Program: BTech / MTech TA / MTech RA / PhD (Tick one)



Also30: Probability and Stochastic Processes Mid Term Exam 1

DATE: 25 SEPTEMBER 2024

Question	Marks Scored
1(a)	
1(b)	
1(c)	
2(a)	
2(b)	
3(a)	
3(b)	
4(a)	
4(b)	
5(a)	
5(b)	
5(c)	
5(d)	
5(e)	
Total	

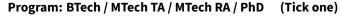
Instructions:

- Fill in your name and roll number on each of the pages.
- · This exam is for a total of 30 MARKS.
- · You may use any result covered in class directly without proving it.
- Hints are provided for some questions.
 However, it is NOT mandatory to solve the question using the approach in the hints.
 If you think you have a better approach in mind than the one given in the hint, feel free to present your approach.
- Show all your working clearly.

 We want to see your thought process, and possibly provide partial credit for the intermediate logical steps.
- Plagiarism will NOT be entertained at any length.
 If you are caught cheating during the exam, your answer script will NOT be evaluated.

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Assume that all random variables appearing in the questions below are defined with respect to \mathscr{F} .





1. Let $A\in \mathscr{F}$ be an event such that $0<\mathbb{P}(A)<1$.

(a) (2 Marks)

Show that for any $B \in \mathscr{F}$,

$$\mathbb{P}(A) \cdot \bigg| \mathbb{P}(B|A) - \mathbb{P}(B) \bigg| = \mathbb{P}(A^c) \cdot \bigg| \mathbb{P}(B|A^c) - \mathbb{P}(B) \bigg|.$$

(b) **(2 Marks)**

Show that for any $B \in \mathscr{F}$,

$$0 \leq \left| \mathbb{P}(A \cap B) - \mathbb{P}(A) \, \mathbb{P}(B) \right| \leq \frac{1}{4}.$$

(c) (2 Marks)

For any $B \in \mathscr{F}$, show that the probability that **exactly** one of the events A or B occurs is given by

$$\mathbb{P}(A) + \mathbb{P}(B) - 2 \, \mathbb{P}(A \cap B).$$



2. (a) (3 Marks)

Let $X_1, X_2, \cdots \overset{\text{i.i.d.}}{\sim}$ Bernoulli(1/2). Let X denote the sum

$$X = \sum_{n=1}^{\infty} \frac{X_n}{2^n}.$$

Show that $X \sim \text{Unif}((0,1))$.

Hint:

Fix $x \in (0, 1)$.

Let $0.x_1x_2x_3\dots$ denote the infinite binary expansion of x, where $x_n\in\{0,1\}$ for each $n\in\mathbb{N}$.

Then, x may be expressed as

$$x = \sum_{n=1}^{\infty} \frac{x_n}{2^n}.$$

Furthermore,

$$\mathbb{P}(\{X \le x\}) = \mathbb{P}\left(\left\{\sum_{n=1}^{\infty} \frac{X_n}{2^n} \le \sum_{n=1}^{\infty} \frac{x_n}{2^n}\right\}\right)$$

$$= \mathbb{P}\left(\left\{\sum_{n=1}^{\infty} \frac{X_n}{2^n} \le \sum_{n=1}^{\infty} \frac{x_n}{2^n}\right\} \cap \left\{X_1 < x_1\right\}\right) + \mathbb{P}\left(\left\{\sum_{n=1}^{\infty} \frac{X_n}{2^n} \le \sum_{n=1}^{\infty} \frac{x_n}{2^n}\right\} \cap \left\{X_1 = x_1\right\}\right).$$

Simplify each of the probability terms in the second line above, and proceed recursively.

(b) (3 Marks)

Fix $q \in (0,1)$. Let $U \sim \mathsf{Unif}((0,1))$, and let

$$X = \lfloor \log_a U \rfloor + 1,$$

where $\lfloor x \rfloor$ denotes the largest integer lesser than or equal to x (for e.g., $\lfloor 0.3 \rfloor = 0$, $\lfloor 4.99 \rfloor = 4$, $\lfloor 2 \rfloor = 2$, and so on). Here, $\log_a U$ denotes the logarithm of U to the base q.

Determine the PMF of X.

Hint:

List down the possible values of $\lfloor \log_a U \rfloor$.

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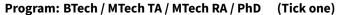
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3. Let X,Y be jointly continuous with the joint PDF

$$f_{X,Y}(x,y) = egin{cases} rac{1}{x}, & 0 \leq y \leq x \leq 1, \\ 0, & ext{otherwise}. \end{cases}$$

- (a) (3 Marks) $\label{eq:definition} \mbox{Determine the PDF of } Z = X + Y.$
- (b) (1 Mark) $\text{Compute } \mathbb{P}(\{Z \leq 1/2\}).$





4. Numbers from [0, 1] are picked uniformly, independently, and sequentially over time.

Let X_n denote the number picked at time n, where $n \in \{0, 1, 2, \ldots\}$.

Let ${\cal N}$ be the random variable defined as

$$N = \min\{n \ge 1 : X_n < X_0\}.$$

That is, N denotes the first time index $n \geq 1$ at which the value of X_n goes below the value of X_0 .

(a) (3 Marks)

For any fixed $n \in \mathbb{N}$, determine $\mathbb{P}(\{N = n\})$.

Hint

The event that N=n is identical to the event that

$$X_1 \geq X_0$$
 and $X_2 \geq X_0$ and \cdots and $X_{n-1} \geq X_0$ and $X_n < X_0$.

(b) **(1 Mark)**

Compute $\mathbb{P}(\{N>2\})$.

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5. Let $X_1, X_2 \stackrel{\text{i.i.d.}}{\sim} \text{Exponential}(\lambda)$.

Two Ph.D. students of IIT Hyderabad (let us call them S_1 and S_2) are on a mission to come up with their own definitions for what it means to "condition on" the event $\{X_1 = X_2\}$.

- First Definition: Student S_1 reasons that $X_1=X_2$ if and only if $\frac{X_1}{X_2}=1$, and therefore finds it apt to define conditioning on the event $\{X_1=X_2\}$ as conditioning on the event $\Big\{\frac{X_1}{X_2}=1\Big\}$.
- Second Definition: Student S_2 reasons that $X_1=X_2$ if and only if $X_1-X_2=0$, and therefore finds it apt to define conditioning on the event $\left\{X_1=X_2\right\}$ as conditioning on the event $\left\{X_1-X_2=0\right\}$.
- (a) (2 Marks)

Show that $\{X_1=X_2\}=\{\omega\in\Omega: X_1(\omega)=X_2(\omega)\}\in\mathscr{F}.$ Furthermore, argue that $\mathbb{P}(\{X_1=X_2\})=0.$

(b) (2 Marks)

Determine the joint PDF of $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_2}$.

(c) (2 Marks)

Determine the conditional PDF of Y_1 , conditioned on the event $\{Y_2 = 1\}$.

(d) (2 Marks)

Determine the joint PDF of Y_1 (as defined above) and $Y_3 = X_1 - X_2$.

(e) (2 Marks)

Determine the conditional PDF of Y_1 , conditioned on the event $\{Y_3 = 0\}$.

Finally, show that the conditional PDFs in parts (c) and (e) are different.

Here, conditioning according to student S_1 's definition leads to a different answer than according to student S_2 's definition. The above problem shows that when conditioning on a zero probability event, one must exercise care to specify the exact definition of conditioning.