AI 5030: Probability and Stochastic Processes

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Homework 2

Topics: Sample Space, Algebra, σ-Algebra

Recall that a set is said to be countable if it is finite or countably infinite.
For all the problems below, unless explicitly stated otherwise, the sample space Ω can be countable or uncountable.

1. Let Ω be a sample space, and let $\mathscr{F}$ be a σ-algebra of subsets of Ω. Argue that $\mathscr{F}$ is closed under countable intersections.
   Hint: Apply De Morgan's laws.

2. Let Ω be a sample space. Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two σ-algebras of subsets of Ω. Show, via an example, that $\mathscr{F} = \mathscr{F}_1 \cup \mathscr{F}_2$ is not necessarily a σ-algebra.
   Note: This exercise shows that union of σ-algebras is not necessarily a σ-algebra.

3. Let Ω be a sample space.
   (a) Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two σ-algebras of subsets of Ω. Show that $\mathscr{F} = \mathscr{F}_1 \cap \mathscr{F}_2$ is also a σ-algebra.
   (b) More generally, let $\mathcal{I}$ be an arbitrary index set (finite, countably infinite, or uncountable), and for each $i \in \mathcal{I}$, let $\mathscr{F}_i$ be a σ-algebra of subsets of Ω. Show that
   $$\mathscr{F} = \bigcap_{i \in \mathscr{I}} \mathscr{F}_i$$
   is also a σ-algebra.
   This exercise shows that intersection of σ-algebras is necessarily a σ-algebra.

4. Let Ω be a sample space, and let $\mathscr{F}$ be a σ-algebra of subsets of Ω. Fix $B \in \mathscr{F}$, and consider the collection
   $$\mathscr{G} = \{A \cap B : A \in \mathscr{F}\}.$$
   That is, $\mathscr{G}$ is a collection of subsets of B formed by taking the intersection of each set in $\mathscr{F}$ with B. Show that $\mathscr{G}$ is a σ-algebra of subsets of B.

5. Let Ω be a sample space. Consider the collection
   $$\mathscr{A}_1 = \{A \subseteq \Omega : A \text{ is finite or } \Omega \setminus A \text{ is finite}\}. \tag{1}$$
   (a) Prove that $\mathscr{A}_1$ is an algebra.
   (b) Construct an example to show that $\mathscr{A}_1$ is not necessarily a σ-algebra.
   Hint: Consider $\Omega = \mathbb{R}$ and $A = \mathbb{Q}$, the set of rational numbers. What do you know about $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$?

6. Let Ω be a sample space. Consider the collection
   $$\mathscr{A}_2 = \{A \subseteq \Omega : A \text{ is countable or } \Omega \setminus A \text{ is countable}\}. \tag{2}$$
   Prove that $\mathscr{A}_2$ is a σ-algebra.
   Hint: Recall that countable means finite or countably infinite. Use the lemma "countable union of countable sets is countable" covered in class.