Name:
Roll Number:
Department:
Program: BTech / MTech TA / MTech RA / PhD (Tick one)

AI5030: Probability and Stochastic Processes

Quiz 5

Date: 14 November 2024

Question	1(a)	1(b)	2	Total
Marks Scored				

Instructions:

- **Fill in your name and roll number on each of the pages.**
- **You may use any result covered in class directly without proving it.**
- **Unless explicitly stated in the question, DO NOT use any result from the homework without proof.**

Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$.
Assume that all random variables appearing in the questions below are defined with respect to $\mathscr{F}$.

1. Let X and Y be jointly continuous, with

$$f_{X,Y}(x,y) = \begin{cases} cy, & -1 \leq x \leq 1, \ 0 \leq y \leq |x|, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) **(1 Mark)**
   Determine the constant c.

   (b) **(2 Marks)**
   Compute $\mathsf{Cov}(X,Y)$.

Solution: We present the solution to each part below.

(a) We have

$$1 = \int_{-1}^{1} \int_{0}^{|x|} cy \, \mathsf{d}y \, \mathsf{d}x = \int_{-1}^{1} c \, \frac{x^2}{2} \, \mathsf{d}x = \frac{c}{3},$$

from which it follows that $c = 3$.

(b) For any $x \in [-1, 1]$, we have

$$f_X(x) = \int_0^{|x|} 3y \, \mathsf{d}y = \frac{3x^2}{2}.$$

Noting that $f_X(x) = f_X(-x)$ for all $x \in [-1,1]$, we get that $\mathbb{E}[X] = 0$. Furthermore,

$$\mathbb{E}[XY] = \int_{-1}^{1} \int_{0}^{|x|} 3xy^2 \, \mathsf{d}y \, \mathsf{d}x = \int_{-1}^{1} x\,|x|^3 \, \mathsf{d}x = 0,$$

where the last equality follows by noting that $x \mapsto x\,|x|^3$ is an odd function.
Thus, it follows that

$$\mathsf{Cov}(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y] = 0.$$

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2. **(2 Marks)**

   Let $X_1, X_2, \ldots \overset{\text{i.i.d.}}{\sim}$ Exponential(1), and let $N \sim$ Geometric$(1/2)$ be independent of $\{X_1, X_2, \ldots\}$. Let $S_N = \sum_{i=1}^{N} X_i$. Compute $\mathbb{E}[S_N \, \mathbf{1}_{\{N=4\}}]$. Justify your steps clearly.

   Solution: We have

   $$\mathbb{E}[S_N \, \mathbf{1}_{\{N=4\}}] = \mathbb{E}[S_4 \, \mathbf{1}_{\{N=4\}}] \overset{(a)}{=} \mathbb{E}[S_4] \cdot \mathbb{E}[\mathbf{1}_{\{N=4\}}] = 4 \cdot \mathbb{P}(\{N=4\}) = \frac{4}{2^4} = \frac{1}{4},$$

   where (a) follows from the fact that $\mathbf{1}_{\{N=4\}}$ is independent of S_4, a consequence of the independence of N and $\{X_1, X_2, \ldots\}$.