Name:
Roll Number:
Department:
Program: BTech / MTech TA / MTech RA / PhD (Tick one)

AI5030: Probability and Stochastic Processes

Quiz 4

Date: 16 October 2024

Question	1	2(a)	2(b)	2(c)	Total
Marks Scored					

Instructions:

- **Fill in your name and roll number on each of the pages.**
- **You may use any result covered in class directly without proving it.**
- **Unless explicitly stated in the question, DO NOT use any result from the homework without proof.**

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Assume that all random variables appearing in the questions below are defined with respect to $\mathscr{F}$.

1. **(2 Marks)**
   Let N be a discrete random variable taking values in $\mathbb{N}$ and having PMF p_N.
   Construct an example for p_N under which $\mathbb{E}[N] = +\infty$ and

$$\lim_{n \to \infty} n \, \mathbb{P}(\{N > n\}) > 0.$$

   Show your working clearly.

   Solution:
   Let N be a discrete random variable

$$\mathbb{P}(\{N > n\}) = \frac{1}{n+1}, \qquad n \in \mathbb{N} \cup \{0\}.$$

   Then, it follows that

$$\mathbb{E}[N] = \sum_{n=0}^{\infty} \mathbb{P}(\{N > n\}) = \sum_{n=0}^{\infty} \frac{1}{n+1} = +\infty,$$

   whereas

$$\lim_{n \to \infty} n \, \mathbb{P}(\{N > n\}) = \lim_{n \to \infty} \frac{n}{n+1} = 1 > 0.$$

   PS: We note here in passing that the PMF of N is given by

$$\begin{aligned}
\mathbb{P}(\{N = n\}) &= \mathbb{P}(\{N \geq n\}) - \mathbb{P}(\{N > n\}) \\
&= \mathbb{P}(\{N > n-1\}) - \mathbb{P}(\{N > n\}) \\
&= \frac{1}{n} - \frac{1}{n+1} \\
&= \frac{1}{n(n+1)}, \qquad n \in \mathbb{N}.
\end{aligned}$$

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2. Suppose that $X(0)$, $X(1)$, and $X(2)$ are independent, discrete random variables defined with respect to $\mathscr{F}$. For each $i \in \{0,1,2\}$, the random variable $X(i)$ takes possible values 1 and 2 with probabilities p_i and $1-p_i$ respectively.

   (a) **(1 Mark)**
   Show, from first principles, that $Y = X(X(0))$ is a random variable with respect to $\mathscr{F}$.

   (b) **(1 Mark)**
   Derive the PMF of Y.

   (c) **(1 Mark)**
   Compute $\mathbb{E}[Y]$.

Solution: We present the solution to each of the parts below.

(a) Note that

$$X(X(0)) = X(1)\,\mathbf{1}_{\{X(0)=1\}} + X(2)\,\mathbf{1}_{\{X(0)=2\}}.$$

Because $X(0)$, $X(1)$, and $X(2)$ are random variables with respect to $\mathscr{F}$ (as per the question), we have $\{X(0)=1\} \in \mathscr{F}$ and $\{X(0)=2\} \in \mathscr{F}$, and hence Y is a simple random variable with respect to $\mathscr{F}$.

(b) Clearly, $Y \in \{1,2\}$. Further, $Y=1$ if and only if either $X(0)=1, X(1)=1$ or $X(0)=2, X(2)=1$. We then have

$$\begin{aligned}
\mathbb{P}(\{Y=1\}) &= \mathbb{P}(\{X(0)=1\} \cap \{X(1)=1\}) + \mathbb{P}(\{X(0)=2\} \cap \{X(2)=1\}) \\
&\overset{(a)}{=} \mathbb{P}(\{X(0)=1\}) \cdot \mathbb{P}(\{X(1)=1\}) + \mathbb{P}(\{X(0)=2\}) \cdot \mathbb{P}(\{X(2)=1\}) \\
&= p_0 \cdot p_1 + (1-p_0) \cdot p_2,
\end{aligned}$$

and $\mathbb{P}(\{Y=2\}) = 1 - \mathbb{P}(\{Y=1\})$. In the above set of equalities, (a) follows from the independence of $X(i), i \in \{0,1,2\}$.

(c) We have

$$\begin{aligned}
\mathbb{E}[Y] &= \mathbb{E}[X(1)\,\mathbf{1}_{\{X(0)=1\}}] + \mathbb{E}[X(2)\,\mathbf{1}_{\{X(0)=2\}}] \\
&\overset{(*)}{=} \mathbb{E}[X(1)] \cdot \mathbb{E}[\mathbf{1}_{\{X(0)=1\}}] + \mathbb{E}[X(2)] \cdot \mathbb{E}[\mathbf{1}_{\{X(0)=2\}}] \\
&= (1 \cdot p_1 + 2 \cdot (1-p_1)) \cdot p_0 + (1 \cdot p_2 + 2 \cdot (1-p_2)) \cdot (1-p_0) \\
&= (2-p_1)p_0 + (2-p_2)(1-p_0).
\end{aligned}$$