Experiment - 3

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Division using Newton Raphson method

Suppose that $f \in C^2[a,b]$. Let $p_0 \in [a,b]$ be an approximation to p such that $f'(p_0) \neq 0$ and $|p-p_0|$ is small.

Consider the first taylor Polynomial for $f(x)$ expanded about p_0 and evaluated at $x=p$

$$\boxed{f(p) = f(p_0) + (p-p_0) f'(p_0) + \frac{(p-p_0)^2}{2} f''(\xi(p))}$$

Since $f(p)=0$, this equation gives

$$\boxed{f(p_0) + (p-p_0) f'(p_0) + \frac{(p-p_0)^2}{2} f''(\xi(p)) = 0}$$

Newton raphson method is derived by assuming that since $|p-p_0|$ is small, the term involving $(p-p_0)^2$ is much smaller.

so $$\boxed{f(p_0) + (p-p_0) f'(p_0) = 0}$$

Solving for P gives

$$p = p_0 - \frac{f(p_0)}{f'(p_0)}$$

This sets the stage for newton's method which starts with an initial approximation p_0 and generates the sequence $\{p_n\}_{n=0}^{\infty}$ by

$$\boxed{p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})} \quad \forall\, n \geq 1}$$

Newton raphson method for division

we have to compute $\boxed{q = \frac{z}{d}}$

→ first find $\frac{1}{d}$ and then multiply this with z

* The method we use for computing $1/d$ is based on newton - raphson iteration to determine the root of $f(x) = 0$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

$$f(x) = \frac{1}{x} - d$$

$$f'(x) = -\frac{1}{x^2}$$

$\therefore$

$$x_{i+1} = x_i - \frac{\frac{1}{x_i} - d}{-\frac{1}{x_i^2}}$$

$$\boxed{x_{i+1} = x_i(2 - x_i d)}$$

Computationally, two multiplications and one subtraction is are required for each iteration.

Error analysis:

let $\delta_i = \frac{1}{d} - x_i$ be the error at the ith iteration. Then

$$\delta_{i+1} = \frac{1}{d} - x_{i+1} = \frac{1}{d} - x_i(2 - x_i d)$$

$$= \frac{1}{d} - 2x_i + x_i^2 d$$

$$= d\left(\frac{1}{d} - x_i\right)^2$$

$$= d(\delta_i)^2$$

$$\delta_{i+1} = d(\delta_i)^2$$

Since $d < 1$ we have $\delta_{i+1} < (\delta_i)^2$ Proving quadratic Convergence.

Choosing the initial estimate

if the initial Value x_0 is Chosen such that $0 < x_0 < \frac{2}{d}$, leading $|\delta_0| < \frac{1}{d}$, convergence is guaranteed.

→ before doing the newton raphson method normalize the d to make sure that $\boxed{d \in \left[\frac{1}{2}, 1\right)}$ this make the Convergence faster.

Table based methods for initial approximation of x_0 in Newton raphson method

A table-based method improves the initial approximation x_0 for $1/d$ by using Pre computed values stored in look up table.

Step 1: Normalize the d to the range $[0.5, 1)$

Step 2: Use the MSb's of d to index the table

→ This means that we divide the range $[0.5, 1)$ into discrete interwals.

→ if we take 6 bits of d we create 64-entry table ($2^6 = 64$)

Step 3: Store Pre comuted Values of $1/d$

for each interwal, store a Pre computed approximation of $1/d$

Step 4: Use the table value for initial x_0

Step 5: At the end apply the Correction to the result.

Generating look up table

Pre compute table for d in $[0.5, 1)$ using 6-bit indexing

→ 6 bit indexing ⇒ $2^6 = 64$ Values

→ divide $[\frac{1}{2}, 1)$ into 64 Parts

→ Using linear approximation find the estimate of $1/d$ Using the below formula

$$\boxed{x_0 = 4(\sqrt{3}-1) - 2d}$$

↳ This formula I found in Computer Arthimatic Algorithms and Hardware designs by Behrooz Parhami

→ Use x_0 as the initial estimate for the iterations

→ Usually with in 2–3 iterations are sufficient.

Example:

input:

Numerator = 1

Denominator = 29

Let us consider fixed point Q(16.16)

Convert the values to fixed point Q_{16}

→ Denominator = 29×2^{16} = 1900544

Now generate the look up table

→ Divide the interval [0.5, 1) in to 64 parts ($N = 2^6 = 64$)

for i=0 to N−1

$$LUT[i] = 0.5 + \frac{i}{64} \times 0.5$$

$$LUT1[i] = 4(\sqrt{3}-1) - 2[LUT[i]]$$

$$fix\_LUT[i] = round(LUT1[i] \times 2^{16})$$

Step 1: Normalize the input to bring in between $(0.5, 1)$ i.e $[2^{15}, 2^{16})$

$d = \text{Denominator} >> 5$

$d = 59392$

no. of shifts = 5

Step 2: find the initial estimate x_0 corresponding to d in the look up table generated above

$$\boxed{\text{index} = \left(\frac{d' - 0.5}{0.5}\right) \times 64}$$

d' is the floating point value

d is the fixed point value of d' computed in Step 1

Compute above formula in fixed point

$$\text{index} = \left(\frac{d - 0.5 \times 2^{16}}{0.5 \times 2^{16}}\right) \times 64$$

$$\text{index} = (d - 32768) \times \frac{64}{2^{15}}$$

$$\text{index} = (d - 32768) >> 9$$

∴ The index of initial estimate for 1/D in look up table is

$$\boxed{\text{index} = (d - 32768) >> 9}$$

index = 52

initial estimate for 1/D is

$x_0 = \text{fix\_LUT}[52]$

$$\boxed{x_0 = 73119}$$

Step 3:-

Compute the 1/D iterately using newton raphson method

$$x_1 = x_0(2 - x_0 \times d) = 72306$$

$$x_2 = x_1(2 - x_1 \times d) = 72315$$

$$x_3 = x_2(2 - x_2 d) = 72316$$

final result = $x_3 \gg$ (no. of shifts)

$= 72316 \gg 5$

$= 2259$

$\therefore$ The final output of $1/29$ in fixed point is 2259

→ Convert it into floating number

$= 2259/2^{16} = 0.0344696$

→ Actual value of $1/29$ is 0.0344827

→ if we compute $1/29$ in fixed point Q16

$$\frac{(1 \ll 16) \ll 16}{29 \ll 16}$$

$= 2259$ (fixed point value)

$= 2259/2^{16} = 0.03446960$

$\therefore$ The value is matched with both fixed point division and newton raphson division method.