Probability and Stochastic Processes

Jointly Continuous Random Variables, Joint PDF, Conditional PDF,

Transformations of Random Variables

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Jointly Continuous Random Variables

Jointly Continuous Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be random variables defined with respect to $\mathscr{F}$.

Definition (Jointly Continuous Random Variables)

X and Y are said to be jointly continuous if $(X, Y) : \Omega \to \mathbb{R}^2$ is a continuous random variable, i.e., there exists a function $f_{X,Y} : \mathbb{R}^2 \to [0, +\infty)$ such that the joint CDF of X and Y may be expressed as

$$F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u, v)\, dv\, du \qquad \forall x, y \in \mathbb{R}.$$

The function $f_{X,Y}$ is called the joint PDF of X and Y.

Remark:

X continuous, Y continuous $\not\Longrightarrow$ X, Y jointly continuous

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Properties of Joint PDF

- $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{X,Y}(u, v)\, dv\, du = 1.$

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Conditional CDF and Conditional PDF

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be jointly continuous random variables defined with respect to $\mathscr{F}$.

Conditional CDF and Conditional PDF

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be jointly continuous random variables defined with respect to $\mathscr{F}$.

Conditional CDF of X conditioned on $\{Y = y\}$: $\quad \mathbb{P}(\{X \leq x\} | \{Y = y\})$.
However, this conditional probability is not defined because $\mathbb{P}(\{Y = y\}) = 0$.

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Remedy:
Fix $y \in \mathbb{R}$ and $\varepsilon > 0$ such that $\mathbb{P}\big(\{Y \in (y - \varepsilon, y + \varepsilon)\}\big) > 0$.
Define conditional probability with respect to the event $\{Y \in (y - \varepsilon, y + \varepsilon)\}$, and let $\varepsilon \downarrow 0$.

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Conditional CDF and Conditional PDF

$$\mathbb{P}(\{X \le x\} | \{Y \in (y-\varepsilon, y+\varepsilon\}) = \frac{\mathbb{P}(\{X \le x\} \cap \{Y \in (y-\varepsilon, y+\varepsilon)\})}{\mathbb{P}(\{Y \in (y-\varepsilon, y+\varepsilon)\})}$$

Conditional CDF and Conditional PDF

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$$= \frac{\int\limits_{-\infty}^{x} \int\limits_{y-\varepsilon}^{y+\varepsilon} f_{X,Y}(u,v)\, dv\, du}{\int\limits_{y-\varepsilon}^{y+\varepsilon} f_Y(v)\, dv}$$

Conditional CDF and Conditional PDF

$$\begin{aligned}
\mathbb{P}(\{X \leq x\} | \{Y \in (y-\varepsilon, y+\varepsilon\}) &= \frac{\mathbb{P}(\{X \leq x\} \cap \{Y \in (y-\varepsilon, y+\varepsilon)\})}{\mathbb{P}(\{Y \in (y-\varepsilon, y+\varepsilon)\})} \\
&= \frac{\int\limits_{-\infty}^{x} \int\limits_{y-\varepsilon}^{y+\varepsilon} f_{X,Y}(u,v)\, dv\, du}{\int\limits_{y-\varepsilon}^{y+\varepsilon} f_Y(v)\, dv} \\
&\approx \frac{\int\limits_{-\infty}^{x} f_{X,Y}(u,y)\, du \cdot 2\varepsilon}{f_Y(y) \cdot 2\varepsilon}
\end{aligned}$$

Conditional CDF and Conditional PDF

$$\mathbb{P}(\{X \leq x\} | \{Y \in (y-\varepsilon, y+\varepsilon\}) = \frac{\mathbb{P}(\{X \leq x\} \cap \{Y \in (y-\varepsilon, y+\varepsilon)\})}{\mathbb{P}(\{Y \in (y-\varepsilon, y+\varepsilon)\})}$$

$$= \frac{\int\limits_{-\infty}^{x} \int\limits_{y-\varepsilon}^{y+\varepsilon} f_{X,Y}(u,v)\,dv\,du}{\int\limits_{y-\varepsilon}^{y+\varepsilon} f_Y(v)\,dv}$$

$$\approx \frac{\int\limits_{-\infty}^{x} f_{X,Y}(u,y)\,du \cdot 2\varepsilon}{f_Y(y) \cdot 2\varepsilon}$$

$$= \int_{-\infty}^{x} \underbrace{\frac{f_{X,Y}(u,y)}{f_Y(y)}}_{\text{conditional PDF}}\,du$$

Conditional CDF for Jointly Continuous Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be jointly continuous random variables defined with respect to $\mathscr{F}$.

Definition (Conditional CDF for Jointly Continuous Random Variables)

The conditional CDF of X, conditioned on the event $\{Y = y\}$, is the function $F_{X|Y=y} : \mathbb{R} \to [0, 1]$ defined as

$$F_{X|Y=y}(x) = \int_{-\infty}^{x} \frac{f_{X,Y}(u, y)}{f_Y(y)}\, du, \qquad x \in \mathbb{R},$$

defined for all $y \in \mathbb{R}$ such that $f_Y(y) > 0$.

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Conditional PDF for Jointly Continuous Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be jointly continuous random variables defined with respect to $\mathscr{F}$.

Definition (Conditional PDF for Jointly Continuous Random Variables)

The conditional PDF of X, conditioned on the event $\{Y = y\}$, is the function $f_{X|Y=y} : \mathbb{R} \to [0, +\infty)$ defined as

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \qquad x \in \mathbb{R},$$

defined for all $y \in \mathbb{R}$ such that $f_Y(y) > 0$.

Independence and Joint Continuity

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be jointly continuous random variables defined with respect to $\mathscr{F}$.

Definition (Joint Continuity and Independence)

X and Y are independent if

$$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y) \qquad \forall x, y \in \mathbb{R}.$$

Remark:

- $X \perp\!\!\!\perp Y \quad \Longleftrightarrow \quad f_{X|Y=y} = f_X$ for all y such that $f_Y(y) > 0$

Conditional PMF Summary

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let X and Y be random variables defined with respect to $\mathscr{F}$.

- If X and Y are jointly discrete,

$$p_{X|Y=y}(x) = \frac{p_{X,Y}(x,y)}{p_Y(y)}, \qquad x \in \mathbb{R},\ p_Y(y) > 0.$$

- For any event $A \in \mathscr{F}$,

$$\mathbb{P}(\{X \in A\} | \{Y = y\}) = \sum_{x \in A} p_{X|Y=y}(x).$$

- For any events $A, B \in \mathscr{F}$,

$$\mathbb{P}(\{X \in A\} | \{Y \in B\}) = \frac{\mathbb{P}(\{X \in A\} \cap \{Y \in B\})}{\mathbb{P}(\{Y \in B\})} = \frac{\sum_{x \in A} \sum_{y \in B} p_{X,Y}(x,y)}{\sum_{y \in B} p_Y(y)}$$

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Conditional PDF Summary

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X and Y be random variables defined with respect to $\mathscr{F}$.

- If X and Y are jointly continuous,

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \qquad x \in \mathbb{R},\ f_Y(y) > 0.$$

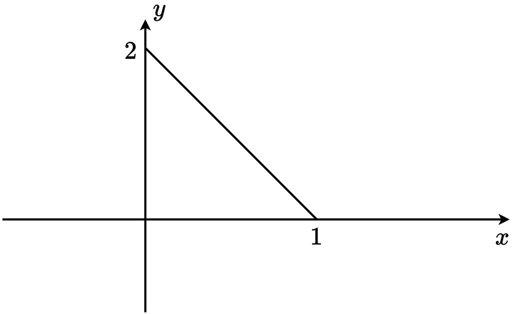
- For any event $A \in \mathscr{F}$,

$$\mathbb{P}(\{X \in A\}|\{Y = y\}) = \int_A f_{X|Y=y}(u)\,du.$$

- For any events $A, B \in \mathscr{F}$,

$$\mathbb{P}(\{X \in A\}|\{Y \in B\}) = \frac{\mathbb{P}(\{X \in A\} \cap \{Y \in B\})}{\mathbb{P}(\{Y \in B\})} = \frac{\int_{x \in A} \int_{y \in B} f_{X,Y}(x,y)\,dy\,dx}{\int_{y \in B} f_Y(y)\,dy}$$

Example



Let $f_{X,Y}(x, y) = 1$ inside the triangle, and 0 elsewhere.
Compute the marginal PDFs of X and Y, and the conditional PDF of X conditioned on $\{Y = y\}$ for various values of y. Argue if X and Y are independent.

Transformations of Random Variables

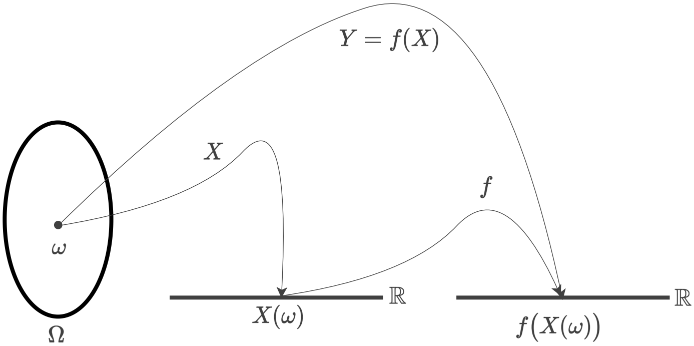
Transformations of Random Variables

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable defined with respect to $\mathscr{F}$.
Consider $f : \mathbb{R} \to \mathbb{R}$.
Define $Y : \Omega \to \mathbb{R}$ as $Y = f(X)$, i.e.,

$$Y(\omega) = f(X(\omega)), \qquad \omega \in \Omega.$$

- For what functions f is $Y = f(X)$ a random variable with respect to $\mathscr{F}$?
- Given the CDF/PMF/PDF of X, what is the CDF/PMF/PDF of $Y = f(X)$?

Picture



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Borel-Measurable Functions – 1

Definition (Borel-Measurable Function)

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be Borel-measurable if

$$f^{-1}(B) = \{x \in \mathbb{R} : f(x) \in B\} \in \mathscr{B}(\mathbb{R}) \qquad \forall B \in \mathscr{B}(\mathbb{R}).$$

Remarks:

- Every continuous function is Borel-measurable. Thus,

$$f(x) = |x|, \quad f(x) = x^2, \quad f(x) = e^x, \quad f(x) = \log x,$$

  are Borel-measurable
- $X : \Omega \to \mathbb{R}$ random variable wrt $\mathscr{F}$ $\qquad$ $f : \mathbb{R} \to \mathbb{R}$ Borel-measurable
  $\implies f(X) : \Omega \to \mathbb{R}$ random variable with respect to $\mathscr{F}$

Borel-Measurable Functions – 2

Definition (Borel-Measurable Function)

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be Borel-measurable if

$$f^{-1}(B) = \{x \in \mathbb{R}^n : f(x) \in B\} \in \mathscr{B}(\mathbb{R}^n) \qquad \forall B \in \mathscr{B}(\mathbb{R}^m).$$

Implication for $m = 1$:

- $X_1, \ldots, X_n$ random variables wrt $\mathscr{F}$, $f : \mathbb{R}^n \to \mathbb{R}$ Borel-measurable
  $\implies f(X_1, \ldots, X_n) : \Omega \to \mathbb{R}$ random variable with respect to $\mathscr{F}$
- Every continuous function $f : \mathbb{R}^n \to \mathbb{R}$ is Borel-measurable. Thus, for instance,

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i$$

  is Borel-measurable

Maximum of Random Variables

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $X_1, \ldots, X_n$ be random variables defined with respect to $\mathscr{F}$, with joint CDF $F_{X_1,\ldots,X_n}$.

- Show that $Y_n = \max\{X_1, \ldots, X_n\}$ is a random variable with respect to $\mathscr{F}$.
- Derive the CDF of Y_n.
- Simplify the CDF of Y_n when $X_1, \ldots, X_n$ are i.i.d..

Minimum of Random Variables

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $X_1, \ldots, X_n$ be random variables defined with respect to $\mathscr{F}$, with joint CDF $F_{X_1,\ldots,X_n}$.

- Show that $Z_n = \min\{X_1, \ldots, X_n\}$ is a random variable with respect to $\mathscr{F}$.
- Derive the CDF of Z_n.
- Simplify the CDF of Z_n when $X_1, \ldots, X_n$ are i.i.d..

Minimum of i.i.d. Exponential Random Variables

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X_1, \ldots, X_n$ be independent, with $X_i \sim \mathrm{Exponential}(\lambda_i)$ for each $i \in \{1, \ldots, n\}$.
Find the distribution of $Z = \min\{X_1, \ldots, X_n\}$.

Sums of Random Variables

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X and Y be random variables with respect to $\mathscr{F}$.

- Show that $X + Y$ is a random variable with respect to $\mathscr{F}$.
- In the cases when X and Y are jointly discrete/continuous, derive the PMF/PDF of $X + Y$.
- Simplify the PMF/PDF when X and Y are independent.

Sum of Two Independent Exponentials

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X \sim \text{Exponential}(\mu_1)$ and $Y \sim \text{Exponential}(\mu_2)$. Assume $X \perp\!\!\!\perp Y$.
Determine the distribution of $Z = X + Y$.

Sum of Random Number of Random Variables

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $\{X_i : i \in \mathbb{N}\}$ be a collection of i.i.d. random variables defined with respect to $\mathscr{F}$ and having a common CDF F.

Let N be a positive integer-valued random variable defined with respect to $\mathscr{F}$ and having the PMF p_N.

Let N be independent of $\{X_i : i \in \mathbb{N}\}$.

Consider the sum

$$S_N := \sum_{i=1}^{N} X_i; \qquad S_N(\omega) = \sum_{i=1}^{N(\omega)} X_i(\omega), \quad \omega \in \Omega.$$

- Show that $S_N : \Omega \to \mathbb{R}$ is a random variable with respect to $\mathscr{F}$.
- Determine the CDF of S_N.