AI 5030: Probability and Stochastic Processes

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Homework 8

Topics: Expectations of Discrete and Continuous Random Variables, Variance, Covariance

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables appearing below are assumed to be defined with respect to $\mathscr{F}$.

1. For any $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the largest integer lesser than or equal to x. Thus, for instance, $\lfloor 3.5 \rfloor = 3$, $\lfloor -8.9 \rfloor = -9$, $\lfloor 2 \rfloor = 2$, and so on.
   Suppose that $X \sim$ Exponential(1). Determine the expected value of $Y = \lfloor X \rfloor$.

2. Let X be a non-negative and continuous random variable with PDF f_X and CDF F_X. Show that
$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(\{X > x\})\, \mathrm{d}x = \int_0^\infty (1 - F_X(x))\, \mathrm{d}x,$$
   where the above integrals are usual Riemann integrals.

   Hint: Write down the formula for expectation in terms of the PDF, and apply change of order of integration.

3. Suppose that X and Y are jointly discrete random variables. The random variable X takes values in $\{-1, 0, 1\}$ with uniform probabilities. Suppose that for each $x \in \{-1, 0, 1\}$,
$$p_{Y|X=x}(y) = \frac{1}{2} \mathbf{1}_{\{|y-x|=1\}}, \qquad y \in \mathbb{R}.$$
   Compute $\mathbb{E}[Y]$.

4. Let $X_1, X_2, \ldots \overset{\text{i.i.d.}}{\sim}$ Exponential(λ), and let $N \sim$ Geometric(p) be independent of $\{X_1, X_2, \ldots\}$. Here, $\lambda > 0$ and $p \in (0, 1)$ are fixed constants. Compute $\mathbb{E}\left[\sum_{i=1}^N X_i\right]$.

5. (a) Let X and Y be jointly continuous with the joint PDF
$$f_{X,Y}(x, y) = \begin{cases} cx^2 + \dfrac{xy}{3}, & 0 \le x \le 1,\ 0 \le y \le 2, \\ 0, & \text{otherwise.} \end{cases}$$
      i. Find the constant c.
      ii. Are X and Y independent?
      iii. Calculate Cov(X, Y).

   (b) Let X and Y be independent random variables distributed uniformly on $[0, 1]$.
   Let $U = \min\{X, Y\}$ and $V = \max\{X, Y\}$. Calculate Cov(U, V).

6. Let $X \sim \mathcal{N}(0, 1)$. Let W be a discrete random variable independent of X and having the PMF
$$\mathbb{P}(\{W = w\}) = \begin{cases} \frac{1}{2}, & w = \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$
   Define a new random variable Y as $Y = WX$.

   (a) Show that $Y \sim \mathcal{N}(0, 1)$.
   (b) Show that X and Y are uncorrelated, but not independent.
   (c) A friend of yours comes to you and claims that $Z = X + Y$ is Gaussian distributed. Is your friend's claim correct?

7. Fix $n \in \mathbb{N}$, $n \ge 2$.
   Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed with finite mean μ and variance σ^2.
   Define the sample mean M_n and sample variance V_n as the random variables
$$M_n := \frac{1}{n} \sum_{i=1}^n X_i, \qquad V_n := \frac{1}{n-1} \sum_{i=1}^n (X_i - M_n)^2.$$

(a) Show that $\mathbb{E}[M_n] = \mu$.

(b) Show that $\mathbb{E}[V_n] = \sigma^2$ (the factor $(n-1)$ in the denominator of V_n is precisely to ensure that the mean of V_n is equal to σ^2).

(c) Show that $\text{Var}(M_n) = \frac{\sigma^2}{n}$.

8. Suppose that X, Y, and Z are three random variables defined with respect to $\mathscr{F}$.
Let the means of Y and Z be μ_Y and μ_Z respectively. Show that

$$\mathbb{E}[\max\{X, \mu_Y\} - \max\{X, \mu_Z\}] \leq |\mu_Y - \mu_Z| \cdot \mathbb{P}\left(\left\{X \in \left[\min\{\mu_Y, \mu_Z\}, \max\{\mu_Y, \mu_Z\}\right]\right\}\right).$$

Hint: Consider the cases $\mu_Y < \mu_Z$ and $\mu_Y \geq \mu_Z$ separately.
For each case, break down the sample space into events of the form $\{X < \mu_Y\}, \{\mu_Y \leq X \leq \mu_Z\}, \{X > \mu_Z\}$.
On each of these events, upper bound the mean value of $\max\{X, \mu_Y\} - \max\{X, \mu_Z\}$.