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Probability and Stochastic Processes

Lecture 03: Properties of Probability Measures, Borel σ-Algebra, Caratheodory's Extension Theorem, Lebesgue Measure

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Recall

Fix a measurable space $(\Omega, \mathscr{F})$.

Definition (Probability Measure)

A function $\mathbb{P} : \mathscr{F} \to [0, 1]$ is called a probability measure if the following properties are satisfied:

1. $\mathbb{P}(\emptyset) = 0$.
2. $\mathbb{P}(\Omega) = 1$.
3. If $A_1, A_2, \ldots$ is a countable collection of disjoint sets, with $A_i \in \mathscr{F}$ for each $i \in \mathbb{N}$, then

$$\mathbb{P}\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mathbb{P}(A_i).$$

Note: $\bigcup_{i\in\mathbb{N}} A_i$ is sometimes written as $\bigcup_{i=1}^{\infty} A_i$. Same holds for intersections.

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Properties of Probability Measures

Exercise

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

- (Finite additivity)
  For any $n \in \mathbb{N}$ and disjoint collection of events $A_1, \ldots, A_n \in \mathscr{F}$,
  $$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i).$$
- For any $A \in \mathscr{F}$,
  $$\mathbb{P}(A^c) = 1 - \mathbb{P}(A).$$
- (Monotonicity)
  If $A \subseteq B$, $A \in \mathscr{F}$, $B \in \mathscr{F}$, then
  $$\mathbb{P}(A) \leq \mathbb{P}(B).$$

Properties of Probability Measures

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

- For any two sets $A, B \in \mathscr{F}$,

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

- (Inclusion-Exclusion principle)
  For any $A_1, \ldots, A_n \in \mathscr{F}$,

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) - \cdots + (-1)^{n+1}\, \mathbb{P}\left(\bigcap_{i=1}^{n} A_i\right).$$

The Liminf Event

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

- If $A_1, A_2, A_3, \ldots, \in \mathscr{F}$, we define the liminf event as

$$\liminf_{n\to\infty} A_n := \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k.$$

- $\omega \in \liminf_{n\to\infty} A_n \implies \omega \in$ all but finitely many A_n

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The Limsup Event

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

- If $A_1, A_2, A_3, \ldots, \in \mathscr{F}$, we define the limsup event as

$$\limsup_{n\to\infty} A_n := \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

- $\omega \in \limsup_{n\to\infty} A_n \implies \omega \in$ infinitely many A_n

Note

In general, we have

$$\liminf_{n\to\infty} A_n \subseteq \limsup_{n\to\infty} A_n.$$

Limit Set

If $\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n$, we call this common set $\lim_{n\to\infty} A_n$.
Examples:

- $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$, where $A_i \in \mathscr{F}$ for each $i \in \mathbb{N}$

$$\limsup_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n = \liminf_{n\to\infty} A_n = \lim_{n\to\infty} A_n$$

- Similarly, for $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$, where $A_i \in \mathscr{F}$ for each $i \in \mathbb{N}$, we have

$$\lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$$

(verify this!)

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Continuity of Probability Measure

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

- If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$, where $A_i \in \mathscr{F}$ for each $i \in \mathbb{N}$, then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathbb{P}\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

- If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \cdots$, where $A_i \in \mathscr{F}$ for each $i \in \mathbb{N}$, then

$$\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = \mathbb{P}\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mathbb{P}(A_n).$$

Union Bound

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

- For any $A_1, A_2, \ldots \in \mathscr{F}$,

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_i\right) \leq \sum_{n=1}^{\infty} \mathbb{P}(A_n)$$

Probability Measures on Discrete Spaces

Discrete Sample Spaces

Let Ω be a non-empty, discrete (countable) sample space.
Thus, Ω may be represented as one of the following:

- $\Omega = \{\omega_1, \ldots, \omega_n\}$ for some $n \in \mathbb{N}$
- $\Omega = \{\omega_1, \omega_2, \ldots\}$

In this case, we simply take $\mathscr{F} = 2^{\Omega}$

Probability Assignment

Given $(\Omega, \mathscr{F})$, we define $\mathbb{P} : \mathscr{F} \to [0, 1]$ as

$$\mathbb{P}(A) = \sum_{\omega \in A} \mathbb{P}(\{\omega\}), \quad A \in \mathscr{F},$$

while making sure that the assignment $\mathbb{P}$ satisfies $\sum_{\omega \in \Omega} \mathbb{P}(\{\omega\}) = 1$.

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Examples

- $\Omega = \{H, T\}, \quad \mathscr{F} = 2^\Omega = \left\{\emptyset, \Omega, \{H\}, \{T\}\right\}$

$$\mathbb{P}(\{H\}) = p = 1 - \mathbb{P}(\{T\}), \quad p \in [0, 1]$$

- $\Omega = \mathbb{N}, \quad \mathscr{F} = 2^\Omega$

$$\mathbb{P}(\{k\}) = \dots \qquad \text{such that} \qquad \sum_{k=1}^{\infty} \mathbb{P}(\{k\}) = 1.$$

  – $\mathbb{P}(\{k\}) = p(1-p)^{k-1}, \quad k \in \Omega \quad$ ($p \in [0, 1]$, Geometric measure)

- $\Omega = \mathbb{N} \cup \{0\}, \quad \mathscr{F} = 2^\Omega$

  – $\mathbb{P}(\{k\}) = e^{-\lambda} \dfrac{\lambda^k}{k!}, \quad k \in \Omega \quad$ ($\lambda > 0$, Poisson measure)

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Probability Measures on Uncountable Spaces

Smallest σ-Algebra

Simple example: $\Omega = \{1, 2, 3, 4, 5\}$

Consider the collection $\mathscr{C} = \Big\{\{1\}, \{2, 3\}\Big\}$. What is $\sigma(\mathscr{C})$?

To construct $\sigma(\mathscr{C})$, we can first construct all σ-algebras that contain $\mathscr{C}$, and then take their intersection.

$$\mathscr{F}_1 = \sigma\left(\Big\{\{1\}, \{2, 3\}, \{4, 5\}\Big\}\right) \qquad \mathscr{F}_2 = \sigma\left(\Big\{\{1\}, \{2, 3\}, \{4\}, \{5\}\Big\}\right)$$

$$\mathscr{F}_3 = \sigma\left(\Big\{\{1\}, \{2\}, \{3\}, \{4, 5\}\Big\}\right) \qquad \mathscr{F}_4 = \sigma\left(\Big\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\Big\}\right) = 2^{\Omega}$$

Then, $\sigma(\mathscr{C}) = \bigcap_{i=1}^{4} \mathscr{F}_i$.

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Smallest σ-Algebra

Definition (Smallest σ-Algebra)

Let Ω be a sample space, and let $\mathscr{C}$ be an arbitrary collection of subsets of Ω. For an arbitrary index set $\mathcal{I}$, let $\{\mathscr{F}_i : i \in \mathcal{I}\}$ be a collection of all σ-algebras containing the sets in $\mathscr{C}$. Then, the smallest σ-algebra generated from $\mathscr{C}$, denoted by $\sigma(\mathscr{C})$, is defined as

$$\sigma(\mathscr{C}) = \bigcap_{i \in \mathcal{I}} \mathscr{F}_i.$$

Remark

If $\mathscr{H}$ is any σ-algebra containing the sets in $\mathscr{C}$, then $\mathscr{H}$ contains the sets in $\sigma(\mathscr{C})$. Mathematically,

$$\mathscr{C} \subseteq \mathscr{H} \implies \sigma(\mathscr{C}) \subseteq \mathscr{H}.$$

Uncountable State Spaces

Suppose that $\Omega = [0, 1], \quad \mathscr{F} = 2^{\Omega}$

Suppose that we want to model the concept of a "uniform probability measure" on Ω

We want: $\quad \mathbb{P}(\{\omega_1\}) = \mathbb{P}(\{\omega_2\})$ for all $\omega_1, \omega_2 \in \Omega$

Suppose that $\mathbb{P}(\{\omega\}) = p > 0$ for all $\omega \in [0, 1]$.
In particular, $\mathbb{P}(\{\omega\}) = p$ for all $\omega \in \mathbb{Q} \cap [0, 1]$.
This implies that

$$\mathbb{P}([0, 1]) \geq \mathbb{P}(\mathbb{Q} \cap [0, 1]) = \sum_{\omega \in \mathbb{Q} \cap [0,1]} \mathbb{P}(\{\omega\}) = \sum_{\omega \in \mathbb{Q} \cap [0,1]} p = +\infty$$

Therefore, we must have $\mathbb{P}(\{\omega\}) = 0$ for all $\omega \in [0, 1]$
However, this does not tell us the value of $\mathbb{P}([\frac{1}{2}, 1])$ (why?)
Remedy: Give up on the requirement $\mathscr{F} = 2^{\Omega}$

Borel σ-Algebra

Consider $\Omega = [0, 1]$

Let $\mathcal{O} = \{(a, b) : 0 \leq a < b \leq 1\} =$ collection of all open sub-intervals of $[0, 1]$

Definition (Borel σ-Algebra)

The smallest σ-algebra containing the sets in $\mathcal{O}$ is called the Borel σ-algebra of subsets of $[0, 1]$, and denoted $\mathscr{B}([0, 1])$. Sets in $\mathscr{B}([0, 1])$ are called Borel sets.

Remarks:

- $(a, b) \in \mathscr{B}([0, 1])$ for all $0 \leq a < b \leq 1$
- $\{x\} \in \mathscr{B}([0, 1])$ for all $x \in [0, 1]$

  Indeed, we may express $\{x\}$ as

$$\{x\} = \bigcap_{n \in \mathbb{N}} A_n, \quad \text{where} \quad A_n = \left(x - \frac{1}{n},\, x + \frac{1}{n}\right) \cap [0, 1]$$

- $(a, b], [a, b), [a, b] \in \mathscr{B}([0, 1])$ for all $0 \leq a < b \leq 1$ (why?)

Uniform Probability Assignment to Sets in $\mathscr{B}([0,1])$

$\Omega = [0,1], \quad \mathscr{F} = \mathscr{B}([0,1])$

Aim: To build a "uniform" probability function $\mathbb{P} : \mathscr{F} \to [0,1]$

- Step 1: Consider the collection $\mathscr{O} = \{(a,b) : 0 \le a < b \le 1\}$
- Step 2: Construct the smallest algebra containing all sets in $\mathscr{O}$. Call this algebra $\mathscr{A}$.
- Step 3: Construct a function $\mathbb{P}_0 : \mathscr{A} \to [0,1]$ satisfying the following properties:
  - $\mathbb{P}_0(\Omega) = 1$
  - $\mathbb{P}_0((a,b)) = b - a$ for all $0 \le a < b \le 1$
  - For any disjoint $A_1, A_2, \ldots \in \mathscr{A}$ such that $\bigcup_{i=1}^{\infty} A_i \in \mathscr{A}$,

$$\mathbb{P}_0\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}_0(A_i).$$

- Step 4: Use Caratheodory's extension theorem to extend $\mathbb{P}_0$ to a measure $\mathbb{P}$ on $\mathscr{F} = \sigma(\mathscr{A}) = \sigma(\mathscr{O})$

Caratheodory's Extension Theorem

Caratheodory's Extension Theorem

Fix a sample space Ω. Let $\mathscr{A}$ be an algebra of subsets of Ω, and let $\mathscr{F} = \sigma(\mathscr{A})$. Suppose that $\mathbb{P}_0 : \mathscr{A} \to [0, 1]$ satisfies

1. $\mathbb{P}_0(\Omega) = 1$, and
2. For all disjoint $A_1, A_2, \ldots \in \mathscr{A}$ such that $\bigcup_{i=1}^{\infty} A_i \in \mathscr{A}$,

$$\mathbb{P}_0\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}_0(A_i).$$

Then, $\mathbb{P}_0$ can be extended uniquely to a measure $\mathbb{P} : \mathscr{F} \to [0, 1]$ such that

$$\mathbb{P}(A) = \mathbb{P}_0(A) \quad \text{for all} \quad A \in \mathscr{A}.$$

The extended measure $\mathbb{P}$ is called the Lebesgue measure, and generally denoted by λ

Uniform Probability Assignment to Sets in $\mathscr{B}([0,1])$

$\Omega = [0,1], \quad \mathscr{F} = \mathscr{B}([0,1]), \quad \lambda : \mathscr{F} \to [0,1]$ - Lebesgue measure

The below properties follow immediately.

- $\lambda(\{x\}) = 0$ for all $x \in [0,1]$
- For all $0 \leq a < b \leq 1$, we have

$$\lambda((a,b)) = \lambda([a,b)) = \lambda((a,b]) = \lambda([a,b]) = b - a$$

- $\lambda(\mathbb{Q} \cap [0,1]) = 0$ (example of a countably infinite set having zero probability)

Lebesgue Measure on $\mathscr{B}(\mathbb{R})$

$\Omega = \mathbb{R}$

Consider $\mathscr{C} = \left\{(a, b) : -\infty \le a < b \le +\infty\right\}, \qquad \sigma(\mathscr{C}) = \mathscr{B}(\mathbb{R})$

$\lambda : \mathscr{B}(\mathbb{R}) \to [0, +\infty]$ – Lebesgue measure

- $\lambda(\mathbb{R}) = +\infty$
- $\lambda(\{x\}) = 0$ for all $x \in \mathbb{R}$
- For all $-\infty < a < b < +\infty$, we have

$$\lambda((a, b)) = \lambda([a, b)) = \lambda((a, b]) = \lambda([a, b]) = b - a$$

- $\lambda(\mathbb{Q}) = 0$

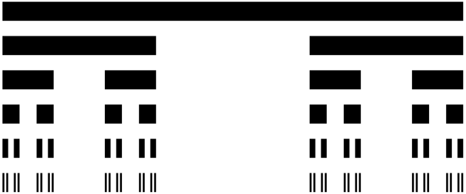
The Cantor Set

Consider the interval $[0, 1]$

- $C_0 = [0, 1]$
- $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$
- $C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$

  $\vdots$

Cantor set $K := \bigcap_{i=1}^{\infty} C_i$



Credits: Jim Belk, Cornell

Properties of Cantor Set

- K is uncountable
- $K \in \mathscr{B}([0, 1])$
- $\lambda(K) = 0$, where λ is the Lebesgue measure

Remark

Cantor set is an example of an uncountable set with zero probability (as measured under the Lebesgue measure).

Reading

- For a proof of the Caratheodory's extension theorem, see [Williams, 1991, Appendix A]
- For interesting problems and exercises on algebra, σ-algebra, and probability measures, see the book by Grimmett and Stirzaker

References

Williams, D. (1991).
Probability with martingales.
Cambridge university press.