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Probability and Stochastic Processes

Open Quiz 02

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Question

Suppose that X, Y have finite variance.
Further, suppose that $X = \mathbb{E}[Y|X]$ and $Y = \mathbb{E}[X|Y]$.
Determine the value of

$$\frac{\mathrm{Var}(X)}{\mathrm{Var}(X) + 2\mathrm{Var}(Y)}.$$

Question

Suppose that X is a continuous random variable with PDF f_X.
Further, suppose that f_X is symmetric, i.e., $f_X(x) = f_X(-x)$ for all $x \in \mathbb{R}$.
Compute the value of

$$\int_0^\infty \mathbb{P}(\{|X - t| \leq 2\})\, \mathrm{d}t.$$

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Question

Suppose that X and Y are continuous random variables, and $X \perp\!\!\!\perp Y$.
Prove formally that $\mathbb{P}(\{X = Y\}) = 0$.

Question

Let X be a random variable taking values in $[0, 1]$. Suppose that $\mathbb{E}[X] = \frac{1}{2}$. Show that

$$\mathrm{Var}(X) \leq \frac{1}{4}.$$

Produce an example for which the above inequality holds with equality.

Question

Random variables X and Y are related as $Y = g(X)$ for some strictly increasing function g. Suppose that $X \sim \mathrm{Exponential}(1)$, and

$$f_Y(y) = \frac{K}{y^{b+1}}, \qquad y \geq 2.$$

Here, $b > 0$ is a fixed constant.
Determine K and the function g in terms of the constant b.

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Question

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ where $\mathbb{P}$ is a uniform distribution on $\Omega = [0, 1]$. Can you think of random variables X, Y such that BOTH of the below conditions are satisfied:

1. $\mathbb{P}(\{X > Y\}) > \frac{1}{2}$.
2. $\mathbb{E}[X] < \mathbb{E}[Y]$.

Question

A coin with an unknown bias p is tossed n times.
Let the probability of seeing the outcomes $(x_1, \ldots, x_n) = \mathbf{x}$ be denoted by $\mathcal{L}(p; \mathbf{x})$.
Find the value of p that maximises $\mathcal{L}(p; \mathbf{x})$.

Question

Let $X_1, X_2, \ldots \overset{\text{i.i.d.}}{\sim} \text{Exponential}(1)$.

Determine $\mathbb{E}[X_1 | X_1 + \cdots + X_{100}]$.

Question

Let $X, Y \stackrel{\text{i.i.d.}}{\sim} \mathrm{Unif}(0,1)$. Let

$$U = \min\{X, Y\}, \qquad V = \max\{X, Y\}.$$

Determine $\mathrm{Cov}(U, V)$.