AI 5030: Probability and Stochastic Processes

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Homework 3

Topics: Probability Measures and Their Properties, Borel σ-Algebra, Lebesgue Measure

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Recall that $\sigma(\mathscr{C})$ denotes the smallest σ-algebra that contains all sets in $\mathscr{C}$.

1. Argue that for any collection of events $A_1, A_2, \ldots \in \mathscr{F}$, the sets

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k \qquad \text{and} \qquad \limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

   are elements of $\mathscr{F}$.

2. Let $(\Omega, \mathscr{F}, \mathbb{P}) = (\mathbb{R}, \mathscr{B}(\mathbb{R}), \lambda)$, where λ denotes the Lebesgue measure.
   Fix $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$, and for each $n \in \mathbb{N}$, let

$$A_n := \left\{ x \in \mathbb{R} : x_1 + \frac{1}{n} < x < x_2 - \frac{1}{n} \right\}.$$

   Compute $\lim_{n\to\infty} A_n$ and its Lebesgue measure.

3. Prove the following inclusion-exclusion principle: for any $n \in \mathbb{N}$ and sets $A_1, \ldots, A_n \in \mathscr{F}$,

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n+1}\, \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right).$$

4. Let $A_1, A_2, \ldots \in \mathscr{F}$ be such that $\mathbb{P}(A_i) = 1$ for all $i \in \mathbb{N}$. Show that $\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = 1$.

   As a corollary, show that if $A, B \in \mathscr{F}$ are such that $\mathbb{P}(A) = 1$ and $\mathbb{P}(B) = 1$, then $\mathbb{P}(A \cap B) = 1$.

5. Fix $n \in \mathbb{N}$. Suppose that an experiment with sample space Ω is performed repeatedly n times.
   For any set $E \in \mathscr{F}$, let $n(E)$ denote the number of times that event E occurs in the n trials of the experiment.
   Let $f : \mathscr{F} \to [0,1]$ be defined as

$$f(E) = \frac{n(E)}{n}, \quad E \in \mathscr{F}.$$

   Show that f satisfies the axioms of probability, and is therefore a valid probability measure on $(\Omega, \mathscr{F})$.

6. Let $(\Omega, \mathscr{F}) = (\mathbb{R}, \mathscr{B}(\mathbb{R}))$.
   In class, we saw that $\mathscr{B}(\mathbb{R}) = \sigma(\mathcal{O})$, where $\mathcal{O} = \{(a,b) : -\infty \le a < b \le +\infty\}$ is the collection of all open sub-intervals of $\mathbb{R}$. The purpose of this exercise is to provide an alternative way to arrive at $\mathscr{B}(\mathbb{R})$.

   (a) Fix $a \in \mathbb{R}$, and for each $n \in \mathbb{N}$, define

$$A_n := \left(-\infty,\ a - \frac{1}{n}\right), \quad B_n := \left(-\infty,\ a + \frac{1}{n}\right), \quad C_n := \left(-\infty,\ a - \frac{1}{n}\right], \quad D_n := \left(-\infty,\ a + \frac{1}{n}\right].$$

   Determine $\bigcap_{n=1}^{\infty} A_n$, $\bigcup_{n=1}^{\infty} A_n$, $\bigcap_{n=1}^{\infty} B_n$, $\bigcup_{n=1}^{\infty} B_n$, $\bigcap_{n=1}^{\infty} C_n$, $\bigcup_{n=1}^{\infty} C_n$, $\bigcap_{n=1}^{\infty} D_n$, and $\bigcup_{n=1}^{\infty} D_n$.

   (b) Consider the collection

$$\mathscr{D} := \left\{ (-\infty, x] : x \in \mathbb{R} \right\}.$$

   Show that any open interval $(a,b) \in \mathcal{O}$ can be expressed in terms of countable unions, complements, and countable intersections of sets in $\mathscr{D}$.
   Hint: use part (a) of the question.

   (c) Use the result in part (b) to argue that $\sigma(\mathcal{O}) = \mathscr{B}(\mathbb{R}) = \sigma(\mathscr{D})$.