Program: BTech / MTech TA / MTech RA / PhD (Tick one)



Al5030: Probability and Stochastic Processes

Quiz 1

DATE: 14 AUGUST 2024

Question	1	2	Total
Marks Scored			

Instructions:

- Fill in your name and roll number on each of the pages.
- · You may use any result covered in class directly without proving it.
- · Unless explicitly stated in the question, DO NOT use any result from the homework without proof.

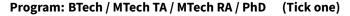
1. (2 Marks)

Show that the collection of all finite-length strings of natural numbers is countably infinite. In your proof, if needed, you may use the fact that \mathbb{N}^d is countably infinite for every $d \in \mathbb{N}$.

Solution:

For any $d \in \mathbb{N}$, let C_d denote the collection of all strings of natural numbers of length d; we note here that a string of natural numbers of length d may be expressed as $a_1a_2\ldots a_d$, where $a_i\in\mathbb{N}$ for each $i\in\{1,2,\ldots,d\}$. It is then easy to see that C_d is equicardinal with \mathbb{N}^d for each $d\in\mathbb{N}$. Indeed, for any $d\in\mathbb{N}$ and $a_1,a_2,\ldots,a_d\in\mathbb{N}$, the mapping $a_1a_2\ldots a_d\mapsto (a_1,a_2,\ldots,a_d)$ is bijective. Using the fact that \mathbb{N}^d is countable for all d, it follows that C_d is countable for each $d\in\mathbb{N}$. Finally, the collection of all finite-length strings of natural numbers may be expressed as $\bigcup_{d=1}^\infty C_d$, and using the facts that countable union of countable sets is countable, we arrive at the desired result.

Name: Roll Number: Department:





2. (3 Marks)

Let Ω be a sample space (finite, countably infinite, or uncountable). Fix $n \in \mathbb{N}$. Let A_1, \ldots, A_n be a partition of Ω , i.e., A_1, \ldots, A_n satisfy the following properties:

- $A_i \subseteq \Omega$ for each $i \in \{1, \ldots, n\}$.
- $A_i \cap A_j = \emptyset$ for all $i \neq j$.
- $\bigcup_{i=1}^n A_i = \Omega$.

Let $\mathscr{C}=\{A_1,\ldots,A_n\}$, and let \mathscr{F} denote the smallest σ -algebra constructed from the sets in \mathscr{C} . Show that $|\mathscr{F}|=2^n$.

Solution:

Because the sets in $\mathscr C$ constitute a partition of Ω , it follows that $\Omega\setminus A_i=\bigcup_{j\neq i}A_j$ for all $i\in\{1,\ldots,n\}$. As a consequence, to construct the smallest σ -algebra from the sets in $\mathscr C$, we may first add the empty set, following which we may add each of the sets in $\mathscr C$, following which we may add the union of sets from $\mathscr C$ chosen two at a time, following which we may add the union of sets from $\mathscr C$ chosen three at a time, and so on. Because there are $\binom{n}{k}$ ways of choosing k sets from $\mathscr C$, for $k=0,\ldots,n$ (here, k=0 corresponds to the empty set, whereas k=n corresponds to Ω), we get that

$$|\mathscr{F}| = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n;$$

the second equality above follows from the binomial theorem.