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Probability and Stochastic Processes

Transformations of Random Variables, Jacobian Formula, Primer on Riemann Integration, Abstract Integrals and Expectations

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Transformations and Jacobian Formula

Sum of Random Number of Random Variables

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $\{X_i : i \in \mathbb{N}\}$ be a collection of i.i.d. random variables defined a common CDF F.
Let N be a positive integer-valued random variable defined with respect to $\mathscr{F}$ and having the PMF p_N. Assume $\mathbb{P}(\{N \in \mathbb{N}\}) = 1$ and $N \perp\!\!\!\perp \{X_1, X_2, \ldots\}$.
Let N be independent of $\{X_i : i \in \mathbb{N}\}$.
Consider the sum

$$S_N := \sum_{i=1}^{N} X_i; \qquad S_N(\omega) = \sum_{i=1}^{N(\omega)} X_i(\omega), \quad \omega \in \Omega.$$

- Show that $S_N : \Omega \to \mathbb{R}$ is a random variable with respect to $\mathscr{F}$.
- Determine the CDF of S_N.

Functions of Independent Random Variables are Independent Random Variables

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X_1, \ldots, X_n$ and Y be random variables with respect to $\mathscr{F}$.
Assume $Y \perp\!\!\!\perp \{X_1, \ldots, X_n\}$.
Let $g : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be Borel-measurable.
Prove that $g(X_1, \ldots, X_n) \perp\!\!\!\perp h(Y)$.

Sum of Geometric Number of Exponential Random Variables

In the previous example, let $X_1, X_2, \ldots \overset{\text{i.i.d.}}{\sim} \mathrm{Exp}(\lambda)$. Let $N \sim \mathrm{Geom}(p)$. Determine the distribution of S_N.

General Transformations and the Jacobian Formula

General Transformations

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a continuous random variable with PDF f_X.

Given $g : \mathbb{R} \to \mathbb{R}$ that is monontone and differentiable with non-zero derivative throughout its domain, what is the PDF of $Y = g(X)$?

General Transformations

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a continuous random variable with PDF f_X.

Given $g : \mathbb{R} \to \mathbb{R}$ that is monontone and differentiable with non-zero derivative throughout its domain, what is the PDF of $Y = g(X)$?

Note that g admits an inverse, say g^{-1}.

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General Transformations: g Monotone Increasing

$$
\begin{aligned}
F_Y(y) &= \mathbb{P}(\{Y \leq y\}) \\
&= \mathbb{P}(\{g(X) \leq y\}) \\
&= \mathbb{P}(\{X \leq g^{-1}(y)\}) \\
&= \int_{-\infty}^{g^{-1}(y)} f_X(x)\, dx \\
&= \int_{-\infty}^{y} \frac{f_X(g^{-1}(v))}{g'(g^{-1}(v))}\, dv
\end{aligned}
$$

make substitution $g(x) = v$

General Transformations: g Monotone Increasing

$$
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F_Y(y) &= \mathbb{P}(\{Y \le y\}) \\
&= \mathbb{P}(\{g(X) \le y\}) \\
&= \mathbb{P}(\{X \le g^{-1}(y)\}) \\
&= \int_{-\infty}^{g^{-1}(y)} f_X(x)\, dx \\
&= \int_{-\infty}^{y} \frac{f_X(g^{-1}(v))}{g'(g^{-1}(v))}\, dv && \text{make substitution } g(x) = v
\end{aligned}
$$

It thus follows that

$$
f_Y(y) = \begin{cases} \dfrac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}, & y \in \mathrm{Range}(g), \\ 0, & y \notin \mathrm{Range}(g). \end{cases}
$$

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General Transformations: g Monotone Decreasing

$$
\begin{aligned}
F_Y(y) &= \mathbb{P}(\{Y \leq y\}) \\
&= \mathbb{P}(\{g(X) \leq y\}) \\
&= \mathbb{P}(\{X \geq g^{-1}(y)\}) \\
&= \int_{g^{-1}(y)}^{+\infty} f_X(x)\,dx \\
&= \int_{y}^{+\infty} \frac{f_X(g^{-1}(v))}{g'(g^{-1}(v))}\,dv
\end{aligned}
$$

make substitution $g(x) = v$

General Transformations: g Monotone Decreasing

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\begin{aligned}
F_Y(y) &= \mathbb{P}(\{Y \leq y\}) \\
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\end{aligned}
$$

make substitution $g(x) = v$

It thus follows that

$$
f_Y(y) = \begin{cases} \dfrac{f_X(g^{-1}(y))}{-g'(g^{-1}(y))}, & y \in \mathrm{Range}(g), \\ 0, & y \notin \mathrm{Range}(g). \end{cases}
$$

General Transformations: g Monotone, Differentiable

When g is monotone and differentiable throughout its domain:

$$f_Y(y) = \begin{cases} \dfrac{f_X(g^{-1}(y))}{\left|g'(g^{-1}(y))\right|}, & y \in \mathrm{Range}(g), \\ 0, & y \notin \mathrm{Range}(g). \end{cases}$$

Examples

- Let $X \sim \mathcal{N}(0, 1)$.
  Derive the PDF of $Y = e^X$ from first principles and using the transformation formula.

General Transformations: g Piecewise Monotone, Differentiable

When g is piecewise monotone and differentiable

Suppose that $I_1, \ldots, I_n$ is a partition of $\mathbb{R}$, and $g : \mathbb{R} \to \mathbb{R}$ is piecewise montone and differentiable with non-zero derivative on I_i for each $i \in \{1, \ldots, n\}$. Let h_i denote the inverse of g on I_i. If X is a continuous random variable with PDF f_X, then the PDF of $Y = g(X)$ is given by

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(h_i(y))}{\left|g'(h_i(y))\right|} \mathbf{1}_{g(I_i)}(y).$$

Example

Suppose $X \sim \mathcal{N}(0, 1)$.
Derive the PDF of $Y = X^2$ from first principles and using the transformation formula.

General Transformations: Multivariate Case

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X_1, \ldots X_n$ be jointly continuous random variables with joint PDF $f_{X_1,\ldots,X_n}$.
Let $Y_i = g_i(X_1, \ldots, X_n)$ for $i \in \{1, \ldots, n\}$, where $g_1, \ldots, g_n$ are smooth[1] functions.
Derive the joint PDF of $Y_1, \ldots, Y_n$.

[1] We will assume that $g_1, \ldots, g_n$ are differentiable with continuous first-order partial derivatives.

Jacobian Matrix and Jacobian

Definition (Jacobian Matrix and Jacobian)

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$g(x_1, \ldots, x_n) = \begin{pmatrix} g_1(x_1, \ldots, x_n) \\ \vdots \\ g_n(x_1, \ldots, x_n) \end{pmatrix}$$

for some smooth functions $g_1, \ldots, g_n$.
The Jacobian matrix of the mapping g at the point $(x_1, \ldots, x_n)$ is defined as

$$J_g(x_1, \ldots, x_n) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{pmatrix}$$

The Jacobian of g at any point $(x_1, \ldots, x_n)$ is simply equal to $\det(J_g(x_1, \ldots, x_n))$.

Jacobi's Transformation Formula

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $X_1, \ldots, X_n$ be jointly continuous with joint PDF $f_{X_1,\ldots,X_n}$.

Jacobi's Transformation Formula

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be one-one, differentiable with continuous first-order partial derivatives, and non-zero Jacobian throughout its domain. Let the individual components of g be denoted by $g_1, \ldots, g_n$. Let $Y_i = g_i(X_1, \ldots, X_n)$ for all $i \in \mathbb{N}$. Then, the joint PDF of $Y = (Y_1, \ldots, Y_n)$ is given by

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = \begin{cases} \dfrac{f_{X_1,\ldots,X_n}(g^{-1}(y_1, \ldots, y_n))}{\left|\det\left(J_g(g^{-1}(y_1, \ldots, y_n))\right)\right|}, & (y_1, \ldots, y_n) \in \text{Range}(g), \\ 0, & (y_1, \ldots, y_n) \notin \text{Range}(g). \end{cases}$$

Jacobi's Transformation Formula

Fix $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $X_1, \ldots, X_n$ be jointly continuous with joint PDF $f_{X_1,\ldots,X_n}$.

Jacobi's Transformation Formula

Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be one-one, differentiable with continuous first-order partial derivatives, and non-zero Jacobian throughout its domain. Let the individual components of g be denoted by $g_1, \ldots, g_n$. Let $Y_i = g_i(X_1, \ldots, X_n)$ for all $i \in \mathbb{N}$. Then, the joint PDF of $Y = (Y_1, \ldots, Y_n)$ is given by

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Remark: For $g : \mathbb{R} \to \mathbb{R}$,

$$J_g(x) = g'(x), \qquad x \in \mathbb{R}.$$

Example

Let X and Y be independent exponential random variables with parameter λ.
Derive the joint PDF of $Y_1 = X_1$ and $Y_2 = X_1 + X_2$.
Also deduce the conditional PDF of Y_1, "conditioned on" the event $\{Y_2 = y\}$.

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Jacobi's Transformation Formula: g Piecewise Differentiable with Non-Zero Jacobian

When g is piecewise differentiable with non-zero Jacobian

Suppose that $I_1, \ldots, I_n$ is a partition of $\mathbb{R}^n$, and $g : \mathbb{R}^n \to \mathbb{R}^n$ is one-one, differentiable with continuous first-order partial derivatives, and has non-zero Jacobian on I_i for each $i \in \{1, \ldots, n\}$. Let h_i denote the inverse of g on I_i. Let $X_1, \ldots, X_n$ be jointly continuous with joint PDF $f_{X_1,\ldots,X_n}$. Let the individual components of g be $g_1, \ldots, g_n$. Let $Y_i = g_i(X_1, \ldots, X_n)$ for each $i \in \mathbb{N}$. Then the joint PDF of $Y = (Y_1, \ldots, Y_n)$ is given by

$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = \sum_{i=1}^{n} \frac{f_{X_1,\ldots,X_n}(h_i(y_1, \ldots, y_n))}{\left|\det\Big(J_g(h_i(y_1, \ldots, y_n))\Big)\right|} \mathbf{1}_{g(I_i)}(y_1, \ldots, y_n).$$

Integration and Expectations

A Primer on Riemann Integration

Let $f : \mathbb{R} \to \mathbb{R}$ be given.
Let $a, b \in \mathbb{R}$, $a < b$, be given.

What is the mathematical interpretation of $\int_a^b f(t)\, dt$?

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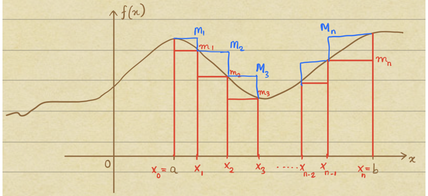
A Primer on Riemann Integration



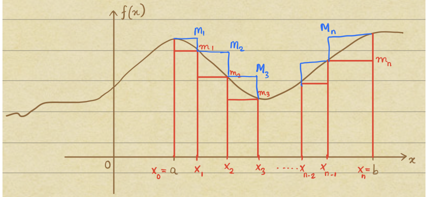
- Create a partition $\mathcal{P}_n = \{x_0, x_1, \dots, x_n\}$, where $x_0 = a$ and $x_n = b$
- Define the lower Riemann sum as the total area under the red rectangles, i.e.,

$$L(f, \mathcal{P}_n) = \sum_{i=1}^{n} m_i \cdot |x_i - x_{i-1}|.$$

- Define the upper Riemann sum as the total area under the blue rectangles, i.e.,

$$U(f, \mathcal{P}_n) = \sum_{i=1}^{n} M_i \cdot |x_i - x_{i-1}|.$$

A Primer on Riemann Integration



- It is easy to see that for all n,

$$L(f, \mathcal{P}_n) \leq L(f, \mathcal{P}_{n+1}), \qquad U(f, \mathcal{P}_n) \geq U(f, \mathcal{P}_{n+1})$$

- If $\lim_{n\to\infty} L(f, \mathcal{P}_n) = \lim_{n\to\infty} U(f, \mathcal{P}_n)$, then this common limit is denoted $\int_a^b f(t)\, dt$

What if $\lim_{n\to\infty} L(f, \mathcal{P}_n) \neq \lim_{n\to\infty} U(f, \mathcal{P}_n)$?

Consider the function

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q}, \\ 0, & x \notin \mathbb{Q}. \end{cases}$$

For the above function:

- $L(f, \mathcal{P}_n) = 0$ for all n
- $U(f, \mathcal{P}_n) = 1$ for all n

Remedy:

A more general theory of integration, proposed by Lebesgue!

Lebesgue's Integration Theory

Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable with respect to $\mathscr{F}$.

Objective

To build the necessary machinery on Lebesgue's theory of integration, so as to be able to interpret an abstract integral of the form

$$\int_A X \, d\mathbb{P}, \qquad A \in \mathscr{F}.$$

Programme:

- Definition of the abstract integral for "simple" random variables
- Definition of the abstract integral for non-negative random variables
- Definition of the abstract integral for arbitrary random variables

Simple Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Let $X : \Omega \to \mathbb{R}$ be a random variable defined with respect to $\mathscr{F}$.

Definition: Simple Random Variable

A random variable X is called a simple random variable if it can be expressed in the form

$$X(\omega) = \sum_{i=1}^{n} a_i \, \mathbf{1}_{A_i}(\omega), \qquad \omega \in \Omega,$$

for some $a_1, \ldots, a_n \geq 0$ and $A_1, \ldots, A_n \in \mathscr{F}$.

Example: Consider $X : \mathbb{R} \to \mathbb{R}$ defined as

$$X(\omega) = \mathbf{1}_{[0,1]}(\omega) + \frac{3}{2} \, \mathbf{1}_{[1,3]}(\omega), \qquad \omega \in \Omega.$$

Here, X can also be represented as $X(\omega) = \mathbf{1}_{[0,3]}(\omega) + \frac{1}{2}\mathbf{1}_{[1,3]}(\omega)$.

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Canonical Representation of a Simple Random Variable

Definition (Canonical Representation of a Simple Random Variable)

A simple random variable X is said to be in canonical representation if

$$X(\omega) = \sum_{i=1}^{n} a_i \, \mathbf{1}_{A_i}(\omega), \qquad \omega \in \Omega,$$

where $a_1, \ldots, a_n \geq 0$ are distinct, and $A_1, \ldots, A_n \in \mathscr{F}$ are disjoint.

Integral of a Simple Random Variable

Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be a random variable with respect to $\mathscr{F}$.

For a simple random variable X in its canonical form

$$X(\omega) = \sum_{i=1}^{n} a_i \, \mathbf{1}_{A_i}(\omega), \qquad \omega \in \Omega,$$

we define $\int_\Omega X \, d\mathbb{P}$ as

$$\int_\Omega X \, d\mathbb{P} := \sum_{i=1}^{n} a_i \, \mathbb{P}(A_i).$$

The quantity $\int_\Omega X \, d\mathbb{P}$ is called the expectation of X under the probability measure $\mathbb{P}$.
Expectation of X is more commonly denoted as $\mathbb{E}[X]$.

Example: The Dirichlet's Function

Let $(\Omega, \mathscr{F}, \mathbb{P}) = ([0,1], \mathscr{B}([0,1]), \lambda)$, λ: Lebesgue measure

Let

$$X(\omega) = \mathbf{1}_{\mathbb{Q}\cap[0,1]}(\omega), \qquad \omega \in \Omega.$$

Then,

$$\int_\Omega X\, d\lambda = 1 \cdot \lambda(\mathbb{Q} \cap [0,1]) = 0.$$

Note: The above function is not Riemann integrable.

Non-Negative Random Variables

Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be any random variable with respect to $\mathscr{F}$ such that

$$X(\omega) \geq 0 \qquad \forall \omega \in \Omega.$$

Let

$$\mathcal{S}(X) := \left\{ q : \Omega \to \mathbb{R} : q \text{ simple} \,,\; q(\omega) \leq X(\omega) \;\; \forall \omega \in \Omega \right\}.$$

Then, the expectation of the non-negative random variable X under $\mathbb{P}$ is defined as

$$\mathbb{E}[X] = \int_\Omega X \, d\mathbb{P} := \sup_{q \in \mathcal{S}(X)} \int_\Omega q \, d\mathbb{P}.$$

Remark: It is possible that $\mathbb{E}[X] = +\infty$.

Expectations and Non-Negative Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

1. For any $A \in \mathscr{F}$, we have
$$\mathbb{E}[\mathbf{1}_A] = \mathbb{P}(A).$$

Expectations and Non-Negative Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

1. For any $A \in \mathscr{F}$, we have
$$\mathbb{E}[\mathbf{1}_A] = \mathbb{P}(A).$$
2. If $X(\omega) \geq 0$ for all $\omega \in \Omega$, then
$$\mathbb{E}[X] \geq 0.$$

Expectations and Non-Negative Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

1. For any $A \in \mathscr{F}$, we have
$$\mathbb{E}[\mathbf{1}_A] = \mathbb{P}(A).$$
2. If $X(\omega) \geq 0$ for all $\omega \in \Omega$, then
$$\mathbb{E}[X] \geq 0.$$
3. If $X(\omega) \geq 0$ for all $\omega \in \Omega$, and $\mathbb{P}(\{X = 0\}) = 1$, then
$$\mathbb{E}[X] = 0.$$

Arbitrary Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be any random variable with respect to $\mathscr{F}$.

Define

$$X_+(\omega) := \max\{X(\omega), 0\}, \quad \omega \in \Omega, \qquad X_-(\omega) := -\min\{X(\omega), 0\}, \quad \omega \in \Omega.$$

We define the expectation of X under $\mathbb{P}$ as

$$\mathbb{E}[X] = \int_\Omega X \, d\mathbb{P} := \mathbb{E}[X_+] - \mathbb{E}[X_-],$$

provided $\min\{\mathbb{E}[X_+], \ \mathbb{E}[X_-]\} < +\infty$.

The Abstract Integral

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Let $X : \Omega \to \mathbb{R}$ be any random variable with respect to $\mathscr{F}$.

For any event $A \in \mathscr{F}$, we define the abstract integral $\int_A X \, d\mathbb{P}$ as

$$\int_A X \, d\mathbb{P} = \int_\Omega (X \cdot \mathbf{1}_A) \, d\mathbb{P},$$

provided the right-hand side is well-defined (i.e., not of the form $\infty - \infty$).

Properties of Expectations

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

1. If $A \in \mathscr{F}$ such that $\mathbb{P}(A) = 0$, then for any random variable X, we have
$$\mathbb{E}[X \cdot \mathbf{1}_A] = 0.$$
2. If $A \in \mathscr{F}$ such that $\mathbb{P}(A) = 1$, then for any random variable X, we have
$$\mathbb{E}[X \cdot \mathbf{1}_A] = \mathbb{E}[X].$$
3. If $\mathbb{P}(\{X \geq Y \geq 0\}) = 1$, then
$$\mathbb{E}[X] \geq \mathbb{E}[Y].$$
4. If $\mathbb{P}(\{X = Y\}) = 1$, then
$$\mathbb{E}[X] = \mathbb{E}[Y].$$
5. If $\mathbb{P}(\{X \geq 0\}) = 1$, and $\mathbb{E}[X] = 0$, then
$$\mathbb{P}(\{X = 0\}) = 1.$$