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Probability and Stochastic Processes

Lecture 01: Functions, Cardinality, Countability

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Functions

Definition (Function)

Given two sets A, B, a function $f : A \to B$ is a rule that maps each element of A to a unique element of B.

- For every $x \in A$,

$$f : x \mapsto f(x) \in B$$

- A is called the domain of f
- B is called the co-domain of f

Note

While every element of A is mapped to some element of B, the converse may not always be true.

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Range of a Function

Definiton (Range)

The range of a function $f : A \to B$, denoted by $R(f)$, is the subset of B defined as

$$R(f) = \Big\{ y \in B : y = f(x) \text{ for some } x \in A \Big\}.$$

- Given $x \in A$, if $f(x) = y$, then y is called the image of x (under f)
- Given $y \in B$, the set $f^{-1}(y) := \{x \in A : f(x) = y\}$ is called the pre-image of y

Image and Pre-Image

- A function $f : A \to B$ is said to be injective if f is *one-one*, i.e., each element of $R(f)$ has a unique pre-image
- A function $f : A \to B$ is said to be surjective if it is *onto*, i.e., range $=$ codomain
- A function $f : A \to B$ is said to be bijective if it is both injective and surjective

Note

- If $f : A \to B$ is bijective, then for each $y \in B$, there exists a unique element $x \in A$ such that $f^{-1}(y) = \{x\}$. In this case, we simply write $f^{-1}(y) = x$.
- Alternatively, if $f : A \to B$ is bijective, we have $f^{-1} : B \to A$.

Cardinality

Definition (Cardinality)

Notation: $|A| =$ cardinality of set A

- Two sets A and B are said to be equicardinal ($|A| = |B|$) if there exists $f : A \to B$ bijective.
- $|B| \geq |A|$ if there exists $f : A \to B$ injective
- $|B| > |A|$ if there exists $f : A \to B$ injective, and A and B are not equicardinal (i.e., no bijective function mapping A to B exists)

Note

$|A|$ is representative of the number of elements in A.

Countability

- A set A is said to be finite if A is empty or $|A| = |\{1, \ldots, n\}| = n$ for some $n \in \mathbb{N}$
- A set A is said to be countably infinite if $|A| = |\mathbb{N}|$, where $\mathbb{N} = \{1, 2, \ldots\}$ denotes the set of natural numbers
- A set A is countable if either $|A| < +\infty$ or $|A| = |\mathbb{N}|$

Remark

If A is countably infinite, then its elements may be listed as $A = \{a_1, a_2, \ldots\}$.

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Examples of Countable Sets

- Set of odd natural numbers, set of even natural numbers
- Set of integers, $\mathbb{Z} = \{0, +1, -1, +2, -2, \ldots\}$
- Set of prime numbers
- Set of rational numbers, $\mathbb{Q}$

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$\mathbb{Q}$ is Countable – Proof

Step 1: $\mathbb{Q} \cap [0, 1]$ is countable. Indeed, note that

$$\mathbb{Q} \cap [0, 1] = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \dots\right\}.$$

Step 2: "Countable union of countable sets is countable."

Lemma

Let $\mathcal{I}$ be a countable index set, and let $\{A_i : i \in \mathcal{I}\}$ be a countable collection of countable sets. Then, $\bigcup_{i \in \mathcal{I}} A_i$ is countable.

Step 3: Complete the proof using the above lemma.

Examples:

1. $f : \mathbb{R} \to \mathbb{R}$, $A \subseteq \mathbb{R}$

$$f(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

- Image of $\frac{1}{2} = \begin{cases} 1, & \text{if } \frac{1}{2} \in A \\ 0, & \text{o.w.} \end{cases}$

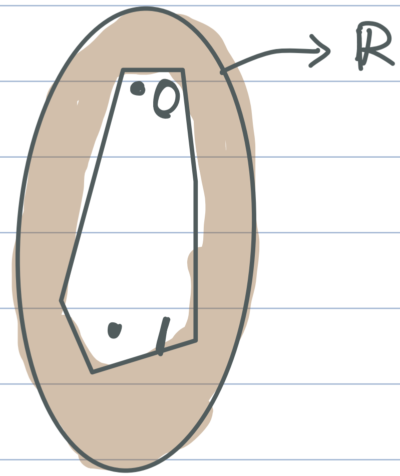
- Pre-image of $\frac{1}{2} = \{x \in \mathbb{R} : f(x) = \frac{1}{2}\} = \{ \ \} = \phi$

emptyset

- Suppose $C \subseteq \mathbb{R}$. Then,

$$f^{-1}(C) = \{x \in \mathbb{R} : f(x) \in C\}$$

$$= \begin{cases} A, & C = \{1\} \\ \mathbb{R} \setminus A, & C = \{0\} \\ \mathbb{R}, & C = \{0,1\} \\ \phi, & C \subseteq \mathbb{R} \setminus \{0,1\} \end{cases}$$



Examples :

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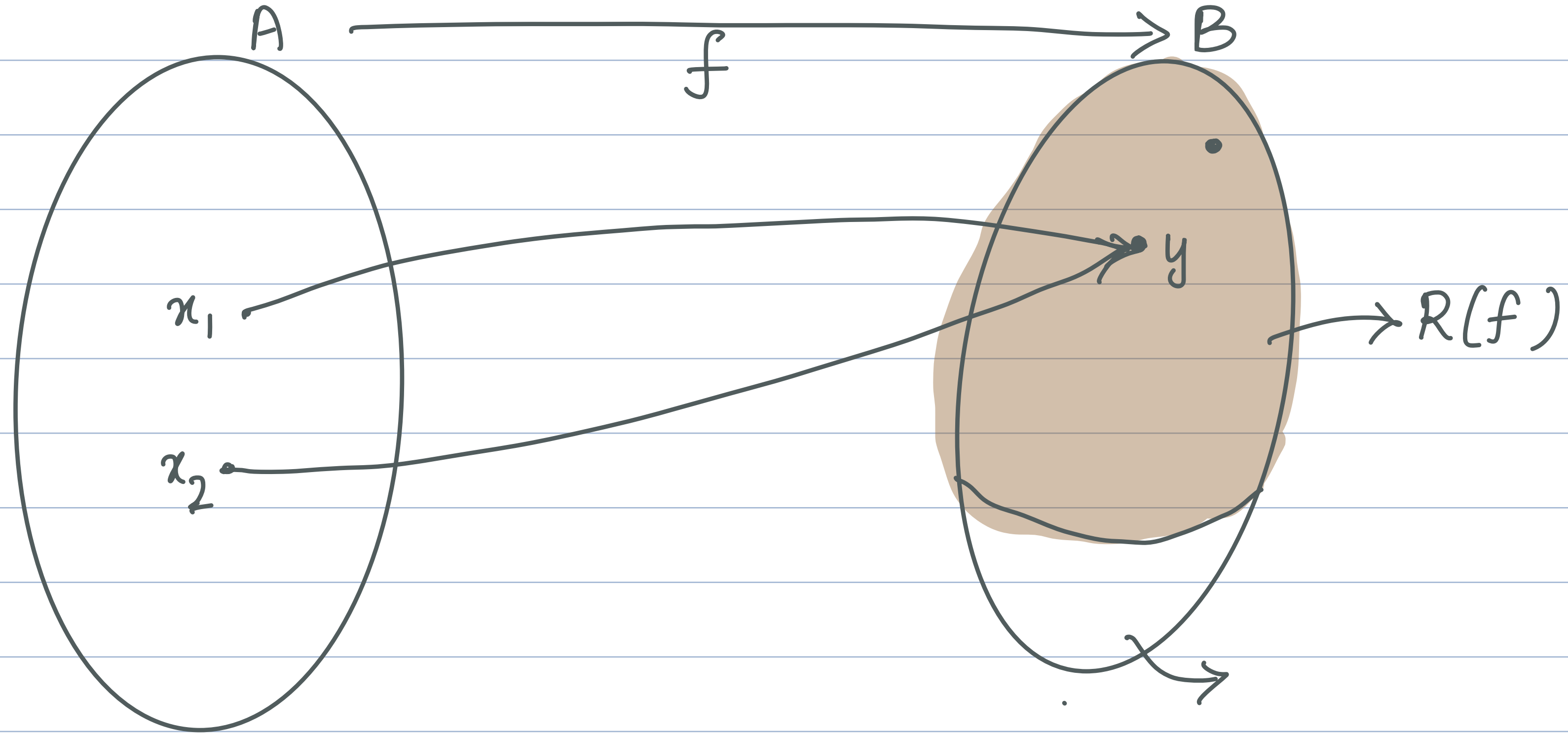
- Pre-image of $\frac{1}{2}$ $= \{x \in \mathbb{R} : f(x) = \frac{1}{2}\} = \{\ \} = \phi$ ← emptyset

- $$f^{-1}([-1,2]) = \{x \in \mathbb{R} : f(x) \in [-1,2]\}$$
$$= \{x \in \mathbb{R} : -1 \leq f(x) \leq 2\}$$
$$= \mathbb{R}.$$

- $$f^{-1}((2,\infty)) = \{x \in \mathbb{R} : f(x) \in (2,\infty)\}$$
$$= \{x \in \mathbb{R} : f(x) > 2\}$$
$$= \phi$$



$$A_y = f^{-1}(y), \quad y \in R(f)$$

$\{A_y : y \in R(f)\}$ - collection of disjoint sets

$$\bigcup_{y \in R(f)} A_y = A$$

Uncountable Sets

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Definition (uncountable sets)

A set A is said to be uncountable if it is not countable, i.e., if $|A| > |\mathbb{N}|$.

A B

$|B| > |A|$ iff i) $\exists$ injection $f : A \to B$

ii) $\nexists$ bijection

Some examples of uncountable sets:

- Unit interval, $[0, 1]$
- Set of all real numbers, $\mathbb{R}$
- Set of all irrational numbers, $\mathbb{R} \setminus \mathbb{Q}$
- Set of all infinite length binary strings, denoted commonly as $\{0, 1\}^{\mathbb{N}}$ or $\{0, 1\}^{\infty}$
- Power set of $\mathbb{N}$ (collection of all subsets of $\mathbb{N}$), denoted $2^{\mathbb{N}}$

$\{0,1\}^{\mathbb{N}}$ is Uncountable – Proof

It suffices to demonstrate that there exists an injective map but no bijective map from $\mathbb{N}$ to $\{0,1\}^{\mathbb{N}}$. $= S$

i) $\exists\ f : \mathbb{N} \to S$ injective

ii) $\nexists$ bijection

$$
\begin{aligned}
1 &\longrightarrow 1000\cdots0 = 1\bar{0} \\
2 &\longrightarrow 01\bar{0} \\
3 &\longrightarrow 001\bar{0} \\
4 & \\
5 & \\
\vdots &
\end{aligned}
$$

$\{0, 1\}^{\mathbb{N}}$ is Uncountable – Proof

It suffices to demonstrate that there exists an injective map but no bijective map from $\mathbb{N}$ to $\{0, 1\}^{\mathbb{N}}$.

Injective map: Define $f : \mathbb{N} \to \{0, 1\}^{\mathbb{N}}$ by

$$f(n) = \text{infinite binary string with `1' in the } n\text{th index.}$$

$\{0,1\}^{\mathbb{N}}$ is Uncountable – Proof

It suffices to demonstrate that there exists an injective map but no bijective map from $\mathbb{N}$ to $\{0,1\}^{\mathbb{N}}$.

Injective map: Define $f : \mathbb{N} \to \{0,1\}^{\mathbb{N}}$ by

$$f(n) = \text{infinite binary string with ‘1’ in the } n\text{th index.}$$

No bijective map: Suppose there exists a bijective map $g : \mathbb{N} \to \{0,1\}^{\mathbb{N}}$. Let

$$g : n \mapsto a_{n1}\, a_{n2}\, a_{n3} \cdots,$$

where $a_{nj} \in \{0,1\}$ for all n, j.

$$\begin{array}{lllll}
1 \longrightarrow & a_{11} & a_{12} & a_{13} & a_{14} \cdots \\
2 \longrightarrow & a_{21} & a_{22} & a_{23} & a_{24} \cdots \\
3 \longrightarrow & a_{31} & a_{32} & a_{33} & a_{34} \cdots \\
4 \longrightarrow & a_{41} & a_{42} & a_{43} & a_{44} \cdots \\
\vdots & & & &
\end{array}$$

$\bar{a}_{11}\ \bar{a}_{22}\ \bar{a}_{33}\ \bar{a}_{44} \cdots$

$\bar{a}_{jj} = 1 - a_{jj}$

$\{0,1\}^{\mathbb{N}}$ is Uncountable – Proof

$2^{\mathbb{N}} \to$ exercise

It suffices to demonstrate that there exists an injective map but no bijective map from $\mathbb{N}$ to $\{0,1\}^{\mathbb{N}}$.

Injective map: Define $f : \mathbb{N} \to \{0,1\}^{\mathbb{N}}$ by

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No bijective map: Suppose there exists a bijective map $g : \mathbb{N} \to \{0,1\}^{\mathbb{N}}$. Let

$$g : n \mapsto a_{n1}\, a_{n2}\, a_{n3}\, \cdots ,$$

where $a_{nj} \in \{0,1\}$ for all n, j.

Cantor's diagonalisation argument: Consider the binary string

$$b = \bar{a}_{11}\, \bar{a}_{22}\, \bar{a}_{33} \cdots ,$$

where $\bar{a}_{jj} = 1 - a_{jj}$ for all $j \in \mathbb{N}$. Then, $\nexists\, n \in \mathbb{N}$ such that $g(n) = b$. Thus, g is not a bijection.

$[0, 1]$ is Uncountable – Proof

Let

$$\mathcal{D} = \left\{ d_1 = \frac{1}{2}, d_2 = \frac{1}{4}, d_3 = \frac{3}{4}, d_4 = \frac{1}{8}, \ldots \right\} \quad - \quad \text{set of dyadic rational numbers}$$

$0 \to 0000\ldots$

$1 \to 1111\ldots$

$\frac{1}{2} = 0.100000\ldots$

$0.a_1 a_2 a_3 a_4 \ldots$

$$\sum_{k=1}^{\infty} \frac{a_k}{2^k}$$

$$\frac{a}{2^b}$$

$\frac{1}{2} = 0.011111\ldots$

$\frac{1}{2} \to 1000\ldots$, $\frac{1}{2} \to 0111\ldots$

$\frac{1}{4} \to 0100\ldots$, $\frac{1}{4} \to 00111\ldots1$

$\frac{1}{8} \to 001000\ldots$, $\frac{1}{8} \to 0001111\ldots$

$[0,1]$ is Uncountable – Proof

Let

$$\mathcal{D} = \left\{ d_1 = \frac{1}{2}, d_2 = \frac{1}{4}, d_3 = \frac{3}{4}, d_4 = \frac{1}{8}, \ldots \right\} \quad - \quad \text{set of dyadic rational numbers}$$

Define $g : \{0,1\}^{\mathbb{N}} \to [0,1]$ defined as

$$g : b = (b_1\, b_2\, \cdots) \mapsto \begin{cases} \sum_{k=1}^{\infty} \frac{b_k}{2^k}, & b \notin \mathcal{D}, \\ d_1, & b = (100\cdots) \\ d_2, & b = (011\cdots) \\ d_3, & b = (0100\cdots) \\ d_4, & b = (0011\cdots) \\ \vdots & \end{cases}$$

Prove that g is a bijection!

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- $2^{\mathbb{N}}$ is uncountable – exercise!
- $\mathbb{R}$ is uncountable
  Consider the function $f : [0,1] \to \mathbb{R}$ defined via

$$f(x) = \tan\left(\pi x - \frac{\pi}{2}\right), \quad x \in [0,1].$$

- $\mathbb{R} \setminus \mathbb{Q}$ is uncountable
  Write $\mathbb{R}$ as

$$\mathbb{R} = \underline{(\mathbb{R} \setminus \mathbb{Q})} \cup \underline{\mathbb{Q}}.$$

Reading Exercise

To be acquainted with the formal proof of the lemma introduced on slide 7, see [Royden and Fitzpatrick, 2010, Section 1.3].

Royden, H. and Fitzpatrick, P. M. (2010).
Real Analysis.
China Machine Press.