AI 5030: Probability and Stochastic Processes

Instructor: Dr. Karthik P. N.

Homework 11

Topic: Multivariate Gaussian

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables appearing below are assumed to be defined with respect to $\mathscr{F}$.

1. Let X and Y have the joint density function

$$f_{X,Y}(x,y) = c \cdot 2e^{-0.5(x^2+4y^2-2xy)}, \quad -\infty < x, y < +\infty.$$

   Find the marginal PDFs of X and Y, and the conditional PDF of X, conditioned on the event $\{Y = y\}$.

2. Suppose that X and Y are jointly Gaussian with means μ_1 and μ_2, variances σ_1^2 and σ_2^2, and correlation coefficient ρ. Find a necessary and sufficient condition for $X+Y$ and $X-Y$ to be independent.

3. Suppose that X and Y have the following joint PDF: for all $x, y \in \mathbb{R}$,

$$f_{X,Y}(x,y) = \frac{1}{2}\left(\frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(x^2 - 2\rho xy + y^2\right)\right) + \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left(x^2 + 2\rho xy + y^2\right)\right)\right).$$

   (a) Determine the marginal PDFs of X and Y. Also compute $\mathbb{E}[XY]$.

   (b) Are X and Y jointly Gaussian? Justify.

   (c) Determine whether X and Y are uncorrelated.

4. Let X and Y be jointly Gaussian with $\mathbb{E}[X] = \mathbb{E}[Y] = 0$, $\mathsf{Var}(X) = \mathsf{Var}(Y) = 1$, and correlation coefficient ρ. Show that $Z = \frac{X}{Y}$ has a Cauchy distribution.
   Note: A Cauchy distribution with parameters μ and γ is given by:

$$f(x) = \frac{\gamma}{\pi\left((x-\mu)^2 + \gamma^2\right)}, \qquad x \in \mathbb{R}.$$

5. Let X_1, X_2, X_3, X_4 be i.i.d. Gaussian random variables with zero mean and unit variance. Show that the PDF of $Y = X_1 X_2 + X_3 X_4$ is given by

$$f(y) = 0.5e^{-|y|}, \quad -\infty < y < \infty.$$

   (Hint: Compute the moment-generating function of Y.)

6. Let X and Y be jointly Gaussian with mean vector $\boldsymbol{\mu} = \mathbf{0}$ and covariance matrix K, where

$$K = \begin{pmatrix} \sigma_X^2 & \rho\,\sigma_X\,\sigma_Y \\ \rho\,\sigma_X\,\sigma_Y & \sigma_Y^2 \end{pmatrix}.$$

   Suppose that $|\rho| < 1$ (thereby implying that K is invertible). Show that the conditional PDF $f_{Y|X=x}(y)$ is the PDF of a one-dimensional Gaussian distribution with mean $x\rho\,\frac{\sigma_Y}{\sigma_X}$ and variance $\sigma_Y^2(1-\rho^2)$.