AI 5030: Probability and Stochastic Processes

Instructor: Dr. Karthik P. N.

Homework 1

Topics: Functions, Cardinality, Countability

1. Show that $2^{\mathbb{N}}$ is uncountable; here, $2^{\mathbb{N}}$ denotes the power set of $\mathbb{N}$.
   Hint: Construct a bijection between $\{0,1\}^{\mathbb{N}}$ and $2^{\mathbb{N}}$, and use the fact from class that $\{0,1\}^{\mathbb{N}}$ is uncountable.

2. Show that $\mathbb{N}^2 := \mathbb{N} \times \mathbb{N} = \{(m,n) : m,n \in \mathbb{N}\}$ is countably infinite.
   Using the principle of mathematical induction, show that $\mathbb{N}^d$ is countably infinite for each $d \in \mathbb{N}$.
   Hint: For the case $n = 2$, consider $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ defined as $g(m,n) = (m+n)^2 + n$. Argue that g is bijective.

3. Show that $\mathbb{N}^{\mathbb{N}} := \underbrace{\mathbb{N} \times \mathbb{N} \times \cdots}_{\text{countably infinitely many cartesian products}}$ is uncountable. (Hint: Construct an injection from $\{0,1\}^{\mathbb{N}}$ to $\mathbb{N}^{\mathbb{N}}$.)

4. Show that for any set A (finite, countably infinite, or uncountable), $|2^A| > |A|$, where 2^A is the power set of A.
   Note: This result demonstrates that there are different levels of infinity. Thus, for instance,
   $$|\mathbb{N}| < |2^{\mathbb{N}}| = |\mathbb{R}| < |2^{\mathbb{R}}| < |2^{2^{\mathbb{R}}}| \cdots$$

5. Fix a countable set A.
   (a) For any $n \in \mathbb{N}$, let B_n denote the collection of all possible n-tuples of the form $(a_1, a_2, \ldots, a_n)$, where $a_k \in A$ for each $k \in \{1, 2, \ldots, n\}$. Show that B_n is countable.
   Hence argue that $\bigcup_{n \in \mathbb{N}} B_n$ is countable.
   (b) A real number $x_0 \in \mathbb{R}$ is called *algebraic* if it is a root of a polynomial with integer coefficients. For example, $x_0 = \sqrt{2}$ is an algebraic number, as it is a root of the polynomial $x^2 - 2 = 0$ (whose coefficients are $1, -2$).
   Using the result in part (a) above, show that the set of all algebraic numbers is countable.
   Hint: Show that the there are only countably many polynomials with integer coefficients.

6. Let $\mathscr{C}$ denote the collection of all finite length binary strings. Is $\mathscr{C}$ countable?