Name:
Roll Number:
Department:
Program: BTech / MTech TA / MTech RA / PhD (Tick one)

AI5030: Probability and Stochastic Processes

Quiz 6

Date: 18 November 2024

Question	1	2	Total
Marks Scored			

Instructions:

- **Fill in your name and roll number on each of the pages.**
- **You may use any result covered in class directly without proving it.**
- **Unless explicitly stated in the question, DO NOT use any result from the homework without proof.**

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Assume that all random variables appearing in the questions below are defined with respect to $\mathscr{F}$.

1. **(2 Marks)**
   Let $U \sim \text{Unif}(0, 2)$. Compute $\mathbb{E}[U|\{U > 1\}]$.

   Solution: The conditional CDF of U, conditioned on the event $\{U > 1\}$, is given by

$$F_{U|\{U>1\}}(u) = \frac{\mathbb{P}(\{U \le u\} \cap \{U > 1\})}{\mathbb{P}(\{U > 1\})} = \frac{\mathbb{P}(\{1 < U \le u\})}{\mathbb{P}(\{1 < U < 2\})} = \frac{F_U(u) - F_U(1)}{F_U(2) - F_U(1)} = \begin{cases} 0, & u \le 1, \\ u - 1, & 1 < u < 2, \\ 1, & u \ge 2. \end{cases}$$

   Then, the conditional PDF of U, conditioned on the event $\{U > 1\}$, is given by

$$f_{U|\{U>1\}}(u) = \begin{cases} 1, & u \in (1, 2), \\ 0, & \text{otherwise}. \end{cases}$$

   We then have

$$\mathbb{E}[U|\{U > 1\}] = \int_1^2 u \, f_{U|\{U>1\}}(u) \, \mathrm{d}u = \int_1^2 u \, \mathrm{d}u = \frac{3}{2}.$$

   Alternatively, noting that U is a non-negative random variable even when conditioned on $\{U > 1\}$, and using the formula for the expectation of a non-negative random variable in terms of its complementary CDF, we get

$$\mathbb{E}[U|\{U > 1\}] = \int_0^\infty (1 - F_{U|\{U>1\}}(u)) \, \mathrm{d}u = \int_0^1 \mathrm{d}u + \int_1^2 (2 - u) \, \mathrm{d}u = \frac{3}{2}.$$

2. **(3 Marks)**

Let X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 3y, & -1 \leq x \leq 1,\ 0 \leq y \leq |x|, \\ 0, & \text{otherwise.} \end{cases}$$

Determine $\mathbb{E}[X|Y]$ and $\mathbb{E}[X^2|Y]$.

Solution:

We note that Y takes values in the interval $[0,1]$. For any given $y \in [0,1]$, we have

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\mathsf{d}x = \int_{-1}^{-y} 3y\,\mathsf{d}x + \int_{y}^{1} 3y\,\mathsf{d}x = 6y(1-y).$$

Noting that $f_Y(y) = 0$ for $y = 0, 1$, we have for any $y \in (0,1)$ that

$$f_{X|\{Y=y\}}(x) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{3y}{6y(1-y)} = \frac{1}{2(1-y)}, & x \in [-1,-y] \cup [y,1], \\ 0, & \text{otherwise.} \end{cases}$$

Then, for any $y \in (0,1)$, we have

$$\mathbb{E}[X|\{Y=y\}] = \int_{-\infty}^{\infty} x\, f_{X|\{Y=y\}}(x)\,\mathsf{d}x = \int_{-1}^{-y} x \cdot \frac{1}{2(1-y)}\,\mathsf{d}x + \int_{y}^{1} x \cdot \frac{1}{2(1-y)}\,\mathsf{d}x = \frac{y^2-1}{2(1-y)} + \frac{1-y^2}{2(1-y)} = 0,$$

from which it follows that $\mathbb{E}[X|Y] = 0$.

Along similar lines, we have for any $y \in (0,1)$ that

$$\mathbb{E}[X^2|\{Y=y\}] = \int_{-\infty}^{\infty} x^2\, f_{X|\{Y=y\}}(x)\,\mathsf{d}x = \int_{-1}^{-y} x^2 \cdot \frac{1}{2(1-y)}\,\mathsf{d}x + \int_{y}^{1} x^2 \cdot \frac{1}{2(1-y)}\,\mathsf{d}x = \frac{1-y^3}{3(1-y)} = \frac{1+y+y^2}{3},$$

from which it follows that

$$\mathbb{E}[X^2|Y] = \frac{1+Y+Y^2}{3}.$$