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Probability and Stochastic Processes

Expectations of Simple Random Variables, Supremum, Infimum, Limit Supremum, and Limit Infimum of a Sequence of Random Variables

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Expectations of Simple Random Variables

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Expectations of Simple Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

- If $A \in \mathscr{F}$ such that $\mathbb{P}(A) = 0$, and X is a simple random variable, then

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- If $A \in \mathscr{F}$ such that $\mathbb{P}(A) = 1$, and X is a simple random variable, then
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- If X, Y are simple, and $X(\omega) \geq Y(\omega) \geq 0$ for all $\omega \in \Omega$, then

$$\mathbb{E}[X] \geq \mathbb{E}[Y].$$

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- Linearity of expectations (for simple random variables)
If X, Y are simple, and $\alpha \geq 0$, then
$$\mathbb{E}[\alpha X] = \alpha \mathbb{E}[X], \qquad \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

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$$\mathbb{E}[\alpha X] = \alpha \mathbb{E}[X], \qquad \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$
- If X is simple, then
$$\begin{aligned} \mathbb{E}[X \cdot \mathbf{1}_{\{X \geq x\}}] &\geq x \cdot \mathbb{P}(\{X \geq x\}), \qquad && x \geq 0, \\ \mathbb{E}[X \cdot \mathbf{1}_{\{X \leq x\}}] &\leq x \cdot \mathbb{P}(\{X \leq x\}), \qquad && x \geq 0, \\ \mathbb{E}[X \cdot \mathbf{1}_{\{X = x\}}] &= x \cdot \mathbb{P}(\{X = x\}), \qquad && x \geq 0. \end{aligned}$$

Expectations of Simple Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

- If X is simple, then the function $\mathbb{Q}_X : \mathscr{F} \to [0, 1]$ defined via

$$\mathbb{Q}_X(A) = \frac{\int_A X \, d\mathbb{P}}{\int_\Omega X \, d\mathbb{P}}, \qquad A \in \mathscr{F},$$

  is a probability measure on $(\Omega, \mathscr{F})$.

- If $A, B \in \mathscr{F}$, $A \subseteq B$, and X is simple, then

$$\int_A X \, d\mathbb{P} \leq \int_B X \, d\mathbb{P}.$$

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$\{$Supremum, Infimum, Limit Supremum, Limit Infimum, Limit$\}$ of Random Variables

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Let $A \subseteq \mathbb{R}$ be a subset of real numbers.

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Supremum of a Sequence of Random Variables

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Lemma

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Infimum of a Sequence of Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
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Equivalently,

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