1. Correction : The limits for X & Y should be $-\infty$ to $+\infty$ (you must have figured this out).

Note that

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} 2c\, e^{-\frac{1}{2}(x^2-2xy+4y^2)}$$

$$= \int_{-\infty}^{\infty} 2c \left(\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left[(x-y)^2+3y^2\right]}\,dx\right) dy$$

$$= \int_{-\infty}^{\infty} 2c\, e^{-3y^2/2} \underbrace{\left(\int_{-\infty}^{\infty} e^{-(x-y)^2/2}\,dx\right)}_{=\sqrt{2\pi}} dy$$

$$= 2\sqrt{2\pi}\cdot c \underbrace{\int_{-\infty}^{\infty} e^{-3y^2/2}}_{\sqrt{2\pi}\cdot\frac{1}{\sqrt{3}}}$$

$$\Rightarrow \quad 2\cdot 2\pi \cdot \frac{1}{\sqrt{3}}\cdot c = 1 \quad \Rightarrow \quad c = \frac{\sqrt{3}}{2\cdot 2\pi}.$$

Thus,

$$f_{X,Y}(x,y) = \frac{\sqrt{3}}{2\pi} e^{-(x^2-2xy+4y^2)/2}, \quad x,y\in\mathbb{R}.$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} e^{-(x-y)^2/2}}_{f_{X|Y=y}(x)} \cdot \underbrace{\frac{1}{\sqrt{2\pi}\cdot\frac{1}{\sqrt{3}}} e^{-3y^2/2}}_{f_Y(y)}, \quad x,y\in\mathbb{R}$$

Thus, we have $Y \sim N(0, 1/3)$. Alternatively, we may write

$$f_{X,Y}(x,y) = \frac{\sqrt{3}}{2\pi} e^{-2(y^2 - xy/2 + x^2/4)}$$

$$= \frac{\sqrt{3}}{2\pi} e^{-2\left[(y-x/4)^2 + \frac{x^2}{4} - \frac{x^2}{16}\right]}$$

$$= \frac{\sqrt{3}}{2\pi} e^{-2\left[(y-x/4)^2\right]} \cdot e^{-3x^2/8}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}\left(\frac{1}{2}\right)} e^{-\frac{(y-x/4)^2}{2\cdot 1/4}}}_{f_{Y|X=x}(y)} \cdot \underbrace{\frac{1}{\sqrt{2\pi}\cdot\frac{2}{\sqrt{3}}} e^{-x^2/2\cdot\left(\frac{4}{3}\right)}}_{f_X(x)}$$

$$\Rightarrow \quad X \sim N\left(0, \frac{4}{3}\right)$$

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2. X, Y jointly Gaussian $\Rightarrow$ $\forall \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $Z = a_1 X + a_2 Y$ is Gaussian

$$\Rightarrow \forall \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

$$a_1(X+Y) + a_2(X-Y) = (a_1+a_2)X + (a_1-a_2)Y \text{ is Gaussian}$$

$$\Rightarrow X+Y \text{ and } X-Y \text{ are jointly Gaussian}$$

$$\Rightarrow X+Y \perp\!\!\!\perp X-Y \iff \mathrm{Cov}(X+Y, X-Y) = 0.$$

Now,

$$\begin{aligned} \mathrm{Cov}(X+Y, X-Y) &= \mathbb{E}[(X+Y)(X-Y)] - \mathbb{E}[X+Y]\cdot\mathbb{E}[X-Y] \\ &= \mathbb{E}[X^2] - \mathbb{E}[Y^2] - \left((\mathbb{E}[X])^2 - (\mathbb{E}[Y])^2\right) \\ &= \mathrm{Var}(X) - \mathrm{Var}(Y) \\ &= \sigma_1^2 - \sigma_2^2. \end{aligned}$$

Thus,

$$X+Y \perp\!\!\!\perp X-Y \iff \sigma_1^2 = \sigma_2^2.$$

3.

a) With prob. $\frac{1}{2}$, $\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}\right)$

With prob $\frac{1}{2}$, $\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}\right)$.

Therefore,

w.p. $\frac{1}{2}$, $X \sim N(0,1)$, $Y \sim N(0,1)$, $\rho_{X,Y} = \rho$

w.p. $\frac{1}{2}$, $X \sim N(0,1)$, $Y \sim N(0,1)$, $\rho_{X,Y} = -\rho$.

Thus,

$X \sim N(0,1)$, $Y \sim N(0,1)$

$\rho_{X,Y} = \frac{1}{2}\cdot\rho + \frac{1}{2}(-\rho) = 0.$

$\Rightarrow \mathbb{E}[XY] = 0 \quad (\because \mathbb{E}[X]=0, \mathbb{E}[Y]=0).$

b) The joint PDF of X and Y is not in multivariate form.
To confirm that X & Y are not jointly Gaussian, we show that

$$Z = X+Y$$

is not Gaussian distributed.

Indeed,

using the formula $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$

w.p. $\frac{1}{2}$, $Z \sim N(0, 2+2\rho)$

w.p. $\frac{1}{2}$, $Z \sim N(0, 2-2\rho)$

using the formula $\mathrm{Var}(X-Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) - 2\,\mathrm{Cov}(X,Y)$

$\Rightarrow \quad M_Z(t) = \frac{1}{2} e^{t^2(2+2\rho)} + \frac{1}{2} e^{t^2(2-2\rho)}$

$\neq e^{ta} e^{t^2\sigma^2/2}$ for any $a \in \mathbb{R}$, $\sigma > 0$.

$\Rightarrow$ Z is not Gaussian.

c) Yes. We have $\rho_{X,Y} = 0$ (as shown above).

$\boxed{\text{Summary: In this example, } X \text{ \& } Y \text{ are individually Gaussian, uncorrelated, but not jointly Gaussian.}}$

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4. This is straight forward from the formula of $f_{X|Y}(x/y)$.

5. Finding the density of Y from first principles or using the theorem is involved because we have 4 rv here. But we are asked to show that Y has the given density. Hence we can do it using mgf. We can find mgf of Y easily.

$E[e^{tY}] = E[e^{tX_1X_2}e^{tX_3X_4}] = E[e^{tX_1X_2}]E[e^{tX_3X_4}]$ by independence. Now

$$E[e^{tX_1X_2}] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{txy}\, e^{-\frac{x^2}{2}}\, e^{-\frac{y^2}{2}}\, dx\, dy$$

which can be calculated easily using the trick of completing squares. Thus we can find mgf of Y. We can also find the mgf corresponding to the given density and thus show that Y has that density.

6. This is straight forward from the formula of $f_{X|Y}(x/y)$.