Experiment - 3

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Division using Newton Raphson method

Suppose that $f \in C^2[a,b]$. Let $p_0 \in [a,b]$ be an approximation to p such that $f'(p_0) \neq 0$ and $|p - p_0|$ is small.

Consider the first taylor polynomial for $f(x)$ expanded about p_0 and evaluated at $x = p$

$$\boxed{f(p) = f(p_0) + (p - p_0) f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p))}$$

Since $f(p) = 0$, this equation gives

$$\boxed{f(p_0) + (p - p_0) f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p)) = 0}$$

Newton raphson method is derived by assuming that since $|p - p_0|$ is small, the term involving $(p - p_0)^2$ is much smaller.

so $$\boxed{f(p_0) + (p - p_0) f'(p_0) = 0}$$

Solving for p gives

$$p = p_0 - \frac{f(p_0)}{f'(p_0)}$$

This sets the stage for newton's method which starts with an initial approximation P_0 and generates the sequence $\{P_n\}_{n=0}^{\infty}$ by

$$\boxed{P_n = P_{n-1} - \frac{f(P_{n-1})}{f'(P_{n-1})} \quad \forall\, n \geq 1}$$

Newton raphson method for division

we have to compute $\boxed{q = \frac{z}{d}}$

→ first find $\frac{1}{d}$ and then multiply this with z

* The method we use for computing $1/d$ is based on newton-raphson iteration to determine the root of $f(x)=0$

$$\boxed{x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}}$$

$$f(x) = \frac{1}{x} - d$$

$$f'(x) = -\frac{1}{x^2}$$

$\therefore$

$$x_{i+1} = x_i - \frac{\frac{1}{x_i} - d}{-\frac{1}{x_i^2}}$$

$$\boxed{x_{i+1} = x_i(2 - x_i d)}$$

Computationally, two multiplications and one subtraction is are required for each iteration.

Error analysis:

let $\delta_i = \frac{1}{d} - x_i$ be the error at the ith iteration. Then

$$\delta_{i+1} = \frac{1}{d} - x_{i+1} = \frac{1}{d} - x_i(2 - x_i d)$$

$$= \frac{1}{d} - 2x_i + x_i^2 d$$

$$= d\left(\frac{1}{d} - x_i\right)^2$$

$$= d(\delta_i)^2$$

$$\delta_{i+1} = d(\delta_i)^2$$

Since $d < 1$ we have $\delta_{i+1} < (\delta_i)^2$ Proving quadratic Convergence.

Choosing the initial estimate

if the initial Value x_0 is Chosen such that $0 < x_0 < \frac{2}{d}$, leading $|\delta_0| < \frac{1}{d}$, convergence is guaranteed.

→ before doing the newton raphson method normalize the d to make sure that $\boxed{d \in [\frac{1}{2}, 1)}$ this make the Convergence faster.

Table based methods for initial approximation of x_0 in Newton raphson method

A table-based method improves the initial approximation x_0 for $1/d$ by using Pre computed values stored in look up table.

Step 1: Normalize the d to the range $[0.5, 1)$

Step 2: Use the MSb's of d to index the table

→ This means that we divide the range $[0.5, 1)$ into discrete intervals.

→ if we take 6 bits of d we create 64-entry table ($2^6 = 64$)

Step 3: Store Pre computed Values of $1/d$

for each interval, store a Pre computed approximation of $1/d$

Step 4: Use the table value for initial x_0

Step 5: At the end apply the Correction to the result.

Generating look up table

Pre compute table for d in $[0.5, 1)$ using 6-bit indexing

→ 6 bit indexing ⇒ $2^6 = 64$ Values

→ divide $[\frac{1}{2}, 1)$ into 64 Parts

→ Using linear approximation find the estimate of $1/d$ using the below formula

$$\boxed{x_0 = 4(\sqrt{3} - 1) - 2d}$$

→ This formula I found in Computer Arthimatic Algorithms and Hardware designs by Behrooz Parhami

→ Use x_0 as the initial estimate for the iterations

→ Usually within 2–3 iterations are sufficient.