Probability and Stochastic Processes

Open Quiz 01

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Question

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space.

1. If $A, B \in \mathscr{F}$ are disjoint, and $\mathbb{P}(B) > 0$, what is $\mathbb{P}(A|B)$?
2. If $A, B \in \mathscr{F}$ are independent, and $\mathbb{P}(B) > 0$, what is $\mathbb{P}(A|B)$?
3. Can two events $A, B \in \mathscr{F}$ be independent and disjoint at the same time? Construct an example / counterexample to prove your point.

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Question

Suppose that $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} \text{Unif}(0, 1)$.
Let $M_n = \max\{X_1, \ldots, X_n\}$ and $m_n = \min\{X_1, \ldots, X_n\}$.
Compute $\mathbb{E}[M_n]$ and $\mathbb{E}[m_n]$.

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Question

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Fix two events $A, B \in \mathscr{F}$. Assume that $0 < \mathbb{P}(B) < 1$.
Let $X = \mathbf{1}_A$ and $Y = \mathbf{1}_B$.
What is $\mathbb{E}[X|Y]$?

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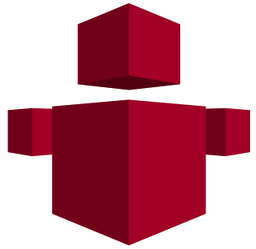
Question

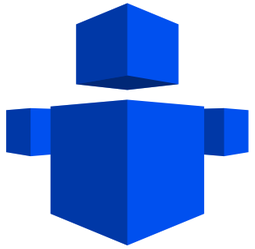
The Director of IITH intends to post an ad about the AI Department on YouTube.
The total amount with the Director is INR 100.
The Director is informed about two ad providers with the following logistics.

Option A



Per ad:
cost INR 10, revenue INR 13

Option B



Per ad:
With probability $\frac{4}{5}$: cost INR 5, revenue INR 8
With probability $\frac{1}{5}$: cost INR 10, revenue INR 15

Let $P = \text{Total Revenue} - \text{Total Cost}$.

1. Determine $\mathbb{E}[P]$ and $\text{Var}(P)$ under Option A.
2. Determine $\mathbb{E}[P]$ and $\text{Var}(P)$ under Option B.

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Question

Fix a measurable space $(\Omega, \mathscr{F})$.
Let events $A, B, C \in \mathscr{F}$ be a partition of Ω.
Let $\mathscr{G} = \sigma(\{A, B, C\})$ be the smallest σ-algebra generated by the sets A, B, C.

1. Write down $\mathscr{G}$ explicitly.
2. What is $|\mathcal{G}|$?
3. Give an example of a function $X : \Omega \to \mathbb{R}$ that is a random variable with respect to $\mathcal{G}$.

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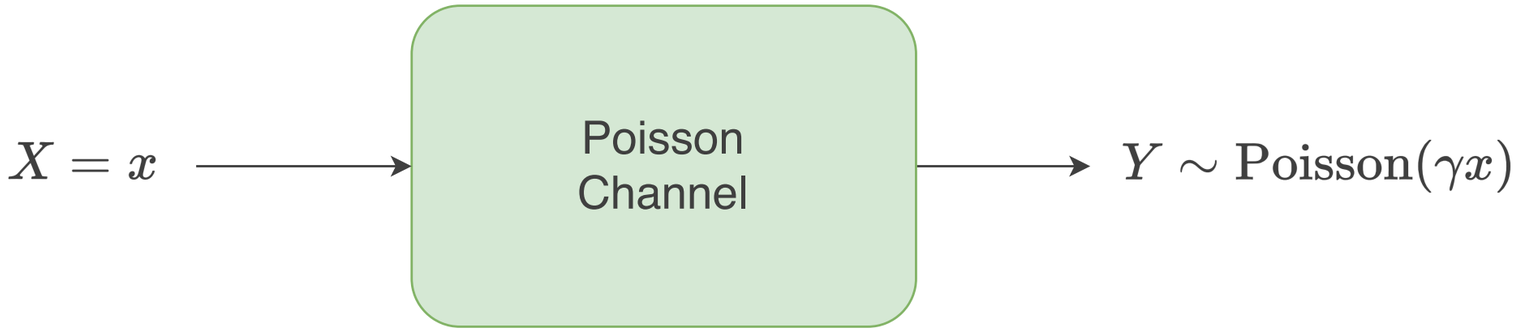
Question

Suppose that $X \sim \text{Exponential}(1)$.
Let Y be related to X as in the below figure.



Here, $\gamma > 0$ is the channel SNR (a fixed constant).
What is the PMF of Y?
You may use the formula

$$\int_0^\infty x^a e^{-bx}\, \mathrm{d}x = \frac{a!}{b^{a+1}}, \qquad \forall a \in \mathbb{N},\ b > 0.$$

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Bonus Question

Let $\mathcal{G}$ be as in the previous question, i.e., $\mathcal{G} = \sigma(\{A, B, C\})$.
Let $\mathcal{S}_{\mathcal{G}}$ denote the collection of all functions that are random variables with respect to $\mathcal{G}$.

1. Show that $\mathcal{S}_{\mathcal{G}}$ is a vector space.
2. Identify the basis vectors of $\mathcal{S}_{\mathcal{G}}$, and thereby compute the dimension of $\mathcal{S}_{\mathcal{G}}$.

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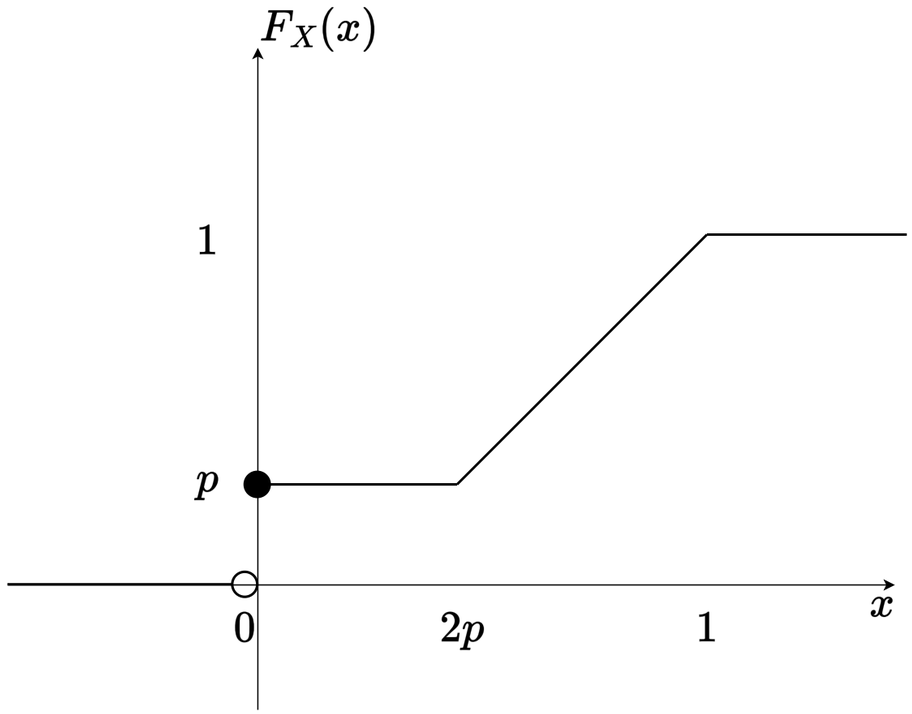
Question

Consider a random variable X with CDF F_X as depicted below.



Sketch the CDFs of $Y = \min\{X, p\}$ and $Z = \max\{X, p\}$.

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Bonus Question

Suppose that $X \sim \text{Exponential}(1)$.
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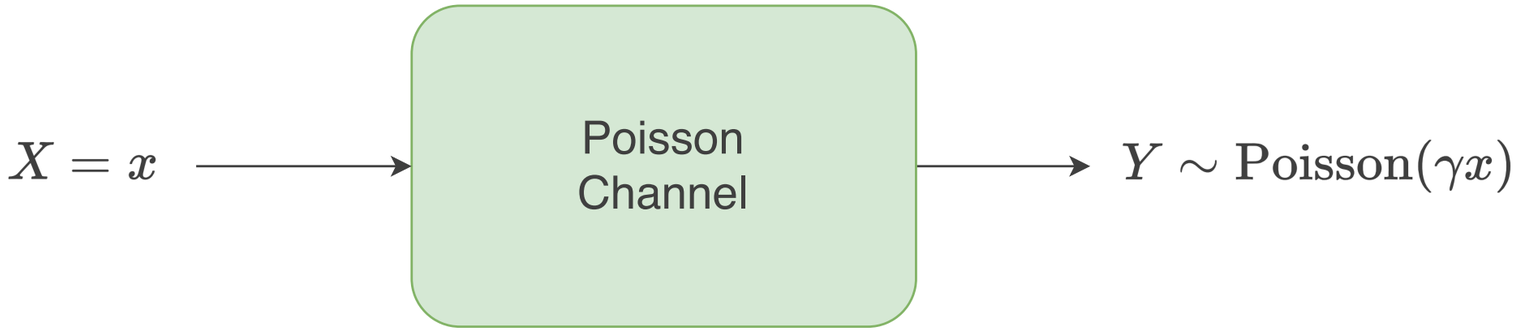

Here, $\gamma > 0$ is the channel SNR (a fixed constant).
What is the **MMSE estimate** of X given Y?

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Question

Suppose X and Y are independent random variables such that $\mathbb{P}(\{X = Y\}) = 1$. Determine $\mathrm{Var}(X)$ and $\mathrm{Var}(Y)$.

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Question

$E[P] = \alpha \cdot 30 + (1-\alpha)\, 58 \quad = 58 - 28\alpha.$

$Var(P) = \alpha^2 \cdot 0 + (1-\alpha)^2 16$

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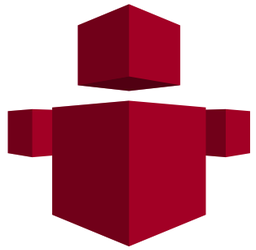
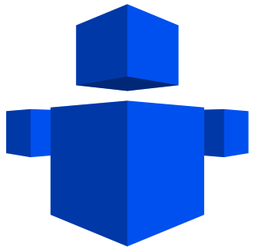
Option A



Per ad:
cost INR 10, revenue INR 13

Option B



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With probability $\frac{1}{5}$: cost INR 10, revenue INR 15

The Director chooses Option A with probability α and Option B with probability $1 - \alpha$.
What value of α maximizes

$$\mathbb{E}[P] - 2\,\mathrm{Var}(P)?$$

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Bonus Question

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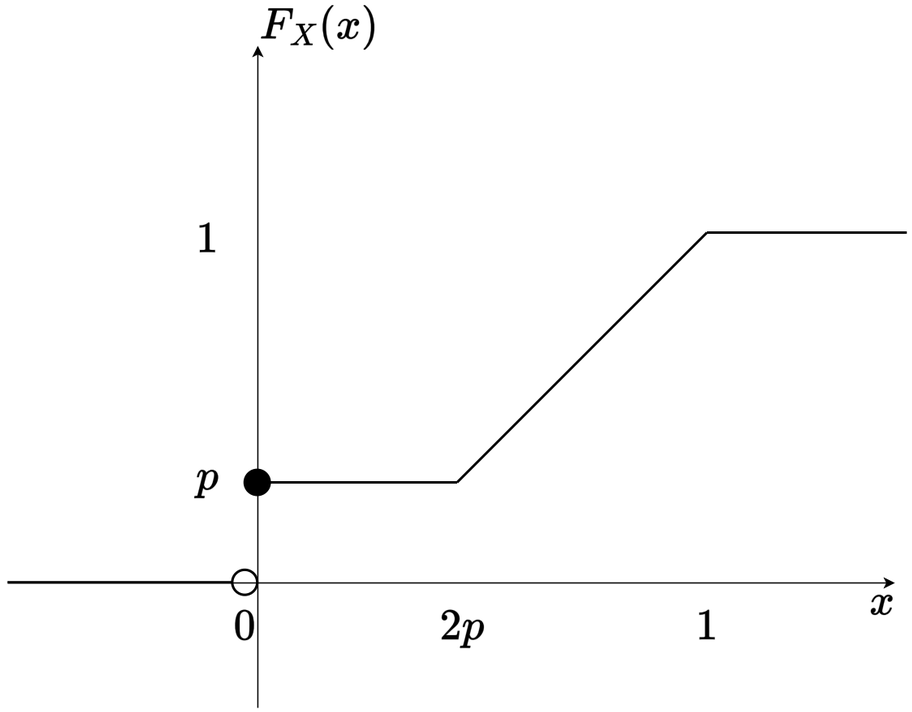

Compute $\mathbb{E}[X]$ and $\mathrm{Var}(X)$.