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Probability and Stochastic Processes

Lecture 02: Sample Space, Algebra, σ-Algebra, Measure

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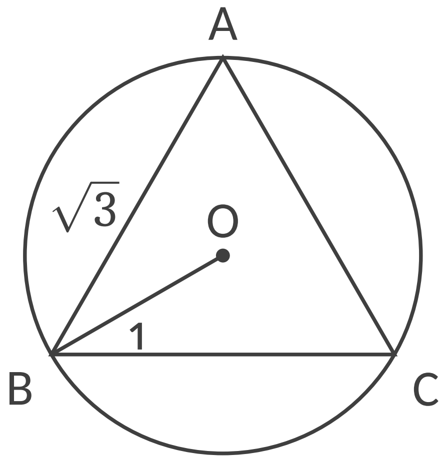
Probability Theory – Humble Beginnings

- Bernoulli (1713) and de Moivre (1718) gave the first definition of probability:

$$\text{probability of an event} = \frac{\#\text{ favourable outcomes}}{\text{total number of outcomes}}.$$

- Cournot (1843):
  "An event with very small probability is morally impossible; an event with very high probability is morally certain."
- French mathematicians of the day were satisfied with the "frequentist" approach to probability, but not the German and English mathematicians of the day
- Frequentist approach could not satisfactorily explain certain paradoxes

Why Axiomatic Theory? Bertrand's Paradox



Take a circle with unit radius and inscribe an equilateral triangle in it. Draw a random chord. What is the probability that the length of the "random chord" is greater than $\sqrt{3}$?

Bertrand's perfectly valid arguments:

- Mid-point of chord should lie inside incircle of radius $1/2$

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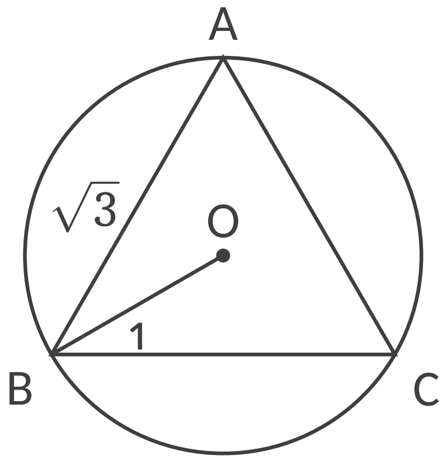

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  Answer: $1/4$

- Angle between chord and tangent at A should be between $\pi/3$ and $2\pi/3$

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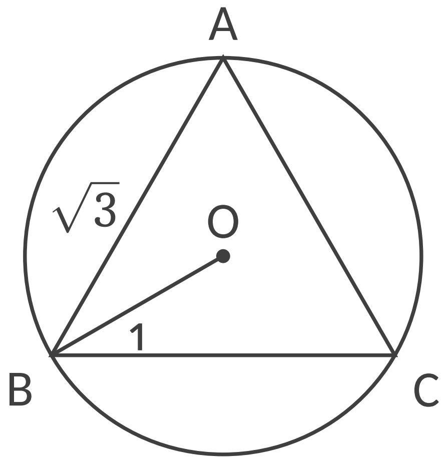

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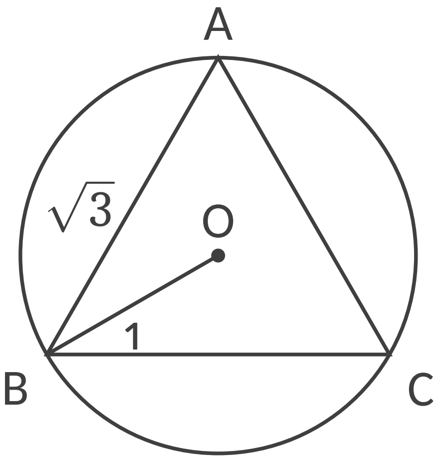

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  Answer: $1/2$

Borel to the Rescue

- Contributions to Measure Theory by Borel (1894) provided a shift in perspective
- Countable unions played a key role in Borel's theory
- Kolmogorov's genius was in applying Borel's theory to formalise the axioms of probability, laying the foundation stone for modern probability theory
- For more details on the history of probability, see [Shafer and Vovk, 2018] and [Kolmogorov, 2004]

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We begin with two universally accepted entities:

- Random experiment
- Outcome (denoted by ω) – source of randomness

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- If our interest is in the velocity with which the coin lands on ground, then $\Omega = [0, \infty) = \mathbb{R}_+$
- If our interest is in the number of times coin flips in air, then $\Omega = \mathbb{N}$

Example: Toss a coin n times, for some $n < \infty$.
Interest: faces that show up

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Event

Informal Definition (Event)

Informally,[a] an event is a subset of outcomes “of interest” to us.

[a] We shall give a more formal definition of an event later.

Example: Toss a coin 3 times; interest is in the faces that show up

Sample space $\Omega =$

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Event A of interest: at least 2 heads show up

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$E = \{HHH, THH, HTH, HHT\} \subset \Omega$

$\{H, T\}^N$ $\{H, T\}^\infty$

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Event A of interest: at least 2 heads show up

$E = \{HHH, THH, HTH, HHT\}$

$\{HHH\}$

Note

If an outcome $\omega \in E$ occurs, we say that the event A occurs.

Algebra

Definition (Algebra)

Let Ω be a sample space.

A collection $\mathscr{A}$ of subsets of Ω is called an algebra if it satisfies the following properties:

1. $\Omega \in \mathscr{A}$.
2. $A \in \mathscr{A} \implies A^c \in \mathscr{A}$ (closure under complements). $\Omega \setminus A$
3. $A, B \in \mathscr{A} \implies A \cup B \in \mathscr{A}$.

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Property 3 above implies, by mathematical induction, that

$$A_1, A_2, \ldots, A_n \in \mathscr{A} \implies \bigcup_{i=1}^{n} A_i \in \mathscr{A} \quad \text{for all } n \in \mathbb{N} \quad \text{(closure under finite unions).}$$

Exercise

Show that an algebra is closed under finite intersections.

Closure under finite intersections:

To show: Given $n \in \mathbb{N}$ and $A_1, \ldots, A_n \in \mathcal{A}$,

$$\bigcap_{i=1}^{n} A_i \in \mathcal{A}$$

$A_1 \cap A_2 \cap \ldots \cap A_n$

$$A_i \in \mathcal{A} \Rightarrow A_i^c \in \mathcal{A} \quad \forall\ i \in \{1, \ldots, n\} \quad (\text{prop. } 2)$$

$$\Rightarrow \underbrace{\bigcup_{i=1}^{n} A_i^c}_{=B} \in \mathcal{A} \quad (\text{finite union closure})$$

$$\Rightarrow \left(\bigcup_{i=1}^{n} A_i^c\right)^c \in \mathcal{A}$$

But note that $\left(\bigcup_{i=1}^{n} A_i^c\right)^c = \bigcap_{i=1}^{n} A_i$ (De Morgan's law)

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σ-Algebra – Motivation

Motivating example: toss a coin until first head shows up

$$\Omega = \{H, TH, TTH, TTTH, \dots\}$$

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Event of interest (A) = # of tosses is even

$A = \{TH, TTTH, \ldots\}$

$\{H, TH\} = \{H\} \cup \{TH\}$

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Remark

Consider an algebra $\mathscr{A}$ of subsets of Ω. Then, $A \notin \mathscr{A}$.

σ-Algebra

Definition (σ-Algebra)

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- $\Omega \in \mathscr{F}$.
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- $A_1, A_2, \ldots \in \mathscr{F} \implies \bigcup_{i \in \mathbb{N}} A_i \in \mathscr{F}$ (closure under countably infinite unions).

$A_1 \cup A_2 \cup A_3 \cup A_4 \cup \cdots$

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- Elements of a σ-algebra are called events

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Remarks:

- Elements of a σ-algebra are called events
- An event $A \in \mathscr{F}$ is also referred to as an $\mathscr{F}$-measurable set

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- An event $A \in \mathscr{F}$ is also referred to as an $\mathscr{F}$-measurable set
- Every σ-algebra is also an algebra, but the converse is not true (see homework)

$\bigcup_{i=1}^{n} A_i = \bigcup_{i\in\mathbb{N}} A_i$, with $A_i = \phi \ \forall \ i \geq n+1$

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- An event $A \in \mathscr{F}$ is also referred to as an $\mathscr{F}$-measurable set
- Every σ-algebra is also an algebra, but the converse is not true (see homework)
- The pair $(\Omega, \mathscr{F})$ is called a measurable space

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σ-Algebra

One can readily verify the following properties of σ-algebras:

- Every σ-algebra is closed under countable intersections
- Let $\mathcal{I}$ be an arbitrary index set, and let $\{\mathscr{F}_i : i \in \mathcal{I}\}$ be a collection of σ-algebras of subsets of a given sample space Ω. Then, $\bigcap_{i\in\mathcal{I}} \mathscr{F}_i$ is a σ-algebra of subsets of Ω.

σ-Algebra – Examples

Fix a sample space Ω.

- The most trivial σ-algebras $\{\emptyset, \Omega\}$ and 2^{Ω}
- For any $A \subset \Omega$, $\{\emptyset, \Omega, A, A^c\}$ is a σ-algebra
- Given a collection $\mathscr{C}$ of subsets of Ω, the smallest σ-algebra containing $\mathscr{C}$ is denoted by $\sigma(\mathscr{C})$.

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$\{\phi\} \subseteq \mathcal{A}$ $\{\phi, \Omega, \{1\}, \{2,3,4,5,6\}\}$

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Example: $\Omega = \{1, 2, \ldots, 6\}$, $\mathscr{C} = \{\{1, 2\}, \{1, 3\}\}$

$\{1\} \in \mathcal{A}$

$\{\{1\}\} \subseteq \mathcal{A}$

$\mathcal{A} = \{\Omega, \phi, \{1,2\}, \{1,3\}, \{3,4,5,6\}, \{2,4,5,6\}, \{1,2,3\}, \{4,5,6\},$
$\{1,3,4,5,6\}, \{2\}, \{1,2,4,5,6\}, \{3\}, \{2,3\}, \{1,4,5,6\},$
$\{2,3,4,5,6\}, \{1\}\}$

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  Example: $\Omega = \{1, 2, \ldots, 6\}, \quad \mathscr{C} = \Big\{\{1, 2\}, \{1, 3\}\Big\}$

$$\sigma(\mathscr{C}) = \Big\{\emptyset, \Omega, \{1, 2\}, \{1, 3\}, \{1\}, \{2, 3, 4, 5, 6\}, \{1, 2, 3\}, \{4, 5, 6\},$$

$$\{3, 4, 5, 6\}, \{2, 4, 5, 6\}, \{2\}, \{3\}, \{1, 3, 4, 5, 6\}, \{1, 2, 4, 5, 6\}, \{2, 3\}, \{1, 4, 5, 6\}\Big\}$$

Measure

Fix a measurable space $(\Omega, \mathscr{F})$.

Definition (Measure)

A function $\mu : \mathscr{F} \to [0, +\infty]$ is called a measure on $(\Omega, \mathscr{F})$ if it satisfies the following properties:

1. $\mu(\emptyset) = 0$.
2. If $A_1, A_2, \ldots$ is a countable collection of disjoint sets, with $A_i \in \mathscr{F}$ for each $i \in \mathbb{N}$, then

$$\mu\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mu(A_i).$$

Property 2 above is called the property of countable additivity.

The triplet $(\Omega, \mathscr{F}, \mu)$ is called a measure space.

$\Omega = \{1,2,3\}$

$\mathcal{F} = \{\phi, \Omega, \{1\}, \{2,3\}\}$

$\phi \to 0$, $\Omega \to 1$, $\{1\} \to 0$, $\{2,3\} \to 1$

Measure

$\mathbb{P} : \mathcal{F} \to [0, 1]$

- When $\mu(\Omega) < +\infty$, the measure μ is called a finite measure
- When $\mu(\Omega) = +\infty$, the measure μ is called an infinite measure
- When $\mu(\Omega) = 1$, the measure μ is called a probability measure, and denoted by $\mathbb{P}$.

$[0,1] \to \{0,1\}^{\mathbb{N}}$

$f : [0,1] \to \mathbb{R}$

Probability Measure

$\Omega = \{0,1\}^{\mathbb{N}} \to$ uncountable

$\mathcal{F}$

Fix a measurable space $(\Omega, \mathscr{F})$.

Definition (Probability Measure)

A function $\mathbb{P} : \mathscr{F} \to [0, 1]$ is called a probability measure if the following properties are satisfied:

1. $\mathbb{P}(\emptyset) = 0.$
2. $\mathbb{P}(\Omega) = 1.$
3. If $A_1, A_2, \ldots$ is a countable collection of disjoint sets, with $A_i \in \mathscr{F}$ for each $i \in \mathbb{N}$, then

$$\mathbb{P}\left(\bigcup_{i\in\mathbb{N}} A_i\right) = \sum_{i\in\mathbb{N}} \mathbb{P}(A_i).$$

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