AI 5030: Probability and Stochastic Processes

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Homework 10

Topics: Probability Generating Functions, Moment Generating Functions, Characteristic Functions, Joint MGF/CF

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. All random variables appearing below are assumed to be defined with respect to $\mathscr{F}$.

1. Let $Y \sim \mathcal{N}(0,1)$. Fix $a > 0$, and for each $\omega \in \Omega$, let

$$Z(\omega) = \begin{cases} Y(\omega), & |Y(\omega)| \leq a, \\ -Y(\omega), & |Y(\omega)| > a. \end{cases}$$

   Show that $Z \sim \mathcal{N}(0,1)$.

   Hint: Show that the MGF of Z is identical to the MGF of Y.

2. Let X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{4}(1 + xy(x^2 - y^2)), & |x| < 1,\ |y| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

   Show that $C_{X,Y}(s) = C_X(s) \cdot C_Y(s)$ for all $s \in \mathbb{R}$, but $X \not\perp\!\!\!\perp Y$.

3. Determine the PMF of a random variable X whose probability generating function has the expression

$$G_X(z) = \frac{\left(\frac{1}{3}z + \frac{2}{3}\right)^4}{z}, \qquad z \in \mathbb{R} \setminus \{0\}.$$

4. Let $X_1, X_2, \ldots \overset{\text{i.i.d.}}{\sim} \text{Poisson}(\lambda)$. Using MGF, compute the distribution of $Y = \sum_{i=1}^n X_i$.

5. Let X be a continuous random variable with PDF f_X. Show that the characteristic function of X is real (i.e., zero imaginary part) if and only if f_X is an even function, i.e.,

$$\text{Im}(C_X(s)) = \mathbb{E}[\sin sX] = 0 \quad \forall s \in \mathbb{R} \qquad \Longleftrightarrow \qquad f_X(x) = f_X(-x) \quad \forall x \in \mathbb{R}.$$

   Hint: For any $z \in \mathbb{C}$, $\text{Im}(z) = 0$ if and only if $\bar{z} = z$, where $\bar{z}$ denotes the complex conjugate of z.

6. Let $X \sim \mathcal{N}(0,1)$. Let W be a discrete random variable independent of X and having the PMF

$$\mathbb{P}(\{W = w\}) = \begin{cases} \frac{1}{2}, & w = \pm 1, \\ 0, & \text{otherwise.} \end{cases}$$

   Define a new random variable Y as $Y = WX$. Using MGF, show that $Y \sim \mathcal{N}(0,1)$.