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Indian Institute of Technology Hyderabad

Probability and Stochastic Processes

Lecture 02: Sample Space, Algebra, σ-Algebra, Measure

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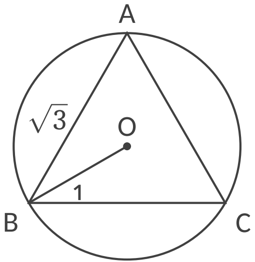
Probability Theory – Humble Beginnings

- Bernoulli (1713) and de Moivre (1718) gave the first definition of probability:

$$\text{probability of an event} = \frac{\text{\# favourable outcomes}}{\text{total number of outcomes}}.$$

- Cournot (1843):
  "An event with very small probability is morally impossible; an event with very high probability is morally certain."
- French mathematicians of the day were satisfied with the "frequentist" approach to probability, but not the German and English mathematicians of the day
- Frequentist approach could not satisfactorily explain certain paradoxes

Why Axiomatic Theory? Bertrand's Paradox



Take a circle with unit radius and inscribe an equilateral triangle in it. Draw a random chord. What is the probability that the length of the "random chord" is greater than $\sqrt{3}$?

Bertrand's perfectly valid arguments:

- Mid-point of chord should lie inside incircle of radius $1/2$
  Answer: $1/4$
- Angle between chord and tangent at A should be between $\pi/3$ and $2\pi/3$
  Answer: $1/3$
- Mid-point of chord should be between O and projection of O onto side BC
  Answer: $1/2$

Borel to the Rescue

- Contributions to Measure Theory by Borel (1894) provided a shift in perspective
- Countable unions played a key role in Borel's theory
- Kolmogorov's genius was in applying Borel's theory to formalise the axioms of probability, laying the foundation stone for modern probability theory
- For more details on the history of probability, see [Shafer and Vovk, 2018] and [Kolmogorov, 2004]

Sample Space

We begin with two universally accepted entities:

- Random experiment
- Outcome (denoted by ω) – source of randomness

Definition (Sample Space)

The sample space (denoted by Ω) of a random experiment is the set of all possible outcomes of the random experiment.

Example: Tossing a coin once

- If our interest is in the face that shows, then $\Omega = \{H, T\}$

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- If our interest is in the velocity with which the coin lands on ground, then $\Omega = [0, \infty) = \mathbb{R}_+$
- If our interest is in the number of times coin flips in air, then $\Omega = \mathbb{N}$

Example: Toss a coin n times, for some $n < \infty$.
Interest: faces that show up
$\Omega = \{H, T\}^n$

Example: Toss a coin infinitely many times.
Interest: faces that show up
$\Omega = \{H, T\}^\infty$

Event

Informal Definition (Event)

Informally,[a] an event is a subset of outcomes "of interest" to us.

[a] We shall give a more formal definition of an event later.

Example: Toss a coin 3 times; interest is in the faces that show up
$\Omega = \{H, T\}^3 = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$
Event A of interest: at least 2 heads show up
$A = \{HHH, THH, HTH, HHT\}$

Note

If an outcome $\omega \in A$ occurs, we say that the event A occurs.

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Algebra

Definition (Algebra)

Let Ω be a sample space.
A collection $\mathscr{A}$ of subsets of Ω is called an algebra if it satisfies the following properties:

1. $\Omega \in \mathscr{A}$.
2. $A \in \mathscr{A} \implies A^c \in \mathscr{A}$ (closure under complements).
3. $A, B \in \mathscr{A} \implies A \cup B \in \mathscr{A}$.

Property 3 above implies, by mathematical induction, that

$$A_1, A_2, \ldots, A_n \in \mathscr{A} \implies \bigcup_{i=1}^{n} A_i \in \mathscr{A} \quad \text{for all } n \in \mathbb{N} \quad \text{(closure under finite unions).}$$

Exercise

Show that an algebra is closed under finite intersections.

σ-Algebra – Motivation

Motivating example: toss a coin until first head shows up

$\Omega = \{H, TH, TTH, TTTH, \dots\}$

Event of interest A = # of tosses is even

$A = \{TH, TTTH, \dots\}$

Remark

The event A cannot be constructed as unions of finitely many subsets of Ω, unless one of the sets under consideration is A itself.

σ-Algebra

Definition (σ-Algebra)

Let Ω be a sample space.
A collection $\mathscr{F}$ of subsets of Ω is called a σ-algebra if it satisfies the following properties:

- $\Omega \in \mathscr{F}$.
- $A \in \mathscr{F} \implies A^c \in \mathscr{F}$ (closed under complements).
- $A_1, A_2, \ldots \in \mathscr{F} \implies \bigcup_{i \in \mathbb{N}} A_i \in \mathscr{F}$ (closure under countably infinite unions).

Remarks:

- Elements of a σ-algebra are called events

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- An event $A \in \mathscr{F}$ is also referred to as an $\mathscr{F}$-measurable set
- Every σ-algebra is also an algebra, but the converse is not true (see homework)
- The pair $(\Omega, \mathscr{F})$ is called a measurable space

σ-Algebra

One can readily verify the following properties of σ-algebras:

- Every σ-algebra is closed under countable intersections
- Let $\mathcal{I}$ be an arbitrary index set, and let $\{\mathscr{F}_i : i \in \mathcal{I}\}$ be a collection of σ-algebras of subsets of a given sample space Ω. Then, $\bigcap_{i\in\mathcal{I}} \mathscr{F}_i$ is a σ-algebra of subsets of Ω.

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σ-Algebra – Examples

Fix a sample space Ω.

- The most trivial σ-algebras $\{\emptyset, \Omega\}$ and 2^{Ω}
- For any $A \subset \Omega$, $\{\emptyset, \Omega, A, A^c\}$ is a σ-algebra
- Given a collection $\mathscr{C}$ of subsets of Ω, the smallest σ-algebra containing $\mathscr{C}$ is denoted by $\sigma(\mathscr{C})$.

  Example: $\Omega = \{1, 2, \ldots, 6\}, \quad \mathscr{C} = \Big\{\{1, 2\}, \{1, 3\}\Big\}$

$$\sigma(\mathscr{C}) = \Big\{\emptyset, \Omega, \{1, 2\}, \{1, 3\}, \{1\}, \{2, 3, 4, 5, 6\}, \{1, 2, 3\}, \{4, 5, 6\},$$

$$\{3, 4, 5, 6\}, \{2, 4, 5, 6\}, \{2\}, \{3\}, \{1, 3, 4, 5, 6\}, \{1, 2, 4, 5, 6\}, \{2, 3\}, \{1, 4, 5, 6\}\Big\}$$

Measure

Fix a measurable space $(\Omega, \mathscr{F})$.

Definition (Measure)

A function $\mu : \mathscr{F} \to [0, +\infty]$ is called a measure on $(\Omega, \mathscr{F})$ if it satisfies the following properties:

1. $\mu(\emptyset) = 0$.
2. If $A_1, A_2, \ldots$ is a countable collection of disjoint sets, with $A_i \in \mathscr{F}$ for each $i \in \mathbb{N}$, then
$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i).$$
Property 2 above is called the property of countable additivity.

The triplet $(\Omega, \mathscr{F}, \mu)$ is called a measure space.

Measure

- When $\mu(\Omega) < +\infty$, the measure μ is called a finite measure
- When $\mu(\Omega) = +\infty$, the measure μ is called an infinite measure
- When $\mu(\Omega) = 1$, the measure μ is called a probability measure, and denoted by $\mathbb{P}$.

Probability Measure

Fix a measurable space $(\Omega, \mathscr{F})$.

Definition (Probability Measure)

A function $\mathbb{P} : \mathscr{F} \to [0, 1]$ is called a probability measure if the following properties are satisfied:

1. $\mathbb{P}(\emptyset) = 0$.
2. $\mathbb{P}(\Omega) = 1$.
3. If $A_1, A_2, \ldots$ is a countable collection of disjoint sets, with $A_i \in \mathscr{F}$ for each $i \in \mathbb{N}$, then

$$\mathbb{P}\left(\bigcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \mathbb{P}(A_i).$$

References

Kolmogorov, A. N. (2004).
From the heritage of A. N. Kolmogorov: The theory of probability.
Theory of Probability & Its Applications, 48(2):191–220.

Shafer, G. and Vovk, V. (2018).
The origins and legacy of Kolmogorov's Grundbegriffe.
arXiv preprint arXiv:1802.06071.