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Probability and Stochastic Processes

Multiple Random Variables, Joint CDF and its Properties, Jointly Discrete Random Variables, Joint PMF, Conditional PMF, Jointly Continuous Random Variables, Joint PDF

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Multiple Random Variables

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Two Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Two Random Variables)

Given two $\mathscr{F}$-measurable random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$, we say $(X, Y) : \Omega \to \mathbb{R}^2$ is a random variable with respect to $\mathscr{F}$ if

$$(X, Y)^{-1}(B) = \{\omega \in \Omega : \big(X(\omega), Y(\omega)\big) \in B\} \in \mathscr{F} \qquad \forall B \in \mathscr{B}(\mathbb{R}^2).$$

$$\begin{tikzcd}
(\Omega, \mathscr{F}) \arrow[r, "{(X,Y)}"] & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2)) \arrow[d] \\
(X,Y)^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}$$

$$\forall B \in \mathscr{B}(\mathbb{R}^2), \quad (X, Y)^{-1}(B) \in \mathscr{F}$$

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Joint Probability Law of Two Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Joint Probability Law of Two Random Variables)

Given two random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ defined with respect to $\mathscr{F}$, their joint probability law $\mathbb{P}_{X,Y} : \mathscr{B}(\mathbb{R}^2) \to [0, 1]$, is the probability measure defined as

$$\mathbb{P}_{X,Y}(B) = \mathbb{P}(\{\omega \in \Omega : \big(X(\omega), Y(\omega)\big) \in B\}), \qquad B \in \mathscr{B}(\mathbb{R}^2).$$

Remarks:

- $\mathbb{P}_{X,Y}$ is called the pushforward of $\mathbb{P}$ under the random variable (X, Y)
- $\mathbb{P}_{X,Y}$ is the probability law of the random variable (X, Y)
- $\mathbb{P}_{X,Y}$ gives the full probabilistic description of (X, Y)

The Picture to Have in Mind

$$
\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "{(X,Y)}"] & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2), \color{red}{\mathbb{P}_{X,Y}}) \arrow[d] \\
(X,Y)^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}
$$

$$\color{red}{\mathbb{P}_{X,Y}}\color{black}(B) = \mathbb{P}((X,Y)^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R}^2)$$

Remarks

- A special class of sets in $\mathscr{B}(\mathbb{R}^2)$ are semi-infinite rectangles of the form

$$(-\infty, x] \times (-\infty, y], \qquad x, y \in \mathbb{R}.$$

- $\mathscr{B}(\mathbb{R}^2) = \sigma\big(\{(-\infty, x] \times (-\infty, y] : \; x, y \in \mathbb{R}\}\big)$

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Joint CDF of Two Random Variables

$$
\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[rr, "{(X,Y)}"] & & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2), \mathbb{P}_{X,Y}) \arrow[d] \\
\{X \le x\} \cap \{Y \le y\} \arrow[u] & & (-\infty, x] \times (-\infty, y] \arrow[ll, "\text{pre-image}"']
\end{tikzcd}
$$

$$\color{red}{F_{X,Y}}\color{black}(x, y) = \mathbb{P}_{X,Y}((-\infty, x] \times (-\infty, y]) = \mathbb{P}(\{X \le x\} \cap \{Y \le y\}), \quad x, y \in \mathbb{R}$$

Definition (Joint CDF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Given random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ with respect to $\mathscr{F}$, their joint CDF $F_{X,Y} : \mathbb{R}^2 \to [0, 1]$ is defined as

$$F_{X,Y}(x, y) = \mathbb{P}_{X,Y}\big((-\infty, x] \times (-\infty, y]\big) = \mathbb{P}(\{X \le x\} \cap \{Y \le y\}), \qquad x, y \in \mathbb{R}.$$

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Notation

- $\{X \leq x\} \cap \{Y \leq y\} = \{X \leq x,\ Y \leq y\}$
- $\mathbb{P}(\{X \leq x\} \cap \{Y \leq y\}) = \mathbb{P}(X \leq x, Y \leq y)$

Joint CDF $\longleftrightarrow$ Joint Probability Law

- If we know $\mathbb{P}_{X,Y} = \{\mathbb{P}_{X,Y}(B) : B \in \mathscr{B}(\mathbb{R}^2)\}$, then we can extract the CDF $F_{X,Y} : \mathbb{R}^2 \to [0,1]$ by using the formula

$$F_{X,Y}(x,y) = \mathbb{P}_{X,Y}\big((-\infty, x] \times (-\infty, y]\big), \qquad x, y \in \mathbb{R}.$$

- Given the joint CDF $F_{X,Y} : \mathbb{R}^2 \to [0,1]$, let

$$\mathbb{P}_{X,Y}\big((-\infty, x] \times (-\infty, y]\big) = F_{X,Y}(x,y), \qquad x, y \in \mathbb{R}.$$

  Then, by Caratheodory's extension theorem, there exists a unique extension of $\mathbb{P}_{X,Y}$ to all Borel subsets of $\mathbb{R}^2$

Properties of Joint CDF

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$
Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be random variables with respect to $\mathscr{F}$ with joint CDF $F_{X,Y}$

- $\lim_{x,y\to-\infty} F_{X,Y}(x,y) = 0, \qquad \lim_{x,y\to+\infty} F_{X,Y}(x,y) = 1$
- (Monotonicity) If $x_1 \le x_2$ and $y_1 \le y_2$, then $F_{X,Y}(x_1, y_1) \le F_{X,Y}(x_2, y_2)$
- $F_{X,Y}$ is continuous from the right and top, i.e., for all $x, y \in \mathbb{R}$,
$$\lim_{u\downarrow 0,\ v\downarrow 0} F_{X,Y}(x+u,\ y+v) = F_{X,Y}(x,y).$$
- $\lim_{y\to\infty} F_{X,Y}(x,y) = F_X(x)$ for all $x \in \mathbb{R}$
  $\lim_{x\to\infty} F_{X,Y}(x,y) = F_Y(y)$ for all $y \in \mathbb{R}$

Marginal Law/CDF from Joint Law/CDF

Given joint CDF/law, we may extract the marginal CDFs/laws
The converse is not possible in general

CDF	Law
$F_{X,Y} = \{F_{X,Y}(x,y) : x, y \in \mathbb{R}\}$	$\mathbb{P}_{X,Y} = \{\mathbb{P}_{X,Y}(B) : B \in \mathscr{B}(\mathbb{R}^2)\}$
$F_X(x) = \lim_{y\to+\infty} F_{X,Y}(x,y)$	$\mathbb{P}_X(A) = \mathbb{P}_{X,Y}(A \times \mathbb{R})$
$F_Y(y) = \lim_{x\to+\infty} F_{X,Y}(x,y)$	$\mathbb{P}_Y(A) = \mathbb{P}_{X,Y}(\mathbb{R} \times A)$

Table: Marginal law/CDF from joint law/CDF.

Multiple Random Variables

Fix a measurable space $(\Omega, \mathscr{F})$.

Definition (Multiple Random Variables)

Given random variables $X_1, \dots, X_n$ defined with respect to $\mathscr{F}$, we say $(X_1, \dots, X_n) : \Omega \to \mathbb{R}^n$ is a random variable with respect to $\mathscr{F}$ if

$$(X_1, \dots, X_n)^{-1}(B) = \{\omega \in \Omega : \big(X_1(\omega), \dots, X_n(\omega)\big) \in B\} \in \mathscr{F} \qquad \forall B \in \mathscr{B}(\mathbb{R}^n).$$

$$\begin{tikzcd}
(\Omega, \mathscr{F}) \arrow[r, "{(X_1, \dots, X_n)}"] & (\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n)) \arrow[d] \\
(X_1, \dots, X_n)^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}$$

$$\forall B \in \mathscr{B}(\mathbb{R}^n), \quad (X_1, \dots, X_n)^{-1}(B) \in \mathscr{F}$$

Joint Probability Law of Multiple Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Joint Probability Law of Multiple Random Variables)

Given two random variables $X_1, \ldots, X_n$ defined with respect to $\mathscr{F}$, their joint probability law is the probability measure $\mathbb{P}_{X_1,\ldots,X_n} : \mathscr{B}(\mathbb{R}^n) \to [0, 1]$ defined as

$$\mathbb{P}_{X_1,\ldots,X_n}(B) = \mathbb{P}(\{\omega \in \Omega : \big(X_1(\omega), \ldots, X_n(\omega)\big) \in B\}), \qquad B \in \mathscr{B}(\mathbb{R}^n).$$

Remarks:

- $\mathbb{P}_{X_1,\ldots,X_n}$ is the probability law of the random variable $(X_1, \ldots, X_n)$
- $\mathbb{P}_{X_1,\ldots,X_n}$ gives the full probabilistic description of $(X_1, \ldots, X_n)$

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Joint Probability Law of Multiple Random Variables

$$
\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[rr, "{(X_1, \dots, X_n)}"] & & (\mathbb{R}^n, \mathscr{B}(\mathbb{R}^n), {\color{red}\mathbb{P}_{X_1,\dots,X_n}}) \arrow[d] \\
(X_1, \dots, X_n)^{-1}(B) \arrow[u] & & B \arrow[ll, "\text{pre-image}"']
\end{tikzcd}
$$

$$
{\color{red}\mathbb{P}_{X_1,\dots,X_n}}(B) = \mathbb{P}((X_1, \dots, X_n)^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R}^n)
$$

Marginal Law/CDF from Joint Law/CDF

Given joint CDF/law, we may extract the marginal CDFs/laws

The converse is not possible in general

CDF	Law
$F_{X_1,\ldots,X_n} = \{F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) : x_1,\ldots,x_n \in \mathbb{R}\}$	$\mathbb{P}_{X_1,\ldots,X_n} = \{\mathbb{P}_{X_1,\ldots,X_n}(B) : B \in \mathscr{B}(\mathbb{R}^n)\}$
$F_{X_1}(x_1) = \lim_{\substack{x_2\to+\infty \\ x_3\to+\infty \\ \vdots \\ x_n\to+\infty}} F_{X_1,\ldots,X_n}(x_1,x_2,\ldots,x_n)$	$\mathbb{P}_{X_1}(A) = \mathbb{P}_{X_1,\ldots,X_n}(A \times \mathbb{R} \times \cdots \times \mathbb{R})$
$F_{X_2}(x_2) = \lim_{\substack{x_1\to+\infty \\ x_3\to+\infty \\ \vdots \\ x_n\to+\infty}} F_{X_1,\ldots,X_n}(x_1,x_2,\ldots,x_n)$	$\mathbb{P}_{X_2}(A) = \mathbb{P}_{X_1,\ldots,X_n}(\mathbb{R} \times A \times \mathbb{R} \times \cdots \times \mathbb{R})$

Table: Marginal law/CDF from joint law/CDF.

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Independence of Random Variables

Independence of Two Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Independence of Two Random Variables)

Two random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ defined with respect to $\mathscr{F}$ are said to be independent if

$$\{X \in B_1\} \perp\!\!\!\perp \{Y \in B_2\} \qquad \forall B_1, B_2 \in \mathscr{B}(\mathbb{R}).$$

That is,

$$\mathbb{P}(\{X \in B_1\} \cap \{Y \in B_2\}) = \mathbb{P}(\{X \in B_1\}) \cdot \mathbb{P}(\{Y \in B_2\}) \qquad \forall B_1, B_2 \in \mathscr{B}(\mathbb{R}).$$

Equivalently,

$$\mathbb{P}_{X,Y}(B_1 \times B_2) = \mathbb{P}_X(B_1) \cdot \mathbb{P}_X(B_2) \qquad \forall B_1, B_2 \in \mathscr{B}(\mathbb{R}).$$

Independence of Two Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Independence of Two Random Variables)

$$X \perp\!\!\!\perp Y \iff \{X \in B_1\} \perp\!\!\!\perp \{Y \in B_2\} \qquad \forall B_1, B_2 \in \mathscr{B}(\mathbb{R}).$$

B_1	B_2	$\{X \in B_1\}$	$\{Y \in B_2\}$	**Implication**
$(-\infty, x]$	$(-\infty, y]$			

Independence of Two Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Independence of Two Random Variables)

$$X \perp\!\!\!\perp Y \iff \{X \in B_1\} \perp\!\!\!\perp \{Y \in B_2\} \qquad \forall B_1, B_2 \in \mathscr{B}(\mathbb{R}).$$

B_1	B_2	$\{X \in B_1\}$	$\{Y \in B_2\}$	**Implication**
$(-\infty, x]$	$(-\infty, y]$	$\{X \leq x\}$	$\{Y \leq y\}$	$F_{X,Y}(x, y) = F_X(x) \cdot F_Y(y)$
$(-\infty, x]$	$(y, +\infty)$			

Independence of Two Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Independence of Two Random Variables)

$$X \perp\!\!\!\perp Y \quad \iff \quad \{X \in B_1\} \perp\!\!\!\perp \{Y \in B_2\} \qquad \forall B_1, B_2 \in \mathscr{B}(\mathbb{R}).$$

B_1	B_2	$\{X \in B_1\}$	$\{Y \in B_2\}$	**Implication**
$(-\infty, x]$	$(-\infty, y]$	$\{X \leq x\}$	$\{Y \leq y\}$	$F_{X,Y}(x, y) = F_X(x) \cdot F_Y(y)$
$(-\infty, x]$	$(y, +\infty)$	$\{X \leq x\}$	$\{Y > y\}$	$\mathbb{P}(\{X \leq x,\ Y > y\}) = \mathbb{P}(\{X \leq x\}) \cdot \mathbb{P}(\{Y > y\})$
$(-\infty, x]$	$\{y\}$			

Independence of Two Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Independence of Two Random Variables)

$$X \perp\!\!\!\perp Y \quad \iff \quad \{X \in B_1\} \perp\!\!\!\perp \{Y \in B_2\} \qquad \forall B_1, B_2 \in \mathscr{B}(\mathbb{R}).$$

B_1	B_2	$\{X \in B_1\}$	$\{Y \in B_2\}$	Implication
$(-\infty, x]$	$(-\infty, y]$	$\{X \leq x\}$	$\{Y \leq y\}$	$F_{X,Y}(x, y) = F_X(x) \cdot F_Y(y)$
$(-\infty, x]$	$(y, +\infty)$	$\{X \leq x\}$	$\{Y > y\}$	$\mathbb{P}(\{X \leq x,\ Y > y\}) = \mathbb{P}(\{X \leq x\}) \cdot \mathbb{P}(\{Y > y\})$
$(-\infty, x]$	$\{y\}$	$\{X \leq x\}$	$\{Y = y\}$	$\mathbb{P}(\{X \leq x,\ Y = y\}) = \mathbb{P}(\{X \leq x\}) \cdot \mathbb{P}(\{Y = y\})$
$(-\infty, x]$	(a, b)			

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Independence of Two Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Independence of Two Random Variables)

$$X \perp\!\!\!\perp Y \iff \{X \in B_1\} \perp\!\!\!\perp \{Y \in B_2\} \qquad \forall B_1, B_2 \in \mathscr{B}(\mathbb{R}).$$

B_1	B_2	$\{X \in B_1\}$	$\{Y \in B_2\}$	**Implication**
$(-\infty, x]$	$(-\infty, y]$	$\{X \leq x\}$	$\{Y \leq y\}$	$F_{X,Y}(x, y) = F_X(x) \cdot F_Y(y)$
$(-\infty, x]$	$(y, +\infty)$	$\{X \leq x\}$	$\{Y > y\}$	$\mathbb{P}(\{X \leq x,\ Y > y\}) = \mathbb{P}(\{X \leq x\}) \cdot \mathbb{P}(\{Y > y\})$
$(-\infty, x]$	$\{y\}$	$\{X \leq x\}$	$\{Y = y\}$	$\mathbb{P}(\{X \leq x,\ Y = y\}) = \mathbb{P}(\{X \leq x\}) \cdot \mathbb{P}(\{Y = y\})$
$(-\infty, x]$	(a, b)	$\{X \leq x\}$	$\{a < Y < b\}$	$\ldots$

Table: Independence of two random variables from various angles.

Independence and Joint CDFs

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be random variables defined with respect to $\mathscr{F}$.

Proposition (Independence and Joint CDFs)

The following statements are equivalent.

1. $\mathbb{P}_{X,Y}(B_1 \times B_2) = \mathbb{P}_X(B_1) \cdot \mathbb{P}_Y(B_2)$ for all $B_1, B_2 \in \mathscr{B}(\mathbb{R})$.

2. $F_{X,Y}(x, y) = F_X(x) \cdot F_Y(y)$ for all $x, y \in \mathbb{R}$.

Independence and Joint CDFs

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be random variables defined with respect to $\mathscr{F}$.

Proposition (Independence and Joint CDFs)

The following statements are equivalent.

1. $\mathbb{P}_{X,Y}(B_1 \times B_2) = \mathbb{P}_X(B_1) \cdot \mathbb{P}_Y(B_2)$ for all $B_1, B_2 \in \mathscr{B}(\mathbb{R})$.
2. $F_{X,Y}(x, y) = F_X(x) \cdot F_Y(y)$ for all $x, y \in \mathbb{R}$.

Interpretation of 2 $\implies$ 1

If the joint probability law products out on the collection

$$\mathscr{D} = \left\{(-\infty,\ x] \times (-\infty,\ y] : x, y \in \mathbb{R}\right\},$$

then it has to product out on $\sigma(\mathscr{D}) = \mathscr{B}(\mathbb{R}^2)$ (by Caratheodory's extension theorem)

Independence of Multiple Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Independence of Multiple Random Variables)

1. Random variables $X_1, \ldots, X_n$, all defined with respect to $\mathscr{F}$, are independent if

$$\mathbb{P}_{X_1,\ldots,X_n}(B_1 \times \cdots \times B_n) = \prod_{i=1}^{n} \mathbb{P}_{X_i}(B_i) \qquad \forall B_1, \ldots, B_n \in \mathscr{B}(\mathbb{R}).$$

2. For an arbitrary index set $\mathcal{I}$, the collection of random variables $\{X_i : i \in \mathcal{I}\}$ is independent if every finite subset of them is independent.

Independent and Identically Distributed (i.i.d.) Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X_1, \ldots, X_n$ be random variables defined with respect to $\mathscr{F}$.

Definition (i.i.d. Random Variables)

$X_1, \ldots, X_n$ are said to be independent and identically distributed (i.i.d.) if

1. $X_1, \ldots, X_n$ are independent.
2. $F_{X_i} = F_{X_j}$ for all $i \neq j$ (identical CDFs).

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Jointly Discrete Random Variables

Jointly Discrete Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Jointly Discrete Random Variables)

Random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ defined with respect to $\mathscr{F}$ are said to be jointly discrete if $(X, Y) : \Omega \to \mathbb{R}^2$ is a discrete random variable.

- X discrete $\iff$

Jointly Discrete Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Jointly Discrete Random Variables)

Random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ defined with respect to $\mathscr{F}$ are said to be jointly discrete if $(X, Y) : \Omega \to \mathbb{R}^2$ is a discrete random variable.

- X discrete $\iff \mathbb{P}_X(E_1) = \mathbb{P}(\{X \in E_1\}) = 1$ for some countable set $E_1 \subset \mathbb{R}$
- Y discrete $\iff$

Jointly Discrete Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Jointly Discrete Random Variables)

Random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ defined with respect to $\mathscr{F}$ are said to be jointly discrete if $(X, Y) : \Omega \to \mathbb{R}^2$ is a discrete random variable.

- X discrete $\Longleftrightarrow \mathbb{P}_X(E_1) = \mathbb{P}(\{X \in E_1\}) = 1$ for some countable set $E_1 \subset \mathbb{R}$
- Y discrete $\Longleftrightarrow \mathbb{P}_Y(E_2) = \mathbb{P}(\{Y \in E_2\}) = 1$ for some countable set $E_2 \subset \mathbb{R}$
- E_1 countable, E_2 countable $\implies E_1 \times E_2$ countable (exercise!)
- $\mathbb{P}_{X,Y}(E_1 \times E_2) = \mathbb{P}(\{X \in E_1\} \cap \{Y \in E_2\}) = 1$

Jointly Discrete Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

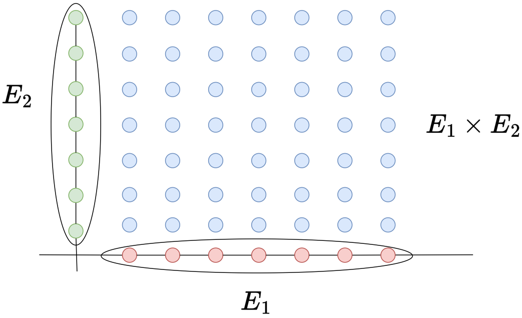
Definition (Jointly Discrete Random Variables)

Random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ defined with respect to $\mathscr{F}$ are said to be jointly discrete if $(X, Y) : \Omega \to \mathbb{R}^2$ is a discrete random variable.

- X discrete $\iff \mathbb{P}_X(E_1) = \mathbb{P}(\{X \in E_1\}) = 1$ for some countable set $E_1 \subset \mathbb{R}$
- Y discrete $\iff \mathbb{P}_Y(E_2) = \mathbb{P}(\{Y \in E_2\}) = 1$ for some countable set $E_2 \subset \mathbb{R}$
- E_1 countable, E_2 countable $\implies E_1 \times E_2$ countable (exercise!)
- $\mathbb{P}_{X,Y}(E_1 \times E_2) = \mathbb{P}(\{X \in E_1\} \cap \{Y \in E_2\}) = 1$

X discrete, Y discrete $\implies (X, Y)$ discrete

Picture of Jointly Discrete Random Variables



Joint PMF

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Joint PMF)

The joint PMF of jointly discrete random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ defined on $\mathscr{F}$ is a function $p_{X,Y} : \mathbb{R}^2 \to [0, 1]$ defined as

$$p_{X,Y}(x, y) = \mathbb{P}(\{X = x\} \cap \{Y = y\}), \qquad x, y \in \mathbb{R}.$$

Note:

$$\mathbb{P}(\{(X, Y) \in E_1 \times E_2\}) = \sum_{x \in E_1} \sum_{y \in E_2} p_{X,Y}(x, y) = 1,$$

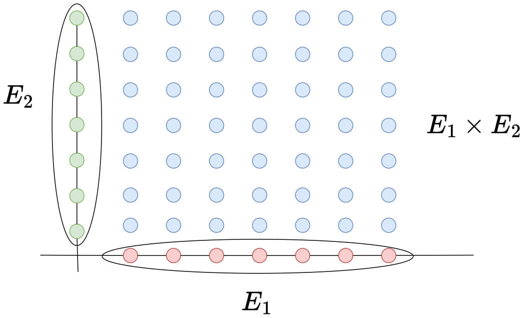
$$\mathbb{P}(\{(X, Y) \in B\}) = \sum_{(x,y) \in B \cap (E_1 \times E_2)} p_{X,Y}(x, y), \quad B \subseteq \mathbb{R}^2.$$

Properties of Joint PMF

- $\sum_{x \in E_1} \sum_{y \in E_2} p_{X,Y}(x, y) = 1.$

- $p_X(x) = \sum_{y \in E_2} p_{X,Y}(x, y), \quad x \in \mathbb{R}$

- $p_Y(y) = \sum_{x \in E_1} p_{X,Y}(x, y), \quad y \in \mathbb{R}$



Conditional PMF

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Conditional PMF)

Let X, Y be jointly discrete random variables defined with respect to $\mathscr{F}$. Fix $y \in \mathbb{R}$ such that $p_Y(y) = \mathbb{P}(\{Y = y\}) > 0$. The conditional PMF of X, conditioned on the event $\{Y = y\}$, is a function $p_{X|Y=y} : \mathbb{R} \to [0, 1]$ defined as

$$p_{X|Y=y}(x) = \frac{\mathbb{P}(\{X = x\} \cap \{Y = y\})}{\mathbb{P}(\{Y = y\})} = \frac{p_{X,Y}(x, y)}{p_Y(y)}, \qquad x \in \mathbb{R},$$

defined for all $y \in \mathbb{R}$ such that $p_Y(y) = \mathbb{P}(\{Y = y\}) > 0$.

Conditional PMF

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Conditional PMF)

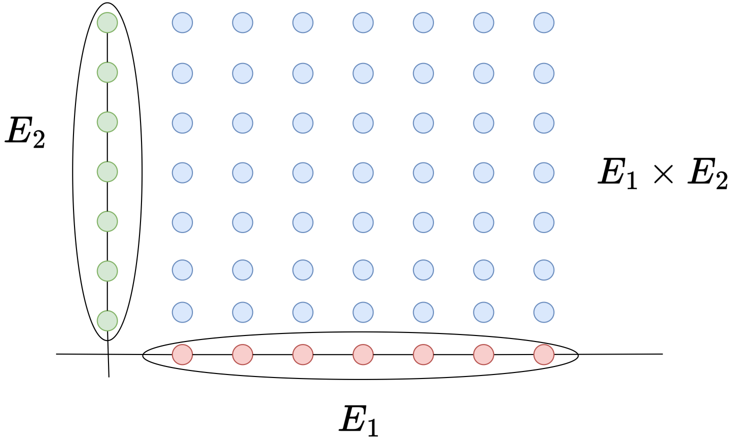
Let X, Y be jointly discrete random variables defined with respect to $\mathscr{F}$. Fix $y \in \mathbb{R}$ such that $p_Y(y) = \mathbb{P}(\{Y = y\}) > 0$. The conditional PMF of X, conditioned on the event $\{Y = y\}$, is a function $p_{X|Y=y} : \mathbb{R} \to [0, 1]$ defined as

$$p_{X|Y=y}(x) = \frac{\mathbb{P}(\{X = x\} \cap \{Y = y\})}{\mathbb{P}(\{Y = y\})} = \frac{p_{X,Y}(x, y)}{p_Y(y)}, \qquad x \in \mathbb{R},$$

defined for all $y \in \mathbb{R}$ such that $p_Y(y) = \mathbb{P}(\{Y = y\}) > 0$.

Remark: In textbooks, $p_{X|Y=y}(x)$ is commonly denoted as $p_{X|Y}(x|y)$

Conditional PMF



Independence of Two Discrete Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Theorem

Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be discrete random variables with respect to $\mathscr{F}$. The following statements are equivalent.

1. $X \perp\!\!\!\perp Y$.
2. $\{X = x\} \perp\!\!\!\perp \{Y = y\}$ for all $x, y \in \mathbb{R}$.
3. $p_{X,Y}(x, y) = p_X(x) \cdot p_Y(y)$ for all $x, y \in \mathbb{R}$.
4. For all $y \in \mathbb{R}$ such that $p_Y(y) > 0$,

$$p_{X|Y=y}(x) = p_X(x) \qquad \forall x \in \mathbb{R}.$$

Jointly Continuous Random Variables

Jointly Continuous Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ be random variables defined with respect to $\mathscr{F}$.

Definition (Jointly Continuous Random Variables)

X and Y are said to be jointly continuous if $(X, Y) : \Omega \to \mathbb{R}^2$ is a continuous random variable, i.e., there exists a function $f_{X,Y} : \mathbb{R}^2 \to [0, +\infty)$ such that the joint CDF of X and Y may be expressed as

$$F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u, v)\, dv\, du \qquad \forall x, y \in \mathbb{R}.$$

The function $f_{X,Y}$ is called the joint PDF of X and Y.

Remark:

X continuous, Y continuous $\not\Longrightarrow$ X, Y jointly continuous