National University of Computer and Emerging Sciences, Lahore Campus

	Course Name:	**Discrete Structure**	**Course Code:**	**CS-1005**
	Degree Program:	**BS (CS), BS(SR), BS(DS)**	**Semester:**	**Fall 2023**
	Exam Duration:	**60 Minutes**	**Total Marks:**	**30**
	Paper Date:		**Weight**	**15 %**
	Sections:	**All**	**No of Page(s):**	**4**
	Exam Type:	**Midterm-2**		

Student Name: ______________________________________ Roll No.________________ Section:_______

Instruction/Notes:
- **i. Attempt all questions neatly on question paper.**
- **ii. Answer sheet is not required.**

<u>Question 1</u>: (marks: 3+4+3)

a) Is the statement "If two sets A and B are uncountable then A − B is uncountable" true? If not true, give a counter example.

a) not true, example : A= [0,1] and B=(0,1) are uncountable but A-B = {0,1} is finite.

b) Determine whether "the set of integers not divisible by 3" is countable or uncountable. If countably infinite, exhibit a one-to-one correspondence between the set of positive integers and that set.

b) This set is countable. The integers in the set are $\pm1, \pm2, \pm4, \pm5, \pm7$, and so on. We can take,

$$A = \{-1, 1, -2, 2, -4, 4, -5, 5, -7, 7, \dots\},$$

Then, the correspondence is given by $1 \leftrightarrow -1, 2 \leftrightarrow 1, 3 \leftrightarrow -2, 4 \leftrightarrow 2, 5 \leftrightarrow -4, 6 \leftrightarrow 4$ and so on.

OR, alternatively,

define a one-to-one correspondence $f: \mathbb{N} \to A$ by :

$$f(n) = \begin{cases} -\dfrac{3(n-1)}{4} - 1 & ; if \quad n \bmod 4 = 1 \\ \dfrac{3(n-2)}{4} + 1 & ; if \quad n \bmod 4 = 2 \\ -\dfrac{3(n-3)}{4} - 2 & ; if \quad n \bmod 4 = 3 \\ \dfrac{3(n-4)}{4} + 2 & ; if \quad n \bmod 4 = 0 \end{cases}$$

c) Prove that there are 100 consecutive positive integers that are not perfect squares. Is your proof constructive or nonconstructive?

c) Proof is constructive as we can prove the existence of such numbers by example. Let's take n = 100. Then $100^2 = 10000$ and $101^2 = 10201$, so the 201 consecutive numbers 10001, 10002, ... , 10200 are not perfect squares.

Question 2: **(marks: 3+3+1+3)**

a) Show that 151 and 951 are relatively prime.
b) Find a linear combination of 151 and 951 that equals 1
c) Hence find the modular inverse 151 mod 951
d) Use Fermat's little Theorem to evaluate 23^{1002} mod 41

a) $951 = 6.151 + 45$
$151 = 3.45 + 16$
$45 = 2.16 + 13$
$16 = 1.13 + 3$
$13 = 4.3 + 1$

b) Since the $gcd(951, 151) = 1$, hence they are relatively prime.
By backward substitution
$1 = 13 - 4.3$
$= 13 - 4.(16 - 1.13)$
$= 5.13 - 4.16$
$= 5.(45 - 2.16) - 4.16$
$= 5.45 - 14.16$
$= 5.45 - 14.(151 - 3.45)$
$= 47.45 - 14.151$
$= 47.(951 - 6.151) - 14.151$
$= 47.951 - 296.151$

c) The modular inverse of $151 \ mod \ 951$ is $-296 \equiv 655 \ mod \ 951$

d) Fermat's little theorem tells us that $23^{40} \equiv 1 \ (mod \ 41)$.

Therefore
$$23^{1002} = (23^{40})^{25}.23^2 \equiv 1^{25}.529$$
$$\equiv 37 (mod 41)$$

<u>Question 3</u>: (marks: 5+5)

a) **Prove using mathematical induction that for any positive integer n,**

$$\mathbf{1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n - 1)}{2}}$$

For any integer $n \geq 1$, let P_n be the statement that

$$1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}.$$

Base Case. The statement P_1 says that

$$1 = \frac{1(3-1)}{2},$$

which is true.

Inductive Step. Fix $k \geq 1$, and suppose that P_k holds, that is,

$$1 + 4 + 7 + \cdots + (3k - 2) = \frac{k(3k-1)}{2}.$$

It remains to show that P_{k+1} holds, that is,

$$1 + 4 + 7 + \cdots + (3(k+1) - 2) = \frac{(k+1)(3(k+1)-1)}{2}.$$

$$\begin{aligned}
1 + 4 + 7 + \cdots + (3(k+1) - 2) &= 1 + 4 + 7 + \cdots + (3(k+1) - 2) \\
&= 1 + 4 + 7 + \cdots + (3k + 1) \\
&= 1 + 4 + 7 + \cdots + (3k - 2) + (3k + 1) \\
&= \frac{k(3k-1)}{2} + (3k+1) \\
&= \frac{k(3k-1) + 2(3k+1)}{2} \\
&= \frac{3k^2 - k + 6k + 2)}{2} \\
&= \frac{3k^2 + 5k + 2)}{2} \\
&= \frac{(k+1)(3k+2)}{2} \\
&= \frac{(k+1)(3(k+1)-1)}{2}.
\end{aligned}$$

Therefore P_{k+1} holds.

Thus, by the principle of mathematical induction, for all $n \geq 1$, P_n holds.

b) Suppose that A is a nonempty set, and f is a function that has A as its domain. Let R be the relation on A consisting of all ordered pairs (x, y) such that $f(x) = f(y)$. Show that R is an equivalence relation on A. What are the equivalence classes of R?

Solution:

b) This relation is reflexive, since obviously $f(x) = f(x)\ for\ all\ x \in A$.

It is symmetric, since if $f(x) = f(y), then\ f(y) = f(x)$.

It is transitive, since if $f(x) = f(y)\ and\ f(y) = f(z)$, then $f(x) = f(z)$.

The equivalence class of x is the set of all $y \in A$ such that $f(y) = f(x)$. This is by definition just the inverse image of $f(x)$. Thus the equivalence classes are precisely the sets $f^{-1}(b)$ for every b in the range of f.