Name:
Roll Number:
Department:
Program: BTech / MTech TA / MTech RA / PhD (Tick one)

AI5030: Probability and Stochastic Processes

Quiz 1

Date: 14 August 2024

Question	1	2	Total
Marks Scored			

Instructions:

- **Fill in your name and roll number on each of the pages.**
- **You may use any result covered in class directly without proving it.**
- **Unless explicitly stated in the question, DO NOT use any result from the homework without proof.**

1. **(2 Marks)**

Show that the collection of all finite-length strings of natural numbers is countably infinite.
In your proof, if needed, you may use the fact that $\mathbb{N}^d$ is countably infinite for every $d \in \mathbb{N}$.

Solution:

For any $d \in \mathbb{N}$, let C_d denote the collection of all strings of natural numbers of length d; we note here that a string of natural numbers of length d may be expressed as $a_1 a_2 \ldots a_d$, where $a_i \in \mathbb{N}$ for each $i \in \{1, 2, \ldots, d\}$. It is then easy to see that C_d is equicardinal with $\mathbb{N}^d$ for each $d \in \mathbb{N}$. Indeed, for any $d \in \mathbb{N}$ and $a_1, a_2, \ldots, a_d \in \mathbb{N}$, the mapping $a_1 a_2 \ldots a_d \mapsto (a_1, a_2, \ldots, a_d)$ is bijective. Using the fact that $\mathbb{N}^d$ is countable for all d, it follows that C_d is countable for each $d \in \mathbb{N}$. Finally, the collection of all finite-length strings of natural numbers may be expressed as $\bigcup_{d=1}^{\infty} C_d$, and using the facts that countable union of countable sets is countable, we arrive at the desired result.

Name:
Roll Number:
Department:
Program: BTech / MTech TA / MTech RA / PhD (Tick one)

2. **(3 Marks)**

   Let Ω be a sample space (finite, countably infinite, or uncountable).
   Fix $n \in \mathbb{N}$. Let $A_1, \ldots, A_n$ be a *partition* of Ω, i.e., $A_1, \ldots, A_n$ satisfy the following properties:

   - $A_i \subseteq \Omega$ for each $i \in \{1, \ldots, n\}$.
   - $A_i \cap A_j = \emptyset$ for all $i \neq j$.
   - $\bigcup_{i=1}^n A_i = \Omega$.

   Let $\mathscr{C} = \{A_1, \ldots, A_n\}$, and let $\mathscr{F}$ denote the smallest σ-algebra constructed from the sets in $\mathscr{C}$. Show that $|\mathscr{F}| = 2^n$.

   Solution:

   Because the sets in $\mathscr{C}$ constitute a partition of Ω, it follows that $\Omega \setminus A_i = \bigcup_{j \neq i} A_j$ for all $i \in \{1, \ldots, n\}$. As a consequence, to construct the smallest σ-algebra from the sets in $\mathscr{C}$, we may first add the empty set, following which we may add each of the sets in $\mathscr{C}$, following which we may add the union of sets from $\mathscr{C}$ chosen two at a time, following which we may add the union of sets from $\mathscr{C}$ chosen three at a time, and so on. Because there are $\binom{n}{k}$ ways of choosing k sets from $\mathscr{C}$, for $k = 0, \ldots, n$ (here, $k = 0$ corresponds to the empty set, whereas $k = n$ corresponds to Ω), we get that

   $$|\mathscr{F}| = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = 2^n;$$

   the second equality above follows from the binomial theorem.