AI 5030: Probability and Stochastic Processes

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Homework 9

Topics: Conditional Expectations, Law of Iterated Expectations

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables appearing below are assumed to be defined with respect to $\mathscr{F}$.

1. Let X and Y be independent Poisson random variables with parameters λ_1 and λ_2 respectively. Determine $\mathbb{E}[X|X+Y]$ (this should be a function of $X+Y$). Hence compute $\mathbb{E}[X]$ using the law of iterated expectations.

   Solution: We know that $X+Y \sim$ Poisson$(\lambda_1+\lambda_2)$. Thus, for any $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
\mathbb{P}(\{X=k\}|\{X+Y=n\}) &= \frac{\mathbb{P}(\{X=k\}\cap\{X+Y=n\})}{\mathbb{P}(\{X+Y=n\})} \\
&= \frac{\mathbb{P}(\{X=k\}\cap\{Y=n-k\})}{\mathbb{P}(\{X+Y=n\})} \\
&\overset{(*)}{=} \frac{\mathbb{P}(\{X=k\})\cdot\mathbb{P}(\{Y=n-k\})}{\mathbb{P}(\{X+Y=n\})} \\
&= \frac{\frac{e^{-\lambda_1}\,\lambda_1^k}{k!}\cdot\frac{e^{-\lambda_2}\,\lambda_2^{n-k}}{(n-k)!}}{\frac{e^{-\lambda_1+\lambda_2}\,(\lambda_1+\lambda_2)^n}{n!}} \\
&= \frac{n!}{k!(n-k)!}\cdot\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\cdot\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}, \qquad k\in\{0,\ldots,n\}.
\end{aligned}$$

   In the above set of equalities, $(*)$ follows because $X \perp\!\!\!\perp Y$. Thus, conditioned on the event $\{X+Y=n\}$, the random variable X is distributed as Binomial$\left(n,\ \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$. Noting that the mean of a binomial random variable with parameters (n,p) is equal to np, it follows that for all $n\in\mathbb{N}\cup\{0\}$,

$$\mathbb{E}[X|\{X+Y=n\}] = n\cdot\frac{\lambda_1}{\lambda_1+\lambda_2},$$

   from which it follows that

$$\mathbb{E}[X|X+Y] = (X+Y)\cdot\frac{\lambda_1}{\lambda_1+\lambda_2}.$$

   Applying $\mathbb{E}[\cdot]$ on both sides of the above equation, and using the law of iterated expectations, we get

$$\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|X+Y]] = \mathbb{E}[X+Y]\cdot\frac{\lambda_1}{\lambda_1+\lambda_2} = (\lambda_1+\lambda_2)\cdot\frac{\lambda_1}{\lambda_1+\lambda_2} = \lambda_1.$$

2. Let X and Y be jointly continuous with the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{2}\,y\,e^{-xy}, & x>0,\ 0<y<2, \\ 0, & \text{otherwise.} \end{cases}$$

   Compute $\mathbb{E}[e^{X/2}|Y]$.

   Solution: For any given $y\in(0,2)$, we first compute the conditional PDF of X, conditioned on the event $\{Y=y\}$. Towards this, we have

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathrm{d}x = \int_0^{\infty}\frac{1}{2}\,y\,e^{-xy}\,\mathrm{d}x = \frac{1}{2}, \qquad y\in(0,2).$$

   Then, for any $y\in(0,2)$, we have

$$f_{X|\{Y=y\}}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = y\,e^{-xy}, \qquad x>0,$$

from which it follows that

$$\mathbb{E}[e^{X/2}|\{Y=y\}] = \int_0^\infty e^{x/2}\, f_{X|\{Y=y\}}(x)\,\mathrm{d}x = \int_0^\infty e^{x/2}\, ye^{-xy}\,\mathrm{d}x = \int_0^\infty ye^{-x(y-1/2)}\,\mathrm{d}x = \begin{cases} \frac{2y}{2y-1}, & \frac{1}{2} < y < 2, \\ +\infty, & 0 < y \le \frac{1}{2}. \end{cases},$$

which in turn implies that

$$\mathbb{E}[e^{X/2}|Y] = \frac{2Y}{2Y-1}\,\mathbf{1}_{\{\frac{1}{2}<Y<2\}} + (+\infty)\,\mathbf{1}_{\{0<Y\le\frac{1}{2}\}}.$$

3. Let X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx(y-x)\,e^{-y}, & 0 \le x \le y < +\infty, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Determine the constant c.
   (b) Determine $\mathbb{E}[X|Y]$.
   (c) Determine $\mathbb{E}[Y|X]$.

Solution: We present the solution to each part below.

(a) Setting

$$1 = \int_0^\infty \int_x^\infty cx(y-x)e^{-y}\,\mathrm{d}y\,\mathrm{d}x,$$

we get $c = 1$.

(b) From question 3 of homework 6, we note that for any $y \ge 0$,

$$f_{X|\{Y=y\}}(x) = \begin{cases} 6x(y-x)y^{-3}, & 0 \le x \le y, \\ 0, & \text{otherwise.} \end{cases}$$

We thus have

$$\mathbb{E}[X|\{Y=y\}] = \int_0^y x\, f_{X|\{Y=y\}}(x)\,\mathrm{d}x = \int_0^y 6x^2(y-x)y^{-3}\,\mathrm{d}x = \frac{y}{2},$$

from which it follows that $\mathbb{E}[X|Y] = \frac{Y}{2}$.
Along similar lines, from question 3 of homework 6, we note that for any $x \in (0,\infty)$,

$$f_{Y|\{X=x\}}(y) = \begin{cases} (y-x)\,e^{-(y-x)}, & y \ge x, \\ 0, & \text{otherwise.} \end{cases}$$

We thus have

$$\mathbb{E}[Y|\{X=x\}] = \int_x^\infty y\, f_{Y|\{X=x\}}(y)\,\mathrm{d}y = \int_x^\infty y\,(y-x)\,e^{-(y-x)}\,\mathrm{d}y = x+2,$$

from which it follows that $\mathbb{E}[Y|X] = X+2$.

4. Suppose that a fair coin is tossed repeatedly until the pattern "HTHH" is observed for the first time in succession. Determine the expected number of coin tosses required.

Hint: Let N denote the number of tosses required. Let $X_n \in \{H,T\}$ denote the outcome of the nth toss for $n \in \mathbb{N}$.
Write $\mathbb{E}[N] = \mathbb{E}[N|\{X_1=H\}]\cdot\mathbb{P}(\{X_1=H\}) + \mathbb{E}[N|\{X_1=T\}]\cdot\mathbb{P}(\{X_1=T\})$. Justify this step.
Express $\mathbb{E}[N|\{X_1=T\}]$ in terms of $\mathbb{E}[N]$. Justify the steps.
Write $\mathbb{E}[N|\{X_1=H\}] = \mathbb{E}[N|\{X_1=H\}\cap\{X_2=H\}]\cdot\mathbb{P}(\{X_2=H\}) + \mathbb{E}[N|\{X_1=H\}\cap\{X_2=T\}]\cdot\mathbb{P}(\{X_2=T\})$.
Again, justify this step.
Express $\mathbb{E}[N|\{X_1=H\}\cap\{X_2=H\}]$ in terms of $\mathbb{E}[N]$. Justify the steps.
Proceed recursively as above.

Solution: We use the following notations:

- Let $\mathbb{E}[N]$ be denoted by α.
- Let $\mathbb{E}[N|\{X_1 = H\}]$ be denoted by α_1.
- Let $\mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\}]$ be denoted by α_2.
- Let $\mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\} \cap \{X_3 = H\}]$ be denoted by α_3.

By the law of iterated expectations, we have

$$\mathbb{E}[N] = \mathbb{E}[\mathbb{E}[N|X_1]].$$

Noting that the outer expectation is with respect to the distribution of X_1, we have

$$\mathbb{E}[N] = \mathbb{E}[N|\{X_1 = H\}] \cdot \mathbb{P}(\{X_1 = H\}) + \mathbb{E}[N|\{X_1 = T\}] \cdot \mathbb{P}(\{X_1 = T\}).$$

We then have

$$\mathbb{E}[N|\{X_1 = T\}] = 1 + \mathbb{E}[N] = 1 + \alpha.$$

On the other hand, using the law of total probability, we have

$$\mathbb{E}[N|\{X_1 = H\}] = \mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = H\}] \cdot \mathbb{P}(\{X_2 = H\}) + \mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\}] \cdot \mathbb{P}(\{X_2 = T\}).$$

We now note that

$$\mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = H\}] = 1 + \mathbb{E}[N|\{X_1 = H\}];$$

the above relation states that despite obtaining consecutive heads on the first two tosses (which seems to throw us off the desired pattern), we may still obtain the desired pattern starting from the second head, discarding the first head (the term '1' on the right-hand side accounts for this first head) and treating the second head as the first head for the remaining tosses. Continuing, we get

$$\begin{aligned}\mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\}] = \mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\} \cap \{X_3 = H\}] \cdot \mathbb{P}(\{X_3 = H\}) \\ + \mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\} \cap \{X_3 = T\}] \cdot \mathbb{P}(\{X_3 = T\}).\end{aligned}$$

We then note that

$$\mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\} \cap \{X_3 = T\}] = 3 + \mathbb{E}[N] = 3 + \alpha;$$

clearly, if the pattern "HTT" is observed on the first three tosses, there is no way to recover the desired pattern from this, and we therefore discard all three tosses (this is the factor '3' on the right-hand side of the above relation) and start afresh. Additionally,

$$\begin{aligned}\mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\} \cap \{X_3 = H\}] = \mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\} \cap \{X_3 = H\} \cap \{X_4 = H\}] \cdot \mathbb{P}(\{X_4 = H\}) \\ + \mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\} \cap \{X_3 = H\} \cap \{X_4 = T\}] \cdot \mathbb{P}(\{X_4 = T\}).\end{aligned}$$

Now, we note that

$$\mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\} \cap \{X_3 = H\} \cap \{X_4 = T\}] = 2 + \mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\}];$$

the above relation captures the scenario in which despite obtaining "HTHT" on the first 4 tosses (which seemingly has thrown us off the desired pattern), we may still recover the desired pattern starting from the "HT" of the third and fourth tosses, treating these outcomes as respectively the first and second outcomes for the remainder of the tosses, while discarding the first two tosses (and accounting for this by adding 2 on the right-hand side). Lastly, we note that

$$\mathbb{E}[N|\{X_1 = H\} \cap \{X_2 = T\} \cap \{X_3 = H\} \cap \{X_4 = H\}] = 4 + 0 = 4;$$

here, the term '0' on the right-hand side represents the end of the coin tossing experiment, as the desired pattern is obtained at this stage.

Combining each of the expressions obtained above, and rewriting them in terms of $\alpha, \alpha_1, \alpha_2, \alpha_3$, we have

$$\begin{aligned}\alpha &= \frac{1}{2}(1 + \alpha) + \frac{\alpha_1}{2}, \\ \alpha_1 &= \frac{1}{2}(1 + \alpha_1) + \frac{\alpha_2}{2},\end{aligned}$$

$$\alpha_2 = \frac{1}{2}(3+\alpha) + \frac{\alpha_3}{2},$$
$$\alpha_3 = \frac{1}{2}(2+\alpha_2) + \frac{4}{2}.$$

Solving the above set of equations, we get $\alpha = \mathbb{E}[N] = 18$.

5. Let X and Y be jointly uniformly distributed over the right-angled triangle with vertices at $(0,0)$, $(1,0)$, and $(0,2)$. Compute $\mathbb{E}[X|\{Y>1\}]$.

Solution: Note that the joint PDF of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} 1, & 0 \le x \le 1,\ 0 \le y \le 2-2x, \\ 0, & \text{otherwise.} \end{cases}$$

To compute $\mathbb{E}[X|\{Y>1\}]$, we first compute the conditional CDF of X, conditioned on $\{Y>1\}$. Towards this, we note that

$$\begin{aligned} F_{X|\{Y>1\}}(x) &= \mathbb{P}(\{X \le x\}|\{Y>1\}) \\ &= \frac{\mathbb{P}(\{X \le x\} \cap \{Y>1\})}{\mathbb{P}(\{Y>1\})}. \end{aligned}$$

On the one hand, we then note that for any $y \in [0,2]$,

$$f_Y(y) = \int_0^{\frac{2-y}{2}} \mathsf{d}x = \frac{2-y}{2},$$

from which we have

$$\mathbb{P}(\{Y>1\}) = \int_1^2 f_Y(y)\,\mathsf{d}y = \int_1^2 \frac{2-y}{2}\,\mathsf{d}y = \frac{1}{4}.$$

On the other hand, we have

$$\mathbb{P}(\{X \le x\} \cap \{Y>1\}) = \begin{cases} 0, & x<0, \\ \int_0^x \int_1^{2-2u} \mathsf{d}v\,\mathsf{d}u, & 0 \le x < \frac{1}{2}, \\ \int_0^1 \int_1^{2-2u} \mathsf{d}v\,\mathsf{d}u, & x \ge \frac{1}{2} \end{cases} = \begin{cases} 0, & x<0, \\ x(1-x), & 0 \le x < \frac{1}{2}, \\ \frac{1}{4}, & x \ge \frac{1}{2}, \end{cases}$$

from which we get

$$F_{X|\{Y>1\}}(x) = \begin{cases} 0, & x<0, \\ 4x(1-x), & 0 \le x < \frac{1}{2}, \\ 1, & x \ge \frac{1}{2}. \end{cases}$$

Differentiating the above conditional CDF with respect to x, we get the conditional PDF expression as below:

$$f_{X|\{Y>1\}}(x) = \begin{cases} 4(1-2x), & 0<x<\frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, we have

$$\mathbb{E}[X|\{Y>1\}] = \int_0^{\frac{1}{2}} x\, f_{X|\{Y>1\}}(x)\,\mathsf{d}x = \int_0^{\frac{1}{2}} (4x-8x^2)\,\mathsf{d}x = \frac{1}{6}.$$

6. Let X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 3y, & -1 \le x \le 1,\ 0 \le y \le |x|, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Determine $\mathbb{E}[Y|\{X \ge Y+0.5\}]$.

(b) Evaluate $\mathbb{E}[Y|X]$, and verify the relation $\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]]$.

Solution: We first note that the range of possible values that Y can take is $[0, 1]$, and for any $y \in [0, 1]$,

$$f_Y(y) = \int_{\{x:|x|\geq y\}} f_{X,Y}(x, y)\, \mathsf{d}x = \int_{-1}^{-y} 3y\, \mathsf{d}x + \int_{y}^{1} 3y\, \mathsf{d}x = 6y(1 - y),$$

from which it follows that

$$\mathbb{E}[Y] = \int_0^1 6y^2(1 - y)\, \mathsf{d}y = \frac{1}{2}.$$

We now provide the solution to each of the parts below.

(a) To obtain the value of $\mathbb{E}[Y|\{X \geq Y + 0.5\}]$, we first compute the conditional PDF of Y, conditioned on the event $A = \{X \geq Y + 0.5\}$. Towards this, we note that

$$\begin{aligned} F_{Y|\{X\geq Y+0.5\}}(y) &= \mathbb{P}(\{Y \leq y\}|A) \\ &= \frac{\mathbb{P}(\{Y \leq y\} \cap A)}{\mathbb{P}(A)}. \end{aligned}$$

On the one hand, we have

$$\mathbb{P}(A) = \mathbb{P}(\{Y \leq X - 0.5\}) = \int_{\frac{1}{2}}^{1} \int_0^{x-0.5} 3y\, \mathsf{d}y\, \mathsf{d}x = \frac{1}{16}.$$

On the other hand, we have

$$\begin{aligned} \mathbb{P}(\{Y \leq y\} \cap A) &= \mathbb{P}(\{Y \leq y\} \cap \{Y \leq X - 0.5\}) \\ &= \mathbb{P}(\{Y \leq \mathsf{min}\{y, X - 0.5\}) \\ &\overset{(*)}{=} \mathbb{P}(\{Y \leq \mathsf{min}\{y, X - 0.5\} \cap \{X - 0.5 < y\}) + \mathbb{P}(\{Y \leq \mathsf{min}\{y, X - 0.5\} \cap \{X - 0.5 \geq y\}) \\ &= \mathbb{P}(\{Y \leq X - 0.5\} \cap \{X - 0.5 < y\}) + \mathbb{P}(\{Y \leq y\} \cap \{X - 0.5 \geq y\}) \\ &= \int_{\frac{1}{2}}^{\frac{1}{2}+y} \int_0^{u-0.5} 3v\, \mathsf{d}v\, \mathsf{d}u + \int_{\frac{1}{2}+y}^{1} \int_0^{y} 3v\, \mathsf{d}v\, \mathsf{d}u = \frac{y^3}{2} + \frac{3y^2}{4} - \frac{3y^3}{2} = \frac{3y^2}{4} - y^3, \end{aligned}$$

where $(*)$ follows from the law of total probability. We thus have

$$F_{Y|A}(y) = \begin{cases} 0, & y < 0, \\ 12y^2 - 16y^3, & 0 \leq y < \frac{1}{2}, \\ 1, & y \geq \frac{1}{2}, \end{cases}$$

from which we get

$$f_{Y|A}(y) = \begin{cases} 24y - 48y^2, & 0 < y < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, we have

$$\mathbb{E}[Y|A] = \int_0^{\frac{1}{2}} y\, f_{Y|A}(y)\, \mathsf{d}y = \int_0^{\frac{1}{2}} (24y^2 - 48y^3)\, \mathsf{d}y = \frac{1}{4}.$$

(b) For any $x \in [-1, 1]$, we have

$$f_X(x) = \int_0^{|x|} 3y\, \mathsf{d}y = \frac{3x^2}{2},$$

Noting that $f_X(x) = 0$ for $x = 0$, we have for all $x \in [-1, 1] \setminus \{0\}$ that

$$f_{Y|\{X=x\}}(y) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \frac{2y}{x^2}, \qquad 0 \leq y \leq |x|.$$

Thus, for any $x \in [-1,1] \setminus \{0\}$, we have

$$\mathbb{E}[Y|\{X=x\}] = \int_0^{|x|} y\,\frac{2y}{x^2}\,\mathsf{d}y = \frac{2|x|}{3},$$

from which it follows that $\mathbb{E}[Y|X] = \frac{2|X|}{3}$.
Finally, using the law of iterated expectations, we have

$$\mathbb{E}[Y] = \mathbb{E}[\mathbb{E}[Y|X]] = \frac{2}{3}\,\mathbb{E}[|X|].$$

We now note that

$$\mathbb{E}[|X|] = \int_{-1}^{1} |x| \cdot \frac{3x^2}{2}\,\mathsf{d}x = \int_0^1 3x^3\,\mathsf{d}x = \frac{3}{4},$$

from which we get $\mathbb{E}[Y] = \frac{2}{3}\,\mathbb{E}[|X|] = \frac{1}{2}$.

7. Define $\mathsf{Var}(X|Y)$ as

$$\mathsf{Var}(X|Y) = \mathbb{E}[(X - \mathbb{E}[X|Y])^2|Y] = \mathbb{E}[X^2|Y] - (\mathbb{E}[X|Y])^2.$$

Verify the relation

$$\mathsf{Var}(X) = \mathbb{E}[\mathsf{Var}(X|Y)] + \mathsf{Var}(\mathbb{E}[X|Y]).$$

Solution: From the given formula for $\mathsf{Var}(X|Y)$, we have

$$\begin{aligned}\mathbb{E}[\mathsf{Var}(X|Y)] &= \mathbb{E}[\mathbb{E}[X^2|Y]] - \mathbb{E}[(\mathbb{E}[X|Y])^2] \\ &= \mathbb{E}[X^2] - \mathbb{E}[(\mathbb{E}[X|Y])^2],\end{aligned}$$

where the last line above follows from the law of iterated expectations. Also, noting from the law of total expectations that $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$, we have

$$\begin{aligned}\mathsf{Var}(\mathbb{E}[X|Y]) &= \mathbb{E}[(\mathbb{E}[X|Y] - \mathbb{E}[X])^2] \\ &= \mathbb{E}[(\mathbb{E}[X|Y])^2] + (\mathbb{E}[X])^2 - 2\,\mathbb{E}[\mathbb{E}[X]\,\mathbb{E}[X|Y]] \\ &= \mathbb{E}[(\mathbb{E}[X|Y])^2] - (\mathbb{E}[X])^2.\end{aligned}$$

Combining the above results, we get

$$\mathbb{E}[\mathsf{Var}(X|Y)] + \mathsf{Var}(\mathbb{E}[X|Y]) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \mathsf{Var}(X).$$