AI 5030: Probability and Stochastic Processes
Instructor: Dr. Karthik P. N.

Homework 4
Topics: Conditional Probability, Independence, Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

1. For any two disjoint sets $A, B \subseteq \Omega$, show that

$$\mathbf{1}_{A\cup B} = \mathbf{1}_A + \mathbf{1}_B,$$

where $\mathbf{1}_E$ denotes the indicator function of the set E.
Use the above result to show that if A and B are any two sets (not necessarily disjoint), then

$$\mathbf{1}_{A\cup B} = \mathbf{1}_A + \mathbf{1}_B - \mathbf{1}_{A\cap B}.$$

Solution: We follow the convention that the functions $f = g \iff f(x) = g(x) \forall x \in \mathcal{D}(f) = \mathcal{D}(g)$ and $(f + g)(x) = f(x) + g(x)$, where $\mathcal{D}(f)$ denotes the domain of the function f.

We recall that $\mathbf{1}_A(\omega) = \begin{cases} 1, & \text{if } \omega \in A \\ 0, & \text{if } \omega \in A^c. \end{cases}$

Now, we compare the evaluations of the indicator functions on an arbitrary $\omega \in \Omega$. When $A, B \subseteq \Omega$ are disjoint, we have three cases:

- $\omega \in A$ but $\omega \notin B$. Here $\mathbf{1}_A(\omega) = 1$, $\mathbf{1}_B(\omega) = 0$ and $\mathbf{1}_{A\cup B}(\omega) = 1$. LHS=RHS=1.
- $\omega \in B$ but $\omega \notin A$. Here $\mathbf{1}_A(\omega) = 0$, $\mathbf{1}_B(\omega) = 1$ and $\mathbf{1}_{A\cup B}(\omega) = 1$. LHS=RHS=1
- $\omega \notin A$ and $\omega \notin B$. Here $\mathbf{1}_A(\omega) = 0$, $\mathbf{1}_B(\omega) = 0$ and $\mathbf{1}_{A\cup B}(\omega) = 0$. LHS=RHS=0.

Hence, we showed that $\mathbf{1}_{A\cup B} = \mathbf{1}_A + \mathbf{1}_B - \mathbf{1}_{A\cap B}$ for any disjoint $A, B \subseteq \Omega$.

Next, when A, B need not be disjoint, we have the following cases.

- $\omega \in A$ but $\omega \notin B$. Here $\mathbf{1}_A(\omega) = 1$, $\mathbf{1}_B(\omega) = 0$, $\mathbf{1}_{A\cup B}(\omega) = 1$, $\mathbf{1}_{A\cap B}(\omega) = 0$. LHS=RHS=1.
- $\omega \in B$ but $\omega \notin A$. Here $\mathbf{1}_A(\omega) = 0$, $\mathbf{1}_B(\omega) = 1$ $\mathbf{1}_{A\cup B}(\omega) = 1$, $\mathbf{1}_{A\cap B}(\omega) = 0$. LHS=RHS=1
- $\omega \notin A$ and $\omega \notin B$. Here $\mathbf{1}_A(\omega) = 0$, $\mathbf{1}_B(\omega) = 0$, $\mathbf{1}_{A\cup B}(\omega) = 0$, $\mathbf{1}_{A\cap B}(\omega) = 0$. LHS=RHS=0.
- $\omega \in A$ and $\omega \in B$. Here $\mathbf{1}_A(\omega) = 1$, $\mathbf{1}_B(\omega) = 1$, $\mathbf{1}_{A\cup B}(\omega) = 1$, $\mathbf{1}_{A\cap B}(\omega) = 1$. LHS=RHS=1.

Hence, proved.

2. Let $\Omega = \{H, T\}^3$ and $\mathscr{F} = 2^\Omega$. Construct a probability measure $\mathbb{P}$ and events $A, B, C \in \mathscr{F}$ such that

   (a) $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B), \quad \mathbb{P}(B \cap C) = \mathbb{P}(B) \cdot \mathbb{P}(C), \quad \mathbb{P}(A \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(C).$

   (b) $\mathbb{P}(A \cap B \cap C) \neq \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C).$

Solution: Consider the assignment of probabilities as depicted in Table 1.

Let A, B, C be events defined as follows.

$$\begin{aligned} A &:= \text{outcome of first coin is head}, \\ B &:= \text{outcome of second coin is head}, \\ C &:= \text{outcome of third coin is head}. \end{aligned}$$

Then, it follows that

$$\mathbb{P}(A \cap B) = \mathbb{P}(\{HHH\} \cup \{HHT\}) = \frac{1}{4},$$

E	$\mathbb{P}(E)$
$\{HHH\}$	$1/4$
$\{HHT\}$	0
$\{HTH\}$	0
$\{HTT\}$	$1/4$
$\{THH\}$	0
$\{THT\}$	$1/4$
$\{TTH\}$	$1/4$
$\{TTT\}$	0

Table 1: Assignment of probabilities to demonstrate that for any 3 events, pairwise independence does not imply joint independence.

while we have

$$\mathbb{P}(A) = \mathbb{P}(\{HHH\} \cup \{HHT\} \cup \{HTH\} \cup \{HTT\}) = \frac{1}{2},$$
$$\mathbb{P}(B) = \mathbb{P}(\{HHH\} \cup \{HHT\} \cup \{THH\} \cup \{THT\}) = \frac{1}{2}.$$

Thus, we have $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B)$, i.e., $A \perp\!\!\!\perp B$. Along similar lines, it can be shown that $\mathbb{P}(C) = 1/2$, $B \perp\!\!\!\perp C$, and $A \perp\!\!\!\perp C$. However, we note that

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(\{HHH\}) = \frac{1}{4} \neq \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C).$$

3. Let $\Omega = [0, +\infty)$ and $\mathscr{F} = \mathscr{B}([0, +\infty))$. Let $X : \Omega \to \mathbb{R}$ be defined as

$$X(\omega) = \sum_{k=1}^{\infty} k \, \mathbf{1}_{[k-1,k)}(\omega) = \mathbf{1}_{[0,1)}(\omega) + 2\,\mathbf{1}_{[1,2)}(\omega) + 3\,\mathbf{1}_{[2,3)}(\omega) + \ldots, \qquad \omega \in \Omega.$$

That is, X takes the constant value 1 on $[0, 1)$, the value 2 on $[1, 2)$, the value 3 on $[2, 3)$, and so on.

(a) Evaluate $X^{-1}([0, 100])$.
(b) Given a natural number $n \in \mathbb{N}$, what is $X^{-1}(\{n\})$?
(c) Evaluate $X^{-1}((-\infty, x])$ for all $x \in \mathbb{R}$, and show that X is a random variable with respect to $\mathscr{F}$.

Solution: Notice that X takes only positive integer values.

(a) $X^{-1}([0, 100]) = \{\omega \in \Omega : 0 \leq X(\omega) \leq 100\} = \bigcup_{i=1}^{100} [i-1, i) = [0, 100)$.
(b) $X^{-1}(\{n\}) = \{\omega \in \Omega : X(\omega) = n\} = [n-1, n)$.
(c) We have

$$X^{-1}((-\infty, x]) = \{\omega \in \Omega : X(\omega) \leq x\} = \begin{cases} \emptyset, & x < 1, \\ [0,1), & 1 \leq x < 2, \\ [0,2), & 2 \leq x < 3, \\ [0,3), & 3 \leq x < 4, \\ \vdots \end{cases}$$

From the above expression, it is clear that $X((-\infty, x]) \in \mathscr{F}$ for all $x \in \mathbb{R}$, and hence X is a random variable with respect to $\mathscr{F}$.

4. Let $X : \Omega \to \mathbb{R}$ be a random variable with respect to $\mathscr{F}$.

(a) Show that $\left(X^{-1}(B)\right)^c = X^{-1}(B^c)$ for any $B \in \mathscr{B}(\mathbb{R})$.

(b) Show that for any two Borel sets $B_1, B_2 \in \mathscr{B}(\mathbb{R})$,

$$X^{-1}(B_1 \cup B_2) = X^{-1}(B_1) \cup X^{-1}(B_2).$$

More generally, for any $B_1, B_2, \ldots \in \mathscr{B}(\mathbb{R})$, show that

$$X^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) = \bigcup_{i=1}^{\infty} X^{-1}(B_i).$$

(c) Consider the collection

$$\mathscr{E} = \big\{E \subseteq \Omega : E = X^{-1}(B) \text{ for some } B \in \mathscr{B}(\mathbb{R})\big\}.$$

That is, each set in $\mathscr{E}$ is the pre-image (under X) of some Borel set B.
Show that $\mathscr{E}$ is a σ-algebra of subsets of Ω.
Hint: Use the results in part (a) and part (b).

Note: To show $A = B$ for any two sets A, B, you need to show $A \subseteq B$ and $B \subseteq A$.

Solution: We provide the solution to each of the parts below.

(a) Fix $B \in \mathscr{B}(\mathbb{R})$. We then note that

$$\begin{aligned}
\omega_0 \in \big(X^{-1}(B)\big)^c &\iff \omega_0 \in \Big\{\omega \in \Omega : X(\omega) \in B\Big\}^c \\
&\iff X(\omega_0) \notin B \\
&\iff X(\omega_0) \in B^c \\
&\iff \omega_0 \in \{\omega \in \Omega : X(\omega) \in B^c\} \\
&\iff \omega_0 \in X^{-1}(B^c),
\end{aligned}$$

thus proving that $\big(X^{-1}(B)\big)^c = X^{-1}(B^c)$.

(b) Let $B_1, B_2, \ldots \in \mathscr{B}(\mathbb{R})$. We then have

$$\begin{aligned}
\omega_0 \in X^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) &\iff \omega_0 \in \left\{\omega \in \Omega : X(\omega) \in \bigcup_{i=1}^{\infty} B_i\right\} \\
&\iff X(\omega_0) \in \bigcup_{i=1}^{\infty} B_i \\
&\iff \exists i_0 \in \mathbb{N} \text{ such that } X(\omega_0) \in B_{i_0} \\
&\iff \exists i_0 \in \mathbb{N} \text{ such that } \omega_0 \in \{\omega \in \Omega : X(\omega) \in B_{i_0}\} \\
&\iff \omega_0 \in \bigcup_{i=1}^{\infty} \{\omega \in \Omega : X(\omega) \in B_i\} \\
&\iff \omega_0 \in \bigcup_{i=1}^{\infty} X^{-1}(B_i),
\end{aligned}$$

thus proving that $X^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) = \bigcup_{i=1}^{\infty} X^{-1}(B_i)$. Setting $B_i = \emptyset$ for all $i \geq 3$, we arrive at the relation $X^{-1}(B_1 \cup B_2) = X^{-1}(B_1) \cup X^{-1}(B_2)$.

(c) To show that $\Omega \in \mathscr{E}$, we note that $\mathbb{R} = (-\infty, +\infty) = \bigcup_{n=1}^{\infty}(-\infty, n] \in \mathscr{B}(\mathbb{R})$, and $\Omega = X^{-1}(\mathbb{R})$.
Suppose that $E \in \mathscr{E}$. Then, there exists $B \in \mathscr{B}(\mathbb{R})$ such that $E = X^{-1}(B)$. Then, we have

$$E^c = \big(X^{-1}(B)\big)^c = X^{-1}(B^c),$$

and noting that $B^c \in \mathscr{B}(\mathbb{R})$, it follows that $E^c \in \mathscr{E}$. Thus, $\mathscr{E}$ is closed under set complements.

Finally, let $E_1, E_2, \ldots \in \mathscr{E}$. Then, by definition, there exist sets $B_1, B_2, \ldots \in \mathscr{B}(\mathbb{R})$ such that $E_i = X^{-1}(B_i)$ for all $i \in \mathbb{N}$. We then have

$$\bigcup_{i=1}^{\infty} E_i = \bigcup_{i=1}^{\infty} X^{-1}(B_i) = X^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right),$$

and noting that $\bigcup_{i=1}^{\infty} B_i \in \mathscr{B}(\mathbb{R})$, it follows that $\bigcup_{i=1}^{\infty} E_i \in \mathscr{E}$. Therefore, $\mathscr{E}$ is closed under countable unions. Together, the above properties demonstrate that $\mathscr{E}$ is a σ-algebra of subsets of Ω.

5. Suppose two fair coins are tossed independently of each other.

   (a) Specify $(\Omega, \mathscr{F}, \mathbb{P})$ for the above experiment.

   (b) Find the probability of the event that both coins turn up heads, conditioned on the event that the first coin turns up head.

   (c) Find the probability of the event that both coins turn up heads, conditioned on the event that at least one of the coins turns up head.

Solution: We provide solution to each of the parts below.

(a) We have $\Omega = \{HH, HT, TH, TT\}$. We simply set $\mathscr{F} = 2^{\Omega}$. To construct $\mathbb{P}$, we note the following requirements:

   i. $\mathbb{P}(\{HH\} \cup \{HT\}) = \mathbb{P}(\{\text{coin } 1 \text{ lands up head}\}) = \frac{1}{2}$ (as coin 1 is fair).
   ii. $\mathbb{P}(\{TH\} \cup \{TT\}) = \mathbb{P}(\{\text{coin } 1 \text{ lands up tail}\}) = \frac{1}{2}$ (as coin 1 is fair).
   iii. $\mathbb{P}(\{TH\} \cup \{HH\}) = \mathbb{P}(\{\text{coin } 2 \text{ lands up head}\}) = \frac{1}{2}$ (as coin 2 is fair).
   iv. $\mathbb{P}(\{TT\} \cup \{HT\}) = \mathbb{P}(\{\text{coin } 2 \text{ lands up tail}\}) = \frac{1}{2}$ (as coin 2 is fair).

   Based on the above requirements, we must have

$$\mathbb{P}(\{HH\}) = \mathbb{P}(\{HT\}) = \mathbb{P}(\{TH\}) = \mathbb{P}(\{TH\}) = \frac{1}{4}.$$

(b) Let E_1 (resp. E_2) denote the event that the first (resp. second) coin turns up head. Then, the desired probability is $\mathbb{P}(E_1 \cap E_2 | E_1)$. By definition, we have

$$\mathbb{P}(E_1 \cap E_2 | E_1) = \frac{\mathbb{P}(\mathbb{E}_1 \cap E_2 \cap E_1)}{\mathbb{P}(E_1)} = \frac{\mathbb{P}(E_1 \cap E_2)}{\mathbb{P}(E_1)} \stackrel{(a)}{=} \frac{\mathbb{P}(E_1) \cdot \mathbb{P}(E_2)}{\mathbb{P}(E_1)} = \mathbb{P}(E_2),$$

   where (a) above follows from the fact that the coin tosses are independent of one another. Note that

$$\mathbb{P}(E_2) = \mathbb{P}(\{HH\} \cup \{TH\}) = \frac{1}{2}.$$

   Therefore, the desired probability is $\mathbb{P}(E_1 \cap E_2 | E_1) = \frac{1}{2}$.

(c) The desired probability is $\mathbb{P}(E_1 \cap E_2 | E_1 \cup E_2)$. Note that

$$\mathbb{P}(E_1 \cap E_2) = \mathbb{P}(\{HH\}) = \frac{1}{4}, \qquad \mathbb{P}(E_1 \cup E_2) = \mathbb{P}(\{HH\} \cup \{HT\} \cup \{TH\}) = \frac{3}{4}.$$

   Therefore, it follows that

$$\mathbb{P}(E_1 \cap E_2 | E_1 \cup E_2) = \frac{\mathbb{P}((E_1 \cap E_2) \cap (E_1 \cup E_2))}{\mathbb{P}(E_1 \cup E_2\}} \stackrel{(a)}{=} \frac{\mathbb{P}(E_1 \cap E_2)}{\mathbb{P}(E_1 \cup E_2)} = \frac{1/4}{3/4} = \frac{1}{3},$$

   where (a) above follows by noting that $E_1 \cap E_2 \subseteq E_1 \cup E_2$.

6. Consider events $A, B, C \in \mathscr{F}$ such A is independent of B and A is independent of C. Show that A is independent of $B \cup C$ if and only if A is independent of $B \cap C$.

Note: To prove an if and only if statement, the "if" and "only if" directions must be proved separately.

Solution: We recall that events $A, B \in \mathscr{F}$ are independent iff $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.

"if": We need to prove that for events $A, B, C \in \mathscr{F}$ such A is independent of B and A is independent of C, then, A is independent of $B \cup C$ if A is independent of $B \cap C$.

$$\begin{aligned}
\mathbb{P}(A \cap (B \cup C)) &= \mathbb{P}\left((A \cap B) \cup (A \cap C)\right) \\
&= \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) - \mathbb{P}\left((A \cap B) \cap (A \cap C)\right) \\
&\overset{(*)}{=} \mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(C) - \mathbb{P}\left((A \cap B) \cap (A \cap C)\right) \ \because A \text{ is independent of } B,\ A \text{ is independent of } C. \\
&= \mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(C) - \mathbb{P}\left((A \cap A) \cap (B \cap C)\right) \\
&= \mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(C) - \mathbb{P}\left(A \cap (B \cap C)\right) \\
&= \mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(C) - \mathbb{P}(A)\mathbb{P}(B \cap C) \ \text{(when } A \text{ is independent of } B \cap C) \\
&= \mathbb{P}(A)\left(\mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C)\right) \\
&\overset{(*)}{=} \mathbb{P}(A)\mathbb{P}(B \cup C),
\end{aligned}$$

"only if": We need to prove that for events $A, B, C \in \mathscr{F}$ such A is independent of B and A is independent of C, then, A is independent of $B \cap C$ if A is independent of $B \cup C$.

$$\begin{aligned}
\mathbb{P}(A \cap (B \cap C)) &= \mathbb{P}((A \cap A) \cap (B \cap C)) \\
&= \mathbb{P}((A \cap B) \cap (A \cap C)) \\
&\overset{(*)}{=} \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) - \mathbb{P}((A \cap B) \cup (A \cap C)) \\
&= \mathbb{P}(A \cap B) + \mathbb{P}(A \cap C) - \mathbb{P}(A \cap (B \cup C)) \\
&= \mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(C) - \mathbb{P}(A \cap (B \cup C)) \ \because A \text{ is independent of } B,\ A \text{ is independent of } C. \\
&= \mathbb{P}(A)\mathbb{P}(B) + \mathbb{P}(A)\mathbb{P}(C) - \mathbb{P}(A)\mathbb{P}(B \cup C) \ \text{(when } A \text{ is independent of } B \cup C) \\
&= \mathbb{P}(A)\left(\mathbb{P}(B) + \mathbb{P}(C) -\right) - \mathbb{P}(B \cup C) \\
&\overset{(*)}{=} \mathbb{P}(A)\mathbb{P}(B \cap C),
\end{aligned}$$

where the equalities marked as $(*)$ follow from the inclusion-exclusion result.