भारतीय प्रौद्योगिकी संस्थान हैदराबाद
Indian Institute of Technology Hyderabad

Probability and Stochastic Processes

Lecture 02: Sample Space, Algebra, σ-Algebra

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Probability Theory – Humble Beginnings

- Bernoulli (1713) and de Moivre (1718) gave the first definition of probability:

$$\text{probability of an event} = \frac{\#\text{ favourable outcomes}}{\text{total number of outcomes}}.$$

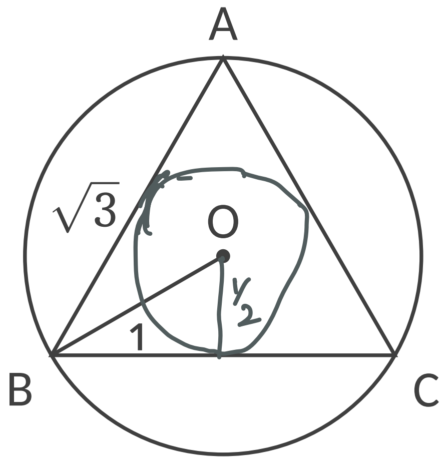
- Cournot (1843):
  "An event with very small probability is morally impossible; an event with very high probability is morally certain."
- French mathematicians of the day were satisfied with the "frequentist" approach to probability, but not the German and English mathematicians of the day
- Frequentist approach could not satisfactorily explain certain paradoxes

Why Axiomatic Theory? Bertrand's Paradox

Take a circle with unit radius and inscribe an equilateral triangle in it. Draw a random chord. What is the probability that the length of the "random chord" is greater than $\sqrt{3}$?

Bertrand's perfectly valid arguments:

- Mid-point of chord should lie inside incircle of radius $1/2$



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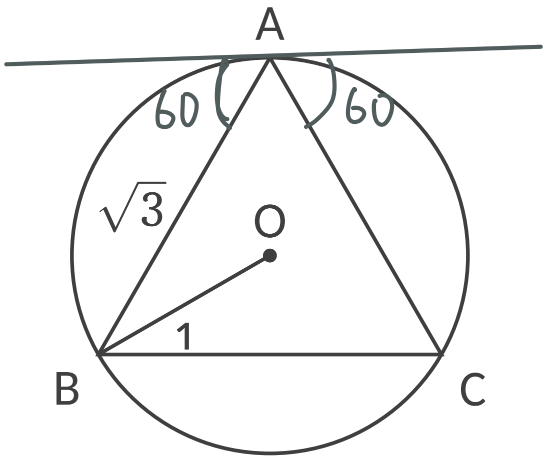

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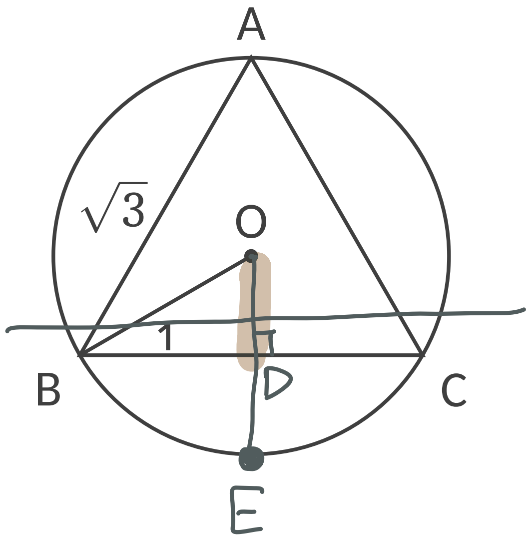

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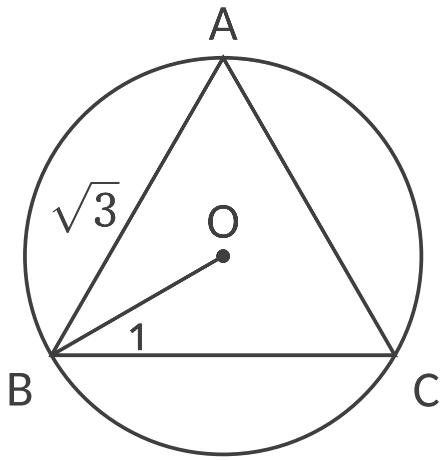

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Borel to the Rescue

- Contributions to Measure Theory by Borel (1894) provided a shift in perspective
- Countable unions played a key role in Borel's theory
- Kolmogorov's genius was in applying Borel's theory to formalise the axioms of probability, laying the foundation stone for modern probability theory
- For more details on the history of probability, see [Shafer and Vovk, 2018] and [Kolmogorov, 2004]

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We begin with two universally accepted entities:

- Random experiment
- Outcome (denoted by ω) – source of randomness

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- If our interest is in the number of times coin flips in air, then $\Omega = \mathbb{N}$

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Example: Toss a coin n times, for some $n < \infty$.
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Note

If an outcome $\omega \in E$ occurs, we say that the event A occurs.

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