Assignment - 1 20/08/24

Span:- if $\bar{v}$ and $\bar{w}$ are the Vectors then the span of $\bar{v}$ and $\bar{w}$ is set of all linear Combinatory

$$a\bar{v} + b\bar{w} \quad a, b \in \mathbb{R}$$

My roll number: EE24MTECH12008

a = 1
b = 2
c = 0
d = 0
e = 8

1) is $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, in the span of the two Vectors $\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 8 \\ -1 \\ 3 \end{bmatrix}$? if so, write the linear Combination of these two Vectors.

if not show it is not

Sol:- let $\bar{V}_1, \bar{V}_2, \bar{V}_3$ are the Vectors

$$\bar{V}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \bar{V}_2 = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \quad \bar{V}_3 = \begin{bmatrix} 8 \\ -1 \\ 3 \end{bmatrix}$$

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Step 1 :- Check $\bar{v}_1, \bar{v}_2, \bar{v}_3$ are linearly dependent or linearly independent.

$$\bar{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \bar{v}_2 = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} \quad \bar{v}_3 = \begin{bmatrix} 8 \\ -1 \\ 3 \end{bmatrix}$$

$$c_1\bar{v}_1 + c_2\bar{v}_2 + c_3\bar{v}_3 = 0$$

$$c_1\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix} + c_3\begin{bmatrix} 8 \\ -1 \\ 3 \end{bmatrix} = 0$$

$$c_1 + 4c_2 + 8c_3 = 0$$
$$2c_1 + 0c_2 - c_3 = 0$$
$$2c_2 + 3c_3 = 0$$

$$c_1 + 4c_2 + 8c_3 = 0$$
$$2c_1 - c_3 = 0$$
$$2c_2 + 3c_3 = 0$$

$$c_3 = 2c_1$$

$$2c_2 = -3c_3$$

$$\cancel{2}c_2 = -3(\cancel{2})(c_1)$$

$$c_2 = -3c_1$$

$$\boxed{c_2 = 0}$$

$$c_1 + 4(-3c_1) + 8(2c_1) = 0$$

$$\boxed{c_3 = 0}$$

$$c_1 - 12c_1 + 16c_1 = 0$$

$$5c_1 = 0$$

$$\boxed{c_1 = 0}$$

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so $c_1\bar{V}_1 + c_2\bar{V}_2 + c_3\bar{V}_3 = 0$

if and only ~~for only~~ if $\boxed{c_1 = c_2 = c_3 = 0}$

So we can say that $\bar{V}_1, \bar{V}_2, \bar{V}_3$ are linearly independent.

$\therefore$ The Vector $\bar{V}_1$ cannot be represented as the linear combination of $\bar{V}_2$ and $\bar{V}_3$ so

The Vector $\bar{V}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ is not there

in the span $\left\{ \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 8 \\ -1 \\ 3 \end{bmatrix} \right\}$

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2) $A = \begin{bmatrix} 1 & 0 & 2 & 8 & 2 & -1 \\ 2 & -1 & -2 & 5 & 4 & 0 \\ 0 & -1 & 0 & 8 & 5 & -1 \\ 4 & -1 & 2 & 0 & 8 & 0 \end{bmatrix}$

dimension of column space of A

& Check for linear dependence

$c_1 + 2c_3 + 8c_4 + 2c_5 - c_6 = 0$

$2c_1 - c_2 - 2c_3 + 5c_4 + 4c_5 = 0$

2) $A = \begin{bmatrix} 1 & 0 & 2 & 8 & 2 & -1 \\ 2 & -1 & -2 & 5 & 4 & 0 \\ 0 & -1 & 0 & 8 & 5 & -1 \\ 4 & -1 & 2 & 0 & 8 & 0 \end{bmatrix}$

(a) → find the no. of linearly independent Vectors in A

The dimension of colown space of A is the no. of linearly independent Vectors in A which span the Vectors

* The rank of the matrix = dimension of the matrix

find the rank (A)

$$= \begin{bmatrix} 1 & 0 & 2 & 8 & 2 & -1 \\ 2 & -1 & -2 & 5 & 4 & 0 \\ 0 & -1 & 0 & 8 & 5 & -1 \\ 0 & -1 & -6 & -32 & 0 & 4 \end{bmatrix} \quad R_4 = R_4 - 4R_1$$

$$= \begin{bmatrix} 1 & 0 & 2 & 8 & 2 & -1 \\ 2 & -1 & -2 & 5 & 4 & 0 \\ 0 & -1 & 0 & 8 & 5 & -1 \\ 0 & 0 & -6 & -40 & -5 & 5 \end{bmatrix} \quad R_4 = R_4 - R_3$$

$$= \begin{bmatrix} 1 & 0 & 2 & 8 & 2 & -1 \\ 0 & -1 & -6 & -11 & 0 & 2 \\ 0 & -1 & 0 & 8 & 5 & -1 \\ 0 & 0 & -6 & -40 & -5 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 & 8 & 2 & -1 \\ 0 & -1 & -6 & -11 & 0 & 2 \\ 0 & 0 & 6 & 19 & 5 & -3 \\ 0 & 0 & -6 & -40 & -5 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 & 8 & 2 & -1 \\ 0 & -1 & -6 & -11 & 0 & 2 \\ 0 & 0 & 6 & 19 & 5 & -3 \\ 0 & 0 & 0 & -21 & 0 & 2 \end{bmatrix}$$

The rank of a matrix is 4 [non zero rows in echelon form of matrix]

* There are 4 linearly independent Vectors.

dimension of the matrix is 4

$\boxed{\text{* The dimension of the Column space of A is } 4}$

b) Dimension of the NULL space of A

$$N(A) = \{ x : A\bar{x} = 0 \ , \ x \in \mathbb{R}^n \}$$

$A\bar{x} = 0$

$$4\times\begin{bmatrix} 1 & 0 & 2 & 8 & 2 & -1 \\ 2 & -1 & -2 & 5 & 4 & 0 \\ 0 & -1 & 0 & 8 & 5 & -1 \\ 4 & -1 & 2 & 0 & 8 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

for solving reduce the matrix A in to echelon form

$$= \begin{bmatrix} 1 & 0 & 2 & 8 & 2 & -1 \\ 2 & -1 & -2 & 5 & 4 & 0 \\ 0 & -1 & 0 & 8 & 5 & -1 \\ 0 & -1 & -6 & -32 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 & 8 & 2 & -1 \\ 2 & -1 & -2 & 5 & 4 & 0 \\ 0 & -1 & 0 & 8 & 5 & -1 \\ 0 & -1 & -6 & -32 & 0 & 4 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 8 & 2 & -1 \\ 0 & -1 & -6 & -11 & 0 & 2 \\ 0 & -1 & 0 & 8 & 5 & -1 \\ 0 & 0 & -6 & -40 & -5 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 8 & 2 & -1 \\ 0 & -1 & -6 & -11 & 0 & 2 \\ 0 & 0 & 6 & 19 & 5 & -3 \\ 0 & 0 & -6 & -40 & -5 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 & 8 & 2 & -1 \\ 0 & -1 & -6 & -11 & 0 & 2 \\ 0 & 0 & 6 & 19 & 5 & -3 \\ 0 & 0 & 0 & -21 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 2 & 8 & 2 & -1 \\ 0 & -1 & -6 & -11 & 0 & 2 \\ 0 & 0 & 6 & 19 & 5 & -3 \\ 0 & 0 & 0 & -21 & 0 & 2 \end{bmatrix}$$

$x_1 + 2x_3 + 8x_4 + 2x_5 - x_6 = 0$

$-x_2 - 6x_3 - 11x_4 + 2x_6 = 0$

$6x_3 + 19x_4 + 5x_5 - 3x_6 = 0$

$-21x_4 + 2x_6 = 0$

let $x_5 = k_1$

let $x_6 = k_2$

$-21x_4 + 2k_2 = 0$

$+21x_4 = +2k_2$

$$\boxed{x_4 = \frac{2}{21}k_2}$$

$$6x_3 + 19x_4 + 5x_5 - 3x_6 = 0$$

$$6x_3 + 19\left[\frac{2}{21}k_2\right] + 5k_1 - 3x_6 = 0$$

$$6x_3 + 19\left[\frac{2}{21}k_2\right] + 5k_1 - 3k_2 = 0$$

$$6x_3 = 3k_2 - \frac{38}{21}k_2 - 5k_1$$

$$\boxed{x_3 = \frac{k_2}{2} - \frac{38k_2}{126} - \frac{5}{6}k_1}$$

$$-x_2 - 6x_3 - 11x_4 + 2x_6 = 0$$

$$-x_2 - 6\left[\frac{k_2}{2\times 63} - \frac{38k_2}{126} - \frac{5k_1}{6}\right]$$
$$-11\left[\frac{2}{21}k_2\right] + 2k_2 = 0$$

$$-x_2 - 6\left[\frac{63k_2 - 38k_2}{126} - \frac{5k_1}{6}\right]$$
$$-\frac{22}{21}k_2 + 2k_2 = 0$$

$$-x_2 - 6\left[\frac{25k_2}{126} - \frac{5k_1}{6}\right]$$
$$-\left[\frac{22}{21}k_2 - 2k_2\right] = 0$$

$\frac{63}{126}$
$\frac{5}{63}$
$\frac{38}{25}$
$\frac{42}{22}$
20

$$-x_2 - \frac{25k_2}{21} + 5k_1 - \frac{20k_2}{21} = 0$$

$$-x_2 + 5k_1 - \frac{\cancel{45}^{15}k_2}{\cancel{21}_{7}} = 0$$

$$-x_2 + 5k_1 - \frac{15k_2}{7} = 0$$

$$\boxed{x_2 = -\frac{15k_2}{7} + 5k_1}$$

$$x_1 + 2x_3 + 8x_4 + 2x_5 - x_6 = 0$$

$$x_1 + 2\left[\frac{25k_2}{126} - \frac{5}{6}k_1\right] + 8\left[\frac{2k_2}{21}\right] + 2k_1 - k_2 = 0$$

$$x_1 + \frac{25k_2}{63} - \frac{5k_1}{3} + \frac{16k_2}{21} + 2k_1 - k_2 = 0$$

$$x_1 - \frac{5k_1}{3} + 2k_1 + \frac{25k_2}{63} + \frac{16k_2}{21} - k_2 = 0$$

$$x_1 + \frac{-5k_1 + 6k_1}{3} + \frac{25k_2 + 48k_2 - 63k_2}{63} = 0$$

$$x_1 + \frac{k_1}{3} + \frac{10k_2}{63} = 0$$

$$\boxed{x_1 = -\frac{k_1}{3} + \frac{10k_2}{63}}$$

$$-\frac{k_1}{3} + \frac{10k_2}{63}$$

$$+5k_1 - \frac{15k_2}{7}$$

$$-\frac{5}{6}k_1 + \frac{25k_2}{126}$$

$$\frac{2}{21}k_2$$

$$k_1$$

$$k_2$$

$$k_1\begin{bmatrix} -\frac{1}{3} \\ +5 \\ -\frac{5}{6} \\ 0 \\ 1 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} +\frac{10}{63} \\ -\frac{15}{7} \\ \frac{25}{126} \\ \frac{2}{21} \\ 0 \\ 1 \end{bmatrix}$$

$$\text{Null Space of } (A) = \text{Span}\left\{ \begin{bmatrix} -\frac{1}{3} \\ +5 \\ -\frac{5}{6} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} +\frac{10}{63} \\ -\frac{15}{7} \\ \frac{25}{126} \\ 2/21 \\ 0 \\ 1 \end{bmatrix} \right\}$$

dimension of null space of A is 2

(c) find a basis for the column space of A

The basis is set of linearly independent vectors in a colown space of A

$$A = \begin{bmatrix} 1 & 0 & 2 & 8 & 2 & -1 \\ 2 & -1 & -2 & 5 & 4 & 0 \\ 0 & -1 & 0 & 8 & 5 & -1 \\ 4 & -1 & 2 & 0 & 8 & 0 \end{bmatrix}$$

reduce the above matrix in to echolon form

$$= \begin{bmatrix} 1 & 0 & 2 & 8 & 2 & -1 \\ 0 & -1 & -6 & -11 & 0 & 2 \\ 0 & 0 & 6 & 19 & 5 & -3 \\ 0 & 0 & 0 & -21 & 0 & 2 \end{bmatrix}$$

* The colowns 1,2,3,4 are the pivot columns so the vectors corresponding to the pivot columns are linearly independent vectors.

The basis of coolumn space of A is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 8 \\ 5 \\ 8 \\ 0 \end{bmatrix} \right\}$$

* any vector in the colowmn space of A can be represented as a linear combination of vectors in the basis.

(d) The basis of null space of A is

$$\left\{ \begin{bmatrix} \frac{1}{3} \\ -5 \\ -\frac{5}{6} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{10}{63} \\ 15/7 \\ 25/126 \\ 2/21 \\ 0 \\ 1 \end{bmatrix} \right\}$$

3) Two Vectors are said to be Collinear, when one can be written as Scalar multiple of the other. Consider two Vectors $\bar{u}$ and $\bar{v}$ that are not collinear. Consider the Vector $\bar{w}$ that doesnot belongs to the linear span of $\bar{u}$ and $\bar{v}$. Proove that $\bar{u}, \bar{v}$ and $\bar{w}$ are linearly independent.

A) Proove that $\bar{u}, \bar{v}, \bar{w}$ are linearly independent

let us Consider

$$\boxed{a\bar{u} + b\bar{v} + c\bar{w} = \bar{0}} \quad (1)$$

for some scalars a, b, c

Consider the Case $c = 0$

$$a\bar{u} + b\bar{v} = 0$$

* $\bar{u}$ and $\bar{v}$ are not collinear so the above linear Combination will be $\bar{0}$ only when $a = 0$ and $b = 0$

Consider the Case $c \neq 0$

if $c \neq 0$ then we can write the eq (1) as

$$\boxed{c\bar{w} = -a\bar{u} - b\bar{v}}$$

The above equation indicates that $\bar{w}$ is in the span of $\bar{u}$ and $\bar{v}$ but given that $\bar{w}$ doesnot belongs to linear span of $\bar{u}$ and $\bar{v}$ which is Contradiction

* So we Prooved that eq (1) is Possible only when $a = b = c = 0$

$\therefore$ $\bar{u}, \bar{v}$ and $\bar{w}$ are linearly independent.

4) A matrix is Said to be upper triangular if $a_{ij} = 0$ for $i > j$. Consider a generic 3×3 upper triangular matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

a) if $a_{11} = b$, $a_{22} = c$, $a_{33} = e$ find the solution of $Ax = 0$

$$\boxed{a_{11} = 2, \quad a_{22} = 0, \quad a_{33} = 8}$$

$$A = \begin{bmatrix} 2 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 8 \end{bmatrix}$$

$$Ax = 0$$

$$\begin{bmatrix} 2 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & a_{12} & a_{13} & 0 \\ 0 & 0 & a_{23} & 0 \\ 0 & 0 & 8 & 0 \end{array}\right]$$

$$\boxed{R_3 = a_{23} R_3 - 8 R_2}$$

$$\sim \left[\begin{array}{ccc|c} 2 & a_{12} & a_{13} & 0 \\ 0 & 0 & a_{23} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 2 & a_{12} & a_{13} & 0 \\ 0 & 0 & a_{23} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\boxed{\begin{aligned} 2x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ a_{23}x_3 &= 0 \end{aligned}}$$

$\boxed{x_3 = 0}$

$x_2 = k$

$2x_1 + a_{12}k = 0$

$$\boxed{x_1 = \frac{-a_{12}k}{2}}$$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{a_{12}k}{2} \\ k \\ 0 \end{bmatrix}$$

$$\bar{x} = k \begin{bmatrix} -\frac{a_{12}}{2} \\ 1 \\ 0 \end{bmatrix}$$

b) $a_{11} = 0$

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 8 \end{bmatrix}.$$

let the three coloumns of a matrix are

$$\bar{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \bar{u} = \begin{bmatrix} a_{12} \\ 0 \\ 0 \end{bmatrix}, \quad \bar{w} = \begin{bmatrix} a_{13} \\ a_{23} \\ 8 \end{bmatrix}$$

Proove that $\bar{v}, \bar{u}, \bar{w}$ are linearly dependent.

$c_1\bar{v} + c_2\bar{u} + c_3\bar{w} = 0 \qquad c_i \in \mathbb{R}$

$$c_1 \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} a_{12} \\ 0 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} a_{13} \\ a_{23} \\ 8 \end{bmatrix} = \bar{0}$$

$c_2 a_{12} + c_3 a_{13} = 0$

$a_{23} c_3 = 0$

$c_3 8 = 0$

$\boxed{c_3 = 0}$ $\qquad c_1 \neq 0$

$\boxed{c_2 = 0}$

c_1 can be any value $\in \mathbb{R}$

The Vector $\bar{v}$ can be expressed as linear Combination of $\bar{u}$ and $\bar{w}$

$\bar{v} = c_1\bar{u} + c_2\bar{w}$

$c_1 = c_2 = 0$

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So the columns of A are linearly dependent.

$\rightarrow a_{22} = 0$

$$A = \begin{bmatrix} 2 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 8 \end{bmatrix}$$

$$\bar{u} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \bar{v} = \begin{bmatrix} a_{12} \\ 0 \\ 0 \end{bmatrix}, \bar{w} = \begin{bmatrix} a_{13} \\ a_{23} \\ 8 \end{bmatrix}$$

$$c_1\bar{u} + c_2\bar{v} + c_3\bar{w} = \bar{0}$$

$$2c_1 + a_{12}c_2 + a_{13}c_3 = 0$$

$$\boxed{c_3 = 0}$$

$$\boxed{c_2 = k}$$

$$\boxed{c_1 = \frac{-a_{12}k}{2} \neq 0}$$

So the vectors $\bar{u}, \bar{v}$ and $\bar{w}$ are linearly dependent

$\rightarrow a_{33} = 0$

$$A = \begin{bmatrix} 2 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\bar{u} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \bar{v} = \begin{bmatrix} a_{12} \\ 0 \\ 0 \end{bmatrix} \quad \bar{w} = \begin{bmatrix} a_{13} \\ a_{23} \\ 0 \end{bmatrix}$$

$$c_1\bar{u} + c_2\bar{v} + c_3\bar{w} = 0$$

$$c_1\begin{bmatrix}2\\0\\0\end{bmatrix}+c_2\begin{bmatrix}a_{12}\\0\\0\end{bmatrix}+c_3\begin{bmatrix}a_{13}\\a_{23}\\0\end{bmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$2c_1 + a_{12}c_2 + a_{13}c_3 = 0$$

$$\boxed{c_3 = 0}$$

$$c_2 = k$$

$$c_1 = \frac{-a_{12}k}{2} \neq 0$$

so the vectors $\bar{u}, \bar{v}$ and $\bar{w}$ are linearly dependent.

(c) if $a_{22} = 0$, find the non zero element in the Null space of A

$$A = \begin{bmatrix}2 & a_{12} & a_{13}\\0 & 0 & a_{23}\\0 & 0 & 8\end{bmatrix}$$

Solve $Ax = 0$ for $a_{22} = 0$

$$\left[\begin{array}{ccc|c}2 & a_{12} & a_{13} & 0\\0 & 0 & a_{23} & 0\\0 & 0 & 8 & 0\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}2 & a_{12} & a_{13} & 0\\0 & 0 & a_{23} & 0\\0 & 0 & 0 & 0\end{array}\right]$$

$2x_1 + a_{12}x_2 + a_{13}x_3 = 0$

$a_{23}x_3 = 0$

$\boxed{x_2 = k}$ $\boxed{x_1 = \frac{-a_{12}k}{2}}$

$\boxed{x_3 = 0}$

The solution of $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \cancel{0}k \\ \cancel{0} \end{bmatrix} = k\begin{bmatrix} -\frac{a_{12}}{2} \\ \cancel{0}1 \\ 0 \end{bmatrix}$

$$\bar{x} = \begin{bmatrix} -\frac{a_{12}}{2} \\ \cancel{0}1 \\ 0 \end{bmatrix}$$

$$\bar{x} = k\begin{bmatrix} -\frac{a_{12}}{2} \\ 1 \\ 0 \end{bmatrix}$$

The null space of $A = \left\{ \begin{bmatrix} -\frac{a_{12}}{2} \\ 1 \\ 0 \end{bmatrix} \right\}$

The non zero element is 1

5) Show that the vector space of polynomials of degree less or equal to d is of dimension d+1. [for instance, the space of quadratic polynomials is of dimension 3] Use this result to show that the vector space of all real functions cannot have finite dimension.

A) Consider the vector space P_d of all polynomials of degree less than or equal to d. This space includes all polynomials of the form

$$\boxed{P(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_d x^d}$$

$a_0, a_1, a_2 \dots a_d \in \mathbb{R}$

→ find the basis of $P(x)$:-

basis is the set of vectors that are linearly independent and span the entire space.

let $P(x) = a_0 + a_1 x + \dots + a_d x^d$

In order to check for linear independence let

$$C_0 \cdot 1 + c_1 x + c_2 x^2 + \dots + c_d x^d = 0$$

The above equation is possible only when

$$\boxed{C_0 = C_1 = C_2 \dots C_d = 0}$$

Hence, the polynomials are linearly independent

Since the set $\{1, x, x^2 \ldots + x^d\}$ is linearly independent. ~~and spans $P(x)$~~

~~Dimension of $P(n) = (d+1)$~~

Check the condition for spanning the vector space

Any Polynomial $P(x)$ in P_d can be written as linear combination of these polynomials.

$$\boxed{P(x) = a_0 \cdot 1 + a_1 x + \ldots + a_d x^d}$$

Since the set $\{1, x, x^2 \ldots x^d\}$ is also spans P_d

So the basis of P_d is $\{1, x, x^2 \ldots x^d\}$

$\boxed{\therefore \text{ The dimension of } P_d \text{ is } d+1}$

Now Proove that the vector space of all real functions cannot have finite dimension;

In order to Proove the above Let us consider the vector space F of all real functions $f: \mathbb{R} \to \mathbb{R}$ is larger than space of polynomials of any finite degree.

Consider the subspace P_d within F, we prooved that $\dim(P_d) = d+1$

→ But F includes not just polynomials, but also functions that cannot expressed of polynomials of any finite degree.

eg:- e^x, $\sin x$, $\cos x$, $\tan x$, $\log(x)$

if F were finite dimensional, let us say with dimension n, then F could only contain a finite number of linearly independent functions. But we can construct infinitely many linearly independent function which is Uncountably many no. of functions.

which is Contradiction

∴ We prooved that vector space of all real functions cannot have finite dimension.

6) Proove that if V and W are two dimensional Subspaces of $\mathbb{R}^3$ then V and W must have non-zero Vector in common.

A) let V and W are two dimensional subspaces

let $\overline{V_1}$ and $\overline{V_2}$ $\in V$

$\overline{W_1}$ and $\overline{W_2}$ $\in W$

$\dim(V) = 2$

$\dim(W) = 2$

$V \cap W$ is the set of all vectors that are both in V and W

The dimension formula for the sum of two subspaces V and W in $\mathbb{R}^n$ is given by

$$\dim(V+W) = \dim(V) + \dim(W) - \dim(V \cap W)$$

$$\dim(V+W) = 2 + 2 - \dim(V \cap W)$$

$$\boxed{\dim(V+W) = 4 - \dim(V \cap W)} \quad —①$$

* The subspace V+W is a subspace of $\mathbb{R}^n$.
The dimension of $\mathbb{R}^n$ is n

$$\boxed{\dim(V+W) \leq 3} \quad —②$$

Consider the eq ① and ②

if $\dim(V \cap W) = 0$

then $\dim(V+W) = 4 - 0$

$= 4$.

This is contradiction.

So the $\boxed{\dim(V \cap W) \neq 0}$

So we prooved that V and W ~~Can to have~~ most have non zero Vector in Common.

7) Suppose A is a $m \times n$ matrix. Give the conditions on the rank of A such that left and right inverses exist, respectively. Can the left and right inverses of a matrix exist simultaneously

A) Let A be an $m \times n$ matrix and B be the matrix of $n \times m$ such that

$$BA = I_n$$

I_n is the $n \times n$ Identity matrix.

Proof for left inverse existance :-

In order to exist $\boxed{BA = I_n}$ A must have all the colomns are linearly independent.

The above statement can be prooved by below

Suppose the colomns of A are linearly dependent then there exist a non zero vector $\bar{x} \in \mathbb{R}^n$ such that

$$A\bar{x} = 0$$

We know that $BA = I_n$

$$A\bar{x} = 0$$

$$BA\bar{x} = B0$$

$$I_n\bar{x} = 0$$

$$\boxed{\bar{x} = 0}$$

which is a contradiction.

$\therefore$ Colomns of A must be linearly independent rank (A) = no. of linearly independent columns of A

$\boxed{\text{rank}(A) = n}$

$\therefore$ A left inverse exists if and only if

$\boxed{\text{ran}(k) = n}$

* in order to accomodate all n independent columns we should have $\underline{\underline{m \geq n}}$

Right inverse

A matrix A has a right inverse if there exist a matrix B such that

$$\boxed{\underset{m\times n}{A}\,\underset{n\times m}{B} = \underset{m\times m}{I_m}}$$

$B \rightarrow n \times m$ matrix

$I_m \rightarrow m \times m$ matrix

Proof :- As we proved for the left inverse here is also same but here the matrix A should have m linearly independent rows.

$\therefore$ The $\boxed{\text{rank}(A) = m}$ for the existance of right inverse.

* in order to accomodate all m independent rows A should have atleast as many columns as rows

$\underline{\underline{n \geq m}}$

Existance for both left and right inversy

This can be prooved by the above two statements

+ for existence of left inverse $BA = I_n$

$$\boxed{\text{rank}(A) = n \text{ and } m \geq n}$$

+ for existance of right inverse $AB = I_m$

$$\boxed{\text{rank}(A) = m \text{ and } n \geq m}$$

if both cases has to exist simultaneously

$$\boxed{\text{rank}(A) = m = n}$$

and $m = n$

$\therefore$ A is a square matrix and $\text{rank}(A) = m = n$ for the existence of both left and right inversy simultaneously

(8) if the Product of two matrices is the zero matrix i.e. $AB=0$ Show that Column space of B is Contained in the null space of A

Proof :-

Coloumn Space of B :- The set of all linear Combinations of columns of B.

The set of all vectors $B\bar{v}$ where $\bar{v}$ is a vector in $\mathbb{R}^p$

$$\boxed{col(B) = \{B\bar{v} \mid v \in \mathbb{R}^p\}}$$

NULL space of A :- The set of all vectors in $\mathbb{R}^n$ such that $A\bar{u}=0$ $\bar{u} \in \mathbb{R}^n$

$$\boxed{NULL(A) = \{\bar{u} \in \mathbb{R}^n \mid A\bar{u}=0\}}$$

Given the Product of two matrices $AB=0$ it means that the Product Contain all the Coloumns with zero Vector.

$$\boxed{A\bar{b}_i = 0 \quad \forall i}$$

where $\bar{b}_i$ is the i^{th} colomn of B

$\therefore$ each Column Vector b_i of B is in the null space of A

$\therefore$ we Prooved that column space of B is contained in the null space of A

Q) Fred I holm's alternative for any A and $\bar{b}$,

One and only one of the following systems has a sollution.

i) $A\bar{x} = \bar{b}$

2) $\{A^T\bar{y} = 0, y^T\bar{b} \neq 0\}$.

Show that it is contradictory for ① and ② both to have

A) Let us assume that both ① and ② had sollutions simultaneously.

→ There exist a solution $\bar{x}$ to $A\bar{x} = b$

→ There exist a non trivial sollution $\bar{y}$ to $A^T\bar{y} = 0$ such that $y^T b \neq 0$

$A\bar{x} = \bar{b}$

→ multiply both sides with y^T

$\Rightarrow y^T A\bar{x} = y^T b$

$\Rightarrow (A^T y)^T \bar{x} = y^T b$

$\Rightarrow$ ~~$0\bar{x} = 0$~~ $0\bar{x} = y^T b$

$\Rightarrow \boxed{y^T\bar{b} = 0}$

but this is contradiction for the assumption that $A^T y = 0$ such that $y^T b \neq 0$

Hence prooved