Name:
Roll Number:
Department:
Program: BTech / MTech TA / MTech RA / PhD (Tick one)

AI5030: Probability and Stochastic Processes

Mid Term Exam 1

Date: 25 September 2024

Question	Marks Scored
1(a)	
1(b)	
1(c)	
2(a)	
2(b)	
3(a)	
3(b)	
4(a)	
4(b)	
5(a)	
5(b)	
5(c)	
5(d)	
5(e)	
Total	

Instructions:

- **Fill in your name and roll number on each of the pages.**
- **This exam is for a total of 30 MARKS.**
- **You may use any result covered in class directly without proving it.**
- **Hints are provided for some questions.**
  However, it is NOT mandatory to solve the question using the approach in the hints.
  If you think you have a better approach in mind than the one given in the hint, feel free to present your approach.
- **Show all your working clearly.**
  We want to see your thought process, and possibly provide partial credit for the intermediate logical steps.
- **Plagiarism will NOT be entertained at any length.**
  If you are caught cheating during the exam, your answer script will NOT be evaluated.

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Assume that all random variables appearing in the questions below are defined with respect to $\mathscr{F}$.

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1. Let $A \in \mathscr{F}$ be an event such that $0 < \mathbb{P}(A) < 1$.

   (a) **(2 Marks)**
   Show that for any $B \in \mathscr{F}$,
   $$\mathbb{P}(A) \cdot \Big|\mathbb{P}(B|A) - \mathbb{P}(B)\Big| = \mathbb{P}(A^c) \cdot \Big|\mathbb{P}(B|A^c) - \mathbb{P}(B)\Big|.$$

   (b) **(2 Marks)**
   Show that for any $B \in \mathscr{F}$,
   $$0 \le \Big|\mathbb{P}(A \cap B) - \mathbb{P}(A)\,\mathbb{P}(B)\Big| \le \frac{1}{4}.$$

   (c) **(2 Marks)**
   For any $B \in \mathscr{F}$, show that the probability that **exactly** one of the events A or B occurs is given by
   $$\mathbb{P}(A) + \mathbb{P}(B) - 2\,\mathbb{P}(A \cap B).$$

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2. (a) **(3 Marks)**

Let $X_1, X_2, \cdots \overset{\text{i.i.d.}}{\sim}$ Bernoulli$(1/2)$.
Let X denote the sum

$$X = \sum_{n=1}^{\infty} \frac{X_n}{2^n}.$$

Show that $X \sim$ Unif$((0,1))$.

Hint:
Fix $x \in (0,1)$.
Let $0.x_1x_2x_3\dots$ denote the infinite binary expansion of x, where $x_n \in \{0,1\}$ for each $n \in \mathbb{N}$.
Then, x may be expressed as

$$x = \sum_{n=1}^{\infty} \frac{x_n}{2^n}.$$

Furthermore,

$$\begin{aligned}\mathbb{P}(\{X \leq x\}) &= \mathbb{P}\left(\left\{\sum_{n=1}^{\infty} \frac{X_n}{2^n} \leq \sum_{n=1}^{\infty} \frac{x_n}{2^n}\right\}\right) \\ &= \mathbb{P}\left(\left\{\sum_{n=1}^{\infty} \frac{X_n}{2^n} \leq \sum_{n=1}^{\infty} \frac{x_n}{2^n}\right\} \bigcap \left\{X_1 < x_1\right\}\right) + \mathbb{P}\left(\left\{\sum_{n=1}^{\infty} \frac{X_n}{2^n} \leq \sum_{n=1}^{\infty} \frac{x_n}{2^n}\right\} \bigcap \left\{X_1 = x_1\right\}\right).\end{aligned}$$

Simplify each of the probability terms in the second line above, and proceed recursively.

(b) **(3 Marks)**

Fix $q \in (0,1)$. Let $U \sim$ Unif$((0,1))$, and let

$$X = \lfloor \log_q U \rfloor + 1,$$

where $\lfloor x \rfloor$ denotes the largest integer lesser than or equal to x (for e.g., $\lfloor 0.3 \rfloor = 0$, $\lfloor 4.99 \rfloor = 4$, $\lfloor 2 \rfloor = 2$, and so on).
Here, $\log_q U$ denotes the logarithm of U to the base q.
Determine the PMF of X.
Hint:
List down the possible values of $\lfloor \log_q U \rfloor$.

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3. Let X, Y be jointly continuous with the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{x}, & 0 \leq y \leq x \leq 1, \\ \\ 0, & \text{otherwise.} \end{cases}$$

   (a) **(3 Marks)**
   Determine the PDF of $Z = X + Y$.

   (b) **(1 Mark)**
   Compute $\mathbb{P}(\{Z \leq 1/2\})$.

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4. Numbers from $[0, 1]$ are picked uniformly, independently, and sequentially over time.
Let X_n denote the number picked at time n, where $n \in \{0, 1, 2, \dots\}$.
Let N be the random variable defined as

$$N = \min\{n \geq 1 : X_n < X_0\}.$$

That is, N denotes the first time index $n \geq 1$ at which the value of X_n goes below the value of X_0.

(a) **(3 Marks)**
For any fixed $n \in \mathbb{N}$, determine $\mathbb{P}(\{N = n\})$.
Hint:
The event that $N = n$ is identical to the event that

$$X_1 \geq X_0 \quad \textit{and} \quad X_2 \geq X_0 \quad \textit{and} \quad \cdots \quad \textit{and} \quad X_{n-1} \geq X_0 \quad \textit{and} \quad X_n < X_0.$$

(b) **(1 Mark)**
Compute $\mathbb{P}(\{N > 2\})$.

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5. Let $X_1, X_2 \overset{\text{i.i.d.}}{\sim}$ Exponential(λ).

   Two Ph.D. students of IIT Hyderabad (let us call them S_1 and S_2) are on a mission to come up with their own definitions for what it means to "condition on" the event $\{X_1 = X_2\}$.

   - **First Definition:** Student S_1 reasons that $X_1 = X_2$ if and only if $\frac{X_1}{X_2} = 1$, and therefore finds it apt to define conditioning on the event $\{X_1 = X_2\}$ as conditioning on the event $\left\{\dfrac{X_1}{X_2} = 1\right\}$.
   - **Second Definition:** Student S_2 reasons that $X_1 = X_2$ if and only if $X_1 - X_2 = 0$, and therefore finds it apt to define conditioning on the event $\{X_1 = X_2\}$ as conditioning on the event $\left\{X_1 - X_2 = 0\right\}$.

   (a) **(2 Marks)**
   Show that $\{X_1 = X_2\} = \{\omega \in \Omega : X_1(\omega) = X_2(\omega)\} \in \mathscr{F}$.
   Furthermore, argue that $\mathbb{P}(\{X_1 = X_2\}) = 0$.

   (b) **(2 Marks)**
   Determine the joint PDF of $Y_1 = X_1 + X_2$ and $Y_2 = \frac{X_1}{X_2}$.

   (c) **(2 Marks)**
   Determine the conditional PDF of Y_1, conditioned on the event $\{Y_2 = 1\}$.

   (d) **(2 Marks)**
   Determine the joint PDF of Y_1 (as defined above) and $Y_3 = X_1 - X_2$.

   (e) **(2 Marks)**
   Determine the conditional PDF of Y_1, conditioned on the event $\{Y_3 = 0\}$.

   Finally, show that the conditional PDFs in parts (c) and (e) are different.

   Here, conditioning according to student S_1's definition leads to a different answer than according to student S_2's definition. The above problem shows that when conditioning on a zero probability event, one must exercise care to specify the exact definition of conditioning.