AI 5030: Probability and Stochastic Processes

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Homework 6

Topics: Joint PDFs, Conditional PDFs, Transformations

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables appearing below are assumed to be defined with respect to $\mathscr{F}$.

1. Virat and Anushka have a date at 7 pm. Each will arrive at the meeting place with a delay that is distributed uniformly randomly between 0 minutes and 60 minutes, independent of the delay of the other. The first to arrive will wait for 15 minutes and leave if the other does not arrive within 15 minutes. Find the probability that both meet.

2. Let $X_1, X_2, X_3 \overset{\text{i.i.d.}}{\sim} \text{Unif}((0,1))$.
   (a) Compute $\mathbb{P}(\{X_1 + X_2 > X_3\})$.
   (b) Derive the CDF of the random variable $X = X_1 X_2$.
   (c) Taking help of the result from part (b), show that $\mathbb{P}(\{X_1 X_2 \le X_3^2\}) = \frac{5}{9}$.

3. Let X and Y be jointly continuous random variables with the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx(y-x)e^{-y}, & 0 \le x \le y < +\infty, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Determine the value of the constant c.
   (b) Show that

$$f_{X|Y=y}(x) = \begin{cases} 6x(y-x)y^{-3}, & 0 \le x \le y, \\ 0, & \text{otherwise,} \end{cases} \qquad f_{Y|X=x}(y) = \begin{cases} (y-x)e^{x-y}, & y \ge x, \\ 0, & \text{otherwise,} \end{cases}$$

4. Suppose that X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cy, & -1 \le x \le 1, \ 0 \le y \le |x|, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Determine the constant c.
   (b) Are X and Y independent?
   (c) Evaluate $\mathbb{P}(\{X \ge Y + 0.5\})$.
   (d) Compute the conditional PDF of X, conditioned on the event $\{Y > 0.5\}$.
   Using the above conditional PDF, evaluate $\mathbb{P}(\{X > 0.75\} | \{Y > 0.5\})$.

5. Let $X_1, X_2 \overset{\text{i.i.d.}}{\sim} \text{Exponential}(\lambda)$. Using the bivariate Jacobian transformation formula, compute the joint PDF of $Y_1 = X_1 + X_2$ and $Y_2 = X_1/X_2$, and show that $Y_1 \perp\!\!\!\perp Y_2$.

6. Let $X_1, X_2 \overset{\text{i.i.d.}}{\sim} \text{Exponential}(\lambda)$. Let $X = X_1$ and $Y = X_1 + X_2$. Determine the joint PDF of X and Y using the bivariate Jacobian transformation formula. Further, for any $y > 0$, show that the conditional PDF of X, conditioned on the event $\{Y = y\}$, is the uniform PDF on the interval $[0, y]$.

7. Let $X, Y \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$. Let $R = \sqrt{X^2 + Y^2}$ and $\Theta = \arctan \frac{Y}{X} = \tan^{-1}\left(\frac{Y}{X}\right)$. Determine the joint PDF of R and Θ, and show that $R \perp\!\!\!\perp \Theta$. What are the marginal PDFs of R and Θ?

8. Let R and Θ be two random variables with the joint PDF

$$f_{R,\Theta}(r,\theta) = re^{-r^2/2} \cdot \frac{1}{2\pi}, \qquad r \ge 0, \quad \theta \in [0, 2\pi].$$

(a) Compute the marginal PDFs of R and Θ, and show that $R \perp\!\!\!\perp \Theta$.

(b) Let $X = R\cos(\Theta)$ and $Y = R\sin(\Theta)$. Show that $X, Y \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$.

9. Let X be a random variable with CDF $F : \mathbb{R} \to [0,1]$.
Define a second function $g_F : (0,1) \to \mathbb{R}$ as

$$g_F(u) = \min\{x \in \mathbb{R} : F(x) \geq u\}, \qquad u \in (0,1).$$

(a) Argue that $F(g_F(u)) \geq u$ for all $u \in (0,1)$.

(b) Let $U \sim \text{Unif}((0,1))$. Show that the random variable $Z = g_F(U)$ has the same CDF as X, i.e., $F_Z = F$. Here, the random variable Z is defined as

$$Z(\omega) = g_F(U(\omega)), \qquad \omega \in \Omega.$$

Remark: This result is the premise for sampling from a custom distribution function F in computer simulations. To generate a sample $X \sim F$, we first generate a random sample U uniformly from $(0,1)$ and set $X = g_F(U)$.
Hint: Fix an arbitrary $z \in \mathbb{R}$.
Using the result in part (a) and the fact that F is non-decreasing, argue that $\{g_F(U) \leq z\} \subseteq \{U \leq F(z)\}$.
On the other hand, using the definition of g_F, show that $\{U \leq F(z)\} \subseteq \{g_F(U) \leq z\}$.
(To show $A \subseteq B$, you need to argue that $\omega \in A$ implies $\omega \in B$.)

(c) Let X be a discrete random variable with the following PMF:

$$p_X(x) = \begin{cases} 0.1, & x = 10, \\ 0.2, & x = 20, \\ 0.3, & x = 30, \\ 0.4, & x = 40, \\ 0, & \text{otherwise.} \end{cases}$$

Explicitly compute the CDF F and the function g_F corresponding to the above PMF.
Write a Python program to generate $N = 100{,}000$ independent samples uniformly from $(0,1)$. Call these samples $\{u_1, u_2, \dots, u_N\}$. For each $i \in \{1, \dots, N\}$, set $x_i = g_F(u_i)$. Plot the histogram of the samples $\{x_1, \dots, x_N\}$, and verify that the histogram matches closely with the PMF p_X.
You may use the NumPy module `random.random()` to generate samples uniformly from $(0,1)$.

(d) How will you generate a random sample from Exponential(1) distribution on a computer?