AI 5030: Probability and Stochastic Processes

Instructor: Dr. Karthik P. N.

Homework 3

Topics: Probability Measures and Their Properties, Borel σ-Algebra, Lebesgue Measure

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Recall that $\sigma(\mathscr{C})$ denotes the smallest σ-algebra that contains all sets in $\mathscr{C}$.

1. Argue that for any collection of events $A_1, A_2, \ldots \in \mathscr{F}$, the sets

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \qquad \text{and} \qquad \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

are elements of $\mathscr{F}$.

Solution: We show below that $\liminf_{n\to\infty} A_n \in \mathscr{F}$. Towards this, let $B_n = \bigcap_{k=n}^{\infty} A_k$ for all $n \in \mathbb{N}$. Clearly, $B_n \in \mathscr{F}$ for all n, as the intersection of countably many sets from $\mathscr{F}$ is an element of $\mathscr{F}$ (this follows from the fact that $\mathscr{F}$ is closed under countable intersections). It then follows that $\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} B_n \in \mathscr{F}$, as $\mathscr{F}$ is closed under countable unions.

Along similar lines as above, it can be proved that $\limsup_{n\to\infty} A_n \in \mathscr{F}$.

2. Consider the measure space $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$, where λ denotes the Lebesgue measure.
Fix $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$, and for each $n \in \mathbb{N}$, let

$$A_n \coloneqq \left\{x \in \mathbb{R} : x_1 + \frac{1}{n} < x < x_2 - \frac{1}{n}\right\}.$$

Compute $\lim_{n\to\infty} A_n$ and its Lebesgue measure.

Solution: Notice that $A_n \subseteq A_{n+1}$ for all $n \in \mathbb{N}$. Therefore, it follows that

$$\lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n = (x_1, x_2).$$

Furthermore, we have $\lambda((x_1, x_2)) = x_2 - x_1$.

Remark:

We may arrive at the result $\lim_{n\to\infty} A_n = (x_1, x_2)$ by arguing that $\lim_{n\to\infty} A_n \neq [x_1, x_2), (x_1, x_2], [x_1, x_2]$. To do this, we can formally argue that $x_1, x_2 \notin \lim_{n\to\infty} A_n$. We present the arguments for x_1, and note that similar arguments hold for x_2. Suppose that $x_1 \in \lim_{n\to\infty} A_n$. That is, $x_1 \in \bigcup_{n=1}^{\infty} A_n$. Then, there exists a natural number $N \in \mathbb{N}$ such that $x_1 \in A_N$. However, $A_N = \{x \in \mathbb{R} : x_1 + 1/N < x < x_2 - 1/N\}$, and therefore $x_1 \notin A_N$.

3. Prove the following inclusion-exclusion principle: for any $n \in \mathbb{N}$ and sets $A_1, \ldots, A_n \in \mathscr{F}$,

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n+1}\, \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right).$$

Solution: The base case when $n = 2$ specialises to the relation

$$\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2),$$

which may be proved by noting that

$$A_1 \cup A_2 = (A_1 \setminus A_2) \cup (A_1 \cap A_2) \cup (A_2 \setminus A_1),$$

and therefore

$$\begin{aligned}\mathbb{P}(A_1 \cup A_2) &= \mathbb{P}(A_1 \setminus (A_1 \cap A_2)) + \mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_2 \setminus (A_1 \cap A_2)) \\ &= \mathbb{P}(A_1) - \mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_1 \cap A_2) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2) \\ &= \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2).\end{aligned}$$

We now assume that the relation in question holds for some $n \in \mathbb{N}$, and prove that it holds for $n+1$. Indeed, we have

$$\begin{aligned}&\mathbb{P}\left(\bigcup_{i=1}^{n+1} A_i\right) = \mathbb{P}\left(\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}\right) \\ &= \mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) + \mathbb{P}(A_{n+1}) - \mathbb{P}\left(\left(\bigcup_{i=1}^{n} A_i\right) \cap A_{n+1}\right) \\ &\overset{(a)}{=} \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{1 \le i < j \le n} \mathbb{P}(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1}\,\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) \\ &\qquad + \mathbb{P}(A_{n+1}) - \mathbb{P}\left(\bigcup_{i=1}^{n} (A_i \cap A_{n+1})\right) \\ &\overset{(b)}{=} \sum_{i=1}^{n+1} \mathbb{P}(A_i) - \sum_{1 \le i < j \le n} \mathbb{P}(A_i \cap A_j) + \sum_{1 \le i < j < k \le n} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n}\,\mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right) + \mathbb{P}(A_{n+1}) \\ &- \left[\sum_{i=1}^{n} \mathbb{P}(A_i \cap A_{n+1}) - \sum_{1 \le i \le j \le n} \mathbb{P}\big((A_i \cap A_{n+1}) \cap (A_j \cap A_{n+1})\big) + \cdots + (-1)^{n+1}\,\mathbb{P}\left(\bigcap_{i=1}^{n} (A_i \cap A_{n+1})\right)\right] \\ &= \sum_{i=1}^{n+1} \mathbb{P}(A_i) - \sum_{1 \le i < j \le n+1} \mathbb{P}(A_i \cap A_j) + \cdots + (-1)^{n+2}\,\mathbb{P}\left(\bigcap_{i=1}^{n+1} A_i\right),\end{aligned}$$

where (a) follows from applying the induction hypothesis to the collection $\{A_1, \ldots, A_n\}$, and (b) follows from applying the induction hypothesis to the collection $\{B_1, \ldots, B_n\}$, with $B_i = A_i \cap A_{n+1}$ for all $i \in \{1, \ldots, n\}$. The desired result thus follows by the principle of mathematical induction.

4. Let $A_1, A_2, \ldots \in \mathscr{F}$ be such that $\mathbb{P}(A_i) = 1$ for all $i \in \mathbb{N}$. Show that $\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = 1$.

   As a corollary, show that if $A, B \in \mathscr{F}$ are such that $\mathbb{P}(A) = 1$ and $\mathbb{P}(B) = 1$, then $\mathbb{P}(A \cap B) = 1$.

   Solution: Note that $\mathbb{P}(A_i^c) = 1 - \mathbb{P}(A_i) = 0$ for all $i \in \mathbb{N}$. Then, using the union bound, we have

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i^c\right) \le \sum_{i=1}^{\infty} \mathbb{P}(A_i^c) = 0,$$

   from which it follows that

$$\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = 1 - \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i^c\right) = 1.$$

   Alternative solution using independence: We first revisit that $\mathbb{P}(A_i) = 1$ implies that A_i is independent of all $A_j \in \mathscr{F}$, where $j \neq i$.
   We consider an arbitrary $j \neq i$. $\mathbb{P}(A_i \cup A_j) = 1 \because 1 \ge \mathbb{P}(A_i \cup A_j) \ge \mathbb{P}(A_i)$. Thus, $\mathbb{P}(A_i \cap A_j) = \mathbb{P}(A_i) + \mathbb{P}(A_j) - \mathbb{P}(A_i \cup A_j) = 1 = \mathbb{P}(A_i)\mathbb{P}(A_j)$, showing that A_i is independent of A_j.
   As A_i is independent of every $A_{j(\neq i)} \in \mathscr{F}$, $\mathbb{P}(\cap_{i=1}^{\infty} A_i) = \Pi_{i=1}^{\infty} \mathbb{P}(A_i) = 1$.

   As a special case, taking A_1 as A, A_2 as B and $A_i = \Omega \in \mathscr{F}\ \forall i \ge 3$, we get that if $\mathbb{P}(A) = \mathbb{P}(B) = 1$, then $\mathbb{P}(A \cap B) = 1$, which proves the corollary.

5. Fix $n \in \mathbb{N}$. Suppose that an experiment with sample space Ω is performed repeatedly n times.
For any set $E \in \mathscr{F}$, let $n(E)$ denote the number of times that event E occurs in the n trials of the experiment.
Let $f : \mathscr{F} \to [0,1]$ be defined as

$$f(E) = \frac{n(E)}{n}, \quad E \in \mathscr{F}.$$

Show that f satisfies the axioms of probability, and is therefore a valid probability measure on $(\Omega, \mathscr{F})$.

Solution: We check the three axioms as follows.

- $f(\phi) = \frac{n(\phi)}{n}$. As a successful experiment always results in a valid (non-null) outcome, $n(\phi) = 0$, so we have that $f(\phi) = 0$.
- $f(\Omega) = \frac{n(\Omega)}{n}$. As $\Omega = \{E_1, E_2, \cdots\}$, the event Ω occurs when any of $E \in \mathscr{F},\ E \neq \phi$, occurs. Thus, $n(\Omega) = n \Rightarrow f(\Omega) = 1$.
- If $E_1, E_2, \ldots$ is a countable collection of disjoint sets, $n(\cup_{i\in\mathbb{N}} E_i) = \sum_{i\in\mathbb{N}} n(E_i)$. Thus, $f(\cup_{i\in\mathbb{N}} E_i) = \frac{n(\cup_{i\in\mathbb{N}} E_i)}{n} = \sum_{i\in\mathbb{N}} \frac{n(E_i)}{n} = \sum_{i\in\mathbb{N}} f(E_i)$.

6. Let $(\Omega, \mathscr{F}) = (\mathbb{R}, \mathscr{B}(\mathbb{R}))$.
In class, we saw that $\mathscr{B}(\mathbb{R}) = \sigma(\mathcal{O})$, where $\mathcal{O} = \{(a,b) : -\infty \leq a < b \leq +\infty\}$ is the collection of all open sub-intervals of $\mathbb{R}$. The purpose of this exercise is to provide an alternative way to arrive at $\mathscr{B}(\mathbb{R})$.

(a) Fix $a \in \mathbb{R}$, and for each $n \in \mathbb{N}$, define

$$A_n \coloneqq \left(-\infty,\ a - \frac{1}{n}\right), \quad B_n \coloneqq \left(-\infty,\ a + \frac{1}{n}\right), \quad C_n \coloneqq \left(-\infty,\ a - \frac{1}{n}\right], \quad D_n \coloneqq \left(-\infty,\ a + \frac{1}{n}\right].$$

Determine $\bigcap_{n=1}^\infty A_n,\ \bigcup_{n=1}^\infty A_n,\ \bigcap_{n=1}^\infty B_n,\ \bigcup_{n=1}^\infty B_n,\ \bigcap_{n=1}^\infty C_n,\ \bigcup_{n=1}^\infty C_n,\ \bigcap_{n=1}^\infty D_n,$ and $\bigcup_{n=1}^\infty D_n$.

(b) Consider the collection

$$\mathscr{D} \coloneqq \Big\{(-\infty, x] : x \in \mathbb{R}\Big\}.$$

Show that any open interval $(a,b) \in \mathcal{O}$ can be expressed in terms of countable unions, complements, and countable intersections of sets in $\mathscr{D}$.
Hint: use part (a) of the question.

(c) Use the result in part (b) to argue that $\sigma(\mathcal{O}) = \mathscr{B}(\mathbb{R}) = \sigma(\mathscr{D})$.

Solution: We present the solution to each of the parts below in order.

(a) Notice that $A_n \subseteq A_{n+1}, B_n \supseteq B_{n+1}, C_n \subseteq C_{n+1}$, and $D_n \supseteq D_{n+1}$ for all $n \in \mathbb{N}$. Therefore,

$$\bigcap_{n=1}^\infty A_n = A_1 = (-\infty,\ a-1), \qquad \bigcup_{n=1}^\infty A_n = (-\infty,\ a).$$

Similarly, we have

$$\bigcap_{n=1}^\infty B_n = (-\infty,\ a], \qquad \bigcup_{n=1}^\infty B_n = B_1 = (-\infty,\ a+1),$$

$$\bigcap_{n=1}^\infty C_n = C_1 = (-\infty,\ a-1], \qquad \bigcup_{n=1}^\infty C_n = (-\infty,\ a),$$

and

$$\bigcap_{n=1} D_n = (-\infty,\ a], \qquad \bigcup_{n=1}^\infty D_n = D_1 = (\infty,\ a+1].$$

(b) Note that $(a,\, b) = (-\infty,\, b) \cap (a,\, +\infty)$. Therefore, it suffices to generate the sets $(-\infty,\, b)$ and $(a,\, +\infty)$ via countable unions, complements, and countable intersections of sets in $\mathscr{D}$. To do so, note that

$$(a,\, +\infty) = ((-\infty,\, a])^c\,.$$

Also, using the result in part (a) above, we have

$$(-\infty, b) = \bigcup_{n=1}^{\infty} \left(-\infty,\, b - \frac{1}{n}\right].$$

(c) This follows by simply noting that (a) any Borel subset of $\mathbb{R}$ may be expressed via countable unions, complements, and countable intersections of open intervals, and (b) any open interval of the form (a, b) may be expressed using countable unions, complements, and countable intersections of sets in $\mathscr{D}$ (see part (b)).