AI 5030: Probability and Stochastic Processes

Instructor: Dr. Karthik P. N.

Homework 4

Topics: Conditional Probability, Independence, Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

1. For any two disjoint sets $A, B \subseteq \Omega$, show that

$$\mathbf{1}_{A \cup B} = \mathbf{1}_A + \mathbf{1}_B,$$

where $\mathbf{1}_E$ denotes the indicator function of the set E.
Use the above result to show that if A and B are any two sets (not necessarily disjoint), then

$$\mathbf{1}_{A \cup B} = \mathbf{1}_A + \mathbf{1}_B - \mathbf{1}_{A \cap B}.$$

2. Let $\Omega = \{H, T\}^3$ and $\mathscr{F} = 2^{\Omega}$. Construct a probability measure $\mathbb{P}$ and events $A, B, C \in \mathscr{F}$ such that
   (a) $\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B), \quad \mathbb{P}(B \cap C) = \mathbb{P}(B) \cdot \mathbb{P}(C), \quad \mathbb{P}(A \cap C) = \mathbb{P}(A) \cdot \mathbb{P}(C)$.
   (b) $\mathbb{P}(A \cap B \cap C) \neq \mathbb{P}(A) \cdot \mathbb{P}(B) \cdot \mathbb{P}(C)$.

3. Let $\Omega = [0, +\infty)$ and $\mathscr{F} = \mathscr{B}([0, +\infty))$. Let $X : \Omega \to \mathbb{R}$ be defined as

$$X(\omega) = \sum_{k=1}^{\infty} k\, \mathbf{1}_{[k-1,k)}(\omega) = \mathbf{1}_{[0,1)}(\omega) + 2\, \mathbf{1}_{[1,2)}(\omega) + 3\, \mathbf{1}_{[2,3)}(\omega) + \ldots, \qquad \omega \in \Omega.$$

That is, X takes the constant value 1 on $[0, 1)$, the value 2 on $[1, 2)$, the value 3 on $[2, 3)$, and so on.
   (a) Evaluate $X^{-1}([0, 100])$.
   (b) Given a natural number $n \in \mathbb{N}$, what is $X^{-1}(\{n\})$?
   (c) Evaluate $X^{-1}((-\infty, x])$ for all $x \in \mathbb{R}$, and show that X is a random variable with respect to $\mathscr{F}$.

4. Let $X : \Omega \to \mathbb{R}$ be a random variable with respect to $\mathscr{F}$.
   (a) Show that $\left(X^{-1}(B)\right)^c = X^{-1}(B^c)$ for any $B \in \mathscr{B}(\mathbb{R})$.
   (b) Show that for any two Borel sets $B_1, B_2 \in \mathscr{B}(\mathbb{R})$,

$$X^{-1}(B_1 \cup B_2) = X^{-1}(B_1) \cup X^{-1}(B_2).$$

   More generally, for any $B_1, B_2, \ldots \in \mathscr{B}(\mathbb{R})$, show that

$$X^{-1}\left(\bigcup_{i=1}^{\infty} B_i\right) = \bigcup_{i=1}^{\infty} X^{-1}(B_i).$$

   (c) Consider the collection

$$\mathscr{E} = \left\{E \subseteq \Omega : E = X^{-1}(B) \text{ for some } B \in \mathscr{B}(\mathbb{R})\right\}.$$

   That is, each set in $\mathscr{E}$ is the pre-image (under X) of some Borel set B.
   Show that $\mathscr{E}$ is a σ-algebra of subsets of Ω.
   Hint: Use the results in part (a) and part (b).

Note: To show $A = B$ for any two sets A, B, you need to show $A \subseteq B$ and $B \subseteq A$.

5. Suppose two fair coins are tossed independently of each other.
   (a) Specify $(\Omega, \mathscr{F}, \mathbb{P})$ for the above experiment.

(b) Find the probability of the event that both coins turn up heads, conditioned on the event that the first coin turns up head.

(c) Find the probability of the event that both coins turn up heads, conditioned on the event that at least one of the coins turns up head.

6. Consider events $A, B, C \in \mathscr{F}$ such A is independent of B and A is independent of C.
Show that A is independent of $B \cup C$ if and only if A is independent of $B \cap C$.

Note: To prove an if and only if statement, the "if" and "only if" directions must be proved separately.