AI 5030: Probability and Stochastic Processes

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Homework 9

Topics: Conditional Expectations, Law of Iterated Expectations

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables appearing below are assumed to be defined with respect to $\mathscr{F}$.

1. Let X and Y be independent Poisson random variables with parameters λ_1 and λ_2 respectively.
   Determine $\mathbb{E}[X|X+Y]$ (this should be a function of $X+Y$).
   Hence compute $\mathbb{E}[X]$ using the law of iterated expectations.

2. Let X and Y be jointly continuous with the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2}\, y\, e^{-xy}, & x > 0,\ 0 < y < 2, \\ 0, & \text{otherwise.} \end{cases}$$

   Compute $\mathbb{E}[e^{X/2}|Y]$.

3. Let X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx(y-x)\, e^{-y}, & 0 \leq x \leq y < +\infty, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Determine the constant c.
   (b) Determine $\mathbb{E}[X|Y]$.
   (c) Determine $\mathbb{E}[Y|X]$.

4. Suppose that a fair coin is tossed repeatedly until the pattern "HTHH" is observed for the first time in succession. Determine the expected number of coin tosses required.
   Hint: Let N denote the number of tosses required. Let $X_n \in \{H, T\}$ denote the outcome of the nth toss for $n \in \mathbb{N}$.
   Write $\mathbb{E}[N] = \mathbb{E}[N|\{X_1 = H\}] \cdot \mathbb{P}(\{X_1 = H\}) + \mathbb{E}[N|\{X_1 = T\}] \cdot \mathbb{P}(\{X_1 = T\})$. Justify this step.
   Express $\mathbb{E}[N|\{X_1 = T\}]$ in terms of $\mathbb{E}[N]$. Justify the steps.
   Write $\mathbb{E}[N|\{X_1 = H\}] = \mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = H\}] \cdot \mathbb{P}(\{X_2 = H\}) + \mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\}] \cdot \mathbb{P}(\{X_2 = T\})$.
   Again, justify this step.
   Express $\mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = H\}]$ in terms of $\mathbb{E}[N]$. Justify the steps.
   Proceed recursively as above.

5. Let X and Y be jointly uniformly distributed over the right-angled triangle with vertices at $(0,0)$, $(1,0)$, and $(0,2)$. Compute $\mathbb{E}[X|\{Y > 1\}]$.

6. Let X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 3y, & -1 \leq x \leq 1,\ 0 \leq y \leq |x|, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Determine $\mathbb{E}[Y|\{X \geq Y + 0.5\}]$.
   (b) Evaluate $\mathbb{E}[Y|X]$, and verify the relation $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]]$.

7. Define $\mathsf{Var}(X|Y)$ as

$$\mathsf{Var}(X|Y) = \mathbb{E}[(X - \mathbb{E}[X|Y])^2|Y] = \mathbb{E}[X^2|Y] - (\mathbb{E}[X|Y])^2.$$

   Verify the relation

$$\mathsf{Var}(X) = \mathbb{E}[\mathsf{Var}(X|Y)] + \mathsf{Var}(\mathbb{E}[X|Y]).$$