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Probability and Stochastic Processes

Discrete Random Variables, Continuous Random Variables, Multiple Random Variables

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Recap: Random Variable and CDF

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Random Variable

$$
\begin{tikzcd}
(\Omega, \mathscr{F}) \arrow[r, "X"] & (\mathbb{R}, \mathscr{B}(\mathbb{R})) \arrow[d] \\
X^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}
$$

$$\forall B \in \mathscr{B}(\mathbb{R}), \quad X^{-1}(B) \in \mathscr{F}$$

Figure: A pictorial representation of the definition of a random variable X

Equivalent Definition of Random Variable

$$
\begin{tikzcd}
(\Omega, \mathscr{F}) \arrow[r, "X"] & (\mathbb{R}, \mathscr{B}(\mathbb{R})) \arrow[d] \\
X^{-1}((-\infty, x]) \arrow[u] & (-\infty, x] \arrow[l, "\text{pre-image}"']
\end{tikzcd}
$$

$$\forall x \in \mathbb{R}, \quad X^{-1}((-\infty, x]) \in \mathscr{F}$$

Figure: Simplified, yet equivalent, definition of random variable

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Probability Law

$$
\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[rr, "X"] & & (\mathbb{R}, \mathscr{B}(\mathbb{R}), {\color{red}\mathbb{P}_X}) \arrow[d] \\
X^{-1}(B) \arrow[u] & & B \arrow[ll, "\text{pre-image}"']
\end{tikzcd}
$$

$$
{\color{red}\mathbb{P}_X}(B) = \mathbb{P} \circ X^{-1}(B) = \mathbb{P}(X^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R})
$$

Figure: Pictorial representation of probability law

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Cumulative Distribution Function (CDF)

$$
\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "X"] & (\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P}_X) \arrow[d] \\
X^{-1}((-\infty, x]) \arrow[u] & (-\infty, x] \arrow[l, "\text{pre-image}"']
\end{tikzcd}
$$

$$F_X(x) = \mathbb{P}_X((-\infty, x]) = \mathbb{P}(X^{-1}((-\infty, x])), \quad x \in \mathbb{R}$$

CDF $\longleftrightarrow$ Probability Law

- If we know $\mathbb{P}_X = \{\mathbb{P}_X(B) : B \in \mathscr{B}(\mathbb{R})\}$, then we can extract the CDF $F_X : \mathbb{R} \to [0, 1]$ by using the formula

$$F_X(x) = \mathbb{P}_X((-\infty, x]), \qquad x \in \mathbb{R}.$$

- Given the CDF $F_X : \mathbb{R} \to [0, 1]$, let

$$\mathbb{P}_X((-\infty, x]) = F_X(x), \qquad x \in \mathbb{R}.$$

Then, by Caratheodory's extension theorem, $\mathbb{P}_X$ can be extended to all $B \in \mathbb{R}$

Properties of CDF

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$

Let $X : \Omega \to \mathbb{R}$ be a random variable with respect to $\mathscr{F}$ with CDF F_X

- $\lim_{x \to -\infty} F_X(x) = 0, \qquad \lim_{x \to +\infty} F_X(x) = 1$

- (Monotonicity) If $x \leq y$, then $F_X(x) \leq F_X(y)$

- (Right-Continuity) F_X is right-continuous, i.e., for all $x \in \mathbb{R}$,

$$\lim_{\varepsilon \downarrow 0} F_X(x + \varepsilon) = F_X(x).$$

Continuity of CDF $\longleftrightarrow$ Zero Mass

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$

Let $X : \Omega \to \mathbb{R}$ be a random variable with respect to $\mathscr{F}$ with CDF F_X

- For any $x \in \mathbb{R}$,

$$\lim_{\varepsilon \downarrow 0} F_X(x - \varepsilon) = \mathbb{P}(\{X < x\}).$$

- F_X is continuous at a point $x \in \mathbb{R}$ if and only if $\mathbb{P}(\{X = x\}) = 0$

Properties of CDF

Lemma

Suppose that $F : \mathbb{R} \to [0, 1]$ satisfies

- $\lim_{x\to-\infty} F(x) = 0, \qquad \lim_{x\to+\infty} F(x) = 1.$
- $x \le y \implies F(x) \le F(y).$
- $\lim_{\varepsilon\downarrow 0} F(x+\varepsilon) = F(x)$ for all $x \in \mathbb{R}$.

Then, there exists a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and an $\mathscr{F}$-measurable random variable $X : \Omega \to \mathbb{R}$ such that $F = F_X$.

Set $\quad \Omega = \mathbb{R}, \quad \mathscr{F} = \mathscr{B}(\mathbb{R}), \quad X(\omega) = \omega, \quad \mathbb{P}_X((-\infty, x]) = F(x), \quad \mathbb{P} = \mathbb{P}_X$

Starting with a function $F : \mathbb{R} \to [0, 1]$ satisfying the above properties, one can generate any probability measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$

Discrete Random Variables

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Discrete Random Variables

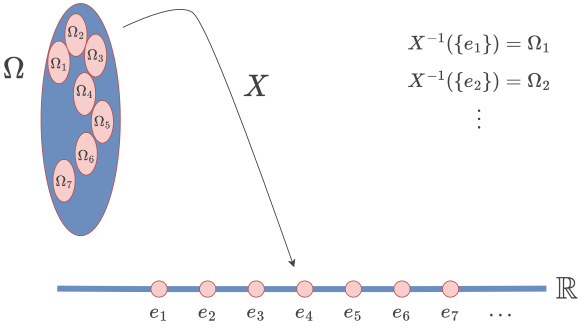
$$
\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "X"] & (\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P}_X) \arrow[d] \\
X^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}
$$

Definition (Discrete Random Variable)

A random variable $X : \Omega \to \mathbb{R}$ is said to be discrete if there exists a countable set $E \subset \mathbb{R}$, say $E = \{e_1, e_2, \ldots\}$, such that $\mathbb{P}_X(E) = 1$.

$$1 = \mathbb{P}_X(E) = \sum_{i=1}^{\infty} \mathbb{P}_X(\{e_i\}) = \sum_{i=1}^{\infty} \mathbb{P}(\{X = e_i\}); \qquad \mathbb{P}_X(B) = \sum_{i:\ e_i \in B} \mathbb{P}_X(\{e_i\}).$$

Discrete Random Variable



$$X^{-1}(\{e_1\}) = \Omega_1$$
$$X^{-1}(\{e_2\}) = \Omega_2$$
$$\vdots$$

$$E = \{e_1, e_2, \dots\}$$

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} \Omega_i\right) = \mathbb{P}_X(E) = 1$$

Probability Mass Function (PMF)

Definition (Probability Mass Function)

Given a random variable $X : \Omega \to \mathbb{R}$, the function $p_X : \mathbb{R} \to [0, 1]$ defined as

$$p_X(x) = \mathbb{P}_X(\{x\}) = \mathbb{P}(\{X = x\}), \qquad x \in \mathbb{R},$$

is called the probability mass function (PMF) of X.

Remark

For a discrete random variable X taking values in a countable set $E = \{e_1, e_2, \ldots\}$, the PMF p_X gives the full probabilistic description of X (i.e., $p_X \longleftrightarrow F_X$), and

$$\sum_{i=1}^{\infty} p_X(e_i) = 1.$$

CDF in Terms of PMF

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Probability Mass Function)

Let $X : \Omega \to \mathbb{R}$ be a discrete random variable taking values in a countable set $E = \{e_1, e_2, \ldots\} \subset \mathbb{R}$. Then,

$$F_X(x) = \sum_{i:\ e_i \leq x} \mathbb{P}(\{X = e_i\}) = \sum_{i:\ e_i \leq x} p_X(e_i), \qquad x \in \mathbb{R}.$$

Examples of Discrete Random Variables

Definition (Discrete Random Variable)

A random variable $X : \Omega \to \mathbb{R}$ is said to be discrete if there exists a countable set $E \subset \mathbb{R}$, say $E = \{e_1, e_2, \ldots\}$, such that $\mathbb{P}_X(E) = 1$.

- $X \sim \text{Bernoulli}(p), \quad p \in [0, 1]$

$$E = \{0, 1\}, \qquad p_X(x) = \begin{cases} p, & x = 1, \\ 1 - p, & x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

- $X \sim \text{unif}(\{1, \ldots, n\})$ for some fixed $n \in \mathbb{N}$

$$E = \{1, \ldots, n\}, \quad p_X(x) = \begin{cases} \frac{1}{n}, & x \in \{1, \ldots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Examples of Discrete Random Variables

Definition (Discrete Random Variable)

A random variable $X : \Omega \to \mathbb{R}$ is said to be discrete if there exists a countable set $E \subset \mathbb{R}$, say $E = \{e_1, e_2, \ldots\}$, such that $\mathbb{P}_X(E) = 1$.

- $X \sim \text{Geometric}(p), \quad p \in (0, 1]$

$$E = \mathbb{N}, \qquad p_X(x) = \begin{cases} p(1-p)^{x-1}, & x \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

- $X \sim \text{Binomial}(n, p)$ for some fixed $n \in \mathbb{N} \cup \{0\}$ and $p \in [0, 1]$

$$E = \{0, \ldots, n\}, \quad p_X(x) = \begin{cases} \binom{n}{x} p^x (1-p)^{n-x}, & x \in \{0, \ldots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

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Examples of Discrete Random Variables

Definition (Discrete Random Variable)

A random variable $X : \Omega \to \mathbb{R}$ is said to be discrete if there exists a countable set $E \subset \mathbb{R}$, say $E = \{e_1, e_2, \ldots\}$, such that $\mathbb{P}_X(E) = 1$.

- $X \sim \text{Poisson}(\lambda), \quad \lambda > 0$

  $$E = \{0, 1, 2, \ldots\}, \qquad p_X(x) = \begin{cases} e^{-\lambda} \frac{\lambda^x}{x!}, & x \in \{0, 1, 2, \ldots\}, \\ 0, & \text{otherwise.} \end{cases}$$

- $$E = \{1, 2, \ldots\}, \qquad p_X(x) = \begin{cases} \frac{6}{\pi^2} \frac{1}{x^2}, & x \in \{1, 2, \ldots\}, \\ 0, & \text{otherwise.} \end{cases}$$

Continuous Random Variables

Continuous Random Variables

$$
\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "X"] & (\mathbb{R}, \mathscr{B}(\mathbb{R}), \mathbb{P}_X) \arrow[d] \\
X^{-1}((-\infty, x]) \arrow[u] & (-\infty, x] \arrow[l, "\text{pre-image}"']
\end{tikzcd}
$$

Definition (Continuous Random Variable)

A random variable $X : \Omega \to \mathbb{R}$ is said to be continuous if there exists a function $f_X : \mathbb{R} \to [0, \infty)$ such that

$$F_X(x) = \mathbb{P}_X((-\infty, x]) = \int_{-\infty}^{x} f_X(t)\, dt, \qquad \forall x \in \mathbb{R}.$$

Continuous Random Variables

Definition (Continuous Random Variable)

A random variable $X : \Omega \to \mathbb{R}$ is said to be continuous if there exists a function $f_X : \mathbb{R} \to [0, \infty)$ such that

$$F_X(x) = \mathbb{P}_X((-\infty, x]) = \int_{-\infty}^{x} f_X(t)\, dt, \qquad \forall x \in \mathbb{R}.$$

Remarks:

- If $X : \Omega \to \mathbb{R}$ is a continuous random variable, its CDF F_X is absolutely continuous (hence continuous)
- The function f_X in the definition is called the probability density function (PDF) of the random variable X
- For a continuous random variable X, its PDF f_X provides the full probabilistic description of X (i.e., $f_X \longleftrightarrow F_X$)

Examples

- $X \sim \text{Uniform}([a,b])$, $$f_X(x) = \begin{cases} \frac{1}{b-a}, & x \in [a,b], \\ 0, & \text{otherwise.} \end{cases}$$
- $X \sim \text{Exponential}(\lambda)$ for some fixed $\lambda > 0$, $$f_X(x) = \begin{cases} \lambda\, e^{-\lambda x}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$
- $X \sim \text{Gaussian}(\mu, \sigma^2)$ for some fixed $\mu \in \mathbb{R}$, $\sigma > 0$,

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\,\sigma^2}\right), \quad x \in \mathbb{R}.$$

- $X \sim \text{Normal} = \text{Gaussian}(0,1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), \quad x \in \mathbb{R}.$$

Some Remarks

- PDF $\neq$ probabilities; PDF can take values greater than 1
  integrals of PDF $=$ probability; total area under PDF $= 1$

- If X is a continuous random variable, then $\mathbb{P}_X(\{x\}) = 0$ for all $x \in \mathbb{R}$
  Furthermore, if $E \subset \mathbb{R}$ is any countable set, then $\mathbb{P}_X(E) = 0$
  Contrast this with discrete random variable!

Multiple Random Variables

Two Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Two Random Variables)

Given two $\mathscr{F}$-measurable random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$, we say $(X, Y) : \Omega \to \mathbb{R}^2$ is a random variable with respect to $\mathscr{F}$ if

$$(X, Y)^{-1}(B) = \{\omega \in \Omega : \big(X(\omega), Y(\omega)\big) \in B\} \in \mathscr{F} \qquad \forall B \in \mathscr{B}(\mathbb{R}^2).$$

$$\begin{tikzcd}
(\Omega, \mathscr{F}) \arrow[rr, "{(X,Y)}"] & & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2)) \arrow[d] \\
(X,Y)^{-1}(B) \arrow[u] & & B \arrow[ll, "\text{pre-image}"']
\end{tikzcd}$$

$$\forall B \in \mathscr{B}(\mathbb{R}^2), \quad (X,Y)^{-1}(B) \in \mathscr{F}$$

Joint Probability Law of Two Random Variables

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Definition (Joint Probability Law of Two Random Variables)

Given two random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ defined with respect to $\mathscr{F}$, their joint probability law $\mathbb{P}_{X,Y} : \mathscr{B}(\mathbb{R}^2) \to [0, 1]$, is the probability measure defined as

$$\mathbb{P}_{X,Y}(B) = \mathbb{P}(\{\omega \in \Omega : \big(X(\omega), Y(\omega)\big) \in B\}), \qquad B \in \mathscr{B}(\mathbb{R}^2).$$

Remarks:

- $\mathbb{P}_{X,Y}$ is called the pushforward of $\mathbb{P}$ under the random variable (X, Y)
- $\mathbb{P}_{X,Y}$ is the probability law of the random variable (X, Y)
- $\mathbb{P}_{X,Y}$ gives the full probabilistic description of (X, Y)

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The Picture to Have in Mind

$$
\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "{(X,Y)}"] & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2), \color{red}{\mathbb{P}_{X,Y}}) \arrow[d] \\
(X,Y)^{-1}(B) \arrow[u] & B \arrow[l, "\text{pre-image}"']
\end{tikzcd}
$$

$$\color{red}{\mathbb{P}_{X,Y}}(B) = \mathbb{P}((X,Y)^{-1}(B)) \quad \forall B \in \mathscr{B}(\mathbb{R}^2)$$

Remarks

- A special class of sets in $\mathscr{B}(\mathbb{R}^2)$ are semi-infinite rectangles of the form

$$(-\infty, x] \times (-\infty, y], \qquad x, y \in \mathbb{R}.$$

- $\mathscr{B}(\mathbb{R}^2) = \sigma\big(\{(-\infty, x] \times (-\infty, y] : \ x, y \in \mathbb{R}\}\big)$

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Joint CDF of Two Random Variables

$$
\begin{tikzcd}
(\Omega, \mathscr{F}, \mathbb{P}) \arrow[r, "{(X,Y)}"] & (\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2), \mathbb{P}_{X,Y}) \arrow[d] \\
\{X \le x\} \cap \{Y \le y\} \arrow[u] & (-\infty, x] \times (-\infty, y] \arrow[l, "\text{pre-image}"']
\end{tikzcd}
$$

$$F_{X,Y}(x, y) = \mathbb{P}_{X,Y}((-\infty, x] \times (-\infty, y]) = \mathbb{P}(\{X \le x\} \cap \{Y \le y\}), \quad x, y \in \mathbb{R}$$

Definition (Joint CDF)

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

Given random variables $X : \Omega \to \mathbb{R}$ and $Y : \Omega \to \mathbb{R}$ with respect to $\mathscr{F}$, their joint CDF $F_{X,Y} : \mathbb{R}^2 \to [0, 1]$ is defined as

$$F_{X,Y}(x, y) = \mathbb{P}_{X,Y}\big((-\infty, x] \times (-\infty, y]\big) = \mathbb{P}(\{X \le x\} \cap \{Y \le y\}), \qquad x, y \in \mathbb{R}.$$

Notation

- $\{X \le x\} \cap \{Y \le y\} = \{X \le x,\ Y \le y\}$
- $\mathbb{P}(\{X \le x\} \cap \{Y \le y\}) = \mathbb{P}(X \le x, Y \le y)$

Joint CDF $\longleftrightarrow$ Joint Probability Law

- If we know $\mathbb{P}_{X,Y} = \{\mathbb{P}_{X,Y}(B) : B \in \mathscr{B}(\mathbb{R}^2)\}$, then we can extract the CDF $F_{X,Y} : \mathbb{R}^2 \to [0, 1]$ by using the formula

$$F_{X,Y}(x, y) = \mathbb{P}_{X,Y}\big((-\infty, x] \times (-\infty, y]\big), \qquad x, y \in \mathbb{R}.$$

- Given the joint CDF $F_{X,Y} : \mathbb{R}^2 \to [0, 1]$, let

$$\mathbb{P}_{X,Y}\big((-\infty, x] \times (-\infty, y]\big) = F_{X,Y}(x, y), \qquad x, y \in \mathbb{R}.$$

  Then, by Caratheodory's extension theorem, there exists a unique extension of $\mathbb{P}_{X,Y}$ to all Borel subsets of $\mathbb{R}^2$