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Indian Institute of Technology Hyderabad

Probability and Stochastic Processes

Monotone Convergence Theorem, Expectations of Non-Negative Random Variables

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Pointwise Convergence and Monotone Convergence Theorem – 1

Pointwise Convergence

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X and $\{X_n\}_{n=1}^{\infty} = \{X_1, X_2, \ldots\}$ be random variables w.r.t. $\mathscr{F}$.

Pointwise Convergence

We say that the sequence of random variables $\{X_n\}_{n=1}^{\infty}$ converges pointwise to the random variable X if

$$\lim_{n\to\infty} X_n(\omega) = X(\omega) \qquad \forall \omega \in \Omega.$$

Notation:

$$X_n \xrightarrow{\text{pointwise}} X.$$

Non-Negative Random Variable as Pointwise Limit of Simple Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Proposition
(Non-Negative Random Variable as Pointwise Limit of Simple Random Variables)

Let X be a non-negative random variable, i.e.,

$$X(\omega) \geq 0 \qquad \forall \omega \in \Omega.$$

Then, there exists a sequence of random variables $\{X_n\}_{n=1}^{\infty}$ such that

- X_n is a simple random variable for each $n \in \mathbb{N}$, and
- $X_n \xrightarrow{\text{pointwise}} X$.

Monotone Convergence Theorem (MCT) – 1

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let X and $\{X_n\}_{n=1}^{\infty}$ be random variables defined w.r.t. $\mathscr{F}$.

Theorem (Monotone Convergence Theorem (MCT) – 1)

Let X be a non-negative random variable.

Suppose that

- X_n is a simple random variable for each $n \in \mathbb{N}$,
- $0 \leq X_1(\omega) \leq X_2(\omega) \leq \cdots$ for all $\omega \in \Omega$ (monotone),
- $X_n \xrightarrow{\text{pointwise}} X$ (convergence).

Then, $0 \leq \mathbb{E}[X_1] \leq \mathbb{E}[X_2] \leq \cdots$, and

$$\mathbb{E}[X] = \lim_{n \to \infty} \mathbb{E}[X_n].$$

Remark: We will state the MCT in greater generality later in the course.

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Expectations of Non-Negative Random Variables

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Expectations of Non-Negative Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

- If $A \in \mathscr{F}$ such that $\mathbb{P}(A) = 0$, and X is a non-negative random variable, then
$$\mathbb{E}[X \cdot \mathbf{1}_A] = 0.$$
- If $A \in \mathscr{F}$ such that $\mathbb{P}(A) = 1$, and X is a non-negative random variable, then
$$\mathbb{E}[X \cdot \mathbf{1}_A] = \mathbb{E}[X].$$
- If X, Y are non-negative random variables, and $X(\omega) \geq Y(\omega) \geq 0$ for all $\omega \in \Omega$, then
$$\mathbb{E}[X] \geq \mathbb{E}[Y].$$

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Expectations of Non-Negative Random Variables

- If X is a non-negative random variable, then

$$\mathbb{E}[X] = 0 \quad \Longleftrightarrow \quad \mathbb{P}(\{X = 0\}) = 1.$$

- Linearity of expectations (for non-negative random variables)
  If X, Y are non-negative random variables, and $\alpha \geq 0$, then

$$\mathbb{E}[\alpha X] = \alpha \mathbb{E}[X], \qquad \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

- If X is a non-negative random variable, then

$$\begin{aligned}
\mathbb{E}[X \cdot \mathbf{1}_{\{X \geq x\}}] &\geq x \cdot \mathbb{P}(\{X \geq x\}), \qquad & x \geq 0,\\
\mathbb{E}[X \cdot \mathbf{1}_{\{X \leq x\}}] &\leq x \cdot \mathbb{P}(\{X \leq x\}), \qquad & x \geq 0,\\
\mathbb{E}[X \cdot \mathbf{1}_{\{X = x\}}] &= x \cdot \mathbb{P}(\{X = x\}), \qquad & x \geq 0.
\end{aligned}$$

Expectation of a Discrete Random Variable

Expectation of a Discrete Random Variable

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let X be a discrete random variable taking values in the countable set $E = \{a_1, a_2, \ldots\}$.

- Suppose X is simple.
  - Then, there exists $n \in \mathbb{N}$ such that
$$E = \{a_1, \ldots, a_n\}, \qquad a_1, \ldots, a_n \geq 0.$$
  - Set $A_i = \{X = a_i\}, \quad i = 1, \ldots, n$. Then, $\{A_1, \ldots, A_n\}$ partition Ω.
  - X can be expressed as
$$X = \sum_{i=1}^{n} a_i \, \mathbf{1}_{A_i}.$$
  - The expectation of X is given by
$$\mathbb{E}[X] = \sum_{i=1}^{n} a_i \, \mathbb{P}(A_i).$$

Expectation of a Discrete Random Variable

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X be a discrete random variable taking values in the countable set $E = \{a_1, a_2, \ldots\}$.

- Suppose X is non-negative.
  - Here,
  $$E = \{a_1, a_2, \ldots\}, \qquad a_i \geq 0 \ \ \forall i \in \mathbb{N}.$$
  - Set $A_i = \{X = a_i\}$, $\quad i = 1, \ldots, n$. Then, $\{A_1, A_2, \ldots\}$ partition Ω.
  - X can be expressed as
  $$X = \sum_{i=1}^{\infty} a_i \, \mathbf{1}_{A_i}.$$
  - For each $n \in \mathbb{N}$, let
  $$X_n := \sum_{i=1}^{n} a_i \, \mathbf{1}_{A_i}.$$

Expectation of a Discrete Random Variable

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X be a discrete random variable taking values in the countable set $E = \{a_1, a_2, \ldots\}$.

- Suppose X is non-negative.
  - Then, $0 \leq X_1 \leq X_2 \leq \cdots$, and $X_n \xrightarrow{\text{pointwise}} X$.
  - Using MCT, we have

$$\mathbb{E}[X] = \lim_{n\to\infty} \mathbb{E}[X_n] = \lim_{n\to\infty} \sum_{i=1}^{n} a_i \, \mathbb{P}(A_i) = \sum_{i=1}^{\infty} a_i \, \mathbb{P}(A_i).$$

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Expectation of a Discrete Random Variable

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X be a discrete random variable taking values in the countable set $E = \{a_1, a_2, \ldots\}$.

- Suppose X is arbitrary.
  - Here, some of the a_i could be negative.
  - Define
$$X_+ = \max\{X, 0\}, \qquad X_- = -\min\{X, 0\}$$
  - If $\min\{\mathbb{E}[X_+], \mathbb{E}[X_-] < +\infty\}$, then
$$\mathbb{E}[X] := \mathbb{E}[X_+] - \mathbb{E}[X_-].$$

Summary

For a discrete random variable X taking values in set $E = \{a_1, a_2, \ldots\}$,

$$\mathbb{E}[X] = \sum_{i=1}^{\infty} a_i \cdot \mathbb{P}(\{X = a_i\}) = \sum_{i=1}^{\infty} a_i \cdot p_X(a_i).$$

Examples

- Suppose $X \sim \mathrm{Unif}\big(\{1, \ldots, n\}\big)$.
  What is $\mathbb{E}[X]$?

- Suppose $X \sim \mathrm{Geom}(p), \qquad p \in (0, 1)$.
  What is $\mathbb{E}[X]$?

- Let $\mathbb{P}(\{X = k\}) = \dfrac{c}{k^2}, \qquad k \in \mathbb{N}$.
  What is $\mathbb{E}[X]$?

- Let $\mathbb{P}(\{X = k\}) = \dfrac{c}{k^2}, \qquad k \in \mathbb{Z}$.
  What is $\mathbb{E}[X]$?

Expectation Over Different Spaces

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let X be a discrete random variable w.r.t. $\mathscr{F}$.
Let $g : \mathbb{R} \to \mathbb{R}$ be Borel-measurable.
Let $Y = g(X)$.

Theorem (Expectation Over Different Spaces)

We have

$$\mathbb{E}[Y] = \int_{\Omega} g(X)\, d\mathbb{P} = \int_{\mathbb{R}} g\, d\mathbb{P}_X = \int_{\mathbb{R}} y\, d\mathbb{P}_Y.$$