

# Remark

- Mathematical induction is a very powerful technique, because we show just two statements, but this can imply infinite number of cases to be correct
- However, the technique does not help us find new theorems. In fact, we have to obtain the theorem (by guessing) in the first place, and induction is then used to formally confirm the theorem is correct

## *Template for Proofs by Mathematical Induction*

1. Express the statement that is to be proved in the form "for all $n \geq b$, $P(n)$" for a fixed integer $b$.
2. Write out the words "Basis Step." Then show that $P(b)$ is true, taking care that the correct value of $b$ is used. This completes the first part of the proof.
3. Write out the words "Inductive Step."
4. State, and clearly identify, the inductive hypothesis, in the form "assume that $P(k)$ is true for an arbitrary fixed integer $k \geq b$."
5. State what needs to be proved under the assumption that the inductive hypothesis is true. That is, write out what $P(k + 1)$ says.
6. Prove the statement $P(k + 1)$ making use the assumption $P(k)$. Be sure that your proof is valid for all integers $k$ with $k \geq b$, taking care that the proof works for small values of $k$, including $k = b$.
7. Clearly identify the conclusion of the inductive step, such as by saying "this completes the inductive step."
8. After completing the basis step and the inductive step, state the conclusion, namely that by mathematical induction, $P(n)$ is true for all integers $n$ with $n \geq b$.