AI 5030: Probability and Stochastic Processes

Instructor: Dr. Karthik P. N.

Homework 5

Topics: CDFs, Joint CDFs, Discrete and Continuous Random Variables, Jointly Discrete Random Variables, Independence of Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables appearing below are assumed to be defined with respect to $\mathscr{F}$.

1. Let X be a random variable. Determine, in each case below, if the function therein can be a valid CDF of X. If not, provide at least one valid justification. For each valid CDF, compute $\mathbb{P}(\{X > 5\})$.

   (a) $F_X(x) = \begin{cases} \frac{e^{-x^2}}{4}, & x < 0, \\ 1 - \frac{e^{-x^2}}{4}, & x \geq 0. \end{cases}$

   Solution: A valid CDF. Follows right continuity, $\lim_{x\to\infty} F_X(x) = 1$ and $\lim_{x\to-\infty} F_X(x) = 0$ and the function is monotonically increasing.
   $\mathbb{P}(\{X > 5\}) = 1 - \mathbb{P}(\{X \leq 5\}) = 1 - F_X(5) = \frac{e^{-25}}{4}$.

   (b) $F_X(x) = \begin{cases} 0, & x < 0, \\ 0.5 + e^{-x}, & 0 \leq x < 3, \\ 1, & x \geq 3. \end{cases}$

   Solution: Not a valid CDF. Follows right continuity, $\lim_{x\to\infty} F_X(x) = 1$ and $\lim_{x\to-\infty} F_X(x) = 0$; but the function is monotonically decreasing when $0 \leq x < 3$. We can also see that $\mathbb{P}(\{X = 0\}) + \mathbb{P}(\{X = 3\}) > 1$.

   (c) $F_X(x) = \begin{cases} 0, & x < 0, \\ 0.5 + \frac{x}{20}, & 0 \leq x < 10, \\ 1, & x \geq 10. \end{cases}$

   Solution: A valid CDF. Follows right continuity, $\lim_{x\to\infty} F_X(x) = 1$ and $\lim_{x\to-\infty} F_X(x) = 0$ and the function is monotonically increasing.
   $\mathbb{P}(\{X > 5\}) = 1 - \mathbb{P}(\{X \leq 5\}) = 1 - F_X(5) = 1 - 0.5 - \frac{5}{20} = 0.25$.

2. Let X be a Geometric random variable. Show that for all $n, k \geq 1$,

$$\mathbb{P}(\{X > n + k\} | \{X > n\}) = \mathbb{P}(\{X > k\}).$$

   This is called the memoryless property of a Geometric distribution.

   Conversely, show that if the random variable X satisfies the above property, then it must be Geometric.

   Solution: Assuming X is a Geometric RV with parameter (probability of success) p. We have that $\mathbb{P}(\{X > n\}) = \sum_{r=n}^{\infty}(1-p)^r p = (1-p)^n$. Then,

$$\begin{aligned} \mathbb{P}(\{X > n + k\} | \{X > n\}) = \frac{\mathbb{P}(\{X > n + k\} \cap \{X > n\})}{\mathbb{P}(\{X > n\})} &\overset{(a)}{=} \frac{\mathbb{P}(\{X > n + k\})}{\mathbb{P}(\{X > n\})} \\ &= \frac{(1-p)^{n+k}}{(1-p)^n} \\ &= (1-p)^k \\ &= \mathbb{P}(\{X > k\}), \end{aligned}$$

   where (a) above follows by noting that for any $\omega \in \Omega$ such that $X(\omega) > n + k$, we have $X(\omega) > n$. Therefore, $\{X > n + k\} \subseteq \{X > n\}$, from which it follows that $\{X > n + k\} \cap \{X > n\} = \{X > n + k\}$.

   Conversely, assume that the relation $\mathbb{P}(\{X > n + k\} | \{X > n\}) = \mathbb{P}(\{X > k\})$ holds for all $n, k \geq 1$. In particular, note that it holds for $k = 1$. This gives us $\mathbb{P}(\{X > n+1\} | \{X > n\}) = \mathbb{P}(\{X > 1\}) \Rightarrow \frac{\mathbb{P}(\{X > n+1\})}{\mathbb{P}(\{X > n\})} = \mathbb{P}(\{X > 1\})$. Applying recursively, we get $\mathbb{P}(\{X > n\}) = \mathbb{P}(\{X > 1\})^n$. Defining $p := 1 - \mathbb{P}(X > 1)$, we have $\mathbb{P}(\{X > n\}) = (1-p)^n$, which matches the CDF of the Geometric distribution.

3. Let X and Y be continuous random variables with PDFs f_X and f_Y respectively. For any $\alpha \in [0,1]$, argue that $\alpha\, f_X + (1-\alpha)\, f_Y$ is a valid PDF. Can you think of a random variable Z whose PDF is $f_Z = \alpha\, f_X + (1-\alpha)\, f_Y$?

Solution: Clearly, the function $f(x) = \alpha f_X(x) + (1-\alpha) f_Y(x)$, $x \in \mathbb{R}$, is non-negative. Furthermore,

$$\int_{-\infty}^{\infty} f(x)\, dx = \alpha \int_{-\infty}^{\infty} f_X(x)\, dx + (1-\alpha) \int_{-\infty}^{\infty} f_Y(x)\, dx = 1,$$

where the last equality follows by noting that $\int_{-\infty}^{\infty} f_X(x)\, dx = 1 = \int_{-\infty}^{\infty} f_Y(x)\, dx$, as f_X and f_Y are valid PDFs.

Let $B \sim$ Ber(α) be a Bernoulli random variable independent of X and Y, i.e., $B \perp\!\!\!\perp X$ and $B \perp\!\!\!\perp Y$. Let Z be a random variable defined as

$$Z(\omega) = \begin{cases} X(\omega), & B(\omega) = 1, \\ Y(\omega), & B(\omega) = 0, \end{cases} \qquad \omega \in \Omega.$$

The CDF of Z is given by

$$\begin{aligned} F_Z(z) &= \mathbb{P}(\{Z \le z\}) \\ &\stackrel{(a)}{=} \mathbb{P}(\{Z \le z\} \cap \{B = 1\}) + \mathbb{P}(\{Z \le z\} \cap \{B = 0\}) \\ &\stackrel{(b)}{=} \mathbb{P}(\{X \le z\} \cap \{B = 1\}) + \mathbb{P}(\{Y \le z\} \cap \{B = 0\}) \\ &\stackrel{(c)}{=} \mathbb{P}(\{X \le z\}) \cdot \mathbb{P}(\{B = 1\}) + \mathbb{P}(\{Y \le z\}) \cdot \mathbb{P}(\{B = 0\}) \\ &= \alpha\, F_X(z) + (1-\alpha)\, F_Y(z), \qquad z \in \mathbb{R}, \end{aligned}$$

where (a) above follows from the law of total probability, (b) above follows from the definition of Z, and (c) above follows from the fact that $B \perp\!\!\!\perp X$ and $B \perp\!\!\!\perp Y$. Differentiating the CDF of Z, we obtain the PDF of Z as

$$f_Z(z) = \alpha\, f_X(z) + (1-\alpha)\, f_Y(z), \qquad z \in \mathbb{R}.$$

4. We learnt in class that random variables X and Y are independent if

$$\{X \le x\} \perp\!\!\!\perp \{Y \le y\} \qquad \forall x, y \in \mathbb{R}. \tag{1}$$

In this exercise, we will show through a series of logical steps that the above definition of independence implies that

$$\{X = x\} \perp\!\!\!\perp \{Y = y\} \qquad \forall x, y \in \mathbb{R}. \tag{2}$$

Fix arbitrary $x, y \in \mathbb{R}$.

(a) Show that if A, B are events such that $A \perp\!\!\!\perp B$, then $A \perp\!\!\!\perp B^c$.

(b) Show that if A, B, C are events such that $A \perp\!\!\!\perp B$ and $A \perp\!\!\!\perp C$, then $A \perp\!\!\!\perp (B \cup C)$ if and only if $A \perp\!\!\!\perp (B \cap C)$.

(c) Let $B_1, B_2, \ldots \in \mathscr{F}$ be such that $B_n \supseteq B_{n+1}$ for all $n \in \mathbb{N}$.
If $A \perp\!\!\!\perp B_n$ for all $n \in \mathbb{N}$, show that $A \perp\!\!\!\perp \bigcap_{n=1}^{\infty} B_n$.

(d) Apply the result in part (c) to the sets $A = \{X \le x\}$ and $B_n = \{Y > y - \frac{1}{n}\}$. Argue using the result in part (a) that $A \perp\!\!\!\perp B_n$ for all n. Hence conclude from the result in part (c) that $\{X \le x\} \perp\!\!\!\perp \{Y \ge y\}$.

(e) Apply the result in part (b) to the sets $A = \{X \le x\}$, $B = \{Y \le y\}$, and $C = \{Y \ge y\}$, and conclude using the results in parts (d) and (a) that $\{X \le x\} \perp\!\!\!\perp \{Y = y\}$. What are $B \cap C$ and $B \cup C$ here?

(f) Repeat the series of logical steps in parts (d) and (e) with $A = \{Y = y\}$ to arrive at $\{X = x\} \perp\!\!\!\perp \{Y = y\}$.

Remark: This exercise shows that if X and Y are independent, then (1) implies (2). However, in general, (2) DOES NOT imply (1). If X and Y are jointly discrete, then (2) implies (1).

Solution: We prove each of the parts below in order.

(a) We have $A = (A \cap B^c) \cup (A \cap B)$, and if $A \perp\!\!\!\perp B$, then $\mathbb{P}(A \cap B) = \mathbb{P}(A)\, \mathbb{P}(B)$. We then note that

$$\begin{aligned} \mathbb{P}(A) &= \mathbb{P}((A \cap B^c) \cup (A \cap B)) \\ \mathbb{P}(A) &= \mathbb{P}(A \cap B^c) + \mathbb{P}(A \cap B) \\ \mathbb{P}(A) &= \mathbb{P}(A \cap B^c) + \mathbb{P}(A)\, \mathbb{P}(B), \end{aligned}$$

from which it follows after re-arranging terms that $\mathbb{P}(A \cap B^c) = \mathbb{P}(A)\, \mathbb{P}(B^c)$, thereby proving that $A \perp\!\!\!\perp B^c$.

(b) See the solution to question 6 of homework 4.

(c) Because $A \perp\!\!\!\perp B_n$ for all $n \in \mathbb{N}$, we have $\mathbb{P}(A \cap B_n) = \mathbb{P}(A)\,\mathbb{P}(B_n)$ for all $n \in \mathbb{N}$. Taking limits as $n \to \infty$, we then get $\lim_{n\to\infty} \mathbb{P}(A \cap B_n) = \lim_{n\to\infty} \mathbb{P}(A)\,\mathbb{P}(B_n) = \mathbb{P}(A)\ \lim_{n\to\infty} \mathbb{P}(B_n)$. Because $B_n \supseteq B_{n+1}$ for all $n \in \mathbb{N}$, we note that $\lim_{n\to\infty} B_n = \bigcap_{n=1}^{\infty} B_n$. Furthermore, we have $A \cap B_n \supseteq A \cap B_{n+1}$ for all $n \in \mathbb{N}$, and therefore $\lim_{n\to\infty} A \cap B_n = \bigcap_{n=1}^{\infty} A \cap B_n = A \cap \bigcap_{n=1}^{\infty} B_n$. Applying continuity of probability, we get

$$\begin{aligned}
\lim_{n\to\infty} \mathbb{P}(A \cap B_n) &= \lim_{n\to\infty} \mathbb{P}(A)\,\mathbb{P}(B_n) \\
\mathbb{P}\big(\lim_{n\to\infty} (A \cap B_n)\big) &= \mathbb{P}(A)\,\mathbb{P}(\lim_{n\to\infty} B_n) \\
\mathbb{P}(A \cap \lim_{n\to\infty} B_n) &= \mathbb{P}(A)\,\mathbb{P}(\lim_{n\to\infty} B_n) \\
\mathbb{P}\left(A \cap \left(\bigcap_{n=1}^{\infty} B_n\right)\right) &= \mathbb{P}(A)\,\mathbb{P}\left(\bigcap_{n=1}^{\infty} B_n\right).
\end{aligned}$$

We have thus proved that $A \perp\!\!\!\perp \bigcap_{n=1}^{\infty} B_n$.

(d) It is clear that, $B_1, B_2, \ldots \in \mathscr{F}$ & $B_n \supseteq B_{n+1}$ for all $n \in \mathbb{N}$. Then, $\lim_{n\to\infty} B_n = \bigcap_{n=1}^{\infty} B_n = \{Y \geq y\}$. Let us consider $C_n = B_n^c = \{Y \leq y - \frac{1}{n}\}$. Because X and Y are independent, it follows that $A \perp\!\!\!\perp C_n$ for all $n \in \mathbb{N}$, where A is as defined in the question. Using the result in part (a), it follows that $A \perp\!\!\!\perp C_n^c = B_n$ for all $n \in \mathbb{N}$. Applying the result in part (c), we get that $A \perp\!\!\!\perp \bigcap_{n=1}^{\infty} B_n = \{Y \geq y\}$.

(e) Here, $A = \{X \leq x\}$, $B = \{Y \leq y\}$, and $C = \{Y \geq y\}$. Then, $B \cup C = \{Y \in \mathbb{R}\} = \Omega$, and $B \cap C = \{Y = y\}$.

$$\begin{aligned}
\mathbb{P}(A \cap (B \cup C)) &= \mathbb{P}(A \cap \Omega) \\
\mathbb{P}(A \cap \Omega) &= \mathbb{P}(A) \\
\mathbb{P}(A \cap \Omega) &= \mathbb{P}(A)\mathbb{P}(\Omega).
\end{aligned}$$

Thus $A \perp\!\!\!\perp (B \cup C)$, and from the question we know that $A \perp\!\!\!\perp B$, $A \perp\!\!\!\perp C$. Therefore, from the result in part (b), we have $A \perp\!\!\!\perp (B \cap C)$, i.e., $\{X \geq x\} \perp\!\!\!\perp \{Y = y\}$.

(f) This follows by following along the lines of the steps presented in the solutions to parts (d) and (e) above.

5. Suppose that two batteries are chosen simultaneously and uniformly at random from the following group of 12 batteries : 3 new, 4 used (yet working), 5 defective. You may assume that all batteries within a particular group are identical. Let X be the number of new batteries chosen, and let Y be the number of used batteries chosen. Determine the joint PMF of X and Y, and compute $\mathbb{P}(\{|X - Y| \leq 1\})$.

Solution: Note that $X + Y \leq 2$ with probability 1. We then have

$$p_{X,Y}(x,y) = \begin{cases}
\dfrac{\binom{5}{2}}{\binom{12}{2}}, & x = 0, y = 0, \\[2ex]
\dfrac{\binom{4}{1} \cdot \binom{5}{1}}{\binom{12}{2}}, & x = 0, y = 1, \\[2ex]
\dfrac{\binom{4}{2}}{\binom{12}{2}}, & x = 0, y = 2, \\[2ex]
\dfrac{\binom{3}{1} \cdot \binom{5}{1}}{\binom{12}{2}}, & x = 1, y = 0, \\[2ex]
\dfrac{\binom{3}{1} \cdot \binom{4}{1}}{\binom{12}{2}}, & x = 1, y = 1, \\[2ex]
\dfrac{\binom{3}{2}}{\binom{12}{2}}, & x = 2, y = 0, \\[2ex]
0, & \text{otherwise.}
\end{cases}$$

From the above joint PMF, it follows that

$$\mathbb{P}(\{|X-Y|\leq 1\}) = p_{X,Y}(0,0) + p_{X,Y}(0,1) + p_{X,Y}(1,0) + p_{X,Y}(1,1) = \frac{57}{66}.$$

6. Let $X \sim$ Uniform$([0,1])$ and $Y = 1 - X$. Derive the joint CDF of X and Y.

Solution: Note that

$$\begin{aligned} F_{X,Y}(x,y) &= \mathbb{P}(\{X \leq x\} \cap \{Y \leq y\}) \\ &= \mathbb{P}(\{X \leq x\} \cap \{1 - X \leq y\}) \\ &= \mathbb{P}(\{1-y \leq X \leq x\}). \end{aligned}$$

Clearly, if $1-y > x$ (equivalently, $x + y < 1$), then $\{1-y \leq X \leq x\} = \emptyset$, and hence $F_{X,Y}(x,y) = 0$. Therefore, it suffices to characterise the joint CDF of X and Y at all points (x,y) such that $1 - y \leq x$, in which case we have

$$F_{X,Y}(x,y) = \mathbb{P}(\{1-y \leq X \leq x\}) = \mathbb{P}(\{1-y < X \leq x\}) = F_X(x) - F_X(1-y).$$

Observe that

$$F_X(x) = \begin{cases} 0, & x < 0, \\ x, & 0 \leq x < 1, \\ 1, & x \geq 1, \end{cases} \qquad F_X(1-y) = \begin{cases} 0, & 1-y < 0 \text{ (or } y > 1), \\ 1-y, & 0 \leq 1-y < 1 \text{ (or } 0 < y \leq 1), \\ 1, & 1-y \geq 1 \text{ (or } y \leq 0). \end{cases}$$

We characterise the joint CDF on a case-by-case basis as below.

(a) Case 1: $x < 0, y > 1, 1-y \leq x$.
In this case, we have
$$F_{X,Y}(x,y) = F_X(x) - F_X(1-y) = 0 - 0 = 0.$$

(b) Case 2: $0 \leq x < 1, y > 1, 1-y \leq x$.
In this case, we have
$$F_{X,Y}(x,y) = F_X(x) - F_X(1-y) = x - 0 = x.$$

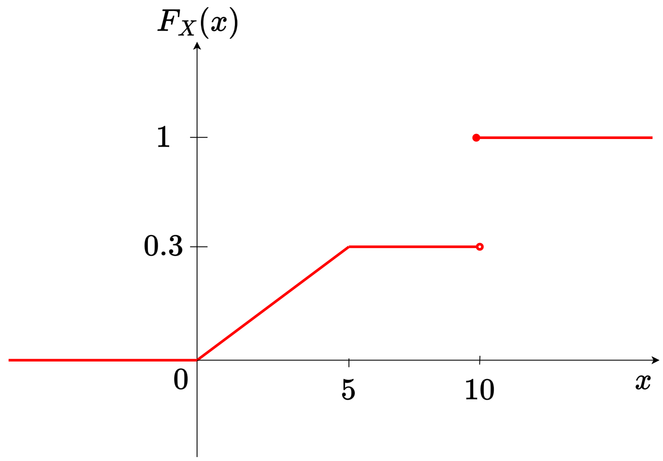
(c) Case 3: $0 \leq x < 1, 0 < y \leq 1, 1-y \leq x$.
In this case,
$$F_{X,Y}(x,y) = F_X(x) - F_X(1-y) = x - (1-y) = x + y - 1.$$

(d) Case 4: $x \geq 1, y > 1, 1-y \leq x$.
In this case,
$$F_{X,Y}(x,y) = F_X(x) - F_X(1-y) = 1 - 0 = 1.$$

(e) Case 5: $x \geq 1, 0 < y \leq 1, 1-y \leq x$.
In this case,
$$F_{X,Y}(x,y) = F_X(x) - F_X(1-y) = 1 - (1-y) = y.$$

(f) Case 6: $x \geq 1, y \leq 0, 1-y \leq x$.
In this case,
$$F_{X,Y}(x,y) = F_X(x) - F_X(1-y) = 1 - 1 = 0.$$

7. Let X be a random variable with CDF F_X as shown in the figure below. Compute $\mathbb{P}(\{X \in [3, 15]\} | \{X > 4\})$.



Solution: We have

$$\begin{aligned}
\mathbb{P}(\{X \in [3, 15]\} | \{X > 4\}) &= \frac{\mathbb{P}(\{X \in [3, 15]\} \cap \{X > 4\})}{\mathbb{P}(\{X > 4\})} \\
&= \frac{\mathbb{P}(\{X \in (4, 15]\})}{1 - \mathbb{P}(\{X \le 4\})} \\
&= \frac{F_X(15) - F_X(4)}{1 - F_X(4)} \\
&= \frac{1 - F_X(4)}{1 - F_X(4)} \\
&= 1,
\end{aligned}$$

where the penultimate line follows by noting that $F_X(15) = 1$.