1. We note that for any $t \in \mathbb{R}$,

$$M_Z(t) = \mathbb{E}[e^{tZ}] = \mathbb{E}[e^{tZ} \mathbb{1}_{\{|Y|>a\}}] + \mathbb{E}[e^{tZ} \mathbb{1}_{\{|Y| \le a\}}]$$

$$= \mathbb{E}[e^{t(-Y)} \mathbb{1}_{\{|Y|>a\}}] + \mathbb{E}[e^{tY} \mathbb{1}_{\{|Y| \le a\}}]$$

using the definition for Z given in question

Now,

$$\mathbb{E}[e^{t(-Y)} \mathbb{1}_{\{|Y|>a\}}] = \mathbb{E}[e^{t(-Y)} \mathbb{1}_{\{|-Y|>a\}}]$$

$$= \int_{-\infty}^{\infty} e^{tx} \mathbb{1}_{\{|x|>a\}} f_{-Y}(x)\, dx \quad \text{[noting that } -Y \text{ takes values in } -\infty \text{ to } +\infty]$$

$$= \int_{-\infty}^{\infty} e^{tx} \mathbb{1}_{\{|x|>a\}} f_Y(x)\, dx \quad [\because Y \text{ and } -Y \text{ have the same PDF}]$$

$$= \mathbb{E}[e^{tY} \mathbb{1}_{\{|Y|>a\}}].$$

Plugging back, we get

$$M_Z(t) = \mathbb{E}[e^{tY} \mathbb{1}_{\{|Y|>a\}}] + \mathbb{E}[e^{tY} \mathbb{1}_{\{|Y| \le a\}}]$$

$$= \mathbb{E}[e^{tY}]$$

$$= M_Y(t).$$

This implies that CDFs of Z and Y match

$$\Rightarrow Z \sim N(0,1).$$

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2. From the given joint PDF expression, it is easy to check that

$$f_X(x) = \frac{1}{2}, \quad x \in (-1,1), \qquad f_Y(y) = \frac{1}{2}, \quad y \in (-1,1).$$

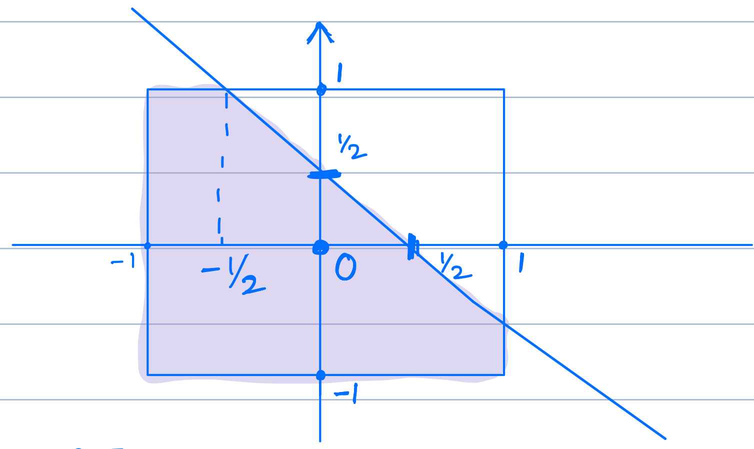
Clearly, $f_{X,Y}(1/2, 1/3) \neq 1/4 = f_X(1/2)\, f_X(1/3)$, so $X \not\perp\!\!\!\perp Y$.

We now evaluate the CDF of $Z = X+Y$.

Note that $X+Y$ takes values between -2 to $+2$.

We will compute $P(X+Y \le z)$ for $z \in (-2,2)$.

As an example,



$$P(X+Y \le 0.5) = \int_{-1}^{-1/2} \int_{-1}^{1} f_{X,Y}(x,y)\, dy\, dx + \int_{-1/2}^{1} \int_{-1}^{0.5-x} f_{X,Y}(x,y)\, dy\, dx. \quad \text{(see Purple region)}$$

By carefully evaluating the CDF, and differentiating it to get the PDF, it can be shown that

$$f_Z(z) = \begin{cases} \frac{2+z}{4}, & z \in (-2,0), \\ \frac{2-z}{4}, & z \in (0,2), \\ 0, & \text{otherwise}. \end{cases}$$

This corresponds to the convolution of the PDFs of X and Y, but $X \not\perp\!\!\!\perp Y$.

Summary: Convolution of PDFs $\not\Rightarrow$ independence.

3.

$$G_X(z) = \frac{(\tfrac{1}{3}z)^4 + 4(\tfrac{1}{3}z)^3(\tfrac{2}{3}) + 6(\tfrac{1}{3}z)^2(\tfrac{2}{3})^2 + 4(\tfrac{1}{3}z)(\tfrac{2}{3})^3 + (\tfrac{2}{3})^4}{z}$$

$$= \frac{z^3}{81} + \frac{8}{81}z^2 + \frac{24}{81}z + \frac{32}{81} + \frac{16}{81}z^{-1}.$$

From the above expression, noting that

$$G_X(z) = \sum_{k=-\infty}^{\infty} z^k \, \mathbb{P}(X=k),$$

we get

$$\mathbb{P}(X=k) = \begin{cases} \frac{1}{81}, & k=3, \\ \frac{8}{81}, & k=2, \\ \frac{24}{81}, & k=1, \\ \frac{32}{81}, & k=0, \\ \frac{16}{81}, & k=-1, \\ 0, & \text{otherwise}. \end{cases}$$

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4. We have

$$M_Y(t) = \mathbb{E}[e^{tY}] = \mathbb{E}\left[e^{t\sum_{i=1}^{n} X_i}\right] \underset{\text{independence}}{=} \prod_{i=1}^{n} \mathbb{E}[e^{tX_i}] \overset{\text{identical distribution}}{=} \left(\mathbb{E}[e^{tX_1}]\right)^n.$$

Now,

$$\mathbb{E}[e^{tX_1}] = \sum_{k=0}^{\infty} e^{tk} \cdot p_{X_1}(k) = \sum_{k=0}^{\infty} e^{tk} \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \cdot \underbrace{\sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!}}_{\text{Taylor series expansion of } e^{\lambda e^t}}$$

$$= e^{-\lambda} \cdot e^{\lambda e^t}$$

$$= e^{\lambda(e^t - 1)}.$$

Then,

$$M_Y(t) = e^{n\lambda(e^t - 1)}$$

$$\Rightarrow \quad Y \sim \text{Poisson}(n\lambda).$$

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5. <u>If part</u> :

Suppose $f_X(x) = f_X(-x) \quad \forall\, x \in \mathbb{R}$.

Then,

$$C_X(s) = \mathbb{E}\left[e^{jsX}\right] = \mathbb{E}\left[\cos sX\right] + j\, \mathbb{E}\left[\sin sX\right].$$

Now,

$$\mathbb{E}\left[\sin sX\right] = \underbrace{\int_{-\infty}^{\infty} \underbrace{\sin(sx)}_{\text{odd function}} \cdot \underbrace{f_X(x)}_{\text{even function}}}_{\text{odd function}}\, dx = 0.$$

$$\Rightarrow \quad C_X \text{ is real}.$$

<u>Only if part</u>

Suppose $C_X(s)$ is real $\forall\, s \in \mathbb{R}$

$$\Rightarrow \quad C_X(s) = \overline{C_X(s)}$$

Note that

$$\bar{C}_X(s) = \mathbb{E}\left[\cos sX\right] - j\, \mathbb{E}\left[\sin sX\right]$$

$$= \mathbb{E}\left[e^{-jsX}\right]$$

$$= \mathbb{E}\left[e^{js(-X)}\right] = C_{-X}(s) \qquad \forall\, s \in \mathbb{R}$$

$\Rightarrow$ X and $-X$ have same Characteristic function

$\Rightarrow$ X and $-X$ have same CDF

$\Rightarrow$ $f_X(x) = f_{-X}(x) \quad \forall\, x \in \mathbb{R}$

$\Rightarrow$ $f_X(x) = f_X(-x) \quad \forall\, x \in \mathbb{R} \quad \left(\because\ f_{-X}(x) = f_X(-x)\right).$

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6. We have

$$M_Y(t) = \mathbb{E}[e^{tY}] = \mathbb{E}[e^{tWX}] = \mathbb{E}[\mathbb{E}[e^{tWX} | W]]$$

$$= \mathbb{E}[e^{tX} | W=1] \cdot \frac{1}{2} + \mathbb{E}[e^{-tX} | W=-1] \cdot \frac{1}{2}$$

law of iterated expectations or law of total probability

$$= \mathbb{E}[e^{tX}] \cdot \frac{1}{2} + \mathbb{E}[e^{-tX}] \cdot \frac{1}{2}$$

$W \perp\!\!\!\perp X$

$$= \mathbb{E}[e^{tX}] \cdot \frac{1}{2} + \mathbb{E}[e^{tX}] \cdot \frac{1}{2}$$

X and −X have same CDF, therefore same MGF

$$= \mathbb{E}[e^{tX}] \qquad \forall\ t \in \mathbb{R}$$

$$\Rightarrow \quad Y \sim N(0,1).$$