

Probability and Stochastic Processes

Expectations over Different Spaces, Absolute Continuity of Measures, Radon–Nikodym Theorem, Expectations of Continuous Random Variables, Variance, Covariance, Uncorrelatedness and Independence

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14/17 October 2024



Expectation Over Different Spaces

Expectation Over Different Spaces

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let X be a discrete random variable w.r.t. \mathscr{F} .

Let $g:\mathbb{R} o \mathbb{R}$ be Borel-measurable.

Let Y = g(X).

Theorem (Expectation Over Different Spaces)

We have

$$\mathbb{E}[Y] = \int_{\Omega} g(X) \, \mathrm{d}\mathbb{P} = \int_{\mathbb{R}} g \, \mathrm{d}\mathbb{P}_X = \int_{\mathbb{R}} \gamma \, \mathrm{d}\mathbb{P}_Y.$$



Proof of Theorem - 1

Suppose g is simple, and Range $(g) = \{y_1, \dots, y_n\}$.

- Y = g(X) is a simple random variable taking values $y_1, \dots, y_n \ge 0$
- We then have

$$\int_{\mathbb{R}} y \, d\mathbb{P}_{Y} = \sum_{i=1}^{n} \gamma_{i} \, \mathbb{P}_{Y}(\{y_{i}\})$$

$$= \sum_{i=1}^{n} \gamma_{i} \, \mathbb{P}(\{\omega \in \Omega : Y(\omega) = y_{i}\})$$

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$$= \sum_{i=1}^{n} \gamma_{i} \, \mathbb{P}(\{\omega \in \Omega : g(X(\omega)) = y_{i}\})$$



Proof of Theorem - 2

Suppose g is non-negative

- Y = g(X) is a non-negative random variable
- There exists a sequence of simple functions $\{g_n\}_{n=1}^{\infty}$ such that $g_n \uparrow g$ pointwise
- $Y_n = g_n(X) \uparrow Y$ pointwise, Y_n simple for all n
- We have

$$egin{aligned} \int_{\Omega} \mathbf{Y} \, \mathrm{d}\mathbb{P} & \stackrel{ ext{MCT}}{=} \lim_{n o \infty} \int_{\Omega} \mathbf{Y}_n \, \mathrm{d}\mathbb{P} \ &= \lim_{n o \infty} \int_{\Omega} g_n(\mathbf{X}) \, \mathrm{d}\mathbb{P} \ &= \lim_{n o \infty} \int_{\mathbb{R}} g_n \, \mathrm{d}\mathbb{P}_{\mathbf{X}} \ &\stackrel{ ext{MCT}}{=} \int_{\mathbb{R}} g \, \mathrm{d}\mathbb{P} \end{aligned}$$



Expectations of Continuous Random Variables

Absolute Continuity of Measures

Consider a measurable space (Ω, \mathscr{F}) .

Let $\mu:\mathscr{F}\to [0,+\infty]$ and $\nu:\mathscr{F}\to [0,+\infty]$ be two measures.

Definition (Absolute Continuity of Measures)

We say ν is absolutely continuous with respect to μ if

$$\mu(\mathbf{A}) = \mathbf{0} \implies \nu(\mathbf{A}) = \mathbf{0}.$$

Notation: $\nu \ll \mu$.

Remark: The above definition applies to probability measures also

Radon-Nikodym Theorem

Consider a measurable space (Ω, \mathscr{F}) .

Let $\mu:\mathscr{F}\to[0,+\infty]$ and $\nu:\mathscr{F}\to[0,+\infty]$ be two measures.

Theorem (Radon-Nikodym Theorem)

Suppose that $\nu \ll \mu$.

Then, there exists a non-negative, measurable function $f:\Omega \to [0,+\infty]$ such that

$$u(\mathtt{A}) = \int_{\mathtt{A}} f \, \mathsf{d} \mu = \int_{\Omega} f \, \mathbf{1}_{\mathtt{A}} \, \mathsf{d} \mu, \qquad orall \mathtt{A} \in \mathscr{F}.$$

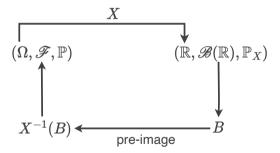
Notation:
$$f = \frac{d\nu}{d\mu}$$
.



Continuous Random Variable - New Definition

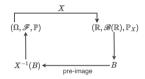
Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $X : \Omega \to \mathbb{R}$ be a random variable with respect to \mathscr{F} .





Continuous Random Variable - New Definition



Definition (Continuous Random Variable)

A random variable X is said to be continuous if $\mathbb{P}_X \ll \lambda$, where λ is the Lebesgue measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$.

Then, by the Radon–Nikodym theorem, there exists a non-negative, measurable function, say $f: \mathbb{R} \to [0, +\infty]$, such that

$$\mathbb{P}_{\!X}(A) = \int_A f \, \mathsf{d} \lambda, \qquad A \in \mathscr{B}(\mathbb{R}).$$

The function f is called the probability density function (PDF) of X.



Expectation of a Continuous Random Variable

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $X : \Omega \to \mathbb{R}$ be a continuous random variable w.r.t. \mathscr{F} , with PDF f_X .

Theorem (Expectation for Continuous Random Variables)

Suppose that $g: \mathbb{R} \to \mathbb{R}$ is measurable. Then,

$$\mathbb{E}[g(X)] = \int_{\mathbb{R}} g f_X \, \mathrm{d} \lambda.$$

In particular,

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f_X \, \mathrm{d}\lambda$$



Proof of Theorem - 1

Assume that g is simple and takes values $y_1, \ldots, y_n \geq 0$.

We have

$$\begin{split} \mathbb{E}[g(X)] &= \int_{\mathbb{R}} g \, d\mathbb{P}_{X} \\ &= \sum_{i=1}^{n} \gamma_{i} \, \mathbb{P}_{X}(\underbrace{\{x \in \mathbb{R} : g(x) = y_{i}\}}) \\ &= \sum_{i=1}^{n} \gamma_{i} \, \mathbb{P}_{X}(B_{i}) \\ &\stackrel{\text{R.N.Thm}}{=} \sum_{i=1}^{n} \gamma_{i} \, \int_{B_{i}} f_{X} \, d\lambda \qquad \qquad = \sum_{i=1}^{n} \int_{\mathbb{R}} \gamma_{i} \mathbf{1}_{B_{i}} f_{X} \, d\lambda = \int_{\mathbb{R}} \sum_{i=1}^{n} \gamma_{i} \mathbf{1}_{B_{i}} f_{X} \, d\lambda = \int_{\mathbb{R}} g f_{X} \, d\lambda. \end{split}$$



Proof of Theorem - 2

Assume that g is non-negative.

- There exists a sequence of simple functions $g_n \uparrow g$ pointwise
- We have

$$\begin{split} \mathbb{E}[g(X)] &\stackrel{\text{MCT}}{=} \lim_{n \to \infty} \mathbb{E}[g_n(X)] \\ &= \lim_{n \to \infty} \int_{\mathbb{R}} g_n f_X \, \mathrm{d}\lambda \\ &\stackrel{\text{MCT}}{=} \int_{\mathbb{R}} \lim_{n \to \infty} g_n f_X \, \mathrm{d}\lambda \qquad \text{(because } g_n f_X \uparrow g f_X \text{ pointwise, as } f_X \geq 0) \\ &= \int_{\mathbb{R}} g f_X \, \mathrm{d}\lambda. \end{split}$$

Examples

- Suppose $X \sim \text{Exponential}(\mu)$. Compute $\mathbb{E}[X]$ and $\mathbb{E}[X^2]$.
- Suppose $X \sim \mathcal{N}(\mu, \sigma^2)$. Compute $\mathbb{E}[X]$, $\mathbb{E}[X^2]$, and $\mathbb{E}[(X \mu)^3]$.
- Suppose $f_X(x)=rac{1}{\pi}\cdotrac{1}{1+x^2},\quad x\in\mathbb{R}.$ Compute $\mathbb{E}[X].$

Exercises

Compute $\mathbb{E}[X], \mathbb{E}[X^2]$ for each of the following cases:

- $X \sim \mathrm{Ber}(p)$.
- $X \sim \text{Poisson}(\lambda)$?
- $X \sim \text{Unif}([a, b])$?



Variance, Covariance, and Correlation

Variance

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let X be random variable with respect to \mathscr{F} .

Let $\mathbb{E}[X]$ be well defined (i.e., not of the form $\infty - \infty$).

Definition (Variance)

The variance of *X* is defined as

$$\operatorname{Var}(X) := \mathbb{E}[(X - \mathbb{E}[X])^2].$$

Remarks:

- $Var(X) \geq 0$.
- The quantity $\sigma_X = \sqrt{\operatorname{Var}(X)}$ is called the standard deviation of X.

A Result on Zero Variance

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let X be random variable with respect to \mathscr{F} .

Let $\mathbb{E}[X]$ be well defined (i.e., not of the form $\infty - \infty$).

Lemma (Zero Variance)

The variance of *X* is zero if and only

$$\mathbb{P}(\{X=c\})=1$$
 for some constant c .

An Alternative Expression for Variance

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let X be random variable with respect to \mathscr{F} .

Alternative Expression for Variance

Let $\mathbb{E}[X]$ be well defined (i.e., not of the form $\infty - \infty$).

1. If
$$\left|\mathbb{E}[X]\right|=+\infty$$
, then $\mathrm{Var}(X)=+\infty$.

2. If
$$\left|\mathbb{E}[X]\right|<+\infty$$
, then

$$\operatorname{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

Remark: Because Var(X) > 0, we always have $\mathbb{E}[X^2] > (\mathbb{E}[X])^2$

Examples

- Compute the variance of $X \sim \text{Ber}(p)$.
- What is the variance of $X \sim \text{Poisson}(\lambda)$?
- What is the variance of $X \sim \text{Unif}([a, b])$?
- What is the variance of $X \sim \text{Exponential}(\mu)$?
- What is the variance of $X \sim \mathcal{N}(\mu, \sigma^2)$?
- Give an example of a random variable X for which $\Big|\mathbb{E}[X]\Big|<+\infty$, but $\mathrm{Var}(X)=+\infty$.

Covariance

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let X, Y be random variables with respect to \mathscr{F} .

Let $\mathbb{E}[X]$, $\mathbb{E}[Y]$ be well defined (i.e., not of the form $\infty - \infty$).

Definition (Covariance)

The covariance of *X* and *Y* is defined as

$$Cov(X, Y) := \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])],$$

provided the expectation on the right-hand side is well defined (i.e., not $\infty - \infty$). Furthermore,

$$Cov(X, Y) = \mathbb{E}[XY] - \mathbb{E}[X] \mathbb{E}[Y],$$

provided the right-hand side is not of the form $\infty - \infty$.

Uncorrelated Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let X, Y be random variables with respect to \mathscr{F} . Let $\mathbb{E}[X], \mathbb{E}[Y]$ be well defined (i.e., not of the form $\infty - \infty$).

Definition (Uncorrelated Random Variables)

X and Y are said to be uncorrelated if

$$Cov(X, Y) = 0.$$

Uncorrelatedness and Independence

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let X, Y be random variables with respect to \mathscr{F} .

Let $\mathbb{E}[X]$, $\mathbb{E}[Y]$ be well defined (i.e., not of the form $\infty - \infty$).

Theorem (Uncorrelatedness and Independence)

If $X \perp \!\!\! \perp Y$, then

$$Cov(X, Y) = 0.$$

The converse is not true in general.